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ON THE STRUCTURE OF SPACES WITH RICCI CURVATURE BOUNDED BELOW. II

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0. Introduction

This paper, the sequel of [4], is the second in a series devoted to the study of the structure of complete connected riemannian manifolds, M^n, whose Ricci curvature has a definite lower bound and of the Gromov-Hausdorff limits, Y, of sequences of such manifolds.

By [4], in the noncollapsed case, off a subset of codimension ≥ 2, such a limit space, Y, is bi-Hölder equivalent to a connected smooth riemannian manifold (for the proof of connectedness, see Section 3 below). Additionally, even in the collapsed case, there exist natural renormalized limit measures, ν, with respect to which Y is infinitesimally Euclidean almost everywhere. We do not know whether "bi-Hölder" can be replaced by "bi-Lipschitz", or "infinitesimally Euclidean" by "locally Euclidean". Nor do we know whether in the collapsed case, the local Hausdorff dimension of the space is the same at all points.

In order to describe the results of the present paper in detail, we will recall some background from [4].

After rescaling the metric, we can assume

$$\mathrm{Ric}_{M^n} \geq -(n-1)\,. \tag{0.1}$$

Sometimes we assume in addition that for some definite $v > 0$,

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$$\mathrm{Vol}(B_1(m)) \geq v > 0\,. \tag{0.2}$$

Let d_{GH} denote Gromov-Hausdorff distance. As indicated above, most of our results are phrased in terms of the structure of pointed Gromov-Hausdorff limits of sequences of such manifolds,
$\{(M_i^n, m_i)\} \stackrel{d_{GH}}{\longrightarrow} (Y, y)$, where

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,. \tag{0.3}$$

In parts of Sections 3, 4, we also assume the noncollapsing condition,

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0\,. \tag{0.4}$$

Our limit spaces carry natural renormalized limit measures, ν, which play a central role in the discussion; see [15] and [4]. These arise as limits of subsequences of renormalized riemannian measures, $\underline{\mathrm{Vol}}_j \to \nu$, where

$$\underline{\mathrm{Vol}}_j(\,\cdot\,) = \frac{1}{\mathrm{Vol}_j(B_1(m_j))}\mathrm{Vol}_j(\,\cdot\,)\,. \tag{0.5}$$

If (0.4) holds (which turns out to be equivalent to the assumption that the limit space, Y, has Hausdorff dimension n) then the measure, ν, is unique and coincides with normalized Hausdorff measure; see [12], [4]. However, in the collapsed case, uniqueness need not hold; see Example 1.24 of [4].

A *tangent cone*, $(Y_y, y_\infty, d_\infty, \nu_\infty)$, at $y \in Y$ is the pointed Gromov-Hausdorff limit as $r_i \to 0$, of some sequence, $\{(Y, y, r_i^{-1}d, \underline{\nu}_i)\}$. Here, d denotes the metric on Y and $\underline{\nu}_i$ is defined as in (0.5). Usually, we just denote a tangent cone by Y_y.

Let $\mathcal{WR} = \cup_k \mathcal{WR}_k$ denote the *weakly regular set* of Y. By definition, $\mathcal{WR}_k$ is the set of points at which *some* tangent cone is isometric to $\mathbf{R}^k$. The *strongly singular set*, $Y \setminus \mathcal{WR}$, is denoted by $\mathcal{SS}$.

Let $B_r^k(0) \subset \mathbf{R}^k$ denote the ball of radius r. We write $y \in (\mathcal{WR}_k)_\epsilon$ if for some $0 < r$, we have $d_{GH}(B_r(y), B_r^k(0)) < \epsilon r$.

Let $\mathcal{R} = \cup_k \mathcal{R}_k$ denote the *regular set* of Y. By definition, $\mathcal{R}_k$ is the set of points at which *every* tangent cone is isometric to $\mathbf{R}^k$. The *singular set*, $Y \setminus \mathcal{R}$, is denoted by $\mathcal{S}$. At present, we do not know of any example for which $\mathcal{WR} \neq \mathcal{R}$, or equivalently, for which $\mathcal{S} \neq \mathcal{SS}$.

We write $y \in (\mathcal{R}_k)_{\epsilon,\delta}$ if for all $0 < r < \delta$, we have $d_{GH}(B_r(y), B_r^k(0)) < \epsilon r$. We put $\cup_\delta (\mathcal{R}_k)_{\epsilon,\delta} = (\mathcal{R}_k)_\epsilon$, the *$\epsilon$-regular set*. Clearly, $\cap_\epsilon (\mathcal{R}_k)_\epsilon = \mathcal{R}_k$.

As indicated above, the volume convergence conjecture of Anderson-Cheeger, $\mathrm{Vol}(M_i^n) \to \mathrm{Vol}(M^n)$, for sequences of manifolds, $M_i^n \xrightarrow{d_{GH}} M^n$, satisfying (0.3), was proved in [12]. This was generalized in Theorem 5.4 of [4], to yield volume convergence for sequences of limit spaces $Y_i^n \xrightarrow{d_{GH}} Y^n$.

In Section 1, we prove a generalization of the original conjecture, for sequences, $M_i^n \xrightarrow{d_{GH}} M^k$, satisfying (0.3), where M^k is a manifold. This generalization is formulated in terms of k-dimensional Hausdorff content $\mathcal{H}_\infty^k$. Thus, we show $\mathcal{H}_\infty^k(M_i^n) \to \mathcal{H}_\infty^k(M^k)$, or equivalently, $\mathcal{H}_\infty^k(M_i^n) \to \mathrm{Vol}(M^k)$. When specialized to the case, $k = n$, this, together with relative volume comparision, [18], yields the result of [12].

At present, the formulation in terms of Hausdorff content cannot be generalized to sequences $Y_i \xrightarrow{d_{GH}} Y^k$. The essential difficulty stems from our lack of knowledge of whether the Hausdorff dimension of the singular set of a limit space can exceed that of the regular set; compare the discussion of polar limit spaces below.

Note that for sequences, $M_i^n \xrightarrow{d_{GH}} M^k$ (even those with uniformly bounded sectional curvature) the renormalized limit measure on M^k need not be unique. Thus, there does not exist a generalization for such sequences in which "Hausdorff content" is replaced by "renormalized volume"; compare Remark 1.49.

For (Z, μ) a measure space, we set

$$\tag{0.6} \fint_Z f\, d\mu = \frac{1}{\mu(Z)} \int_Z f\, d\mu \,.$$

The main technical result of Section 1 is Theorem 1.2, which (in nonquantitative form) asserts the following.

Let $(M_i^n, m_i) \xrightarrow{d_{GH}} (\mathbf{R}^k, 0)$ and $\mathrm{Ric}_{M_i^n} \geq -\delta_i$, where $\delta_i \to 0$. Then there exist Lipschitz maps, $\Phi_i : B_1(m_i) \to B_1^k(0)$, with $|d\Phi_i| \leq c(n)$, such that

$$\tag{0.7} \fint_{B_r(m_i)} |d\Phi_i - 1| \to 0 \,,$$

and

$$\tag{0.8} \fint_{B_r^k(0)} |\tilde{V}_i(z) - 1| \to 0 \,,$$

where, $\tilde{V}(z)$, the renormalized volume of the fibre, is defined by

$$\tilde{V}_i(z) = \frac{\mathrm{Vol}_{\mathbf{R}^k}(B_r^k(0))}{\mathrm{Vol}_n(B_r(m_i))}\mathrm{Vol}_{n-k}(\Phi_i^{-1}(z))\,. \tag{0.9}$$

Here and sometimes below, we attach a subscript to Vol, in order to emphasize the relevant dimension or space.

Since in (0.8), we understand $\tilde{V}_i(z) = 0$ if $\Phi_i^{-1}(z)$ is empty, it follows that for i sufficiently large, the range of Φ_i has almost full measure.

We point out that there exist sequences, $M_i^4 \xrightarrow{d_{GH}} T^3$, for which the maps, Φ_i, cannot be chosen to be fibrations; see [1].

As a particular consequence, we find that there exists $\epsilon(n) > 0$, such that for any limit space, Y, satisfying (0.3), the Hausdorff dimension satisfies, $\dim Y \geq \underline{k}$, where $\underline{k}$ denotes the largest, k, such that $(\mathcal{WR}_k)_{\epsilon(n)} \neq \emptyset$. Moreover, for *polar* limit spaces, those for which the base point of every iterated tangent cone is a pole, equality holds.

Theorem 1.2 leaves open the possibility that $\mathcal{H}^k(A) = \infty$, for every subset, $A \subset \mathcal{R}_k$, for which $\nu(A) > 0$. In actuality, there is a subset of $\mathcal{R}_k$ of full measure with respect to ν, on which $\mathcal{H}^k$ and ν are mutually absolutely continuous; see [5]. From this assertion, (whose proof is entirely different from that of Theorem 1.2) we can also obtain the applications mentioned in the previous paragraph.

In Section 2, we define lower dimensional "Hausdorff" measures associated to a renormalized limit measure, ν, on a collapsed limit space; compare [14]. For all β, we define a measure, $\nu_{-\beta}$, the Hausdorff measure associated to ν in codimension β. If $\nu_{-\beta}(U) = \infty$, for all $\beta > \beta'$, we say $\mathrm{codim}_\nu U \leq \beta'$.

We show that $\mathrm{codim}_\nu \mathcal{SS} \geq 1$. Conjecturally, we have $\mathrm{codim}_\nu \mathcal{S} \geq 1$ as well. Recall in this connection that in Section 2 of [4], it was shown that $\nu(\mathcal{S}) = 0$. Moreover, in the noncollapsed case, where $\nu = \mathcal{H}^n$, the normalized n-dimensional Hausdorff measure, we have $\dim \mathcal{S} \leq n-2$; see [4], Section 7. However, in the collapsed case, the estimate, $\mathrm{codim}_\nu \mathcal{S} \geq 1$, would be optimal in general. Indeed, the space $[0,1]$, with the measure, $\nu = \mathcal{H}^1$, occurs as such a limit space. For this space, the regular set is the open interval, $(0,1)$.

In Section 3 we show that in the noncollapsed case, $(\overset{\circ}{\mathcal{R}})_\epsilon$, the interior of $\mathcal{R}_\epsilon$, is connected. In particular, for all $z \in \mathcal{R}$, there exists $\mathcal{C}(z) \subset \mathcal{R}$, with $\nu(Y \setminus \mathcal{C}(z)) = 0$, such that for all $w \in \mathcal{C}(z)$ and $\epsilon > 0$, there exists a minimal geodesic from z to w which is contained in $(\overset{\circ}{\mathcal{R}})_\epsilon$. This result is obtained as a consequence of (a precise version of) the following fact:

Removing a closed subset, B, for which $\nu_{-1}(B) = 0$, cannot disconnect a (possibly collapsed) space, Y, which is the limit of a sequence of manifolds satisfying (0.3). This property is well known for smooth manifolds.

In Section 4, using the results of Section 3, we show that the isometry group of a noncollapsed limit space is a Lie group. It was conjectured in [4] that this holds in the collapsed case as well.

In Section 5, we show that a (possibly collapsed) limit space which contain a one 1-dimensional piece and which satisfies an additional condition, is itself 1-dimensional. In this extremely special case, this assertion provides an affirmative answer to a number of questions which were raised at the begining of this introduction.

1. Generalized volume convergence; the collapsing case

In this section we prove a generalization in the collapsing case, of the volume convergence conjecture of Anderson-Cheeger and deduce some consequences. The original conjecture was proved in [12]. The main technical theorem of this section is Theorem 1.2 which we formulate in terms of $\tilde{V}(z)$, the renormalized volume function for the fibres; see (0.9) for the definition of $\tilde{V}(z)$.

In what follows, we will denote by $\Psi(u_1, \ldots, u_k \mid \ldots)$, any nonnegative function depending on the numbers, $u_1, \ldots, u_k$, and some additional parameters, such that when these additional parameters are fixed, we have

$$\lim_{u_1,\ldots,u_k \to 0} \Psi(u_1, \ldots, u_k | \ldots) = 0\,. \tag{1.1}$$

Theorem 1.2 *Let*

$$Ric_{M^n} \geq -(n-1)\epsilon\, r^2\,, \tag{1.3}$$

and assume that for some $m \in M^n$, we have

$$d_{GH}(B_{\ell\, r}(m), B_{\ell\, r}(0)) < \epsilon\, r\,. \tag{1.4}$$

Then there is a harmonic map, $\Phi : B_{3r}(m) \to \mathbf{R}^k$, with $\Phi(B_r(m)) \subset B^k_r(0)$ and

$$Lip\, \Phi \leq c(n), \tag{1.5}$$

such that

$$\fint_{B_r(m)} |d\Phi - 1| \leq \Psi(\epsilon, \ell^{-1}\,|\,n)\,, \tag{1.6}$$

$$\fint_{B^k_r(0)} |\tilde{V}(z) - 1| \leq \Psi(\epsilon, \ell^{-1}\,|\,n)\,. \tag{1.7}$$

In particular,

$$\frac{Vol_{\mathbf{R}^k}(B^k_r(0) \setminus \Phi(B_r(m)))}{Vol_{\mathbf{R}^k}(B^k_r(0))} \leq \Psi(\epsilon, \ell^{-1}\,|\,n)\,. \tag{1.8}$$

Before proceeding to the proof of Theorem 1.2, we will discuss a preliminary result which is valid on arbitrary complete riemannian manifolds. After Lemma 1.14, the assumptions of Theorem 1.2 will once again be in force.

If $\mathbf{b}_1, \cdots, \mathbf{b}_k \in C^\infty(M^n)$, we define $\Phi : M^n \to \mathbf{R}^k$ by

$$\Phi = (\mathbf{b}_1, \ldots, \mathbf{b}_k)\,. \tag{1.9}$$

Our goal is to study the function, $\tilde{V}(z)$, or equivalently, its unrenormalized version,

$$V(z) = \mathrm{Vol}_{n-k}(\Phi^{-1}(z))\,. \tag{1.10}$$

For technical reasons, we will introduce a weighted volume function, $J(z)$, the key properties of which, are easier to establish; compare Lemma 1.14 below and Section 2 of [13], where a weighted volume was also employed. We then deduce the properties of $V(z)$ from those of $J(z)$.

For all $\epsilon > 0$, we choose a smooth nondecreasing function, $\chi_\epsilon : \mathbf{R}_+ \to \mathbf{R}_+$, in such a way that $\max|\chi'_\epsilon|$ is independent of ϵ and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } \ t \leq \epsilon/2\,, \\ (1 - 2\epsilon t) + 2\epsilon^2, & \text{if } \ \epsilon \leq t \leq 1 + \epsilon\,, \\ 1, & \text{if } \ t \geq 1 + 2\epsilon\,. \end{cases} \tag{1.11}$$

In the application, we take $\epsilon > 0$ sufficiently small and from now on, we just write χ for χ_ϵ.

We set

$$J(z) = \begin{cases} \int_{\Phi^{-1}(z)} \chi(\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)) & \text{if } \Phi^{-1}(z) \neq \emptyset\,, \\ 0 & \text{if } \Phi^{-1}(z) = \emptyset\,. \end{cases} \tag{1.12}$$

Here $(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)$ denotes the matrix whose (s,t)-th entry is $\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle$.

Note that

$$0 \leq J \leq V\,. \tag{1.13}$$

It follows easily from the implicit function theorem that $J(z)$ is a smooth function; see (1.17). The following lemma, which provides an estimate for the gradient of $J(z)$, will be employed in proving Lemma 1.31, at the very end of the proof of Theorem 1.2.

Lemma 1.14. *Let $\mathbf{b}_1, \cdots, \mathbf{b}_k$ be functions with bounded gradient,*

$$|\nabla \mathbf{b}_i| \leq C\,. \tag{1.15}$$

Assume that $\Phi^{-1}(z)$ is compact for all z (such that $\Phi^{-1}(z)$ is nonempty). Then there exists $\bar{C} = \bar{C}(C,k) > 0$ such that

$$|\nabla J(z)| \leq \bar{C} \sum_{j=1}^{k} \int_{\Phi^{-1}(z)} |Hess_{\mathbf{b}_j}|\,. \tag{1.16}$$

Proof. For $z \in \mathbf{R}^k$, let $\underline{\Phi}$ denote the subset of $\Phi^{-1}(z)$ consisting of those points at which the vectors, $\nabla \mathbf{b}_1, \ldots, \nabla \mathbf{b}_k$, are linearly independent. At $m \in \underline{\Phi}^{-1}(z)$, we denote by $(a_{i,j})$, the inverse of the matrix, $(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)$, and by $\widehat{\mathrm{tr}}(\mathrm{Hess}_{\mathbf{b}_i})$, the trace of the restriction to $\underline{\Phi}^{-1}(z)_m$, of the bilinear form $\mathrm{Hess}_{\mathbf{b}_i}$. Here $\underline{\Phi}^{-1}(z)_m$ denotes the tangent space of $\underline{\Phi}^{-1}(z)$ at m.

The proof is now an direct consequence of the formula,

$$\begin{aligned} \frac{\partial J}{\partial z_j} = & \int_{\underline{\Phi}^{-1}(z)} \chi'(\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)) \sum_{i=1}^{k} a_{i,j} \nabla \mathbf{b}_i\,(\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)) \\ & + \int_{\underline{\Phi}^{-1}(z)} \chi(\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)) \sum_{i=1}^{k} a_{i,j} \widehat{\mathrm{tr}}(\mathrm{Hess}_{\mathbf{b}_i})\,. \end{aligned} \tag{1.17}$$

To see that (1.17) holds, note that at points of $\underline{\Phi}^{-1}(z)$, the vector field orthogonal to $\underline{\Phi}^{-1}(z)$, which projects to $\frac{\partial}{\partial z_i}$ is $\sum_i a_{i,j} \nabla \mathbf{b}_i$. Note also that in (1.11), the set of points at which the integrand in (1.12) does not vanish, is contained in the interior of the set $\bigcup_z \underline{\Phi}^{-1}(z)$. Hence, in (1.12), the right-hand side can be rewritten as an integral over $\underline{\Phi}^{-1}(z)$. By differentiating this expression under the integral sign in the direction of $\sum_i a_{i,j}\nabla\mathbf{b}_i$, we obtain (1.17). The first term in (1.17) arises from the derivative of $\chi(\det(\langle\nabla\mathbf{b}_s, \nabla\mathbf{b}_t\rangle))$. The second term arises (via the first variation formula) from the derivative of the area element on $\underline{\Phi}^{-1}(z)$ in the direction of $\sum_i a_{i,j}\nabla\mathbf{b}_i$. To see this, observe that virtually by definition, $\widehat{\mathrm{tr}}(\mathrm{Hess}_{\mathbf{b}_i})$ is the inner product of the mean curvature vector to $\underline{\Phi}^{-1}(z)$ with the vector $\nabla\mathbf{b}_i$. q.e.d.

Proof of Theorem 1.2. We begin by defining functions, $\mathbf{b}_i$, as in [12], [4].

After rescaling the metric, we can assume that $r = 1$. Also, without loss of generality, we can assume $\ell > 3$. Let $(e_i)_{i=1,\dots,k}$ be the standard basis for $\mathbf{R}^k$. By (1.4) there exists an ϵ-Gromov-Hausdorff approximation, F, from $B^k_\ell(0)$ to $B_\ell(m) \subset M^n$. By means of this approximation, we can define k points in M^n by $q_i = F(\ell\, e_i, x)$. Put

$$b_i(\cdot) = \overline{\cdot, q_i} - \overline{m, q_i} \tag{1.19}$$

and let $\mathbf{b}_i$ denote the function on $B_3(m)$ such that

$$\Delta\mathbf{b}_i = 0\,, \tag{1.20}$$

$$\mathbf{b}_i \,|\, \partial B_3(m) = b_i \,|\, \partial B_3(m). \tag{1.21}$$

As in (1.9), we set $\Phi = (\mathbf{b}_1, \dots, \mathbf{b}_k)$. If necessary, by slightly rescaling the functions, $\mathbf{b}_i$, without loss of generality, we can and will assume that

$$B_{1-\Psi}(m) \subset \Phi^{-1}(B^k_1(0)) \subset B_1(m)\,, \tag{1.22}$$

where $\Psi = \Psi(\epsilon, \ell^{-1} \,|\, n)$, and that $\Phi^{-1}(z)$ is compact for $z \in B^k_1(0)$ with $\Phi^{-1}(z) \neq \emptyset$; compare Lemmas 2.5 and 6.15 of [4].

It follows from the Cheng-Yau gradient estimate, [8], that (1.5) holds.

From [3] (compare also [12]) it follows that

(1.23)

$$\fint_{B_1(m)} \left\{\sum_j \big|\, |\nabla\mathbf{b}_i| - 1\,\big|^2 + \sum_{i\neq j} |\langle\nabla\mathbf{b}_i,\, \nabla\mathbf{b}_j\rangle| + \sum_i |\mathrm{Hess}_{\mathbf{b}_i}|^2\right\} \leq \Psi\,(\epsilon,\, \ell^{-1} \,|\, n)\,.$$

In particular, (1.23) gives (1.6). Thus, it suffices to prove (1.7) (which implies (1.8)).

For convenience of notation, we put

$$f_{z,r} = \fint_{B_r(z)} f \, d\mu \,. \tag{1.24}$$

Relation (1.7), follows by adding the following three inequalities (and multiplying through by $\mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0))(\mathrm{Vol}_n(B_1(m)))^{-1}$).

(1.25)
$$\left| J_{0,1} - (\mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0)))^{-1}\mathrm{Vol}_n(B_1(m)) \right| \le \Psi(\epsilon, \ell^{-1} \,|\, n)\, \mathrm{Vol}_n(B_1(m)) \,,$$

$$0 \le \fint_{B_1^k(0)} |V - J| \le \Psi(\epsilon, \ell^{-1} \,|\, n)\, \mathrm{Vol}_n(B_1(m)) \,, \tag{1.26}$$

$$\fint_{B_1^k(0)} |J - J_{0,1}| \le \Psi(\epsilon, \ell^{-1} \,|\, n)\mathrm{Vol}_n(B_1(m)) \,. \tag{1.27}$$

To see (1.25), note that by the coarea formula,

(1.28)
$$J_{0,1} = \frac{1}{\mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0))} \int_{\Phi^{-1}(B_1^k(0))} \chi(\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)) \sqrt{\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)} \,,$$

which, together with (1.22), (1.23), gives (1.25).

Similarly, the coarea formula gives

$$V_{0,1} = \frac{1}{\mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0))} \int_{\Phi^{-1}(B_1^k(0))} \sqrt{\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)} \,, \tag{1.29}$$

which, together with (1.22), (1.23), implies

(1.30)
$$\left| V_{0,1} - (\mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0)))^{-1}\mathrm{Vol}_n(B_1(m)) \right| \le \Psi(\epsilon, \ell^{-1} \,|\, n)\, \mathrm{Vol}(B_1(m)) \,.$$

From (1.13), (1.25), (1.29) we get (1.26).

The proof of (1.27), relies on the following lemma, which represents a "reverse Poincaré inequality" for the function $J(z)$.

Lemma 1.31.

$$\fint_{B_1^k(0)} |\nabla J(z)| \le \Psi(\epsilon, \ell^{-1} \,|\, n) J_{0,1} \,. \tag{1.32}$$

Proof. Observe that from (1.16) together with the coarea formula, we get

$$
\begin{aligned}
\int_{B_1^k(0)} \left|\frac{\partial J}{\partial z_i}\right| &\leq \bar{C} \sum_{j=1}^{k} \int_{B_1^k(0)} \int_{\Phi^{-1}(z)} |\mathrm{Hess}_{\mathbf{b}_j}| \\
&\leq \bar{C} \sum_{j=1}^{k} \int_{B_1(m)} |\mathrm{Hess}_{\mathbf{b}_j}| \sqrt{\det(\langle \nabla \mathbf{b}_s, \nabla \mathbf{b}_t \rangle)}\ .
\end{aligned} \tag{1.33}
$$

Thus, (1.32) follows easily from (1.23) together with the Schwarz inequality. q.e.d.

From (1.32) and the Poincaré inequality for $B_1^k(0)$, we obtain

$$
⨍_{B_1^k(0)} |J - J_{0,1}| \leq \Psi(\epsilon, \ell^{-1} \,|\, n)\, J_{0,1}\,, \tag{1.34}
$$

which, together with (1.25) gives (1.27). This suffices to complete the proof of Theorem 1.2. q.e.d.

Remark 1.35. Relation (1.7) of Theorem 1.2 can be viewed as providing a sharpening of Proposition 1.35 of [4], which implies the uniqueness of renormalized limit measures on $\mathbf{R}^k$, arising from sequences for which $\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i$, where $\delta_i \to 0$. In the context of Theorem 1.2, Proposition 1.35 would leave open the possiblity that the function, $\tilde{V}(z)$, oscillates rapidly and is only close to being constant in the sense of measures. In particular, Theorem 1.40 below does not follow from Proposition 1.35; compare (1.48).

Corollary 1.36. *There exists $\epsilon(n) > 0$ such that if $(M_i^n, m_i)\} \xrightarrow{d_{GH}} (Y, y)$ satisfies (0.3) and $(\mathcal{WR}_k)_\epsilon \neq \emptyset$, for some $\epsilon \leq \epsilon(n)$, then $\mathcal{H}^k(Y) > 0$.*

Proof. Let $y \in (\mathcal{WR}_k)_\epsilon$. If $\epsilon \leq \epsilon(n)$, for $\epsilon(n)$ sufficiently small, then for i sufficiently large, there exist Lipschitz maps, $\Phi_i : B_r(m_i) \to \mathbf{R}^k$, with uniformly bounded Lipschitz constants, as in Theorem 1.2. We can assume that some subsequence, $\{\Phi_j\}$, converges to a Lipschitz map, $\Phi : B_r(y) \to \mathbf{R}^k$, $B_r(y) \subset Y$. Since $\Phi(\overline{B_s(y)})$ is compact, for all $0 < s < r$, and Φ_i is almost surjective for i sufficiently large, a straightforward limiting argument shows that

$$
\frac{\mathrm{Vol}_{\mathbf{R}^k}(B_r^k(0) \setminus \Phi(B_r(y)))}{\mathrm{Vol}_{\mathbf{R}^k}(B_r^k(0))} \leq \Psi(\epsilon, \ell^{-1} \,|\, n)\,. \tag{1.37}
$$

Now, from the fact that Φ is Lipschitz, it follows easily that $\mathcal{H}^k(B_r(y))$ is positive. This completes the proof. q.e.d.

Recall that in [4], a limit space satisfying (0.3) is called *polar*, if the base point of every iterated tangent cone is a pole, i.e., every infinite geodesic which emanates from the base point is a ray. At present, we do not know of an explicit example of limit space which is not polar.

Let $\mathcal{D}_k$ denote the set of points, y, such that no tangent cone splits of a factor, $\mathbf{R}^k$ isometrically. By Theorem 4.7 of [4], if Y is polar, then $\dim \mathcal{D}_k \le k$. As pointed out in Section 4 of [4], from this result and Corollary 1.36, we immediately obtain the following consequence.

Theorem 1.38. *The Hausdorff dimension of a polar limit space is an integer.*

We close this section with the generalization of the results of [12], [4] which was described in Section 0 and at the beginning of this section.

Recall that if Z is a metric space and $U \subset Z$, the k-dimensional spherical Hausdorff content, $\mathcal{H}^k_\infty(U)$, of U is defined as follows; see [14]. Let $\mathcal{B} = \{B_{r_i}(q_i)\}$ denote a covering of U. Put

$$\mathcal{H}^k_\infty(U) = \mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0)) \inf_{\mathcal{B}} \sum_i r_i^k \,. \tag{1.39}$$

Theorem 1.40. *Let $(M_i^n, m_i) \xrightarrow{d_{GH}} (M^k, m)$ satisfy (0.3), with M^k a manifold. Then for any ball, $B_r(\underline{m}) \subset M^k$, and sequence, $\underline{m}_i \to \underline{m}$,*

$$\mathcal{H}^k_\infty(B_r(\underline{m}_i)) \to \mathit{Vol}(B_r(\underline{m}))\,. \tag{1.41}$$

Proof. By standard covering and rescaling arguments, it suffices to show that under the assumptions of Theorem 1.2, we have

$$\mathcal{H}^k_\infty(B_1(m)) \ge \mathrm{Vol}_{\mathbf{R}^k}(B_1^k(0)) - \Psi(\epsilon, \ell^{-1} \,|\, n)\,; \tag{1.42}$$

the opposite inequality is clear.

Fix $\eta > 0$. As in the proof of Lemma 2.5 of [4] (compare also [12]) (1.23) implies that there exists $E_\eta \subset B_{1-\eta}(m)$, with

$$\mathrm{Vol}_n(E_\eta) \ge (1 - \Psi(\epsilon, \ell^{-1} \,|\, \eta, n))\mathrm{Vol}_n(B_{1-\eta}(m))\,, \tag{1.43}$$

such that for all $p \in E_\eta$, $r < \eta/6$, and $q_1, q_2 \in B_r(p)$,

$$|\overline{\Phi(q_1), \Phi(q_2)} - \overline{q_1, q_2}| \le \Psi(\epsilon, \ell^{-1} \,|\, \eta, n)\,. \tag{1.44}$$

Clearly, (1.44) implies

$$(1 - \Psi(\epsilon, \ell^{-1} \,|\, \eta, n))\mathcal{H}^k_\infty(E_\eta)) \geq \mathrm{Vol}_{\mathbf{R}^k}(\Phi(E_\eta))\,. \tag{1.45}$$

Thus (by letting $\epsilon \to 0$ and then $\eta \to 0$) it suffices to show

$$\mathrm{Vol}_{\mathbf{R}^k}(\Phi(E_\eta)) \geq (1 - \Psi(\epsilon, \ell \,|\, \eta, n))\mathrm{Vol}_{\mathbf{R}^k}(B_{1-\eta}(0))\,. \tag{1.46}$$

Let $A \subset B^k_1(0)$ denote an arbitrary subset. From (1.5), and the coarea formula, it follows that

$$\int_A V(z) \leq (c(n))^k \mathrm{Vol}_n(\Phi^{-1}(A))\,. \tag{1.47}$$

Hence, by (1.7),

$$\mathrm{Vol}_{\mathbf{R}^k}(A) \leq (c(n))^k \frac{\mathrm{Vol}_n(\Phi^{-1}(A))}{\mathrm{Vol}_n(B_1(m))} \mathrm{Vol}_{\mathbf{R}^k}(B_1(0)) - \Psi(\epsilon, \ell \,|\, n)\,. \tag{1.48}$$

If in (1.48), we take $A \subset \Phi(B_{1-\eta}(m))$ to be the subset of points, z, such that $\Phi^{-1}(z) \subset B_{1-\eta}(0) \setminus E_\eta$, then from (1.8), (1.43), (1.48), we obtain (1.46). q.e.d.

Remark 1.49. Even though Theorem 1.2 implies a statement about convergence of renormalized volumes, it does not follow that $\mathcal{H}^k_\infty(B_r(\underline{m_i}))$ could be replaced by the renormalized volume $\underline{\mathrm{Vol}}(B_r(\underline{m_i}))$ in Theorem 1.40. As mentioned in Section 0, such a statement would be false in general, even for sequences for which the sectional curvatures of the M^n_i are uniformly bounded. Note in this connection that the hypothesis of Theorem 1.2 is much stronger that that of Theorem 1.40. Note also that the notion of k-dimensional Hausdorff content does not involve any sort of renormalization.

2. The strongly singular set

In this section, we introduce a family of lower dimensional Hausdorff measures associated to a Borel measure, μ; compare [14]. We then specialize to the case $\mu = \nu$, a renormalized limit measure on a possibly collapsed limit space, Y. We show that for the notion of codimension defined by this family of measures, the strongly singular set satisfies $\mathrm{codim}_\nu \mathcal{SS} \geq 1$. Recall that $\mathcal{SS} = Y \setminus \mathcal{WR}$, where $\mathcal{WR}$ denotes the weakly regular set.

Let Z be a metric space and let μ be a Borel measure on Z. For $\beta \in \mathbf{R}$, we define the associated Hausdorff measure in codimension β as follows. Fix $\delta > 0$ and $U \subset Z$. Let $\mathcal{B} = \{B_{r_i}(q_i)\}$ be a covering of U with $r_i < \delta$, for all i. Put

$$(\mu_{-\beta})_\delta(U) = \inf_{\mathcal{B}} \sum_i r_i^{-\beta} \mu(B_{r_i}(q_i)) . \tag{2.1}$$

As usual, $(\mu_{-\beta})_\delta(U)$ is a nonincreasing function of δ and we put

$$\mu_{-\beta}(U) = \lim_{\delta \to 0} (\mu_{-\beta})_\delta(U) . \tag{2.2}$$

Clearly, $\mu_{-\beta}$ is a metric outer measure. Thus, by standard measure theory, the Borel sets are $\mu_{-\beta}$-measurable; see e.g. [14].

We say that μ satisfies a *doubling condition* if for all $s' > 0$, there exists $\kappa = \kappa(s')$, such that, $\mu(B_{2s}(z)) \le 2^\kappa \mu(B_s(z))$, for all $z \in Z$ and $0 < s \le s'$. It is more standard to require that the constant, κ, can be chosen independent of s', but this stipulation would play no role here.

If μ satisfies a doubling condition in our sense, it follows from the standard argument given in Section 1 of [4] that we have $\mu_0 = \mu$. If μ is a Hausdorff measure, $\mathcal{H}^k$, then (up to normalization) so is $\mu_{-\beta}$, for any β. For $U \subset Z$, we say $\operatorname{codim}_\mu U \ge \beta'$, if $\mu_{-\beta}(U) = 0$, for all $\beta < \beta'$.

Let Y be the pointed Gromov-Hausdorff limit of a sequence of manifolds, $\{(M_i^n, m_i)\}$, such that (0.3) holds. Let ν be a renormalized limit measure on Y as in Section 1 of [4].

Theorem 2.3. *The set, $\mathcal{SS} \subset Y$, satisfies $\operatorname{codim}_\nu \mathcal{SS} \ge 1$.*

Proof. The proof follows the pattern of that of Theorem 4.7 of [4]. We begin by recalling some definitions.

If *every* tangent cone at $y \in Y$ splits off a factor, $\mathbf{R}^k$, isometrically, then y is called *k-Euclidean.* We denote the set of k-Euclidean points by $\mathcal{E}_k$. We put $\mathcal{WD}_k = Y \setminus \mathcal{E}_{k+1}$.

Let $y \in Y$ and let $\rho_y(z) = \overline{z, y}$ denote the distance function from y. We say that a point, $z \in Y$, is *not* a *restricted cut point* of y, if for all $\epsilon > 0$, there exists $r(z, \epsilon) > 0$, such that for $0 < r < r(z, \epsilon)$, there exists a space, X_r, $(0, x_r) \in \mathbf{R} \times X_r$ (the isometric product) and a pointed ϵr–Gromov–Hausdorff approximation, $\phi_r : B_r(z) \to B_r((0, x_r))$, such that

$$|\rho_y - t \circ \phi_r| < \epsilon r \qquad (\text{on } B_r(z)) . \tag{2.4}$$

Here t denotes the coordinate function on $\mathbf{R} \times X_r$ corresponding to the factor, $\mathbf{R}$.

Let $\mathcal{WD}_0(y)$ denote the set of restricted cut points of y. Note that $\mathcal{WD}_0 \subset \cap_y \mathcal{WD}_0(y)$. We put $\mathcal{E}_1(y) = Y \setminus \mathcal{WD}_0(y)$.

The following proposition is an improved version of Proposition 2.13 of [4]. The conclusion of that proposition, is the assertion, $\nu(\mathcal{WD}_0(y)) = 0$; see (2.14) of [4]. We observe that a trivial modification of the proof enables one to strengthen the conclusion to $\nu_{-\beta}(\mathcal{WD}_0(y)) = 0$, for all $\beta < 1$; see (2.6) below.

Proposition 2.5. *If Y is not a single point, then for all $y \in Y$ and $\beta < 1$.*

$$\nu_{-\beta}(\mathcal{WD}_0(y)) = 0\,. \tag{2.6}$$

Proof. We will use the notation of [4]. Fix $0 \leq \beta < 1$. Note that (2.22) of [4] can be strengthened to the assertion that the set, $\bigcup_{k \geq k_0}^{\infty} W(p,j,k,\tau_\ell)$, of [4], admits a covering by balls, $\{B_{\tau_\ell 2^{-(j+k)}}(q_{i_k})\}$, where $k \geq k_0$, such that

$$\sum_{i_k=1}^{N_k} (\tau_\ell 2^{-k})^{-\beta} \nu\left(\overline{(B_{\tau_\ell 2^{-k}}(q_{i_k})}\right) \leq c(n, 2^{-j}, 2^j, \tau_\ell, \beta) 2^{-k(1-\beta)}\,. \tag{2.7}$$

This follows immediately from (2.18) of [4], provided we divide both sides of that equation by $(\tau\eta)^\beta$. By using (2.7) in place of (2.22) of [4], we get in place of (2.23) of [4],

$$\nu_{-\beta}\left(\bigcap_{k_0=1}^{\infty} \bigcup_{k \geq k_0}^{\infty} W(p,j,k,\tau_\ell)\right) = 0\,. \tag{2.8}$$

Hence, we can strengthen (2.24) of [4] to

$$\nu_{-\beta}(\mathcal{WD}_0(y) \cap A_{2^{-j},2^j}(y)) = 0\,, \tag{2.9}$$

and letting $j \to \infty$, we obtain (2.6). q.e.d.

Now we can finish the proof of Theorem 2.3.

Assume that the conclusion of Theorem 2.3 is false. Then for some β, with $0 < \beta < 1$, we have $\nu_{-\beta}(\mathcal{SS}(Y)) = \infty$. In view of Proposition 2.5, a density argument completely analogous to that used in the proof Theorem 4.7 of [4], shows that $\nu'_{-\beta}(\mathcal{SS}(Y_y)) = \infty$, for some tangent cone, Y_y, which splits isometrically as $\mathbf{R} \times X$, for some X and some renormalized limit measure, ν', which admits a corresponding splitting. Here, (2.6) plays the role of the assumption made in Theorem 4.7, to the effect that the space in question is polar.

The above argument can be repeated starting with Y_y in place of Y. As in Theorem 4.7, the resulting second order tangent cone splits off a factor, $\mathbf{R}^2$, isometrically. After $n+1$ repetitions, we obtain an an iterated tangent cone which splits off a factor, $\mathbf{R}^{n+1}$, isometrically. But this is impossible for limit spaces satisfying (0.3). q.e.d.

3. Connectedness properties of $\mathcal{R}$; the noncollapsed case

In this section, we consider pointed Gromov-Hausdorff limit spaces, $\{(M_i^n, m_i, \underline{\mathrm{Vol}}_i)\} \xrightarrow{d_{GH}} (Y, y, \nu)$, satisfying (0.3). In the main application, Theorem 3.9, we add the noncollapsing condition (0.4).

Recall that if M^n is a smooth manifold and $B \subset M^n$ is closed, with $\mathcal{H}^{n-1}(B) = 0$, then $M^n \setminus B$ is arcwise connected. The following lemma enables us to extend this result to possibly collapsed limit spaces.

Lemma 3.1. *For all $d, \epsilon > 0$ there exists $C(n, d, \epsilon) > 0$, such that the following holds. Let M^n satisfy (0.1) and let*

$$E = \bigcup_j \overline{B_{r_j}(q_j)}\,, \tag{3.2}$$

$$B_\epsilon(x_1) \cup B_\epsilon(x_2) \subset \overline{B_d(m)} \setminus E\,. \tag{3.3}$$

Then if every minimal geodesic, $\gamma : [0, \ell] \to M^n$, with $\gamma(0) = x_1$, $\gamma(\ell) \in B_\epsilon(x_2)$, intersects E, we have

$$0 < c(n, d, \epsilon) \leq \sum_j r_j^{-1}\, \underline{Vol}(B_{r_j}(q_j))\,. \tag{3.4}$$

Proof. As in [19], by observing the ball, $B_\epsilon(x_2)$, from the point, x_1, we get

$$\begin{aligned}(3.5)\qquad \mathrm{Vol}(B_\epsilon(x_2)) &\leq C(n,d,\epsilon)\,\mathrm{Vol}(\partial E)\\ &\leq C(n,d,\epsilon)\sum_j \mathrm{Vol}(\partial B_{r_j}(q_j))\\ &\leq C(n,d,\epsilon)\sum_j r_j^{-1}\mathrm{Vol}(B_{r_j}(q_j))\,,\end{aligned}$$

where in the last step, we have used relative volume comparison. Since by (0.5) and the relative volume comparison theorem,

$$(3.6)\qquad 0 < C(n,d,\epsilon) \leq \underline{\mathrm{Vol}}(B_\epsilon(x_2))\,,$$

if we divide both sides of (3.6) by $\mathrm{Vol}(B_1(m))$, the claim follows. q.e.d.

Theorem 3.7. *Let Y satisfy (0.3) and let B be a closed subset of Y, with $\nu_{-1}(B) = 0$, for some renormalized limit measure ν. Let $y_1 \in Y \setminus B$. Then for ν-almost all $y_2 \in Y \setminus B$, there exists a minimal geodesic, from y_1 to y_2 which lies in $Y \setminus B$.*

Proof. Let $\{(M_i^n, m_i, \underline{\mathrm{Vol}}_i)\} \xrightarrow{d_{GH}} (Y, y, \nu)$. It suffices to assume $B \subset B_d(y)$, for some $d < \infty$. Then, since $\nu_{-1}(B) = 0$, it follows that for all $\eta > 0$, there exists, $\{B_{r_j}(w_j)\}$, with $1 \leq j \leq N_\eta$, such that $B \subset E_\eta = \bigcup_j \overline{B_{r_j}(w_j)}$ and $\Sigma_j r_j^{-1}\nu(\overline{B_{r_j}(w_j)}) \leq \eta$.

For $\epsilon > 0$ and $y_1 \in Y$, with $\overline{y_1, E_\eta} \geq \epsilon$, let $A_{\eta,\epsilon} \subset B_d(y) \setminus B$, denote the set of points, y_2, such that there exists a minimal geodesic from y_1 to y_2 lying at distance $\geq \epsilon$ from E_η. Clearly, the sets, E_η and $A_{\eta,\epsilon}$ are compact.

For some fixed sequence of Gromov-Hausdorff approximations, let $m_{i,1}, q_{i,j} \in M_i^n$, be such that $m_{i,1} \to y_1$, $q_{i,j} \to w_j$. Put $E_{\eta,i} = \bigcup_j^{N_\eta} \overline{B_{r_j}(q_{i,j})}$. For $\psi > 0$, let $A_{i,\eta,\epsilon+\psi}$ denote the set of points, $m_{i,2}$ such that there exists a minimal geodesic segment from $m_{i,1}$ to $m_{i,2}$ lying at distance $\geq \epsilon + \psi$ from $E_{\eta,i}$. Then by a standard compactness argument, for i sufficiently large, we have (relative to suitable Gromov-Hausdorff approximations) $A_{i,\eta,\epsilon+\psi} \subset T_\psi(A_{\eta,\epsilon})$, where $T_\psi(\,\cdot\,)$, denotes the tubular neighborhood of radius ψ. Moreover, since $A_{\eta+\epsilon}$ is compact, after passing to a subsequence, we have $\limsup_{i\to\infty} \underline{\mathrm{Vol}_i}(A_{i,\eta,\epsilon+\psi}) \leq \nu(A_{\eta,\epsilon+\psi})$. From Lemma 3.1, we get, $\underline{\mathrm{Vol}}(B_d(m_i)) - \Psi(\eta\,|\,\epsilon,\psi,n) \leq \underline{\mathrm{Vol}}_i(A_{i,\eta,\epsilon+\psi})$.

Thus, by letting, $\eta \to 0$, $\psi \to 0$ and then $\epsilon \to 0$, the theorem follows.
q.e.d.

Example 3.8. As observed in Example 1.24 of [4], the space $[0, \infty)$, with the measure, ν, given by integration of the 1-form, $r\,dr$, occurs as the limit of a collapsing sequence of 2-dimensional manifolds satisfying (0.3). In this case, $\nu_{-1}(\{0\}) = 0$, even though for the standard 1-dimensional Hausdorff measure, we have $\mathcal{H}^0(\{0\}) = 1$.

In the noncollapsed case, the singular set $\mathcal{S}$, satisfies $\dim \mathcal{S} \le n-2$; see Theorem 6.2 of [4]. In addition, for all $\epsilon > 0$, there exists $\epsilon' > 0$, such that $(\mathcal{R})_{\epsilon'} \subset \overset{\circ}{(\mathcal{R})}_{\epsilon}$, the interior of $(\mathcal{R})_\epsilon$; compare the proof of Corollary 3.10 below. Thus, from Theorem 3.7, we obtain:

Theorem 3.9. *Let Y^n, satisfy (0.3), (0.4). Then for all $z \in \mathcal{R}$, there exists $\mathcal{C}(z) \subset \mathcal{R}$, with $\nu(Y \setminus \mathcal{C}(z)) = 0$, such that for all $w \in \mathcal{C}(z)$ and $\epsilon > 0$, there exists a minimal geodesic from z to w which is contained in $\overset{\circ}{(\mathcal{R})}_{\epsilon}$. Moreover, for all $\epsilon, \psi > 0$ and all $y_1, y_2 \in \overset{\circ}{(\mathcal{R})}_{\epsilon}$, there exists a curve, $c : [0, \ell] \to \overset{\circ}{\mathcal{R}}_{\epsilon}$ from y_1 to y_2, with length, $L(c) \le \overline{y_1, y_2} + \psi$.*

From Theorem 3.9, we get the following corollary. Let $(\mathcal{R})_{\epsilon,\delta} \equiv (\mathcal{R}_n)_{\epsilon,\delta}$ be defined as in Section 0.

Corollary 3.10. *Let Y^n, satisfy (0.3), (0.4). Then for all $z, w \in \mathcal{R}$ and $\epsilon > 0$, there exists $\delta > 0$, such that z, w lie in the same component of $(\mathcal{R})_{\epsilon,\delta}$.*

Proof. By Theorem 3.9, for all $\epsilon' > 0$, there exists a continuous curve, $c_{\epsilon'} \subset (\mathcal{R})_{\epsilon'}$, from z to w. It follows from Theorem A.1.5 of [4] (which depends on the conjectures of Anderson-Cheeger proved in [12]) that for $0 < \epsilon' < \epsilon$ sufficiently small, there exists $\delta > 0$, such that for such a curve, $c_{\epsilon'}$, we have $c_{\epsilon'} \subset (\mathcal{R})_{\epsilon,\delta}$. q.e.d.

4. Isometry groups of noncollapsed limit spaces

In this section we will show that the isometry group of a limit space satisfying (0.3), (0.4) is a Lie group. Conjecturally, this holds even in the collapsed case. Note that Fukaya-Yamaguchi have proved that the isometry group of an Alexandrov space is a Lie group; see [16] and compare also [20].

Theorem 4.1. *If (Y^n, y) is a pointed Gromov-Hausdorff limit of a sequence of manifolds, $\{(M_i^n, m_i)\}$, satisfying (0.3), (0.4), then the isometry group, Isom(Y), is a Lie group.*

Proof. The theorem is an immediate consequence of Theorem 4.5 below, together with Corollary 3.10. q.e.d.

Let Z be an arbitrary metric space. Denote by d (or $\overline{\cdot,\cdot}$) the metric on Z and as usual, let d_0 denote the standard metric on $\mathbf{R}^k$.

For $H \subset \mathrm{Isom}(Z)$, put

$$\rho_H(z) = \sup_{h \in H} \overline{h(z), z}\,, \tag{4.2}$$

$$D_{H,r}(z) = \sup_{w \in B_{\frac{1}{2}r}(z)} \rho_H(w)\,. \tag{4.3}$$

If every closed ball in Z is the closure of its interior, for example, if Z is a length space, then for fixed H, the function $D_{H,r}(z)$ is continuous in r and z.

We will use the standard (and obvious) fact that there exists no nontrivial subgroup, $H \subset \mathrm{Isom}(\mathbf{R}^n)$, with

$$D_{H,1}(0) \le \frac{1}{20}\,. \tag{4.4}$$

Theorem 4.5. *Let Z be a locally compact metric space such that every closed ball is the closure of its interior. Assume*
(a) $\mathcal{R} = \cup_{i=1}^N \mathcal{R}_k$ is dense in Z.
(b) For all $\epsilon > 0$, $k = 1, \dots, N$, there exists $z_1(\epsilon), \dots, z_{N(\epsilon)}(\epsilon) \in \mathcal{R}_k$, such that for all $w \in \mathcal{R}_k$, there exist β, $\delta > 0$, such that z_β and w lie in the same component of $(\mathcal{R}_k)_{\epsilon,\delta}$.
Then the isometry group, Isom(Z), is a Lie group.

Proof. By [17] and [21] it suffices to show that Isom(Z) does not contain a sequence of small subgroups. Assume to the contrary, that there exists a sequence, $\{H_i\}$, of nontrivial subgroups, such that for all $R > 0$ and $z \in Z$,

$$\lim_{i \to \infty} D_{H_i,R}(z) = 0\,. \tag{4.6}$$

To obtain a contradiction, it suffices to show that for all $\epsilon > 0$, there exist $z(\epsilon)$, $i(\epsilon)$, $\tau(\epsilon) > r(\epsilon) > 0$, with $z(\epsilon) \in \mathcal{R}_{\epsilon,\tau(\epsilon)}$, such that

$$D_{H_{i(\epsilon)},r(\epsilon)}(z(\epsilon)) = \frac{1}{20} r(\epsilon)\,. \tag{4.7}$$

For in that case, by taking a sequence, $\epsilon_j \to 0$, and a suitable subsequence, $\epsilon_\ell \to 0$, we can assume that for some k and $H_{i(\epsilon_\ell)}$ satisfying (4.4), we have in the sense of equivariant Gromov-Hausdorff convergece,

$$(B_{r(\epsilon_\ell)}(z_{\epsilon_\ell}), (r(\epsilon_\ell))^{-1} d, H_{i(\epsilon_\ell)}) \xrightarrow{d_{GH}} (B_1^k(0), d_0, H)\,, \tag{4.8}$$

for some $H \subset \mathrm{Isom}(\mathbf{R}^k)$ satisfying (4.3). This would contradict (4.4).

Fix $\epsilon > 0$. To simplify the notation, in what follows, we will suppress the dependence of the relevant quantities on the particular choice of ϵ.

Choose $\eta > 0$, such that for $z_1, \ldots, z_N$ as in (b), we have $z_1, \ldots, z_N \in \mathcal{R}_{\epsilon,\eta}$. By (4.6), there exists i, such that for all α,

$$D_{H_i,\eta}(z_\alpha) \leq \frac{1}{20}\eta\,. \tag{4.9}$$

By (a), there exists $\theta > 0$, $w \in \mathcal{R}_{\epsilon,\theta}$, such that

$$D_{H_i,\theta}(w) \geq \frac{1}{20}\theta \tag{4.10}$$

For this w, choose β as in (b) and $\lambda < \min(\eta, \theta)$, such that z_β and w lie in the same component of $\mathcal{R}_{\epsilon,\lambda}$.

Suppose,

$$D_{H_i,\lambda}(z_\beta) \geq \frac{1}{20}\lambda. \tag{4.11}$$

Then by (4.9), the continuity of $D_{H_i,\cdot}(z_\beta)$ and the Intermediate Values Theorem, there exists r, with $\lambda \leq r \leq \eta$, such that if we take $z(\epsilon) = z_\beta$, $i(\epsilon) = i$, $\tau(\epsilon) = \eta$, $r(\epsilon) = r$, then (4.7) holds.

Similarly, if

$$D_{H_i,\lambda}(w) \leq \frac{1}{20}\lambda, \tag{4.12}$$

there exists r, with $\lambda \leq r \leq \theta$, such that if we take, $z(\epsilon) = w$, $i(\epsilon) = i$, $\tau(\epsilon) = \theta$, $r(\epsilon) = r$, then (4.7) holds.

Thus, we can assume

$$D_{H_i,\lambda}(z_\beta) \leq \frac{1}{20}\lambda \leq D_{H_i,\lambda}(w)\,. \tag{4.13}$$

Since z_β and w lie in the same component of $(\mathcal{R}_k)_{\epsilon,\lambda}$, it follows from (4.13) and the Intermediate Values Theorem, that there exists $z(\epsilon)$ lying in this component, such that if we take $\epsilon = i$, $\tau(\epsilon) = \lambda$, $r(\epsilon) = \lambda$, then (4.7) holds for some $z(\epsilon) \in (\mathcal{R}_k)_{\epsilon,\lambda}$. This suffices to complete the proof. q.e.d.

5. Limit spaces with 1-dimensional pieces

In this section we show that if a limit space contains a 1-dimensional piece and satisfies an additional condition, then it is actually a 1-dimensional manifold with possibly nonempty boundary. Hence, such a limit space is isometric to $(-\infty,\infty)$, $[0,\infty)$, or to $[0,\ell]$, for some ℓ, or to a circle. This enables one to rule out certain candidates for limit spaces which could not be eliminated by arguments based solely on the splitting theorem and to confirm a number of basic conjectures concerning (possibly collapsed) limit spaces in this extremely special case.

Let Z be a connected length space. A minimal geodesic segment, $\gamma : [-2\ell, 2\ell] \to Z$, is called a 1-*dimensional piece* of Z if $B_\epsilon(\gamma(s)) = \gamma((s-\epsilon, s+\epsilon))$, for all $(s-\epsilon, s+\epsilon) \subset [-2\ell, 2\ell]$. In particular, for any minimal geodesic, $\sigma : [0, L] \to Z$, with $\sigma(0) = \gamma(0)$, $\sigma(L) \in B_\ell(\gamma(2\ell))$, $2\ell \leq L$, we have $\sigma \,|\, [0, 2\ell] = \gamma \,|\, [0, 2\ell]$.

If Z contains a 1-dimensional piece, but is *not* 1-dimensional, then a segment, γ, as above, when suitably extended in at least one direction, must branch i.e., for all $\epsilon > 0$, there exists a minimal geodesic segment, $\sigma : [0, 2\ell+\epsilon] \to Z$ as above, with $\sigma(2\ell+\delta) \neq \gamma(2\ell+\delta)$, for some $0 < \delta < \epsilon$. In particular, there exist *distinct* points, $p_1, p_2 \in B_\ell(\gamma(2\ell))$, such that $\overline{\gamma(0), p_1} = \overline{\gamma(0), p_2} > 2\ell$.

We say that γ as above *branches weakly at* $\gamma(2\ell)$, if for all $p_1, p_2 \in B_\ell(\gamma(2\ell))$ with $\overline{\gamma(0), p_1}$, $\overline{\gamma(0), p_2} > 2\ell$, there exist minimal geodesic segments, $\sigma_i : [0, L_i] \to Z$, from $\gamma(0)$ to p_i, such that $\sigma_1(s) = \sigma_2(s)$, for some $s > 2\ell$.

Theorem 5.1. *Let Y satisfy (0.3). If Y contains a 1-dimensional piece, $\gamma : [-2\ell, 2\ell] \to Y$, which branches at $\gamma(2\ell)$, then γ branches* *weakly at* $\gamma(2\ell)$.

Proof. If the assertion is false, there exist p_1, p_2, with $2\ell < \overline{\gamma(0), p_1} = \overline{\gamma(0), p_2} = L$, such that for all minimal geodesics, $\sigma_i : [0, L] \to Z$, from $\gamma(0)$ to p_i, we have $\sigma_1(s) \neq \sigma_2(s)$, for all $2\ell < s \leq L$.

For $0 < r < R$, let $A_{r,R}(p)$ denote the open annulus, $B_R(p) \setminus \overline{B_r(p)}$. Let $p \in B_\ell(\gamma(2\ell))$ satisfy $\overline{\gamma(0), p} = L > 2\ell$ and let ϵ be sufficiently small.

Since $A_{L-2\ell, L-2\ell+2\epsilon}(p) \supset B_\epsilon(\gamma(2\ell - \epsilon))$, by observing $B_\epsilon(\gamma(2\ell - \epsilon))$ from the point, p_i and applying directionally restricted relative volume comparison, it follows that

$$\frac{\nu(A_{L-2\ell-2\epsilon, L-2\ell}(p_i) \cap B_{2\epsilon}(\gamma(2\ell)))}{\nu(B_\epsilon(\gamma(2\ell - \epsilon)))} \geq 1 - \Psi\,, \tag{5.2}$$

where $\Psi = \Psi(\epsilon \,|\, L - 2\ell, n)$.

Since $A_{L-2\ell-2\epsilon, L-2\ell}(p_i) \cap B_\epsilon(\gamma(2\ell)) \subset A_{2\ell, 2\ell+2\epsilon}(\gamma(0))$ and by assumption, $A_{L-2\ell-2\epsilon, L-2\ell}(p_1) \cap A_{L-2\ell-2\epsilon, L-2\ell}(p_2) \cap B_{2\epsilon}(\gamma(2\ell)) = \emptyset$, we get

$$\frac{\nu(A_{2\ell, 2\ell+2\epsilon}(\gamma(0)) \cap B_{2\epsilon}(\gamma(2\ell)))}{\nu(B_\epsilon(\gamma(2\ell - \epsilon)))} \geq 2 - \Psi\,. \tag{5.3}$$

On the other hand, since $A_{2\ell-2\epsilon, 2\ell}(\gamma(0)) \cap B_\ell(\gamma(2\ell)) = B_\epsilon(\gamma(2\ell - \epsilon))$, by observing $A_{2\ell, 2\ell+2\epsilon}(\gamma(0)) \cap B_{2\epsilon}(\gamma(2\ell))$ from $\gamma(0)$ and applying relative volume comparison, we get with (5.3),

$$\nu(B_\epsilon(\gamma(2\ell - \epsilon))) \geq (2 - \Psi)\nu(B_\epsilon(\gamma(2\ell - \epsilon)))\,. \tag{5.4}$$

For ϵ sufficiently small, this is a contradiction. q.e.d.

Example 5.5. As shown in [4], the metric horn Y^5, with metric $dr^2 + (\frac{1}{2} r^{1+\epsilon})^2 g^{S^4}$, arises as the limit of a collapsing sequence, $\{(M_i^8, g_i)\}$. Let Y_j^5 denote the space obtained by attaching at the origin, a line segment, $[-j, 0]$, to the space, Y^5. It follows from Theorem 5.1, that for no $j > 0$ does the space, Y_j^5, arise as the limit of a sequence of manifolds satisfying (0.4).

Remark 5.6. The nonexistence of limit spaces, Y_j^5, discussed in Example 5.5, actually follows from an earlier unpublished result of the authors. It was announced in Example 8.77 [4]. The argument in the proof of Theorem 5.1 was suggested by the referee of [4].

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