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THE AFFINE SOBOLEV INEQUALITY

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1. Introduction

The Sobolev inequality is one of the fundamental inequalities connecting analysis and geometry. The literature related to it is vast (see, for example, [1], [5], [7], [3], [6], [11], [12], [19], [22], [20], [21], [23], [25], [27], [28], [37], and [45]). In this paper, a new inequality that is stronger than the Sobolev inequality is presented. A remarkable feature of the new inequality is that it is independent of the norm chosen for the ambient Euclidean space.

The Sobolev inequality in the Euclidean space $\mathbb{R}^n$ states that for any C^1 function $f(x)$ with compact support there is

$$\int_{\mathbb{R}^n} |\nabla f| \, dx \geq n\omega_n^{1/n} \|f\|_{\frac{n}{n-1}}, \tag{1.1}$$

where $|\nabla f|$ is the Euclidean norm of the gradient of f, $\|f\|_p$ is the usual L_p norm of f in $\mathbb{R}^n$, and ω_n is the volume enclosed by the unit sphere S^{n-1} in $\mathbb{R}^n$. The best constant in the inequality is attained at the characteristic functions of balls.

It is known that the sharp Sobolev inequality (1.1) is equivalent to the classical isoperimetric inequality (see, for instance, [2], [8], [13], [14], [33], [35], [40], and [41]). We prove an affine Sobolev inequality which is stronger than (1.1). This inequality is proved by using a generalization of the Petty projection inequality to compact sets that is established in this paper (see [30], [31], [16], [26], [38] and [42] for the classical Petty projection inequality of convex bodies).

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Theorem 1.1. *If f is a C^1 function with compact support in $\mathbb{R}^n$, then*

$$\frac{1}{n}\int_{S^{n-1}} \|\nabla_u f\|_1^{-n}\, du \le c_n \|f\|_{\frac{n}{n-1}}^{-n}, \tag{1.2}$$

where $\nabla_u f$ is the partial derivative of f in direction u, du is the standard surface measure on the unit sphere, and the constant $c_n = \left(\frac{\omega_n}{2\omega_{n-1}}\right)^n$ is best.

The inequality (1.2) is $GL(n)$ invariant while the inequality (1.1) is only $SO(n)$ invariant. Thus, inequality (1.2) does not depend on the Euclidean norm of $\mathbb{R}^n$. The best constant in (1.2) is attained at the characteristic functions of ellipsoids. Applying the Hölder inequality and Fubini's theorem to the left-hand side of (1.2), one can easily see that inequality (1.2) is stronger than the Sobolev inequality (1.1). For radial functions, the inequality (1.2) reduces to (1.1). It is worth noting that the left-hand side of (1.2) is a natural geometric invariant. Specifically, for a C^1 function $f(x)$, there is an important norm of $\mathbb{R}^n$ given by

$$\|u\| = \|\nabla_u f\|_1 = \int_{\mathbb{R}^n} |\langle u, \nabla f(x)\rangle| dx, \quad u \in \mathbb{R}^n,$$

where $\langle\ ,\ \rangle$ is the usual inner product in $\mathbb{R}^n$. The volume of the unit ball of this norm is exactly the left-hand side of (1.2).

We will also prove a generalization of the Gagliardo-Nirenberg inequality.

Theorem 1.2. *Let $\{u_i\}_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$ and let $\{\lambda_i\}_1^m$ be a sequence of positive numbers satisfying*

$$|x|^2 = \sum_{i=1}^m \lambda_i \langle x, u_i\rangle^2, \quad x \in \mathbb{R}^n.$$

If f is a C^1 function with compact support in $\mathbb{R}^n$, then

$$\prod_{i=1}^m \|\nabla_{u_i} f\|_1^{\frac{\lambda_i}{n}} \ge 2\|f\|_{\frac{n}{n-1}}.$$

2. Basics of convex bodies

A convex body is a compact convex set with nonempty interior in $\mathbb{R}^n$.

A convex body K is uniquely determined by its support function defined by

$$h_K(u) = \max_{x \in K} \langle u, x \rangle, \quad u \in S^{n-1}.$$

If K contains the origin in its interior, the polar body K^* of K is given by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

Denote by $V(K)$ the volume of K. The mixed volume $V(K, L)$ of convex bodies K and L is defined by

$$V(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon}.$$

There is a unique finite measure S_K on S^{n-1} so that

$$V(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS_K(u).$$

The measure S_K is called the surface area measure of K. When K has a C^2 boundary ∂K with positive curvature, the Radon-Nykodim derivative of S_K with respect to the Lebesgue measure on S^{n-1} is the reciprocal of the Gauss curvature of ∂K.

An important inequality of mixed volume is the Minkowski inequality,

$$V(K, L)^n \geq V(K)^{n-1} V(L), \tag{2.1}$$

with equality if and only if K and L are homothetic.

For a convex body K, let $K|u^\perp$ be the projection of K onto the 1-codimensional subspace $u^\perp$ orthogonal to u. The projection function $\mathrm{v}(K, u)$ of K is defined by

$$\mathrm{v}(K, u) = \mathrm{vol}_{n-1}(K|u^\perp), \quad u \in S^{n-1}.$$

The projection function $\mathrm{v}(K, u)$ of K defines a new convex body ΠK whose support function is given by

$$h_{\Pi K}(u) = \mathrm{v}(K, u) = \mathrm{vol}_{n-1}(K|u^\perp), \quad u \in S^{n-1}.$$

The convex body ΠK is called the projection body of K. The volume of the polar of the projection body $\Pi^* K$ is given by

$$V(\Pi^* K) = \frac{1}{n} \int_{S^{n-1}} \mathrm{v}(K, u)^{-n} du.$$

The Petty projection inequality is (see [30] and [36])

$$V(K)^{n-1}V(\Pi^* K) \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n, \tag{2.2}$$

with equality if and only if K is an ellipsoid.

It was shown in [32] that the Petty projection inequality is stronger than the classical isoperimetric inequality of convex bodies.

Let $u_1, u_2, \dots, u_n$ be an orthonormal basis of $\mathbb{R}^n$. For a convex body K in $\mathbb{R}^n$, the Loomis-Whitney inequality is (see [29] and [8, p. 95])

$$V(K)^{n-1} \leq \prod_{i=1}^{n} \mathrm{v}(K, u_i).$$

The Loomis-Whitney inequality was generalized by Ball [4]. Let $\{u_i\}_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$, and let $\{c_i\}_1^m$ be a sequence of positive numbers for which

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n,$$

where $u_i \otimes u_i$ is the rank-1 orthogonal projection onto the span of u_i, and I_n is the identity on $\mathbb{R}^n$. Then, for a convex body K in $\mathbb{R}^n$, Ball proved the inequality

$$V(K)^{n-1} \leq \prod_{i=1}^{m} \mathrm{v}(K, u_i)^{c_i}. \tag{2.3}$$

The condition on u_i and c_i is equivalent to

$$|x|^2 = \sum_{i=1}^{m} c_i \langle x, u_i \rangle^2, \quad x \in \mathbb{R}^n.$$

For details of convex bodies, see [16], [38] and [42].

3. Inequalities for compact domains

In this section, we generalize inequalities (2.1)–(2.3) to compact domains. In this paper, a compact domain is the closure of a bounded open set. The generalization of the Minkowski inequality to compact domains can be obtained from the Brunn-Minkowski inequality. This appears to be standard. See [10] and [8].

If M and N are compact domains in $\mathbb{R}^n$, then the Brunn-Minkowski inequality is

$$V(M+N)^{\frac{1}{n}} \geq V(M)^{\frac{1}{n}} + V(N)^{\frac{1}{n}}, \tag{3.1}$$

with equality if and only if M and N are homothetic.

Let M be a compact domain with piecewise C^1 boundary ∂M, and let K be a convex body in $\mathbb{R}^n$. The *mixed volume* of M and K, $V(M,K)$, is defined by

$$V(M,K) = \frac{1}{n}\int_{\partial M} h_K(\nu(x))dS_M(x), \tag{3.2}$$

where dS_M is the surface area element of ∂M, and $\nu(x)$ is the exterior unit normal vector of ∂M at x.

If K is the unit ball B_n in $\mathbb{R}^n$, then $nV(M,B_n)$ is the surface area $S(M)$ of M.

Lemma 3.1. *If M is a compact domain with piecewise C^1 boundary ∂M, and K is a convex body in $\mathbb{R}^n$, then*

$$nV(M,K) = \lim_{\varepsilon\to 0^+} \frac{V(M+\varepsilon K) - V(M)}{\varepsilon}. \tag{3.3}$$

When M is not convex, the limit of the right-hand side of (3.3) may not exist. Equation (3.3) holds when M is a convex body or is a compact domain with piecewise C^1 boundary. We give a proof of Lemma 3.1 in the Appendix.

Lemma 3.2. *If M is a compact domain with piecewise C^1 boundary, and K is a convex body in $\mathbb{R}^n$, then*

$$V(M,K)^n \geq V(M)^{n-1}V(K), \tag{3.4}$$

with equality if and only if M and K are homothetic.

Proof. For $\varepsilon \geq 0$, consider the function

$$f(\varepsilon) = V(M+\varepsilon K)^{\frac{1}{n}} - V(M)^{\frac{1}{n}} - \varepsilon V(K)^{\frac{1}{n}}.$$

From the Brunn-Minkowski inequality (3.1), the function $f(\varepsilon)$ is non-negative and concave. By Lemma 3.1, we have

$$\lim_{\varepsilon \to 0^+} \frac{f(\varepsilon) - f(0)}{\varepsilon} = V(M)^{\frac{1-n}{n}} V(M, K) - V(K)^{\frac{1}{n}} \geq 0.$$

This proves the inequality (3.4). If the equality holds, $f(\varepsilon)$ must be linear, and M and K are homothetic. q.e.d.

Lemma 3.2 was proved in [10] when M has a $C^{1,1}$ boundary.

Let M be a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary ∂M and exterior unit normal vector $\nu(x)$. For any continuous function f on S^{n-1}, define a linear functional μ_M on the space of continuous functions $C(S^{n-1})$ on S^{n-1} by

$$\mu_M(f) = \int_{\partial M} f(\nu(x))\, dS_M(x), \tag{3.5}$$

where dS_M is the surface area element of M. The linear functional μ_M is a non-negative linear functional on $C(S^{n-1})$. Since the sphere is compact, μ_M is a finite measure on S^{n-1}. The measure μ_M is called the *surface area measure* of the compact domain M.

The Minkowski existence theorem states that for every finite non-negative measure μ on S^{n-1} such that

$$\int_{S^{n-1}} u d\mu(u) = 0, \text{ and } \int_{S^{n-1}} |\langle u, v\rangle| d\mu(v) > 0, \quad u \in S^{n-1}, \tag{3.6}$$

there exists a unique convex body K (up to translation) whose surface area measure is μ. See [38], pp. 389-393.

We verify that the surface area measure μ_M of a compact domain M defined above satisfies (3.6). By Green's formula, for any C^1 vector field $\xi(x)$ in $\mathbb{R}^n$, there is

$$\int_{\partial M} \langle \xi(x), \nu(x)\rangle dS_M(x) = \int_M \operatorname{div} \xi(x) dx.$$

Choose $\xi(x) = e_i$, $i = 1, 2, \cdots, n$, the coordinate vectors, then $\operatorname{div} \xi = \operatorname{div} e_i = 0$. Therefore, Green's formula yields

$$\int_{\partial M} \langle e_i, \nu(x)\rangle dS_M(x) = 0.$$

Let $f(u) = \langle e_i, u\rangle$. Then (3.5) gives

$$\int_{S^{n-1}} \langle e_i, u\rangle d\mu_M(u) = \int_{\partial M} \langle e_i, \nu(x)\rangle dS_M(x) = 0.$$

Since M has non-empty interior, one has

$$\int_{\partial M} |\langle u, \nu(x)\rangle| dS_M(x) > 0,$$

that is,

$$\int_{S^{n-1}} |\langle u, v\rangle| d\mu_M(v) > 0.$$

Hence, μ_M satisfies the condition (3.6).

Let M be a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary ∂M. A *convexification* $\breve{M}$ of M is a convex body whose surface area measure $S_{\breve{M}}$ is defined by

$$S_{\breve{M}} = \mu_M. \tag{3.7}$$

Note that a convex body is determined by its surface area measure only up to translation. Therefore, the convexification $\breve{M}$ is unique up to translation.

Let M be a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary ∂M. The *projection function* $\mathrm{v}(M, u)$ of M on S^{n-1} is defined by

$$\begin{aligned} \mathrm{v}(M, u) &= \frac{1}{2}\int_{\partial M} |\langle u, \nu(x)\rangle| dS_M(x) \\ &= \frac{1}{2}\int_{S^{n-1}} |\langle u, v\rangle| d\mu_M(v), \quad u \in S^{n-1}, \end{aligned}$$

where $\nu(x)$ is the exterior unit normal vector of M at x.

The following lemma is obvious.

Lemma 3.3. *If M is a compact domain with piecewise C^1 boundary, and $\breve{M}$ is a convexification of M, then*

$$\begin{aligned} \mathrm{v}(M, u) &= \mathrm{v}(\breve{M}, u), \quad u \in S^{n-1}, \\ V(M, K) &= V(\breve{M}, K), \end{aligned}$$

for any convex body K in $\mathbb{R}^n$.

Lemma 3.4. *If M is a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary, and $\breve{M}$ is its convexification, then*

$$V(\breve{M}) \geq V(M), \tag{3.8}$$

with equality if and only if M is convex.

Proof. From Lemma 3.3, for any convex body K, we have

$$V(M, K) = V(\breve{M}, K).$$

Let $K = \breve{M}$. Then

$$V(M, \breve{M}) = V(\breve{M}),$$

and the generalized Minkowski inequality (3.4) yields

$$V(\breve{M}) = V(M, \breve{M}) \geq V(M)^{\frac{n-1}{n}} V(\breve{M})^{\frac{1}{n}}.$$

This proves (3.8). q.e.d.

The convexification in $\mathbb{R}^n$ was introduced in [9]. Lemma 3.4 for polytopes was proved in [9]; see also [43].

Let M be a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary ∂M. The *projection body* ΠM of M is defined by

$$h_{\Pi M}(u) = \mathrm{v}(M, u), \quad u \in S^{n-1}.$$

It is easily seen that ΠM is an origin-symmetric convex body. Let ℓ_u be a line parallel to the unit vector u, and let $d\ell_u$ be the volume element of the subspace $u^\perp$ orthogonal to u. Then

$$h_{\Pi M}(u) = \frac{1}{2} \int \#(M \cap \ell_u) d\ell_u.$$

This is the projection that counts (geometric) multiplicity. For the projection bodies of more general compact sets, see [39].

The volume of the polar projection body $\Pi^* M$ is

$$V(\Pi^* M) = \frac{1}{n} \int_{S^{n-1}} \mathrm{v}(M, u)^{-n} du.$$

Note that the arithmetic average of the projection function $\mathrm{v}(M, u)$ over S^{n-1} is the surface area of M, up to a constant factor. One can view the $SL(n)$-invariant $V(\Pi^* M)^{-1/n}$ as an affine surface area of M.

Lemma 3.5. *If M is a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary, then*

$$V(M)^{n-1}V(\Pi^* M) \le \left(\frac{\omega_n}{\omega_{n-1}}\right)^n, \tag{3.9}$$

with equality if and only if M is an ellipsoid.

Proof. By (3.8), Lemma 3.3, and the Petty projection inequality (2.2) for convex bodies, we have

$$V(M)^{n-1}V(\Pi^* M) \le V(\breve{M})^{n-1}V(\Pi^* \breve{M}) \le \left(\frac{\omega_n}{\omega_{n-1}}\right)^n,$$

with equalities if and only if M is convex and $\breve{M}$ is an ellipsoid, and hence M is an ellipsoid. q.e.d.

As noted earlier for convex bodies, the generalized Petty projection inequality (3.9) is stronger than the classical isoperimetric inequality for compact domains. From the Hölder inequality, one can easily see

$$S(M) \ge \frac{n\omega_n^{1+\frac{1}{n}}}{\omega_{n-1}} V(\Pi^* M)^{-\frac{1}{n}},$$

which and (3.9) imply the classical isoperimetric inequality

$$S(M) \ge n\omega_n^{1/n} V(M)^{\frac{n-1}{n}}.$$

Lemma 3.6. *Let $\{w_i\}_1^m$ be a sequence of non-zero vectors in $\mathbb{R}^n$ which are not contained in one hyperplane. Then for any compact domain M in $\mathbb{R}^n$*

$$\prod_{i=1}^m \mathrm{v}(M, w_i)^{\lambda_i} \ge c\, V(M)^{n-1}, \tag{3.10}$$

where $\lambda_i = \langle A^{-1}w_i, w_i\rangle$, $c = (\det A)^{\frac{1}{2}} / \prod_{i=1}^m \left(\frac{|w_i|}{\sqrt{\lambda_i}}\right)^{\lambda_i}$, and A is the positive definite matrix given by $\langle Ax, x\rangle = \sum_{i=1}^m \langle x, w_i\rangle^2$.

Proof. First, we show the case that M is a convex body K. Let Q be a non-singular matrix so that $A = Q^T Q$, and let $y = Qx$. Then

$$|y|^2 = \langle Ax, x\rangle = \sum_{i=1}^m \langle w_i, x\rangle^2 = \sum_{i=1}^m \lambda_i \langle u_i, y\rangle^2,$$

where $u_i = \lambda_i^{-\frac{1}{2}} Q^{-T} w_i$.

It can be easily verified that

$$\mathrm{v}(Q^T K, w_i)|w_i| = \det(Q)\mathrm{v}(K, Q^{-T} w_i)|Q^{-T} w_i|.$$

From (2.3), we have

$$\begin{aligned} V(K)^{n-1} &\leq \prod_{i=1}^m \mathrm{v}(K, u_i)^{\lambda_i} = \prod_{i=1}^m \mathrm{v}(K, Q^{-T} w_i)^{\lambda_i} \\ &= \prod_{i=1}^m \mathrm{v}(Q^T K, w_i)^{\lambda_i} \left(\frac{|w_i|}{|Q^{-T} w_i| \det Q} \right)^{\lambda_i} \\ &= \prod_{i=1}^m \mathrm{v}(Q^T K, w_i)^{\lambda_i} \left(\frac{|w_i|}{\sqrt{\lambda_i} \det Q} \right)^{\lambda_i}. \end{aligned}$$

Using the fact that $V(Q^T K) = V(K) \det Q$ and $\sum_{i=1}^m \lambda_i = n$ we obtain the inequality (3.10) for convex M.

When M is not convex, let $\breve{M}$ be a convexification of M. By Lemma 3.3, M and $\breve{M}$ have the same projection function. From (3.8) and the convex case it follows that

$$\begin{aligned} cV(M)^{n-1} \leq cV(\breve{M})^{n-1} &\leq \prod_{i=1}^m \mathrm{v}(\breve{M}, w_i)^{\lambda_i} \\ &= \prod_{i=1}^m \mathrm{v}(M, w_i)^{\lambda_i}. \qquad \text{q.e.d.} \end{aligned}$$

4. The affine Sobolev inequality

In this section , we prove the results stated in the Introduction.

Theorem 4.1. *If f is a C^1 function with compact support in $\mathbb{R}^n$, then*

$$\frac{1}{n} \int_{S^{n-1}} \|\nabla_u f\|_1^{-n}\, du \leq \Big(\frac{\omega_n}{2\omega_{n-1}}\Big)^n \|f\|_{\frac{n}{n-1}}^{-n}. \tag{4.1}$$

Proof. For $t > 0$, consider the level sets of f in $\mathbb{R}^n$,

$$\begin{aligned} M_t &= \{x \in \mathbb{R}^n : |f(x)| > t\}, \\ S_t &= \{x \in \mathbb{R}^n : |f(x)| = t\}. \end{aligned}$$

Since f is of class C^1, for almost all $t > 0$, S_t is a C^1 submanifold which has non-zero normal vector ∇f. Let dS_t be the surface area element of S_t. Then one has the formula of volume elements,

$$dx = |\nabla f|^{-1} dS_t dt. \tag{4.2}$$

We have

$$\begin{aligned} \|\nabla_u f\|_1 &= \int_{\mathbb{R}^n} |\nabla_u f(x)|\, dx \\ &= \int_0^\infty \int_{S_t} |\langle \nabla f, u\rangle| |\nabla f|^{-1} dS_t dt \\ &= 2\int_0^\infty \mathrm{v}(M_t, u) dt. \end{aligned} \tag{4.3}$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx &= \int_{\mathbb{R}^n} \left(\int_0^{|f|} \frac{n}{n-1} t^{\frac{1}{n-1}} dt \right) dx \\ &= \frac{n}{n-1} \int_0^\infty t^{\frac{1}{n-1}} \left(\int_{M_t} dx \right) dt \\ &= \frac{n}{n-1} \int_0^\infty t^{\frac{1}{n-1}} V(M_t) dt. \end{aligned} \tag{4.4}$$

Since $V(M_t)$ is decreasing with respect to t, it follows that

$$\begin{aligned} t^{\frac{1}{n-1}} V(M_t) &= \left(t V(M_t)^{\frac{n-1}{n}} \right)^{\frac{1}{n-1}} V(M_t)^{\frac{n-1}{n}} \\ &\leq \left(\int_0^t V(M_\tau)^{\frac{n-1}{n}} d\tau \right)^{\frac{1}{n-1}} V(M_t)^{\frac{n-1}{n}} \\ &= \frac{n-1}{n} \frac{d}{dt} \left(\int_0^t V(M_\tau)^{\frac{n-1}{n}} d\tau \right)^{\frac{n}{n-1}}, \end{aligned}$$

so that

$$\int_0^\infty t^{\frac{1}{n-1}} V(M_t) dt \leq \frac{n-1}{n} \left(\int_0^\infty V(M_t)^{\frac{n-1}{n}} dt \right)^{\frac{n}{n-1}}. \tag{4.5}$$

Combining (4.4) and (4.5) gives

$$\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx \leq \left(\int_0^\infty V(M_t)^{\frac{n-1}{n}} dt \right)^{\frac{n}{n-1}}. \tag{4.6}$$

By the generalized Petty projection inequality (3.9), we obtain

$$\begin{aligned}V(M_t)^{\frac{n-1}{n}} &\leq \frac{\omega_n}{\omega_{n-1}} V(\Pi^* M_t)^{-\frac{1}{n}} \\ &= \frac{\omega_n}{\omega_{n-1}} \left(\frac{1}{n} \int_{S^{n-1}} \mathrm{v}(M_t, u)^{-n} du \right)^{-\frac{1}{n}}.\end{aligned} \tag{4.7}$$

From (4.6) and (4.7), it follows that

$$\|f\|_{\frac{n}{n-1}} \leq \frac{\omega_n}{\omega_{n-1}} \int_0^\infty \left(\frac{1}{n} \int_{S^{n-1}} \mathrm{v}(M_t, u)^{-n} du \right)^{-\frac{1}{n}} dt.$$

Thus Minkowski's inequality for integrals yields

$$\int_0^\infty \left(\int_{S^{n-1}} \mathrm{v}(M_t, u)^{-n} du \right)^{-\frac{1}{n}} dt \leq \left(\int_{S^{n-1}} \left(\int_0^\infty \mathrm{v}(M_t, u) dt \right)^{-n} du \right)^{-\frac{1}{n}}.$$

By (4.3) and the last two inequalities, we finally obtain

$$\|f\|_{\frac{n}{n-1}} \leq \frac{\omega_n}{2\omega_{n-1}} \left(\frac{1}{n} \int_{S^{n-1}} \|\nabla_u f\|_1^{-n} \, du \right)^{-\frac{1}{n}},$$

which proves the Theorem. q.e.d.

We observe that inequality (4.1) is stronger than the Sobolev inequality (1.1). Indeed, the Hölder inequality and Fubini's theorem together yield

$$\begin{aligned}\left(\frac{1}{n\omega_n} \int_{S^{n-1}} \|\nabla_u f\|_1^{-n} \, du \right)^{-\frac{1}{n}} &\leq \frac{1}{n\omega_n} \int_{S^{n-1}} \|\nabla_u f\|_1 \, du \\ &= \frac{1}{n\omega_n} \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle \nabla f, u \rangle| dx du \\ &= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \int_{S^{n-1}} |\langle \nabla f, u \rangle| du dx \\ &= \frac{2\omega_{n-1}}{n\omega_n} \int_{\mathbb{R}^n} |\nabla f| dx.\end{aligned} \tag{4.8}$$

Thus inequalities (4.1) and (4.8) are combined to give the Sobolev inequality (1.1).

Let us show that the generalized Petty projection inequality (3.9) can be proved by using the inequality (4.1).

For compact domain M and for small $\varepsilon > 0$, we define

$$f_\varepsilon(x) = \begin{cases} 0 & \operatorname{dist}(x, M) \geq \varepsilon, \\ 1 - \frac{\operatorname{dist}(x,M)}{\varepsilon} & \operatorname{dist}(x, M) < \varepsilon. \end{cases}$$

If ε is small and $\operatorname{dist}(x, M) < \varepsilon$, then there exists a unique $x' \in \partial M$ so that

$$\operatorname{dist}(x, M) = |x' - x|.$$

Let

$$\nu(x') = \frac{x' - x}{|x' - x|}.$$

Consider

$$M_\varepsilon = \{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x, M) < \varepsilon\},$$

and its closure $\bar{M}_\varepsilon$. One has

$$\nabla f_\varepsilon(x) = \begin{cases} \varepsilon^{-1}\nu(x') & x \in M_\varepsilon, \\ 0 & x \notin \bar{M}_\varepsilon. \end{cases}$$

It follows that

$$\int_{\mathbb{R}^n} |\langle \nabla f_\varepsilon, u\rangle| \, dx = \varepsilon^{-1} \int_{M_\varepsilon} |\langle \nu(x'), u\rangle| \, dx.$$

Let $t = \operatorname{dist}(x, M)$, $0 < t < \varepsilon$. Then $dx = dS_M dt + o(\Delta t)$. Therefore, as $\varepsilon \to 0$, we have

$$\varepsilon^{-1} \int_{M_\varepsilon} |\langle \nu(x'), u\rangle| \, dx \longrightarrow \int_{\partial M} |\langle \nu(x'), u\rangle| \, dS_M(x') = 2\mathrm{v}(M, u).$$

On the other hand, since f_ε converges to the characteristic function χ_M of M, we have

$$\int_{\mathbb{R}^n} |f_\varepsilon|^{\frac{n}{n+1}} dx \longrightarrow V(M).$$

It follows that (4.1) implies (3.9).

We have seen that the affine Sobolev inequality (4.1) is equivalent to the generalized Petty projection inequality (3.9). The constant in the inequality (4.1) is sharp. It is attained at the characteristic functions of ellipsoids.

5. A generalization of the Gagliardo-Nirenberg inequality

Let f be a C^1 function with compact support in $\mathbb{R}^n$. Gagliardo [15] and Nirenberg [34] proved the inequality

$$\prod_{i=1}^{n} \|\nabla_{x_i} f\|_1^{\frac{1}{n}} \geq 2\|f\|_{\frac{n}{n-1}}. \tag{5.1}$$

This inequality implies the Sobolev embedding theorem. See [1, p. 38]. We give a generalization of inequality (5.1) which is equivalent to the inequality (3.10) for compact domains.

Theorem 5.1. *Let $\{w_i\}_1^m$ be a sequence of vectors in $\mathbb{R}^n$ not contained in one hyperplane. If $f(x)$ is a C^1 function with compact support in $\mathbb{R}^n$, then*

$$\prod_{i=1}^{m} \|\nabla_{w_i} f\|_1^{\frac{\lambda_i}{n}} \geq c\,\|f\|_{\frac{n}{n-1}}, \tag{5.2}$$

where the constants $\lambda_i = \langle A^{-1}w_i, w_i\rangle$ and $c = 2\left(\det A \prod_{i=1}^m \lambda_i^{\lambda_i}\right)^{\frac{1}{2n}}$ depend only on the sequence of vectors, and A is the positive definite matrix given by $\langle Ax, x\rangle = \sum_{i=1}^m \langle x, w_i\rangle^2$.

Proof. We use the notation in Theorem 4.1. From (4.6), we get

$$\|f\|_{\frac{n}{n-1}} \leq \int_0^\infty V(M_t)^{\frac{n-1}{n}}\,dt.$$

Using (3.10), $\sum_{i=1}^m \lambda_i = n$, and the Hölder inequality, we have

$$\begin{aligned} c^{\frac{1}{n}} \int_0^\infty V(M_t)^{\frac{n-1}{n}}\,dt &\leq \int_0^\infty \prod_{i=1}^m \mathrm{v}(M_t, w_i)^{\frac{\lambda_i}{n}}\,dt \\ &\leq \prod_{i=1}^m \left(\int_0^\infty \mathrm{v}(M_t, w_i)dt\right)^{\frac{\lambda_i}{n}}, \end{aligned}$$

where the constant c is from (3.10). Thus

$$c^{\frac{1}{n}}\|f\|_{\frac{n}{n-1}} \leq \prod_{i=1}^m \left(\int_0^\infty \mathrm{v}(M_t, u_i)dt\right)^{\frac{c_i}{n}}. \tag{5.3}$$

Similar to (4.3), one has

$$\|\nabla_{w_i} f\|_1 = 2|w_i| \int_0^\infty \mathrm{v}(M_t, w_i)dt.$$

From this and (5.3), inequality (5.2) follows. q.e.d.

Corollary 5.2. *Let $\{u_i\}_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$ and let $\{\lambda_i\}_1^m$ be a sequence of positive numbers satisfying*

$$|x|^2 = \sum_{i=1}^m \lambda_i \langle x, u_i\rangle^2, \quad x \in \mathbb{R}^n.$$

If f is a C^1 function with compact support in $\mathbb{R}^n$, then

$$\prod_{i=1}^m \|\nabla_{u_i} f\|_1^{\frac{\lambda_i}{n}} \geq 2\|f\|_{\frac{n}{n-1}}.$$

Similar to the equivalence of (3.9) and (4.1), the geometric inequality (3.10) is equivalent to the analytic inequality (5.2). A similar argument can also be carried out. The constant of the inequality (5.2) is best; it is attained at the characteristic functions of parallelepipeds.

6. Appendix

Proof of Lemma 3.1. Let k be a positive integer. Since ∂M is compact and of class C^1 piecewise, one can choose $\delta > 0$ such that

$$\begin{gathered}|\langle x - x', \nu(x')\rangle| \leq k^{-1}|x - x'|, \quad |\nu(x) - \nu(x')| \leq k^{-1}, \\ |h_K(\nu(x)) - h_K(\nu(x'))| < k^{-1}, \quad x, x' \in \partial M, \ |x - x'| < \delta.\end{gathered}$$

For $x \in \partial M$, consider a point $y \notin M$ but $y \in x + \varepsilon K$. We estimate the distance of y to ∂M. Let $x' \in \partial M$ be the point which attains the distance. Then

$$\begin{aligned}|y - x'| &= |\langle y - x', \nu(x')\rangle| = |\langle y - x, \nu(x')\rangle + \langle x - x', \nu(x')\rangle| \\ &= |\langle y - x, \nu(x)\rangle + \langle y - x, \nu(x') - \nu(x)\rangle + \langle x - x', \nu(x')\rangle| \\ &\leq |\langle y - x, \nu(x)\rangle| + |y - x||\nu(x') - \nu(x)| + |\langle x - x', \nu(x')\rangle|.\end{aligned} \tag{6.1}$$

Let d be the diameter of K. Obviously, $|x-x'|<2\varepsilon d$. Choose ε so that $2\varepsilon d<\delta$. Then

$$\begin{aligned}&|\langle x-x',\nu(x')\rangle|\leq k^{-1}2\varepsilon d,\quad |y-x||\nu(x')-\nu(x)|\leq\varepsilon dk^{-1},\\&\quad\langle y-x,\nu(x)\rangle\leq\varepsilon h_K(\nu(x)).\end{aligned}$$

If $\langle y-x,\nu(x)\rangle<0$, then

$$|\langle y-x,\nu(x)\rangle|<|\langle x-x',\nu(x)|\leq k^{-1}2\varepsilon d.$$

When k is large enough, we have

$$|\langle y-x,\nu(x)\rangle|\leq\varepsilon h_K(\nu(x))\leq\varepsilon h_K(\nu(x'))+\frac{\varepsilon}{k}.$$

Therefore

$$|y-x'|\leq\varepsilon h_K(\nu(x'))+\varepsilon\frac{3d+1}{k}. \tag{6.2}$$

Let y' be a point in $x+\varepsilon K$ so that

$$\langle y'-x,\nu(x)\rangle=\varepsilon h_K(\nu(x)).$$

Similar to (6.1), we have

$$|y'-x'|\geq|\langle y'-x,\nu(x)\rangle|-|y-x||\nu(x')-\nu(x)|-|\langle x-x',\nu(x')\rangle|.$$

It follows that

$$|y'-x'|\geq\varepsilon h_K(\nu(x'))-\varepsilon\frac{3d+1}{k}. \tag{6.3}$$

Consider the regions

$$\begin{aligned}&D_\varepsilon=\{x:x\in M+\varepsilon K,\ \text{but } x\notin M\},\\&D_\varepsilon^{\pm}=\{x+t\nu(x):x\in\partial M,0\leq t\leq\varepsilon(h_K(\nu(x))\pm(3d+1)/k\}.\end{aligned}$$

In view of (6.2) and (6.3), we have shown

$$D_\varepsilon^-\subseteq D_\varepsilon\subseteq D_\varepsilon^+.$$

From the equations $V(M+\varepsilon K)-V(K)=V(D_\varepsilon)$ and

$$\lim_{\varepsilon\to0^+}\frac{V(D_\varepsilon^\pm)}{\varepsilon}=\int_{\partial M}\left(h_K(\nu(x))\pm\frac{3d+1}{k}\right)dS_M,$$

we obtain the inequalities

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+}\frac{V(M+\varepsilon K)-V(M)}{\varepsilon} &\le \lim_{\varepsilon\to 0^+}\frac{V(D_\varepsilon^+)}{\varepsilon} \\
&= nV(M,K)+\frac{3d+1}{k}S(M), \\
\liminf_{\varepsilon\to 0^+}\frac{V(M+\varepsilon K)-V(M)}{\varepsilon} &\ge \lim_{\varepsilon\to 0^+}\frac{V(D_\varepsilon^-)}{\varepsilon} \\
&= nV(M,K)-\frac{3d+1}{k}S(M).
\end{aligned}$$

These prove the lemma. q.e.d.

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