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# A HOLOMORPHIC CASSON INVARIANT FOR CALABI-YAU 3-FOLDS, AND BUNDLES ON $K3$ FIBRATIONS

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## Abstract

We briefly review the formal picture in which a Calabi-Yau $n$-fold is the complex analogue of an oriented real $n$-manifold, and a Fano with a fixed smooth anticanonical divisor is the analogue of a manifold with boundary, motivating a holomorphic Casson invariant counting bundles on a Calabi-Yau 3-fold. We develop the deformation theory necessary to obtain the virtual moduli cycles of [31], [7] in moduli spaces of stable sheaves whose higher obstruction groups vanish. This gives, for instance, virtual moduli cycles in Hilbert schemes of curves in $\mathbb{P}^3$, and Donaldson– and Gromov-Witten– like invariants of Fano 3-folds. It also allows us to define the holomorphic Casson invariant of a Calabi-Yau 3-fold $X$, prove it is deformation invariant, and compute it explicitly in some examples. Then we calculate moduli spaces of sheaves on a general $K3$ fibration $X$, enabling us to compute the invariant for some ranks and Chern classes, and equate it to Gromov-Witten invariants of the "Mukai-dual" 3-fold for others. As an example the invariant is shown to distinguish Gross' diffeomorphic 3-folds. Finally the Mukai-dual 3-fold is shown to be Calabi-Yau and its cohomology is related to that of $X$.


## 1. Introduction

This paper is a continuation of the ideas presented in [12], [40]. There a formal picture was outlined in which the complex analogue of a real *oriented* $n$-manifold is a *Calabi-Yau* $n$-fold with a fixed holomorphic $n$-form playing the role of a "complex orientation", while a Fano with fixed smooth anticanonical divisor is the analogue of a manifold with boundary; the boundary being the (Calabi-Yau) divisor. I have since discovered this picture was known and used in low dimensions by the Yale school of Frenkel, Khesin, Todorov and others (see for instance [13],

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[14], [26]); we concentrate on the central, three dimensional theory of [12]. The delay in publication is due to a complete reworking of the deformation theory of Section 3. While [40] and earlier forms of this paper initially used an older version of [31], then the derived category language of [7], here we present more elementary sheaf deformation theory that allows us to use the more down-to-earth definition of obstruction theory in the published form of [31] (without mention of the derived categories or $T^1$-lifting of older drafts of this paper). This deformation theory is folklore but scattered and hard to find, and either highly abstract or not done in enough generality for this application (namely in global families, as the complex structure on the manifold is allowed to vary, arbitrary order deformations and obstructions are considered, and the determinant is fixed). So we are forced to give a full account, for which the unpublished manuscript [15] has been useful.

In Section 2 we review the classical Casson invariant and Chern-Simons functional, before describing their holomorphic analogues. We give two formulae for the holomorphic Chern-Simons functional, one in an algebro-geometric framework, the other illustrating the complex analogue of a manifold with boundary. We then discuss the holomorphic Casson invariant, and a version of it that so far has not been made rigorous but is reviewed as it motivates a number of calculations (and predicts the correct result for them).

Section 3 discusses the technicalities involved in defining the invariant via gauge theory, and then tackles them using algebraic geometry, and the virtual moduli cycle theory of [31], [7]. The result required to apply this machinery to count stable sheaves is that the tangent-obstruction complex [31] or cut-off cotangent complex [7] (these are dual to each other) of the moduli space admits a certain two-step locally free resolution. We obtain one in all the cases we might hope for, namely whenever the higher obstruction groups $\mathrm{Ext}^i_0(\mathcal{E},\mathcal{E})$, $i \geq 3$, of the sheaves $\mathcal{E}$ vanish.

This is the case, for instance, for ideal sheaves of curves in $\mathbb{P}^3$, so we obtain a virtual moduli cycle of the correct dimension in the corresponding Hilbert schemes. The result also applies to a Calabi-Yau 3-fold, of course, allowing us to make our definition. Finally in this section we give some examples, including one conjectured in [12] that is fitted into this scheme and worked out in full, by computing all reflexive sheaves of certain Chern classes on a $(2,4)$ complete intersection in $\mathbb{P}^5$.

Section 4 deals with bundles on $K3$ fibrations. The results here may be of interest to physicists, being the "F-theory" dual picture to the geometry of $K3$-fibred 4-folds. Calculating moduli spaces of sheaves on threefolds is extremely difficult, and we are forced to consider only Chern classes satisfying certain constraints, though this allows us to work on a general $K3$ fibred Calabi-Yau $X$ (without reducible or multiple fibres). Thus we obtain a calculation of the invariant in some cases (where it is 1), and obtain Gromov-Witten invariants of the "Mukai-dual" 3-fold in others. We then show this dual 3-fold is Calabi-Yau and determine its cohomology in terms of that of $X$. Finally we show that these results allow us to distinguish Gross' diffeomorphic Calabi-Yau 3-folds.

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## 2. The holomorphic Casson invariant

We begin by describing Taubes' version [39] of the Casson invariant in purely formal terms, ignoring such issues as the structure group and reducible connections; we shall be able to bypass these in the holomorphic situation by considering only bundles for which semistability implies stability (such as bundles with rank and degree coprime).

For us, then, the Casson invariant of a real 3-manifold $M$ counts flat connections with structure group $G$ on a fixed vector bundle $E$. Formally the curvature $F_A$ of a connection $A$ defines a closed one-form

$$a \mapsto \frac{1}{4\pi^2}\int_M \operatorname{tr}(a \wedge F_A), \qquad a \in \Omega^1(\operatorname{ad} E), \tag{2.1}$$

on the space of connections $\mathscr{A}$. This is gauge invariant, and so descends to the space of gauge equivalence classes $\mathscr{B}$. Fixing a basepoint $A_0 \in \mathscr{A}$ this one-form is in fact the exterior derivative of a locally defined function, the Chern-Simons functional:

$$CS(A) = \frac{1}{4\pi^2}\int_M \mathrm{tr}\left(\frac{1}{2}d_{A_0}a\wedge a + \frac{1}{3}a\wedge a\wedge a\right), \qquad A = A_0 + a,$$

which is independent of gauge transformations connected to the identity, and well defined modulo $\mathbb{Z}$ on $\mathscr{B}$. In particular, at a zero of the one form, i.e., a flat connection, we see that the deformation complex of a flat connection is self-dual – this is the statement that the Hessian of $CS$ is symmetric – as then are its cohomology groups $H^i(\mathrm{ad}\,E;A) \cong H^{3-i}(\mathrm{ad}\,E;A)^*$ by Poincaré duality. Therefore the virtual dimension of the moduli space of flat connections

$$\sum_{i=0}^{3}(-1)^{i+1}\dim H^i(\mathrm{ad}\,E;A)$$

is zero, and we could hope to count them. Formally, flat connections are the zeroes of the covector field $F_A$ (2.1) on $\mathscr{B}$, i.e., critical points of $CS$, and we are trying to make sense of the Euler characteristic of the infinite dimensional space $\mathscr{B}$.

This formal picture translates wholesale onto a Calabi-Yau 3-fold (which for us means a smooth, compact, Kähler 3-fold $X$ with a trivialisation $\theta \in H^{3,0}$ of the canonical bundle $K_X \cong \mathcal{O}_X$). Naively, we replace $x$ by $z$, $d$ by $\bar\partial$, Poincaré duality by Serre duality, and integrating against the complex volume form $\theta$ on $X$ instead of against the orientation of $M$.

So now we consider the space $\mathscr{A}$ of $\bar\partial$-operators (or "half-connections") on a fixed $C^\infty$-bundle $E \to X$, and the closed one-form given by $F_A^{0,2}$:

$$a \mapsto \frac{1}{4\pi^2}\int_X \mathrm{tr}(a\wedge F_A^{0,2})\wedge\theta, \qquad a\in\Omega^{0,1}(\mathrm{ad}\,E).$$

Again this is gauge invariant and descends to the space of gauge equivalence classes $\mathscr{B}$. Fixing a basepoint $A_0 \in \mathscr{A}$ this one-form is the exterior derivative of a locally defined holomorphic function, the holomorphic Chern-Simons functional:

$$CS(A) = \frac{1}{4\pi^2}\int_X \mathrm{tr}\left(\frac{1}{2}\bar\partial_{A_0}a\wedge a + \frac{1}{3}a\wedge a\wedge a\right)\wedge\theta, \quad A = A_0 + a, \tag{2.2}$$

which is independent of gauge transformations connected to the identity. The periods under large gauge transformations are more complicated (and will usually be dense), but this will not concern us. All statements will be made "mod periods"; what we should really do is consider $CS$ to be an element of some Albanese torus and formulate statements there, as discussed in [40], but this would take us too far afield. The zeroes of the one-form, i.e., the critical points of $CS$, are the integrable holomorphic structures on the bundle $E$ (i.e., $\bar{\partial}^2 = F_A^{0,2} = 0$ instead of $d^2 = F_A = 0$), and the deformation complex of a holomorphic connection is self-dual – again the statement that the Hessian of $CS$ is symmetric – as then are its cohomology groups $H^{0,i}(\operatorname{ad} E; A) \cong H^{0,3-i}(\operatorname{ad} E; A)^*$, by Serre duality and the fixed trivialisation $\theta$ of the canonical bundle $K_X$. Therefore the virtual dimension of the moduli space of holomorphic bundles

$$\sum_{i=0}^{3}(-1)^{i+1}\dim H^{0,i}(\operatorname{ad} E; A)$$

is zero, and we could hope to count the bundles to formally compute the Euler characteristic of $\mathscr{B}$.

Of course to get some kind of *compact* space of objects to count we have to consider either Hermitian-Yang-Mills connections in the gauge theory set-up, or stable holomorphic bundles in algebraic geometry; this is in some sense most of them.

Similar holomorphic analogues of all the main gauge theories also exist [12], [40] and have also now been studied by physicists [1], [5], and there is work of Tyurin [44] on related topics. Also, formally manipulating the holomorphic analogue of the Chern-Simons path integral using $CS$ (2.2) gives a holomorphic linking number for complex curves in a Calabi-Yau manifold [40] but since I have discovered the more professional treatment of [14], [27], here we concentrate solely on defining and calculating the holomorphic analogue of the Casson invariant. Firstly, however, we give two formulae for $CS$ to make it more familiar and further illustrate the holomorphic analogy.

The first is in an algebro-geometric spirit, and well known on complex curves as Abel-Jacobi theory: fixing a complex curve $\Sigma$ the appropriate Chern-Simons functional computes holonomy of $\bar{\partial}$-operators $A = \bar{\partial} + a$ on a topologically trivial line bundle, for simplicity. It is

$$a \mapsto \int_\Sigma a \wedge \omega, \qquad \omega \in H^{1,0}(\Sigma),$$

modulo periods. Again these periods are dense and the function is only well defined modulo a discrete lattice when considered as a function of all $\omega$s at once, as an element of the torus

$$\left(H^{1,0}\right)^* / H_1(\Sigma;\mathbb{Z}).$$

In this case we also have the alternative formula

$$\int_\Sigma a \wedge \omega = \int_\gamma \omega \qquad \text{modulo periods,}$$

where $\gamma$ is a path connecting the points that are the zeroes and poles of a section that is meromorphic with respect to the holomorphic structure $\bar\partial + a$. That is, if the holomorphic structure defined by $a$ on the line bundle corresponds to the divisor $\sum a_i(p_i)$, with $a_i \in \mathbb{Z}$ and $p_i \in \Sigma$, then $\partial\gamma = \sum a_i(p_i)$. The principle of the result is that the Euler class of the line bundle is represented holomorphically by both the divisor, which is $\partial\gamma$, and the curvature $da$, and $\partial$ and $d$ are adjoints.

In a special case there is an analogous formula on a Calabi-Yau 3-fold, for a rank 2 holomorphic bundle $E$. Similarly the principle is that $dCS = p_1 \wedge \theta$ and, when $\det E$ is trivial, $p_1 = c_2$ represents the Euler class of $E$, so we are interested in a homology class $\Delta$, the analogue of $\gamma$, with boundary the zero set of a section. (Here and below we use $CS$ to denote both the functional (2.2) and the integrand.)

**Proposition 2.3.** [40] *Suppose that $A_0$ and $A = A_0 + a$ are integrable ($F^{0,2} = 0$) $\bar\partial$-operators on $E$ with trivial determinant, and $(E, A)$, $(E, A_0)$ admit holomorphic sections $s$, $s_0$, with transverse zero sets $(s)_0$, $(s_0)_0$. Then $CS$ defined by (2.2) may also be described as follows. As the zero sets are homologous, write $(s)_0 - (s_0)_0 = \partial\Delta$ for some singular 3-chain $\Delta$. Then, modulo periods, $CS(A) - CS(A_0) = \int_\Delta \theta$.*

The second formula is the complex analogue of the classical formula for the Chern-Simons functional of a connection $A$ on a bundle $E \to M$ on a real 3-manifold: bound $M$ by a 4-manifold $N$ and extend $(E, A)$ to a bundle and connection $(\mathbb{E}, \mathbb{A})$ on $N$. Then

$$CS(A) - CS(A_0) = \int_N p_1(\mathbb{A}) - p_1(\mathbb{A}_0),$$

where $p_1(\mathbb{A}) = (1/4\pi^2)\,\mathrm{tr}\, F_{\mathbb{A}} \wedge F_{\mathbb{A}}$ is the Chern-Weil differential form representing the first Pontryagin class of $\mathbb{E}$. This can be rephrased in a way that will make the complex analogy more apparent in terms of the long exact homology and cohomology sequences of the pair $(N, M)$, by the commutative pairings

$$
\begin{array}{ccccc}
 & [CS(A)] & \mapsto & [p_1(\mathbb{A}) - p_1(\mathbb{A}_0)] & \\
\to & H^3(M) & \xrightarrow{d} & H^4(N,M) & \to 0 \\
 & \otimes & & \otimes & \\
\leftarrow & H_3(M) & \xleftarrow{\partial} & H_4(N,M) & \leftarrow 0. \\
 & [M] & \leftarrow\!\shortmid & [N] &
\end{array}
$$

That is, the fundamental class of $M$ is in the image of the lower map, coming from the fundamental class of $N$, so to find $\int_M CS(A)$ we can map the class $[CS(A)]$ into $H^4(N,M)$ and evaluate on $[N]$ to give the result.

The holomorphic analogue replaces the exact sequence of the pair $(N,M)$ by the sheaf cohomology sequence of the pair $(Y,X)$,

$$0 \to K_Y \xrightarrow{s} \mathcal{O}_Y \to \mathcal{O}_X \to 0, \tag{2.4}$$

in the case that $X$ is an anticanonical divisor in a 4-fold $Y$. Here then we think of $X$ as being bounded by the "Fano" $Y$; we use the term Fano loosely to mean a variety $Y$ with a section $s$ of its anticanonical bundle $K_Y^{-1}$ with a smooth zero set $X$, which is its "boundary" – it is Calabi-Yau by the adjunction formula. In fact about a point of $X \subset Y$, choosing a local coordinate $z$ whose zero locus is $X$, to leading order $s^{-1} = \frac{1}{2\pi i}\theta\frac{dz}{z}$ uniquely defines a holomorphic 3-form $\theta$ on $X$ ([18] p 147). Then we obtain

**Theorem 2.5.** *Suppose that the Calabi-Yau 3-fold $X$ is a smooth effective anticanonical divisor in a 4-fold $Y$ defined by $s \in H^0(K_Y^{-1})$. If $E \to X$ is a bundle that extends to a bundle $\mathbb{E} \to Y$, then for a $\bar{\partial}$-operator $A$ on $E$, let $\mathbb{A}$ be any $\bar{\partial}$-operator on $\mathbb{E}$ extending $A$. We have, modulo periods,*

$$CS(A) = \frac{1}{4\pi^2}\int_Y \operatorname{tr} F_{\mathbb{A}}^{0,2} \wedge F_{\mathbb{A}}^{0,2} \wedge s^{-1}.$$

*Proof.* Notice that $H^4(\mathcal{O}_Y) = H^0(K_Y)^* = 0$: if $t \in H^0(K_Y)$ then $s.t$ is a holomorphic function on $Y$ vanishing on $X$, thus $t = 0$. So the sequence (2.4) gives us the following commutative diagram of Serre duality pairings,

$$
\begin{array}{ccccccc}
 & [CS(A)] & \mapsto & [p_1(\mathbb{A})-p_1(\mathbb{A}_0)]\wedge s^{-1} & & \\
\to & H^3(\mathcal{O}_X) & \longrightarrow & H^4(K_Y) & \longrightarrow & 0 \\
 & \otimes & & \otimes & & \\
\leftarrow & H^0(\mathcal{O}_X) & \longleftarrow & H^0(\mathcal{O}_Y) & \longleftarrow & 0, \\
 & [1] & \leftarrow\!\shortmid & [1] & &
\end{array}
\tag{2.6}
$$

(the first pairing is by integrating against $\theta$) since the upper map takes a holomorphic $(0,3)$-form on $X$, extends it to a $C^\infty$ form on $Y$, and takes $\bar\partial(\,\cdot\,)\wedge s^{-1}$ of the result. Setting $CS(A_0)=\int_Y p_1(\mathbb{A}_0)$ to fix constants gives the result. q.e.d.

Just as the real case $CS(A)=\int_N p_1(\mathbb{A})$ can be proved directly by Stokes' theorem, the above amounts to an application of Stokes' theorem *and* the Cauchy residue theorem, hence reducing dimensions by two, as also observed in [26]. If $\nu_\delta(X)$ denotes a small tubular neighbourhood of $X\subset Y$ then, by Stokes' theorem,

$$\int_Y p_1(\mathbb{A})\wedge s^{-1}=\int_Y d\,(CS(\mathbb{A})\wedge s^{-1})=\lim_{\delta\to 0}\int_{\partial\nu_\delta} CS(\mathbb{A})\wedge s^{-1},$$

which can be integrated first over the fibres of the circle bundle $\partial\nu_\delta\to X$, and then along $X$, by Fubini's theorem. As $s^{-1}\sim\frac{1}{2\pi i}\theta\frac{dz}{z}$, integration over the fibres gives, by the Cauchy reside formula, $\int_X CS(A)\wedge\theta$ in the limit of $\delta\to 0$.

Hence, just as $d$ is adjoint to the boundary operator $\partial$ in real geometry, $\bar\partial(\,\cdot\,)\wedge s^{-1}$ is adjoint to this complex boundary operation of taking the anticanonical divisor $(s)_0$ with its induced complex volume form. An application of this to holomorphic linking is given in [27] – the $\bar\partial$-Green's function for the current represented by a complex curve (weighted by a holomorphic one-form) is represented, as a current, by any Fano surface containing it as an anticanonical divisor. Thus integrating this over another curve (against a one-form), to give Atiyah's holomorphic linking number, is the same as intersecting this second curve with the complex surface, and weighting intersection numbers by ratios of the holomorphic volume forms at intersection points.

Finally we mention briefly the holomorphic analogue of Casson's original approach to counting flat connections by splitting a 3-manifold $M$ (in fact a homology sphere) across a Riemann surface $\Sigma$,

$$M=M_1\cup_\Sigma M_2.$$

The orientation of $M$ induces a symplectic structure on $\Sigma$, and so one on (the smooth locus of) its moduli space of flat connections $\mathcal{M}_\Sigma$. Then those connections on $\Sigma$ that extend to flat connections on $M_1$ form a Lagrangian submanifold in $\mathcal{M}_\Sigma$, the image of the restriction map $\mathcal{M}_{M_1} \to \mathcal{M}_\Sigma$. Similarly for $\mathcal{M}_{M_2}$. These are both of half dimension so we expect them to intersect in a finite number of points – the flat connections on $\Sigma$ that extend to both $M_1$ and $M_2$, i.e., the flat connections on $M$. Casson overcomes the technical difficulties to define just such an invariant, counting (one half of) all the flat connections except the trivial one.

Although not yet rigorous there is a holomorphic analogue of this [12], [26] following work of Tyurin [43]. We review it briefly because it motivates some examples and is verified in all of them. Our complex analogue of gluing across a boundary is to take two Fano 3-folds $X_i$ with a common anticanonical divisor $S$, and form the normal crossings space

$$X = X_1 \cup_S X_2,$$

which is a singular Calabi-Yau. The (singular) holomorphic volume form on $X_i$, with poles along $S$, induces a complex symplectic structure on the surface $S$ (this is just the adjunction formula) and so on its moduli space of (stable) bundles $\mathcal{M}_S$ [32]. Then those bundles on $S$ that extend to holomorphic bundles on $X_1$ form a complex Lagrangian submanifold given by the restriction map $\mathcal{M}_{X_1} \to \mathcal{M}_S$ (at least where this is defined, i.e., where stability is preserved on $S$); similarly for $X_2$. As before intersecting these

$$\mathcal{M}_{X_1} \cap \mathcal{M}_{X_2} \tag{2.7}$$

in $\mathcal{M}_S$ we expect to get a finite number of holomorphic bundles that extend to both $X_i$, i.e., bundles on $X$. In the examples we consider $X$ will be smoothable and the number of bundles will be preserved on smoothing to give the holomorphic Casson invariant of the smooth 3-fold.

## 3. Virtual moduli cycles

To count (stable) holomorphic bundles on a Calabi-Yau 3-fold, there are two things we require of the moduli problem – compactness and transversality. In gauge theory such results are easier in lower dimensions. In two real dimensions moduli of stable bundles are both compact and of the right dimension. In three dimensions this can be achieved after a perturbation [39] (leaving aside problems with reducibles) for flat connections. In dimension four we need both perturbations to achieve transversality, and a compactification to take account of the non-compactness caused by conformal invariance [11]. Just recently we now have the results of Tian [42] in higher dimensions, proving just about everything that one would like to be true, giving a natural analogue of the Donaldson-Uhlenbeck compactification of four dimensions. For a Kähler 3-fold this involves ideal instanton singularities along holomorphic curves in the 3-fold, but also some codimension 3 singularities that are harder to deal with.

What is missing, however, is a transversality result. Staying within the confines of algebraic Calabi-Yau manifolds we cannot hope to get a moduli space of the correct dimension; there is no result along the lines of Donaldson's generic smoothness result for moduli spaces on algebraic surfaces. One reason is that, in the rank two case for instance, relating the deformation theory of a bundle to that of a zero set of a section via the Serre construction, points are unobstructed on a surface but curves can be obstructed on a 3-fold (there is a fuller discussion of this and other such issues in [40], and an example in [41]).

So we would like more perturbations. There is an elliptic perturbation of the Hermitian-Yang-Mills equation valid on any almost-complex symplectic manifold, which I learnt from Donaldson:

$$\begin{aligned} F_A^{0,2} &= \bar\partial^* u, \\ i\Lambda F_A^{1,1} &= \lambda I. \end{aligned}$$

(The problem is that the Hermitian-Yang-Mills equations appear over determined, but are not because of the Bianchi identity $\bar\partial F_A^{0,2} = 0$ on a Kähler manifold. This can be formalised by introducing the (0,3)-form $u$ as above, and then $\bar\partial^* u$ vanishes. It need not be zero in the almost complex case.)

It seems that Tian's work [42] should also apply to these equations (with the singularities now along pseudo-holomorphic curves and at points) so long as we can get a bound on $\|F_A\|^2_{L^2}$ similar to that given by characteristic class formulae in the integrable case. A Weitzenböck formula shows that this is the case if, for instance, the scalar curvature is everywhere positive. Thus, at present, this would work best for Fano 3-folds and the Calabi-Yau case is borderline. Either way a transversality result, proving that for generic almost complex structures the moduli space of solutions is of the correct dimension, seems a long way off, as does understanding the codimension three singularities.

So we turn to algebraic geometry where we have the now standard compactification of the moduli space of stable bundles by semistable sheaves, due to Gieseker, Maruyama and Simpson. This moduli space will invariably be singular and of too high a dimension, but it is often clear what the contribution of a particular component to the "number of bundles" should be – the Euler number of its cotangent bundle in the case it is smooth, two if it is a scheme-theoretic double point, etc. In the general case there is the machinery of [31], [7] to produce a "virtual moduli cycle" of the correct dimension (zero, for us) inside any moduli space satisfying certain conditions; we briefly outline the picture.

Suppose a variety $M$ (which will eventually be our moduli space $\mathcal{M}$) sits inside a smooth ambient $n$-fold $Z$, cut out by a section $s$ of a rank $r$ vector bundle $E \to Z$. Then the "virtual dimension" of $M$ is $(n-r)$ – the dimension it would be were $s$ transverse. If it is not but, for instance, $s$ lies in a subbundle $E' \subset E$ and is a transverse section of $E'$, then it is clear the "correct" $(n-r)$-cycle we should take is the Euler class of the cokernel bundle $E/E'$ over $M$ – this is homologous to the zero set of a transverse perturbation of $s$ if one exists. In the general case dealt with by Fulton-MacPherson intersection theory [17], $s$ induces a cone in $E|_M$, which can be thought of as "$s$ made vertical", i.e., the limit of the images of $\lambda s$ as $\lambda \to \infty$. We may then intersect this with the zero set $M$ inside $E$ to get a cycle in $\mathcal{M}$ (whose image in $Z$ represents the top Chern class of $E$, as required).

The point here is that we worked entirely on $M$ and not in the ambient space $Z$, and so we might hope the method is applicable to moduli problems where the ambient space $Z$ does not exist. Instead the deformation theory of the moduli problem often gives us the infinitesimal version of $(Z, E, s)$ on $M$, namely the derivative of $s$, yielding the exact sequence

$$0 \to TM \to TZ|_M \xrightarrow{ds} E|_M \to \mathrm{ob} \to 0, \tag{3.1}$$

for some cokernel ob which in the moduli problem becomes the obstruction sheaf. In the general case we require a global version of this, namely a two term locally free resolution

$$0 \to \mathcal{T}_1 \to E_1 \to E_2 \to \mathcal{T}_2 \to 0, \tag{3.2}$$

of the tangent-obstruction functors to be introduced presently (3.26, 3.28).

Here $E_1$ and $E_2$ play the roles of $TZ|_M$ and $E|_M$ in the above motivation (these last two have the same fibre rank at each point of $\mathcal{M}$, and hence are locally free) and have difference in ranks equal to the virtual dimension of the moduli problem. It is shown in [31], [7] that such data on $\mathcal{M}$ is in fact sufficient to obtain a cone in the vector bundle $E_1$ which can be intersected with the zero set $\mathcal{M}$ to give a virtual moduli cycle with the correct properties. The precise statement is given below.

First then, we need to develop the necessary sheaf deformation theory. An earlier version of this paper used the cotangent complex approach of [7], and Lehn's description [29] of equations cutting out the Quot scheme to calculate it; the deformation theory here is more classical and natural, even if it is a little longer, dealing with higher order deformations.

An excellent reference for the sheaf theory we use is [24]; we shall assume familiarity with Gieseker stability (here always referred to just as stability), slope stability, the fact that

$$\text{slopestable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \text{slopesemistable},$$

and that for rank and degree coprime the circle is completed by slope semistability implying slope stability. Also refer to [24] for the fact that for either form of stability

$$\text{(slope) stable} \Rightarrow \text{simple},$$

i.e., the only endomorphisms of the sheaf are the scalars $\mathbb{C}\,.\,\mathrm{id}$. Finally, stable sheaves are pure, i.e., torsion-free on restriction to their support.

## 3.1 Sheaf deformation theory

In this section all schemes will be complex and quasi-projective, and all sheaves coherent. We begin by recalling the definition of the trivial thickening

$$S * \mathfrak{n} \tag{3.3}$$

of a scheme $S$ by a coherent sheaf of $\mathcal{O}_S$-modules $\mathfrak{n}$. Namely make the $\mathcal{O}_S$-module $\mathcal{O}_S \oplus \mathfrak{n}$ into a sheaf of rings by stipulating that $\mathfrak{n}^2 = 0$, giving the trivial ring extension $\mathcal{O}_S * \mathfrak{n}$; the associated scheme is $S * \mathfrak{n}$. Then sheaf deformation theory is built on the following standard result.

**Lemma 3.4.** *Let $\mathfrak{n}$ be a sheaf on a scheme $S$, and let $S * \mathfrak{n}$ be the trivial extension of (3.3) above. Then deformations of a sheaf $\mathcal{E}$ on $X \times S$, flat over $S$, to a sheaf on $X \times (S * \mathfrak{n})$, flat over $S * \mathfrak{n}$, are in 1-1 correspondence with* $\operatorname{Ext}^1_{X\times S}(\mathcal{E}, \mathcal{E} \otimes \mathfrak{n})$.

*Proof.* Given such a deformation $\mathcal{F}$, tensoring with the sequence

$$0 \to \mathfrak{n} \to \mathcal{O}_{S*\mathfrak{n}} \to \mathcal{O}_S \to 0, \tag{3.5}$$

gives a sequence, exact by flatness,

$$0 \to \mathcal{E} \otimes \mathfrak{n} \to \mathcal{F} \to \mathcal{E} \to 0 \tag{3.6}$$

on $X \times (S * \mathfrak{n})$, using the fact that the left and right hand terms of (3.5) are $\mathcal{O}_S$-modules. The sequence (3.6) defines the class in $\operatorname{Ext}^1_{X\times S}(\mathcal{E}, \mathcal{E} \otimes \mathfrak{n})$.

Conversely such a class gives a sequence of $\mathcal{O}_{X\times S}$-modules (3.6), which defines an $\mathcal{O}_{X\times(S*\mathfrak{n})}$-module $\mathcal{F}$ since there is an obvious action of $\mathcal{O}_S * \mathfrak{n}$ on $\mathcal{F}$: the $\mathfrak{n}$-action is given by projecting $\mathcal{F} \to \mathcal{E}$ and tensoring with $\mathfrak{n}$, mapping to $\mathcal{E} \otimes \mathfrak{n} \subset \mathcal{F}$. Flatness over $X \times (S * \mathfrak{n})$ also follows from the sequence (3.6) and the following Lemma. q.e.d.

**Lemma 3.7.** *Let $S \subset Y$ be a subscheme with ideal $\mathfrak{n} \subset \mathcal{O}_Y$ such that $\mathfrak{n}^2 = 0$. Then an $\mathcal{O}_Y$-module $\mathcal{F}$ is flat over $Y$ if and only if $\mathcal{F} \otimes \mathfrak{n} \to \mathcal{F}$ is injective and $\mathcal{F}|_S = \mathcal{F} \otimes \mathcal{O}_S$ is flat over $S$.*

*Proof.* We must show ([21] III 9.1A) that $\mathcal{F} \otimes J \to \mathcal{F}$ is injective for any ideal $J \subset \mathcal{O}_Y$. Given such a $J$, we have an exact sequence

$$0 \to J \cap \mathfrak{n} \to J \to J' \to 0,$$

where $J'$ is annihilated by $\mathfrak{n}$ so is naturally an ideal in $\mathcal{O}_S$. Thus the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & \mathcal{F} \otimes (J \cap \mathfrak{n}) \arrow[r] \arrow[d] & \mathcal{F} \otimes J \arrow[r] \arrow[d] & \mathcal{F}|_S \otimes J' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \otimes \mathfrak{n} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}|_S \arrow[r] & 0
\end{tikzcd}$$

is exact by our hypotheses. To show the central vertical map is injective, then, it is sufficient to show that the first vertical map is injective. But this map is $\mathcal{F}|_S \otimes (J \cap \mathfrak{n}) \to \mathcal{F}|_S \otimes \mathfrak{n}$, through which $\mathcal{F}|_S \otimes (J \cap \mathfrak{n}) \to \mathcal{F}|_S$ factors, and $\mathcal{F}|_S$ is assumed flat so this last map is an injection. q.e.d.

Next we consider obstructions to deformations. We will repeatedly use the following set-up.

- Let $S \subset Y \subset Y_1$ all be schemes *over* $S$, and denote the ideals of $S \subset Y$, $S \subset Y_1$, $Y \subset Y_1$ by $\mathfrak{n}$, $\mathfrak{m}$ and $\mathscr{I}$ respectively. Assume also that $\mathfrak{m} \,.\, \mathscr{I} = 0$, giving an exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \mathscr{I} \to \mathfrak{m} \to \mathfrak{n} \to 0. \tag{3.8}$$

Notice we do *not* assume that $Y = S * \mathfrak{n}$ this time, or even that $\mathfrak{n}^2 = 0$.

Given a sheaf $\mathcal{E}$ over $X \times Y$ (flat over $Y$ and restricting to $\mathcal{E}_0$ over $S$) we get an exact sequence

$$0 \to \mathcal{E} \otimes \mathfrak{n} \to \mathcal{E} \to \mathcal{E}_0 \to 0, \tag{3.9}$$

giving a class $e \in \mathrm{Ext}^1_{X\times S}(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n})$ (we will consider all terms as $\mathcal{O}_S$-modules using the projections $Y \to S$, $Y_1 \to S$, so all Exts will be over $X \times S$ from now on).

We would like to lift this to an $\mathcal{F}$ on $Y_1$ to give a sequence

$$0 \to \mathcal{E} \otimes \mathfrak{m} \to \mathcal{F} \to \mathcal{E}_0 \to 0, \tag{3.10}$$

(here $\mathcal{F} \otimes \mathfrak{m} \cong \mathcal{E} \otimes \mathfrak{m}$ since $\mathfrak{m} \,.\, \mathscr{I} = 0$) defining $f \in \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{m})$. $f$ is a lift of $e$ in the sequence

$$\mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{m}) \to \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n}) \xrightarrow{\partial} \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}), \tag{3.11}$$

obtained by applying $\mathrm{Hom}\,(\mathcal{E}_0, \;.\;)$ to the sequence $\mathcal{E} \otimes$ (3.8):

$$0 \to \mathcal{E}_0 \otimes \mathscr{I} \to \mathcal{E} \otimes \mathfrak{m} \to \mathcal{E} \otimes \mathfrak{n} \to 0. \tag{3.12}$$

(Exactness follows from the flatness of $\mathcal{E}$ over $Y$.)

**Proposition 3.13.** *$\mathcal{E}$ over $X \times Y$ as above extends to a sheaf $\mathcal{F}$ over $X \times Y_1$, flat over $Y_1$, if and only if there is a lift $f \in \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{m})$* (3.10) *of $e \in \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n})$* (3.9)*, i.e., if and only if $\partial e \in \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ in the above sequence* (3.11) *is zero.*

*Proof.* We are left with showing that the existence of a lift $f$ of $e$ gives such an $\mathcal{F}$. $f$ gives a sequence (3.9) *of* $\mathcal{O}_S$*-modules* lifting (3.10), and so a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}\otimes\mathfrak{m} & \xrightarrow{\iota} & \mathcal{F} & \longrightarrow & \mathcal{E}_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\pi} & & \| & & \\ 0 & \longrightarrow & \mathcal{E}\otimes\mathfrak{n} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0\,. & & & & \end{array}$$

Here $\mathcal{F}$ is an $\mathcal{O}_S$-module, and we would like to make it an $\mathcal{O}_{Y_1}$-module, where $\mathcal{O}_{Y_1}\cong\mathcal{O}_S\oplus\mathfrak{m}$ via the maps $S\rightleftarrows Y_1$. So we define the action $\mathcal{F}\otimes\mathfrak{m}\to\mathcal{F}$ by $\iota\circ(\pi\otimes\mathrm{id})$ in the above diagram.

Flatness of $\mathcal{F}$ over $Y_1$ then follows from Lemma 3.7 on noting that $\mathcal{F}|_Y=\mathcal{E}$ is flat over $Y$, and $\mathscr{I}^2=0$ since $\mathscr{I}\subset\mathfrak{m}$ and $\mathfrak{m}\,.\,\mathscr{I}=0$.

q.e.d.

We note in passing that $\partial$ is cup product with the element $e'\in\mathrm{Ext}^1(\mathcal{E}\otimes\mathfrak{n},\mathcal{E}_0\otimes\mathscr{I})$ defining the extension (3.12), so $\mathcal{E}$ extends to $\mathcal{F}$ if and only if $e'\cup e\in\mathrm{Ext}^2(\mathcal{E}_0,\mathcal{E}_0\otimes\mathscr{I})$ is zero.

## The trace map

We have now pretty much found the tangent-obstruction (in the sense of [31]) complex of the moduli problem for stable sheaves, as we shall see below. Unfortunately we are more interested in the moduli problem for sheaves of fixed determinant, for which we need the machinery of the Mukai-Artamkin trace map (in a little more generality than [32], [2], for higher order deformations).

Given a coherent sheaf $\mathcal{F}$ on a quasi-projective scheme $X$ and an affine open cover $\mathcal{U}=\{U_i : i=1,\dots,n\}$ of $X$, denote the Čech complex by

$$\check{C}^p(\mathcal{F})=\prod_{i_0<\dots<i_p}\Gamma(\mathcal{F}|_{U_{i_0}\cap\dots\cap U_{i_p}}),$$

with the usual Čech differential $\delta:\check{C}^p\to\check{C}^{p+1}$ ([21] III 4.1). This computes the sheaf cohomology of $\mathcal{F}$, and as the construction is functorial in $\mathcal{F}$ it can also be applied to a complex of sheaves $\mathcal{F}^\bullet$ to give a double complex whose associated total complex has cohomology the hypercohomology $\mathbb{H}^*(\mathcal{F}^\bullet)$.

If $X$ is smooth then any sheaf $\mathcal{E}$ has a finite locally free resolution $E^\bullet \to \mathcal{E} \to 0$, and the trace map is defined as follows (by standard arguments it will be independent, up to quasi-isomorphism, of the choices $\mathcal{U}$ and $E^\bullet$).

Given any sheaf $\mathcal{I}$, form the complex $\mathcal{H}om^\bullet(E^\bullet, E^\bullet \otimes \mathcal{I})$ with

$$\mathcal{H}om^i(E^\bullet, E^\bullet \otimes \mathcal{I}) = \oplus_j \mathcal{H}om(E^j, E^{i+j} \otimes \mathcal{I})$$

and differential $d\phi = d^{i+j} \circ \phi - (-1)^i \phi \circ d^j$ for $\phi \in \mathcal{H}om(E^j, E^{i+j} \otimes \mathcal{I})$. This admits cochain maps

$$\mathcal{H}om^\bullet(E^\bullet, E^\bullet \otimes \mathcal{I}) \rightleftarrows \mathcal{I}, \tag{3.14}$$

with the upper map given by

$$\mathrm{tr} = \sum_i (-1)^i \ \mathrm{tr}^i \otimes \mathrm{id}_{\mathcal{I}}$$

(where $\mathrm{tr}^i : \mathcal{H}om(E^i, E^i) \to \mathcal{O}$ is the usual trace map on locally free sheaves), and the lower map

$$\mathrm{id} = \sum_i \mathrm{id}_{E^i} \otimes \mathrm{id}_{\mathcal{I}}\,.$$

That these are cochain maps follows from the easy computations $\mathrm{tr} \circ d = 0$ and $d \circ \mathrm{id} = 0$.

Notice that $\mathrm{tr} \circ \mathrm{id} = \sum (-1)^i \ \mathrm{tr}^i \circ \mathrm{id}_{E^i} = \sum (-1)^i \,\mathrm{rk}\,(E^i) = \mathrm{rk}\,(\mathcal{E})$, so, for $\mathrm{rk}\,(\mathcal{E}) > 0$, we have a splitting

$$\mathcal{H}om^\bullet(E^\bullet, E^\bullet \otimes \mathcal{I}) = \mathcal{H}om_0^\bullet(E^\bullet, E^\bullet \otimes \mathcal{I}) \,\oplus\, \mathcal{I},$$

where $\mathcal{H}om_0^\bullet$ is the kernel of tr.

Thus (3.14) induces cochain maps between Čech complexes

$$\check{C}^\bullet(\mathcal{H}om^\bullet(E^\bullet, E^\bullet \otimes \mathcal{I})) \rightleftarrows \check{C}^\bullet(\mathcal{I}),$$

inducing maps tr and id on cohomology

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{E} \otimes \mathcal{I}) \rightleftarrows H^i(\mathcal{I}),$$

such that $\mathrm{tr} \circ \mathrm{id} = \mathrm{rk}\,\mathcal{E}$. Thus for $\mathrm{rk}\,(\mathcal{E}) > 0$, there is a splitting $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E} \otimes \mathcal{I}) = \mathrm{Ext}^i_0(\mathcal{E}, \mathcal{E} \otimes \mathcal{I}) \,\oplus\, H^i(\mathcal{I})$ with $\mathrm{Ext}_0$ the kernel of tr.

To work towards showing that taking the trace of deformations and obstructions of a sheaf gives the deformations and obstructions of the determinant of the sheaf, we first need this (rather elementary) fact for locally free sheaves. Of course, phrasing the deformation theory of holomorphic vector bundles in terms of connections or transition functions this is simple; the work is then in showing the deformation theory coincides with the abstract sheaf deformation theory of the last section. This involves simple but very large computations with transition functions as in [15]. Here we prefer to work directly with our definitions above; this then makes the proof below a little long, but the reader could take it on trust.

**Proposition 3.15.** *Take $S \subset Y \subset Y_1$ to be as in (3.8). Suppose we have a rank $r$* locally free *sheaf $E$ on $X \times Y$, a flat deformation of $E_0$ on $X \times S$ giving an extension $e \in \operatorname{Ext}^1(E_0, E \otimes \mathfrak{n})$ (3.9). Assuming first that $\mathfrak{n}^2 = 0$, then the extension defined similarly by the determinant $\Lambda^r E$ is*

$$\operatorname{tr}(e) \in H^1(\mathfrak{n}) = \operatorname{Ext}^1(\Lambda^r E_0, \Lambda^r E_0 \otimes \mathfrak{n}).$$

*Likewise, for any $\mathfrak{n}$, given the obstruction $\partial e \in \operatorname{Ext}^2(E_0, E_0 \otimes \mathscr{I})$ of Proposition 3.13 to extending $E$ over $X \times Y$ to $F$ over $X \times Y_1$ (flat over $Y_1$), the obstruction to extending $\Lambda^r E$ is given by*

$$\operatorname{tr}(\partial e) \in H^2(\mathscr{I}) = \operatorname{Ext}^2(\Lambda^r E_0, \Lambda^r E_0 \otimes \mathscr{I}).$$

*Proof.* We begin by giving explicit descriptions of $e$ and the obstruction $\partial e$.

Applying $\operatorname{Hom}(E_0, \ .\ )$ to the extension (of $\mathcal{O}_{X \times S}$-modules)

$$0 \to E \otimes \mathfrak{n} \to E \to E_0 \to 0 \tag{3.16}$$

gives a connecting homomorphism $\operatorname{Hom}(E_0, E_0) \to \operatorname{Ext}^1(E_0, E \otimes \mathfrak{n})$ under which the image of $\mathrm{id}_{E_0}$ is the extension class $e \in \operatorname{Ext}^1(E_0, E \otimes \mathfrak{n})$.

So consider the exact sequence of complexes below given by applying the exact functor $\mathcal{H}om(E_0, \ .\ )$ (recall that $E_0$ is a locally free $\mathcal{O}_S$-module) to the sequence (3.16) and taking Čech complexes:

$$\begin{aligned} 0 \ &\to \check{C}^\bullet(\mathcal{H}om(E_0, E \otimes \mathfrak{n})) \xrightarrow{\iota} \check{C}^\bullet(\mathcal{H}om(E_0, E)) \\ &\to \check{C}^\bullet(\mathcal{H}om(E_0, E_0)) \to 0. \end{aligned} \tag{3.17}$$

$\mathrm{id}_{E_0}$ gives a closed element of $\check{C}^0(\mathcal{H}om(E_0, E_0))$ (i.e., a global section). Lift this to some $a \in \check{C}^0(\mathcal{H}om(E_0, E))$, giving, for each $U$, $V$ in some affine open cover $\mathcal{U}$ of $X$, $a_U$, $a_V$ such that over $U \cap V$, $a_V - a_U =: \iota(e_{U \cap V})$ defines the (closed) element

$$e \in \check{C}^1(\mathcal{H}om(E_0, E \otimes \mathfrak{n}))$$

that represents $e \in \operatorname{Ext}^1(E_0, E \otimes \mathfrak{n})$.

To identify $\partial e$ we use the similar exact sequence of Čech complexes

$$\begin{aligned}0 &\to \check{C}^\bullet(\mathcal{H}om(E_0, E_0 \otimes \mathscr{I})) \xrightarrow{\iota} \check{C}^\bullet(\mathcal{H}om(E_0, E \otimes \mathfrak{m})) \\ &\to \check{C}^\bullet(\mathcal{H}om(E_0, E \otimes \mathfrak{n})) \to 0,\end{aligned}$$

associated to the sequence (3.11).

That is, choose lifts $b_{UV} \in \operatorname{Hom}_{UV}(E_0, E \otimes \mathfrak{m})$ of $e_{UV}$. Then over $U \cap V \cap W$, $b_{UV} + b_{VW} + b_{WU} = \iota((\partial e)_{UVW})$ defines $(\partial e)_{UVW} \in \operatorname{Hom}_{UVW}(E_0, E_0 \otimes \mathscr{I})$ giving the obstruction class $\partial e \in \operatorname{Ext}^2(E_0, E_0 \otimes \mathscr{I})$.

We now use these explicit calculations to repeat the above exercise on the induced deformation of determinants

$$0 \to \Lambda^r E \otimes \mathfrak{n} \to \Lambda^r E \to \Lambda^r E_0 \to 0. \tag{3.18}$$

Here for $\Lambda^r E$ we have taken the $r$th exterior power of $E$ *as a sheaf of* $\mathcal{O}_{X\times Y}$*-modules* (*not* as a sheaf of $\mathcal{O}_{X\times S}$-modules) but then we consider the result and the above sequence as $\mathcal{O}_{X\times S}$-modules. Thus we have the analogue

$$\begin{aligned}0 &\to \check{C}^\bullet(\mathcal{H}om(\Lambda^r E_0, \Lambda^r E \otimes \mathfrak{n})) \xrightarrow{\iota} \check{C}^\bullet(\mathcal{H}om(\Lambda^r E_0, \Lambda^r E)) \\ &\to \check{C}^\bullet(\mathcal{H}om(\Lambda^r E_0, \Lambda^r E_0)) \to 0\end{aligned}$$

of (3.17).

Use $\Lambda^r a_U \in \operatorname{Hom}_U(\Lambda^r E_0, \Lambda^r E)$ to lift

$$\mathrm{id}_{\Lambda^r E_0} = \Lambda^r \,\mathrm{id}_{E_0} \in \operatorname{Hom}_U(\Lambda^r E_0, \Lambda^r E_0).$$

Over $U \cap V$, $\Lambda^r a_U - \Lambda^r a_V = \iota(\epsilon_{UV})$ defines

$$\epsilon_{UV} \in \operatorname{Hom}_{UV}(\Lambda^r E_0, \Lambda^r E \otimes \mathfrak{n}),$$

the extension class $\epsilon \in \operatorname{Ext}^1(\Lambda^r E_0, \Lambda^r E \otimes \mathfrak{n}) = H^1(\mathfrak{n})$ of (3.18).

Over $U \cap V$ there is a splitting of (3.16) induced by $a_U$:

$$E \cong (E \otimes \mathfrak{n}) \oplus E_0, \tag{3.19}$$

with respect to which $a_U = 0 \oplus \mathrm{id}_{E_0}$, $a_V = e_{U\cap V} \oplus \mathrm{id}_{E_0}$, and (3.18) splits as $\Lambda^r E \cong (\Lambda^r E \otimes \mathfrak{n}) \oplus \Lambda^r E_0$ (we are omitting some $|_{U\cap V}$s for clarity).

So in this splitting, over $U \cap V$,

$$\epsilon_{UV} = \Lambda^r(\mathrm{id} \oplus e_{UV}) - \mathrm{id},\ \ \epsilon_{UW} = \Lambda^r\,(\mathrm{id} \oplus e_{UW}) - \mathrm{id},$$

while $\epsilon_{VW} = \Lambda^r(\mathrm{id} \oplus e_{UW}) - \Lambda^r(\mathrm{id} \oplus e_{UV})$.

For the first part of the Proposition we want to show that, for $\mathfrak{n}^2 = 0$, these $\epsilon$ s are (the images under $\iota$ of) the traces of the corresponding $e$ s (i.e., $\epsilon_{UV} = \mathrm{tr}(e_{UV})$, etc.). But this is clear from the expansion

$$\Lambda^r(\mathrm{id}_{E_0} \oplus e) = \mathrm{id}_{\Lambda^r E_0} + \mathrm{tr}_U(e) + 2\,\mathrm{tr}_U(\Lambda^2 e) + \dots,$$

for $e:\ E_0 \to E \otimes \mathfrak{n}$, where $\mathrm{tr}_U(\Lambda^k e)$ is a map of the right kind, i.e., an element of $(\Lambda^r E_0)^* \otimes \Lambda^r E \otimes \mathfrak{n}$, on defining $\mathrm{tr}_U$ by the composition

$$\begin{aligned}\Lambda^k E_0^* \otimes \Lambda^k E \otimes \mathfrak{n} &\to \Lambda^k E_0^* \otimes \Lambda^k E_0 \otimes \mathfrak{n} \to \mathfrak{n} \\ &\to \Lambda^r E_0^* \otimes \Lambda^r E_0 \otimes \mathfrak{n} \to \Lambda^r E_0^* \otimes \Lambda^r E \otimes \mathfrak{n}\end{aligned}$$

of the projection, trace, identity and $\Lambda^r a_U$ maps respectively. This is just a glorified version of the expansion of the determinant in terms of the elementary symmetric polynomials of $e$, but over the ring $\mathcal{O}_Y$. For $\mathfrak{n}^2 = 0$ it is just the usual trace, independent of $a_U$, since any two $a$ s differ by something in the ideal $\mathfrak{n}$. In this case all the higher order terms in the above expansion become zero anyway leaving just $\mathrm{tr}(e)$, giving the required result.

To identify the obstruction we pick the obvious lifts of the $\epsilon$ s. That is, over $U \cap V$ in the splitting (3.19), $\epsilon_{UV} = \mathrm{tr}_U(e_{UV}) + 2\ \mathrm{tr}_U(\Lambda^2 e_{UV}) + \dots$, so we set $\beta_{UV} = \mathrm{tr}_U(b_{UV}) + 2\ \mathrm{tr}_U(\Lambda^2 b_{UV}) + \dots$. ($\mathrm{tr}_U$ is defined by the same formula as before but with $\mathfrak{m}$ replacing $\mathfrak{n}$. Then $\beta_{UV}$ is actually skew symmetric with respect to $U$ and $V$, so is well defined, though it takes a calculation relating $\mathrm{tr}_U$ and $\mathrm{tr}_V$ to check it.)

The class of the obstruction we are seeking is given, on $U \cap V \cap W$, by

$$\beta_{UV} + \beta_{VW} + \beta_{WU} = \mathrm{tr}_U(b_{UV}) + \mathrm{tr}_V(b_{VW}) + \mathrm{tr}_W(b_{WU}) + \dots. \tag{3.20}$$

We claim that only the terms with a linear dependence on the $b$ s are non-zero up to coboundaries; this can be checked by a large local calculation or the following cheat. We know that $b_{VW} = -b_{UV} - b_{WU} + i$, where $i$ lies in the ideal $\mathscr{I}$ ($i$ is of course $\iota(\partial e)_{UVW}$).) If we had chosen the different lift $-b_{UV} - b_{WU}$ of $e_{VW}$ then (3.20) would of course give zero up to coboundaries. Thus (3.20), considered as a polynomial in $i$, has zero constant term, and any term of order $\geq 2$ (in the $b$ s or $i$)

involves $i$ multiplying something in $\mathfrak{m}$, which vanishes since $\mathfrak{m} \,.\, \mathscr{I} = 0$. Thus the obstruction class is the just the linear part of (3.20), which is $\operatorname{tr}((\partial e)_{UVW})$ since $b_{UV} + b_{VW} + b_{WU} = \iota((\partial e)_{UVW})$. q.e.d.

Similar explicit calculations over patches of a cover $\mathcal{U}$ also give the standard results that, for line bundles $L_i$ with deformation and obstruction classes $e(L_i)$ and $\partial(e(L_i))$ as above, we have

$$\begin{aligned} e(\otimes_i L_i) &= \sum_i e(L_i) \in H^1(\mathfrak{n}), \\ \text{and} \qquad \partial(e(\otimes_i L_i)) &= \sum_i \partial(e(L_i)) \in H^2(\mathfrak{n}). \end{aligned} \tag{3.21}$$

Also recall that for any sheaf $\mathcal{E}$ with a finite locally free resolution $E^\bullet$, the determinant of $\mathcal{E}$ is defined to be the line bundle

$$\det \mathcal{E} = \bigotimes_i (\det E^i)^{(-1)^i}, \tag{3.22}$$

which is independent of the resolution.

So as before let $X$ be a smooth quasi-projective variety and let $S \subset Y \subset Y_1$ be as in (3.8). Fix a sheaf $\mathcal{E}$ on $X \times Y$ that is flat over $Y$ and restricts to $\mathcal{E}_0$ on $X \times S$. Then since $X$ is smooth, $\mathcal{E}$ has such a finite locally free resolution $E^\bullet$ that restricts on $X \times S$, by flatness, to a finite locally free resolution $E_0^\bullet$ of $\mathcal{E}_0$.

**Theorem 3.23.** *In the above situation, denote by* $e \in \operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n})$ *and* $\partial e \in \operatorname{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ *the deformation and obstruction classes of* $\mathcal{E}$ *(3.9, 3.13). Then the obstruction class of* $\det \mathcal{E}$ *is* $\operatorname{tr}(\partial e) \in H^2(\mathscr{I}) = \operatorname{Ext}^2(\det \mathcal{E}_0, \det \mathcal{E}_0 \otimes \mathscr{I})$*, and, if* $\mathfrak{n}^2 = 0$*, the deformation class of* $\det \mathcal{E}$ *is* $\operatorname{tr}(e) \in H^1(\mathfrak{n}) = \operatorname{Ext}^1(\det \mathcal{E}_0, \det \mathcal{E}_0 \otimes \mathfrak{n})$.

*Proof.* Again we explicitly chase Čech cocycles representing the extension and deformation classes. The exact sequence $0 \to \mathfrak{n} \to \mathcal{O}_{X \times Y} \to \mathcal{O}_{X \times S} \to 0$ gives the exact diagram of resolutions

$$\begin{tikzcd} 0 \arrow[r] & E^\bullet \otimes \mathfrak{n} \arrow[r] \arrow[d] & E^\bullet \arrow[r] \arrow[d] & E_0^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{E} \otimes \mathfrak{n} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}_0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0\,. & \end{tikzcd}$$

In turn this gives an exact sequence of Čech complexes

$$\begin{aligned}0 \to \check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E^\bullet \otimes \mathfrak{n})) &\xrightarrow{\iota} \check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E^\bullet)) \\ &\to \check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E_0^\bullet)) \to 0,\end{aligned}$$

where the first complex computes $\mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n})$, etc. $\mathrm{id} \in \check{C}^0(\mathcal{H}om^\bullet(E_0^\bullet, E_0^\bullet))$ is $\oplus_i \mathrm{id}_i$ with $\mathrm{id}_i \in \check{C}^0(\mathcal{H}om^0(E_0^i, E_0^i))$. Lift this to $a \in \check{C}^0(\mathcal{H}om^\bullet(E_0^\bullet, E^\bullet))$ as $\oplus_i a_i$ with $a_i \in \check{C}^0(\mathcal{H}om^0(E_0^i, E^i))$.

We now apply the total differential $d+\delta$ to $a$, with $d$ the differential on $\mathcal{H}om^\bullet$ and $\delta$ the Čech differential, keeping track of degrees. We get

$$(d+\delta)\, a = \iota\,(\oplus_i e_i + \oplus_i f_i),$$

where

$$e_i \in \check{C}^1(\mathcal{H}om^0(E_0^i, E^i \otimes \mathfrak{n}))$$

and

$$f_i \in \check{C}^0(\mathcal{H}om(E_0^i, E^{i-1} \otimes \mathfrak{n})) \oplus \check{C}^0(\mathcal{H}om(E_0^{i+1}, E^i \otimes \mathfrak{n})).$$

Thus $\iota(e_i) = \delta(a_i)$ (and $\iota(f_i) = d(a_i)$).

So $\oplus_i e_i + \oplus_i f_i$ is closed under $d+\delta$ and represents the class $e \in \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n})$ of the deformation $\mathcal{E}$, whereas $e_i$ is closed under $\delta$ and represents the class in $\mathrm{Ext}^1(E_0^i, E^i \otimes \mathfrak{n})$ of the deformation $E^i$. Suppose that $\mathfrak{n}^2 = 0$ so that $E \otimes \mathfrak{n} = E_0 \otimes \mathfrak{n}$, etc. and tr is defined. Then since tr only acts on $\mathcal{H}om^0$ components of the complex (3.14), we see that

$$\begin{aligned}\mathrm{tr}(e) &= \mathrm{tr}(\oplus_i e_i + \oplus_i f_i) = \mathrm{tr}(\oplus_i e_i) \\ &= \sum_i (-1)^i \,\mathrm{tr}(e(E^i)) = \sum (-1)^i \, e(\det E^i),\end{aligned}$$

by Proposition 3.15. By (3.21) this is $e(\det \mathcal{E})$, as required.

To deal with obstructions use $0 \to \mathscr{I} \to \mathfrak{m} \to \mathfrak{n} \to 0$ to give the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & E_0^\bullet \otimes \mathscr{I} \arrow[r] \arrow[d] & E^\bullet \otimes \mathfrak{m} \arrow[r] \arrow[d] & E^\bullet \otimes \mathfrak{n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_0 \otimes \mathscr{I} \arrow[r] \arrow[d] & \mathcal{E} \otimes \mathfrak{m} \arrow[r] \arrow[d] & \mathcal{E} \otimes \mathfrak{n} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0,
\end{tikzcd}$$

giving the exact sequence of Čech complexes

$$0 \to \check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E_0^\bullet \otimes \mathscr{I})) \xrightarrow{\iota} \check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E^\bullet \otimes \mathfrak{m})) \\ \to \check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E^\bullet \otimes \mathfrak{n})) \to 0.$$

Lift our class $e = \oplus_i e_i + \oplus_i f_i$ to $\oplus_i b_i + \oplus_i c_i$ in $\check{C}^\bullet(\mathcal{H}om^\bullet(E_0^\bullet, E^\bullet \otimes \mathfrak{n}))$, and apply $\delta + d$ to give, in different degrees,

$$\oplus_i \delta(b_i) \; + \; \oplus_i(\delta(c_i) + d(b_i)) \; + \; \oplus_i d(c_i).$$

This is $\iota(\partial e)$, by definition, with the component of $\partial e$ in $\check{C}^2(\mathcal{H}om^0)$ being a sum $\oplus_i o_i$ of terms such that $\iota(o_i) = \delta(b_i)$. Thus $o_i$ is, by the definition of $b_i$, the obstruction $\partial\, e(E^i)$ to the extension of $E^i$, and since tr only acts on $\mathcal{H}om^0$ components of the complex (3.14), i.e., on only the $\oplus_i \partial e(E^i)$ parts of $\partial e$, we have

$$\operatorname{tr}(\partial e) = \operatorname{tr}(\oplus_i \partial\, e(E^i)) = \sum_i (-1)^i \operatorname{tr}(\partial\, e(E^i)) = \sum_i (-1)^i \partial\, e(\det E^i),$$

by Proposition 3.15. But by (3.21) this is $\partial\, e(\det \mathcal{E})$, as required. q.e.d.

We are finally in a position to find the "tangent-obstruction complex" of our moduli problem, as defined in [31] (though our moduli problem is contravariant, not covariant).

Fix a smooth quasi-projective scheme $X$ and Chern classes $c_i \in H^{2i}(X)$, and consider the moduli functor $\mathcal{M}$ that assigns to any scheme $S$ the set of isomorphism classes of sheaves $\mathcal{E}$ on $X \times S$, flat over $S$, whose restriction to each fibre is *stable and has Chern classes* $c_i$. Here two sheaves are considered isomorphic if they differ by tensoring with a line bundle on $S$. This moduli functor has a coarse moduli space which we also denote by $\mathcal{M} = \mathcal{M}(X, c_i)$ [24], such that any sheaf $\mathcal{E}$ as above induces a morphism $f : S \to \mathcal{M}$. Similarly for the sub-moduli problem $\mathcal{M}_L$ for sheaves of fixed determinant $L$ (3.22), with its moduli space $\mathcal{M}_L \subset \mathcal{M}$. We shall call this standard data:

- $S$ an affine scheme,
- $\mathcal{E}_0$ on $X \times S$, stable on each fibre of $p : X \times S \to S$ and flat over $S$,
- Chern classes $c_i(\mathcal{E}_0) \in H^{2i}(X)$, a rank $r(\mathcal{E}) \in H^0(X)$, and a line bundle $L$ on $X$ with $c_1(L) = c_1$,
- The corresponding classifying morphism $f : S \to \mathcal{M} = \mathcal{M}(X, c_i)$,

• An $\mathcal{O}_S$-module $\mathcal{I}$.

(3.24)

Here we have also included the arbitrary $\mathcal{O}_S$-module $\mathcal{I}$, along which we will consider deformations.

**Definition 3.25.** [31] Given standard data (3.24) as above, the tangent functor of the moduli functor assigns to each $\mathcal{I}$ an $\mathcal{O}_S$-module $\mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I})$ such that the set of sheaves on $X \times (S * \mathcal{I})$ restricting to $\mathcal{E}_0$ on $X \times S$ is isomorphic to

$$\Gamma_S(\mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I})).$$

We also require that given $\mathcal{E}_0$ on $X \times S_2$, a morphism of schemes $f : S_1 \to S_2$ and a homomorphism $f^*\mathcal{I}_2 \to \mathcal{I}_1$ induce a canonical homomorphism

$$f^*\mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I}_2) \to \mathcal{T}^1_{f^*\mathcal{E}_0}(\mathcal{I}_1)$$

compatible with base-change as in *[31]*.

Denote by $\mathcal{E}xt_p$ the right derived functors of $\mathcal{H}om_p = p_* \mathcal{H}om$, where $p : X \times S \to S$. The functoriality and compatability with base-change properties will follow from those of $\mathcal{E}xt_p$ in our case. For $S$ affine, so that $H^1$ of any $\mathcal{O}_S$-module vanishes, the Leray spectral sequence shows that $\mathrm{Ext}^1_{X\times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes p^*\mathcal{I}) = \Gamma_S(\mathcal{E}xt^1_p(\mathcal{E}_0, \mathcal{E}_0 \otimes p^*\mathcal{I}))$ and similarly, for its trace-free counterpart, $\mathrm{Ext}^1_{X\times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes p^*\mathcal{I})_0 = \Gamma_S(\mathcal{E}xt^1_p(\mathcal{E}_0, \mathcal{E}_0 \otimes p^*\mathcal{I}))_0$. Thus we have, by Lemma 3.4 and Theorem 3.23,

**Proposition 3.26.** *The tangent functor for the above sheaf moduli problem $\mathcal{M}$ assigns to any $\mathcal{O}_S$-module $\mathcal{I}$ the $\mathcal{O}_S$-module*

$$\mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I}) = \mathcal{E}xt^1_p(\mathcal{E}_0, \mathcal{E}_0 \otimes p^*\mathcal{I}).$$

*For the moduli functor $\mathcal{M}_L$ the same applies to the trace-free $\mathcal{O}_S$-module $(\mathcal{T}^1_{\mathcal{E}_0})_0(\mathcal{I}) = \mathcal{E}xt^1_p(\mathcal{E}_0, \mathcal{E}_0 \otimes p^*\mathcal{I})_0$.*

**Definition 3.27.** [31] Given standard data (3.24) as above, an obstruction sheaf for the moduli functor is an $\mathcal{O}_S$-module $\mathcal{T}^2_{\mathcal{E}_0}$ satisfying the following conditions.

Let $S \subset Y \subset Y_1$ be as in (3.8), and take a sheaf $\mathcal{E}$ over $X \times Y$ that is flat over $Y$, stable on the $X$-fibres, and restricts to $\mathcal{E}_0$ on $S$. Then there is an obstruction class

$$\mathrm{ob}\,(\mathcal{E}, Y_0, Y_1) \in \Gamma_S(\mathcal{T}^2_{\mathcal{E}_0} \otimes \mathscr{I})$$

whose vanishing is necessary and sufficient for there to be an $\mathcal{O}_{Y_1}$-flat extension of $\mathcal{E}$ to $\mathcal{E}_1$ over $X \times Y_1$. This should be functorial and canonical under base-change (as in *[31] 1.2*, but with contravariance replacing covariance).

**Theorem 3.28.** *Given any affine scheme $S$ and a sheaf $\mathcal{E}_0$ on $X \times S$ as above such that* $\dim \operatorname{Ext}^i(\mathcal{E}_0|_{X_s}, \mathcal{E}_0|_{X_s})$ *is constant for all $s \in S$ and $i \geq 3$, the sheaves*

$$\mathcal{T}^2_{\mathcal{E}_0} = \mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0) \qquad \text{and} \qquad (\mathcal{T}^2_{\mathcal{E}_0})_0 = \mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0)_0$$

*are obstruction sheaves for both $\mathcal{M}$ and $\mathcal{M}_L$.*

*Proof.* Proposition 3.13 shows that

$$\operatorname{ob}(\mathcal{E}, Y_0, Y_1) := \partial(e_{\mathcal{E}}) \in \operatorname{Ext}^2_{X \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes p^* \mathscr{I})$$

defines an obstruction class, and Theorem 3.23 shows that in fact it lies in the subspace $\operatorname{Ext}^2_{X \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes p^* \mathscr{I})_0$ since deformations of the line bundle $\det \mathcal{E}$ are unobstructed ($\operatorname{Pic} X$ is smooth since $X$ is smooth). By the Leray spectral sequence $\operatorname{Ext}^2_{X \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes p^* \mathscr{I}) = \Gamma_S(\mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}))$ (and similarly for the trace-free Exts, for which the rest of the proof is also similar and is omitted).

There is a spectral sequence

$$\operatorname{Tor}_j(\mathcal{E}xt^i_p(\mathcal{E}_0, \mathcal{E}_0), \mathscr{I}) \Longrightarrow \mathcal{E}xt^{i-j}_p(\mathcal{E}_0, \mathcal{E}_0 \otimes p^* \mathscr{I})$$

(by standard nonsense and the flatness of $\mathcal{E}_0$ so that $\operatorname{Tor}_j(\mathcal{E}_0, p^* \mathscr{I}) = 0$, $j > 0$). By base-change (e.g. [3]) the hypothesis of the Theorem implies that $\mathcal{E}xt^i_p(\mathcal{E}_0, \mathcal{E}_0)$ is locally free for $i \geq 3$, so its Tors vanish (as do those of $\mathcal{H}om_p(\mathcal{E}_0, \mathcal{E}_0) \cong \mathcal{O}_S$). So the spectral sequence for $\mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0 \otimes p^* \mathscr{I})$ degenerates to $\mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0) \otimes \mathscr{I}$, and $\mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0)$ is an obstruction sheaf as defined in (3.27).

The necessary base-change for $\mathcal{T}^2_{\mathcal{E}_0}$ follows from base change for $\mathcal{E}xt_p$. q.e.d.

The condition required to employ the machinery of [31], [7] to obtain a virtual cycle is then the following, as promised in (3.2).

**Definition 3.29.** [31] $\mathcal{M}$ has a *perfect* tangent-obstruction complex if there is a complex $E_1 \to E_2$ of locally free sheaves on $\mathcal{M}$ resolving the tangent obstruction functors (3.26, 3.28), in the following sense. Given standard data (3.24) we require that the cohomologies of the complex

$$f^* E_1 \otimes \mathcal{I} \to f^* E_2 \otimes \mathcal{I}$$

are $\mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I})$ and $\mathcal{T}^2_{\mathcal{E}_0} \otimes \mathcal{I}$, in degrees 1 and 2, respectively. Similarly for $\mathcal{M}_L$ using the $(\mathcal{T}^i_{\mathcal{E}_0})_0$ s.

**Remarks.** The first condition is easily seen to be equivalent to the exactness of $E_2^* \to E_1^* \to \Omega_{\mathcal{M}} \to 0$, because of course the tangent functor assigns to data (3.24) the $\mathcal{O}_S$-module

$$\mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I}) = \mathcal{H}om_S(f^*\Omega_{\mathcal{M}}, \mathcal{I}),$$

by the standard deformation theory of the morphism $f$. One can also prove the equality $\mathcal{H}om_S(f^*\Omega_{\mathcal{M}}, \mathcal{I}) = \mathcal{E}xt^1_S(f^*\mathcal{E}, f^*\mathcal{E} \otimes \mathcal{I})$ directly using (3.26) to identify $\Omega_{\mathcal{M}}$ with $\mathcal{E}xt^{n-1}_p(\mathcal{E}, \mathcal{E} \otimes K_X)$ (recovering the result of [29]; here $\mathcal{E}$ is a local universal bundle on $X \times \mathcal{M} \xrightarrow{p} \mathcal{M}$ as will be described below) and using some base-change and relative Serre duality [23].

There is a weaker notion of perfect in ([31]; comments following Corollary 3.6) which we will need later – namely that $E_1$ need only exist locally on $\mathcal{M}$, with $E_2$ still a global vector bundle on $\mathcal{M}$ surjecting onto $\mathcal{T}^2$.

**Theorem 3.30.** *Let $X$ be a smooth, polarised, complex projective variety, and fix Chern classes $c_i \in H^{2i}(X)$ and a line bundle $L$ on $X$ with $c_1(L) = c_1$. Let $\mathcal{M}$ denote the corresponding moduli space of* stable *sheaves, and $\mathcal{M}_L$ the subscheme of those with determinant $L$ (3.22). If the numbers*

$$\dim \operatorname{Ext}^i(\mathcal{E}, \mathcal{E}), \quad i \geq 3,$$

*are constant over $\mathcal{M} \ni \mathcal{E}$ (e.g. if $\operatorname{Ext}^i_0(\mathcal{E}, \mathcal{E}) = 0 \ \forall \mathcal{E} \in \mathcal{M}, \ \forall i \geq 3$), then the tangent-obstruction complex of $\mathcal{M}$ given by $\mathcal{T}^1_{\mathcal{E}_0}$, $\mathcal{T}^2_{\mathcal{E}_0}$ (3.26, 3.28) is perfect. Similarly, for rank $r > 0$, the tangent-obstruction complex of $\mathcal{M}$ given by $\mathcal{T}^1_{\mathcal{E}_0}$, $(\mathcal{T}^2_{\mathcal{E}_0})_0$ is also perfect, as is $(\mathcal{T}^1_{\mathcal{E}_0})_0$, $(\mathcal{T}^2_{\mathcal{E}_0})_0$ for $\mathcal{M}_L$.*

**Remarks.** The required 2-step resolution is given in [31] for $X$ a surface, but the method does not generalise to higher dimensions. The proof below can be seen (see [40]) as first working out the tangent-obstruction theory of the Quot scheme (3.35) and then passing to the quotient $\mathcal{M}$ of the relevant subset of Quot by the appropriate projective linear group (3.36) – but we shall work purely algebraically.

We work in the universal case of $S$ being (an open subset of) $\mathcal{M}$ and resolve the sheaves $\mathcal{E}xt^i_p(\mathcal{E}, \mathcal{E} \otimes \mathcal{I})$, $i = 1, 2$, where $\mathcal{I}$ is an $\mathcal{O}_{\mathcal{M}}$-module. Pulling back via maps $f : S \to \mathcal{M}$ gives all that we need; the only problem is that the universal sheaf $\mathcal{E}$ on $X \times \mathcal{M} \xrightarrow{p} \mathcal{M}$ may of course not exist. However we shall ignore this irritation since the usual methods (see e.g. [24] 10.2) circumvent it – $\mathcal{E}$ exists locally on open subsets $X \times S$ ($S \subset \mathcal{M}$) and choices differ by line bundles pulled up from $S$. But $\mathcal{E}xt_p^*(\mathcal{E}, \mathcal{E} \otimes \mathcal{I})$ is invariant under twisting by such line bundles and so exists uniquely and globally on $X \times \mathcal{M}$. Thus all sequences in the proof below are really local, except those involving $\mathcal{E}xt_p$ s, which patch together globally.

*Proof.* We denote by $\mathcal{O}(1)$ the pull-back to $X \times \mathcal{M}$ of the polarisation on $X$, by $P(n) = \chi(\mathcal{E}(n))$ the Hilbert polynomial associated to the Chern classes $c_i$, and by $\mathcal{E}$ the (local) universal bundle (see the Remarks above). Choose $n_1 \gg 0$ such that $\mathcal{E}(n_1)$ is generated by its fibrewise sections and has no other cohomology, i.e., we have a sequence

$$0 \to \mathscr{K} \to p^*\left(p_*\mathcal{E}(n_1)\right)(-n_1) \to \mathcal{E} \to 0, \tag{3.31}$$

for some kernel $\mathscr{K}$, and $R^i p_*(\mathcal{E}(n_1)) = 0 \ \ \forall i \geq 1$. $p_*(\mathcal{E}(n_1))$ is locally free as it has fibres of fixed dimension $P(n_1)$, and $\mathscr{K}$ is flat over $\mathcal{M}$ because the other two terms are.

Now twist with $n_2$ sufficiently large such that $\mathscr{K}(n_2)$ and $\mathcal{E}(n_2)$ are generated by fibrewise sections with no $R^i p_*$, and take cohomology:

$$0 \to p_*(\mathscr{K}(n_2)) \to p_*\mathcal{E}(n_1) \otimes \mathcal{V} \to p_*(\mathcal{E}(n_2)) \to 0. \tag{3.32}$$

$\mathcal{V}$ denotes $p_*(\mathcal{O}(n_1 - n_2))$, and to pull the sheaf $p_*\mathcal{E}(n_1)$ through $p_*p^*$ we have used its local freeness.

Define an $\mathcal{O}_{\mathcal{M}}$-flat sheaf $\mathcal{K}$ by

$$0 \to \mathcal{K} \to p^*p_*\mathscr{K}(n_2) \to \mathscr{K}(n_2) \to 0.$$

Then applying $\mathcal{H}om_p(\,\cdot\,, \mathcal{E}(n_2) \otimes p^*\mathcal{I})$, for any $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{I}$, yields

$$\begin{aligned} 0 &\to \mathcal{H}om_p(\mathscr{K}, \mathcal{E} \otimes p^*\mathcal{I}) \to (p_*\mathscr{K}(n_2))^* \otimes p_*\mathcal{E}(n_2) \otimes \mathcal{I} \\ &\to \mathcal{H}om_p(\mathcal{K}, \mathcal{E}(n_2) \otimes p^*\mathcal{I}) \to \mathcal{E}xt_p^1(\mathscr{K}, \mathcal{E} \otimes p^*\mathcal{I}) \to 0, \end{aligned} \tag{3.33}$$

where the final zero comes from the choice of $n_2 \gg 0$, and the second term is produced by the projection formula for the $\mathcal{O}_{\mathcal{M}}$-flat sheaves $p^*p_*\mathscr{K}(n_2)$ and $\mathcal{E}(n_2)$.

The higher terms in the above long exact sequence (3.33), with $\mathcal{I}$ trivial, give

$$0 \to \mathcal{E}xt_p^i(\mathcal{K}, \mathcal{E}(n_2)) \to \mathcal{E}xt_p^{i+1}(\mathscr{K}, \mathcal{E}) \to 0, \quad i \geq 1,$$

while the long exact $\mathcal{H}om_p(\,\cdot\,,\mathcal{E})$ sequence of (3.31) yields

$$0 \to \mathcal{E}xt^j_p(\mathscr{K},\mathcal{E}) \to \mathcal{E}xt^{j+1}_p(\mathcal{E},\mathcal{E}) \to 0, \quad j \geq 1,$$

by the choice of $n_1 \gg 0$. Thus

$$\mathcal{E}xt^i_p(\mathcal{K},\mathcal{E}(n_2)) \cong \mathcal{E}xt^{i+1}_p(\mathscr{K},\mathcal{E}) \cong \mathcal{E}xt^{i+2}_p(\mathcal{E},\mathcal{E}), \tag{3.34}$$

for all $i \geq 1$.

The last term is locally free, by base-change and the constancy of dim Exts in the hypothesis. Therefore so is the first term for $i \geq 1$, so its $i = 0$ counterpart

$$E_2 := \mathcal{H}om_p(\mathcal{K},\mathcal{E}(n_2))$$

is also locally free by base-change [3]. Here we have used the facts that $X$ is smooth and that the sheaves concerned are flat over $\mathcal{M}$, which also implies that $\mathrm{Tor}_i(\mathcal{E}(n_2),p^*\mathcal{I}) = 0$ for $i > 0$ giving a spectral sequence $\mathrm{Tor}_i(\mathcal{E}xt^j_p(\mathcal{K},\mathcal{E}(n_2)),\mathcal{I}) \Longrightarrow \mathcal{E}xt^{i-j}_p(\mathcal{K},\mathcal{E}(n_2)\otimes p^*\mathcal{I})$ as in the proof of Theorem 3.28. By (3.34) this vanishes for $i \geq 1$, $j \geq 1$ since $\mathcal{E}xt^{j+2}_p(\mathcal{E},\mathcal{E})$ is locally free for $j \geq 1$, so the spectral sequence degenerates to give $\mathcal{H}om_p(\mathcal{K},\mathcal{E}(n_2)\otimes p^*\mathcal{I}) \cong E_2 \otimes \mathcal{I}$. Therefore (3.33) has become

$$\begin{aligned} 0 &\to \mathcal{H}om_p(\mathscr{K},\mathcal{E}\otimes p^*\mathcal{I}) \to E_1' \otimes \mathcal{I} \to E_2 \otimes \mathcal{I} \\ &\to \mathcal{E}xt^2_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \to 0, \end{aligned} \tag{3.35}$$

with $E_2$ a vector bundle on $\mathcal{M}$; here we have defined

$$E_1' := (p_*\mathscr{K}(n_2))^* \otimes p_*\mathcal{E}(n_2).$$

Applying $\mathcal{H}om_p(\,\cdot\,,\mathcal{E}\otimes p^*\mathcal{I})$ to (3.31) yields

$$\begin{aligned} 0 &\to \mathcal{H}om_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \to (p_*\mathcal{E}(n_1))^* \otimes p_*\mathcal{E}(n_1) \otimes \mathcal{I} \\ &\to \mathcal{H}om_p(\mathscr{K},\mathcal{E}\otimes p^*\mathcal{I}) \to \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \to 0. \end{aligned}$$

The first two terms are $\mathcal{I}$ (by base-change since $\mathcal{E}$ is stable, and so simple, on the fibres) and $\mathcal{E}nd_{\mathcal{M}}(p_*\mathcal{E}(n_1)) \otimes \mathcal{I}$ with the identity map between them, giving

$$0 \to \mathcal{E}nd_0(p_*\mathcal{E}(n_1)) \otimes \mathcal{I} \to \mathcal{H}om_p(\mathscr{K},\mathcal{E}\otimes p^*\mathcal{I}) \to \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \to 0.$$

Fit this into the sequence (3.35)

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & & & & \\
 & \mathcal{E}nd_0(p_*\mathcal{E}(n_1))\otimes\mathcal{I} \arrow[d] \arrow[dr] & & & & \\
0 \arrow[r] & \mathcal{H}om_p(\mathscr{K},\mathcal{E}\otimes p^*\mathcal{I}) \arrow[r] \arrow[d] & E_1'\otimes\mathcal{I} \arrow[r] & E_2\otimes\mathcal{I} \arrow[r] & \mathcal{E}xt^2_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \arrow[r] & 0 \\
 & \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd} \tag{3.36}$$

and divide by the two injections of $\mathcal{E}nd_0(p_*\mathcal{E}(n_1))$ to give the required sequence

$$0\to\mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I})\to E_1\otimes\mathcal{I}\to E_2\otimes\mathcal{I}\to\mathcal{E}xt^2_p(\mathcal{E},\mathcal{E})\otimes\mathcal{I}\to 0, \tag{3.37}$$

where I claim that $E_1$ is locally free. To prove this it is enough to show the above map of vector bundles $\mathcal{E}nd_0(p_*\mathcal{E}(n_1))\to(p_*\mathscr{K}(n_2))^*\otimes p_*\mathcal{E}(n_2)=E_1'$ is nowhere zero. But this follows easily by repeating all of the above analysis at a single point of $\mathcal{M}$, instead of relative to $\mathcal{M}$, and using base-change: the same exact sequences show that the map on fibres $\mathrm{End}_0(H^0(\mathcal{E}(n_1))\to H^0(\mathscr{K}(n_2))^*\otimes H^0(\mathcal{E}(n_2))$ is an injection.

Finally, take the cokernel of the map $R^1p_*\mathcal{O}\to E_1$, and/or the kernel of the map $E_2\to R^2p_*\mathcal{O}$, in the following diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \arrow[r] & E_1\otimes\mathcal{I} \arrow[r] & E_2\otimes\mathcal{I} \arrow[r] \arrow[dr] & \mathcal{E}xt^2_p(\mathcal{E},\mathcal{E}\otimes p^*\mathcal{I}) \arrow[r] \arrow[d, "\mathrm{tr}"] & 0 \\
 & R^1p_*\mathcal{O}\otimes\mathcal{I} \arrow[u, "\mathrm{id}"] \arrow[ur] & & & R^2p_*\mathcal{O}\otimes\mathcal{I} \arrow[d] & \\
 & 0 \arrow[u] & & & 0. &
\end{tikzcd} \tag{3.38}$$

For rank $r>0$ these maps are injective and surjective respectively, and so give locally free resolutions of the trace-free tangent-obstruction complexes. q.e.d.

**Corollary 3.39.** *Let $X$ be a smooth projective 3-fold with trivial or anti-effective canonical bundle, let $\mathcal{M}$ denote the projective moduli* *space of semistable (with respect to the projective polarisation) sheaves of some fixed rank $r$ and Chern classes, and $\mathcal{M}_L$ those sheaves of fixed determinant $L$. Suppose that all such sheaves are stable (e.g. if rank and degree are coprime). Then for $r > 0$ there is a virtual cycle $Z_0 \subset \mathcal{M}_L$, defined by the tangent-obstruction functors $(\mathcal{T}^1)_0$, $(\mathcal{T}^2)_0$, of dimension the virtual dimension*

$$vd = \sum_{i=0}^{3} (-1)^{i+1} \dim \mathrm{Ext}_0^i(\mathcal{E}, \mathcal{E}).$$

*Its class in the Chow group $A_{vd}(X)$ is independent of the resolution (3.30). For any $r$ there are similar classes $Z \subset \mathcal{M}$ using the tangent-obstruction functors $\mathcal{T}^1$, $(\mathcal{T}^2)_0$ (of dimension $vd + h^{0,1}(X)$) and $\mathcal{T}^1$, $\mathcal{T}^2$ (dimension $vd + h^{0,1}(X) - h^{0,2}(X)$). If $\mathcal{M}$ is smooth then the appropriate obstruction sheaf is locally free, and the virtual cycle is its top Chern class.*

*Proof.* $\mathrm{Hom}_0(\mathcal{E}, \mathcal{E}) = 0$ for any stable sheaf $\mathcal{E}$ in the moduli space, so by Serre duality $\mathrm{Ext}_0^3(\mathcal{E}, \mathcal{E}) \cong \mathrm{Hom}_0(\mathcal{E}, \mathcal{E} \otimes K_X)^*$ and the assumptions on the canonical bundle $K_X$, $\mathrm{Ext}_0^3(\mathcal{E}, \mathcal{E})$ vanishes also. Thus by Theorem 3.30 we may apply ([31] 3.7) to give the required virtual cycle. q.e.d.

For $r > 0$ we consider the trace-free obstruction class $(\mathcal{T}^2)_0$, since $\mathcal{T}^2$ contains a trivial $R^2 p_* \mathcal{O}$ factor whose Chern class vanishes making the class of the virtual moduli cycle zero. For a Calabi-Yau manifold the above cycle $Z_0$ has dimension zero and its length will give our definition of the holomorphic Casson invariant. For this to be a sensible definition, however, we would like it to be deformation invariant. So we need to work out the deformation and obstruction theory of a sheaf $\mathcal{E}$ on $X$ as the complex structure of $X$ varies over an affine curve. We will do this in the next section; meanwhile since $\mathbb{P}^3$ has no complex deformations we can use the following Corollary to give an application of the virtual cycle.

**Corollary 3.40.** *Hilbert schemes of curves in a 3-fold $X$ with trivial or negative canonical bundle (such as $\mathbb{P}^3$) have a virtual moduli cycle.*

*Proof.* We use the particular moduli space of rank 1 sheaves of trivial determinant that contains the ideal sheaves of the curves. All we need to show is that any sheaf $\mathcal{E}$ of the same Hilbert polynomial and determinant is also an ideal sheaf (these are then automatically stable). But since $\mathcal{E}$, being stable, is torsion free, it is contained in its double dual, which is rank one, reflexive, and so a line bundle ([34] pp 154–156). So it equals its own determinant $\mathcal{O}$, so $\mathcal{E} \subset \mathcal{O}$ is a sheaf of ideals. q.e.d.

Corollary (3.39) allows us to define Donaldson-like invariants for such a 3-fold $X$ by doing intersection theory on moduli spaces of sheaves using characteristic classes of universal sheaves, in the usual way. For instance we may now define (deformation invariant, by the results of the next section) Gromov-Witten-like invariants (with *integral coefficients*) using Corollary (3.40): choose $n$ homology classes $\alpha_i$ in $X$ of total codimension equal to the virtual dimension of a fixed Hilbert scheme $H$ of curves in $X$, plus two. There is a universal curve $C$ over $H$ with a *flat* morphism to $H$, and similarly the projection of the $n$th fibre product to $H$,

$$C \times_H \ldots \times_H C \to H,$$

is flat. Thus ([17] 1.7) we may pull back the virtual cycle in $H$ to a cycle $Z$ in the total space and push it forward via the universal evaluation map

$$ev : C \to X \qquad \text{inducing} \qquad ev^n : C \times_H \ldots \times_H C \to X^n,$$

to $X^n$. Then the invariant is the integer intersection number (in the *smooth* variety $X^n$)

$$ev_*^n(Z) \,.\, (\times_{i=1}^n \alpha_i)\,. \tag{3.41}$$

Similarly for a Calabi-Yau 3-fold $X$ we may simply count the number of points in the virtual moduli cycle (i.e., take its length as a scheme) to get a count of curves in $X$.

These invariants differ from the Gromov-Witten invariants since Hilbert schemes contain many nasty components representing things other than curves. An example that will also be relevant later (explained to me by Jun Li) is given by two disjoint $\mathbb{P}^1$ s in 3-space coming together at a single point in a flat family. Consideration of the Euler characteristic of the structure sheaf, or just looking at the equations defining the subscheme, shows that the limiting curve must have a fat point at the intersection point (pointing in the direction in which the curves came together). In a separate flat family this point can break off to give a $\mathbb{P}^1$ and a distinct point in the same Hilbert scheme.

**Remark.** A naive approach to creating a virtual fundamental class would simply be to take the top Chern class of the obstruction sheaf. This is correct if $\mathcal{M}$ is smooth and so we have an obstruction *bundle*. In general Pidstrigatch [35] and Siebert [37] have shown the correct formula is the natural generalization of this given (3.1, 3.2), namely the $vd$-dimensional part of

$$Z = c\,(E_1 - E_0) \cap [c_F(\mathcal{M})],$$

where $c_F(\mathcal{M})$ is Fulton's total Chern class of the scheme $\mathcal{M}$ ([17] 4.2.6).

## Deformation invariance

Fix a quasi-projective scheme $\mathcal{X}$ with a flat map to a smooth affine curve $C$, with projective fibres $\iota : X_t \hookrightarrow \mathcal{X}$ over $t \in C$. Given a stable (in particular, simple) sheaf $\mathcal{E}$ on $X_0$ we study deformations of its (stable and simple) pushforward $\iota_*\mathcal{E}$ to $\mathcal{X}$, thus allowing it to move onto other fibres $X_t$ (it is easy to see, using stability or simplicity, that this is all it can do: its support must remain over a finite number of points in $C$ and stability, which is an open property, prevents it from splitting over more than one fibre; the deformation theory below will show this for instance).

We will need here and later a technical result, part of whose proof was worked out with the generous help of Brian Conrad.

**Lemma 3.42.** *Suppose $\iota : D \subset Z$ is a Cartier divisor in a quasi-projective scheme $Z$, with normal bundle $\nu = \mathcal{O}_D(D)$. Then for coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on $D$ there is a long exact sequence*

$$\to \operatorname{Ext}^i_D(\mathcal{E},\mathcal{F}) \to \operatorname{Ext}^i_Z(\iota_*\mathcal{E},\iota_*\mathcal{F}) \to \operatorname{Ext}^{i-1}_D(\mathcal{E},\mathcal{F}\otimes\nu) \xrightarrow{\delta} \operatorname{Ext}^{i+1}_D(\mathcal{E},\mathcal{F}) \to \ .$$

*Proof.* The sheaf sequence $0 \to \mathcal{O}(-D) \to \mathcal{O} \to \iota_*\mathcal{O}_D \to 0$ yields

$$\mathcal{H}om\,(\mathcal{O},\iota_*\mathcal{F}) \to \mathcal{H}om\,(\mathcal{O}(-D),\iota_*\mathcal{F}) \to \mathcal{E}xt^1(\iota_*\mathcal{O}_D,\iota_*\mathcal{F}) \to 0.$$

The first map, multiplication by the section of $\mathcal{O}(D)$ defining $D$, is zero since $\iota_*\mathcal{F}$ is supported on $D$. Thus we see that $\mathcal{E}xt^1(\iota_*\mathcal{O}_D,\iota_*\mathcal{F}) \cong \iota_*(\mathcal{F}\otimes\nu)$, so $\mathcal{E}xt^1(\iota_*E,\iota_*\mathcal{F}) \cong \iota_*(\mathcal{H}om\,(E,\mathcal{F})\otimes\nu)$ for any locally free sheaf $E$ on $D$. Thus, for a locally free sheaf $E$ that is sufficiently negative (so that $\operatorname{Ext}^1 = H^0(\mathcal{E}xt^1)$ in what follows) we have

$$\operatorname{Ext}^1(\iota_*E,\iota_*\mathcal{F}) \cong \operatorname{Hom}\,(E,\mathcal{F}\otimes\nu), \tag{3.43}$$

with higher Exts zero.

Now take a (not necessarily finite) locally free resolution of $E^\bullet \to \mathcal{E} \to 0$ of $\mathcal{E}$ on $D$ with the $E^i$s ($i = 0, 1, \ldots$) sufficiently negative with respect to $\iota_*\mathcal{F}$ as above (this is possible since $D$ is projective and we may take $E^{i+1} = H^0(E^i(N)) \otimes \mathcal{O}_D(-N)$ for large $N$). Take also an injective resolution $0 \to \iota_*\mathcal{F} \to I^\bullet$ on $Z$ ($\bullet = 0, 1, \ldots$). Then the cohomology of the complex $\mathrm{Hom}^\bullet(\iota_*\mathcal{E}, I^\bullet)$ computes $\mathrm{Ext}^\bullet(\iota_*\mathcal{E}, \iota_*\mathcal{F})$. But $I^\bullet$ is a complex of injectives bounded from below so respects quasi-isomorphisms like $\iota_*E^\bullet \to \iota_*\mathcal{E} \to 0$ (since $\iota_*$ is exact). Thus the double complex $\mathrm{Hom}^\bullet(\iota_*E^\bullet, I^\bullet)$ also computes $\mathrm{Ext}^\bullet(\iota_*\mathcal{E}, \iota_*\mathcal{F})$.

Now, by (3.43) above, the associated single complex $\mathrm{Hom}^\bullet(\iota_*E^i, I^\bullet)$ (for *fixed* $i$, and with differential $\delta_i^j : \mathrm{Hom}\,(\iota_*E^i, I^j) \to \mathrm{Hom}\,(\iota_*E^i, I^{j+1})$) has cohomology only in degrees 0 and 1. Thus, truncating all of these complexes (as $i$ varies) simultaneously by setting terms in degree $\bullet \geq 2$ to zero, and replacing the degree one term by $\ker \delta_i^1$, we get a quasi-isomorphic complex $B^{i,\bullet}$, sitting in an exact sequence of complexes

$$0 \to \ker \delta_i^0 \to B^{i,\bullet} \to \mathrm{coker}\, \delta_i^0\, [-1] \to 0.$$

(Here $[-1]$ means shift the complex one place to the right.) So by (3.43) this sequence is just

$$0 \to \mathrm{Hom}\,(E^i, \mathcal{F}) \to B^{i,\bullet} \to \mathrm{Hom}\,(E^i, \mathcal{F} \otimes \nu) \to 0. \tag{3.44}$$

The above complexes compute $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$, $\mathrm{Ext}^i(\iota_*\mathcal{E}, \iota_*\mathcal{F})$ and $\mathrm{Ext}^{i-1}(\mathcal{E}, \mathcal{F} \otimes \nu)$ respectively, so taking the long exact sequence in cohomology of the exact sequence of total complexes gives the required result. q.e.d.

**Remark.** The usual arguments show that the sequence is independent of choice of resolutions, and that the maps are the natural ones. The map $\mathrm{Ext}^1_D(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^1_Z(\iota_*\mathcal{E}, \iota_*\mathcal{F})$ pushes forward by $\iota_*$ (which is exact) an extension of $\mathcal{E}$ by $\mathcal{F}$ on $D$ to the corresponding extension of $\iota_*\mathcal{E}$ by $\iota_*\mathcal{F}$ on $Z$. The map $\mathrm{Ext}^1_Z(\iota_*\mathcal{E}, \iota_*\mathcal{F}) \to \mathrm{Hom}_D(\mathcal{E}, \mathcal{F} \otimes \nu)$ is a little harder to describe – taking an extension class on $Z$ and a (local) section of $\mathcal{E}$ on $D$, we must produce a section of $\mathcal{F} \otimes \nu$. But pulling back the extension (of $\iota_*\mathcal{E}$ by $\iota_*\mathcal{F}$) by the section of $\mathcal{E}$ gives a section of $\mathcal{E}xt^1(\iota_*\mathcal{O}_D, \iota_*\mathcal{F})$ which is shown in the proof to be canonically isomorphic to $\iota_*(\mathcal{F} \otimes \nu)$.

So in our usual set up (3.8) of $S \subset Y \subset Y_1$, consider the obstructions and deformations of a sheaf $\mathcal{F} := \iota_*\mathcal{E}$ on $\mathcal{X} \times Y$, where $\mathcal{E}$ is a stable sheaf on $X_0 \times Y$ (flat over $Y$), and $\iota$ denotes the inclusions $X_0 \subset \mathcal{X}$ and $X_0 \times Y \subset \mathcal{X} \times Y$ of the fibre of $\mathcal{X}$ over $0 \in C$. Restricting to $S \subset Y$ gives $\mathcal{F}_0 = \iota_* \mathcal{E}_0$.

The sequence (3.11) for the deformations of $\mathcal{F} = \iota_* \mathcal{E}$, and the corresponding sequence for $\mathcal{E}$ in one lower degree, are linked by the sequence of Lemma 3.42 (applied to the sheaves $\mathcal{E}$ and $\mathcal{E} \otimes \mathscr{I}$ on the Cartier divisor $X_0 \times Y \subset \mathcal{X} \times Y$ and written vertically below) to form the diagram:

$$\begin{tikzcd}
 & \mathrm{Ext}^1_{X_0\times S}(\mathcal{E}_0, \mathcal{E}\otimes\mathfrak{n}) \arrow[r, "\partial_0"] \arrow[d, "\iota_*"] & \mathrm{Ext}^2_{X_0\times S}(\mathcal{E}_0, \mathcal{E}_0\otimes\mathscr{I}) \arrow[d, "\iota_*"] \\
\mathrm{Ext}^1_{\mathcal{X}\times S}(\mathcal{F}_0, \mathcal{F}\otimes\mathfrak{m}) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{X}\times S}(\mathcal{F}_0, \mathcal{F}\otimes\mathfrak{n}) \arrow[r, "\partial"] \arrow[d] & \mathrm{Ext}^2_{\mathcal{X}\times S}(\mathcal{F}_0, \mathcal{F}_0\otimes\mathscr{I}) \arrow[d, "\phi"] \\
\mathrm{Hom}_{X_0\times S}(\mathcal{E}_0, \mathcal{E}\otimes\mathfrak{m}) \arrow[r] & \mathrm{Hom}_{X_0\times S}(\mathcal{E}_0, \mathcal{E}\otimes\mathfrak{n}) \arrow[r] & \mathrm{Ext}^1_{X_0\times S}(\mathcal{E}_0, \mathcal{E}_0\otimes\mathscr{I}).
\end{tikzcd}$$

Here we have used the fact that the normal bundle to $X_0 \times S$ in $\mathcal{X} \times S$ is of course trivial. Since our extension class $e_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}_0, \mathcal{F} \otimes \mathfrak{n})$ is in the image of $\iota_*$ in the above diagram, the obstruction $\partial(e_{\mathcal{F}}) = \iota_* \partial_0(e_{\mathcal{E}})$ is in the kernel of $\phi$. Extending the right hand vertical sequence upwards therefore shows the following.

**Theorem 3.45.** *The obstruction map $\partial$ of (3.13) takes values in* coker$(\delta)$*, with $\delta$ the last map in the following sequence (3.42) relating the first order deformations of $\mathcal{F}_0 = \iota_* \mathcal{E}_0$ to those of $\mathcal{E}_0$:*

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{X_0\times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}) &\to \mathrm{Ext}^1_{\mathcal{X}\times S}(\iota_*\mathcal{E}_0, \iota_*\mathcal{E}_0 \otimes \mathscr{I}) \\ \to \mathrm{Hom}_{X_0\times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}) &\xrightarrow{\delta} \mathrm{Ext}^2_{X_0\times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}). \end{aligned} \tag{3.46}$$

*The first map pushes deformations on $X_0 \times S$ forward to $\mathcal{X} \times S$. For $\mathcal{E}_0$ simple (e.g. stable) the penultimate term is just $H^0(\mathscr{I})$, and $\delta$ is the obstruction to first order deformations of $\mathcal{E}_0$ off the fibre $X_0 \times S \subset \mathcal{X} \times S$.*

We want to repeat this result for trace-free determinants. Of course there can be an obstruction to deforming the determinant of a sheaf to a nearby fibre, so we need to assume this vanishes by fixing a determinant that extends to all of $\mathcal{X}$. So choose line bundle $L$ on $\mathcal{X}$, and study stable sheaves on fibres $X_t$ of $\mathcal{X}$ whose determinant *on* $X_t$ is $L|_{X_t}$. We will also now insist that rank $\mathcal{E}_0 = r > 0$.

Then we showed above (3.45) that the obstruction to extending $\mathcal{F}$ is the image in $\mathrm{Ext}^2(\mathcal{F}_0, \mathcal{F}_0 \otimes \mathscr{I})$ of the obstruction $\partial(e_{\mathcal{E}}) \in \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ to extending $\mathcal{E}$ from $X_0 \times Y$ to $X_0 \times Y_1$. By Theorem (3.23) this last obstruction in fact lies in the trace-free part $\mathrm{Ext}^2_0(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$. So to get the trace-free analogue of the sequence (3.45) we want to show that the map

$$\mathrm{Hom}_{X_0 \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}) \xrightarrow{\delta} \mathrm{Ext}^2_{X_0 \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$$

of (3.45) in fact factors through $\mathrm{Ext}^2_0$. But this map fits into the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{X_0 \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}) & \xrightarrow{\delta} & \mathrm{Ext}^2_{X_0 \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}) \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ H^0_{X_0 \times S}(\mathscr{I}) & \xrightarrow{\delta} & H^2_{X_0 \times S}(\mathscr{I}). \end{array}$$

where the bottom row is the corresponding sequence for $\det \mathcal{E}$, so the map $\delta$ is the obstruction to extending $\det \mathcal{E}$, which vanishes by our assumption that $\det \mathcal{E}$ is the restriction of a global line bundle $L$ on $\mathcal{X}$. Now this diagram commutes by the following observation of Brian Conrad, for which I am very grateful. Namely, the map $\mathrm{Hom}\,(\mathcal{E}, \mathcal{F}) \xrightarrow{\delta} \mathrm{Ext}^2(\mathcal{E}, \mathcal{F})$ of Lemma 3.42, is covariant in $\mathcal{F}$ and contravariant in $\mathcal{E}$, making it covariant in $\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{H}om^\bullet(E^\bullet, \mathcal{F})$, where $E^\bullet$ is a finite locally free resolution of $\mathcal{E}$. Therefore applying the trace map to $\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E})$ gives the commutativity of the above diagram.

So we can improve Theorem 3.45 to

**Theorem 3.47.** *Taking $\mathcal{E}_0$ to have rank $r > 0$ and determinant $L|_{X_0}$, the restriction of a global line bundle $L$ on $\mathcal{X}$, Theorem 3.45 holds with $\mathrm{Ext}^2_{X_0 \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ replaced by its trace-free part $\mathrm{Ext}^2_{X_0 \times S}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})_0$.*

**Corollary 3.48.** *Use the projections $p : X_0 \times S \to S$ and $q : \mathcal{X} \times S \to S$, and let $\mathcal{E}$ be a stable sheaf on $X_0 \times Y$ (flat over $Y$) with restriction $\mathcal{E}_0$ to $X_0 \times S$ such that $\dim \mathrm{Ext}^i_0(\mathcal{E}_0|_{X_s}, \mathcal{E}_0|_{X_s}) = 0$ for all $s \in S$ and $i \geq 3$. Then*

$$\begin{aligned} \mathcal{T}^2_{\mathcal{F}_0} &:= \mathrm{coker}\,\{\mathcal{O}_S \xrightarrow{\delta} \mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0)\} \\ &= \mathrm{image}\,\{\mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0) \to \mathcal{E}xt^2_q(\mathcal{F}_0, \mathcal{F}_0)\} \end{aligned}$$

*is an obstruction sheaf for $\mathcal{F} = \iota_* \mathcal{E}$. If $r > 0$ and $\det \mathcal{E}_0$ is the restriction to $X_0 \times S$ of a global line bundle $L$ on $\mathcal{X} \times S$ (pulled back from $\mathcal{X}$), then*

$$\begin{aligned} (\mathcal{T}^2_{\mathcal{F}_0})_0 &:= \mathrm{coker}\,\{\mathcal{O}_S \xrightarrow{\delta} \mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0)_0\} \\ &= \mathrm{image}\,\{\mathcal{E}xt^2_p(\mathcal{E}_0, \mathcal{E}_0)_0 \to \mathcal{E}xt^2_q(\mathcal{F}_0, \mathcal{F}_0)\} \end{aligned}$$

*is also an obstruction sheaf for $\mathcal{F} = \iota_*\mathcal{E}$. Also,*

$$(\mathcal{T}^1_{\mathcal{F}_0})_0(\mathcal{I}) := \operatorname{coker}\{R^1p_*\mathcal{O} \otimes \mathcal{I} \to \mathcal{E}xt^1_q(\mathcal{F}_0, \mathcal{F}_0 \otimes \mathscr{I})\}$$

*(where the map is the identity into $\mathcal{E}xt^1_p(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ followed by the inclusion into $\mathcal{E}xt^1_q(\mathcal{F}_0, \mathcal{F}_0 \otimes \mathscr{I})$) is a tangent functor for deformations of $\mathcal{F}_0$ with fixed determinant $L$ on the $X_t$ fibres.*

*Thus tangent-obstruction functors of the moduli problems for $\mathcal{E}$ on $X_0$ and $\mathcal{F}$ on $\mathcal{X}$ fit into the exact sequences of $\mathcal{O}_S$-modules*

$$0 \to \mathcal{T}^1_{\mathcal{E}_0}(\mathcal{I}) \to \mathcal{T}^1_{\mathcal{F}_0}(\mathcal{I}) \to \mathcal{I} \to \mathcal{T}^2_{\mathcal{E}_0} \otimes \mathcal{I} \to \mathcal{T}^2_{\mathcal{F}_0} \otimes \mathcal{I} \to 0, \tag{3.49}$$

*and*

(3.50)
$$0 \to (\mathcal{T}^1_{\mathcal{E}_0})_0(\mathcal{I}) \to (\mathcal{T}^1_{\mathcal{F}_0})_0(\mathcal{I}) \to \mathcal{I} \to (\mathcal{T}^2_{\mathcal{E}_0})_0 \otimes \mathcal{I} \to (\mathcal{T}^2_{\mathcal{F}_0})_0 \otimes \mathcal{I} \to 0,$$

*for any $\mathcal{O}_S$-module $\mathcal{I}$.*

*Proof.* Use the results of (3.28, 3.26) for $\mathcal{E}_0$ and $\mathcal{F}_0 = \iota_*\mathcal{E}_0$, using the condition that the higher Ext groups of $\mathcal{E}_0$ vanish to show the same for $\mathcal{F}_0$ (this follows from the exact sequence (3.42) as usual.) Then Theorems 3.45, 3.47 give the required exact sequences. q.e.d.

**Theorem 3.51.** *The tangent-obstruction functors of $\mathcal{E}$ on $X_0$ (3.26, 3.28) and of $\mathcal{F} = \iota_*\mathcal{E}$ on $\mathcal{X}$ (3.48) are compatible in the sense of Li-Tian ([31] Definition 3.8).*

*Proof.* The compatibility of Li-Tian says roughly that for an obstructed sheaf $\mathcal{E}$ on $X_0$ to extend without obstruction inside $\mathcal{X}$, the obstruction must cancel the obstruction to extending $\iota_*\mathcal{E}_0$ in the direction of the base $C$.

The precise statement is that there should be an exact sequence (3.49) (or (3.50) in the trace-free case) such that the following holds. Let $S \subset Y \subset Y_1$ be as in (3.8), and take a sheaf $\mathcal{E}$ on $X_0 \times Y$, giving a corresponding class $e \in \operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n})$. Suppose that the obstruction class $\partial_0(e) \in \operatorname{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ to extending it to $\mathcal{X}_0 \times Y_1$, has vanishing image in $\operatorname{Ext}^2(\mathcal{F}_0, \mathcal{F}_0 \otimes \mathscr{I})$. Thus $\mathcal{F} = \iota_*(\mathcal{E})$ extends to a sheaf on

$\mathcal{X} \times Y_1$, giving a class $f \in \operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F} \otimes \mathfrak{m})$ and a diagram

$$\begin{tikzcd}
 & & & e \in \operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n}) \arrow[d, "\iota_*"] \\
 & & f \in \operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F} \otimes \mathfrak{m}) \arrow[r] \arrow[d, "\phi"] & \operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F} \otimes \mathfrak{n}) \arrow[d, "\psi"] \\
0 \arrow[r] & \operatorname{Hom}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}) \arrow[r] \arrow[d, "\delta"] & \operatorname{Hom}(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{m}) \arrow[r] & \operatorname{Hom}(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n}) \\
e \in \operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E} \otimes \mathfrak{n}) \arrow[r, "\partial_0"] & \operatorname{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I}), & &
\end{tikzcd}$$

where all horizontal maps come from sequences of the form (3.11), and vertical maps from (3.42). Since $\iota_*(e)$ is mapped to zero under $\psi$, $\phi(f)$ is in the image of a unique class $c \in \operatorname{Hom}(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$. Then Definition 3.8 of [31] requires that $\delta(c) = -\partial_0(e)$.

In our situation this holds by abstract homological algebra. We first need to lift the big commutative diagram of Ext groups that the above diagram is a part of to the level of short exact sequences of complexes. Given an $\mathcal{O}_S$-module $J$, we use the complex $B_J^\bullet$, whose cohomology is $\operatorname{Ext}^\bullet(\mathcal{F}_0, \mathcal{F} \otimes J)$, from the proof of Lemma 3.42. This is functorial in $J$, and we denote the other two complexes in (3.44) by $A_J^\bullet$ and $C_J^\bullet$ (computing the cohomology of $\operatorname{Ext}^\bullet(\mathcal{E}_0, \mathcal{E} \otimes J)$ and $\operatorname{Ext}^{\bullet-1}(\mathcal{E}_0, \mathcal{E} \otimes \nu \otimes J)$ respectively).

Thus we get a (horizontal) sequence of sequences of the form (3.44) (written vertically) by setting $J$ to be the different modules in the exact sequence $0 \to \mathscr{I} \xrightarrow{j} \mathfrak{m} \xrightarrow{\pi} \mathfrak{n} \to 0$, yielding the exact commutative diagram of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^\bullet_{\mathscr{I}} \arrow[r, "j_A"] \arrow[d, "\iota_{\mathscr{I}}"] & A^\bullet_{\mathfrak{m}} \arrow[r, "\pi_A"] \arrow[d, "\iota_{\mathfrak{m}}"] & A^\bullet_{\mathfrak{n}} \arrow[r] \arrow[d, "\iota_{\mathfrak{n}}"] & 0 \\
0 \arrow[r] & B^\bullet_{\mathscr{I}} \arrow[r, "j_B"] \arrow[d, "p_{\mathscr{I}}"] & B^\bullet_{\mathfrak{m}} \arrow[r, "\pi_B"] \arrow[d, "p_{\mathfrak{m}}"] & B^\bullet_{\mathfrak{n}} \arrow[r] \arrow[d, "p_{\mathfrak{n}}"] & 0 \\
0 \arrow[r] & C^\bullet_{\mathscr{I}} \arrow[r, "j_C"] \arrow[d] & C^\bullet_{\mathfrak{m}} \arrow[r, "\pi_C"] \arrow[d] & C^\bullet_{\mathfrak{n}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

We now follow the previous diagram chase around this diagram at the level of complexes. So we start with $e \in A^1_{\mathfrak{n}}$ with $\delta e = 0$ (we will denote all coboundary operators by $\delta$) which we want to lift to $f \in B^1_{\mathfrak{m}}$ with $\delta f = 0$.

Lift $e$ to $\beta \in A^1_{\mathfrak{m}}$ (using the fact that $\pi_A$ is onto). This is then assumed not coclosed; its coboundary is by definition $\delta(\beta) = j_A(\partial_0(e))$ where $\partial_0(e) \in A^2_{\mathscr{I}}$ represents the obstruction class $\partial_0(e) \in \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ to lifting $e$. Pushing forward $\tilde{f} := \iota_{\mathfrak{m}}(\beta)$ to a class in $B^1_{\mathfrak{m}}$, we see that a coclosed class $f \in B^1_{\mathfrak{m}}$ lifting $\iota_{\mathfrak{n}}(e) \in B^1_{\mathfrak{n}}$ exists if and only if there is an $\alpha \in B^1_{\mathscr{I}}$ with $\delta(j_B(\alpha)) = -\delta(\tilde{f})$; the required $f$ is then $f = \tilde{f} + j_B(\alpha)$. We are assuming $f$, and so $\alpha$, exist.

Thus, since $j_B(\delta(\alpha)) = -\delta(\tilde{f}) = -\delta(\iota_{\mathfrak{m}}(\beta)) = -\iota_{\mathfrak{m}} j_A(\partial_0(e)) = -j_B\,\iota_{\mathscr{I}}(\partial_0(e))$, and since $j_B$ is an injection, we have

$$\delta(\alpha) = -\iota_{\mathscr{I}}(\partial_0(e)). \tag{3.52}$$

In particular then, $p_{\mathscr{I}}(\alpha)$ is coclosed, defining a class $c \in \mathrm{Hom}\,(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathscr{I})$ whose coboundary (by which we mean now the connecting homomorphism $\delta$ in the cohomology sequence of the left hand column that arises in the previous diagram) $\delta(c)$ is $-\partial_0(e)$ by (3.52).

Thus we are left with showing that $c$ is the same $c$ as defined above, i.e., that $j_C(c)$ is $p_{\mathfrak{m}}(f)$. But $j_C(c) = j_C p_{\mathscr{I}}(\alpha) = p_{\mathfrak{m}} j_B(\alpha) = p_{\mathfrak{m}}(f - \tilde{f}) = p_{\mathfrak{m}}(f)$ as required, since $\tilde{f} = \iota_{\mathfrak{m}}(\beta)$ is in the kernel of $p_{\mathfrak{m}}$. q.e.d.

**Corollary 3.53.** *The virtual moduli cycles of Corollary 3.39 are deformation invariant in the following sense. Given a family $\mathcal{X} \to C$ of smooth projective varieties $X_t$, $t \in C$, consider the family of moduli spaces $\mathcal{M}_t$ of stable sheaves $\mathcal{E}$ as in (3.39) on the fibre $X_t$ (containing the virtual moduli cycle $Z_t$). These form the moduli space $\mathcal{M} \to C$ of sheaves $(\iota_t)_*\mathcal{E}$. Then under the same conditions as Corollary 3.39, $\mathcal{M}$ has a virtual moduli cycle $\mathcal{Z}$ of dimension* $\dim Z_t + 1$, *and as elements of the Chow group they satisfy $\iota_t^!\mathcal{Z} = Z_t$. (Here $\iota_t^!$ is the Gysin homomorphism (*[17] *6.2) of the inclusion $\iota_t : \{t\} \to C$.) There is also the corresponding result for sheaves of fixed determinant $L_t$, with $L$ a fixed line bundle on all of $\mathcal{X}$, for rank $r > 0$.*

*Proof.* We first need to show that the tangent-obstruction functors of $\mathcal{M} \to C$ are perfect in the weaker sense of the Remark preceding Theorem 3.30. The $E_2$ of that Theorem exists on all of $\mathcal{M}$ and surjects onto $\mathcal{T}^2_{\mathcal{E}_0}$, and this in turn surjects onto $\mathcal{T}^2_{\mathcal{F}_0}$. So all we need is a local vector bundle $E_1$ giving a resolution of $\mathcal{T}^1_{\mathcal{F}_0}$, $\mathcal{T}^2_{\mathcal{F}_0}$ over an open set of $\mathcal{M}$.

The map $\mathcal{O}_{\mathcal{M}} \to \mathcal{T}^2_{\mathcal{E}_0}$ of (3.49) locally factors through $E_2 \to \mathcal{T}^2_{\mathcal{E}_0}$ via a local lift $\mathcal{O} \to E_2$. Combining this lift with the map $E_1 \to E_2$ of Theorem 3.30 into a map $E_1 \oplus \mathcal{O} \to E_2$ gives an exact sequence, for any

$\mathcal{O}_S$-module $\mathcal{I}$,

$$0 \to \mathcal{T}^1_{\mathcal{F}_0}(\mathcal{I}) \to (E_1 \oplus \mathcal{O}) \otimes \mathcal{I} \to E_2 \otimes \mathcal{I} \to \mathcal{T}^2_{\mathcal{F}_0} \otimes \mathcal{I} \to 0,$$

by combining (3.37) with (3.49). Thus we can use [31] to produce the virtual moduli cycle $Z$ (and similarly for the trace-free versions, using (3.38) and (3.50) instead).

$\iota_t^! \mathcal{Z} = Z_t$ follows from ([31] 3.9), given the compatibility of Theorem 3.51. q.e.d.

## The holomorphic Casson invariant

**Definition 3.54.** Fix a smooth projective Calabi-Yau 3-fold $X$, and a rank $r$ and Chern classes $c_i$ such that the moduli space $\mathcal{M}$ of semistable sheaves with this data (and fixed determinant $L$ if rank$> 0$) contains only stable sheaves (for instance if the rank and degree are coprime). Then we define the holomorphic Casson invariant $\lambda_{\{c_i\}}(X)$ to be the length as a scheme of the zero dimensional projective virtual moduli cycle $Z_0 \subset \mathcal{M}_L$ of Corollary 3.39. It is invariant under deformations of $X$ in any projective family to which $L$ extends (e.g. if $h^{0,2}(X) = 0$ this is immediate).

The deformation invariance comes from the relation $\iota_t^! Z = Z_t$ of (3.53) and the resulting "conservation of number" ([17] 10.2). This invariant is clearly similar in nature to the Gromov-Witten invariants of $X$. In fact we might expect to recover GW invariants either by considering moduli of ideal sheaves of curves as in (3.40, 3.41), or by relating rank two bundles to curves via zero sets of their sections (and vice-versa by the Serre construction). However, we have already remarked that the first case gives something slightly different to GW invariants. As for rank two bundles, under the Serre construction spheres and tori tend to correspond to unstable bundles, and for a higher genus curve to correspond to a bundle its tangent bundle must extend to a line bundle on $X$, cutting down the space of admissible curves. Since the space of curves has expected dimension zero anyway we tend to find (for instance in the examples below) that rank two bundles correspond to non-generic high-dimensional families of curves. In this case deforming the curve corresponds to deforming the section, not the bundle, so GW invariants do not arise.

We might also like to count fewer singular sheaves, i.e., to only count bundles. We would expect from Donaldson theory to have to include sheaves with codimension two singularities (limits of stable bundles, where "bubbling" occurs), but in the rank two case one might hope to be able to ignore sheaves with codimension three singularities. However again the example above of a flat family of curves producing a distinct point shows that bundles can degenerate to sheaves with codimension three singularities, and so such sheaves can lie in the same connected component as bundles.

In all of the examples we consider, however, we will be able to show there are no such singular sheaves, and in fact GW invariants will arise in a completely different way in the last section.

**Examples.** Simple examples of the invariant are given by considering ideal sheaves of 1, 2 or 3 points in a Calabi-Yau 3-fold $X$. The moduli space is then $X$, $\mathrm{Hilb}^2X$ or $\mathrm{Hilb}^3X$, and so smooth, with the invariant giving the Euler number of the cotangent bundle (since this is the obstruction bundle), i.e., $-\chi(X)$, $\chi(\mathrm{Hilb}^2X)$, and $-\chi(\mathrm{Hilb}^3X)$ respectively.

A deeper example is motivated by Donaldson's reinterpretation [12] of work of Mukai ([32] 0.9) as an example of the Tyurin-style Casson invariant of (2.7).

Fix a smooth quadric $Q_0$ in $\mathbb{P}^5$, in a fixed $\mathbb{P}^2$-family of quadrics spanned by $Q_0$, $Q_1$ and $Q_2$, say. The singular quadrics in the family lie on the sextic curve

$$C = \{[\lambda_0; \lambda_1; \lambda_2] \in \mathbb{P}^2 : \det(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2) = 0\} \subset \mathbb{P}^2$$

where the quadratic form defining the quadric becomes singular.

For every point of $\mathbb{P}^2 \setminus C$ we get two tautological rank 2 bundles $A$ and $B$ over the corresponding quadric (thinking of it as a Grassmannian of 2-planes in $\mathbb{C}^4$, $A$ and $B$ are defined by the tautological sequence $0 \to A^* \to \mathbb{C}^4 \to B \to 0$), and so also over the $K3$ surface

$$S = Q_0 \cap Q_1 \cap Q_2.$$

In fact these $A$ and $B$ bundles give a double cover $\mathcal{M}$ of $\mathbb{P}^2$ branched along the sextic curve $C$ (the $A$ and $B$ bundles coincide on the singular quadrics) as the moduli space of bundles of the same topological type over $S$. $\mathcal{M}$ is a (complex symplectic) $K3$ surface, notice.

Similarly the Fano $X_1 = Q_0 \cap Q_1$ lies in the pencil spanned by $Q_0$ and $Q_1$, a line $\mathbb{P}^1$ in our $\mathbb{P}^2$-family. The 2-fold branched cover of this line induced by $\mathcal{M} \to \mathbb{P}^2$, i.e., the set of $A$ and $B$ bundles on the quadrics in this pencil, is the moduli space for $X_1$. Similarly for $X_2 = Q_0 \cap Q_2$ and the cover of the line $\langle X_0, X_2 \rangle \subset \mathbb{P}^2$.

So we have an example of the Tyurin-Casson invariant (2.7), with two complex Lagrangians (the curves covering the $\mathbb{P}^1$s) as the moduli spaces of bundles on the two Fanos, injecting into the complex symplectic moduli space of bundles on the common anticanonical divisor $S$. Their intersection, namely the double cover of the intersection point $\{Q_0\}$ of the lines in $\mathbb{P}^2$, corresponds to the two stable bundles $A_{Q_0}$ and $B_{Q_0}$ on the singular Calabi-Yau that is the union of $X_1$ and $X_2$.

Deforming this singular quartic in $Q_0$ to a smooth Calabi-Yau we would like, then, to prove the following.

**Theorem 3.55.** *Let $Q_0$ be a smooth quadric in $\mathbb{P}^5$, and let $X$ be a smooth quartic hypersurface in $Q_0$. Then the bundles $A$ and $B$ on $Q_0$ restrict to stable, isolated bundles of the same topological type on $X$, and they are the only semistable sheaves in the moduli space. Thus the corresponding holomorphic Casson invariant is 2.*

*Proof.* Standard exact sequences and geometry on the Grassmannian $\mathrm{Gr}\,(2,4) \cong Q_0$ show that the bundles are stable and isolated ([40] 2.3.1). The more difficult part is to show that any semistable sheaf $\mathcal{E}$ of the same Chern classes is either $A$ or $B$. We will do this by controlling $\mathcal{E}$'s cohomology by studying it on hyperplane sections, and then using Riemann-Roch to produce sections (c.f. [22]).

Let $\widehat{\mathcal{E}}$ denote the double dual (or "reflexive hull") of $\mathcal{E}$. Let $H = \mathbb{P}^4 \cap X$ be a smooth hyperplane section, with $\mathbb{P}^4 \subset \mathbb{P}^5$ sufficiently generic that $\widehat{\mathcal{E}}|_H$ is the double dual of $\mathcal{E}|_H$ and so a bundle $F$, say. Then the Riemann-Roch formula for $F$ is

$$2h^0(F) - h^1(F) = 12 - c_2(F), \tag{3.56}$$

by Serre duality, $K_H = \mathcal{O}_H(1)$, and an unpleasant computation. Thus $F$ has at least 4 sections, as $c_2(F) = c_2(\widehat{\mathcal{E}}) \,.\, \omega \leq c_2(\mathcal{E}) \,.\, \omega = 4$ (recall that passing to double duals lowers $c_2$ because of the exact sequence embedding a sheaf inside its double dual).

Also, as $H$ is generic, we may assume its only line bundles are the $\mathcal{O}(n)$ bundles, by Noether-Lefschetz theory (see e.g. [19]). $\widehat{\mathcal{E}}$ is slope semistable and so slope semistable on restriction to the generic hyperplane (see e.g. [22] 3.2). Thus $F(-1)$ cannot have any sections, so the four sections of $F$ must vanish only on points, giving us a Koszul resolution ([18] p 688) of the form

$$0 \to \mathcal{O} \to F \to \mathscr{I}_c(1) \to 0, \tag{3.57}$$

where $\mathscr{I}_c$ is the ideal sheaf of functions vanishing at $c = c_2(F) \leq 4$ points. Taking sections and using the Riemann-Roch formula (3.56) gives

$$h^0(\mathscr{I}_c(1)) = h^0(F) - 1 \geq 5 - c/2.$$

However $h^0(\mathcal{O}_H(1)) = 5$, and the $c$ points impose at least $\min(2, c)$ conditions on the sections of $\mathcal{O}_H(1)$ as they are the restriction of the sections of $\mathcal{O}_{\mathbb{P}^4}(1)$ on the $\mathbb{P}^4$ hyperplane in $\mathbb{P}^5$. Thus $5 - c/2 \leq h^0(\mathscr{I}_c(1)) \leq 5 - \min(2, c)$, i.e., $c \geq 2\min(2, c)$, whose only integral solutions for $0 \leq c \leq 4$ are $c = 0$ and $c = 4$. We can rule out $c = 0$ by stability (either by the Bogomolov inequality or the fact that (3.57) would split); thus $c = c_2(F) = 4$. Therefore $\widehat{\mathcal{E}}$ has the same second Chern class as $\mathcal{E}$ on $X$ (since $H^4(X; \mathbb{C})$ is generated by a class non-zero on $H$), and can only differ from it in codimension three.

Notice also that as we must have $h^0(\mathscr{I}_4(1)) = 3$, the points lie on a web of hyperplanes in the $\mathbb{P}^4$ hyperplane, i.e., *on a line in* $\mathbb{P}^4$.

To control the cohomology of $F$ we use (3.57) to give, for $t \geq 2$,

$$\begin{array}{ccccccc} 0 \to & H^1(F(t-1)) & \to H^1(\mathscr{I}_4(t)) \to & H^2(\mathcal{O}_H(t-1)) & \to 0, \\ & \downarrow \wr & & \downarrow \wr & \\ & H^1(F(1-t))^* & & H^0(\mathcal{O}_H(2-t))^* & \end{array}$$

by stability. But the sequence $0 \to \mathscr{I}_4(t) \to \mathcal{O}_H(t) \to \mathcal{O}_4(t) \to 0$ shows that $h^1(\mathscr{I}_4(t)) = 0 \ \ \forall t \geq 3$ since we can find a polynomial of any degree$\geq 3$ taking any prescribed values at 4 points on a line (in $\mathbb{P}^4$). Similarly $h^1(\mathscr{I}_4(2)) = 1$, and we get $H^1(F(-n)) = 0 \ \ \forall n \geq 1$.

Thus we can now pass up to $X$ using the sequence $0 \to \widehat{\mathcal{E}}(-1) \to \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}|_H \to 0$, which is exact because $\widehat{\mathcal{E}}$ is torsion free, giving

$$H^1(\widehat{\mathcal{E}}(-n-1)) \to H^1(\widehat{\mathcal{E}}(-n)) \to H^1(F(-n)).$$

Since $H^1(\widehat{\mathcal{E}}(-n))$ vanishes for large $n$ it therefore vanishes for all $n \geq 1$; in particular $H^2(\widehat{\mathcal{E}}) = 0$. This and stability simplify the Riemann-Roch formula for $\widehat{\mathcal{E}}$ to

$$h^0(\widehat{\mathcal{E}}) - h^1(\widehat{\mathcal{E}}) = 4 + c_3(\widehat{\mathcal{E}})/2.$$

The third Chern class of a rank two reflexive sheaf on a smooth 3-fold is always nonnegative and vanishes if and only if the sheaf is locally free

([22] 2.6), so $\widehat{\mathcal{E}}$ has at least 4 sections which do not vanish on divisors, by stability. Thus we have a presentation ([22] 4.1)

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathscr{I}_C(1) \to 0,$$

for some degree four curve $C$. Computing the third Chern class of such an extension to be zero shows that $\widehat{\mathcal{E}}$ is locally free with the same Chern classes as $\mathcal{E}$, i.e., $\mathcal{E} \cong \widehat{\mathcal{E}}$ is locally free. (The point here is that the extension is locally free, and not just reflexive, because the determinant of $\widehat{\mathcal{E}}$, restricted to $C$, is isomorphic to the determinant of the normal bundle to $C$ with the isomorphism set up by the determinant of the section. Thus the section vanishes transversally along $C$, the bove sequence becomes the Koszul resolution (3.57) for this section, and $\mathcal{E}$ is locally free. In fact, the Serre construction ([34] p. 93) uniquely constructs a sheaf $\mathcal{E}$ (which is then locally free) from the above resolution when the appropriate determinants are equal; it is only when they differ that reflexive sheaves arise from Hartshorne's generalization of the Serre construction [22].)

But now $h^0(\mathscr{I}_C(1)) \geq 3$, so $C$ lies in a web of hyperplanes in $X \subset Q_0 \subset \mathbb{P}^5$. Thus it lies on a linear $\mathbb{P}^2$ plane $P \subset \mathbb{P}^5$. Since $C$ lies in the quadric $Q_0$, the plane $P$ must do so too, otherwise the quadric would intersect $P$ in a conic curve containing the degree four curve $C$, which is impossible.

But the planes in $Q_0$ are precisely the standard planes in $\mathrm{Gr}\,(2,4)$ – zero sets of sections of $A$ and $B$. $P$ uniquely defines either the $A$ or the $B$ bundle on $Q_0$ via the Serre construction (see [34] p. 93).

$$0 \to \mathcal{O} \to A/B \to \mathscr{I}_P(1) \to 0,$$

by the extension data $\mathrm{Ext}^1(\mathscr{I}_P(1), \mathcal{O}_{Q_0}) \cong H^0(\mathcal{O}_P) \ni 1$. This extension restricts, on the quartic $X$, to $1 \in H^0(\mathcal{O}_C) \cong \mathrm{Ext}^1(\mathscr{I}_C(1), \mathcal{O}_X)$, defining our bundle $\mathcal{E}$

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathscr{I}_C(1) \to 0$$

by uniqueness, since $H^0(\mathcal{O}_C) \cong \mathbb{C}$ ($C$ is a curve in $\mathbb{P}^2$, so is connected). Therefore $\mathcal{E}$ is one of $A$ or $B$ restricted to $X$. q.e.d.

## 4. $K3$ fibrations

We now turn to calculating the invariants on $K3$-fibred Calabi-Yau manifolds with no reducible or multiple fibres, using the nice properties of moduli of bundles and sheaves on a $K3$ due to Mukai ([24] Chapter 6). These do not quite generalise to the singular $K3$ fibres. Although these have a (trivial) dualising sheaf as the fibres are complete intersections ([21] III 7.11) and so the usual Serre duality holds ([21] III 7.6), the Serre duality we have been using [32],

$$\operatorname{Ext}^i(\mathcal{E},\mathcal{E})^* \cong \operatorname{Ext}^{2-i}(\mathcal{E},\mathcal{E}), \tag{4.1}$$

only holds for sheaves $\mathcal{E}$ with a *finite* locally free resolution. On the singular fibres there will be sheaves with unbounded locally free resolutions for which the result does not hold.

We will consider stability on $X$ using a *suitable* polarisation $\omega$, in the sense of Friedman. This means that the fibres are small so that semistable sheaves restrict, on the generic fibre, to semistable sheaves. The basic idea is to add a large multiple of the fibre divisor $f$ to any fixed polarisation, so that the sign of the degree $\omega\,.\,\omega\,.\,c_1$ of a possibly destabilizing subsheaf is the same as the sign of the degree $f\,.\,\omega\,.\,c_1$ on a generic fibre. Since such subsheaves that we need to consider form a bounded family we can do this:

**Proposition 4.2.** *Let $X$ be a surface-fibred projective 3-fold. Choose rank and Chern classes such that slope semistability implies slope stability on smooth fibres (e.g. if rank and degree are coprime). Then we may add a sufficient number of fibre classes to the polarisation such that sheaves of the given rank and Chern classes are stable if and only if they are stable on the generic fibre, and there are no strictly semistable sheaves. Such a polarisation is called suitable.*

*Proof.* This should really be proved directly, but a cheat goes as follows. Choose $N \gg 0$ such that any slope semistable sheaf $\mathcal{E}$ (of the given Chern classes) restricts to a slope semistable sheaf on a generic hyperplane $H$ in the linear system $|\mathcal{O}(N)|$ ([24] 7.2.1). Now there is an $M$ such that $(\omega + Mf)|_H$ is a suitable polarisation on $H \to C$ by ([24] 5.3) (here $\omega$ denotes the Kähler form on $X$, and $f$ the class of a fibre). Thus $\mathcal{E}$ is slope semistable on the generic fibre $H_t$ of $H \to C$, and so slope semistable on the generic fibre $X_t$ of $X \to C$ (we are using the easy fact that the (slope/semi) stability of a sheaf on a hyperplane section implies the same on the whole space). By assumption, then, $\mathcal{E}$ is slope stable on $X_t$.

Reversing the argument, by increasing $N$ if necessary, if $\mathcal{E}$ is slope stable on $X_t$, then by a theorem of Bogomolov ([24] 7.3.5) $\mathcal{E}$ is stable on $H_t$. Since $(\omega + Mf)|_H$ is a suitable polarisation, this implies that $\mathcal{E}$ is slope stable on $H$, and so on $X$. q.e.d.

**Definition 4.3.** Let $X$ be a smooth polarised 3-fold, $K3$-fibred over a smooth curve $C$, let $c_i \in H^{i,i}_{\mathbb{Z}}(X_t)$, $i = 1, 2$ be Chern classes in the cohomology of a generic fibre, and let $r \in \mathbb{Z}_{\geq 0}$ be a rank. Then we say that $(X, r, c_i)$ is *admissible* if and only if

- $X \to C$ has no reducible or non-reduced fibres,
- the polarisation on $X$ is chosen to be suitable for $(r, c_i)$ using Proposition 4.2,
- $c_1$ is the restriction of the first Chern class of a global line bundle on $X$, i.e., is in the image of $H^{1,1}_{\mathbb{Z}}(X) \to H^{1,1}_{\mathbb{Z}}(X_t)$,
- $\gcd(r, c_1 \,.\, \omega, \frac{1}{2}c_1^2 - c_2) = 1$ (where $\omega$ is the Kähler form of the induced polarisation on the fibre), and
- on any fibre $X_t$, slope semistability of sheaves with Chern classes $(r, c_i)$ implies slope stability (e.g. if rank $r$ and degree $c_1 \,.\, \omega|_{X_t}$ are coprime).

We then set $d = 2rc_2 - (r-1)c_1^2 - 2(r^2-1)$, the dimension $\dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = \sum_i (-1)^{i+1} \dim \mathrm{Ext}^i_0(\mathcal{E}, \mathcal{E})$ (by Serre duality and the simpleness of $\mathcal{E}$) of any nonempty moduli space of stable sheaves $\mathcal{E}$ of Chern classes $(r, c_i)$ on a smooth fibre. $d$ is even.

The penultimate condition ensures that a universal sheaf exists on the product of any fibre and its moduli space ([24] 4.6.6), and also that semistable sheaves on the fibres are in fact stable ([24] 4.6.8 – here is where we use the assumption that the fibres are reduced and irreducible to ensure the ranks of possible destabilising subsheaves are integers.) Thus the fibre moduli spaces are *fine*. But in general this may not be enough to ensure slope stability, so we require the last condition. Under these conditions we will often just talk about stability. For some results we will restrict to fibrations $X \to C$ whose singular fibres have only ordinary/rational double point (ODP/RDP) singularities, since we then understand something about reflexive sheaves on such $K3$ fibres (e.g. [25], [28]).

**Definition 4.4.** We say the above data (4.3) is *very admissible* if each singular fibre of $X \to \mathbb{P}^1$ contains only a single ODP, and $r$ and $c_1 \,.\, \omega$ are coprime.

**Theorem 4.5.** *Choose an admissible triple $(X, r, c_i)$ as in Definition 4.3. Then there is a projective scheme $\mathcal{M} \to C$ with fibres $\mathcal{M}_t$ that are the moduli spaces of stable sheaves of Chern classes $(r, c_i)$ on $X_t$. Suppose that for some smooth fibre $X_t$, $\mathcal{M}_t \neq \emptyset$. Then $\mathcal{M} \to C$ is surjective with generic fibres smooth of dimension $d$, and there is a universal sheaf $T$ on $X \times_C \mathcal{M}$ (unique up to twisting by the pull-back of a line bundle on $C$).*

*Suppose now that the singular fibres of $X \to C$ have only RDPs. If $d = 2$ and $r \geq 2$ then all elements of $\mathcal{M}$ are reflexive on their supporting fibre (and so locally free on smooth fibres). If $d = 0$ and $r \geq 1$, $T$ is locally free, each $\mathcal{M}_t$ is a single reduced point, $\mathcal{M} \cong C$, and so $T$ is a* bundle on $X$.

*Proof.* Simpson's construction of a projective proper $\mathcal{M} \to C$ is now standard [24], given the assumption that there are no semistable sheaves. If we have a stable sheaf $\mathcal{E}$ in the moduli space of a smooth fibre $X_t$ then its deformations are unobstructed since by Serre duality $\mathrm{Ext}^2_0(\mathcal{E}, \mathcal{E}) \cong \mathrm{End}_0(\mathcal{E})^* = 0$ and $\det E$ is unobstructed by the assumptions on $c_1$. Thus it may deformed off $X_t$ (3.45), making $\mathcal{M} \to C$ onto over an open set of $C$, and so onto all of $C$ by properness. Since $\mathrm{Ext}^2_0(\mathcal{E}, \mathcal{E}) = 0$, $\mathcal{M}_t$ is smooth of the correct dimension $d$.

Consider sheaves $\mathcal{E}_t \in \mathcal{M}_t$ to be torsion sheaves on $X$, by pushing them forward to $(\iota_t)_* \mathcal{E}_t$, where $\iota_t : X_t \to X$ is the inclusion. Then $\mathcal{M} \to C$ is part of the moduli space of sheaves on $X$ of the same Hilbert polynomial (in fact deformation theory and stability show it an entire component of the moduli space). There is a universal sheaf on $X \times \mathcal{M}$ as the numerical conditions of ([24] 4.6.6) are satisfied by $(\iota_t)_* \mathcal{E}_t$: they are satisfied by $\mathcal{E}_t$ on $X_t$, as $\gcd(r, c_1 . \omega, \frac{1}{2}c_1^2 - c_2) = 1$, and they only depend on the Hilbert polynomial, which is the same for $\mathcal{E}_t$ and $(\iota_t)_* \mathcal{E}_t$.

The universal sheaf on $X \times \mathcal{M}$ is supported on the image of the diagonal map $X \times_C \mathcal{M} \to X \times \mathcal{M}$, and so defines $T$ on $X \times_C \mathcal{M}$.

The statements about reflexivity of sheaves of rank $r \geq 1$ are standard arguments of Mukai ([24] 6.1.6, 6.1.9) for smooth fibres – taking the double dual of a sheaf does not affect $c_1$ but decreases $c_2$ (look at the exact sequence of the sheaf injecting its double dual, which it does if $r \geq 1$ since it must then be torsion-free by stability), which decreases $d$ (4.3) by $2r$ times as much. Since the double dual is also stable it sits inside a moduli space of the correct dimension, thus $d(\widehat{\mathcal{E}})$ must be greater than or equal to zero. Thus if $d = 0$, or $d = 2$ and $r \geq 2$, the sheaf must be its own double dual and so reflexive.

For a sheaf $\mathcal{E}$ on a singular fibre $X_t$ with only RDPs we again take its double dual $\widehat{\mathcal{E}}$, and then pull this up to the minimal desingularisation $\pi: \widetilde{X}_t \to X_t$ of the fibre (another $K3$), and divide by torsion. This gives a vector bundle $\widetilde{\mathcal{E}}$ on $\widetilde{X}_t$ which we show is stable with respect to the (degenerate) polarisation $\mathcal{O}(1)$ pulled up from $X_t$. Any subsheaf $\mathcal{F} \hookrightarrow \widetilde{\mathcal{E}}$ can be pushed down to a subsheaf of $\pi_*\widetilde{\mathcal{E}}$, and $\pi_*\widetilde{\mathcal{E}} \cong \widehat{\mathcal{E}}$, $R^1\pi_*\widetilde{\mathcal{E}} = 0$ (see e.g. [25]). So for $n \gg 0$,

$$\begin{aligned}\chi(\mathcal{F}(n)) &= \chi(\pi_*\mathcal{F}(n)) - \chi(R^1\pi_*\mathcal{F}) \\ &\leq \chi(\pi_*\mathcal{F}(n)) \leq \chi(\widehat{\mathcal{E}}(n)) \\ &= \chi(\pi_*\widetilde{\mathcal{E}}(n)) = \chi(\widetilde{\mathcal{E}}(n))\end{aligned}$$

by the stability of $\widehat{\mathcal{E}}$, demonstrating the stability of $\widetilde{\mathcal{E}}$. Thus it is also stable for nearby nondegenerate Kähler forms.

In particular $\widetilde{\mathcal{E}}$ is simple, so that its topological invariant $d(\widetilde{\mathcal{E}})$ (4.3) gives the dimension $\operatorname{Ext}^1(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}})$ of the moduli space it sits in. This must be greater than or equal to zero, so the previous argument (in the smooth case) goes through as before to show that $\mathcal{E}$ is locally free for $d(\mathcal{E}) = 0,\ r \geq 1$ and reflexive for $d(\mathcal{E}) = 2,\ r \geq 2$, if we can show the inequality

$$d(\widetilde{\mathcal{E}}) \leq d(\widehat{\mathcal{E}}), \tag{4.6}$$

with equality if and only if $\widehat{\mathcal{E}}$ is locally free (notice that we already know that $d(\widehat{\mathcal{E}}) \leq d(\mathcal{E})$ with equality if and only if $\mathcal{E}$ is reflexive).

So we want to compare the topological invariant $d$ on smooth fibres with $d(\widehat{\mathcal{E}})$ on the resolution of the singular fibre $X_t$. This is more-or-less contained in the work of Langer [28], and I am grateful to him for explaining it to me. He defines a second Chern class for reflexive sheaves such as $\widehat{\mathcal{E}}$ on singular surfaces such as $X_t$, which we will denote by $c_2^L(\widehat{\mathcal{E}})$. This is not the same as what we will denote by $c_2$, namely the class which gives the right contribution to the Riemann-Roch formula (the deformation invariant $c_2(\widehat{\mathcal{E}}) := c_2(\mathcal{E}) - l$, where $l$ is the length of the torsion sheaf cokernel of $\mathcal{E} \hookrightarrow \widehat{\mathcal{E}}$, and $c_2(\mathcal{E})$ is measured on a nearby smooth fibre).

Langer's definition (building on work of Wahl) is given in terms of the sheaf $\widetilde{\mathcal{E}} = (\pi^*\widehat{\mathcal{E}}\,/\,\text{torsion})$ upstairs, as ([28] Section 3)

$$c_2^L(\widehat{\mathcal{E}}) := c_2(\widetilde{\mathcal{E}}) - \sum_y c_2(\widetilde{\mathcal{E}}, y),$$

where $y$ runs over the RDPs of $X_t$, and $c_i(\widetilde{\mathcal{E}}, y)$ is the local $i$th Chern class on its resolution ([28] 2.2, 2.3). He defines ([28] 2.7) $a_y(\widehat{\mathcal{E}})$ to be the local difference between $c_2^L$ and the $c_2$ we use that fits into the Riemann-Roch formula; that is

$$c_2(\widehat{\mathcal{E}}) = c_2^L(\widehat{\mathcal{E}}) - \sum_y a_y(\widehat{\mathcal{E}}).$$

The last result of ([28] Section 6) is that for an RDP $y$,

$$a_y(\widehat{\mathcal{E}}) = \frac{1}{2}c_1^2(\widetilde{\mathcal{E}}, y) - c_2(\widetilde{\mathcal{E}}, y),$$

giving

$$c_2(\widetilde{\mathcal{E}}) = c_2(\widehat{\mathcal{E}}) + \frac{1}{2}\sum_y c_1^2(\widetilde{\mathcal{E}}, y).$$

Also, the local first Chern class satisfies

$$c_1^2(\widetilde{\mathcal{E}}) - c_1^2(\widehat{\mathcal{E}}) = \sum_y c_1^2(\widetilde{\mathcal{E}}, y),$$

so that putting these two formulae together gives

$$\begin{aligned}\big[2rc_2 - (r-1)c_1^2 \ - 2(r^2-1)\big](\widetilde{\mathcal{E}})& \\ - \big[2rc_2 - (r-1)c_1^2 - 2(r^2-1)\big](\widehat{\mathcal{E}})& \\ = \sum_y c_1^2(\widetilde{\mathcal{E}}, y).&\end{aligned} \tag{4.7}$$

Thus $d(\widetilde{\mathcal{E}}) - d(\widehat{\mathcal{E}}) = \sum_y c_1^2(\widetilde{\mathcal{E}}, y)$ is the sum of squares of divisors like $-c_1(\widetilde{\mathcal{E}}, y)$, a positive multiple of which is effective and supported entirely on the exceptional set. This is therefore negative, and zero if and only if $c_1(\widetilde{\mathcal{E}}, y) = 0$ for all $y$, if and only if $\widehat{\mathcal{E}}$ is locally free at all singular points $y$ [25].

Finally, we want to show that if $d = 0$ then $\mathcal{M}_t$ is a single point. This is an argument of Mukai for smooth fibres ([24] 6.1.6) which generalises to singular fibres with RDPs since we have just shown that the sheaves are locally free, so the duality $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E})^* \cong \mathrm{Ext}^{2-i}(\mathcal{E}, \mathcal{E})$ (4.1) holds, which is all that is used. q.e.d.

Alternatively, in the $d = 0$ case, we can give an easier proof of the existence of $T$: using the same diagonal map $X \times_C \mathcal{M} \hookrightarrow X \times \mathcal{M}$ and Luna's étale slice theorem for the quotient map from the relevant Quot scheme to $\mathcal{M}$, $T$ exists locally over $C$. It is simple, so is patched together by nonzero scalars on overlaps of open sets. Thus the obstruction to the patchings satisfying the cocycle condition lies in $H^2(\mathcal{O}_C^*)$, which is zero. We start by analysing the holomorphic Casson invariant in this $d = 0$ case.

**Theorem 4.8.** *Let $(X, r, c_i)$ be as in Definition 4.3, $K3$-fibred over $C = \mathbb{P}^1$, with $d = 0$, $r \geq 1$. If the fibres have only RDPs then the bundle $T \to X$ constructed in Theorem 4.5 is slope stable, isolated, and the only semistable sheaf with the same Chern classes.*

**Remarks.** Though the method of this proof and that of Theorem 4.19 below could be applied to other such "adiabatic limit" problems for stable bundles in any dimension, the conditions we require (that the fibre moduli spaces should be empty for stable sheaves of the same rank and $c_1$, lower $c_2$, and any $c_i$ $i \geq 3$) are so stringent that they are probably only effective on $K3$– (or, with some modifications to take account of the fundamental group, $T^4$–) fibred 3-folds, and Fano-surface-fibred 3-folds. Again, allowing for the fundamental group (i.e., fixing determinants), we can extend the above to any base curve $C$, but we restrict to $\mathbb{P}^1$ for simplicity and because it is the case relevant to Calabi-Yau 3-folds.

Notice we are not claiming that the moduli spaces are so simple for *any* classes on the total space satisfying $2rc_2 - (r-1)c_1^2 - 2(r^2-1) = 0$ on the fibres. Even when a stable bundle with these classes exists on a fibre, we could modify $T$ by giving it ideal-sheaf singularities contained in fibres, or make it unstable on some fibres, or add codimension 3 singularities, to produce stable sheaves in different, more complicated moduli spaces. What is remarkable about $T$'s moduli space is that no such singular phenomena can occur.

**Corollary 4.9.** *Any such $K3$-fibred 3-fold admits isolated bundles if the generic fibre does (with $c_1$ a class coming from the total space), and with respect to some polarisation these bundles are stable and the only point in their moduli space of sheaves. In particular, the relevant holomorphic Casson invariants of $K3$-fibred Calabi-Yau 3-folds are one. The same is true for the dual Chern classes $(-1)^i c_i \in H^{i,i}(X_t)$.*

*Proof.* The point here is to show that the Casson invariant is defined, i.e., that semistability implies stability. But this follows from the choice of polarisation (4.2), implying that slope semistability implies slope stability, which is stronger.

As for deformations, in a polarised family of Calabi-Yau manifolds the fibration structure survives (the obstructions to the survival of a generic fibre $X_t$ in a deformation lie in $H^1(\nu_{X_t|X}) \cong H^1(\mathcal{O}_{K3}) = 0$), and the suitability of the polarisation is preserved (the volume of the fibres is unchanged, for instance).

For the dual Chern classes we take the dual bundle on a smooth fibre and repeat the construction, giving the unique bundle $T^*$ in the moduli space. q.e.d.

*Proof of Theorem.* Denoting the projection map by $\pi : X \to \mathbb{P}^1$, I claim that $\mathcal{E}xt^1_\pi(T,T)$ is zero by standard base-change arguments. Namely: pick a finite, very negative, locally free resolution $E^\bullet \to T$, so that $\operatorname{Ext}^i(E^j|_{X_t}, T|_{X_t}) = 0$ for $i > 0$ and $t \in \mathbb{P}^1$. Then $\mathcal{E}xt^*_\pi(T,T)$ is the cohomology of the complex $F^\bullet = p_* \mathcal{H}om(E^\bullet, T)$. $T$ is flat over $\mathbb{P}^1$ so we can restrict to any fibre $X_t$ to give a resolution of $T|_{X_t}$; thus $F^\bullet|_{X_t} = \operatorname{Hom}_{X_t}(E^\bullet|_{X_t}, T|_{X_t})$ computes $\operatorname{Ext}^*_{X_t}(T|_{X_t}, T|_{X_t})$. Since the higher Exts vanish by construction, and since $T$ is flat over $\mathbb{P}^1$, the dimension of each $\operatorname{Hom}_{X_t}(E^\bullet|_{X_t}, T|_{X_t})$ is constant in $t$. Thus $F^\bullet$ is a complex of *locally frees*. By stability, its zeroth cohomology on each fibre is canonically $\operatorname{End}(T|_{X_t}) = \mathbb{C}\,.\,\mathrm{id}$, so we get a nowhere vanishing map $\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\sim} \ker(F^0 \xrightarrow{d^0} F^1)$, and the image $d^0(F^0/\mathcal{O}_{\mathbb{P}^1}) \subset F^1$ is locally free. Therefore $\mathcal{E}xt^1_\pi(T,T)$ is the kernel of the map $\widehat{d^1}$ that $d^1$ induces on the locally free cokernel of $d^0$. Similarly $\operatorname{Ext}^1(T|_{X_t}, T|_{X_t})$ is the kernel of $\widehat{d^1}|_{X_t}$ on $\operatorname{coker} d^0|_{X_t}$. But this is zero for $X_t$ a smooth fibre, so that $\widehat{d^1}$ is generically an injection of vector bundles. Thus its kernel, as a map of sheaves, is zero, and $\mathcal{E}xt^1_\pi(T,T)$ vanishes.

So the Leray spectral sequence for $\pi_*\mathcal{E}nd_0(T)$ and $\mathcal{E}xt^i_\pi(T,T)$ shows that $T$ is isolated. (Alternatively, the proof below extended to $X \times \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ shows there is a unique bundle on this thickened space.) Stability follows from fibrewise stability and the choice of polarisation.

What we want to prove is that any stable sheaf $\mathcal{E}$ of the same Chern classes is isomorphic to $T$. The basic idea is that, firstly, taking double duals decreases $c_2$ and the fibrewise moduli spaces of lower $c_2$ are empty, and secondly, that if sheaves have fibres on which they are unstable, this only ever increases $c_2$. Playing these two phenomena off against each other ensures we have no non-reflexive sheaves, and no unstable fibres.

We first replace the stable sheaf $\mathcal{E}$ by its double dual $\widehat{\mathcal{E}}$, which on the generic fibre is the double dual of the restriction of $\mathcal{E}$ to that fibre, and hence slope stable and locally free there.

We have a sequence

$$0 \to \mathcal{E} \to \widehat{\mathcal{E}} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a sheaf supported on a subvariety $Z$ of codimension two or higher. Therefore

$$c_2(T).\,\omega = c_2(\mathcal{E}).\,\omega = c_2(\widehat{\mathcal{E}}).\,\omega + \int_Z \mathrm{rk}_Z(\mathcal{F})\omega \geq c_2(\widehat{\mathcal{E}}).\,\omega, \tag{4.10}$$

with a similar inequality $c_2(\widehat{\mathcal{E}}|_{X_t}) \leq c_2(T|_{X_t})$ on a generic fibre $X_t$. But the dimension $d$ of the moduli space on a smooth fibre decreases by $2r \geq 2$ for every decrease in $c_2$, and must remain non-negative on the generic fibre where $\widehat{\mathcal{E}}|_{X_t}$ is stable. Since $d = 0$ we see that $\widehat{\mathcal{E}}|_{X_t}$ is reflexive and in the same moduli space as $T|_{X_t}$, and so it *is* $T|_{X_t}$, recalling that the moduli space is a single point.

Therefore, using stability, on the generic fibre $\mathrm{Hom}\,(T|_{X_t}, \widehat{\mathcal{E}}|_{X_t})$ is a copy of $\mathbb{C}$, so

$$\pi_* \,\mathcal{H}om\,(T, \widehat{\mathcal{E}})$$

is a rank one torsion free sheaf on $\mathbb{P}^1$, i.e., $\mathcal{O}(-n)$ for some $n$. This gives the exact sequence

$$0 \to T(-n) \xrightarrow{\phi} \widehat{\mathcal{E}} \to Q \to 0, \tag{4.11}$$

where $T(-n)$ denotes the twist of $T$ with the pullback of $\mathcal{O}(-n)$ from $\mathbb{P}^1$, and $Q$ is a sheaf supported on some finite number $d$ of possibly singular fibres $\{X_{t_i}\}_{i=1}^d$. (Some of these fibres might be infinitely close; $d$ is the total number counted with multiplicities: the length of the scheme in $\mathbb{P}^1$ over which the union of the fibres sit.) Write $Q = \oplus_{i=1}^d \iota_* Q_i$, where $Q_i$ is a sheaf supported on $X_{t_i}$.

Since $\widehat{\mathcal{E}}$ is torsion free we have the sequence

$$0 \to \widehat{\mathcal{E}}(-1) \to \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}|_{X_{t_i}} \to 0,$$

inducing

$$0 \to \mathrm{Hom}(T(-n+1), \widehat{\mathcal{E}}) \to \mathrm{Hom}(T(-n), \widehat{\mathcal{E}}) \to \mathrm{Hom}(T(-n), \widehat{\mathcal{E}}|_{X_{t_i}}).$$

Therefore if $\phi$ in (4.11) is zero on $X_{t_i}$ it comes from an element of the first group in the above sequence. Thus, by reducing $n$ if necessary, we may assume that $\phi|_{X_{t_i}} \neq 0$, so that the rank of $Q_i$ is at most $r-1$ on its support.

Since $\widehat{\mathcal{E}}$ is reflexive on a smooth variety, it has homological dimension at most one, i.e., $\mathcal{E}xt^i(\widehat{\mathcal{E}}, \mathcal{F}) = 0 \ \forall i \geq 2$, for all coherent sheaves $\mathcal{F}$. Also $T(-n)$ is locally free, so from (4.11) we see that $\mathcal{E}xt^i(Q, \mathcal{F}) = 0 \ \forall i \geq 2$, and $Q$ has homological dimension at most one. Therefore it is supported in exactly codimension one; i.e., the rank of each $Q_i$ is at least 1 (since there are no reducible fibres). That is $1 \leq r_i \leq r-1$, where $r_i = \operatorname{rk} Q_i$.

Now the slope stability of $T|_{X_{t_i}}$, and the sequence

$$T|_{X_{t_i}} \to \widehat{\mathcal{E}}\,|_{X_{t_i}} \to Q_i \to 0,$$

imply that the slope of $Q_i$ is less than that of $T|_{X_{t_i}}$. That is,

$$c_1(Q_i)\,.\,\omega|_{X_{t_i}} < \frac{r_i}{r} c_1(T)\,.\,\omega_1\,.\,\omega, \tag{4.12}$$

where $\omega_1$ is the pullback via $\pi$ of the standard Kähler form on $\mathbb{P}^1$.

By the Grothendieck-Riemann-Roch theorem (or more elementary considerations), $ch(\iota_* Q_i) = \iota_* ch(Q_i)$, where $\iota : X_{t_i} \to X$ is the inclusion of a fibre (in particular it has a trivial normal bundle), and $ch$ is the Chern character. Therefore

$$c_1(\iota_* Q_i) = r_i\,\omega_1 \qquad \text{and} \qquad c_2(\iota_* Q_i) = -\iota_* c_1(Q_i),$$

since $c_1(\iota_* Q_i)^2 = 0$. (The pushforward on cohomology $\iota_*$ increases the (complex) degree by one in cohomology; it is Poincaré dual to the inclusion on homology.)

Combining this with (4.11), and denoting the total Chern class by $c$, we have

$$\begin{aligned} c(\widehat{\mathcal{E}}\,) &= c(T(-n)) \prod_{i=1}^{d} c(\iota_* Q_i) \\ &= \Big(1 + c_1(T) - rn\omega_1 + c_2(T) - (r-1)n\omega_1 c_1(T)\Big) \\ &\qquad \qquad . \left(1 + \sum_{i=1}^{d} r_i \omega_1 - \sum_{i=1}^{d} \iota_* c_1(Q_i)\right) \end{aligned}$$

up to second degree in cohomology. Since $\mathcal{E}$ and $T$ have the same Chern classes, this gives

$$\begin{aligned} c_2(\widehat{\mathcal{E}}\,) - c_2(\mathcal{E}) = &\left(\sum_{i=1}^{d}(r_i) - rn\right)\omega_1 \\ &+ \left(\sum_{i=1}^{d}(r_i) - (r-1)n\right)\omega_1 c_1(T) \\ &- \sum_{i=1}^{d}\iota_* c_1(Q_i). \end{aligned}$$

The degree one piece thus gives

$$\sum_{i=1}^{d} r_i = rn,$$

which with (4.12) yields

$$\sum_{i=1}^{d} c_1(Q_i)\,.\,\omega|_{X_{t_i}} < n\,c_1(T)\,.\,\omega_1\,.\,\omega, \tag{4.13}$$

unless the number of fibres $d$ is zero, in which case both sides vanish. Taking the cup product of the second order piece with $\omega$, however, gives

$$n\,c_1(T)\,.\,\omega_1\,.\,\omega - \sum_{i=1}^{d} c_1(Q_i)\,.\,\omega|_{X_{t_i}} = (c_2(\widehat{\mathcal{E}}\,) - c_2(\mathcal{E}))\,.\,\omega,$$

which, by (4.10), is nonpositive. Therefore (4.13) cannot hold, and so $d = 0 = n$, $Q = 0$, and (4.11) becomes

$$0 \to T \to \widehat{\mathcal{E}} \to 0.$$

Thus the Chern classes of $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are the same (they are both equal to that of $T$) and $\mathcal{E} \cong \widehat{\mathcal{E}} \cong T$. q.e.d.

**Examples**

- Consider the Calabi-Yau 3-fold that is a smooth (2,2,3) divisor in a product of projective spaces,

$$X_{2,2,3} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2.$$

The projection $\pi_1$ to the first $\mathbb{P}^1$ exhibits $X$ as a $K3$ fibration, with fibre a (2,3) divisor in $\mathbb{P}^1 \times \mathbb{P}^2$. In turn this fibre is a double cover of $\mathbb{P}^2$ branched over a sextic – the zero locus of the discriminant of the quadratic on $\mathbb{P}^1$ (which has coefficients that are cubics on $\mathbb{P}^2$) defining the $K3$ fibre. It is well known that the pullback of the tangent bundle of $\mathbb{P}^2$ is an isolated slope stable bundle on any such $K3$ with respect to the pullback of the polarisation on $\mathbb{P}^2$ (the proof in [11] 9.1.8 works even for the singular covers). So taking the polarisation $\pi_1^*\mathcal{O}(N)\otimes\pi_3^*\mathcal{O}(1),\ \ N \gg 0$, in the obvious notation, we see that

$$\pi_3^*\, T\,\mathbb{P}^2 \to X$$

is slope stable and unique in its moduli space of sheaves, giving a holomorphic Casson invariant of one. The same is true of the pullback of $T\mathbb{P}^2$ to a more general double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a smooth (4,6)-divisor.

- On $K3 \times T^2$, fixing determinants to be trivial in the $T^2$ direction, the method of the proof of the theorem shows that, given an isolated slope stable bundle on $K3$ with $\gcd\left(r, c_1 \,.\, \omega, \frac{1}{2}c_1^2 - c_2\right) = 1$, we get a holomorphic Casson invariant of $r^2$ by pulling the bundle back from $K3$ and twisting by line bundles on $T^2$ whose $r$th power is trivial. (Notice that here we are taking a polarisation with the $T^2$ fibres large, the opposite of [16].) It would be especially interesting to study the higher dimensional moduli spaces. The parts of the moduli space arising as pullbacks from $K3$ are smooth and the corresponding invariant would be the Euler characteristic of the moduli space, to within a sign. Generating functions of these give modular forms, by the work of Vafa and Witten [45], so it would be interesting to compute the corrections given by other parts of the moduli space. For instance, the next section will show there are no corrections for the 2-dimensional moduli spaces and we get a Casson invariant

$$\lambda(K3 \times T^2) = r^2\chi(\mathcal{M}_{K3}).$$

- It would be fashionable, and by now almost expected, to find modular forms arising from the Casson invariants on more general Calabi-Yau manifolds. An easy, cheating, way to produce them is to take the moduli spaces of stable bundles on Fano surfaces in [45], and push these forward to give torsion sheaves on any Calabi-Yau 3-fold that is modeled locally on the total space of the canonical bundle over the Fano. Deformation theory (3.42) shows that all sheaves in the same component of the moduli space are of the same form, so the holomorphic Casson invariant is the Euler characteristic of the moduli space if it is smooth, and some appropriate modification of it if not. (In fact this gives a way of rigorously defining the Euler characteristics of moduli spaces in [45] so long as there are no semistable sheaves.) Thus generating functions of the invariants (for Chern classes on the Calabi-Yau that are pushforwards from the Fano surface) do indeed give modular forms in the cases studied in [45].

- We can also consider the two examples of Mark Gross [20], [36]. Let $E_1$ be the trivial rank four bundle $\mathcal{O}^{\oplus 4}$ over $\mathbb{P}^1$, $E_2 = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$, and $P_i = \mathbb{P}(E_i)$. Let the $X_i$ be anticanonical divisors in $P_i$ (Gross shows they can be chosen smooth). They are clearly both fibred $\pi : X_i \to \mathbb{P}^1$ by $K3$ s that are quartics in $\mathbb{P}^3$. Their second cohomologies are generated by $t = c_1(\mathcal{O}_{\mathbb{P}(E_i)}(1))$ and the fibre class $f = \pi^*\omega_1$. We use a polarisation $t + Nf,\ N \gg 0$.

The $X_i$ are in fact diffeomorphic via a diffeomorphism taking $t$ and $f$ on $X_1$ to $t$ and $f$ on $X_2$. They are not, however, deformation equivalent as projective or even symplectic manifolds (by work of P. M. H. Wilson in the first case, and Gross and Ruan in the second [36]). We might hope to be able to prove the projective statement (or even both if Tian's work mentioned in Section 3 can be developed to give symplectic invariants) using the holomorphic Casson invariant. The most obvious bundle to take is, in the first case,

$$\pi^* T\,\mathbb{P}^3 \to X_1.$$

**Proposition 4.14.** *The restriction of $T\,\mathbb{P}^3$ to* any *irreducible, reduced quartic in $\mathbb{P}^3$ is slope stable (with respect to the restriction of the Fubini-Study metric), isolated, and unique in its moduli space.*

*Proof.* While it is well known that $T\,\mathbb{P}^3$ is stable on the generic quartic this is not enough for us – it is a priori possible that the unique stable bundle on each quartic is generically $T\,\mathbb{P}^3$ but something else on a closed subset of quartics. The other statements follow by the usual methods if we can show that $T\,\mathbb{P}^3$ is stable, since it is locally free and so satisfies Serre duality even on the singular quartics.

So fix a (possibly singular) quartic $S \subset \mathbb{P}^3$. Suppose firstly that $T\,\mathbb{P}^3|_S$ is (slope) destabilised by a rank two subsheaf. Then the quotient map gives us a sequence

$$T\,\mathbb{P}^3(-1)|_S \to \mathcal{L} \to 0$$

on $S$, with $\mathcal{L}$ a torsion free (without loss of generality) rank one sheaf. $\mathcal{L}$ is generated by its sections since $T\,\mathbb{P}^3(-1)$ is, and of degree less than or equal to 1. If $\mathcal{L}$ were trivial then the dual of the above sequence would give $\Omega_{\mathbb{P}^3}(1)|_S$ a section which the restriction sequence from $\mathbb{P}^3$ shows it does not have. Therefore $\mathcal{L}$ must have degree 1 and at least 2 sections to be generated by them.

Sections of $\mathcal{L}$ vanish on a degree one curve in $S \subset \mathbb{P}^3$, which must therefore be a line. Taking two sections of $\mathcal{L}$ we get two homologous distinct $\mathbb{P}^1$ s in $S$ which is a contradiction, since they must have self-intersection $-2$ by adjunction.

Since $S$ is irreducible and reduced it is now sufficient to consider $T\,\mathbb{P}^3(-1)$ being destabilised by a rank one subsheaf. Dualising, we get a sequence

$$\Omega_{\mathbb{P}^3}(2)|_S \to \mathcal{L} \to 0,$$

for some other torsion free (without loss of generality) rank one sheaf $\mathcal{L}$. Again this shows that $\mathcal{L}$ is generated by its sections, and is of degree $\leq 2$ this time. But degree 2 curves in $\mathbb{P}^3$ are also $\mathbb{P}^1$ s and a similar argument applies. q.e.d.

Therefore, by Theorem 4.8, $\pi^*\,T\,\mathbb{P}^3 \to X_1$ is the unique stable sheaf in its moduli space with total Chern class $1 + 4t + 6t^2 + 4t^3$. Thus we have a holomorphic Casson invariant of one, but it does not distinguish the $X_i$s since for the corresponding (under the diffeomorphism) Chern class $1 + 4t + 6t^2 + 4t^3$ on $X_2$ the invariant is also one, given by the relative tangent bundle down the fibres of $\mathbb{P}(E_2)$:

$$T_\pi\,\mathbb{P}(E_2)|_{X_2} \to X_2.$$

So to distinguish the $X_i$ s we have to consider two dimensional moduli spaces on the $K3$ fibres.

## The fibre dimension two case

To study bundles with $d = 2$ dimensional moduli spaces on the fibres we follow the same method as before, considering sections of the fibration of fibrewise moduli spaces – the "Mukai-dual" 3-fold $\mathcal{M} \to \mathbb{P}^1$ constructed by applying Mukai duality (replacing a $K3$ by a 2-dimensional moduli space of sheaves on it – another $K3$) fibrewise as in Theorem 4.5. We will show that $\mathcal{M}$ is Calabi-Yau.

In this two dimensional situation a new feature can occur, namely the sections can change homology class. However, just as introducing codimension two singularities and unstable fibres drives $c_2$ higher, we shall find that changing the section to one of higher degree does the same (where degree is measured appropriately). Thus for sections of the smallest possible degree this cannot happen in a fixed moduli space.

Relating the deformation theories of the bundles and the sections will show the Casson invariant of $X$ is equal to the appropriate Gromov-Witten invariant of $\mathcal{M}$. Thus we see Gromov-Witten invariants arising, but not in the way we might have expected (with rank 2 bundles corresponding to curves via zero sets of their sections), and in fact for arbitrary rank.

Throughout this section we will work with $K3$ fibrations whose singular fibres have only single ODPs. Many of the results are true for arbitrary reduced irreducible fibres (and in fact all of them should be, but I cannot prove it). We start with some technicalities improving on Theorem 4.5.

**Theorem 4.15.** *Fix $(X \to \mathbb{P}^1, r, c_i)$ as in Definition 4.4, with $d = 2$ and $r \geq 2$. Then the moduli space $\mathcal{M} \to \mathbb{P}^1$ is smooth with fibres $\mathcal{M}_t$ having only ODPs as singularities. The singular points represent the reflexive non locally free sheaves on $X_t$; all other points correspond to vector bundles.*

*Proof.* These results were conjectured in an earlier version of this paper based on comparing monodromy in $\mathcal{M} \to \mathbb{P}^1$ around singular fibres with the corresponding monodromy in $X \to \mathbb{P}^1$ (using Mukai's natural isomorphisms between the cohomologies of the fibres $X_t$ and $\mathcal{M}_t$ as in ([HL] 6.1.14); see Theorem 4.24 below). Since then, however, a remarkable proof of the smoothness of $\mathcal{M}$ using Fourier-Mukai transforms has appeared [10] (this applies directly in our case, even without assuming RDP fibres: the assumptions on Chern classes (4.3) make the fibrewise moduli spaces of "Fourier-Mukai type" in the terminology of [10] – the Mukai vector of the sheaves is primitive of square zero). The remaining results can be deduced from standard theory of reflexive sheaves on ODPs, and I would like to thank Akira Ishii for explaining this theory to me.

From Theorem 4.5 all sheaves in $\mathcal{M}$ are reflexive, and locally free if supported on a smooth fibre. So consider a reflexive, non locally free sheaf $\mathcal{E}$ on a singular fibre $X_t$. As in the proof of Theorem 4.5, consider $\widetilde{\mathcal{E}} = \pi^*\mathcal{E}/\,$torsion on the minimal resolution $\pi : \widetilde{X}_t \to X_t$. We proved that $\widetilde{\mathcal{E}}$ was simple and had $d(\widetilde{\mathcal{E}}) < d(\mathcal{E}) = 2$, so that it sits in a moduli space of dimension zero:

$$\operatorname{Ext}^1(\widetilde{\mathcal{E}},\widetilde{\mathcal{E}}) = 0.$$

In fact from equation (4.7) we see that the local first Chern class satisfies $c_1^2(\widetilde{\mathcal{E}}, y) = -2$, where $y$ is the ODP (for instance, dimension considerations force it to be between $-1$ and $-3$, and it must be even since this is the only ODP, and the intersection form on $K3$ is even).

Thus standard theory of reflexive modules (see e.g. [25] Example 3.2) implies that $\widetilde{\mathcal{E}}$, restricted to the exceptional $-2$-sphere $Z \subset \widetilde{X}_t$, is

$$\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

The local deformations of the corresponding reflexive module over the ODP are then given by ([25] Theorem 4.3 (2)) pushing down bundles whose restriction to $Z$ is any deformation of

$$\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \tag{4.16}$$

(or $\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, but this gives the same local reflexive module). Pushdowns of deformations of the above bundle give the 2 dimensional local moduli space of deformations of the sheaf. This has an ODP (the original reflexive sheaf) away from which the sheaves are all locally free ([25] Theorem 4.6), corresponding to the deformation $\mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ of (4.16) on the resolution.

Now $H^2(\mathcal{E}nd\,\mathcal{E}) \cong \operatorname{Hom}(\mathcal{E}nd\,\mathcal{E}, K_{X_t})^*$ by Serre duality, and this is $\mathbb{C}$ ($K_{X_t}$ is trivial and $\mathcal{E}nd(\mathcal{E})^* \cong \mathcal{E}nd\,\mathcal{E}$ since it is reflexive – it is the pushforward of the restriction of itself to the smooth locus of a normal surface). (Thanks to Akira Ishii for this argument.)

The local-to-global Ext spectral sequence yields

$$0 \to H^1(\mathcal{E}nd\,\mathcal{E}) \to \operatorname{Ext}^1(\mathcal{E},\mathcal{E}) \to H^0(\mathcal{E}xt^1(\mathcal{E},\mathcal{E})) \to H^2(\mathcal{E}nd\,\mathcal{E}),$$

where the last term is all trace $H^2(\mathcal{O}_{X_t}) \cong \mathbb{C}$ that survives in the spectral sequence for $\operatorname{Ext}^2(\mathcal{E},\mathcal{E})$. Thus the penultimate map is onto, and global deformations of $\mathcal{E}$ map *onto* local deformations $\mathcal{E}xt^1(\mathcal{E},\mathcal{E})$ of the module over the ODP. But both have dimension 3 (global deformations of $\mathcal{E}$ are the deformations of a singular point in a surface $\mathcal{M}_t$ in a smooth 3-fold $\mathcal{M}$, so have Zariski tangent space of dimension 3), so we get a local isomorphism from the global deformations to the local model described above. In particular we see that there are locally free deformations of

$\mathcal{E}$, the moduli space is 2 dimensional, and the non locally free sheaf is isolated in its moduli space and represents an ODP. q.e.d.

We need to fix an appropriate polarisation on $\mathcal{M} \to \mathbb{P}^1$. In fact, all that will concern us will be its restriction to the fibres, so we consider a polarised $K3$ surface $S$ with at worst an ODP singularity $y \in S$, and $\mathcal{M}_S$ a moduli space of semistable sheaves of fixed determinant such that rank and degree are coprime. Let $H$ be a generic smooth hyperplane section of $S$ missing its singularity $y$, $p: S \times \mathcal{M}_S \to \mathcal{M}_S$ and $q: S \times \mathcal{M}_S \to S$ be the projections, and let $\mathcal{E}$ be a (local) universal sheaf. Then there is an obvious class $[\mathcal{E}nd_0(\mathcal{E}|_H)]$ in the $K$-group $K^0(S \times \mathcal{M}_S)$ which is just $\mathcal{E}nd_0\mathcal{E}$ if $\mathcal{E}$ is locally free, and more generally the kernel of the trace map on the restriction to $H$ of the tensor product of a finite locally free resolution of $\mathcal{E}|_{S\backslash\{y\}}$ with the dual complex. This is uniquely defined even though $\mathcal{E}$ is only defined locally up to tensoring with a line bundle pulled back from $\mathcal{M}$.

Likewise the push down

$$p_! : K^0(S \times \mathcal{M}_S) \to K^0(\mathcal{M}_S)$$

([24] 2.1.11) is $p_! = p_* - R^1p_* + R^2p_*$, or $p_*$ on a finite resolution by sheaves with no higher cohomology (this always exists [21] III 2.7). Then the result we need is the following.

**Lemma 4.17.** *In the above set-up and notation, the determinant line bundle*

$$(\det p_!\, \mathcal{E}nd_0\mathcal{E}|_H)^*$$

*is ample on $\mathcal{M}_S$.*

*Proof.* Fix a point $x \in S\backslash\{x\}$ and let $|_{\{x\}\times\mathcal{M}_S}$ denote $\otimes\, q^*\mathcal{O}_x$ in the K-group (resolve by locally frees away from $y$ and restrict to $x$). Let $\chi = \chi(\mathcal{E}|_H) = \chi(\mathcal{E}) - \chi(\mathcal{E}(-1))$. Then it is a result of Jun Li ([30], [24] Section 8.2) that

$$\det p_!\, (\chi\mathcal{E}|_{\{x\}\times\mathcal{M}_S} - r\mathcal{E}|_H)$$

is ample on the moduli space of slope stable sheaves (which for us is all of $\mathcal{M}_S$) *for $S$ smooth.* But the proof goes through for surfaces with isolated singularities for rank and degree coprime: for a fixed *bundle* $\mathcal{E}$ the restriction to the generic curve in $|\mathcal{O}(a)|$, $a \gg 0$ is semistable (Flenner's theorem ([24] 7.1) applies to normal varieties) and so stable if $a$ is coprime to the rank, and the generic curve misses the singularities and is smooth. Now proceed as in ([24] 8.2) to deduce ampleness of Jun

Li's line bundle on the locus of locally free sheaves. As the non locally free sheaves are isolated in the moduli space (4.15) this is sufficient.

Thus it is sufficient to show that the two line bundles have the same first Chern class.

But by Grothendieck-Riemann-Roch we have

$$\begin{aligned} c_1\, p_!\,(\chi\mathcal{E}|_{\{x\}\times\mathcal{M}_S} - r\mathcal{E}|_H) &= p_*\big[ch\,(\chi\mathcal{E}|_{\{x\}\times\mathcal{M}_S} - r\mathcal{E}|_H)\ \mathrm{Td}(H)\big]_2 \\ &= p_*\big(-r\,ch_2(E|_H)\big) \\ &\qquad + c_1(\mathcal{E}|_{\{x\}\times\mathcal{M}_S})\left(\chi - \frac{1}{2}r\,(2-2g)\right), \end{aligned}$$

where $g$ is the genus of $H \subset S$. Therefore $\chi - \frac{1}{2}r\,(2-2g)$ is the degree of $\mathcal{E}|_H$, i.e., $p_*\,c_1(\mathcal{E})$, so we obtain

$$\begin{aligned} c_1\, p_!\,(\chi\mathcal{E}|_{\{x\}\times\mathcal{M}_S} - r\mathcal{E}|_H) &= p_*\big(-r\,ch_2(\mathcal{E}|_H) + \frac{1}{2}c_1(\mathcal{E}|_H)^2\big) \\ &= p_*\big(r\,c_2(\mathcal{E}|_H) - \frac{1}{2}(r-1)\,c_1(\mathcal{E}|_H)^2\big) \\ &= -\frac{1}{2}\,p_*\,ch_2(\mathcal{E}nd_0\mathcal{E}|_H). \end{aligned} \tag{4.18}$$

But since $ch_1(\mathcal{E}nd_0\mathcal{E}|_H) = 0$, Grothendieck-Riemann-Roch equates the last term with $\frac{1}{2}c_1\big((p_!\,(\mathcal{E}nd_0\mathcal{E}|_H))^*\big)$. q.e.d.

So consider again the relative moduli space $\mathcal{M} \to \mathbb{P}^1$, with a (local) universal sheaf $T$. Let $p$ be the projection $p: X \times_{\mathbb{P}^1} \mathcal{M} \to \mathcal{M}$, and pick a smooth hyperplane $H$ of $X$, flat over $\mathbb{P}^1$. Then the first Chern class $\Omega$ of the line bundle $(\det p_!\,(\mathcal{E}nd_0 T|_H))^*$ is, by the above lemma, ample on the fibres $\mathcal{M}_t$.

So denoting by $\omega_1$ the pull-back to $\mathcal{M}$ of the Fubini-Study form on $\mathbb{P}^1$, $N\omega_1 + \Omega$ is ample on $\mathcal{M}$ for $N$ sufficiently large. $N$ will be unimportant for us (we are concerned with sections of $\mathcal{M} \to \mathbb{P}^1$ which all have fixed degree measured against $\omega_1$), so we fix one such $N$ and measure the degree of sections against the above polarisation.

In the following, for any section $t: \mathbb{P}^1 \to \mathcal{M}$ we will denote by $\tilde{t}$ the induced section $\tilde{t}: X \to X \times_{\mathbb{P}^1} \mathcal{M}$.

**Theorem 4.19.** *Let $(X, r, c_i)$ satisfy Definition 4.4. Denote by $T$ a universal sheaf on $X \times_{\mathbb{P}^1} \mathcal{M}$ as produced by (4.5), and choose a section $\sigma$ of $\mathcal{M} \to \mathbb{P}^1$ of strictly minimal degree with respect to the polarisation $N\omega_1 + \Omega$ described above. Then all stable sheaves with the same Chern classes as $\tilde{\sigma}^* T$ are pull-backs of $T$ by sections in the same homology class, and these are locally free.*

**Remarks.** The theorem should be true in greater generality, without the assumptions of Definition 4.4. For fibrations with worse singular fibres one can extend the proof, getting slightly weaker results. In fact one can show, using [10] to study the obvious Fourier-Mukai transform on a singular fibre $X_t \times \mathcal{M}_t$, that smooth points of the moduli space are in one-one correspondence with locally free sheaves. Understanding when the singularities of $\mathcal{M}_t$ are isolated is more difficult, and stops one proving the ampleness (4.17) of Jun Li's line bundle. But in examples this is obvious, so the results still apply.

As mentioned earlier the theorem also applies to 3-folds fibred by surfaces with negative canonical bundle, and arbitrary base curves if we take account of the fundamental group. The $K3$-fibred case is most relevant to us, however.

*Proof.* We want to study an arbitrary stable sheaf $\mathcal{E}$ with the same Chern classes as $\tilde{\sigma}^* T$. We follow the proof of Theorem 4.8, adapting the argument where necessary.

As in (4.8) the restriction of the double dual $\widehat{\mathcal{E}}$ to the generic fibre must lie in the same moduli space as $T$, giving us a rational (and so regular) section $s : \mathbb{P}^1 \to \mathcal{M}$. Again, as before, this gives us a sequence

$$0 \to \tilde{s}^* T(-n) \xrightarrow{\phi} \widehat{\mathcal{E}} \to Q \to 0,$$

for some $n$, with $Q = \oplus_{i=1}^d \iota_* Q_i$ supported on a finite number of fibres $\{X_{t_i}\}_{i=1}^d$, rank $(Q_i) = r_i$, $1 \leq r_i \leq r$, and $\phi|_{X_{t_i}} \neq 0$.

Computing Chern classes as in (4.8) gives, for $c_1$,

$$c_1(\tilde{\sigma}^* T) - c_1(\tilde{s}^* T) = \left( \sum_{i=1}^d (r_i) - rn \right) \omega_1, \tag{4.20}$$

which, with the stability inequality (4.12), yields

$$\begin{aligned} &\sum_{i=1}^d c_1(Q_i) \,.\, \omega|_{X_{t_i}} \\ &\qquad \leq \left[ n\,\omega_1 + \frac{1}{r}\big(c_1(\tilde{\sigma}^* T) - c_1(\tilde{s}^* T)\big) \right] . \, c_1(\tilde{s}^* T) \,.\, \omega, \end{aligned} \tag{4.21}$$

with the inequality strict if $d \neq 0$. For $c_2$ we obtain

$$
\begin{aligned}
c_2(\widehat{\mathcal{E}}\,) = &- \sum_{i=1}^{d} \iota_* c_1(Q_i) \\
&+ \left( \sum_{i=1}^{d} (r_i) - (r-1)n \right) \omega_1 \,.\, c_1(\tilde{s}^* T) \\
&+ c_2(\tilde{s}^* T).
\end{aligned}
$$

Proceeding as before, with (4.21) this yields

$$
\begin{aligned}
0 &\geq (c_2(\widehat{\mathcal{E}}\,) - c_2(\mathcal{E})) \,.\, \omega \\
&\geq \left(1 - \frac{1}{r}\right) \big[c_1(\tilde{\sigma}^* T) - c_1(\tilde{s}^* T)\big] \,.\, c_1(\tilde{s}^* T) \,.\, \omega \\
&\quad + \big[c_2(\tilde{s}^* T) - c_2(\tilde{\sigma}^* T)\big]\, \omega,
\end{aligned}
$$

with a strict inequality for $d \neq 0$.

Now $\big[c_1(\tilde{\sigma}^* T) - c_1(\tilde{s}^* T)\big]^2 = 0$ from (4.20) (this also shows that the inequality is independent of twisting $T$ by a line bundle on $\mathbb{P}^1$, as it should be), so we may rewrite the first term in the above inequality to arrive at

$$
0 \geq \left(\tilde{s}^* - \tilde{\sigma}^*\right) \omega \,.\, \left[ c_2(T) - \frac{1}{2} \left(1 - \frac{1}{r}\right) c_1(T)^2 \right].
$$

The idea now is that the quantity in the square brackets is the discriminant ([24] 3.4) of $T$ (divided by $2r$), which, by stability, should be positive in some sense. In fact, letting $H$ be a smooth hyperplane section dual to $\omega$, flat over $\mathbb{P}^1$ as before, we have

$$
0 \geq \int_X (\tilde{\sigma}^* - \tilde{s}^*)\ \omega \,.\, ch_2\left(\mathcal{E}nd_0 T\right) = \int_H (\tilde{\sigma}^* - \tilde{s}^*)\ ch\left(\mathcal{E}nd_0 T|_H\right)\ \mathrm{Td}(T_p).
$$

Here $p$ denotes the projection $p : H \times_{\mathbb{P}^1} \mathcal{M} \to \mathcal{M}$, with relative tangent bundle the class $T_p$ in K-theory, and the other terms in $(ch \,.\, \mathrm{Td})$ do not contribute as $ch_1 = 0$ and $ch_0\ \mathrm{Td}(T_p)$ is the same under $\tilde{s}^*$ and $\tilde{\sigma}^*$. Also $|_H$ is meant in the sense of multiplication by $\mathcal{O}_{H \times_{\mathbb{P}^1} \mathcal{M}}$ in K-theory, as before.

Thus, by Grothendieck-Riemann-Roch, we have

$$
0 \geq \int_{\mathbb{P}^1} (\sigma^* - s^*)\, p_*\big(ch(\mathcal{E}nd_0 T|_H)\ \mathrm{Td}(T_p)\big) = \int_{\mathbb{P}^1} (\sigma^* - s^*)\ ch\left(p_!\, \mathcal{E}nd_0 T|_H\right),
$$

where $p_! = p_* - R^1 p_*$. Thus, finally,

$$0 \geq \int_{\mathbb{P}^1} (s^* - \sigma^*)\ c_1\big((\det p_!\, \mathcal{E}nd_0 T|_H)^*\big).$$

Letting $\Omega = c_1\big((\det p_!\, \mathcal{E}nd_0 T|_H)^*\big)$, which is ample on the fibres, $N\omega_1 + \Omega$ is a polarisation for $N \gg 0$. $s$ and $\sigma$ are both sections of $\mathcal{M} \to \mathbb{P}^1$ and so have the same degree with respect to $\omega_1$, and so we arrive at

$$0 \geq \int_{\mathbb{P}^1} (s^* - \sigma^*)\, \Omega = \int_{\mathbb{P}^1} (s^* - \sigma^*)\ (N\omega_1 + \Omega).$$

Thus if, as in the conditions of the theorem, $\sigma$ is of strictly minimal degree then we must have $s$ and $\sigma$ in the same homology class, and, as the inequality is now non-strict, $d = 0 = Q$. By (4.20) $n = 0$, and so $\tilde{s}^*\, T \cong \widehat{\mathcal{E}} \cong \mathcal{E}$.

Finally, sections $s : \mathbb{P}^1 \to \mathcal{M}$ miss the singular points of fibres where the derivative of the projection to $\mathbb{P}^1$ vanishes. Thus $s$ takes its image in the smooth points of $\mathcal{M}$, which correspond to locally free sheaves on the fibres by (4.15). Thus $\tilde{s}^*\, T$ is locally free. q.e.d.

**Corollary 4.22.** *Under the conditions of Theorem 4.19, the holomorphic Casson invariant of $X$ equals the algebraic Gromov-Witten invariant $GW_{\mathcal{M}}([\sigma])$ of $\mathcal{M}$ (as defined in* [31], [6]*) counting curves in the same homology class as $\sigma$.*

*Proof.* Let $\mathcal{N}$ denote the moduli space of curves in the homology class (which is isomorphic to the space of stable maps as the curves are sections – they can have no bubbles in fibres by the assumptions on degree – and so regularly embedded). Use $s : \mathcal{N} \times \mathbb{P}^1 \hookrightarrow \mathcal{N} \times \mathcal{M}$ to denote the universal curve, with $\tilde{s} : \mathcal{N} \times X \hookrightarrow \mathcal{N} \times \mathcal{M} \times_{\mathbb{P}^1} X$ the induced map. Let the pull-back of the universal sheaf on $X \times_{\mathbb{P}^1} \mathcal{M}$ to $\mathcal{N} \times X \times_{\mathbb{P}^1} \mathcal{M}$ be denoted by $\mathcal{E}$. Then the sheaf $\tilde{s}^*\mathcal{E}$ on $\mathcal{N} \times X$ exhibits $\mathcal{N}$ as the relevant moduli space of sheaves on $X$ as produced by the above Theorem, depicted by the diagram:

$$\begin{array}{ccc}
\tilde{s}^*\mathcal{E} & & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{N} \times X & \overset{\tilde{s}}{\hookrightarrow} & \mathcal{N} \times \mathcal{M} \times_{\mathbb{P}^1} X \\
\downarrow p & & \downarrow \pi \\
\mathcal{N} \times \mathbb{P}^1 & \overset{s}{\hookrightarrow} & \mathcal{N} \times \mathcal{M} \\
\downarrow q & & \downarrow \rho \\
\mathcal{N} & & \mathcal{N} \times \mathbb{P}^1.
\end{array}$$

We must show that the tangent-obstruction functor of the sheaves is the same as that of the curves in the Gromov-Witten theory of [31]. Since the curves are sections they miss the singularities of the projection $\mathcal{M} \to \mathbb{P}^1$ and so, by the above discussion, lie in the open set of $\mathcal{M}$ corresponding to locally free sheaves on fibres, which therefore satisfy the Serre duality (4.1) on their $\mathrm{Ext}_0$ s. Thus the only non-zero such $\mathrm{Ext}_0$ is $\mathrm{Ext}^1_{X_t}(\mathcal{F}_t, \mathcal{F}_t)_0$ (for $\mathcal{F}_t$ a sheaf on the fibre $X_t$), which is the fibrewise tangent space to $\mathcal{M} \to \mathbb{P}^1$ at $\mathcal{F}_t$, i.e., the normal bundle to any section through $\mathcal{F}_t$.

For smoothly embedded rational curves the deformation theory of [31] reduces to $R^i q_* \nu_s \otimes \mathcal{I}$, $i = 0, 1$, where $\nu_s$ is the normal bundle to $s$ and $\mathcal{I}$ is an arbitrary $\mathcal{O}_{\mathcal{N}}$-module. This is then

$$R^i q_*(s^* T_\rho) = R^i q_*(s^* \mathcal{E}xt^1_\pi(\mathcal{E}, \mathcal{E} \otimes \pi^* \mathcal{I})_0) = R^i q_*(\mathcal{E}xt^1_p(\tilde{s}^* \mathcal{E}, \tilde{s}^* \mathcal{E} \otimes p^* \mathcal{I})),$$

where for the last equality we have base-changed around the square in the above diagram using the flatness of $\mathcal{E}$ and $(\mathcal{E}xt^1_\pi)_0$ (since the other $(\mathcal{E}xt^i_\pi)_0$ s vanish). By the Leray spectral sequence this yields $\mathcal{E}xt^{i+1}_{q \circ p}(\tilde{s}^* \mathcal{E}, \tilde{s}^* \mathcal{E} \otimes (q \circ p)^* \mathcal{I})_0$, $i = 0, 1$, which gives the deformation theory of the sheaf moduli problem, as required. q.e.d.

**Remark.** Degenerating the base $\mathbb{P}^1$ to a curve with one node degenerates $\mathcal{M}$ (which we show below is Calabi-Yau) to a normal crossings space of two "Fanos" (their anticanonical bundles are effective) joined across a common anticanonical divisor $S$ – the fibre of $\mathcal{M}$ over the node. Then the Tyurin-Casson invariant picture sketched in (2.7) becomes, in terms of the sections of $\mathcal{M} \to \mathbb{P}^1$, a formal picture mentioned in [40] (where it was also motivated by relating bundles to curves, but in that case via zero sets of rank 2 bundles). Namely, to count curves in such a singular Calabi-Yau, one should count those in each Fano component which meet in the "boundary" $S$. Analogously to the Tyurin picture we find [40] that the image of the map taking a curve in one Fano component to its intersection with $S$, which generically lies in the complex symplectic space $\mathrm{Hilb}^n(S)$, is a complex Lagrangian. (Here $n$ is the intersection number of the curve with $S$.) Intersecting the two Lagrangians should give something like the GW invariant of a smoothing of the Calabi-Yau. It would be nice to rigorise this and the Tyurin-Casson invariant (of course one is a special case of the other, by considering the case of ideal sheaves of curves).

We next show that $\mathcal{M}$ is in fact a Calabi-Yau 3-fold. This result has now also been given a new proof in [10], using Fourier-Mukai transforms, without the need for assumptions on the singular fibres of $X$. Notice that although (4.19) did not hold for rank $r \leq 1$, this result does, provided that $\mathcal{M}$ is smooth and sheaves supported on singular fibres without locally free resolutions are isolated. Thus, in the case that the fibres $X_t$ are themselves elliptically fibred, we may take as $\mathcal{M}_t$ the moduli of torsion sheaves that are rank one, degree zero sheaves supported on elliptic fibres (so that $\mathcal{M}_t$ is a compactified Jacobian of $X_t \to \mathbb{P}^1$). In a hyperkähler rotated complex structure (and for some appropriate choice of "B-field") this should be the mirror $K3$ to $X_t$ ([38] 4.1) and $\mathcal{M}$ is a obtained from $X$ by fibrewise mirror symmetry and hyperkähler rotation. However $\mathcal{M}$ is *not* the mirror of $X$ (its Hodge numbers are not flipped, for instance; see below), though it is "T-dual" to $X$.

**Theorem 4.23.** *Let $\mathcal{M}$ be a relative moduli space of slope stable sheaves on $X$ with data (4.3), $d = 2$, $r \geq 2$, and single ODP singularities in the fibres of $X \to \mathbb{P}^1$. Then $\mathcal{M}$ is a Calabi-Yau 3-fold.*

*Proof.* As before consider $\mathcal{M}$ to be the moduli space of torsion sheaves $(\iota_t)_*\mathcal{E}$, where $\mathcal{E}$ is a sheaf on $X_t \subset X$. Let $T$ denote the universal sheaf on $X \times_{\mathbb{P}^1} \mathcal{M}$, and let $\iota : X \times_{\mathbb{P}^1} \mathcal{M} \to X \times \mathcal{M}$ be the inclusion and $p : X \times \mathcal{M} \to \mathcal{M}$ the projection. Then, as $\mathcal{M}$ is smooth (4.15), it has a tangent sheaf given by $\mathcal{E}xt^1_p(\iota_*T, \iota_*T)$.

The holomorphic 3-form is the pairing on the tangent bundle given by the cup product pairings

$$\mathcal{E}xt^1_p(\iota_*T, \iota_*T) \otimes \mathcal{E}xt^1_p(\iota_*T, \iota_*T) \otimes \mathcal{E}xt^1_p(\iota_*T, \iota_*T) \to \mathcal{E}xt^3_p(\iota_*T, \iota_*T) \cong \mathcal{O}_{\mathcal{M}},$$

in the spirit of Mukai's symplectic structure for moduli spaces on $K3$ surfaces [32]. Notice its dependence on the holomorphic 3-form on $X$ is through the final isomorphism. To show that this pairing is non-degenerate everywhere and that $\mathcal{M}$ is Calabi-Yau it is enough to show it does not vanish on any divisor in $\mathcal{M}$. But by (4.15) it is enough to show this at points of $\mathcal{M}$ corresponding to locally free sheaves since their complement is isolated; these then satisfy Serre duality on their Exts. As then all of the sheaves in the above pairing are locally free and we can localise at a point to get, at the level of tangent spaces, the cup product pairing

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^3(\mathcal{F}, \mathcal{F}) \cong \mathbb{C}$$

at the point $\mathcal{F} = (\iota_t)_*\mathcal{E} \in \mathcal{M}$, where $\mathcal{E}$ is a sheaf on the fibre $X_t$.

So now to show the pairing is non-degenerate it is enough to show, by Serre duality, that the pairing

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \otimes \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F},\mathcal{F})$$

is onto. By Lemma 3.42 above the inclusion $X_t \subset X$ induces a long exact sequence

$$\ldots \to \mathrm{Ext}^i_{X_t}(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^i_X(\mathcal{F},\mathcal{F}) \to T_t\mathbb{P}^1 \otimes \mathrm{Ext}^{i-1}_{X_t}(\mathcal{E},\mathcal{E}) \to \ldots,$$

which yields, for $X_t$ a $K3$ and $\mathcal{E}$ stable and satisfying Serre duality,

$$0 \to \mathrm{Ext}^1_{X_t}(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^1_X(\mathcal{F},\mathcal{F}) \to T_t\mathbb{P}^1 \to 0,$$

and

$$0 \to \mathrm{Ext}^2_{X_t}(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to T_t\mathbb{P}^1 \otimes \mathrm{Ext}^1_{X_t}(\mathcal{E},\mathcal{E}) \to 0.$$

The pairing respects these sequences in the obvious way. Firstly the cup product of the image of two elements of $\mathrm{Ext}^1_{X_t}(\mathcal{E},\mathcal{E}) \subset \mathrm{Ext}^1_X(\mathcal{F},\mathcal{F})$ is just (the image of) the cup product on $X_t$, with image in $\mathrm{Ext}^2_{X_t}(\mathcal{E},\mathcal{E}) \subset \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})$, and this is non-degenerate. Secondly, pairing an element $x$ of $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{F})$ with (the image of) an element $y$ of $\mathrm{Ext}^1_{X_t}(\mathcal{E},\mathcal{E})$ and projecting $\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to T_t\mathbb{P}^1 \otimes \mathrm{Ext}^1_{X_t}(\mathcal{E},\mathcal{E})$ gives the same as projecting $x$ to $T_t\mathbb{P}^1$ and tensoring with $y$. Again this is onto, so we are done.

q.e.d.

We now study the cohomology of such an $\mathcal{M}$, putting Mukai's isomorphism of Hodge structures ([24] 6.1.14) between $X_t$ and $\mathcal{M}_t$ into a family $X \times_{\mathbb{P}^1} \mathcal{M}$. We avoid problems with the singularities of fibres of $X$ by working on $X \times \mathcal{M}$.

Fix $X$ and a rank and Chern classes satisfying the conditions of Theorem 4.19, pick a universal sheaf $\mathcal{E}$ on $X \times \mathcal{M}$, and let $\pi$ and $p$ be the projections to $X$ and $\mathcal{M}$ respectively.

Then define maps

$$f : H^*(X;\mathbb{C}) \to H^*(\mathcal{M};\mathbb{C}), \qquad f' : H^*(\mathcal{M};\mathbb{C}) \to H^*(X;\mathbb{C}),$$

by

$$\begin{aligned} f(c) &= p_*\Big(ch^\vee(\mathcal{E})\sqrt{\mathrm{Td}(X\times\mathcal{M})}\,.\,\pi^*(c)\Big), \\ f'(c) &= \pi_*\Big(-ch(\mathcal{E})\sqrt{\mathrm{Td}(X\times\mathcal{M})}\,.\,p^*(c)\Big), \end{aligned}$$

where $ch^\vee = \sum(-1)^i ch_i$.

**Theorem 4.24.** *$f \circ f'$ is the identity.*

*Proof.* We simply mimic the exposition of Mukai's proof for $K3$ surfaces in ([24] 6.1.13). Label two copies of $\mathcal{M}$ by $\mathcal{M}_1$ and $\mathcal{M}_2$, and denote by $\mathcal{E}_i$ the pull back of the universal sheaf on $X \times \mathcal{M}_i$ to $X \times \mathcal{M}_1 \times \mathcal{M}_2$. We must transfer a class $c$ from $\mathcal{M}_1$ to $X$ via $f'$, then back to $\mathcal{M}_2$ via $f$. Pulling everything back to $X \times \mathcal{M}_1 \times \mathcal{M}_2$ via the commutativity of various push-down pull-back squares, and pushing down $X$ before $\mathcal{M}_1$ (as in [24] 6.1.13), we can reduce to the following diagram

$$\begin{array}{c} X \times \mathcal{M}_1 \times \mathcal{M}_2 \\ \downarrow \tilde{p} \\ \mathcal{M}_1 \times \mathcal{M}_2 \\ q_2 \swarrow \quad \searrow q_1 \\ \mathcal{M}_2 \qquad \mathcal{M}_1\,, \end{array}$$

arriving at

$$ff'(c) = q_{2\,*}\tilde{p}_*\Big[-ch^{\vee}(\mathcal{E}_1)ch(\mathcal{E}_2)\sqrt{\mathrm{Td}(\mathcal{M}_1 \times \mathcal{M}_2)}\ \mathrm{Td}(X)\,.\,q_1^*(c)\Big]\,,$$

where we have supressed some obvious pull-back maps for clarity. By multiplicativity of the Chern character this yields

$$q_{2\,*}\Big[\tilde{p}_*\big(-\,ch(\mathbf{R}\mathcal{H}om(\mathcal{E}_1,\mathcal{E}_2)\ \mathrm{Td}(X)\big)\sqrt{\mathrm{Td}(\mathcal{M}_1 \times \mathcal{M}_2)}\,.\,q_1^*(c)\Big]\,. \tag{4.25}$$

Now $\tilde{p}_*(ch(\mathbf{R}\mathcal{H}om(\mathcal{E}_1,\mathcal{E}_2)\,\mathrm{Td}(X)) = ch(\mathbf{R}\mathcal{H}om_{\tilde{p}}(\mathcal{E}_1,\mathcal{E}_2))$, by the Grothendieck-Riemann-Roch theorem. But we know that

$$\mathrm{Ext}^i_X(\mathcal{F}_1,\mathcal{F}_2) = 0, \quad \forall i, \tag{4.26}$$

for sheaves $\mathcal{F}_1 \neq \mathcal{F}_2$ in $\mathcal{M}$ *unless both are non locally free on their support* (where again we are considering $\mathcal{M}$ to be a moduli space of torsion sheaves supported on fibres $X_t$). This is because if either is locally free then the Serre duality (4.1) holds on the fibres and the usual arguments go through ([24] 6.1.8, 6.1.10) to give the vanishing of all $\mathrm{Ext}^i_{X_t}$s on the fibre; thus by (3.42) all $\mathrm{Ext}^i_X$s vanish too.

So the $\mathrm{Ext}^i$s ($i \leq 3$) vanish outside a subvariety of codimension three (the union of the diagonal $X \times \Delta \subset X \times \mathcal{M}_1 \times \mathcal{M}_2$ and the codimension six (4.15) locus of pairs of non locally free sheaves) for all $i \leq 3$, from which it follows ([33] 2.26) that

$$\mathcal{E}xt^i_{\tilde{p}}(\mathcal{E}_1,\mathcal{E}_2) \equiv 0, \quad i < 3,$$

and $\mathcal{E}xt^3_{\tilde{p}}(\mathcal{E}_1, \mathcal{E}_2)$ is concentrated on $\Delta$. On $\Delta$ we have $\operatorname{Ext}^3(\mathcal{F}, \mathcal{F}) \cong H^3(\mathcal{O}_X)$ via the trace map, inducing an isomorphism

$$\mathbf{R}\mathcal{H}om_{\tilde{p}}(\mathcal{E}_1, \mathcal{E}_2)\,[3] \cong \mathcal{E}xt^3_{\tilde{p}}(\mathcal{E}, \mathcal{E}) \xrightarrow{\operatorname{tr}} R^3\tilde{p}_*\mathcal{O}_{X\times\Delta} \cong \mathcal{O}_\Delta.$$

Again by Grothendieck-Riemann-Roch we have

$$ch(\iota_*\mathcal{O}_\Delta)\sqrt{\operatorname{Td}(\mathcal{M}_1 \times \mathcal{M}_2)} = \iota_*(ch(\mathcal{O}_\Delta)),$$

so plugging everything into (4.25) (and noting that the shift by [3] introduces a minus sign in $ch$) gives

$$ff'(c) = q_{2*}(ch(\mathcal{O}_\Delta)\, q_1^*(c)) = c. \qquad \text{q.e.d.}$$

**Corollary 4.27.** *Let $\mathcal{M}$ be a relative moduli space of slope stable sheaves on $X$ with data (4.4), and $d = 2$, $r \geq 2$. Then the groups $I^k = \bigoplus_{i-j=k} H^{i,j}$ are isomorphic on $X$ and $\mathcal{M}$; thus their Hodge numbers $h^{i,j}$ are the same.*

*Proof.* Since the Fourier-Mukai machinery of [10] now gives a quick proof of this result we will only give a sketch of our previous argument.

Note that all we want to do is reverse the roles of $X$ and $\mathcal{M}$, using the universal sheaf on $X \times_{\mathbb{P}^1} \mathcal{M}$, restricted to fibres $\{x_t\} \times \mathcal{M}_t$, to exhibit $X$ as a fibrewise moduli space of sheaves on $\mathcal{M}$ and deduce (4.24) that $f' \circ f = \mathrm{id}$.

Pick a smooth fibre $X_t$ and consider the universal bundle on $X_t \times \mathcal{M}_t$. Firstly, by results of Mukai, (see e.g. [9] Theorem 1.1), the corresponding Fourier-Mukai transform is an equivalence. By using the corresponding transform for Hermitian-Yang-Mills connections in differential geometry (see for example [4]), one can show that the transform maps slope stable sheaves to slope stable sheaves, making $X_t$ a moduli space of slope stable sheaves on $\mathcal{M}_t$ of "Fourier-Mukai type" (that is the Mukai vector is primitive and of square zero; equivalently, $\gcd(r, c_1 \,.\, \omega, \frac{1}{2}c_1^2 - c_2) = 1$ for an appropriate choice of polarisation).

Thus even on singular fibres semistable sheaves are stable and so simple, so long as we can show that $X$ really does parametrise sheaves on $\mathcal{M}$ over these fibres; i.e., we need to show that the universal sheaf on $X \times \mathcal{M}$ is *flat over* $X$.

But this follows by mimicking the argument of ([8] Lemma 5.1) one dimension up on $K3$ fibrations instead of elliptic fibrations. (All that [8] uses is the fact that the sheaves parametrised by $\mathcal{M}$ are flat over $\mathcal{M}$

and have a locally free resolution of length 2 on $X$. In our case flatness over $\mathcal{M}$ is again immediate, and since the sheaves are reflexive on the fibres by Theorem 4.5 they have depth 2 and so homological dimension 1 by the Auslander-Buchsbaum theorem ([24] 1.1).)

Using (Gieseker) stability we get the condition (4.26) so that the proof of Theorem 4.24 goes through with $X$ and $\mathcal{M}$ exchanged.

Since $ch$ and Td are of pure $(p,p)$ Hodge types, the $I^k$s are preserved. Calculating their dimensions in terms of $h^{0,1}$, $h^{1,1}$ and $h^{1,2}$, by symmetries of the Hodge diamond, we see that these three numbers are preserved. q.e.d.

Theorem 4.24 is of course the fibrewise Fourier-Mukai transform of [10] at the level of K-theory. Theorem 4.19 can also be interpreted in terms of this transform; the transform of $\mathcal{O}_{[\sigma]}$, where $[\sigma]$ is the image of a minimal degree section of $\mathcal{M} \to \mathbb{P}^1$, is just $\tilde{\sigma}^* T$. Thus one might try to use [10] to study moduli of sheaves on $X$; the difficulties are then displaced to understanding when the transform gives stable sheaves, and not more exotic elements of the derived category. Nonetheless, one might try to study moduli by transforming the structure sheaves of connected curves in a class that intersects the fibre class once. The singular nodal sections with components lying in fibres will give rise to fibres on which instabilities and singularities of the sheaves lie.

## Examples

We have already mentioned how the two dimensional results apply to $K3 \times T^2$. Now we again study Gross' examples, this time considering two dimensional moduli spaces, doing a family version of an example in ([24] 5.3.7) and extending the proof of stability to *all* quartics in $\mathbb{P}^3$.

**Proposition 4.28.** *Fix any irreducible, reduced quartic surface $S \subset \mathbb{P}^3$, and a closed point $x \in S$. Then the rank two sheaf $E = E_x$ defined by the sequence*

$$0 \to E \to H^0(\mathscr{I}_x(1)) \to \mathscr{I}_x(1) \to 0,$$

*is slope stable with respect to the restriction of the Fubini-Study metric. $E_x$ is a bundle for $x$ a smooth point of $S$, and the above gives an isomorphism between the appropriate moduli space and $S$.*

*Proof.* The dual $E^*$ of such a sheaf is generated by its sections, and a destabilising rank one (torsion free, without loss of generality) quotient sheaf $\mathcal{L}$ is therefore also generated by its sections and of degree $\leq 2$. Therefore the proof of (4.14) applies to obtain a contradiction. q.e.d.

So in this case, for the two Gross 3-folds $X_i$, the relative moduli spaces $\mathcal{M}_i$ are in fact isomorphic to $X_i$. It is easy to see that the polarisation on $\mathcal{M}_i$ is that on $X$ plus $N\omega_1$, since this gives the right fibre polarisation on $\mathcal{M}_t$ for the generic fibre $X_t$ of Picard number one. Unfortunately this example only satisfies the conditions of Definition 4.3, not Definition 4.4, so Theorem 4.19 does not really apply. But the extra conditions of Definition 4.4 were only used to show that Jun Li's line bundle (4.17) was ample on the singular fibres of $\mathcal{M} \to \mathbb{P}^1$, and that the smooth points of the fibres corresponded to locally free sheaves, both of which, as mentioned earlier, should be true in general anyway. I cannot prove it in the general case, but it clearly holds in this example, so we may use Theorem 4.19.

Thus the holomorphic Casson invariants are given by counting the number of lowest degree sections of

$$\left(X_i \to \mathbb{P}^1\right) \subset \left(\mathbb{P}(E_i) \to \mathbb{P}^1\right)$$

with respect to the form $t = c_1(\mathcal{O}_{\mathbb{P}(E_i)}(1))$, which is ample on the fibres. For $X_2$ the subbundle $\mathcal{O}(1) \subset E_2 = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$ gives a section of $\mathbb{P}(E_2) \to \mathbb{P}^1$ which lies in $X_2$, is of lowest degree $(-1)$ with respect to $t$, and is unique. Thus we get a Casson invariant of one, counting a locally free sheaf.

For the moduli space that corresponds to this one under the diffeomorphism between the $X_i$, however, we must consider sections of $X_1 \to \mathbb{P}^1$ of degree $-1$ with respect to the Kähler form of $\mathbb{P}^3$ on $X \subset \mathbb{P}^1 \times \mathbb{P}^3$. There clearly are none.

(The Casson invariant of $X_1$ that corresponds to lowest degree sections is for degree 0 sections, i.e., those of the form $\mathbb{P}^1 \times \{x\} \subset X_1$. Write the equation defining the (2,4) divisor $X_1 \cong \mathcal{M}_1$ as $x^2 f + xyg + y^2 h = 0$, where $f$, $g$, $h$ are quartic polynomials on $\mathbb{P}^3$. Then $\mathbb{P}^1 \times \{x\}$ lies in $S$ if and only if $x$ lies in the intersection of the three quartic surfaces in $\mathbb{P}^3$. Thus we may deform to a case where there are precisely 64 isolated sections in the moduli space, and therefore the Casson invariant is 64, and again all the sheaves are locally free.)

Thus we have finally recovered the result that the $X_i$ are not deformation equivalent, as polarised varieties. Although the result is both weaker than the results using Gromov-Witten invariants [36] and essentially a more complicated rerun of the same proof (going as it does via the same Gromov-Witten invariants), it is nonetheless encouraging that there are examples of the invariant which are calculable yet contain highly non-trivial information.

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