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A NEW CONSTRUCTION OF COMPACT 8-MANIFOLDS WITH HOLONOMY Spin(7)

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1. Introduction

In Berger's classification [1] of holonomy groups of Riemannian manifolds there are two special cases, the exceptional holonomy groups G_2 in 7 dimensions and Spin(7) in 8 dimensions. Bryant [2] and Bryant and Salamon [3] showed that such metrics exist locally, and wrote down explicit, complete metrics with holonomy G_2 and Spin(7) on noncompact manifolds.

The first examples of metrics with holonomy G_2 and Spin(7) on *compact* 7- and 8-manifolds were constructed by the author in [10], [11], [12]. The survey paper [13] provides a good introduction to these constructions. Here is a brief description of the method used in [10] to construct compact 8-manifolds with holonomy Spin(7), divided into four steps.

(a) We start with a flat Spin(7)-structure (Ω_0, g_0) on the 8-torus T^8, and a finite group Γ of isometries of T^8 preserving (Ω_0, g_0). Then T^8/Γ is an *orbifold*, a singular manifold with only quotient singularities.

(b) For certain Γ one can resolve the singularities of T^8/Γ in a natural way, using complex geometry. This gives a nonsingular, compact 8-manifold M, and a projection $\pi : M \to T^8/\Gamma$.

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(c) We write down a 1-parameter family of Spin(7)-structures (Ω_t, g_t) on M for $t \in (0, \epsilon)$, such that (Ω_t, g_t) has small torsion when t is small, and converges to the singular Spin(7)-structure $\pi^*(\Omega_0, g_0)$ as $t \to 0$.

(d) Using analysis we prove that for small t, the Spin(7)-structure (Ω_t, g_t) can be deformed to a nearby Spin(7)-structure $(\tilde{\Omega}_t, \tilde{g}_t)$ on M, with zero torsion. Then $\tilde{g}_t$ has holonomy Spin(7).

In this paper we will describe a new method for constructing compact 8-manifolds with holonomy Spin(7), in which one starts not with a torus T^8 but with a *Calabi–Yau 4-orbifold* Y with isolated singular points $p_1, \ldots, p_k$. We use algebraic geometry to find a number of suitable complex orbifolds Y, which in the simplest cases are hypersurfaces in *weighted projective spaces* $\mathbb{CP}^5_{a_0,\ldots,a_5}$.

Then, instead of a finite group Γ, we suppose we have an antiholomorphic, isometric involution $\sigma : Y \to Y$, whose only fixed points are $p_1, \ldots, p_k$. This involution does not preserve the SU(4)-structure on Y, but it does preserve the induced Spin(7)-structure. We think of σ as breaking the structure group of Y from SU(4) down to Spin(7). Define $Z = Y/\langle\sigma\rangle$. Then Z is an orbifold with isolated singular points $p_1, \ldots, p_k$, and the Calabi–Yau structure on Y induces a torsion-free Spin(7)-structure on Z.

If the singularities of Z are of a suitable kind, we can resolve them to get a compact 8-manifold M with holonomy Spin(7), as in steps (b)–(d) above. To perform the resolution we need to find *Asymptotically Locally Euclidean* Spin(7)*-manifolds* corresponding to the singularities of Z, which are a special class of noncompact Spin(7)-manifolds asymptotic to quotient singularities $\mathbb{R}^8/G$.

Our construction then yields new examples of compact 8-manifolds M with holonomy Spin(7). We calculate the Betti numbers $b^k(M)$ in each case. They turn out to be rather different to the Betti numbers arising from the previous construction in [10]. In particular, in this new construction the middle Betti number b^4 tends to be rather large, as big as 11 662 in one example, whereas the manifolds of [10] all satisfied $b^4 \leq 162$.

Sections 2 and 3 introduce the holonomy group Spin(7) and Calabi–Yau orbifolds, and §4 defines the idea of ALE Spin(7)-manifold, and gives a number of examples. Section 5 then proves our main result, that given a Calabi–Yau 4-orbifold Y and an antiholomorphic involution

$\sigma : Y \to Y$ satisfying certain conditions, we can construct a compact 8-manifold M with holonomy Spin(7).

We explain in §6 how to use the construction in practice, and ways of computing the Betti numbers of the resulting 8-manifolds M. Sections 7–10 apply the construction to generate new examples of compact 8-manifolds with holonomy Spin(7), and we finish in §11 with a discussion of our results.

The material in this paper will be discussed in the author's book [14], which pays much attention to the exceptional holonomy groups, and also gives a more sophisticated version of the original construction [10] of compact 8-manifolds with holonomy Spin(7).

2. Background on the holonomy group *Spin*(7)

We now collect together some facts we will need about the holonomy group Spin(7), taken from the books by Salamon [18, Ch. 12] and the author [14, Ch. 10]. First we define Spin(7) as a subgroup of $\mathrm{GL}(8,\mathbb{R})$.

Definition 2.1. Let $\mathbb{R}^8$ have coordinates $(x_1,\dots,x_8)$. Write $\mathrm{d}\mathbf{x}_{ijkl}$ for the 4-form $\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k\wedge\mathrm{d}x_l$ on $\mathbb{R}^8$. Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\begin{aligned}\Omega_0 =&\mathrm{d}\mathbf{x}_{1234}+\mathrm{d}\mathbf{x}_{1256}+\mathrm{d}\mathbf{x}_{1278}+\mathrm{d}\mathbf{x}_{1357}-\mathrm{d}\mathbf{x}_{1368}\\ &-\mathrm{d}\mathbf{x}_{1458}-\mathrm{d}\mathbf{x}_{1467}-\mathrm{d}\mathbf{x}_{2358}-\mathrm{d}\mathbf{x}_{2367}-\mathrm{d}\mathbf{x}_{2457}\\ &+\mathrm{d}\mathbf{x}_{2468}+\mathrm{d}\mathbf{x}_{3456}+\mathrm{d}\mathbf{x}_{3478}+\mathrm{d}\mathbf{x}_{5678}.\end{aligned} \tag{1}$$

The subgroup of $\mathrm{GL}(8,\mathbb{R})$ preserving Ω_0 is Spin(7). It is a compact, connected, simply-connected, semisimple, 21-dimensional Lie group, which is isomorphic as a Lie group to the double cover of SO(7). This group also preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric $g_0 = \mathrm{d}x_1^2+\cdots+\mathrm{d}x_8^2$ on $\mathbb{R}^8$.

Let M be an 8-manifold. For each $p\in M$, define $\mathcal{A}_pM$ to be the subset of 4-forms $\Omega\in\Lambda^4T_p^*M$ for which there exists an isomorphism between T_pM and $\mathbb{R}^8$ identifying Ω and the 4-form Ω_0 of (1). Let $\mathcal{A}M$ be the bundle with fibre $\mathcal{A}_pM$ at each $p\in M$. Then $\mathcal{A}M$ is a subbundle of Λ^4T^*M with fibre $\mathrm{GL}(8,\mathbb{R})/\operatorname{Spin}(7)$. It is not a vector subbundle, and has codimension 27 in Λ^4T^*M. We say that a 4-form Ω on M is *admissible* if $\Omega|_p\in\mathcal{A}_pM$ for each $p\in M$.

Now the conventional definition of a Spin(7)-structure on an 8-manifold M (which we will not use) is a principal subbundle Q of the frame bundle $\mathcal{F}$ with structure group Spin(7). There is a 1-1 correspondence between Spin(7)-structures Q in this sense, and admissible 4-forms $\Omega \in C^\infty(\mathcal{A}M)$ on M. Each Spin(7)-structure Q induces a 4-form Ω, a metric g and an orientation on M, corresponding to Ω_0, g_0 and the orientation on $\mathbb{R}^8$.

Definition 2.2. Let M be an 8-manifold, Ω an admissible 4-form on M, and g the associated metric. We shall abuse notation by referring to the pair (Ω, g) as a Spin(7)-*structure* on M. Let ∇ be the Levi-Civita connection of g. We call $\nabla\Omega$ the *torsion* of (Ω, g), and we say that (Ω, g) is *torsion-free* if $\nabla\Omega = 0$. A triple (M, Ω, g) is called a Spin(7)-*manifold* if M is an 8-manifold, and (Ω, g) a torsion-free Spin(7)-structure on M.

Let (Ω, g) be a Spin(7)-structure on an 8-manifold M. Then (Ω, g) is torsion-free if and only if $\mathrm{d}\Omega = 0$. If (Ω, g) is torsion-free then g is Ricci-flat, and M is spin and has a constant positive spinor. If M is compact and $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ then the positive Dirac operator

$$D_+ : C^\infty(S_+) \to C^\infty(S_-)$$

has kernel $\mathbb{R}$ and cokernel 0. Thus D_+ has index 1.

But the index of D_+ is the $\hat{A}$-*genus* $\hat{A}(M)$, and is given by

$$24\hat{A}(M) = -1 + b^1(M) - b^2(M) + b^3(M) + b^4_+(M) - 2b^4_-(M), \tag{2}$$

where $b^k = b^k(M)$ are the Betti numbers of M. Thus a compact 8-manifold M with holonomy Spin(7) must satisfy $b^3 + b^4_+ = b^2 + b^4_- + 25$. As in [10, Th. C], one can use this to show:

Theorem 2.3. *Let (M, Ω, g) be a compact* Spin(7)*-manifold. Then* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *if and only if M is simply-connected, and $b^3 + b^4_+ = b^2 + b^4_- + 25$.*

The following result [10, Th. D] describes the moduli space of holonomy Spin(7) metrics.

Theorem 2.4. *Let M be a compact 8-manifold admitting metrics with holonomy* Spin(7)*. Then the moduli space of metrics with holonomy* Spin(7) *on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $1 + b^4_-(M)$.*

Our next proposition follows from the ideas of [14, §10.6].

Proposition 2.5. *Let M be an 8-manifold. Then there exists a tubular open neighbourhood $\mathcal{T}M$ of $\mathcal{A}M$ in Λ^4T^*M which is a fibration* *over M, a smooth map of fibre bundles $\Theta : \mathcal{T}M \to \mathcal{A}M$, and positive constants ρ, C, such that*

(i) *If (Ω, g) is a* Spin(7)*-structure and ξ a 4-form on M with $|\xi - \Omega|_g \le \rho$, then $\xi \in C^\infty(\mathcal{T}M)$.*

(ii) *Suppose (Ω, g) is a* Spin(7)*-structure on M, and ξ a 4-form on M with $|\xi - \Omega|_g \le \rho$. Write $\Omega' = \Theta(\xi)$, and let (Ω', g') be the associated* Spin(7)*-structure. Then $|\xi - \Omega'|_{g'} \le |\xi - \Omega|_g$. If (Ω, g) is also torsion-free, then $\bigl|\nabla'(\xi - \Omega')\bigr|_{g'} \le C\bigl|\nabla(\xi - \Omega)\bigr|_g$.*

Here ∇, ∇' are the Levi-Civita connections of g and g', and $|\,.\,|_g$, $|\,.\,|_{g'}$ the norms defined using g and g'.

This is an entirely local result, involving calculations at a point, and ρ, C are independent of M. The inequality $|\xi - \Omega'|_{g'} \le |\xi - \Omega|_g$ in part (ii) should be understood as saying that $\Omega' = \Theta(\xi)$ is the Spin(7)-form closest to ξ. That is, $\mathcal{T}M$ is a small open neighbourhood of $\mathcal{A}M$ in $\Lambda^4 T^*M$, and Θ is the projection from $\mathcal{T}M$ to the nearest point in $\mathcal{A}M$. But as we have not fixed a metric on M, we do not have a way to measure distance in $\Lambda^4 T^*M$, and so we use the metrics g, g' associated to the Spin(7)-forms Ω, Ω' to do this.

Our final result is proved in [10, Th. A & Th. B], and also in [14, Ch. 13].

Theorem 2.6. *Let λ, μ, ν be positive constants. Then there exist positive constants κ, K such that whenever $0 < t \le \kappa$, the following is true.*

Let M be a compact 8-manifold, and (Ω, g) a Spin(7)*-structure on M. Suppose that ϕ is a smooth 4-form on M with $\mathrm{d}\Omega + \mathrm{d}\phi = 0$, and*

(i) *$\|\phi\|_{L^2} \le \lambda t^{9/2}$ and $\|\mathrm{d}\phi\|_{L^{10}} \le \lambda t$,*

(ii) *the injectivity radius $\delta(g)$ satisfies $\delta(g) \ge \mu t$, and*

(iii) *the Riemann curvature $R(g)$ satisfies $\bigl\|R(g)\bigr\|_{C^0} \le \nu t^{-2}$.*

Then there exists a smooth, torsion-free Spin(7)*-structure $(\tilde\Omega, \tilde g)$ on M with $\|\tilde\Omega - \Omega\|_{C^0} \le K t^{1/2}$.*

Here is how to interpret this result. As $\nabla\Omega = 0$ if and only if $\mathrm{d}\Omega = 0$ and $\mathrm{d}\phi + \mathrm{d}\Omega = 0$, the torsion $\nabla\Omega$ is determined by $\mathrm{d}\phi$. Thus we can think of ϕ as a *first integral of the torsion* of (Ω, g). So $\|\phi\|_{L^2}$ and $\|\mathrm{d}\phi\|_{L^{10}}$ are both measures of the torsion of (Ω, g). As t is small, part (i) of the theorem says that (Ω, g) has *small torsion* in a certain sense.

Parts (ii) and (iii) say that the injectivity radius of g should not be too small, and its curvature not too large. When a metric becomes singular, in general its injectivity radius goes to zero and its curvature becomes infinite. So we can interpret (ii) and (iii) as saying that g is not too close to being singular.

Thus, the theorem as a whole says that if the torsion of (Ω, g) is small enough, and g is not too singular, then we can deform (Ω, g) to a nearby, torsion-free Spin(7)-structure $(\tilde{\Omega}, \tilde{g})$ on M. We can hence use Theorem 2.3 to show that if M is simply-connected and $b^3 + b^4_+ = b^2 + b^4_- + 25$, then $\tilde{g}$ has holonomy Spin(7).

We prove Theorem 2.6 using analysis: we write the condition that $(\tilde{\Omega}, \tilde{g})$ be torsion-free as a nonlinear elliptic p.d.e., which can be approximated by a linear elliptic p.d.e. when $\tilde{\Omega} - \Omega$ is small. Then we use tools such as Sobolev spaces, the Sobolev Embedding Theorem and elliptic regularity to show that this nonlinear elliptic p.d.e. has a smooth solution.

3. Calabi–Yau manifolds and orbifolds

We now give a brief introduction to Calabi–Yau geometry, and the relation between Calabi–Yau 4-folds and Spin(7)-manifolds. Some suitable references are Salamon [18, Ch. 8] and the author [14, Ch. 6].

Definition 3.1. A *Calabi–Yau manifold* or *orbifold* is a compact Kähler manifold or orbifold (Y, J, g) of dimension m, with $\mathrm{Hol}(g) = \mathrm{SU}(m)$.

Now Calabi–Yau manifolds and orbifolds are nearly the same thing as Ricci-flat Kähler manifolds and orbifolds, as we see in the next proposition. It follows from elementary properties of holonomy groups and Kähler geometry.

Proposition 3.2. *Any Calabi–Yau orbifold (Y, J, g) is Ricci-flat. Conversely, let (Y, J, g) be a compact Ricci-flat Kähler orbifold of dimension m, with singular set S. Suppose that $Y \setminus S$ is simply-connected and $h^{p,0}(Y) = 0$ for $0 < p < m$. Then* $\mathrm{Hol}(g) = \mathrm{SU}(m)$, *so Y is a Calabi–Yau orbifold.*

But using Yau's proof of the Calabi conjecture [20], one can show that suitable complex orbifolds admit Ricci-flat Kähler metrics.

Theorem 3.3. *Let (Y, J) be a compact complex orbifold admit-*

ting Kähler metrics, with $c_1(Y) = 0$. *Then there is a unique Ricci-flat Kähler metric in each Kähler class on* Y.

Now the action of $\mathrm{SU}(m)$ on $\mathbb{C}^m$ fixes the complex m-form $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$. It follows by general principles of Riemannian holonomy that any Riemannian manifold or orbifold with holonomy $\mathrm{SU}(m)$ admits a complex m-form θ corresponding to $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$ which is constant under the Levi-Civita connection ∇. So we get:

Proposition 3.4. *Let* (Y, J, g) *be a Calabi–Yau manifold or orbifold of dimension* m, *with Kähler form* ω. *Then there exists a constant* $(m, 0)$-*form* θ *on* Y, *such that near every point* $p \in Y$ *we can choose complex coordinates* $(z_1, \ldots, z_m)$ *in which*

$$\begin{aligned} g &= |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \\ \omega &= \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \textit{and} \quad \theta &= \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m \end{aligned} \tag{1}$$

at p. *This form* θ *is unique up to multiplication by* $e^{i\phi}$ *for some* $\phi \in [0, 2\pi)$.

We call θ the *holomorphic volume form* of Y. Now we restrict our attention to complex dimension 4. Here is a criterion for a complex 4-orbifold to be Calabi–Yau.

Proposition 3.5. *Let* (Y, J) *be a compact complex* 4-*orbifold with* $c_1(Y) = 0$, *admitting Kähler metrics. Suppose* $Y \backslash S$ *is simply-connected, where* S *is the singular set of* Y, *and* $h^{2,0}(Y) = 0$. *Then each Kähler class on* Y *contains a unique metric* g *such that* (Y, J, g) *is a Calabi–Yau* 4-*orbifold.*

Proof. As $\pi_1(Y \setminus S) = 0$ we have $b^1(Y) = 0$, so that $h^{1,0}(Y) = 0$. Since $\pi_1(Y \setminus S) = 0$ and $c_1(Y) = 0$ the canonical bundle K_Y of Y is trivial, and this implies that $h^{p,0}(Y) = h^{4-p,0}(Y)$. Thus $h^{3,0}(Y) = 0$. But we are given that $h^{2,0}(Y) = 0$. Hence $h^{p,0}(Y) = 0$ for $0 < p < 4$, and the proposition follows from Proposition 3.2 and Theorem 3.3. q.e.d.

A Calabi–Yau 4-fold Y has holonomy $\mathrm{SU}(4)$, and so carries a natural torsion-free $\mathrm{SU}(4)$-structure. Since $\mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$, this $\mathrm{SU}(4)$-structure induces a $\mathrm{Spin}(7)$-structure on Y, which is also torsion-free.

Proposition 3.6. *Suppose* (Y, J, g) *is a Calabi–Yau* 4-*orbifold, with Kähler form* ω *and holomorphic volume form* θ. *Define a* 4-*form* Ω *on*

Y by $\Omega = \frac{1}{2}\omega\wedge\omega + \mathrm{Re}(\theta)$. Then (Ω, g) is a torsion-free $\mathrm{Spin}(7)$*-structure on Y.*

Proof. Let p be a point in Y. Then by Proposition 3.4 we can choose complex coordinates $(z_1, \ldots, z_4)$ near p such that g, ω and θ are given by (1) at p, with $m = 4$. Define real coordinates $(x_1, \ldots, x_8)$ on Y near p such that $(z_1, \ldots, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8)$. Then from (1) we see that g, ω and $\mathrm{Re}(\theta)$ are given at p by

$$g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2, \quad \omega = \mathrm{d}\mathbf{x}_{12} + \mathrm{d}\mathbf{x}_{34} + \mathrm{d}\mathbf{x}_{56} + \mathrm{d}\mathbf{x}_{78}$$

and

$$\begin{aligned}\mathrm{Re}(\theta) = &\mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1368} - \mathrm{d}\mathbf{x}_{1458} - \mathrm{d}\mathbf{x}_{1467} \\ &- \mathrm{d}\mathbf{x}_{2358} - \mathrm{d}\mathbf{x}_{2367} - \mathrm{d}\mathbf{x}_{2457} + \mathrm{d}\mathbf{x}_{2468},\end{aligned}$$

where $\mathrm{d}\mathbf{x}_{ij\ldots l} = \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \cdots \wedge \mathrm{d}x_l$.

It follows from this equation that $\Omega = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\theta)$ coincides with the 4-form Ω_0 defined in (1). As this holds for all $p \in Y$, we see that (Ω, g) is a $\mathrm{Spin}(7)$-*structure* on Y, in the sense of Definition 2.2. Now $\nabla\omega = \nabla\theta = 0$, where ∇ is the Levi-Civita connection of g, and so $\nabla\Omega = 0$. But $\nabla\Omega$ is the *torsion* of (Ω, g), so that (Ω, g) is torsion-free, as we want. q.e.d.

Thus Calabi-Yau 4-folds are also Spin(7)-manifolds.

4. ALE Spin(7)-manifolds

ALE manifolds, or *Asymptotically Locally Euclidean manifolds*, are a class of noncompact Riemannian manifolds with one end modelled asymptotically on a quotient singularity $\mathbb{R}^n/G$.

Definition 4.1. Let G be a finite subgroup of $\mathrm{SO}(n)$ which acts freely on $\mathbb{R}^n \setminus \{0\}$. Let X be a noncompact n-manifold and $\pi : X \to \mathbb{R}^n/G$ a continuous, surjective map, such that $\pi^{-1}(0)$ is a compact subset of X, and $\pi : X \setminus \pi^{-1}(0) \to (\mathbb{R}^n/G) \setminus \{0\}$ is a diffeomorphism. Then we call (X, π) a *real resolution* of $\mathbb{R}^n/G$.

A metric g on X is called *Asymptotically Locally Euclidean*, or *ALE*, if

$$\nabla^l(\pi_*(g) - g_0) = O(r^{-n-l}) \quad \text{on } \{x \in \mathbb{R}^n/G : r(x) > R\}, \text{ for all } l \geq 0.$$

Here g_0 is the Euclidean metric on $\mathbb{R}^n/G$, r is the radius function on $\mathbb{R}^8/G$, and $R > 0$ is a constant. We say that (X, g) is *asymptotic to* $\mathbb{R}^n/G$.

One reason ALE manifolds are interesting is that if you have an ALE manifold (X, g_X) asymptotic to $\mathbb{R}^n/G$, and a compact Riemannian orbifold (Y, g_Y) with isolated singularities modelled on $\mathbb{R}^n/G$, then you can glue X and Y together to get a nonsingular, compact Riemannian manifold (M, g_M). We think of this as resolving the singularities of Y using X.

This technique is particularly valuable when X and Y both have special holonomy, so that $\mathrm{Hol}(g_X)$ and $\mathrm{Hol}(g_Y)$ both lie in some holonomy group $H \subset \mathrm{SO}(n)$, as then we can hope to construct a metric g_M on M with $\mathrm{Hol}(g_M) \subseteq H$. So ALE manifolds (X, g_X) with $\mathrm{Hol}(g_X) \subseteq H$ are ingredients in a construction for compact manifolds with holonomy H.

In fact the only interesting candidates for the holonomy group H are $U(m)$ and $\mathrm{SU}(m)$ for $m \geq 2$, and Spin(7). Kronheimer [16], [17] constructed and classified all ALE 4-manifolds with holonomy SU(2). Calabi [4, p. 285] found an explicit family of ALE manifolds with holonomy $\mathrm{SU}(m)$ asymptotic to $\mathbb{C}^m/\mathbb{Z}_m$, and more generally the author [15], [14, Ch. 8] gave existence theorems for ALE manifolds with holonomy $\mathrm{SU}(m)$. No examples of ALE 8-manifolds with holonomy Spin(7) are known, at the time of writing.

However, we can construct compact 8-manifolds with holonomy Spin(7) using only ALE 8-manifolds whose holonomy is a proper subgroup of Spin(7) such as SU(4) or $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$, and many examples of these can be found using the results of [15]. To discuss these, it is useful to define the idea of *ALE* Spin(7)*-manifold*, as in [14, Ch. 13].

Definition 4.2. Let G be a finite subgroup of Spin(7) which acts freely on $\mathbb{R}^8 \setminus \{0\}$, let (X, π) be a real resolution of $\mathbb{R}^8/G$, and (Ω, g) a torsion-free Spin(7)-structure on X. We call (X, Ω, g) an *ALE* Spin(7)*-manifold* if

$$\nabla^l(\pi_*(\Omega) - \Omega_0) = O(r^{-8-l}) \quad \text{on } \{x \in \mathbb{R}^8/G : r(x) > R\}, \text{ for all } l \geq 0.$$

Here Ω_0 is the Spin(7) 4-form on $\mathbb{R}^8/G$ given in (1), r the radius function on $\mathbb{R}^8/G$, and $R > 0$ a constant.

In the rest of the section we give some examples of ALE Spin(7)-manifolds.

4.1 An example of an ALE Spin(7)-manifold

We define a finite group $G \subset \mathrm{Spin}(7)$, such that $\mathbb{R}^8/G$ has an isolated singularity at 0, and construct two topologically distinct ALE Spin(7)-manifolds (X_1, Ω_1, g_1) and (X_2, Ω_2, g_2) asymptotic to $\mathbb{R}^8/G$. These will be used in §5 as part of a construction of compact 8-manifolds with holonomy Spin(7).

Let $\mathbb{R}^8$ have coordinates $(x_1, \ldots, x_8)$ and Spin(7)-structure (Ω_0, g_0), as in Definition 2.1. Use the complex coordinates

$$(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8)$$

to identify $\mathbb{R}^8$ with $\mathbb{C}^4$. Then $g_0 = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_4|^2$, and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}(\theta_0)$, where ω_0 is the Kähler form of g_0 and $\theta_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4$ the complex volume form on $\mathbb{C}^4$.

Define $\alpha, \beta : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\begin{aligned} \alpha &: (z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4), \\ \beta &: (z_1, \ldots, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3). \end{aligned} \tag{4}$$

Then $\alpha \in \mathrm{SU}(4) \subset \mathrm{Spin}(7)$ and $\beta \in \mathrm{Spin}(7)$, and α, β satisfy $\alpha^4 = \beta^4 = 1$, $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha^3$. Let $G = \langle \alpha, \beta \rangle$. Then G is a finite nonabelian subgroup of Spin(7) of order 8 which acts freely on $\mathbb{R}^8 \setminus \{0\}$.

Now $\mathbb{C}^4/\langle\alpha\rangle$ is a complex singularity, as $\alpha \in \mathrm{SU}(4)$. Let (Y_1, π_1) be the blow-up of $\mathbb{C}^4/\langle\alpha\rangle$ at 0. Then Y_1 is the unique crepant resolution of $\mathbb{C}^4/\langle\alpha\rangle$. The action of β on $\mathbb{C}^4/\langle\alpha\rangle$ lifts to a free antiholomorphic map $\beta : Y_1 \to Y_1$ with $\beta^2 = 1$. Define $X_1 = Y_1/\langle\beta\rangle$. Then X_1 is a nonsingular 8-manifold, and the projection $\pi_1 : Y_1 \to \mathbb{C}^4/\langle\alpha\rangle$ pushes down to $\pi_1 : X_1 \to \mathbb{R}^8/G$.

By [15, Th. 3.3, Th. 3.4] there exist ALE Kähler metrics g_1 on Y_1 with holonomy SU(4), which were in fact written down explicitly by Calabi [4, p. 285]. Each such g_1 is invariant under the action of β on Y_1. Let ω_1 be the Kähler form of g_1, and $\theta_1 = \pi_1^*(\theta_0)$ the holomorphic volume form on Y_1. Then Proposition 3.6 defines a torsion-free Spin(7)-structure (Ω_1, g_1) on Y_1 with $\Omega_1 = \frac{1}{2}\omega_1 \wedge \omega_1 + \mathrm{Re}(\theta_1)$.

As $\beta^*(\omega_1) = -\omega_1$ and $\beta^*(\theta_1) = \bar{\theta}_1$, we see that β preserves (Ω_1, g_1). Thus (Ω_1, g_1) pushes down to a torsion-free Spin(7)-structure (Ω_1, g_1) on X_1. Then (X_1, Ω_1, g_1) is an *ALE* Spin(7)*-manifold* asymptotic to $\mathbb{R}^8/G$. The Betti numbers of X_1 are $b^1 = b^2 = b^3 = 0$ and $b^4 = 1$, and $\pi_1(X_1) = \mathbb{Z}_2$.

4.2 A second ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$

Define new complex coordinates $(w_1,\ldots,w_4)$ on $\mathbb{R}^8$ by

$$(w_1,w_2,w_3,w_4)=(-x_1+ix_3,x_2+ix_4,-x_5+ix_7,x_6+ix_8).$$

Then $g_0=|\mathrm{d}w_1|^2+\cdots+|\mathrm{d}w_4|^2$ and $\Omega_0=\frac{1}{2}\omega_0'\wedge\omega_0'+\mathrm{Re}(\theta_0')$, where ω_0' is the Kähler form of g_0 with respect to the complex structure induced by the w_j, and $\theta_0'=\mathrm{d}w_1\wedge\cdots\wedge\mathrm{d}w_4$ is the complex volume form on $\mathbb{C}^4$.

As the action of SU(4) on $\mathbb{R}^8=\mathbb{C}^4$ induced by the w_j preserves g_0,ω_0' and θ_0', it preserves (Ω_0,g_0). Thus the action of SU(4) on $\mathbb{R}^8$ compatible with the coordinates w_j is a subgroup of Spin(7). Note that this is a *different* SU(4) subgroup of Spin(7) to that considered above, induced by the z_j. In the coordinates w_j, we find that α,β act by

$$\begin{aligned}\alpha&:(w_1,\ldots,w_4)\mapsto(\bar w_2,-\bar w_1,\bar w_4,-\bar w_3),\\ \beta&:(w_1,\ldots,w_4)\mapsto(iw_1,iw_2,iw_3,iw_4).\end{aligned}\tag{5}$$

Observe that (4) and (5) are the same, except that the rôles of α,β are reversed. Therefore we can use the ideas above again.

Let Y_2 be the crepant resolution of $\mathbb{C}^4/\langle\beta\rangle$. The action of α on $\mathbb{C}^4/\langle\beta\rangle$ lifts to a free antiholomorphic involution of Y_2. Let $X_2=Y_2/\langle\alpha\rangle$. Then X_2 is nonsingular, and as above there exists a torsion-free Spin(7)-structure (Ω_2,g_2) on X_2, making (X_2,Ω_2,g_2) into an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

Now (X_1,Ω_1,g_1), (X_2,Ω_2,g_2) are clearly isomorphic as Spin(7)-manifolds, but they should be regarded as *topologically distinct* ALE manifolds, because the isomorphism between them acts nontrivially on $\mathbb{R}^8/G$. Thus, we have found two topologically distinct ALE Spin(7)-manifolds (X_1,Ω_1,g_1), (X_2,Ω_2,g_2) asymptotic to the same singularity $\mathbb{R}^8/G$.

4.3 Other examples of ALE Spin(7)-manifolds

We can use the ideas above to construct other ALE Spin(7)-manifolds too. Here we very briefly describe two infinite families of ALE Spin(7)-manifolds X_1^n, X_2^n for $n=1,3,5,\ldots$. For simplicity they will not be used in the rest of the paper, although they easily could be.

Identify $\mathbb{R}^8$ and $\mathbb{C}^4$ as in §4.1. Let $n\geq 1$ be an odd integer, and define $\alpha,\beta,\gamma:\mathbb{C}^4\to\mathbb{C}^4$ by

$$\begin{aligned}\alpha&:(z_1,\ldots,z_4)\mapsto(\mathrm{e}^{2\pi i/n}z_1,\mathrm{e}^{-2\pi i/n}z_2,\mathrm{e}^{2\pi i/n}z_3,\mathrm{e}^{-2\pi i/n}z_4),\\ \beta&:(z_1,\ldots,z_4)\mapsto(iz_1,iz_2,iz_3,iz_4),\\ \gamma&:(z_1,\ldots,z_4)\mapsto(\bar z_2,-\bar z_1,\bar z_4,-\bar z_3).\end{aligned}$$

Then $\alpha, \beta \in \mathrm{SU}(4)$ and $\gamma \in \mathrm{Spin}(7)$, and $G^n = \langle \alpha, \beta, \gamma \rangle$ is a finite nonabelian subgroup of Spin(7) of order $8n$ which acts freely on $\mathbb{R}^8 \setminus \{0\}$. Note that G^1 coincides with the group G of §4.1-§4.2.

We can construct a family of ALE Spin(7)-manifolds asymptotic to $\mathbb{R}^8/G^n$ as follows. The complex singularity $\mathbb{C}^4/\langle \alpha, \beta \rangle$ has a unique crepant resolution Y_1^n, which can be described explicitly using toric geometry. The action of γ on $\mathbb{C}^4/\langle \alpha, \beta \rangle$ lifts to a free antiholomorphic involution $\gamma : Y_1^n \to Y_1^n$, so that $X_1^n = Y_1^n/\langle \gamma \rangle$ is a nonsingular 8-manifold with a projection $\pi_1^n : X_1^n \to \mathbb{R}^8/G^n$.

By the results of [15], there exist ALE Kähler metrics g_1^n on Y_1^n with holonomy SU(4). We can choose g_1^n to be γ-invariant, and then the induced Spin(7)-structure (Ω_1^n, g_1^n) on Y_1^n is also γ-invariant, and pushes down to X_1^n, making $(X_1^n, \Omega_1^n, g_1^n)$ into an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G^n$. Using the idea of §4.2, we can also construct a second ALE Spin(7)-manifold $(X_2^n, \Omega_2^n, g_2^n)$ asymptotic to $\mathbb{R}^8/G^n$.

5. Proof of the construction

Starting with a Calabi–Yau 4-orbifold Y with isolated singularities of a certain kind, and an antiholomorphic involution σ on Y, we will now construct a compact 8-manifold M by resolving $Z = Y/\langle \sigma \rangle$, and prove that there exist torsion-free Spin(7)-structures $(\tilde{\Omega}, \tilde{g})$ on M, which have holonomy Spin(7) if M is simply-connected.

5.1 A class of Spin(7)-orbifolds Z

We set out below the ingredients in our construction, and the assumptions they must satisfy.

Condition 5.1. Let (Y, J) be a compact complex 4-orbifold with $c_1(Y) = 0$, admitting Kähler metrics. Let σ be an antiholomorphic involution on Y. That is, $\sigma : Y \to Y$ is a diffeomorphism satisfying $\sigma^2 = \mathrm{id}$ and $\sigma^*(J) = -J$. Define $\alpha : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\alpha : (z_1, z_2, z_3, z_4) \longmapsto (iz_1, iz_2, iz_3, iz_4). \tag{6}$$

Then $\alpha^4 = 1$, so that $\langle \alpha \rangle \cong \mathbb{Z}_4$, and $\mathbb{C}^4/\langle \alpha \rangle$ has an isolated singular point at 0. We require that the singular set of Y should be k isolated points $p_1, \ldots, p_k$ for some $k \geq 1$, each modelled on $\mathbb{C}^4/\langle \alpha \rangle$, and that the fixed set of σ in Y is exactly $\{p_1, \ldots, p_k\}$. We also suppose that $Y \setminus \{p_1, \ldots, p_k\}$ is simply-connected, and $h^{2,0}(Y) = 0$.

In the rest of the section we assume that Condition 5.1 holds.

Proposition 5.2. *There is a σ-invariant metric g_Y on Y making (Y, J, g_Y) into a Calabi–Yau orbifold. We can choose the holomorphic volume form θ_Y on (Y, J, g_Y) such that $\sigma^*(\theta_Y) = \bar{\theta}_Y$. Let (Ω_Y, g_Y) be the torsion-free* Spin(7)*-structure on Y from Proposition 3.6. Then (Ω_Y, g_Y) is σ-invariant.*

Proof. Let g' be a Kähler metric on Y. Then $\sigma^*(g')$ is also a Kähler metric on Y, and so $g'' = g' + \sigma^*(g')$ is a σ-invariant Kähler metric on Y. Let κ be the Kähler class of g''. Then κ is σ-invariant, regarded as an equivalence class of metrics on Y. By Condition 5.1 we know that $c_1(Y) = 0$ and $h^{2,0}(Y) = 0$, and that $Y \setminus S$ is simply-connected, where $S = \{p_1, \ldots, p_k\}$ is the singular set of Y. Thus by Proposition 3.5, the Kähler class κ contains a unique metric g_Y such that (Y, J, g_Y) is a Calabi–Yau orbifold. As κ is σ-invariant we see that g_Y is σ-invariant, by uniqueness of g_Y.

Proposition 3.4 shows that there exists a holomorphic volume form θ on Y. Since σ is antiholomorphic, it is easy to show that $\sigma^*(\theta) = e^{i\phi}\bar{\theta}$, for some $\phi \in [0, 2\pi)$. Define $\theta_Y = e^{i\phi/2}\theta$. Then θ_Y is a holomorphic volume form for (Y, J, g_Y), and $\sigma^*(\theta_Y) = \bar{\theta}_Y$, as we want.

Let (Ω_Y, g_Y) be as in Proposition 3.6. Then $\Omega_Y = \frac{1}{2}\omega_Y \wedge \omega_Y + \mathrm{Re}(\theta_Y)$, where ω_Y is the Kähler form of g_Y. As $\sigma^*(g_Y) = g_Y$ and $\sigma^*(J) = -J$ we have $\sigma^*(\omega_Y) = -\omega_Y$, and $\sigma^*(\mathrm{Re}(\theta_Y)) = \mathrm{Re}(\theta_Y)$ as $\sigma^*(\theta_Y) = \bar{\theta}_Y$. Thus Ω_Y and g_Y are both σ-invariant. q.e.d.

In our next result, if Y is an orbifold and $p \in Y$ an orbifold point modelled on $\mathbb{R}^n/G$, then we say that the *tangent space* T_pY to Y at p is $\mathbb{R}^n/G$, in the obvious way. The proof looks complicated, but it is really only linear algebra.

Proposition 5.3. *For each $j = 1, \ldots, k$ we can identify the tangent space $T_{p_j}Y$ to Y at p_j with $\mathbb{C}^4/\langle\alpha\rangle$ so that g_Y is identified with $|\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_4|^2$ at p_j, θ_Y is identified with $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4$ at p_j, and $\mathrm{d}\sigma : T_{p_j}Y \to T_{p_j}Y$ is identified with the map $\beta : \mathbb{C}^4/\langle\alpha\rangle \to \mathbb{C}^4/\langle\alpha\rangle$ given by*

$$\beta : (z_1, \ldots, z_4)\langle\alpha\rangle \longmapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3)\langle\alpha\rangle. \tag{7}$$

Proof. Since J, g_Y and θ_Y form a Calabi–Yau structure on Y, there certainly exists an isomorphism $\iota : T_{p_j}Y \to \mathbb{C}^4/\langle\alpha\rangle$ which identifies g_Y with $|\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_4|^2$ and θ_Y with $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4$. This ι is unique up to the action of SU(4) on $\mathbb{C}^4/\langle\alpha\rangle$. That is, if $B \in$ SU(4), then

$B \circ \iota : T_{p_j}Y \to \mathbb{C}^4/\langle\alpha\rangle$ also identifies g_Y with $|dz_1|^2 + \cdots + |dz_4|^2$ and θ_Y with $dz_1 \wedge \cdots \wedge dz_4$.

Now $d\sigma : T_{p_j}Y \to T_{p_j}Y$ is complex antilinear, and so ι identifies $d\sigma$ with the map $\gamma : \mathbb{C}^4/\langle\alpha\rangle \to \mathbb{C}^4/\langle\alpha\rangle$ given by

$$\gamma : \left\{ i^k \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} : k = 0,1,2,3 \right\} \longmapsto \left\{ i^k A \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{pmatrix} : k = 0,1,2,3 \right\}, \tag{8}$$

for some 4×4 complex matrix A. In fact A is only defined up to multiplication by a power of i.

As $d\sigma$ preserves g_Y and takes θ_Y to $\bar{\theta}_Y$ on $T_{p_j}Y$, it follows that γ preserves $|dz_1|^2 + \cdots + |dz_4|^2$ and takes $dz_1 \wedge \cdots \wedge dz_4$ to $d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_4$ on $\mathbb{C}^4/\langle\alpha\rangle$. These imply that $A\bar{A}^t = I$ and $\det(A) = 1$, and so $A \in \mathrm{SU}(4)$. Also, $\gamma^2 = I$ as $\sigma^2 = \mathrm{id}$, and this implies that $A\bar{A} = i^k I$ for $k = 0,1,2$ or 3. And because σ fixes only $p_1, \ldots, p_k$ in Y, the only fixed point of γ in $\mathbb{C}^4/\langle\alpha\rangle$ is 0.

So A lies in SU(4) and satisfies $A\bar{A} = i^k I$. When we replace ι by $B \circ \iota$ for $B \in \mathrm{SU}(4)$, the matrix A is replaced by BAB^t. We wish to show that we can choose $B \in \mathrm{SU}(4)$ such that the maps β of (7) and γ of (8) coincide. That is, we must show that there exists $B \in \mathrm{SU}(4)$ and $l = 0,1,2$ or 3 such that

$$i^l BAB^t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{9}$$

Now $A\bar{A} = i^k I$ shows that A and $\bar{A}$ commute, and so $A\bar{A} = \bar{A}A = \overline{A\bar{A}}$. Thus $i^k I$ is a real matrix, which implies that $k = 0$ or 2, and $A\bar{A} = \pm I$. By studying the eigenvectors of A, one can prove that there exists $B \in \mathrm{SU}(4)$ such that BAB^t is one of

$$I, \; -I, \; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We exclude the first three possibilities because γ fixes $(1,0,0,0)\langle\alpha\rangle$ in $\mathbb{C}^4/\langle\alpha\rangle$, contradicting the fact that the only fixed point of γ in $\mathbb{C}^4/\langle\alpha\rangle$ is 0. Putting $l = 0$ in the fourth case and $l = 3$ in the fifth, we see that

(9) holds. Thus $B \circ \iota$ identifies $T_{p_j}Y$ with $\mathbb{C}^4/\langle\alpha\rangle$ and satisfies all the conditions of the proposition, and the proof is complete. q.e.d.

Now §4.1 defined a finite group $G = \langle\alpha,\beta\rangle$ acting on $\mathbb{R}^8$, and the definitions (6) and (7) of α and β above coincide with (4) in §4.1. Thus the singularities of $Z = Y/\langle\sigma\rangle$ are all modelled on $\mathbb{R}^8/G$, and we easily prove:

Corollary 5.4. *Define $Z = Y/\langle\sigma\rangle$. Then Z is a compact, real 8-dimensional orbifold. The* Spin(7)*-structure (Ω_Y, g_Y) on Y pushes down to give a torsion-free* Spin(7)*-structure (Ω_Z, g_Z) on Z. The singularities of Z are k points $p_1, \ldots, p_k$. For each $j = 1, \ldots, k$ there is an isomorphism $\iota_j : \mathbb{R}^8/G \to T_{p_j}Z$ which identifies the* Spin(7)*-structures (Ω_0, g_0) on $\mathbb{R}^8/G$ and (Ω_Z, g_Z) on $T_{p_j}Z$. Here G and (Ω_0, g_0) are defined in §4.1.*

5.2 Desingularizing Z to get a compact 8-manifold M

So far we have constructed a Spin(7)-orbifold (Z, Ω_Z, g_Z) with finitely many singular points $p_1, \ldots, p_k$, each modelled on the singularity $\mathbb{R}^8/G$ of §4.1. But in §4.1 and §4.2 we wrote down two ALE Spin(7)-manifolds X_1 and X_2 asymptotic to $\mathbb{R}^8/G$. We shall now resolve each singular point p_j in Z using either X_1 or X_2 to get a compact 8-manifold M. We include a parameter $t \in (0, 1]$ in the construction.

Definition 5.5. For each j let ι_j be as in Corollary 5.4, and let $\exp_{p_j} : T_{p_j}Z \to Z$ be the *exponential map*, which is well-defined as Z is complete. Then $\exp_{p_j} \circ \iota_j$ maps $\mathbb{R}^8/G$ to Z. Choose $\zeta > 0$ small, and let $B_{2\zeta}(\mathbb{R}^8/G)$ be the open ball of radius 2ζ about 0 in $\mathbb{R}^8/G$. Define $U_j \subset Z$ by $U_j = \exp_{p_j} \circ \iota_j\big(B_{2\zeta}(\mathbb{R}^8/G)\big)$, and $\psi_j : B_{2\zeta}(\mathbb{R}^8/G) \to U_j$ by $\psi_j = \exp_{p_j} \circ \iota_j$. Let $\zeta > 0$ be chosen small enough that U_j is open in Z and $\psi_j : B_{2\zeta}(\mathbb{R}^8/G) \to U_j$ is a diffeomorphism for $1 \le j \le k$, and that $U_i \cap U_j = \emptyset$ when $i \ne j$.

Proposition 5.6. *There is a smooth 3-form σ_j on $B_{2\zeta}(\mathbb{R}^8/G)$ for $1 \le j \le k$ and a constant $C_1 > 0$, such that $\psi_j^*(\Omega_Z) - \Omega_0 = \mathrm{d}\sigma_j$ and $|\nabla^l\sigma_j| \le C_1 r^{3-l}$ on $B_{2\zeta}(\mathbb{R}^8/G)$, for $l = 0, 1, 2$. Here $|\,.\,|$ and ∇ are defined using the metric g_0 on $B_{2\zeta}(\mathbb{R}^8/G)$, and $r : B_{2\zeta}(\mathbb{R}^8/G) \to [0, 2\zeta)$ is the radius function.*

Proof. The derivative of $\exp_{p_j}$ at 0 is the identity map on $T_{p_j}Z$. Thus the derivative of ψ_j at 0 is $\iota_j : \mathbb{R}^8/G \to T_{p_j}Z$, and so $\psi_j^*(\Omega_Z)|_0 = \iota_j^*(\Omega_Z) = \Omega_0|_0$, since ι_j identifies Ω_0 and Ω_Z. Therefore $\psi_j^*(\Omega_Z) = \Omega_0$ at

0 in $B_{2\zeta}(\mathbb{R}^8/G)$. As $\psi_j^*(\Omega_Z)-\Omega_0$ is a 4-form on a subset of $\mathbb{R}^8/G$, we can pull it back to $\mathbb{R}^8$, and regard $\psi_j^*(\Omega_Z)-\Omega_0$ as a 4-form on the ball $B_{2\zeta}(\mathbb{R}^8)$ of radius 2ζ in $\mathbb{R}^8$.

Then $\psi_j^*(\Omega_Z)-\Omega_0$ is a smooth G-invariant 4-form on $B_{2\zeta}(\mathbb{R}^8)$ which vanishes at 0. But G contains $-1:\mathbb{R}^8\to\mathbb{R}^8$, and any 4-form invariant under this map -1 has zero first derivative at 0. Hence $\psi_j^*(\Omega_Z)-\Omega_0$ vanishes to first order at 0 in $B_{2\zeta}(\mathbb{R}^8)$, and so by Taylor's Theorem we can show that $\bigl|\psi_j^*(\Omega_Z)-\Omega_0\bigr|=O(r^2)$ and $\bigl|\nabla\psi_j^*(\Omega_Z)\bigr|=O(r)$ on $B_{2\zeta}(\mathbb{R}^8)$.

Now Ω_Z and Ω_0 are closed, so that $\psi_j^*(\Omega_Z)-\Omega_0$ is closed, and as $B_{2\zeta}(\mathbb{R}^8/G)$ is contractible we can write $\psi_j^*(\Omega_Z)-\Omega_0=\mathrm{d}\sigma_j$ for some smooth 3-form σ_j on $B_{2\zeta}(\mathbb{R}^8/G)$. Since $\psi_j^*(\Omega_Z)-\Omega_0$ vanishes to first order at 0 we can easily arrange that σ_j vanishes to second order at 0, and therefore $|\nabla^l\sigma_j|=O(r^{3-l})$ for $l=0,1,2$, using Taylor's Theorem as above. Thus there exists $C_1>0$ such that $|\nabla^l\sigma_j|\le C_1r^{3-l}$ on $B_{2\zeta}(\mathbb{R}^8/G)$, for $l=0,1,2$ and $j=1,\dots,k$. q.e.d.

Definition 5.7. Let the ALE Spin(7)-manifolds (X_n,Ω_n,g_n) and projections $\pi_n:X_n\to\mathbb{R}^8/G$ be as in §4.1 and §4.2 for $n=1,2$. For each $t\in(0,1]$ and $n=1,2$ let $X_n^t=X_n$, define a Spin(7)-structure (Ω_n^t,g_n^t) on X_n^t by $\Omega_n^t=t^4\Omega_n$ and $g_n^t=t^2g_n$, and define $\pi_n^t:X_n^t\to\mathbb{R}^8/G$ by $\pi_n^t=t\pi_n$. Then (X_n^t,Ω_n^t,g_n^t) is an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

Using the ideas of [15] or the explicit formula of Calabi [4, p. 285] we can show that there exist $C_2>0$ and a smooth 3-form τ_n^t on $\mathbb{R}^8/G\setminus B_{t\zeta}(\mathbb{R}^8/G)$, satisfying

$$(\pi_n^t)_*(\Omega_n^t)=\Omega_0+\mathrm{d}\tau_n^t \quad\text{and}\quad \bigl|\nabla^l\tau_n^t\bigr|\le C_2t^8r^{-7-l}\quad\text{for } l=0,1,2 \tag{10}$$

on $\mathbb{R}^8/G\setminus B_{t\zeta}(\mathbb{R}^8/G)$, where $|\,.\,|$ and ∇ are defined using the metric g_0.

For $j=1,\dots,k$, choose n_j to be 1 or 2. There are 2^k ways of defining the n_j. We shall resolve each singular point p_j in Z using $X_{n_j}^t$ to get a 1-parameter family of resolutions (M^t,π^t) of Z.

Definition 5.8. For each $j=1,\dots,k$, define open subsets M_0^t in Z and M_j^t in $X_{n_j}^t$ for $1\le j\le k$ by

$$M_0^t=Z\setminus\bigcup_{j=1}^k\psi_j\bigl(\overline{B}_{t^{4/5}\zeta}(\mathbb{R}^8/G)\bigr)\quad\text{and}\quad M_j^t=(\pi_{n_j}^t)^{-1}\bigl(B_{2t^{4/5}\zeta}(\mathbb{R}^8/G)\bigr).$$

That is, M_0^t is the complement in Z of the closed balls of radius $t^{4/5}\zeta$ about p_j for $1 \le j \le k$, and M_j^t is the inverse image of $B_{2t^{4/5}\zeta}(\mathbb{R}^8/G)$ in $X_{n_j}^t$.

Define an equivalence relation '$\sim$' on the disjoint union $\coprod_{j=0}^k M_j^t$ by $x \sim y$ if either (a) $x = y$,

(b) $x \in M_j^t$ and $y \in U_j \cap M_0^t$ and $\psi_j \circ \pi_{n_j}^t(x) = y$, for some $j = 1, \dots, k$, or

(c) $y \in M_j^t$ and $x \in U_j \cap M_0^t$ and $\psi_j \circ \pi_{n_j}^t(y) = x$, for some $j = 1, \dots, k$.

Define the *resolution* M^t of Z to be $\coprod_{j=0}^k M_j^t / \sim$. It is easy to see that M^t is a compact 8-manifold. Define a projection $\pi^t : M^t \to Z$ by $\pi^t([x]) = x$ when $x \in M_0^t$, and $\pi^t([x]) = \psi_j \circ \pi_{n_j}^t(x)$ when $x \in M_j^t$ for some $j = 1, \dots, k$, where $[x]$ is the equivalence class of x under $\sim$. Then π^t is well-defined, continuous and surjective, and $\pi^t : M^t \setminus \bigcup_{j=1}^k (\pi^t)^{-1}(p_j) \to Z \setminus \{p_1, \dots, p_k\}$ is a diffeomorphism.

Since the resolutions (M^t, π^t) of Z form a smooth connected family, they are all diffeomorphic to the same compact 8-manifold M. We can regard M_j^t as an open subset of M^t for $j = 0, \dots, k$, and then the M_j^t form an *open cover* of M^t. If $1 \le i, j \le k$ and $i \ne j$ then $M_i^t \cap M_j^t = \emptyset$. The overlap $M_0^t \cap M_j^t$ is naturally isomorphic to an *annulus* in $\mathbb{R}^8/G$, with inner radius $t^{4/5}\zeta$ and outer radius $2t^{4/5}\zeta$. The reason for including the factors $t^{4/5}$ will be explained shortly.

We now calculate the fundamental group of M^t.

Proposition 5.9. *If $n_j = 1$ for $j = 1, \dots, k$ then $\pi_1(M^t) \cong \mathbb{Z}_2$. Otherwise, M^t is simply-connected.*

Proof. Since $Y \setminus \{p_1, \dots, p_k\}$ is simply-connected by Condition 5.1 and σ acts freely on $Y \setminus \{p_1, \dots, p_k\}$, we see that the fundamental group of $Z \setminus \{p_1, \dots, p_k\}$ is $\mathbb{Z}_2$. The natural inclusion of $Z \setminus \{p_1, \dots, p_k\}$ in M^t induces a homomorphism from $\pi_1(Z \setminus \{p_1, \dots, p_k\})$ to $\pi_1(M^t)$, which is easily shown to be surjective. Also, as $X_{n_j}^t$ is X_1 or X_2 we have $\pi_1(X_{n_j}^t) \cong \mathbb{Z}_2$.

Therefore, $\pi_1(M_t)$ is $\mathbb{Z}_2$ if the generator of $\pi_1(Z \setminus \{p_1, \dots, p_k\})$ projects to the nonzero element of $\pi_1(X_{n_j}^t)$ for all $1 \le j \le k$, and $\pi_1(M^t)$ is trivial otherwise. But calculation shows that the generator of $\pi_1(Z \setminus \{p_1, \dots, p_k\})$ is nonzero in $\pi_1(X_{n_j}^t)$ if and only if $n_j = 1$. q.e.d.

This shows that of the 2^k possible ways of choosing the n_j, one possibility gives $\pi_1(M^t) = \mathbb{Z}_2$, and the remaining $2^k - 1$ possibilities all give simply-connected M^t.

5.3 A Spin(7)-structure (Ω^t, g^t) on M^t with small torsion

Each open subset M^t_j in M^t carries a torsion-free Spin(7)-structure, (Ω_Z, g_Z) for $j = 0$ and $(\Omega^t_{n_j}, g^t_{n_j})$ for $1 \le j \le k$. We shall join these Spin(7)-structures together with a partition of unity to get a Spin(7)-structure (Ω^t, g^t) on M^t and estimate its torsion.

Definition 5.10. Let $\eta : [0,\infty) \to [0,1]$ be a smooth function with $\eta(x) = 0$ for $x \le \zeta$ and $\eta(x) = 1$ for $x \ge 2\zeta$. Define a 4-form ξ^t on M^t by $\xi^t = \Omega_Z$ in $M^t_0 \setminus \bigcup_{j=1}^k M^t_j$, and $\xi^t = \Omega^t_{n_j}$ in $M^t_j \setminus M^t_0$ for $1 \le j \le k$, and

$$\xi^t = \Omega_0 + \mathrm{d}\big(\eta(t^{-4/5}r)\sigma_j\big) + \mathrm{d}\big((1-\eta(t^{-4/5}r))\tau^t_{n_j}\big) \quad \text{in } M^t_0 \cap M^t_j \tag{11}$$

for $1 \le j \le k$, where we identify $M^t_0 \cap M^t_j$ with an annulus in $\mathbb{R}^8/G$ in the natural way. Since $\Omega_Z = \Omega_0 + \mathrm{d}\sigma_j$ and $\Omega^t_{n_j} = \Omega_0 + \mathrm{d}\tau^t_{n_j}$ in $M^t_0 \cap M^t_j$, it follows that ξ^t is smooth, and as Ω_Z, $\Omega^t_{n_j}$ and Ω_0 are closed, ξ^t is closed.

Lemma 5.11. *There exists $C_3 > 0$ such that for each $j = 1, \dots, k$ and $t \in (0,1]$, this 4-form ξ^t satisfies*

$$\big|\xi^t - \Omega_0\big| \le C_3 t^{8/5} \quad \text{and} \quad \big|\nabla(\xi^t - \Omega_0)\big| \le C_3 t^{4/5} \tag{12}$$

in $M^t_0 \cap M^t_j$, where $|.|$ and ∇ are defined using the metric g_0.

Proof. Expanding (11) we find that

$$\begin{aligned}\xi^t - \Omega_0 =& \eta(t^{-4/5}r)\mathrm{d}\sigma_j + (1-\eta(t^{-4/5}r))\mathrm{d}\tau^t_{n_j} \\ &+ t^{-4/5}\eta'(t^{-4/5}r)\mathrm{d}r \wedge (\sigma_j - \tau^t_{n_j})\end{aligned}$$

in $M^t_0 \cap M^t_j$. Since $t^{4/5}\zeta \le r \le 2t^{4/5}\zeta$, Proposition 5.6 and (10) show that

$$\begin{aligned}&\big|\sigma_j\big| \le 8C_1\zeta^3 t^{12/5}, \quad \big|\mathrm{d}\sigma_j\big| \le 4C_1\zeta^2 t^{8/5}, \quad \big|\nabla\mathrm{d}\sigma_j\big| \le 2C_1\zeta t^{4/5}, \\ &\big|\tau^t_{n_j}\big| \le C_2\zeta^{-7} t^{12/5}, \quad \big|\mathrm{d}\tau^t_{n_j}\big| \le C_2\zeta^{-8} t^{8/5} \quad \text{and} \quad \big|\nabla\mathrm{d}\tau^t_{n_j}\big| \le C_2\zeta^{-9} t^{4/5}.\end{aligned}$$

Combining these with the previous equation and using the facts that $|\mathrm{d}r| = 1$ and η' is bounded independently of t, we soon prove (12).

q.e.d.

We can now explain why we chose the power $t^{4/5}$ in Definition 5.8. Suppose we had defined M^t and ξ^t using t^α in place of $t^{4/5}$, for some $\alpha \in [0,1]$. Then in the calculation above the σ_j and $\tau^t_{n_j}$ terms would contribute $O(t^{2\alpha})$ and $O(t^{8-8\alpha})$ to $\xi^t - \Omega_0$ respectively, and so $\xi^t - \Omega_0$ would be $O(t^{2\alpha}) + O(t^{8-8\alpha})$. This is smallest when $2\alpha = 8 - 8\alpha$, that is, when $\alpha = 4/5$. So the power $t^{4/5}$ minimizes the size of $\xi^t - \Omega_0$.

Now we can define the Spin(7)-structures (Ω^t, g^t) on M^t.

Definition 5.12. Let ρ be as in Proposition 2.5, and choose $\epsilon \in (0,1]$ such that $C_3\epsilon^{8/5} \le \rho$. Suppose $t \in (0,\epsilon]$. Then

$$\left|\xi^t - \Omega_0\right| \le C_3 t^{8/5} \le \rho$$

in $M^t_0 \cap M^t_j$ for $1 \le j \le k$ by (12), and so ξ^t lies in $\mathcal{T}M^t$ on $M^t_0 \cap M^t_j$ by part (i) of Proposition 2.5. But ξ^t is Ω_Z or $\Omega^t_{n_j}$ outside the overlaps $M^t_0 \cap M^t_j$, and thus $\xi^t \in C^\infty(\mathcal{T}M^t)$. For each $t \in (0,\epsilon]$ define $\Omega^t = \Theta(\xi^t)$, where Θ is given in Proposition 2.5. Then $\Omega^t \in C^\infty(\mathcal{A}M^t)$, and so Ω^t extends to a Spin(7)-structure (Ω^t, g^t) on M^t. Define a 4-form ϕ^t on M^t by $\phi^t = \xi^t - \Omega^t$. Then $\mathrm{d}\Omega^t + \mathrm{d}\phi^t = 0$, as $\mathrm{d}\xi^t = 0$ on M^t.

Here ξ^t is a 4-form which does not lie in $\mathcal{A}M^t$, but is close to $\mathcal{A}M^t$ for small t, and Ω^t is the section of $\mathcal{A}M^t$ closest to ξ^t. What is really happening is that the Spin(7)-structure (Ω^t, g^t) is equal to $(\Omega^t_{n_j}, g^t_{n_j})$ in $M^t_j \backslash M^t_0$ and to (Ω_Z, g_Z) outside M^t_j for $j = 1, \ldots, k$, and (Ω^t, g^t) interpolates smoothly between these two possibilities on the annulus $M^t_j \cap M^t_0$.

5.4 Existence of torsion-free Spin(7)-structures on M

Next we shall show that (Ω^t, g^t) can be deformed to a torsion-free Spin(7)-structure on M when t is small.

Theorem 5.13. *In the situation above, there exist constants $\lambda, \mu, \nu > 0$ such that for all $t \in (0,\epsilon]$ we have*

(i) $\|\phi^t\|_{L^2} \le \lambda t^{24/5}$ *and* $\|\mathrm{d}\phi^t\|_{L^{10}} \le \lambda t^{36/25}$*;*

(ii) *the injectivity radius $\delta(g^t)$ satisfies $\delta(g^t) \ge \mu t$; and*

(ii) *the Riemann curvature $R(g^t)$ satisfies $\left\|R(g^t)\right\|_{C^0} \le \nu t^{-2}$.*

Here all norms are calculated using the metric g^t on M^t.

Proof. Outside the overlaps $M^t_0 \cap M^t_j$ for $1 \le j \le k$ we either have $\xi^t = \Omega^t = \Omega_Z$ or $\xi^t = \Omega^t = \Omega^t_{n_j}$. In both cases $\phi^t = \xi^t - \Omega^t = 0$, and so ϕ^t is zero outside the $M_0^t \cap M_j^t$. In $M_0^t \cap M_j^t$ we apply part (ii) of Proposition 2.5 with $\Omega = \Omega_0$ and $\xi = \xi^t$, to get

$$\left|\phi^t\right|_{g^t} \le |\xi^t - \Omega_0|_{g_0} \quad \text{and} \quad \left|\nabla^{g^t}\phi^t\right|_{g^t} \le C\left|\nabla^{g_0}(\xi^t - \Omega_0)\right|_{g_0}.$$

Combining this with (12) gives

$$\left|\phi^t\right|_{g^t} \le C_3 t^{8/5} \quad \text{and} \quad \left|\mathrm{d}\phi^t\right|_{g^t} \le \left|\nabla^{g^t}\phi^t\right|_{g^t} \le CC_3 t^{4/5}.$$

Now each $M_0^t \cap M_j^t$ is an annulus in $\mathbb{R}^8/G$ with inner radius $t^{4/5}\zeta$ and outer radius $2t^{4/5}\zeta$, and the metric g^t on $M_0^t \cap M_j^t$ is close to the flat metric g_0 on $\mathbb{R}^8/G$. Therefore we can find $C_4 > 0$ independent of t such that $\sum_{j=1}^k \mathrm{vol}\big(M_0^t \cap M_j^t\big) \le C_4 t^{32/5}$. Hence

$$\int_{M^t} |\phi^t|^2 \mathrm{d}V \le (C_3 t^{8/5})^2 C_4 t^{32/5} \quad \text{and}$$
$$\int_{M^t} |\mathrm{d}\phi^t|^{10} \mathrm{d}V \le (CC_3 t^{4/5})^{10} C_4 t^{32/5}.$$

Taking roots gives part (i) of the theorem, with $\lambda = C_3 \max(C_4^{1/2}, CC_4^{1/10})$.

Parts (ii) and (iii) are elementary. The metric $g_{n_j}^t$ is made by scaling g_{n_j} by a factor t. Thus $\delta(g_{n_j}^t) = t\delta(g_{n_j})$ and $\|R(g_{n_j}^t)\|_{C^0} = t^{-2}\|R(g_{n_j})\|_{C^0}$. We make g^t by gluing together the $g_{n_j}^t$ on the patches M_j^t for $j = 1, \ldots, k$ and g_Z on M_0^t. It is clear that for small t, the dominant contributions to $\delta(g^t)$ and $\|R(g^t)\|_{C^0}$ come from $\delta(g_{n_j}^t)$ and $\|R(g_{n_j}^t)\|_{C^0}$ for some j, and these are proportional to t and t^{-2}. This proves (ii) and (iii) for some $\mu, \nu > 0$, and the theorem is complete.

q.e.d.

Finally we can prove our main result.

Theorem 5.14. *Suppose Condition 5.1 holds, and let M be the compact 8-manifold defined in Definition 5.8. Then there exist torsion-free* Spin(7)*-structures $(\tilde{\Omega}, \tilde{g})$ on M. If $\pi_1(M) = \{1\}$ then* $\mathrm{Hol}(\tilde{g}) = \mathrm{Spin}(7)$*, and if $\pi_1(M) = \mathbb{Z}_2$ then* $\mathrm{Hol}(\tilde{g}) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$.

Proof. Let λ, μ, ν be as in Theorem 5.13. Then Theorem 2.6 gives a constant $\kappa > 0$. Choose $t > 0$ with $t \le \epsilon \le 1$ and $t \le \kappa$. Let (Ω, g) be the Spin(7)-structure (Ω^t, g^t) on $M = M^t$, and ϕ the 4-form ϕ^t. Then $\mathrm{d}\Omega + \mathrm{d}\phi = 0$ by Definition 5.12, and parts (i)–(iii) of Theorem 5.13 imply (i)–(iii) of Theorem 2.6, as $t \le 1$.

Therefore all the hypotheses of Theorem 2.6 hold, and the theorem shows that there exists a torsion-free Spin(7)-structure $(\tilde{\Omega}, \tilde{g})$ on M.

It remains to identify the holonomy group $\mathrm{Hol}(\tilde{g})$ of $\tilde{g}$. Now we can regard the Spin(7)-orbifold (Z, Ω_Z, g_Z) as the limit as $t \to 0$ of the Spin(7)-manifolds $(M, \tilde{\Omega}, \tilde{g})$. Because of this, it is not difficult to show that $\mathrm{Hol}(g_Z) \subseteq \mathrm{Hol}(\tilde{g})$.

Now $\mathrm{Hol}(g_Z) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$, and thus

$$\mathbb{Z}_2 \ltimes \mathrm{SU}(4) \subseteq \mathrm{Hol}(\tilde{g}) \subseteq \mathrm{Spin}(7).$$

If $\pi_1(M) = \{1\}$ then $\mathrm{Hol}(\tilde{g})$ is connected. But the only connected Lie subgroup of Spin(7) containing $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$ is Spin(7), so $\mathrm{Hol}(\tilde{g}) = \mathrm{Spin}(7)$. If $\pi_1(M) = \mathbb{Z}_2$ then $\mathrm{Hol}(\tilde{g}) \neq \mathrm{Spin}(7)$ by Theorem 2.3. This forces $\mathrm{Hol}^0(\tilde{g}) = \mathrm{SU}(4)$, and it is then easy to see that $\mathrm{Hol}(\tilde{g}) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$. q.e.d.

Since by Proposition 5.9 we can always choose the n_j so that M is simply-connected, we can always arrange for $\tilde{g}$ to have holonomy Spin(7). When $\pi_1(M) = \mathbb{Z}_2$, the complex orbifold Y has a crepant resolution $\tilde{Y}$, which admits Kähler metrics $\tilde{g}$ with holonomy SU(4), making it into a Calabi–Yau manifold. The action of σ on Y lifts to a *free* action of σ on $\tilde{Y}$, and so $M = \tilde{Y}/\langle\sigma\rangle$ is a compact 8-manifold. If we choose $\tilde{g}$ to be σ-invariant then it pushes down to M, and has holonomy $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$.

6. How to apply the construction

We now explain ways of finding orbifolds Y and involutions $\sigma : Y \to Y$ satisfying Condition 5.1, and how to calculate the Betti numbers of the resulting 8-manifolds M with holonomy Spin(7).

6.1 Finding suitable Calabi–Yau 4-orbifolds Y

To apply the construction of §5 we need a source of compact Kähler 4-orbifolds Y with $c_1(Y) = 0$ and isolated singularities modelled on $\mathbb{C}^4/\mathbb{Z}_4$. Fortunately, physicists and algebraic geometers have been studying Calabi–Yau manifolds for many years, mainly in complex dimension 3. Several powerful methods have been developed for constructing Calabi–Yau manifolds, and we will adapt some of these to our problem.

The main idea we shall use is borrowed from Candelas, Lynker and Schrimmrigk [5], who constructed a large number of Calabi–Yau 3-folds as crepant resolutions of hypersurfaces in weighted projective spaces $\mathbb{CP}^4_{a_0,\dots,a_4}$. We shall explain their methods, beginning with *weighted projective spaces*, which are an important class of complex orbifolds.

Definition 6.1. Let $m \geq 1$ be an integer, and $a_0, a_1, \ldots, a_m$ positive integers with highest common factor 1. Let $\mathbb{C}^{m+1}$ have complex coordinates on $(z_0, \ldots, z_m)$, and define an action of the complex Lie group $\mathbb{C}^*$ on $\mathbb{C}^{m+1}$ by

$$(z_0, \ldots, z_m) \stackrel{u}{\longmapsto} (u^{a_0} z_0, \ldots, u^{a_m} z_m), \qquad \text{for } u \in \mathbb{C}^*. \tag{13}$$

Define the *weighted projective space* $\mathbb{CP}^m_{a_0,\ldots,a_m}$ to be $\bigl(\mathbb{C}^{m+1} \setminus \{0\}\bigr)/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{m+1} \setminus \{0\}$ with the action (13). Then $\mathbb{CP}^m_{a_0,\ldots,a_m}$ is compact and Hausdorff, and has the structure of a *complex orbifold.*

Let $[z_0, \ldots, z_m]$ be a point in $\mathbb{CP}^m_{a_0,\ldots,a_m}$, and let k be the highest common factor of the set of those a_j for which $z_j \neq 0$. If $k = 1$ then $[z_0, \ldots, z_m]$ is a nonsingular point of $\mathbb{CP}^m_{a_0,\ldots,a_m}$, and if $k > 1$ then $[z_0, \ldots, z_m]$ is an orbifold point with orbifold group $\mathbb{Z}_k$.

We call a polynomial $f(z_0, \ldots, z_m)$ *weighted homogeneous of degree* d if

$$f(u^{a_0} z_0, \ldots, u^{a_m} z_m) = u^d f(z_0, \ldots, z_m) \quad \text{for all } u, z_0, \ldots, z_m \in \mathbb{C}.$$

Let f be such a polynomial, and define a hypersurface Y in $\mathbb{CP}^m_{a_0,\ldots,a_m}$ by

$$Y = \bigl\{[z_0, \ldots, z_m] \in \mathbb{CP}^m_{a_0,\ldots,a_m} : f(z_0, \ldots, z_m) = 0\bigr\}.$$

Then we call Y a *hypersurface of degree* d in $\mathbb{CP}^m_{a_0,\ldots,a_m}$.

We say that f is *transverse* if $f(z_0, \ldots, z_m) = 0$ and $\mathrm{d}f(z_0, \ldots, z_m) = 0$ have no common solutions in $\mathbb{C}^{m+1} \setminus \{0\}$. If f is transverse then the only singular points of Y are also singular points of $\mathbb{CP}^m_{a_0,\ldots,a_m}$, and Y is an *orbifold*, all of whose orbifold groups are cyclic. Note that for given weights $a_0, \ldots, a_m$ and degree d, there may not exist any transverse polynomials f.

So let Y be a hypersurface of degree d in $\mathbb{CP}^m_{a_0,\ldots,a_m}$, defined by a transverse polynomial. Using the *adjunction formula*, we find that $c_1(Y) = 0$ if and only if $d = a_0 + \cdots + a_m$. In this case it is easy to show that Y is a *Calabi–Yau orbifold.* Candelas et al. [5] considered the case $m = 4$, and used a computer to search for Calabi–Yau 3-orbifolds of this kind, finding some 6000 examples. They then resolved the singularities of each to get a Calabi–Yau 3-manifold.

As we are interested in Calabi–Yau 4-orbifolds, we shall consider hypersurfaces Y in $\mathbb{CP}^5_{a_0,\ldots,a_5}$. Here is a simple class of such Y.

Example 6.2. Let $a_0, \ldots, a_5$ be positive integers with highest common factor 1, and let $d = a_0 + \cdots + a_5$. Usually we order the a_j with $a_0 \le a_1 \le \cdots \le a_5$. Suppose that a_j divides d for $j = 0, \ldots, 5$, and define $k_j = d/a_j$. Define a hypersurface Y in $\mathbb{CP}^5_{a_0,\ldots,a_5}$ by

$$Y = \bigl\{[z_0, \ldots, z_5] \in \mathbb{CP}^5_{a_0,\ldots,a_5} : z_0^{k_0} + \cdots + z_5^{k_5} = 0\bigr\}.$$

Since $a_j k_j = d$ we see that $z_0^{k_0} + \cdots + z_5^{k_5}$ is a weighted homogeneous polynomial of degree d, and it is also transverse.

Therefore Y is a complex orbifold, with singularities only at the intersection of Y with the singular set of $\mathbb{CP}^5_{a_0,\ldots,a_5}$. Since the degree d of Y satisfies $d = a_0 + \cdots + a_5$, we have $c_1(Y) = 0$. Also Y admits Kähler metrics, as $\mathbb{CP}^5_{a_0,\ldots,a_5}$ is Kähler. So Y is a compact complex orbifold with $c_1(Y) = 0$, admitting Kähler metrics.

Now to apply the construction of §5, the singular points of Y must satisfy Condition 5.1. This is a strong restriction on $a_0, \ldots, a_5$, which admits only a few solutions. However, we can get many other suitable orbifolds Y by generalizing our construction a bit. Here are four ways to do this.

- **Defining Y by a different polynomial.** We could define Y using some more general transverse weighted homogenous polynomial of degree d in $z_0, \ldots, z_5$, instead of $z_0^{k_0} + \cdots + z_5^{k_5}$. The requirement that a_j divides d for $j = 0, \ldots, 5$ is then replaced by some other condition on the a_j and d.
- **Dividing by a finite group.** Let W be a Calabi–Yau hypersurface in $\mathbb{CP}^5_{a_0,\ldots,a_5}$, and G a finite group acting on W preserving its Calabi–Yau structure. Then $Y = W/G$ is a Calabi–Yau orbifold.
- **Partial crepant resolutions.** Let W be a Calabi–Yau hypersurface in $\mathbb{CP}^5_{a_0,\ldots,a_5}$ which has some singularities of the kind we want, together with other singularities that we don't want. We let Y be a partial crepant resolution of W, which resolves the singularities that we don't want, leaving those that we do.
- **Complete intersections in $\mathbb{CP}^m_{a_0,\ldots,a_5}$.** Rather than a hypersurface in $\mathbb{CP}^5_{a_0,\ldots,a_5}$, we take Y to be a *complete intersection* of $m-4$ hypersurfaces in $\mathbb{CP}^m_{a_0,\ldots,a_m}$, for some $m > 5$.

We can also use combinations of these four techniques — for instance, we can take Y to be a partial crepant resolution of W/G, where W is a hypersurface in $\mathbb{CP}^5_{a_0,\ldots,a_5}$, and G a finite group acting on W.

6.2 Antiholomorphic maps $\sigma : Y \to Y$

Suppose we have chosen an orbifold Y as above, with isolated singular points $p_1, \dots, p_k$. The next ingredient in our construction is an antiholomorphic involution $\sigma : Y \to Y$, which should fix only $p_1, \dots, p_k$. For example, suppose Y is a hypersurface in $\mathbb{CP}^5_{a_0,\dots,a_5}$. Then to find σ we would look for an antiholomorphic involution $\sigma : \mathbb{CP}^5_{a_0,\dots,a_5} \to \mathbb{CP}^5_{a_0,\dots,a_5}$ with $\sigma(Y) = Y$, and restrict σ to Y.

The most obvious such σ maps $[z_0, \dots, z_5] \mapsto [\bar{z}_0, \dots, \bar{z}_5]$. But this will not do, as its fixed points are not isolated in Y. To get isolated fixed points we need to try something more subtle. Here is an example of the kind of thing we mean.

Example 6.3. In the situation of Example 6.2, suppose that $a_0, \dots, a_3$ are odd and a_4, a_5 even with $a_0 = a_1$, $a_2 = a_3$ and $a_4 = a_5$. Define $\sigma : \mathbb{CP}^5_{a_0,\dots,a_5} \to \mathbb{CP}^5_{a_0,\dots,a_5}$ by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

As σ swaps the pairs z_0, z_1 and z_2, z_3 and z_4, z_5, we need $a_0 = a_1$, $a_2 = a_3$ and $a_4 = a_5$ for σ to be well-defined. Clearly σ is antiholomorphic, and $\sigma(Y) = Y$.

Now σ^2 acts by

$$\sigma^2 : [z_0, \dots, z_5] \mapsto [-z_0, -z_1, -z_2, -z_3, z_4, z_5].$$

But putting $u = -1$ in (13) gives $[-z_0, -z_1, -z_2, -z_3, z_4, z_5] = [z_0, \dots, z_5]$, as $a_0, \dots, a_3$ are odd and a_4, a_5 even. Thus $\sigma^2 = 1$, and $\sigma : Y \to Y$ is an antiholomorphic involution.

It is not difficult to show that the fixed points of σ in $\mathbb{CP}^5_{a_0,\dots,a_5}$ are

$$\big\{[0,0,0,0,1,\mathrm{e}^{i\theta}] \in \mathbb{CP}^5_{a_0,\dots,a_5} : \theta \in [0, 2\pi)\big\}.$$

Now $[0,0,0,0,1,\mathrm{e}^{i\theta}]$ lies in Y if $1 + e^{k_5 i\theta} = 0$. The solutions to this equation are $\mathrm{hcf}(k_4, k_5)$ isolated points in Y.

Observe the trick we have used here: if $a_j = a_{j+1}$ then we can choose σ to act on the coordinates z_j, z_{j+1} by $(z_j, z_{j+1}) \mapsto (\bar{z}_{j+1}, -\bar{z}_j)$. All the fixed points of σ will then satisfy $z_j = z_{j+1} = 0$. By doing this with two pairs of coordinates, say z_0, z_1 and z_2, z_3, the fixed points of σ satisfy $z_0 = z_1 = z_2 = z_3 = 0$. Thus they will be of complex codimension 4 in Y, and will be *isolated*, as we want.

This trick can also be adapted to more general situations, in which Y is a quotient by a finite group, or a partial crepant resolution, and so on. Note that as σ^2 maps $(z_j, z_{j+1}) \mapsto (-z_j, -z_{j+1})$, care must be taken to ensure that $\sigma^2 = 1$.

6.3 Calculating the Euler characteristic of Y

To determine the Betti numbers of the 8-manifold M that we construct, we will need to know the *Euler characteristic* of Y. Now there are two different notions of the Euler characteristic of an orbifold, defined by Satake [19, §3.3]. The version we are interested in is the *ordinary Euler characteristic* $\chi(Y)$, which is an integer and satisfies $\chi(Y) = \sum_{j=0}^{2n}(-1)^j b^j(Y)$. There is also the *orbifold Euler characteristic* $\chi_V(Y)$, which is a rational number that crops up naturally in problems involving characteristic classes.

In the next example we explain an elementary and fairly crude method for finding $\chi(Y)$ in the case that Y is a hypersurface in $\mathbb{CP}^m_{a_0,\dots,a_m}$, of the kind considered in Example 6.2. It is also possible to calculate $\chi_V(Y)$ using Chern classes and get $\chi(Y)$ by adding on contributions from the singular set (see for instance Hosono et al. [9, §2]), but we will not discuss this.

Example 6.4. Let $a_0, \dots, a_m$, $k_0, \dots, k_m$ and d be positive integers with $a_j k_j = d$ for $j = 0, \dots, m$. For each $j = 0, \dots, m$, define $Y_j \subset \mathbb{CP}^j_{a_0,\dots,a_j}$ by

$$Y_j = \{[z_0, \dots, z_j] \in \mathbb{CP}^j_{a_0,\dots,a_j} : z_0^{k_0} + \dots + z_j^{k_j} = 0\},$$

and define $\pi_j : Y_j \to \mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}}$ by $\pi_j : [z_0, \dots, z_j] \mapsto [z_0, \dots, z_{j-1}]$.

Suppose for simplicity that a_i divides a_j for $0 \le i < j \le m$. Then for each j, π_j is a k_j-fold branched cover of $\mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}}$, branched over Y_{j-1}. That is, if $p \in \mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}}$ then $\pi_j^{-1}(p)$ is one point when $p \in Y_{j-1}$ and k_j points when $p \notin Y_{j-1}$. It follows that

$$\begin{aligned} \chi(Y_j) &= k_j \cdot \chi(\mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}}) + (1 - k_j)\chi(Y_{j-1}) \\ &= k_j j + (1 - k_j)\chi(Y_{j-1}), \end{aligned} \tag{14}$$

since $\chi(\mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}}) = j$. This equation gives $\chi(Y_j)$ in terms of $\chi(Y_{j-1})$. Hence by induction we can write $\chi(Y_m)$ in terms of $\chi(Y_0)$. But $Y_0 = \emptyset$ so that $\chi(Y_0) = 0$, and thus we determine $\chi(Y_m)$.

If a_i does not divide a_j for some $0 \leq i < j \leq m$, then π_j is also branched over other parts of $\mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}}$. Let $p = [z_0, \dots, z_{j-1}]$ be in $\mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}} \setminus Y_{j-1}$, and let I be the set of i in $\{0, \dots, j-1\}$ for which $z_i \neq 0$. Define $l = \mathrm{hcf}(a_i : i \in I)$ and $m = \mathrm{hcf}(l, a_j)$. Then it turns out that $\pi_j^{-1}(p)$ is $k_j m/l$ points in Y_j. Clearly $k_j m/l = k_j$ if $l = m$, that is, if l divides a_j.

Thus π_j is also branched over subsets of $\mathbb{CP}^{j-1}_{a_0,\dots,a_{j-1}} \setminus Y_{j-1}$ corresponding to subsets $I \subseteq \{0, \dots, j-1\}$ for which $l = \mathrm{hcf}(a_i : i \in I)$ does not divide a_j. To calculate $\chi(Y_j)$ in this case we must modify (14) by adding in contributions from each such I. We will explain this when we meet it in examples later.

6.4 How to find topological invariants of Y, Z and M

To calculate the cohomology and fundamental group of our complex orbifolds Y we will need the following result, a form of the *Lefschetz Hyperplane Theorem*. It is proved in Griffiths and Harris [8, p. 156] and Goresky and MacPherson [6, p. 153].

Theorem 6.5. *Let M be a compact, m-dimensional complex manifold, N a nonsingular hypersurface in M, and L the holomorphic line bundle over M associated to the divisor N. Suppose L is positive. Then:*

(a) *the map $H^k(M, \mathbb{C}) \to H^k(N, \mathbb{C})$ induced by the inclusion $N \hookrightarrow M$ is an isomorphism for $0 \leq k \leq m-2$ and injective for $k = m-1$, and*

(b) *the map of homotopy groups $\pi_k(N) \to \pi_k(M)$ induced by the inclusion $N \hookrightarrow M$ is an isomorphism for $0 \leq k \leq m-2$ and surjective for $k = m-1$.*

The result also holds if M and N are orbifolds instead of manifolds, and N is a nonsingular hypersurface in the orbifold sense.

Here is a procedure for calculating the fundamental group and Betti numbers of Y, Z and M. The most difficult part is finding the Euler characteristic $\chi(Y)$, which we have already explained above.

(a) Calculate $\pi_1(Y)$, $H^2(Y, \mathbb{C})$ and $H^3(Y, \mathbb{C})$ explicitly. This can usually be done using Theorem 6.5. If Y is a hypersurface in $\mathbb{CP}^5_{a_0,\dots,a_5}$ then $\pi_1(Y) = \{1\}$, $H^2(Y, \mathbb{C}) = \mathbb{C}$ and $H^3(Y, \mathbb{C}) = 0$. Verify that $\pi_1\bigl(Y \setminus \{p_1, \dots, p_k\}\bigr) = \{1\}$ and $h^{2,0}(Y) = 0$, as in Condition 5.1.

(b) Compute the Euler characteristic $\chi(Y)$ of Y, as in §6.3.

(c) Calculate $H^2(Z,\mathbb{C})$ and $H^3(Z,\mathbb{C})$ from $H^2(Y,\mathbb{C})$ and $H^3(Y,\mathbb{C})$. Note that $H^j(Z,\mathbb{C})$ is the σ-invariant part of $H^j(Y,\mathbb{C})$. Since σ swaps $H^{p,q}(Y)$ and $H^{q,p}(Y)$, it follows that $b^3(Z) = \frac{1}{2}b^3(Y)$.

(d) Compute the Euler characteristic $\chi(Z)$ of Z. If σ fixes k points in Y, then this is given by $\chi(Z) = \frac{1}{2}\bigl(\chi(Y) + k\bigr)$.

(e) From (c) we know $b^2(Z)$ and $b^3(Z)$, and $b^1(Z) = 0$ as $\pi_1(Z)$ is finite. Thus we can calculate $b^4(Z)$ using the formula $b^4(Z) = \chi(Z) - 2 - 2b^2(Z) + 2b^3(Z)$.

(f) Now M was constructed in §5 by gluing $X_{n_1}, \ldots, X_{n_k}$ into Z, where $n_j = 1$ or 2 and X_1, X_2 are defined in §4. It is easy to show that the Betti numbers of X_1 and X_2 are $b^1 = b^2 = b^3 = 0$ and $b^4 = 1$. Therefore the Betti numbers $b^j(M)$ satisfy

$$b^j(M) = b^j(Z) \text{ for } j = 1,2,3, \text{ and } b^4(M) = b^4(Z) + k. \tag{15}$$

Also, Proposition 5.9 gives $\pi_1(M)$.

(g) As M has metrics with holonomy Spin(7) or $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$ by Theorem 5.14, we know that $\hat{A}(M) = 1$. Thus (2) gives

$$b^2(M) - b^3(M) - b^4_+(M) + 2b^4_-(M) + 25 = 0.$$

So we can calculate $b^4_\pm(M)$ using the equations

$$\begin{aligned} b^4_+(M) &= \tfrac{1}{3}\bigl(b^2(M) - b^3(M) + 2b^4(M) + 25\bigr), \\ b^4_-(M) &= \tfrac{1}{3}\bigl(-b^2(M) + b^3(M) + b^4(M) - 25\bigr). \end{aligned} \tag{16}$$

6.5 A way of checking the answers

If you make a mistake at some stage in these calculations, which is quite easy to do, then you are likely not to notice unless your values for $b^4_\pm(M)$ are not integers. Thus it is desirable to have some method for checking the answers. Here is a way of doing this. All of our examples have been checked for consistency in this way and others, but for brevity we will leave out the calculations.

Suppose we can compute the Hodge number $h^{3,1}(Y)$, using complex geometry. Then we can compute $b^4_-(Z)$ using the formula

$$b^4_-(Z) = h^{3,1}(Y) + b^2(Y) - b^2(Z) - 1.$$

But as X_1 and X_2 have $b^4_- = 1$, as in (15) we have $b^4_-(M) = b^4_-(Z) + k$. This gives an independent way of finding $b^4_-(M)$, which can be compared with your answer in part (g) above.

Now there is a complicated method for computing $h^{3,1}(Y)$ involving spectral sequences, and also a much simpler method called the 'polynomial deformation method' which *does not always give the right answer.* Both are discussed by Green and Hübsch [7]. Here is a sketch of the polynomial deformation method.

For simplicity suppose that Y is a hypersurface of degree d in $\mathbb{CP}^5_{a_0,\dots,a_5}$. As Y is a Calabi-Yau orbifold, $h^{3,1}(Y)$ is the dimension of the moduli space of complex structures on Y. We assume (this is *not* necessarily true) that every small deformation of Y is also a hypersurface of degree d in $\mathbb{CP}^5_{a_0,\dots,a_5}$, and that two nearby isomorphic hypersurfaces Y, Y' of degree d are related by an automorphism of $\mathbb{CP}^5_{a_0,\dots,a_5}$.

If these assumptions hold, then $h^{3,1}(Y) = m - n$, where m is the dimension of the space of hypersurfaces of degree d in $\mathbb{CP}^5_{a_0,\dots,a_5}$, and n is the dimension of the automorphism group of $\mathbb{CP}^5_{a_0,\dots,a_5}$. Both m and n are readily computed from $a_0, \dots, a_5$ and d.

7. A simple example

Let Y be the hypersurface of degree 12 in $\mathbb{CP}^5_{1,1,1,1,4,4}$ given by

$$Y = \bigl\{[z_0, \dots, z_5] \in \mathbb{CP}^5_{1,1,1,1,4,4} : z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + z_4^3 + z_5^3 = 0\bigr\}.$$

Then $c_1(Y) = 0$, as $12 = 1+1+1+1+4+4$, and Y is Kähler as $\mathbb{CP}^5_{1,\dots,4}$ is Kähler. Calculation shows that Y has three singular points $p_1 = [0,0,0,0,1,-1]$, $p_2 = [0,0,0,0,1,e^{\pi i/3}]$ and $p_3 = [0,0,0,0,1,e^{-\pi i/3}]$, satisfying Condition 5.1.

We use the method of §6.3 to calculate the Euler characteristic $\chi(Y)$.

Proposition 7.1. *The orbifold Y defined above has $\chi(Y) = 4887$.*

Proof. Define Y_j and π_j as in §6.3, where $Y_5 = Y$. Then Y_1 is the set of 12 points $[z_0, z_1]$ in $\mathbb{CP}^1$ with $z_0^{12} + z_1^{12} = 0$, and so $\chi(Y_1) = 12$. Now $\pi_2 : Y_2 \to \mathbb{CP}^1$ is a 12-fold branched cover branched over Y_1, so by (14) we have

$$\chi(Y_2) = 12\chi(\mathbb{CP}^1) - 11\chi(Y_1) = 12 \cdot 2 - 11 \cdot 12 = -108.$$

Similarly, $\pi_3 : Y_3 \to \mathbb{CP}^2$ is a 12-fold branched cover branched over Y_2, so that

$$\chi(Y_3) = 12\chi(\mathbb{CP}^2) - 11\chi(Y_2) = 12 \cdot 2 - 11 \cdot (-108) = 1224.$$

And $\pi_4 : Y_4 \to \mathbb{CP}^3$ is a 3-fold branched cover of $\mathbb{CP}^3$ branched over Y_3, giving

$$\chi(Y_4) = 3\chi(\mathbb{CP}^3) - 2\chi(Y_3) = 3 \cdot 4 - 2 \cdot 1224 = -2436.$$

Finally, $\pi_5 : Y \to \mathbb{CP}^4_{1,1,1,1,4}$ is a 3-fold branched cover of $\mathbb{CP}^4_{1,1,1,1,4}$ branched over Y_4, and so

$$\chi(Y) = 3\chi(\mathbb{CP}^4_{1,1,1,1,4}) - 2\chi(Y_4) = 3 \cdot 5 - 2 \cdot (-2436) = 4887,$$

as we want. q.e.d.

Proposition 7.2. *The Betti numbers of Y are*

$$b^0(Y) = 1, \;\; b^1(Y) = 0, \;\; b^2(Y) = 1, \;\; b^3(Y) = 0 \;\; \textit{and} \;\; b^4(Y) = 4883.$$

Also $Y \setminus \{p_1, p_2, p_3\}$ is simply-connected and $h^{2,0}(Y) = 0$.

Proof. Theorem 6.5 shows that $H^k(Y, \mathbb{C}) \cong H^k(\mathbb{CP}^5_{1,\dots,4}, \mathbb{C})$ for $0 \leq k \leq 3$. Since $b^k(\mathbb{CP}^5_{1,\dots,4})$ is 1 for k even with $0 \leq k \leq 10$ and 0 otherwise, this shows that $b^0(Y) = b^2(Y) = 1$ and $b^1(Y) = b^3(Y) = 0$, and so $b^4(Y) = 4883$ as $\chi(Y) = 4887$.

Theorem 6.5 also gives $\pi_1(Y) \cong \pi_1(\mathbb{CP}^5_{1,\dots,4})$, so Y is simply-connected. As the nonsingular set of $\mathbb{CP}^5_{1,\dots,4}$ is simply-connected, we can strengthen this to show that $Y \setminus \{p_1, p_2, p_3\}$ is simply-connected. The isomorphism $H^k(Y, \mathbb{C}) \cong H^k(\mathbb{CP}^5_{1,\dots,4}, \mathbb{C})$ above identifies $H^{p,q}(Y)$ with $H^{p,q}(\mathbb{CP}^5_{1,\dots,4})$, and so $h^{p,q}(Y) = h^{p,q}(\mathbb{CP}^5_{1,\dots,4})$ for $p + q \leq 3$. Hence $h^{2,0}(Y) = 0$. q.e.d.

Now define a map $\sigma : Y \to Y$ by

$$\sigma : [z_0, \dots, z_5] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

As in Example 6.3, we find that σ is an antiholomorphic involution of Y, and that the fixed points of σ are exactly p_1, p_2, p_3. Thus Condition 5.1 holds for Y and σ. So we can apply the construction of §5, and resolve the orbifold $Z = Y/\langle\sigma\rangle$ to get a compact 8-manifold M. Choosing $n_j = 2$ for at least one $j = 1, 2, 3$, Proposition 5.9 shows that M is simply-connected, and Theorem 5.14 shows that M admits metrics with holonomy Spin(7).

Theorem 7.3. *This compact* 8*-manifold* M *has Betti numbers*

$$b^0 = 1, \ b^1 = b^2 = b^3 = 0, \ b^4 = 2446, \ b^4_+ = 1639 \ \textit{and} \ b^4_- = 807.$$

There exist metrics with holonomy Spin(7) *on* M*, which form a smooth family of dimension* 808.

Proof. We first calculate the Betti numbers of Z. As σ fixes 3 points in Y, by properties of the Euler characteristic we find that $\chi(Z) = \frac{1}{2}(\chi(Y)+3)$. But $\chi(Y) = 4887$ by Proposition 7.1, so $\chi(Z) = 2445$. As $H^k(Z,\mathbb{C})$ is the σ-invariant part of $H^k(Y,\mathbb{C})$ we see from Proposition 7.2 that $b^0(Z) = 1$ and $b^1(Z) = b^3(Z) = 0$. Also $H^2(Y,\mathbb{C})$ is generated by $[\omega_Y]$ and $\sigma^*(\omega_Y) = -\omega_Y$, so σ acts as -1 on $H^2(Y,\mathbb{C})$, and $H^2(Z,\mathbb{C}) = 0$.

Thus $b^0(Z) = 1$, $b^1(Z) = b^2(Z) = b^3(Z) = 0$ and $\chi(Z) = 2445$, giving $b^4(Z) = 2443$. Equation (15) then gives the Betti numbers of M, and (16) gives $b^4_\pm$. Theorem 5.14 shows that there exist torsion-free Spin(7)-structures $(\tilde{\Omega},\tilde{g})$ on M, with $\mathrm{Hol}(\tilde{g}) = \mathrm{Spin}(7)$ as M is simply-connected. By Theorem 2.4 the moduli space of metrics on M with holonomy Spin(7) is a smooth manifold of dimension $1 + b^4_-(M) = 808$. q.e.d.

7.1 A variation on this example

Here is a variation on the above, using the idea of *partial crepant resolution* mentioned in §6.1. Let Y be as above, but define $\sigma' : Y \to Y$ by

$$\sigma' : [z_0,\ldots,z_5] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5].$$

Then σ' is an antiholomorphic involution of Y, which fixes the point $p_1 = [0,0,0,0,1,-1]$ in Y, and no other points. In particular, σ' swaps over the other two singular points p_2, p_3.

Thus Y and σ' do not satisfy Condition 5.1, because the fixed set of σ' is not the same as the singular set $\{p_1,p_2,p_3\}$ of Y. To rectify this we resolve the singular points p_2,p_3. Let Y' be the blow-up of Y at p_2 and p_3. This is a crepant resolution of Y, and so is also a Calabi–Yau orbifold.

Then Y' has just the one singular point p_1. The action of σ' on Y lifts to Y', with sole fixed point p_1. Thus Condition 5.1 holds for Y' and σ'. Therefore we can apply the construction of §5 to Y' and σ', so that $Z' = Y'/\langle\sigma'\rangle$ is a compact Spin(7)-orbifold with one singular point p_1 modelled on $\mathbb{R}^8/G$. Choosing $n_1 = 2$ we get a resolution M' of

Z', which is a compact, simply-connected 8-manifold admitting metrics with holonomy Spin(7).

We shall calculate the topological invariants of Y' and M'.

Proposition 7.4. *The Betti numbers of Y' are*

$$b^0 = 1,\ \ b^1 = 0,\ \ b^2 = 3,\ \ b^3 = 0\ \ \textit{and}\ \ b^4 = 4885,\ \ \textit{so that}\ \ \chi(Y') = 4893.$$

Also, $Y' \setminus \{p_1\}$ is simply-connected and $h^{2,0}(Y') = 0$.

Proof. By definition Y' is the blow-up of Y at p_2, p_3. Each blow-up fixes b^1 and b^3 and adds 1 to b^2 and b^4. So the Betti numbers of Y' follow from Proposition 7.2. As $Y \setminus \{p_1, p_2, p_3\}$ is simply-connected and $h^{2,0}(Y) = 0$, we see that $Y' \setminus \{p_1\}$ is simply-connected and $h^{2,0}(Y') = 0$.
q.e.d.

Here is the analogue of Theorem 7.3:

Theorem 7.5. *This compact 8-manifold M' has Betti numbers*

$$b^0 = 1,\ \ b^1 = 0,\ \ b^2 = 1,\ \ b^3 = 0,\ \ b^4 = 2444,\ \ b^4_+ = 1638\ \ \textit{and}\ \ b^4_- = 806.$$

There exist metrics with holonomy Spin(7) *on M', which form a smooth family of dimension* 807.

Proof. As σ fixes 1 point in Y' we have $\chi(Z') = \frac{1}{2}(\chi(Y') + 1)$, so $\chi(Z') = 2447$ by the previous proposition. Since $H^k(Z', \mathbb{C})$ is the σ-invariant part of $H^k(Y', \mathbb{C})$ we have $b^0(Z') = 1$ and $b^1(Z') = b^3(Z') = 0$. Now $b^2(Y') = 3$, and $H^2(Y', \mathbb{C})$ is generated by $[\omega_{Y'}]$ and the cohomology classes dual to the two exceptional divisors $\mathbb{CP}^3$ introduced by blowing up p_2 and p_3. But σ' swaps p_2 and p_3, so σ'_* swaps the corresponding classes in $H^2(Y', \mathbb{C})$, and $\sigma'_*(\omega_{Y'}) = -\omega_{Y'}$ by definition. Therefore $H^2(Y', \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}^2$, where σ'_* acts as 1 on $\mathbb{C}$ and -1 on $\mathbb{C}^2$. Hence $H^2(Z', \mathbb{C}) \cong \mathbb{C}$, and $b^2(Z') = 1$.

Thus $b^0(Z') = b^2(Z') = 1$, $b^1(Z') = b^3(Z') = 0$ and $\chi(Z') = 2447$, giving $b^4(Z') = 2443$. Equation (15) then gives the Betti numbers of M, and (16) gives $b^4_\pm$. Theorem 5.14 shows that there exist torsion-free Spin(7)-structures $(\tilde{\Omega}, \tilde{g})$ on M, with $\mathrm{Hol}(\tilde{g}) = \mathrm{Spin}(7)$ as M is simply-connected. By Theorem 2.4 the moduli space of metrics on M with holonomy Spin(7) is a smooth manifold of dimension $1 + b^4_-(M) = 807$.
q.e.d.

Observe that the Betti numbers of M and M' in Theorems 7.3 and 7.5 are very similar. It is an interesting question whether one can regard M and M' as two different resolutions of some singular Spin(7)-manifold M_0, not necessarily an orbifold. We leave this as a research exercise for the reader; the answer is not as simple as it looks.

8. Examples from hypersurfaces in $\mathbb{CP}^5_{a_0,\dots,a_5}$

Here are three more examples based on hypersurfaces in $\mathbb{CP}^5_{a_0,\dots,a_5}$.

8.1 A hypersurface of degree 16 in $\mathbb{CP}^5_{1,1,1,1,4,8}$

Let Y be the hypersurface of degree 16 in $\mathbb{CP}^5_{1,1,1,1,4,8}$ given by

$$Y = \big\{[z_0,\dots,z_5] \in \mathbb{CP}^5_{1,1,1,1,4,8} : z_0^{16} + z_1^{16} + z_2^{16} + z_3^{16} + z_4^4 + z_5^2 = 0\big\}.$$

Then $c_1(Y) = 0$. We find that Y has two singular points $p_1 = [0,0,0,0,1,i]$ and $p_2 = [0,0,0,0,1,-i]$, both satisfying Condition 5.1.

Following Propositions 7.1 and 7.2, we find that $\chi(Y) = 9498$, and

Proposition 8.1. *The Betti numbers of Y are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0 \quad \textit{and} \quad b^4 = 9494.$$

Also $Y \setminus \{p_1, p_2\}$ is simply-connected and $h^{2,0}(Y) = 0$.

Define an antiholomorphic involution $\sigma : Y \to Y$ by

$$\sigma : [z_0,\dots,z_5] \longmapsto [\bar z_1, -\bar z_0, \bar z_3, -\bar z_2, \bar z_4, -\bar z_5].$$

The fixed points of σ are exactly the singular points p_1, p_2 of Y. Thus Condition 5.1 holds for Y and σ, and we can apply the construction of §5. Resolving $Z = Y/\langle\sigma\rangle$ gives a compact 8-manifold M. We choose at least one of n_1, n_2 to be 2, so that M is simply-connected. Then as in Theorem 7.3, we get:

Theorem 8.2. *This compact 8-manifold M has Betti numbers*

$$b^0 = 1,\ b^1 = b^2 = b^3 = 0,\ b^4 = 4750,\ b^4_+ = 3175 \ \textit{and}\ b^4_- = 1575.$$

There exist metrics with holonomy Spin(7) *on M, which form a smooth family of dimension* 1576.

8.2 A hypersurface of degree 24 in $\mathbb{CP}^5_{1,1,1,1,8,12}$

Let Y be the hypersurface of degree 24 in $\mathbb{CP}^5_{1,1,1,1,8,12}$ given by

$$Y = \{[z_0, \dots, z_5] \in \mathbb{CP}^5_{1,1,1,1,8,12} : z_0^{24} + z_1^{24} + z_2^{24} + z_3^{24} + z_4^3 + z_5^2 = 0\}.$$

Then $c_1(Y) = 0$. We find that Y has one singular point $p_1 = [0,0,0,0,-1,1]$, which satisfies Condition 5.1.

Following Proposition 7.1, we find that $\chi(Y) = 23\,325$. Care is needed to get the right answer here. Define $\pi_5 : Y \to \mathbb{CP}^4_{1,1,1,1,8}$ by $\pi_5 : [z_0, \dots, z_5] \mapsto [z_0, \dots, z_4]$, and $Y_4 \subset \mathbb{CP}^4_{1,1,1,1,8}$ by

$$Y_4 = \{[z_0, \dots, z_4] \in \mathbb{CP}^4_{1,1,1,1,8} : z_0^{24} + z_1^{24} + z_2^{24} + z_3^{24} + z_4^3 = 0\}.$$

Then π_5 is a double cover of $\mathbb{CP}^4_{1,1,1,1,8}$ branched over Y_4 *and the point* $[0,0,0,0,1]$ in $\mathbb{CP}^4_{1,1,1,1,8}$. Hence we get

$$\chi(Y) = 2\chi(\mathbb{CP}^4_{1,1,1,1,8}) - \chi(Y_4) - \chi([0,0,0,0,1]) = 9 - \chi(Y_4).$$

If we had not observed that π_5 is also branched over $[0,0,0,0,1]$, then we would have got $\chi(Y) = 23\,326$, which is incorrect.

As in Proposition 7.2, we show:

Proposition 8.3. *The Betti numbers of Y are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0 \quad \textit{and} \quad b^4 = 23\,231.$$

Also $Y \setminus \{p_1\}$ is simply-connected and $h^{2,0}(Y) = 0$.

Define an antiholomorphic involution $\sigma : Y \to Y$ by

$$\sigma : [z_0, \dots, z_5] \longmapsto [\bar z_1, -\bar z_0, \bar z_3, -\bar z_2, \bar z_4, \bar z_5].$$

The fixed points of σ are exactly the singular point p_1 of Y. Thus Condition 5.1 holds for Y and σ, and choosing the simply-connected resolution M of $Z = Y/\langle\sigma\rangle$, in the usual way we get:

Theorem 8.4. *This compact 8-manifold M has Betti numbers*

$$b^0 = 1,\ b^1 = b^2 = b^3 = 0,\ b^4 = 11\,662,\ b^4_+ = 7783 \ \textit{and}\ b^4_- = 3879.$$

There exist metrics with holonomy Spin(7) *on M, which form a smooth family of dimension* 3880.

This is the example with the largest value of b^4 known to the author.

8.3 A hypersurface of degree 40 in $\mathbb{CP}^5_{1,1,5,5,8,20}$

Here is a more complicated example, in which the hypersurface in $\mathbb{CP}^5_{a_0,\dots,a_5}$ has other singularities which must first be resolved. Let W be the hypersurface of degree 40 in $\mathbb{CP}^5_{1,1,5,5,8,20}$ given by

$$W = \big\{[z_0,\dots,z_5] \in \mathbb{CP}^5_{1,1,5,5,8,20} : z_0^{40} + z_1^{40} + z_2^8 + z_3^8 + z_4^5 + z_5^2 = 0\big\}.$$

Then $c_1(W) = 0$. The singularities of W are the disjoint union of the single point $p_1 = [0,0,0,0,-1,1]$ and the nonsingular curve Σ of genus 3 given by

$$\Sigma = \big\{[0,0,z_2,z_3,0,z_5] \in \mathbb{CP}^5_{1,1,5,5,8,20} : z_2^8 + z_3^8 + z_5^2 = 0\big\}.$$

The singular point at p_1 satisfies Condition 5.1. The singularity at each point of Σ is modelled on $\mathbb{C} \times \mathbb{C}^3/\mathbb{Z}_5$, where the generator β of $\mathbb{Z}_5$ acts on $\mathbb{C}^3$ by

$$\beta : (z_0, z_1, z_4) \mapsto (e^{2\pi i/5} z_0, e^{2\pi i/5} z_1, e^{-4\pi i/5} z_4).$$

Now the singularity $\mathbb{C}^3/\mathbb{Z}_5$ normal to Σ in W has a unique crepant resolution X, which can be described using toric geometry. Let Y be the partial crepant resolution of W which resolves the singularities at Σ using X, but leaves the singular point p_1 unchanged.

Proposition 8.5. *The Betti numbers of Y are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 3, \quad b^3 = 12, \quad \text{and} \quad b^4 = 7453.$$

Also $Y \setminus \{p_1\}$ is simply-connected and $h^{2,0}(Y) = 0$.

Proof. Calculating the Betti numbers of W in the usual way gives

$$b^0(W) = 1,\ b^1(W) = 0,\ b^2(W) = 1,\ b^3(W) = 0,\ b^4(W) = 7449. \tag{17}$$

As W is modelled on $\mathbb{C} \times \mathbb{C}^3/\mathbb{Z}_5$ at each point of Σ, the resolution Y is modelled on $\mathbb{C} \times X$. Since $b^2(X) = b^4(X) = 2$, the Betti numbers of Y satisfy

$$b^k(Y) = b^k(W) + 2b^{k-2}(\Sigma) + 2b^{k-4}(\Sigma).$$

But Σ has genus 3, and so its Betti numbers are $b^0(\Sigma) = b^2(\Sigma) = 1$ and $b^1(\Sigma) = 6$. Combining this with (17) gives the Betti numbers of Y. The last part follows as in Proposition 7.2. q.e.d.

Define $\sigma : W \to W$ by

$$\sigma : [z_0, \ldots, z_5] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5].$$

The only fixed point of σ is p_1. Moreover, σ lifts to the resolution Y of W, and $\sigma : Y \to Y$ is an antiholomorphic involution which fixes only p_1 in Y. Thus Condition 5.1 holds for Y and σ, and we can apply the construction of §5, and resolve $Z = Y/\langle\sigma\rangle$ to get a simply-connected 8-manifold M. Proceeding in the usual way, the end result is

Theorem 8.6. *This compact* 8*-manifold* M *has Betti numbers*

$$b^0 = 1,\ \ b^1 = b^2 = 0,\ \ b^3 = 6,\ \ b^4 = 3730,\ \ b^4_+ = 2493 \ \text{ and } \ b^4_- = 1237.$$

There exist metrics with holonomy Spin(7) *on* M*, which form a smooth family of dimension* 1238.

Note that $b^3 > 0$ in this example; this is because the resolution of the singular curve Σ contributes $H^1(\Sigma, \mathbb{C}) \otimes H^2(X, \mathbb{C}) = \mathbb{C}^6 \otimes \mathbb{C}^2 = \mathbb{C}^{12}$ to $H^3(Y, \mathbb{C})$. Half of this $\mathbb{C}^{12}$ is σ-invariant, and so pushes down to $H^3(Z, \mathbb{C})$ and lifts to $H^3(M, \mathbb{C})$.

9. A hypersurface in $\mathbb{CP}^5_{1,1,1,1,2,2}$ over $\mathbb{Z}_2$

Let W be the hypersurface of degree 8 in $\mathbb{CP}^5_{1,1,1,1,2,2}$ given by

$$W = \big\{[z_0, \ldots, z_5] \in \mathbb{CP}^5_{1,1,1,1,2,2} : z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^4 + z_5^4 = 0\big\}.$$

Then $c_1(W) = 0$. We find that W has four singular points $p_1, \ldots, p_4$ modelled on $\mathbb{C}^4/\{\pm 1\}$, given by

$$[0,0,0,0,1,\mathrm{e}^{\pi i/4}], \qquad [0,0,0,0,1,\mathrm{e}^{3\pi i/4}],$$
$$[0,0,0,0,1,\mathrm{e}^{5\pi i/4}], \qquad [0,0,0,0,1,\mathrm{e}^{7\pi i/4}].$$

Define $\beta : W \to W$ by

$$\beta : [z_0, \ldots, z_5] \mapsto [iz_0, iz_1, iz_2, iz_3, z_4, z_5].$$

Then $\beta^2 = 1$, as $[z_0, \ldots, z_5] = [-z_0, -z_1, -z_2, -z_3, z_4, z_5]$ in $\mathbb{CP}^5_{1,1,1,1,2,2}$. The fixed set of β is the four points $p_1, \ldots, p_4$ together with the compact complex surface S in W, given by

$$S = \big\{[z_0, z_1, z_2, z_3, 0, 0] \in \mathbb{CP}^5_{1,1,1,1,2,2} : z_0^8 + z_1^8 + z_2^8 + z_3^8 = 0\big\}.$$

Thus $W/\langle\beta\rangle$ is a compact complex orbifold. Its singular set is the disjoint union of $p_1,\dots,p_4$ and S. Each singular point p_j is modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where the generator α of $\mathbb{Z}_4$ acts on $\mathbb{C}^4$ by (6). Each singular point in S is locally modelled on $\mathbb{C}^2\times\mathbb{C}^2/\{\pm1\}$.

Let Y be the blow-up of $W/\langle\beta\rangle$ along S. Because the singularities normal to S are modelled on $\mathbb{C}^2/\{\pm1\}$, this is a *partial crepant resolution.* So Y is a compact complex orbifold with isolated singular points $p_1,\dots,p_4$, modelled on $\mathbb{C}^4/\langle\alpha\rangle$. Now $c_1(W)=0$, so $c_1(W/\langle\beta\rangle)=0$, and as Y is a partial crepant resolution of $W/\langle\beta\rangle$ we see that $c_1(Y)=0$.

Proposition 9.1. *The Betti numbers of Y are*

$$b^0=1,\quad b^1=0,\quad b^2=2,\quad b^3=0\quad\textit{and}\quad b^4=1806.$$

Also $Y\setminus\{p_1,\dots,p_4\}$ is simply-connected and $h^{2,0}(Y)=0$.

Proof. As in Proposition 7.1, we find $\chi(W)=2708$ and $\chi(S)=304$. Thus

$$\begin{aligned}\chi(W/\langle\beta\rangle)=&\tfrac{1}{2}\big(\chi(W)+\chi(4\text{ points})+\chi(S)\big)\\=&\tfrac{1}{2}(2708+4+304)=1508.\end{aligned}$$

Using Theorem 6.5 we find that W has $b^0=b^2=1$ and $b^1=b^3=0$, and it soon follows that $W/\langle\beta\rangle$ also has $b^0=b^2=1$ and $b^1=b^3=0$. Since $\chi(W/\langle\beta\rangle)=1508$ we see that $b^4(W/\langle\beta\rangle)=1504$.

Now Y is the blow-up of $W/\langle\beta\rangle$ along S, so that each point of S is replaced by a copy of $\mathbb{CP}^1$. It can be shown that the Betti numbers of Y satisfy

$$b^k(Y)=b^k(W/\langle\beta\rangle)+b^{k-2}(S). \tag{18}$$

But S can be thought of as an octic in $\mathbb{CP}^3$, and by the usual method we find that the Betti numbers of S are $b^0=1$, $b^1=0$, $b^2=302$, $b^3=0$ and $b^4=1$. Combining these with (18) and the Betti numbers of $W/\langle\beta\rangle$ above gives the Betti numbers of Y. The last part follows as usual. q.e.d.

Define an antiholomorphic involution $\sigma:W\to W$ by

$$\sigma:[z_0,\dots,z_5]\longmapsto[\bar z_1,-\bar z_0,\bar z_3,-\bar z_2,\bar z_5,\bar z_4].$$

The fixed points of σ are exactly the singular points $p_1,\dots,p_4$ of W. Also σ commutes with β, and acts freely on S. Hence σ pushes down to an antiholomorphic involution of $W/\langle\beta\rangle$, and lifts to the blow-up Y, to give an antiholomorphic involution $\sigma : Y \to Y$ with fixed points $p_1, \dots, p_4$.

Thus Condition 5.1 holds for Y and σ, and in the usual way we choose a simply-connected resolution M of $Z = Y/\langle\sigma\rangle$ satisfying:

Theorem 9.2. *This compact* 8*-manifold* M *has Betti numbers*

$$b^0 = 1,\ b^1 = b^2 = b^3 = 0,\ b^4 = 910,\ b^4_+ = 615 \text{ and } b^4_- = 295.$$

There exist metrics with holonomy Spin(7) *on* M*, which form a smooth family of dimension* 296.

9.1 A variation on this example

We shall use the idea of §7.1 to make a second 8-manifold M' from the orbifold Y above. Let W and Y be as in §9.1, but define $\sigma' : W \to W$ by

$$\sigma' : [z_0, \dots, z_5] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, i\bar{z}_5].$$

Then σ' pushes down to $W/\langle\beta\rangle$ and lifts to Y as above. However, this time σ' fixes the singular points $p_1 = [0,0,0,0,1,e^{\pi i/4}]$ and $p_2 = [0,0,0,0,1,e^{5\pi i/4}]$ in Y, but it swaps round $p_3 = [0,0,0,0,1,e^{3\pi i/4}]$ and $p_4 = [0,0,0,0,1,e^{7\pi i/4}]$.

Thus, Condition 5.1 does not hold for Y and σ', as the fixed set $\{p_1, p_2\}$ of σ' does not coincide with the singular set $\{p_1, \dots, p_4\}$ of Y. So let Y' be the blow-up of Y at p_3 and p_4. Then Y' is a partial crepant resolution of Y, as the singularities at p_3, p_4 are modelled on $\mathbb{C}^4/\mathbb{Z}_4$. The singularities of Y' are p_1, p_2, and σ' lifts to an antiholomorphic involution of Y' fixing only p_1 and p_2.

We find the Betti numbers of Y' by adding contributions to those of Y, as in §7.1. Applying the construction of §5 to Y' and σ' gives a simply-connected 8-manifold M', such that

Theorem 9.3. *This compact* 8*-manifold* M' *has Betti numbers*

$$b^0 = 1,\ b^1 = 0,\ b^2 = 1,\ b^3 = 0,\ b^4 = 908,\ b^4_+ = 614 \text{ and } b^4_- = 294.$$

There exist metrics with holonomy Spin(7) *on* M'*, which form a smooth family of dimension* 295.

10. Complete intersections in $\mathbb{CP}^6_{a_0,\dots,a_6}$

We now try starting with the intersection of two hypersurfaces in $\mathbb{CP}^6_{a_0,\dots,a_6}$.

10.1 The intersection of two octics in $\mathbb{CP}^6_{1,1,1,1,4,4,4}$

Let Y be the complete intersection of two octics in $\mathbb{CP}^6_{1,1,1,1,4,4,4}$ given by

$$Y = \big\{[z_0,\dots,z_5] \in \mathbb{CP}^6_{1,1,1,1,4,4,4} : z_0^8 + z_1^8 + 2iz_2^8 - 2iz_3^8 + z_4^2 - z_5^2 = 0,\\ 2iz_0^8 - 2iz_1^8 + z_2^8 + z_3^8 + z_4^2 - z_6^2 = 0\big\}.$$

Then $c_1(Y) = 0$. We find that Y has 4 singular points

$$p_1 = [0,0,0,0,1,1,1], \qquad p_2 = [0,0,0,0,1,-1,-1],\\ p_3 = [0,0,0,0,1,1,-1] \quad \text{and} \quad p_4 = [0,0,0,0,1,-1,1],$$

satisfying Condition 5.1.

By adapting the method of §6.3 we can show that $\chi(Y) = 2580$, and applying Theorem 6.5 twice we find that $b^k(Y) = b^k(\mathbb{CP}^6_{1,\dots,4})$ for $0 \le k \le 3$. Thus we prove:

Proposition 10.1. *The Betti numbers of Y are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0 \quad \textit{and} \quad b^4 = 2576,$$

Also $Y \setminus \{p_1,\dots,p_4\}$ is simply-connected and $h^{2,0}(Y) = 0$.

Define an antiholomorphic involution $\sigma : Y \to Y$ by

$$\sigma : [z_0,\dots,z_6] \longmapsto [\bar z_1, -\bar z_0, \bar z_3, -\bar z_2, \bar z_4, \bar z_5, \bar z_6].$$

The fixed points of σ are exactly the singular points $p_1,\dots,p_4$ of Y, and Condition 5.1 holds for Y and σ. Proceeding in the usual way, we set $Z = Y/\langle\sigma\rangle$ and resolve Z to get a simply-connected 8-manifold M, which satisfies:

Theorem 10.2. *This compact 8-manifold M has Betti numbers*

$$b^0 = 1,\ b^1 = b^2 = b^3 = 0,\ b^4 = 1294,\ b^4_+ = 871 \ \textit{and}\ b^4_- = 423.$$

There exist metrics with holonomy Spin(7) *on M, which form a smooth family of dimension* 424.

10.2 A variation on this example

Now let Y be as in §10.1, but define $\sigma' : Y \to Y$ by

$$\sigma' : [z_0, \ldots, z_6] \longmapsto [\bar{z}_3, -\bar{z}_2, \bar{z}_1, -\bar{z}_0, \bar{z}_4, \bar{z}_6, \bar{z}_5].$$

Then σ' is an antiholomorphic involution, with fixed points p_1 and p_2, which swaps round p_3 and p_4. Following the method of §7.1, define Y' to be the blow-up of Y at p_3 and p_4. Then Y' is a Calabi–Yau orbifold, σ' lifts to Y', and Condition 5.1 holds for Y' and σ'.

As usual we set $Z' = Y'/\langle\sigma'\rangle$ and resolve Z' to get a simply-connected 8-manifold M', such that we have

Theorem 10.3. *This compact 8-manifold M' has Betti numbers*

$$b^0 = 1,\ b^1 = 0,\ b^2 = 1,\ b^3 = 0,\ b^4 = 1292,\ b^4_+ = 870 \text{ and } b^4_- = 422.$$

There exist metrics with holonomy Spin(7) *on M', which form a smooth family of dimension* 423.

10.3 The intersection of two 12-tics in $\mathbb{CP}^6_{3,3,3,3,4,4,4}$

Let $P(z_4, z_5, z_6)$ and $Q(z_4, z_5, z_6)$ be generic homogeneous cubic polynomials with real coefficients, and define W to be the complete intersection of two 12-tics in $\mathbb{CP}^6_{3,3,3,3,4,4,4}$ given by

$$W = \big\{[z_0, \ldots, z_5] \in \mathbb{CP}^6_{3,3,3,3,4,4,4} : z_0^4 + z_1^4 + z_2^4 + z_3^4 + P(z_4, z_5, z_6) = 0,\\ iz_0^4 - iz_1^4 + 2iz_2^4 - 2iz_3^4 + Q(z_4, z_5, z_6) = 0\big\}.$$

Then $c_1(W) = 0$. As P and Q are generic, the singular set of W is the disjoint union of the 9 points $p_1, \ldots, p_9$ given by

$$\big\{[0, 0, 0, 0, z_4, z_5, z_6] \in \mathbb{CP}^6_{3,3,3,3,4,4,4} : P(z_4, z_5, z_6) = Q(z_4, z_5, z_6) = 0\big\},$$

and the curve Σ of genus 33 given by

$$\Sigma = \big\{[z_0, z_1, z_2, z_3, 0, 0, 0] \in \mathbb{CP}^6_{3,3,3,3,4,4,4} : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0,\\ iz_0^4 - iz_1^4 + 2iz_2^4 - 2iz_3^4 = 0\big\}.$$

Each point p_j satisfies Condition 5.1, and each point of Σ is modelled on $\mathbb{C} \times \mathbb{C}^3/\mathbb{Z}_3$, where the action of $\mathbb{Z}_3$ on $\mathbb{C}^3$ is generated by

$$\beta : (z_4, z_5, z_6) \mapsto (e^{2\pi i/3} z_4, e^{2\pi i/3} z_5, e^{2\pi i/3} z_6).$$

Define an antiholomorphic involution $\sigma : W \to W$ by

$$\sigma : [z_0, \dots, z_6] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5, \bar{z}_6].$$

Then the fixed points of σ are some subset of $\{p_1, \dots, p_9\}$. Exactly which subset depends on the choice of P and Q, but σ must fix an odd number of the p_j, as the remaining p_j are swapped in pairs.

So let σ fix $2k+1$ of the p_j, for some $k = 0, \dots, 4$, and number the p_j such that σ fixes $p_1, \dots, p_{2k+1}$ and swaps $p_{2k+2}, \dots, p_9$ in pairs. Define Y_k to be the blow-up of W along Σ and at the points $p_{2k+2}, \dots, p_9$. Then Y_k is a partial crepant resolution of W. Thus Y_k is a Calabi–Yau orbifold, with singular points $p_1, \dots, p_{2k+1}$. Also σ lifts to Y_k to give an antiholomorphic involution $\sigma : Y_k \to Y_k$ with fixed points $p_1, \dots, p_{2k+1}$.

It can be shown that we can choose P and Q so that k takes any value in $\{0, 1, 2, 3, 4\}$. For example, if $P = z_4^3 - z_5^3$ and $Q = z_4^3 - z_6^3$ then σ fixes only $p_1 = [0, 0, 0, 0, 1, 1, 1]$, so that $k = 0$, but if $P = z_4^2 z_5 - z_5^3$ and $Q = z_4^2 z_6 - z_6^3$ then σ fixes the 9 points $[0, 0, 0, 0, 1, z_5, z_6]$ for $z_5, z_6 \in \{1, 0, -1\}$, and $k = 4$.

Combining the methods used to prove Propositions 8.5 and 10.1, we get

Proposition 10.4. *The Betti numbers of Y_k are $b^0 = 1$, $b^1 = 0$, $b^2 = 10 - 2k$, $b^3 = 66$, $b^4 = 395 - 2k$, $b^4_+ = 262$ and $b^4_- = 133 - 2k$. Also $Y_k \setminus \{p_1, \dots, p_{2k+1}\}$ is simply-connected, and $h^{2,0}(Y_k) = 0$.*

In the usual way we resolve $Z_k = Y_k / \langle \sigma \rangle$ to get M_k, which satisfies

Theorem 10.5. *For each $k = 0, \dots, 4$ there is a compact 8-manifold M_k with Betti numbers $b^0 = 1$, $b^1 = 0$, $b^2 = 4 - k$, $b^3 = 33$, $b^4 = 200 + 2k$, $b^4_+ = 132 + k$ and $b^4_- = 68 + k$. There exist metrics with holonomy* Spin(7) *on M_k, which form a smooth family of dimension $69 + k$.*

These examples have the largest value of b^3 and the smallest values of b^4 that the author has found using this construction.

11. Conclusions

In Table 1 we give the Betti numbers (b^2, b^3, b^4) of the compact 8-manifolds with holonomy Spin(7) that we constructed in §7–§10. There are 14 sets of Betti numbers, none of which coincide with any in [10], so we have found at least 14 topologically distinct new examples of compact 8-manifolds with holonomy Spin(7).

Table 1: Betti numbers (b^2, b^3, b^4) of compact Spin(7)-manifolds

$(4,33,200)$	$(3,33,202)$	$(2,33,204)$	$(1,33,206)$	$(0,33,208)$
$(1,0,908)$	$(0,0,910)$	$(1,0,1292)$	$(0,0,1294)$	$(1,0,2444)$
$(0,0,2446)$	$(0,6,3730)$	$(0,0,4750)$	$(0,0,11\,662)$	

The examples of §7–§10 are by no means all the manifolds that can be produced using the methods of this paper, but only a selection chosen for their simplicity and to illustrate certain techniques. Readers are invited to look for other examples themselves; the author would be particularly interested in examples which have especially large or small values of b^4.

We have also chosen to restrict our attention in §5–§10 to orbifolds Y all of whose singularities are modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where the generator α of $\mathbb{Z}_4$ acts as in (6). This is not a necessary restriction, and there are other types of singularities for Y and Z for which the construction would work, such as the $\mathbb{R}^8/G^n$ considered in §4.3, and which occur in suitable orbifolds Y. However, the author has not found many such Y; the $\mathbb{C}^4/\mathbb{Z}_4$ singularities do seem to be the easiest to construct.

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