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EXTREMAL METRICS ON TORIC SURFACES: A CONTINUITY METHOD

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Abstract

The paper develops an existence theory for solutions of the Abreu equation, which include extremal metrics on toric surfaces. The technique employed is a continuity method, combined with "blow-up" arguments. General existence results are obtained, assuming a hypothesis (the "M-condition") on the solutions, which is shown to be related to the injectivity radius.

1. Introduction

This is the first in a series of papers which continue the study in [**7**], [**8**] of the Kahler geometry of toric varieties. The purpose of the present paper is to introduce an analytical condition (the "M-condition") and show that it controls sequences of extremal metrics on toric surfaces. To set the scene for our discussion we consider the following data:

- an open polygon $P \subset \mathbf{R}^2$, with compact closure $\overline{P}$;
- a map σ which assigns to each edge E of P a strictly positive weight $\sigma(E)$;
- a smooth function A on $\overline{P}$.

The datum σ yields a measure $d\sigma$ on the boundary ∂P—on each edge E we take $d\sigma$ to be a constant multiple of the standard Lebesgue measure with the constant normalised so that the mass of the edge is $\sigma(E)$. Equally, the datum σ specifies an affine-linear defining function λ_E for each edge E, i.e., the edge lies in the hyperplane $\lambda_E^{-1}(0)$. We choose an inward-pointing normal vector v at a point of E with

$$|i_v d\mu| = d\sigma_E$$

where $d\mu$ is the fixed standard area form on $\mathbf{R}^2$ and we specify λ_E by the condition that $\nabla_v \lambda_E = 1$.

For a continuous function f on $\overline{P}$ we set

$$L_{A,\sigma} f = \int_{\partial P} f d\sigma - \int_P A f d\mu.$$

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We require our data (P, σ, A) to satisfy the condition that $L_{A,\sigma} f$ vanishes for all affine-linear functions f—in other words, that ∂P and P have the same mass and centre of mass with respect to the measures $d\sigma$ and $A d\mu$ respectively. Notice that given P and σ there is a unique affine-linear function A_σ such that (P, σ, A_σ) satisfies this requirement.

Now let u be a convex function on $\overline{P}$, smooth in the interior. We say that u satisfies the *Guillemin boundary conditions* if

- any point x_0 in the interior of an edge E is contained in a neighbourhood N_{x_0} on which
$$u = \lambda_E \log \lambda_E + f$$
where f is smooth in $N_{x_0} \cap \overline{P}$ and with strictly positive second derivative on $N_{x_0} \cap E$;
- if x_0 is a vertex of P, the intersection of two edges E, E', then there is a neighbourhood N_{x_0} on which
$$u = \lambda_E \log \lambda_E + \lambda_{E'} \log \lambda_{E'} + f$$
where f is smooth in $N_{x_0} \cap \overline{P}$.

(Note that these boundary conditions depend on the weights via the affine-linear defining functions. Thus we can extend the concept to unbounded polygons with specified defining functions.)

With this material in place, we can recall that the basic question we wish to address is the existence of a smooth solution u to the fourth order partial differential equation (*Abreu's equation*)
$$u^{ij}_{ij} = -A,$$
in P, satisfying the Guillemin boundary conditions. (Here we use the summation convention, and u^{ij} is the inverse of the Hessian of u. Our general practice is to use upper indices (x^1, x^2) for the co-ordinates on $\mathbf{R}^2$, although we switch to lower indices when this is more convenient.) If such a function u exists, it is an absolute minimum of the functional
$$\mathcal{F}(f) = -\int_P \log\det(f_{ij}) + L_{A,\sigma} f,$$
over all convex functions f on $\overline{P}$, smooth in the interior. In [**7**] we were led to conjecture that a solution exists if and only if the linear functional has the property that $L_{A,\sigma} f \geq 0$ for all convex f having L^1 boundary values, with strict inequality if f is not affine-linear. We showed in [**7**] that this is a necessary condition for the existence of a solution and the problem is to establish the sufficiency. We will write $\mathcal{C}(P)$ for the set of pairs (A, σ) which satisfy this positivity condition.

The motivation for this problem stems from the case when P is a "Delzant polygon", corresponding to a compact symplectic 4-manifold X with a torus action. Such a polygon comes with a preferred choice of σ–we will refer to the pair (P, σ) as a "Delzant weighted polygon".

The convex functions u satisfying the Guillemin boundary conditions correspond to invariant Kahler metrics on X. In general for a strictly convex smooth function u on a polygon P we let g be the Riemannian metric on P defined by the Hessian u_{ij} and $\hat{g}$ be its extension to $P \times \mathbf{R}^2$ given by

$$\hat{g} = u_{ij} dx^i dx^j + u^{ij} d\theta_i d\theta_j. \tag{1}$$

This is a Kahler metric, with Kahler form $dx^i d\theta_i$, invariant under translations in the $\mathbf{R}^2$ variables. In particular $\hat{g}$ descends to a metric (which we denote by the same symbol) on $P \times \mathbf{R}^2/2\pi\mathbf{Z}^2$. If the polygon is Delzant then, with the preferred choice of σ, this metric extends to a smooth metric on a compact 4-manifold X. The expression $-u^{ij}_{ij}$ gives one half the scalar curvature of the metric $\hat{g}$, [**1**]. When $A = A_\sigma$ our problem is equivalent to the existence of an *extremal Kahler metric* (in the given cohomology class) on X. In particular, if it happens that A_σ is constant (i.e., if the centre of mass of $(\partial P, d\sigma)$ coincides with the centre of mass of P) our problem is equivalent to the existence of a constant scalar curvature Kahler metric. The positivity condition described above is related to algebro-geometric notions of "stability".

In [**7**] we obtained a rather weak existence result by the variational method applied to the functional $\mathcal{F}$. In the present paper we change our approach to the continuity method. In Section 2 we set up the framework for this. We show that solutions persist under small perturbations of the data (P, A, σ). Given any polygon P_1 and $(A_1, \sigma_1) \in \mathcal{C}(P_1)$ we show that there is a path (P_t, A_t, σ_t) for $t \in [0, 1]$ such that $(A_t, \sigma_t) \in \mathcal{C}(P_t)$ for each t and a solution to our problem exists when $t = 0$. This is rather trivial if one allows arbitrary functions A_t but we show that if A_1 is linear (respectively, constant) we can arrange that the A_t are also linear (respectively, constant). Thus in the standard fashion our problem comes down to establishing closedness with respect to t, that is to say to establishing *a priori* estimates for a solution u in terms of given data (P, A, σ).

In [**8**] we studied this problem in the interior of the polygon and showed that, roughly speaking, singularities cannot develop there. The goal of this paper, and its sequels, is to extend these estimates, in appropriate form, up to the boundary. Now we will introduce the central notion of this paper. Let u be a smooth convex function defined on some convex set $\Omega \subset \mathbf{R}^n$ and let p, q be distinct points in Ω. Let ν be the unit vector pointing in the direction from p to q. We write

$$V(p, q) = (\nabla_\nu u)(q) - (\nabla_\nu u)(p), \tag{2}$$

where ∇_ν denotes the derivative in the direction ν. Thus $V(p, q)$ is positive by the convexity condition. Let $I(p, q)$ be the line segment

$$I(p, q) = \left\{ \frac{p+q}{2} + t(p-q) : -3/2 \leq t \leq 3/2 \right\}.$$

Definition 1. For $M > 0$ we say that u satisfies the M-condition if for any p, q such that $I(p,q) \subset \Omega$ we have $V(p,q) \leq M$.

It is easy to see that if the domain is a polygon P as above, and if u satisfies Guillemin boundary conditions, then u satisfies the M-condition for some M. Our main result is

Theorem 1. *Let $(P^{(\alpha)}, \sigma^{(\alpha)}, A^{(\alpha)})$ be a sequence of data sets converging to (P, A, σ). Suppose that for each α there is a solution $u^{(\alpha)}$ to the problem defined by $(P^{(\alpha)}, \sigma^{(\alpha)}, A^{(\alpha)})$. If there is an $M > 0$ such that each $u^{(\alpha)}$ satisfies the M-condition then there is a solution of the problem defined by (P, A, σ).*

While it is crucial for our continuity method that we do not restrict attention to Delzant polygons, it is easier to outline the proof of Theorem 1 in this special situation. In Section 3 we develop a variety of arguments which ultimately show that the M- condition gives a lower bound on the injectivity radius of the metric on the 4-dimensional manifold, in terms of the maximal size of the curvature (see Proposition 10 below). If the curvature were to become large, in the sequence, then after rescaling we are able to obtain "blow up limits" which have zero scalar curvature. In the special situation when we are actually working with compact 4-manifolds these limits could be obtained as a consequence of general results in Riemannian geometry but we give proofs (in Section 4) adapted to our particular circumstances, in order to handle general polygons and also in order to make the paper self-contained. Then we show that these blow-up limits do not exist. There are essentially two cases to consider. In one case we can appeal to a more general theorem of Anderson, but we also give an independent proof for the particular result we need. In the other case we use a maximum principle argument, based on a result which we prove in the Appendix. Thus we conclude, from the nonexistence of these blow-up limits, that in fact the curvature was bounded in the sequence, which leads to the desired convergence.

The upshot of all this is that we can prove the existence conjecture of [**7**] if we can establish an *a priori* M-condition on solutions. More precisely, for given data (P, σ, A) and a choice of base point $p_0 \in P$ we can define

$$\lambda(P, \sigma, A) = \sup \int_{\partial P} f d\sigma,$$

where the supremum runs over positive convex functions f vanishing at p_0 and with $L_{A,\sigma} f = 1$. We showed in [**7**] that, for data in $\mathcal{C}(P)$, this $\lambda(P, \sigma, A)$ is finite and the remaining problem is to show that solutions to our problem satisfy an M-condition, where M will depend, among other things, on $\lambda(P, \sigma, A)$. This will be taken up in the sequels to the present paper (although the author envisages that the actual argument will be rather more complicated than this outline suggests).

2. The continuity method

2.1. Connectedness. For a given polygon P we have defined $\mathcal{C}(P)$ to be the set of (A,σ) such that $L_{A,\sigma}$ is strictly positive on the non-affine convex functions. Clearly $\mathcal{C}(P)$ is itself a convex set. We now define a "canonical weight function" σ_P as follows. Let p_0 be the centre of mass of P, with the standard Lebesgue measure on $\mathbf{R}^2$ and for each edge E of P let cE be the triangle with base E and vertex p_0. Obviously, up to sets of measure 0, the polygon P is decomposed into a disjoint union of these triangles. Now define

$$\sigma_P(E) = \text{Area } (cE).$$

To simplify notation, and without loss of generality, suppose $p_0 = 0$. Clearly the mass of the boundary, in the measure $d\sigma_P$, is the same as the area of P. Further, if q, q' are the endpoints of an edge E the centre of mass of cE is $\frac{1}{3}(q+q')$ while the centre of mass of E is $\frac{1}{2}(q+q')$. Summing over the edges it follows that the centre of mass of ∂P is also at 0. Hence the linear function A_{σ_P} associated to these canonical weights is the constant function 1.

Lemma 1. *The pair $(\sigma_P, 1)$ is in $\mathcal{C}(P)$.*

This is essentially a result of Zhou and Zhu, (Thm. 0.1 of [**19**]), but since the proof is very simple we include it here. Take standard polar co-ordinates (r,θ) on $\mathbf{R}^2$. By elementary calculus one finds that the measure $d\sigma_P$ is given by the 1-form $\frac{1}{2}r^2 d\theta$, restricted to the boundary. Let f be a convex function on the closure of P. Since $L_{(\sigma_P,1)}(f)$ is unchanged by the addition of an affine-linear function, we can suppose without loss of generality that f achieves its minimum value at the origin, and that the minimum value is zero. Now let the boundary be given by the equation $r = R(\theta)$. Then we have, by convexity,

$$f(r,\theta) \leq \frac{r}{R(\theta)} f(R(\theta),\theta).$$

Thus

$$\int_P f d\mu = \int_0^{2\pi}\int_0^{R(\theta)} f(r,\theta) r dr d\theta \leq \int_0^{2\pi}\int_0^{R(\theta)} \frac{r^2}{R(\theta)} f(R(\theta),\theta) dr d\theta.$$

Integrating with respect to r,

$$\int_P f d\mu \leq \frac{1}{3}\int_0^{2\pi} f(R(\theta),\theta)\ R(\theta)^2 d\theta,$$

whereas

$$\int_{\partial P} f d\sigma_P = \frac{1}{2}\int_0^{2\pi} f(R(\theta),\theta)\ R(\theta)^2 d\theta.$$

So

$$L_{(\sigma_P,1)}(f) \geq \frac{1}{6} \int_0^{2\pi} f(R(\theta),\theta)\ R(\theta)^2 d\theta,$$

and this is clearly strictly positive if f is not identically zero. The argument extends immediately to the case when f only has L^1 boundary values.

We define the notion of a "continuous path of polygons" P_t in the obvious way: the polygons should have the same number of edges and the vertices should vary continuously. Similarly, there is an obvious definition of a continuous 1-parameter family of data sets (σ_t, A_t) corresponding to P_t.

Proposition 1. *Let P_t , $t \in [0,1]$ be a continuous path of polygons and suppose we have $(A_0,\sigma_0) \in \mathcal{C}(P_0), (A_1,\sigma_1) \in \mathcal{C}(P_1)$. Then these can be joined by a continuous 1-parameter family with $(\sigma_t, A_t) \in \mathcal{C}(P_t)$. If A_0, A_1 are affine-linear we can suppose that each A_t is affine-linear, and if A_0, A_1 are constant we can suppose that each A_t is constant.*

First, if $\sigma_0 = \sigma_{P_0}, \sigma_1 = \sigma_{P_1}, A_0 = 1, A_1 = 1$ we can take $\sigma_t = \sigma_{P_t}$, $A_t = 1$ for all $t \in [0,1]$. These lie in $\mathcal{C}(P_t)$ by the preceding lemma, and obviously form a continuous family. Now, by composing paths, we can reduce to the case when $P_1 = P_0$ and $\sigma_0 = \sigma_{P_0}$, $A_0 = 1$. Here we just use the linear interpolation, applying the convexity of $\mathcal{C}(P_0)$. If A_1 is affine-linear (respectively constant) then each A_t will be affine-linear (respectively constant), and the proof is complete.

2.2. Openness. Let P_t , $t \in [0,1]$ be a continuous 1-parameter family of polygons and σ_t a 1-parameter family of weights. Each edge E of P_0 varies in a 1-parameter family $E(t)$ of edges and we have affine-linear defining functions $\lambda_{E(t)} : \mathbf{R}^2 \to \mathbf{R}$. We can choose a continuous 1-parameter family of diffeomorphisms $\chi_t : P_0 \to P_t$ such that, near to each edge E,

$$\lambda_{E(t)} \circ \chi_t = \lambda_E.$$

(This implies that χ_t is affine-linear near each vertex of P_0.) Then, for small t, a function u_t on P_t satisfies the Guillemin boundary conditions for (P_t, σ_t) if and only if $\tilde{u}_t = u_t \circ \chi_t$ satisfies the boundary conditions for (P_0, σ_0). In a 1-parameter family, we say that u_t varies continuously with t if the functions $\tilde{u}_t - u_0$ (which are smooth functions on P_0) are continuous in t, along with all their multiple derivatives.

In this subsection we prove

Proposition 2. *Let (P_t, σ_t, A_t) be a continuous 1-parameter family of data and suppose a solution u_0 to our problem exists when $t = 0$. Then for small t there is a solution u_t, and u_t varies continuously with t.*

Of course, this will be proved by linearising and applying the implicit function theorem. On the face of it, this might seem a substantial task, in view of the singular behaviour of the solutions required by the boundary conditions, but we will explain that the superficial technical difficulties evaporate when the problem is set up in a suitable way.

We begin by reviewing the relation between complex and symplectic co-ordinates in this theory, and the role of the Legendre transform. In this Subsection it will be more convenient to use lower indices x_1, x_2 for our co-ordinates on the plane. Consider a convex function u on a convex open subset U of $[0,\infty)^2 \subset \mathbf{R}^2$ which satisfies Guillemin boundary conditions along the intersection of U with the axes, so

$$u = x_1 \log x_1 + x_2 \log x_2 - x_1 - x_2 + f(x_1, x_2),$$

where f is smooth on U. We suppose that the derivative ∇u maps the set $U \cap (0,\infty)^2$ *onto* the dual space, in which case the convexity condition implies that it is a diffeomorphism. Then the Legendre transform $\phi(\xi_1, \xi_2)$ is defined on the dual space by the formulas

$$\xi_a = \log x_a + \frac{\partial f}{\partial x_a},$$

and

$$\phi(\xi_1, \xi_2) = \sum x_a \xi_a - u(x_1, x_2) = -f(x_1, x_2) + \sum x_a \frac{\partial f}{\partial x_a}.$$

The basic fact that we need is that there is a 1-1 correspondence between pairs (u, U) as above and smooth $S^1 \times S^1$-invariant functions Φ on $\mathbf{C}^2$ with $i\overline{\partial}\partial\Phi > 0$. This is given by

$$\Phi(z_1, z_2) = \phi(\log |z_1|^2, \log |z_2|^2).$$

Further, if a family u_t varies continously with respect to an additional parameter (in the sense of C^∞ convergence of the functions f_t on compact subsets of their domains) then the transforms Φ_t vary continously in t (in the sense of C^∞ convergence on compact subsets of $\mathbf{C}^2$).

Now let (P, σ) be a weighted polygon and q be a vertex of P; the intersection of two edges E, E'. The linear parts of the functions $\lambda_E, \lambda_{E'}$ give a preferred set of linear coordinates on $\mathbf{R}^2$. If q' is another vertex the two sets of coordinates differ by an element $G(q, q') \in GL(2, \mathbf{R})$. We next review the "standard" case when all the $G(q, q')$ lie in $GL(2, \mathbf{Z})$, i.e., when (P, σ) is a "Delzant" weighted polygon. In this case we construct a complex surface $X^{\mathbf{C}}$ from the data in the following way. For each vertex q we take a copy $\mathbf{C}^2_q$ of $\mathbf{C}^2$ and we identify points using the $G(q, q')$ acting multiplicatively on the open subsets $(\mathbf{C}^*_q)^2 \equiv (\mathbf{C}^*)^2$. Thus if $G(q, q') = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we identify $(z_1, z_2) \in \mathbf{C}^2_q$ with $(z_1', z_2') \in \mathbf{C}^2_{q'}$ where

$$z_1' = z_1^a z_2^b \ , \ z_2' = z_1^c z_2^d.$$

In this way we get a complex surface $X^{\mathbf{C}}$, with a $(\mathbf{C}^*)^2$-action, containing an open dense orbit $X_0^{\mathbf{C}}$ which is identified with each of the $(\mathbf{C}_q^*)^2$. We denote the quotient space $X^{\mathbf{C}}/(S^1 \times S^1)$ by X. Any point v in $\mathbf{R}^2$ defines a map $\chi_v : X_0^{\mathbf{C}} \to \mathbf{R}^+$. In the chart $(\mathbf{C}_q^*)^2$ this is given by $(z_1, z_2) \mapsto |z_1|^\alpha |z_2|^\beta$, where v has components (α, β) in the coordinates $\lambda_E, \lambda_{E'}$. Suppose we have a function u on P which satisfies Guillemin boundary conditions. For each vertex q we translate to make q the origin, and identify $\overline{P}$ with a convex subset of $[0,\infty)^2$ using the maps $\lambda_E, \lambda_{E'}$. We take the Legendre transform ϕ_q and pass to logarithmic coordinates to obtain a smooth function Φ_q on $\mathbf{C}_q^2$. This yields a collection of functions (Φ_q) in our charts which satisfy:

1) $i\overline{\partial}\partial\Phi_q > 0$,
2) Φ_q is invariant under the action of $S^1 \times S^1$,
3) $\Phi_q - \Psi_{q'} = \log \chi_{q-q'}$ on $X_0^{\mathbf{C}}$.

Conversely, given such a collection Φ_q, we can recover u, up to the addition of an affine linear function on P. Further, the derivative of ϕ_q defines a homeomorphism from $X = X^{\mathbf{C}}/S^1 \times S^1$ to $\overline{P}$.

Next we move on to the case of a general weighted polygon (P, σ). While we cannot construct a space $X^{\mathbf{C}}$, we will see that most of the ideas above extend. We define a space X by taking for each vertex q a copy $[0,\infty)_q^2$ of $[0,\infty)^2$ and identify (r_1, r_2) in $(0,\infty)_q^2$ with $(r_1^a r_2^b, r_1^c r_2^d)$ in $(0,\infty)_{q'}^2$. Of course we can identify $[0,\infty)_q^2$ with a quotient of $\mathbf{C}_q^2$ by $S^1 \times S^1$. This space X has a dense open subset X_0 on which there are maps $\chi_v : X_0 \to (0,\infty)$. A function u on P satisfying Guillemin boundary conditions again yields a collection of functions Ψ_q on $\mathbf{C}_q^2$, with the same properties (1), (2), (3) as before, and u defines a homeomorphism from X to $\overline{P}$.

Here we digress to consider a general situation. Suppose we have a compact topological space Z which is covered by open "charts" $Z_\alpha \subset Z$. Suppose that for each α there is a homeomorphism from Z_α to B_α/G_α, where B_α is the unit ball in some Euclidean space and G_α is a compact Lie group, acting isometrically on the Euclidean space. We suppose we have sheaves $\mathcal{L}_k^p$ on Z which restrict, in the charts, to the G_α-invariant locally L_k^p functions on the Euclidean spaces (including $k = \infty$, with the obvious interpretation). In the case when the G_α are finite groups this is essentially the notion of an orbifold, but as far as the author knows there is not a standard terminology for the general situation. The usual machinery of global analysis transfers without difficulty to this situation. Thus if we suppose we have a local linear operator D taking functions (say) on Z to functions on Z, given in the charts by a collection of G_α-equivariant elliptic differential operators we can reproduce all the results of the Fredholm alternative, invertibility on Sobolev spaces etc. Similarly, for nonlinear operators we can apply the usual implicit function theorem arguments, and we will not take the space to formalise this further.

The point of the preceding remarks is that the space X is equipped with exactly this kind of structure. It is covered by open sets which are identified with quotients $\mathbf{C}^2_q/(S^1 \times S^1)$, and it is easy to see that there are unique sheaves $\mathcal{L}^p_k$ as above. Thus, while a general weighted polygon does not define a complex surface $X^{\mathbf{C}}$, it does define a space X in which we can apply the standard analytical machinery. If we fix a function u, and hence an identification between X and $\overline{P}$, one easily shows that the "smooth" functions $\mathcal{L}^p_\infty$ on X are identified with the smooth functions on the manifold with corners $\overline{P}$, but the situation for general p, k is not so clear and in any case we can avoid this issue by working systematically in the equivariant charts.

With all these preliminaries in place, we move on to our deformation problem. First consider the case where we fix the data (P, σ) and vary the function A. Of course we need to stay within the class where the mass and centre of mass of $(P, Ad\mu)$ agree with those of $(\partial P, \sigma)$. Working in a chart $\mathbf{C}^2_q$, we are in the standard situation, considering the scalar curvature $S(\Phi)$ of the metric determined by a Kahler potential Φ, with $\Phi = \Phi_q$. It is well known that this is a nonlinear elliptic differential operator. The linearisation has the form

$$S(\Phi + \eta) = S(\Phi) + \mathcal{D}^*\mathcal{D}(\eta) + \nabla S.\nabla\eta + O(\eta^2). \tag{3}$$

Here $\mathcal{D}$ is the Lichnerowicz operator $\overline{\partial}_T\nabla$, where $\overline{\partial}_T$ is the $\overline{\partial}$-operator on vector fields, and $\mathcal{D}^*$ is the formal adjoint. However there is a subtlety here, because the equation we want to solve is $S(\Phi) = A$ and while A is a prescribed function on the polygon P the identification between X and $\overline{P}$ also depends on Φ, so schematically we have an equation $S(\Phi) = A(\Phi)$. Simple calculations show that the dependence of A on Φ precisely cancels out the "extra" term in (3). In other words, if we vary our function A on P to $A + \alpha$ then the linearisation of the equation in the chart $\mathbf{C}^2_q$ is just $\mathcal{D}^*\mathcal{D}\eta = \alpha$, where α is regarded as a function on $\mathbf{C}^2_q$ via the identification furnished by Φ_q. This is rather clear from the "moment map" point of view (compare the discussion in [**7**]), and we will not take more space to discuss the calculations here. The upshot is that we can solve the nonlinear equation, for small variations of A, provided we avoid the obstructions from the cokernel of the linearisation $\mathcal{D}^*\mathcal{D}$, which is the same as the kernel of $\mathcal{D}$. But this kernel consists exactly of the pull-back of the affine-linear functions on $\overline{P}$ and the constraint is just that the mass and centre of mass of α vanish, which is true by hypothesis.

The case where we deform the data (P, σ) is a little more complicated. Consider a 1-parameter family (P_t, σ_t) of small deformations of (P_0, σ_0) (in reality the nature of the parameter space is irrelevant). Choose a family of diffeomorphisms χ_t as above, and let $u_t = u_t \circ \chi_t^{-1}$. Then u_t is convex on P_t and satisfies Guillemin boundary conditions, for small t. Fix a vertex q of P_0 where edges E, E' meet. There is no loss in supposing that q is the origin and that $\lambda_{E,0}, \lambda_{E',0}$ are the standard coordinate functions (x_1, x_2). The chart $\mathbf{C}^2_q$ is regarded as a fixed space, independent of t, and for small t we have a function $\Phi_{q,t}$ on $\mathbf{C}^2_q$ obtained from the Legendre transform of u_t. Unwinding the definitions, $\Phi_{q,t}(z_1, z_2) = \phi_{q,t}(\log|z_1|, \log|z_2|)$ where $\psi_{q,t}$ is the Legendre transform of a function u_t^* on a convex set $U_t \subset [0,\infty)^2$. The function u_t^* has the form $u \circ W_t^{-1}$, where W_t is a diffeomorphism from U_0 to $U(t)$ which we write as $\tilde{W}(x_1, x_2) = (\tilde{x}_1(x_1, x_2), \tilde{x}_2(x_1, x_2))$. This diffeomorphism has the property that $\tilde{x}_i = x_i$ when x_i is small, in particular it is the identity in a neighbourhood of the origin, so $u_t^* = u$ near the origin. It is clear then that u_t^* converges to u as $t \to 0$, in C^∞ on compact sets. Thus the corresponding functions $\Phi_{q,t}$ converge, in C^∞ on compact sets by the remarks above.

The conclusion of the discussion above is the following. Let X_t be the space associated with (P_t, σ_t). For small t and each vertex q of P_0 we have an atlas of "charts"

$$\pi_{q,t} : \mathbf{C}^2_q \to X_t$$

covering X_t. In these charts the equation $S(\Phi) = A$ we want to solve is given by a continuously varying family of nonlinear elliptic PDE for invariant functions. Thus, as before, we can adapt the usual theory from the manifold case to construct solutions.

2.3. A starting point. It is clear that any two plane polygons with the same number of edges can be joined by a continuous path. The next issue we need to address is the existence of *some* data set for which a solution to our problem exists. This is trivial if we allow arbitrary functions A, but for later developments we want to be able to restrict to the cases where A is constant.

Proposition 3. *For each $r \geq 3$ there is a polygon P with r vertices and a set of weights σ such that there is a solution to our problem for the data $(P, \sigma, 1)$.*

One possible approach to this is to consider the canonical weights σ_P associated to any polygon P. In this case a solution to the constant scalar curvature equation must actually satisfy a second order equation of Monge-Ampere type, corresponding (in the local complex differential geometry) to a Kahler-Einstein metric. This equation, expressed on P, is

$$\log\det(u_{ij}) = u - x^i u_i.$$

Then one can hope to extend the proof by Wang and Zhu [**18**] of the existence of Kahler-Einstein metrics on toric Fano varieties to the case of a general polygon P. However instead we will outline another approach by adapting arguments of Arezzo and Pacard [**3**],[**4**].

Suppose first that (P, σ) is a Delzant weighted polygon, with $A_\sigma = 1$ (which is, in other langauge, the vanishing of the "Futaki invariant"). One way in which this vanishing condition can occur is if P is symmetrical about the origin under the map $x \mapsto -x$, and for simplicity let us suppose that this is the case. Suppose we know that the polarised variety X corresponding to P admits a constant scalar curvature metric. Now Arezzo and Pacard study the following general problem: if we know that a complex surface Z admits a constant scalar metric, find a constant scalar curvature metric on the blow-up $\hat{Z}$ of Z at some finite set of points $z_1, \ldots z_q$ in Z. In this problem there is a positive real parameter associated to each point: the integral of the class of the Kahler form on the corresponding exceptional divisor. Arezzo and Pacard show that one can find such a metric, for small values of these parameters, modulo obstructions coming from the kernel $\mathcal{H}$ of the operator $\mathcal{D}$ on Z. Thus there is a smooth map $F : [0, \infty)^q \to \mathcal{H}$ with $F(0) = 0$ and the zeros of F in $(0, \delta)^q$ give constant scalar curvature metrics. Now in our case we choose a pair of vertices $q, -q$ of P. These correspond to points, Q and $-Q$ say, in X, which are fixed points of the torus action. Then the blow up $\hat{X}$ is another toric surface.

The translation of the blow-up construction to the language of polygons is well-known. Choose coordinates, as in the previous subsection, so that q is the origin and $\lambda_E, \lambda_{E'}$ are the standard coordinate functions x^1, x^2. Then for small ϵ we form a new polygon by removing the triangle

$$\{(x^1, x^2) : x^1 > 0, x^2 > 0, x^1 + x^2 \geq \epsilon\}$$

from P. This operation corresponds to blowing up the point Q, and ϵ to the blow-up parameter mentioned above. The boundary measure on the new polygon is fixed as follows. On the portion of the boundary which coincides with the boundary of P the measure is the same as the original one. On the "new" piece of boundary, corresponding to $x^1 + x^2 = \epsilon$ in the coordinates above, the measure is chosen so that the mass of the new edge is the same as each of the portions of the original edges which were removed. Of course when we blow up both points $Q, -Q$ we "cut off" two triangles, one with a vertex at q and one with a vertex at $-q$. If we choose the blow-up parameters to be equal then the new polygon P_ϵ has the same symmetry under $x \mapsto -x$. In this situation the obstructions arising from the kernel of $\mathcal{D}$—i.e., from the affine-linear functions on P, are forced to vanish by the symmetry and it follows directly from the results of Arrezzo and Pacard that there is a solution of our problem on P_ϵ, for small enough ϵ and suitable weights.

This argument comes close to solving our problem. We can start with the square, corresponding to the manifold $S^2 \times S^2$ with a standard constant scalar curvature metric. Then cut off two opposite corners to get a solution for a hexagon, symmetric about the origin. Then cut off two opposite corners of this to get a solution for an octagon, and so on. Thus we find r-gons admitting solutions for any *even* value of r.

Perhaps this argument can be extended by some elementary trick to cover odd values of r but, lacking this, we go back to appeal to the core idea underlying Arezzo and Pacard's construction, adapted to the toric situation. They take the standard zero scalar curvature "Burns metric" on the blow up of $\mathbf{C}^2$ at the origin, which is asymptotically Euclidean, scale this by a small factor and glue it to the original metric on Z to obtain an "approximate solution" on the blow up. Then the heart of the matter is to study the problem of deforming this to a genuine solution, via an implicit function theorem and analysis of the linearised equation. Just as in the previous subsection, in the toric case the the space $X^{\mathbf{C}}$ itself plays no real role here and everything can be formulated in terms of corresponding operations on the space X, using identical local formulae in our equivariant charts. Further, also as in the previous subsection, the obstructions to finding a solution can be completely understood in terms of the centre of mass of the measure σ.

Let P be a polygon with at least 4 vertices and centre of mass at the origin. Let q be a vertex of P and let F, F' be two edges of P which do *not* contain q. Let σ be a weight function on ∂P such that $A_\sigma = 1$ and suppose that there is a solution u of our problem for the data P, σ, i.e., a constant scalar curvature metric. Take two positive real parameters λ, μ and consider the family of weight functions $\sigma(\lambda, \mu)$ on ∂P with

$$\sigma_{\lambda,\mu}(F) = \lambda\sigma(F)\ ,\ \sigma_{\lambda,\mu}(F') = \mu\sigma(F')$$

and with $\sigma_{\lambda,\mu}$ equal to σ on all the other edges. Then the centre of mass of $(\partial P, \sigma_{\lambda,\mu})$ yields a map from $\mathbf{R}^+ \times \mathbf{R}^+$ to $\mathbf{R}^2$, and it is easy to see that the derivative has rank 2 at the point $\lambda = \mu = 1$. Now take another small parameter ϵ and define a polygon P_ϵ by cutting off a small triangle at q, using this parameter, in the manner discussed above. For each λ, μ we get a weight function $\hat{\sigma}_{\lambda,\mu}$ for P_ϵ. Let $v(\lambda, \mu, \epsilon) \in \mathbf{R}^2$ be the difference of the centre of mass of P_ϵ and $(\partial P_\epsilon, \hat{\sigma}_{\lambda,\mu})$. The implicit function theorem implies that there are smooth functions $\lambda(\epsilon), \mu(\epsilon)$ such that $\lambda(0) = \mu(0) = 1$ and

$$v(\lambda(\epsilon), \mu(\epsilon), \epsilon) = 0.$$

(Of course, what is involved here is just elementary geometry, and one could write these functions down explicitly if desired.) This means that, when $\lambda = \lambda(\epsilon), \mu = \mu(\epsilon)$ the data $(P_\epsilon, \hat{\sigma}_{\lambda,\mu})$ satisfies the obvious necessary condition to have a constant scalar curvature metric, i.e., $A_{\hat{\sigma}_{\lambda,\mu}}$ is constant. Adapting the proof of Arrezzo and Pacard one can show that there is indeed a solution, for small enough ϵ. Using this repeatedly we get r-gons admitting solutions for all $r \geq 4$. When $r = 3$ we can use the standard solution coming from the Fubini Study metric on $\mathbf{CP}^2$ and thus complete the proof of Proposition 3.

3. Geometric estimates

3.1. Riemannian geometry in the polygon. Throughout this section we consider a function u on a polygon $P \subset \mathbf{R}^2$ as before, satisfying Guillemin boundary conditions determined by a weight function σ. We consider the Riemannian metric g on P defined by the Hessian u_{ij}, along with its extension $\hat{g}$ to $P \times \mathbf{R}^2$. Then P can be regarded as a totally geodesic submanifold of $P \times \mathbf{R}^2$. Suppose, momentarily, that the data (P, σ) is Delzant, so corresponds to a genuine 4-manifold $X^{\mathbf{C}}$, a compactification of $P \times T^2$. Then there is an isometric involution of $X^{\mathbf{C}}$ (given by $\theta_i \mapsto -\theta_i$) with fixed set a smooth surface Σ which can be obtained by gluing 4 copies of $\overline{P}$ along suitable edges, and the metric g extends smoothly to Σ. It is easy to see from this that, in any case, the metric g extends to a Riemannian metric on $\overline{P}$, equipped with a suitable smooth structure (as a 2-manifold with corners), and that the edges are geodesics. Thus $\overline{P}$ is geodesically convex, in that any two points can be joined by a minimal geodesic, and any geodesic can be extended until it reaches the boundary. A main theme of this subsection is to relate the Riemannian geometry and the Euclidean geometry in P. We write Dist_g for the distance function defined by g and $\mathrm{Dist}_{\mathrm{Euc}}$ for the Euclidean distance. Recall from [**8**], Sec. 5.2 that the tensor

$$F^{ij}_{kl} = u^{ij}_{kl} = \frac{\partial^2 u^{ij}}{\partial x^k \partial x^l}$$

defined by the function u is equivalent to the Riemann curvature tensor of the metric $\hat{g}$. We define

$$|F|^2 = F^{ij}_{kl} F^{ab}_{cd} u_{ia} u_{jb} u^{kc} u^{ld}.$$

Then the absolute value of the sectional curvatures of $\hat{g}$ are bounded by $|F|$. In this section we will explore the interaction between the M-condition and a bound on $|F|$. A crucial fact that we will use later in the paper is that if $u^{ij}_{ij} = -A$ then

$$\int_P |F|^2 d\mu_{\mathrm{Euc}} - \int_P A^2 d\mu_{\mathrm{Euc}} \tag{4}$$

is an invariant of the data (P, σ), see [**8**], Corollary 5.

Lemma 2. *Suppose u satisfies the M-condition. Let I be a line segment in $\overline{P}$ with mid-point p and let p' be an end point of I. Then the Riemannian length of the segment pp' is at most*

$$\frac{1}{(\sqrt{2}-1)} \sqrt{M} \sqrt{|p - p'|_{\mathrm{Euc}}}.$$

We can suppose that p' is the origin and that p is $(L,0)$, so $|p-p'|_{Euc}=L$ and the segment of the x^1-axis from 0 to $2L$ lies in $\overline{P}$. We apply the definition of the M-condition to the pair of points p,q, where $q=(L/2,0)$. This gives

$$\int_{L/2}^{L} u_{11}(t,0)dt \leq M.$$

The Riemannian length of the straight line segment from q to p is

$$\int_{L/2}^{L} \sqrt{u_{11}}(t,0)dt,$$

which is at most

$$\sqrt{(L/2)}\left(\int_{L/2}^{L} u_{11}(t,0)dt\right)^{1/2};$$

hence the Riemannian length of this segment is at most $\sqrt{LM/2}$. Replacing p by $2^{-r}p$ and summing over r we see that the Riemannian length of the segment from 0 to p is at most

$$\sqrt{(ML)}\sum_{r=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{r},$$

from which the result follows.

Corollary 1. *Suppose that u satisfies the M condition and that p is a point of P. Then*

$$\mathrm{Dist}_g(p,\partial P) \leq \frac{1}{\sqrt{2}-1}\sqrt{M}\sqrt{\mathrm{Dist}_{\mathrm{Euc}}(p,\partial P)}.$$

To see this we take p' to be the point on ∂P closest to p, in the Euclidean metric. If $p''=2p-p'$ then the segment $p'p''$ lies in $\overline{P}$ and we can apply the Lemma above.

Next we derive a crucial result which relates the restriction of u to lines and the curvature tensor F.

Lemma 3. *At each point of P,*

$$\left(\frac{\partial}{\partial x^1}\right)^2 (u_{11}^{-1}) \leq |F|.$$

One way of approaching this is to observe that the restriction of the function u to a slice $\{x^2=\text{constant}\}$ represents the metric on a symplectic quotient, and then to exploit the fact that curvature increases in holomorphic quotient bundles. However we will not explain this further and instead give a direct proof. Observe that the quantity

$$\left(\frac{\partial}{\partial x^1}\right)^2 (u_{11}^{-1})$$

is unchanged by rescaling x^1. This means that, by rescaling x^1 and making a different choice of x^2, we can suppose that at the point p_0 in question u_{ij} is the standard Euclidean tensor. Then the square of the norm of the curvature tensor at this point is

$$|F|^2 = \sum_{i,j,k,l} \left(u^{ij}_{kl}\right)^2 ,$$

and so $u^{11}_{11} \leq |F|$. Now, at a general point of P we have

$$u^{11} = \frac{u_{22}}{u_{11}u_{22} - u_{21}^2},$$

which gives

$$u^{11} - u_{11}^{-1} = \frac{u_{12}^2}{u_{11}(u_{11}u_{22} - u_{12}^2)}.$$

Since u_{12} vanishes at the point p_0 we have

$$\left(\frac{\partial}{\partial x^1}\right)^2 (u^{11} - u_{11}^{-1}) = 2\frac{(u_{121})^2}{u_{11}^2 u_{22}} = 2(u_{121})^2 \geq 0$$

at p_0. So

$$\left(\frac{\partial}{\partial x^1}\right)^2 u_{11}^{-1} \leq u^{11}_{11} \leq |F|.$$

Lemma 4. *Let p be a point of P and $\nu = (\nu^i)$ a unit vector. Suppose the segment $\{p + t\nu : -3R \leq t \leq 3R\}$ lies in P, that $|F| \leq 1$ in P and that u satisfies the M-condition. Then*

$$u_{ij}\nu^i\nu^j \leq \mathrm{Max}\left(\frac{2M}{\pi R}, 2\left(\frac{M}{\pi}\right)^2\right).$$

We can suppose that ν is the unit vector in the x^1 direction and that p is the origin. Let $H(t) = u_{11}(t, 0)$. We apply the definition of the M-condition to obtain

$$\int_{-R}^{R} H(t)dt \leq M.$$

By the previous Lemma,

$$\frac{d^2}{dt^2}H(t)^{-1} \leq 1.$$

Suppose $H(0)^{-1} = \epsilon$. Then for any $t \in [-R, R]$ we have

$$H(\pm t) \leq \epsilon \pm Ct + t^2/2$$

where $C = H'(0)$. Since $H(\pm t) > 0$ we must have $\epsilon \pm Ct + t^2/2 > 0$. Then, adding the two terms, we have

$$H(t) + H(-t) \geq \frac{1}{\epsilon + Ct + t^2/2} + \frac{1}{\epsilon - Ct + t^2/2} \geq \frac{2}{\epsilon + t^2/2}.$$

This gives

$$\int_{-R}^{R} H(t)dt \geq \int_{-R}^{R} \frac{dt}{\epsilon + t^2/2} = 2\epsilon^{-1/2} \int_{0}^{R\epsilon^{-1/2}} \frac{dt}{1+t^2/2}.$$

So we have

$$M \geq \frac{2\sqrt{2}}{\sqrt{\epsilon}} \tan^{-1}\left(\frac{R}{\sqrt{2\epsilon}}\right).$$

Now use the fact that

$$\frac{4}{\pi}\tan^{-1}(z) \geq \mathrm{Min}(1, z)$$

and a little manipulation to obtain the stated bounds on $\epsilon^{-1} = u_{11}(0,0)$.

The results in the rest of this subsection depend upon a special feature of the Riemannian metric g, and its relation to the metric $\hat{g}$. Consider the 1-forms $\epsilon_i = dx^i$ on $P \times \mathbf{R}^2$. Under the isomorphism between cotangent vectors and tangent vectors defined by the symplectic form these correspond to the Killing fields $\frac{\partial}{\partial \theta_i}$. These two Killing fields span a covariant constant subspace of the tangent space, on the other hand they are Jacobi fields along any geodesic in P. Thus we conclude that the 1-forms $\epsilon_i = dx^i$ satisfy a Jacobi equation of the schematic form

$$\nabla_t^2 \epsilon_i = F * \epsilon_i,$$

along any geodesic. Expressed in different notation, if e_1, e_2 is a parallel frame of cotangent vectors along a geodesic and if we write $\epsilon^i = \sum G_{ij} e_j$, then the matrix $G(t)$ satisfies an equation of the form

$$\frac{d^2}{dt^2} G = -RG,$$

where R is a symmetric matrix with $|\,(R_{ij})\,| \leq |F|$. If we express things in terms of the vector fields $\frac{\partial}{\partial \theta_i}$ on the 4-manifold this is almost the same as the standard discussion, as in [**11**], of the Fermi fields associated to the orbits of the isometric action.

We need a simple comparison result for Jacobi fields.

Lemma 5. *Suppose that $R(t)$ is a symmetric $k \times k$ matrix-valued function on an interval $(0, a)$ with $R(t) \geq -1$. Suppose that $\epsilon_1(t), \ldots, \epsilon_k(t)$ are k-vector solutions of the Jacobi equation $\epsilon'' = -R\epsilon$ which are linearly independent at each point in the interval and with $(\epsilon_i', \epsilon_j) = (\epsilon_i, \epsilon_j')$. Then $\frac{|\epsilon_1(t)|}{\sinh t}$ is a decreasing function of t.*

The author does not find precisely this result stated in standard textbooks, so we give a proof, although this follows familiar lines. Fix a point $t_0 \in (0, a)$ and consider the derivative of $|\epsilon_1(t)|/\sinh t$ at $t = t_0$. Clearly we can suppose that $\epsilon_i(t_0)$ is the standard orthonormal frame for the k-vectors. In particular $|\epsilon_1(t_0)| = 1$ and we want to show that $(\epsilon_1', \epsilon_1) \leq \cosh t_0/\sinh t_0$ at $t = t_0$. Express $\epsilon_i(t)$ in terms of the fixed orthonormal frame by $\epsilon_i = \sum G_i^j e_j$ so G is a solution of the matrix equation $G'' = -RG$ with $G(t_0) = 1$. Set $S = G'G^{-1}$ so that S satisfies the Ricatti equation $S' + S^2 = -R$. The hypothesis that $(\epsilon_i', \epsilon_j) = (\epsilon_i, \epsilon_j')$ implies that $S(t)$ is symmetric for all t. At $t = t_0$ we have $(\epsilon_1', \epsilon_1) = S_{11}$, the $(1,1)$ entry of the matrix S, so it suffices to prove that $S(t_0) \leq \cosh t_0/\sinh t_0$, or equivalently that all the eigenvalues of $S(t_0)$ are bounded above by $\cosh t_0/\sinh t_0$. Now each eigenvalue $\lambda(t)$ of $S(t)$ satisfies a scalar Ricatti differential inequality

$$\lambda' + \lambda^2 \leq 1$$

(see [**11**], [**13**]: by standard arguments we may ignore the complications that might occur from multiple eigenvalues). Suppose that $\lambda(t_0) > \cosh t_0/\sinh t_0$. Then we can find $\tau \in (0, t_0)$ such that $\lambda(t_0) = \cosh(t_0 - \tau)/\sinh(t_0 - \tau)$. Now the function $\mu(t) = \cosh(t-\tau)/\sinh(t-\tau)$ satisfies the equation $\mu' + \mu^2 = 1$. So $\lambda' + \lambda^2 \leq \mu' + \mu^2$ in the interval $(\tau, t_0]$ and $\lambda(t_0) = \mu(t_0)$. It follows that $\lambda(t) \geq \mu(t)$ for $t \in (\tau, t_0)$ and since $\mu(t) \to \infty$ as t tends to τ from above we obtain a contradiction.

Notice that Lemma 5 contains as a special case the familiar Rauch comparison result: if $|\epsilon_1| \sim t$ as $t \to 0$ then $|\epsilon(t)| \leq \sinh t$ for all $t < a$. Notice also that the hypothesis $(\epsilon_i', \epsilon_j) = (\epsilon_i, \epsilon_j')$ is satisfied in our situation, as one sees by a standard manipulation involving the Lie brackets of the $\frac{\partial}{\partial \theta_i}$.

Lemma 6. *Let E be an edge of the polytope P and suppose that the defining function λ_E (determined by σ) is x^1. Then if u satisfies Guillemin boundary conditions and $|F| \leq 1$ throughout P, we have*

$$u^{11}(p) \leq \sinh^2 \mathrm{Dist}_g(p, E)$$

for any p in P.

To see this we consider a geodesic parametrised by $t \geq 0$, starting at time 0 on the boundary component E. Near the boundary we can describe the geometry in terms of a 4-manifold with a group action in the familiar way. The vector field $\frac{\partial}{\partial \theta_1}$ is smooth in the 4-manifold and vanishes at $t = 0$. The condition that x^1 is the normalised defining function just asserts that this vector field is the generator of a circle action of period 2π. It follows that

$$\lim_{t \to 0} t^{-1} |\frac{\partial}{\partial \theta_1}| \leq 1$$

(with equality when the geodesic is orthogonal to the edge E). Then, by the above, $\sqrt{u^{11}} = |\frac{\partial}{\partial \theta_1}| \leq \sinh t$ and the result follows.

Corollary 2. *Let E be an edge of P with defining function λ_E. Then if $|F| \leq 1$ we have*

$$\lambda_E(p) \leq \cosh(\mathrm{Dist}_g(p, E)) - 1.$$

Notice that this is an affine-invariant statement. There is no loss in supposing that, as above, $\lambda_E = x^1$. Then for a geodesic starting from a point of E, parametrised by arc length, we have

$$|\frac{dx^1}{dt}| \leq |dx^1|_g = \sqrt{u^{11}} \leq \sinh t;$$

hence $x^1 \leq \cosh t - 1$.

Lemma 7. *Suppose that $|F| \leq 1$ and that p is a point in P with $dist_g(p, \partial P) \geq \alpha > 0$. Then if q is a point with $\mathrm{Dist}_g(p,q) = d$, we have*

$$\left(u^{ij}(q)\right) \leq \frac{\sinh^2(\alpha+d)}{\sinh^2\alpha}\left(u^{ij}(p)\right).$$

If $d < \alpha$, we have

$$\left(u^{ij}(q)\right) \geq \frac{\sinh^2(\alpha-d)}{\sinh^2\alpha}\left(u^{ij}(p)\right).$$

(Here the notation $\left(A^{ij}\right) \leq \lambda\left(B^{ij}\right)$ means that for any vector ν_i we have $\nu_i\nu_j A^{ij} \leq \lambda\nu_i\nu_j B^{ij}$.) To prove the Lemma, observe that it suffices by affine invariance to prove the corresponding inequalities for the matrix entry $u^{11} = |\epsilon_1|^2$. For the first inequality we consider a minimal geodesic γ from $p = \gamma(0)$ to $q = \gamma(d)$ and extend it "backwards" to $t > -\alpha$. Then replacing t by $t + \alpha$ we are in the situation considered in Lemma 5 and we obtain

$$\frac{|\epsilon_1(p)|}{\sinh\alpha} \geq \frac{|\epsilon_1(q)|}{\sinh(\alpha+d)}.$$

For the second inequality we extend the geodesic "forwards" to the interval $[0,\alpha]$ and argue similarly.

Suppose that $p = (p^1, p^2)$ is a point of P and $r > 0$. Put

$$E_{p,r} = \{(x^1,x^2) \in \mathbf{R}^2 : u_{ij}(p)(x^i - p^i)(x^j - p^j) \leq r^2\}.$$

So $E(p,r)$ is the interior of the ellipse defined by the parameter r and the quadratic form $u_{ij}(p)$.

The Euclidean area of $E(p,r)$ is $\pi r^2 \det(u_{ij}(p))^{-1/2}$.

Lemma 8. *Suppose that $|F| \leq 1$ and that p is a point in P with $\mathrm{dist}_g(p, \partial P) \geq \alpha > 0$. Then for any $\beta < \alpha$ the β-ball in P, with respect to the metric g satisfies*

$$E(p, c\beta) \subset B_g(p,\beta) \subset E(p, C\beta),$$

where $c = \sinh(\alpha-\beta)/\sinh\alpha$ and $C = \sinh(\alpha+\beta)/\sinh\alpha$. In particular, the Euclidean area of the β ball for the metric g is bounded below by

$$\mathrm{Area}_{\mathrm{Euc}} B_g(p,\beta) \geq \pi c^2\beta^2 \det(u_{ij})(p)^{-1/2}.$$

There is no loss in supposing that the matrix $u^{ij}(p)$ is the identity matrix, so we have to show that the ball $B_g(p,\beta)$ defined by the metric g contains a Euclidean disc of radius $c\beta$, and is contained in a Euclidean disc of radius $C\beta$. We know by Lemma 7 that on the ball $B_g(p,\beta)$ we have

$$c^2 \leq (u^{ij}) \leq C^2.$$

Thus $C^{-2} \leq (u_{ij}) \leq c^{-2}$, and the Euclidean length of a path in $B_g(p,\rho)$ is at least c^{-1} times the length calculated in the metric g, and at most C^{-1} times that length. The second statement immediately tells us that $B_g(p,\beta)$ lies in $E(p,C\beta)$. In the other direction, suppose q is a point in the Euclidean disc of radius $c\beta$ centred on p. We claim that q lies in the (closed) g ball $B_g(p,\beta)$. For if not there is point q' in the open line segment pq such that the distance from q' to p is β and the line segment pq' lies in $B_g(p,\beta)$. But the Euclidean length of this line segment is strictly less than $c\beta$ so the length in the metric g is less than β, a contradiction.

3.2. The injectivity radius. We continue to consider a convex function u, satisfying Guillemin boundary conditions, on a polygon P, as in the previous subsection. The present subsection has two purposes. In one direction we discuss coordinates in neighbourhoods of boundary points obtained from geodesic coordinates in four dimensions. In another direction, we want to relate these ideas to the standard notion of the injectivity radius. Since we will want sometimes to work with incomplete manifolds we should clarify our definitions. By the statement that "the injectivity radius at a point p is at least r" we mean that the exponential map at p is defined on tangent vectors of length r, and yields an embedding of the Euclidean r-ball. In fact the discussion of the injectivity radius need only enter our main proof in a rather minor way, but it is useful to explain how the arguments fit into the wider world of Riemannian geometry.

First we consider the vertices. Let q be a vertex of P, so we have an Riemannian 4-manifold X_q^c, which is not complete. The torus action on X_q^c gives a constraint on the exponential map.

Lemma 9. *If $|F| \leq 1$ in P then the injectivity radius of X_q^c at q is at least $\pi/2$.*

The exponential map is equivariant with respect to the standard torus action on the tangent space at q. Suppose the exponential map is defined for some $r' < r$ and let ξ be a unit vector in the Lie algebra of the torus, corresponding to a vector field v_ξ on X_q^c. Then the length of the vector field v_ξ is bounded below on the boundary of the r' ball. However, if the exponential map is not defined on the r ball then as we let r' approach its maximal possible value there is some choice of ξ such that the length of v_ξ goes to zero on the boundary (since the corresponding points in $\overline{P}$ must be approaching another edge).

Now suppose that the r ball is not embedded by the exponential map. Then there is a nontrivial geodesic starting and ending at q, of length less than π. But the vector fields $\frac{\partial}{\partial\theta_i}$ give Jacobi fields along this geodesic, vanishing at the endpoints. By a standard comparison theorem these vector fields must vanish identically along the geodesic which means that the initial tangent vector of the geodesic is fixed by the torus action. Since there are no such fixed tangent vectors we have a contradiction.

Now we note a general fact of Riemannian geometry.

Lemma 10. *Let $g_{ij} = \delta_{ij} + \eta_{ij}$ be a Riemannian metric on the Euclidean ball B of radius π in $\mathbf{R}^n$, with sectional curvature bounded in absolute value by 1. Suppose that $|\,(\eta_{ij})\,| \leq \epsilon\,(\delta_{ij})$, for some $\epsilon < 1$. Then the injectivity radius at the origin is at least $\sqrt{1-\epsilon}$.*

First, the g-distance from the origin to the boundary of the ball B is at least $\pi\sqrt{1-\epsilon}$, so the exponential map is defined as stated. Since the curvature is less than 1, we only need to check that there are no geodesic loops starting and ending at the origin, of length less than $2\sqrt{1-\epsilon}$. Suppose γ is a geodesic loop, of length L, and for $s < 1$ let γ_s be the loop $\gamma_s(t) = s\gamma(t)$. Then the length of γ_s is at most $L' = \sqrt{\frac{1+\epsilon}{1-\epsilon}}L$. For small s the loop γ_s can be lifted to a loop over the exponential map. The argument on page 100 of [**6**] (proof of a Theorem of Klingenberg) shows that this is true for all s, provided that $L' < \pi$, which will be the case if $L < 2\sqrt{1-\epsilon}$. But, as in the argument cited, γ itself lifts to a ray under the exponential map, giving a contradiction.

Now we consider an interior point q of the polygon. We can think of this as a point in the Riemannian 4-manifold $P \times \mathbf{R}^2$, with the metric $\hat{g}$ and we write $I(q, \hat{g})$ for the injectivity radius at that point. We can also consider q as a point in the quotient space $P \times \mathbf{R}^2/\mathbf{Z}^2$ and we write $I'(q, \hat{g})$ for the injectivity radius there.

Lemma 11. *Suppose $|F| \leq 1$ in P.*

1) *For any $\alpha > 0$ there is an $i(\alpha) > 0$ such that if $\mathrm{Dist}_g(q, \partial P) \geq \alpha$ then $I(q, \hat{g}) \geq i(\alpha)$.*
2) *If u satisfies an M condition then there is an $i(\alpha, M) > 0$ such that if $\mathrm{Dist}_g(q, \partial P) \geq \alpha$ then $I'(q, \hat{g}) \geq i'(\alpha, M)$.*

To prove the first item we apply Lemma 8. We can suppose that the Hessian u_{ij} at the point q is the standard form δ_{ij}. Then Lemma 8 tells us that that the metric $\hat{g}$ is close to Euclidean—in the given coordinates x^i, θ_j—over a ball of a definite size determined by α. Then we can apply Lemma 10. To prove the second item we just need to check that the quotient by $\mathbf{Z}^2$ does not create any short loops. Since the metric in the fibre direction is given by $u^{ij} d\theta_i d\theta_j$, this is the same as showing that for any non-zero integer vector ν_i the quantity $u^{ij}\nu_i\nu_j$ is not small. But we know, by combining Lemmas 2 and 4, that $u_{ij} \leq C$, where C depends on M, α. This implies that $u^{ij}\nu_i\nu_j \geq C^{-1}(\nu_1^2 + \nu_2^2) \geq C^{-1}$.

To take stock of our progress so far, consider the case when P corresponds to a compact 4-manifold X^c. Then Lemmas 9 and 11 give lower bounds on the injectivity radius at points of X^c which correspond to either vertices or to interior points of P. Our remaining task is to consider the points which lie on the boundary edges. For this we introduce a numerical invariant of a weighted polygon (P, σ). Let q be a point in the interior of an edge E and let d be the Euclidean distance from q to the end points of E. Set

$$\mu(q) = \min_{E'} \frac{\lambda_{E'}(q)}{d},$$

where E' runs over the set of edges *not equal* to E. Now let $\mu = \mu_{P,\sigma}$ be the minimum of $\mu(q)$ over all such boundary points q. It is easy to see that $\mu_{P,\sigma} > 0$.

For each (open) edge E of P we define a Riemannian 4-manifold X^c_E as follows. We choose coordinates such that the defining function λ_E is x^1 and take the quotient of $P \times \mathbf{R}^2$ by Λ, where Λ is the copy of $\mathbf{Z}$ embedded as $\mathbf{Z} \times \{0\}$ in $\mathbf{R}^2$. This gives a manifold with an action of $S^1 \times \mathbf{R}$. Then, just as in the construction of the manifolds X^c_q associated to vertices q, we can adjoin a copy of $E \times \mathbf{R}$, fixed under the circle action, and the metric extends smoothly. If q is a point on the interior of E we write $I(q, \hat{g})$ for the injectivity radius about the corresponding point in X^c_E. If (P, σ) is Delzant we can also consider q as a point in the compact manifold X^c and we write $I'(q, \hat{g})$ for the injectivity radius there.

Lemma 12. *Suppose that $|F| \leq 1$ in P and that u satisfies an M condition. Then for any $\alpha > 0$ there is an $i(\alpha, \mu, M) > 0$ such that $I(q, \hat{g}) \geq i(\alpha, \mu, M)$ if the distance in the metric g from q to the set of vertices is at least α. If (P, σ) is Delzant then there is an $i'(\alpha, \mu, M) > 0$ such that $I'(q, \hat{g}) \geq i'(\alpha, \mu, M)$.*

Not surprisingly, the proof of this Lemma–for an edge point–is a combination of the arguments used in the cases of vertices and interior points. The first thing is to see that the exponential map at q in X^c_E is defined on a ball of a definite size (depending on α, μ, M). This is the same as showing that the distance in the metric g from q to any other edge E' of P is not small. But we know by Lemma 2 that the Euclidean distance d from q to the end points of E is not small, hence by the definition of μ, $\lambda_{E'}(q)$ is bounded below by a quantity depending on μ, M, α.Then Corollary 2 implies that the distance in the metric g from q to E' is not too small. The remaining task is to show, as in the proof of Lemma 9, that there are no short geodesic loops in X^c_E

starting at q. Now there is a circle action on X^c_E which fixes the point q and the argument used in the proof of Lemma 9 shows that any short geodesic loop must lie in the fixed set of the action, which is $E \times \mathbf{R}$. The Riemannian metric on $E \times \mathbf{R}$ is defined by the restriction of u to E. The arguments used in the proof of Lemma 11 apply, in an obvious way, to give a lower bound on the injectivity radius in $E \times \mathbf{R}$, so we see that there are no short geodesic loops and the proof of the lower bound on $I(q, \hat{g})$ is complete.

In the case when (P, σ) is Delzant a neighbourhood of q in X^c is quotient of X^c_E by an action of $\mathbf{Z}$ and we again we need to show that this does not create any short loops. This just comes down an upper bound on the second derivative of u along the edge, which is furnished by Lemma 4 and the M-condition.

Proposition 4. *Suppose that (P, σ) is Delzant, that u satisfies an M condition and $|F| \leq 1$ in P. Then there is an r, depending only on M and $\mu_{P,\sigma}$, such that the injectivity radius of the Riemannian 4-manifold X^c is at least r.*

By applying Lemma 10 it suffices to show that for any $\kappa > 0$ there is an r' such that for each point p of X^c we can find another point p' such that the injectivity radius at p' is at least r' and the distance from p to p' is at most $\kappa r'$. If p is close to a vertex we take p' to be the vertex and use Lemma 9. If p is close to an edge but not close to any vertex we take p' to be a nearby point on the edge, and use Lemma 12. If p is not close to any edge we take $p' = p$ and use Lemma 11.

We conclude this section with another simple observation, similar to Lemma 9, which will be useful later.

Lemma 13. *Suppose that q is a point on an edge E of P and γ is a geodesic starting at q which is orthogonal to E at q. If p is the point a distance d from q along the geodesic, where $d < \pi/2$, then* $\mathrm{Dist}_g(p, E) = d$.

In the case when (P, σ) is Delzant this is essentially a standard result. By the same argument as in Lemma 9, a geodesic segment with endpoints on E of length less than π must lie in E. This means that the exponential map on the normal bundle of the 2-sphere corresponding to E is an embedding on vectors of length less than π, from which the assertion follows. The reader can easily check that the proof works in just the same way for a general (P, σ).

4. Convergence of sequences

4.1. Elliptic estimates. In this subsection we assemble some results of a rather standard nature, the general theme being that the derivatives of the scalar curvature of a Kahler metric control those of the full curvature tensor. Similar, but more sophisticated, results are contained in [**2**], [**14**].

Throughout this subsection we suppose that (M, g, J) is a Kahler surface with scalar curvature S and let p be a point of M. We suppose that the exponential map at p is defined on the unit ball and for $\rho \leq 1$ let B_ρ be the ρ ball in centred at p.

We begin with a simple result, which will be the essential thing we need for our main argument

Proposition 5. *Suppose that* $|\mathrm{Riem}| \leq 1$ *on* B_1*. Then for any* $\alpha \in (0,1)$ *and* $\rho < 1$ *there is a Holder bound, for points* p' *with* $d(p,p') \leq \rho$*,*

$$|\ |\mathrm{Riem}(p')| - |\mathrm{Riem}(p)|\ | \leq C_{\alpha,\rho}(1 + \|\nabla S\|_{L^\infty}) d(p,p')^\alpha.$$

By pulling back the metric we can suppose that the exponential map is an embedding on the unit ball. By a covering argument it suffices to prove the result for some ρ and then by rescaling we can suppose that $|\mathrm{Riem}|$ is as small as we please.

Various approaches to the proof are possible. We will base our argument on a general perturbation result for linear elliptic equations. Suppose that D_0 is a constant-coefficient first order elliptic operator over $\mathbf{R}^n$ (i.e., with injective symbol) and E is a perturbation term, defined over the unit ball, of the form

$$E(f) = \sum \epsilon_i \frac{\partial f}{\partial x_i} + Tf.$$

(Here we are considering operators on vector-valued functions, so the coefficients will be matrices in general.) Fix an exponent $p > 1$ and suppose that

- ϵ_i are sufficiently small;
- we have L^q bounds on T

where q and the allowable size of the ϵ_i depend on D_0 and p. Then by considering $D_0 + E$ as a perturbation of D_0 we obtain an elliptic estimate of the form

$$\|f\|_{L^p_1(B_{1/2})} \leq C\left(\|(D_0 + E)f\|_{L^p(B)} + \|f\|_{L^p(B)}\right),$$

where C depends on the L^q bounds on the coefficients T. The proof is essentially the same as [**9**] Theorem 9.11, together with the remark on p. 241.

To apply this we work in geodesic coordinates on our Kahler surface. A bound on the curvature gives a C^1 bound on the metric coefficients g_{ij} in these co-ordinates. Since the metric is Kahler the almost-complex structure J is covariant constant, hence, when written as a tensor in these coordinates, the coefficients are also bounded in C^1. We use the following identities connecting the curvature tensors, written in a schematic form

$$\overline{\partial}\mathrm{Riem} = 0 \quad , \quad \overline{\partial}^* \mathrm{Riem} = \pi(\nabla \mathrm{Ric}),$$

$$\overline{\partial}\mathrm{Ric} = 0 \quad , \quad \overline{\partial}^* \mathrm{Ric} = \pi(\nabla S).$$

Here π denotes certain natural contractions on tensors of the appropriate type. Then we can apply the discussion above to the elliptic operator $\overline{\partial} \oplus \overline{\partial}^*$ defined by the Kahler metric. We express this, in geodesic coordinates, as a perturbation of the constant coefficient model. When the curvature is small the relevant terms ϵ_i, T are small in L^∞. Now the general elliptic estimate above yields

$$\|\mathrm{Riem}\|_{L^p_1(B_{1/2})} \leq C(\|\nabla S\|_{L^p} + 1),$$

and we get a C^α bound on $|\mathrm{Riem}|$ from the Sobolev embedding theorem.

Next we extend this to higher derivatives.

Proposition 6. *With notation as above suppose that* $|\mathrm{Riem}| \leq 1$ *on* B_1*. Then for any* $l \geq 1$ *there are constants* $C_{l,\rho}$ *such that*

$$|\nabla^l \mathrm{Riem}| \leq C_{l,\rho}(\|\nabla^{l+1} S\|_{L^\infty(B)} + 1),$$

on B_ρ*.*

We only outline a proof, since this is somewhat standard. We can apply the perturbation argument as above to the $\overline{\partial} + \overline{\partial}^*$- operator mapping from L^p_{k+1} to L^p_k provided we know that the coefficients ϵ_i, T are controlled in L^p_k. (Here p is chosen sufficiently large.) Since T depends on the first derivatives of the metric tensor g and the complex structure J, in coordinates, we need $g, J \in L^p_{k+1}$. To achieve this we work in harmonic coordinates [**12**], in which the L^p_{k+1} norm of the metric tensor is controlled by the L^p_{k-1} norm of the curvature tensor. Since the tensor J is covariant constant we also get an L^p_{k+1} bound on its representative in these coordinates. Now we bootstrap, starting from the L^p_1 bound on the curvature tensor which was already obtained in the proof of Proposition 5. In harmonic coordinates we can consider the $\overline{\partial} \oplus \overline{\partial}^*$ operator mapping L^p_3 to L^p_2 and obtain L^p_3 bounds on the curvature tensor, in terms of derivatives of the scalar curvature, and so on.

Now consider a more specialised situation in which we have a pair of holomorphic vector fields v_1, v_2 on an embedded ball B_1 in the Kahler manifold X. Suppose that the Riemannian gradient of the scalar curvature can be expressed as $\nabla S = A_1 v_1 + A_2 v_2$ where A_1, A_2 are functions on the manifold. Suppose in turn that all derivatives of A_1, A_2 can be expressed in a similar way:

$$\nabla A_i = \sum A_{ij} v_j,$$

$$\nabla A_{ij} = \sum A_{ijk} v_k,$$

and so on.

Proposition 7. *In this situation, if* $|\mathrm{Riem}| \leq 1$ *on* B_1 *then we have* $|\nabla^l \mathrm{Riem}| \leq C_{l,\rho}$ *on* $B(\rho)$*, where* $C_{l,\rho}$ *depends on the* L^∞ *norms of the vector fields* v_1, v_2 *and the functions* $A_{i_1 \dots i_k}$ *over the ball* B_1*, for* $k \leq l+1$*.*

To prove this we exploit the first order elliptic equation $\overline{\partial} v_i = 0$ for the vector fields and build this into our bootstrapping argument. First, the L^∞ norm of ∇S is obviously controlled by the L^∞ norms of A_i, v_i. So in harmonic coordinates we control the L^p_3 norm of the metric and obtain elliptic estimates for the $\overline{\partial}$-operator mapping L^p_3 to L^p_2 and we get an L^p_3 bound on v_i. Now we can write

$$\nabla^2 S = \sum A_{ij} v_i \otimes v_j + \sum A_i \nabla v_i$$

and we get an L^p_2 bound on $\nabla^2 S$ and so on.

4.2. Bounded curvature. Now we show that to prove Theorem 1 it suffices to bound the curvature tensors of the solutions.

Proposition 8. *Suppose that* $(P^{(\alpha)}, \sigma^{(\alpha)}, A^{(\alpha)})$ *are data-sets converging to a limit* (P, σ, A) *and that* $u^{(\alpha)}$ *are solutions. If there are fixed* M, K *such that* $u^{(\alpha)}$ *satisfies the* M*-condition and* $|F(u^{(\alpha)})| \leq K$*, for all* α*, then there is a solution* $u^{(\infty)}$ *for the data* (P, σ, A)*.*

Of course, the solution $u^{(\infty)}$ will be obtained as a limit of the $u^{(\alpha)}$, provided that these are suitably normalised with respect to the addition of affine-linear functions. Although the domains of definition $P^{(\alpha)}$ are different, it obviously makes sense to talk about a subsequence of the $u^{(\alpha)}$ converging on compact subsets of P, and this is what we show first. (In fact we already have this interior convergence from the results of [**8**]—without assuming the curvature bound—but we will give an independent argument since it will be pave the way for the proofs in 4.4 below.) To simplify the presentation we just consider the case when the $P^{(\alpha)}, \sigma^{(\alpha)}$ are all the same (P, σ) and only $A^{(\alpha)}$ varies with α. The reader will easily see that the general case is not essentially different. We simplify notation by sometimes writing u and A for $u^{(\alpha)}$ and $A^{(\alpha)}$.

By Lemma 2, there is some fixed D such that for any point p in P there is a vertex q such that the Riemannian distance from p to q is less than D. Then Lemma 6 gives a universal bound

$$u^{ij} \leq C.$$

On the other hand Lemma 4 gives a bound

$$u_{ij} \leq C/d_{\mathrm{Euc}},$$

where d_{Euc} is the Euclidean distance to the boundary of P. So we deduce that u_{ij} is bounded above and below on compact subsets of the interior. On such sets the definition of the curvature tensor u^{ij}_{kl} immediately gives a C^2 bound on the u_{ij}, so we can suppose that the u_{ij} converge in $C^{3,\alpha}$. From this it is entirely straightforward to deduce the C^∞ convergence, on compact subsets of P. Thus the essential issue is to show that the limit satisfies the Guillemin boundary conditions. To see this, fix a point q on the boundary of P. There are two cases to consider, either q is a vertex or lies on the interior of an edge E.

Case 1 : *q is a vertex.*
The function $u = u^{(\alpha)}$ defines an $S^1 \times S^1$-invariant metric on $X_q \cong \mathbf{C}^2$. By Lemma 9, the geodesic ball of some fixed small radius about the origin is embedded. This geodesic ball maps to neighbourhood of q in $\overline{P}$ which is contained in a Euclidean neighbourhood of one fixed size, and contains a Euclidean neighbourhood of another fixed size. We are in the framework of Proposition 7, with $v_i = I\frac{\partial}{\partial \theta_i}$ and $A_i = \frac{\partial A}{\partial x^i}, A_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}$ and so on. Thus the norm of v_i in the Riemannian metric is $\sqrt{u^{11}}$ and this is bounded. Similarly for v_2. All the derivatives of A are bounded so we can apply Proposition 7 to deduce that all covariant derivatives of the curvature tensor are bounded in this ball. We pass to geodesic coordinates in which we have data $(g^{(\alpha)}, J^{(\alpha)})$. Then in these geodesic coordinates all derivatives of the metric tensors are bounded and we can suppose that the metrics converge in C^∞, likewise for the complex structures since these are covariant constant. The limit is a smooth Kahler metric (g^∞, J^∞) on a small ball in $\mathbf{R}^4$, invariant under the fixed, standard, action of $S^1 \times S^1$. For each α, the functions x^1_α, x^2_α which map the ball to neighbourhoods of q in $\overline{P}$ are characterised as moment maps for the action with respect to the symplectic forms $\omega^{(\alpha)}$ determined by $(g^{(\alpha)}, J^{(\alpha)})$. It follows that these also converge. By Guillemin's analysis of the structure of invariant Kahler metrics we know that the limit (g^∞, J^∞) corresponds to a function u^∞ on a neighborhood of q in $\overline{P}$, satisfying Guillemin boundary conditions, and it is clear from the convergence of the data $(g^{(\alpha)}, J^{(\alpha)}, x^1_\alpha, x^2_\alpha)$ that the second derivative of this coincides with the limit we have already found on the interior. Thus we see that this interior limit satisfies Guillemin boundary conditions in a neighbourhood of the vertex q.

Case 2 : *q is in the interior of an edge.*
We suppose that P is defined near q by the equation $x^1 > 0$. The argument is similar to that above. By Lemma 12 we get exponential coordinates on balls for the $\hat{g}$ metric on X^c_E whose image in $\overline{P}$ contains a fixed Euclidean neighbourhood of q. Arguing just as in the previous case, we get bounds on the covariant derivatives of the metric tensors and can suppose that in geodesic coordinates these converge, along with the complex structures. So we have $(g^\alpha, J^{(\alpha)}) \to (g^{\infty)}, J^\infty)$ say. For each α we have a pair of $J^{(\alpha)}$-holomorphic, commuting vector fields $v_1^{(\alpha)}, v_2^{(\alpha)}$ and Iv_1 is a Killing field generating a circle action fixing q.

Just as in the previous case, the exponential map is equivariant for this action so the limiting metric $g^{(\infty)}$ is also preserved by the same fixed circle action. For the other sequence of vector fields $v_2^{(\alpha)}$ we have to argue differently. We know that these are bounded in L^∞ so it follows from the ellipticity of the $\overline{\partial}$-operator that we can suppose (after perhaps taking a subsequence) that these converge. What we have to see is that the limit $v_2^{(\infty)}$ is not a multiple of v_1. But this is the case, since v_1 vanishes at q while the length of $v_2(q)$ is u^{22} which is bounded below by Lemma 4. So we obtain, in the limit in geodesic coordinates over a small neighbourhood of q

- a Kahler metric $g^{(\infty)}, J^{(\infty)}$;
- a pair of linearly independent, commuting, holomorphic vector fields $v_1^{(\infty)}, v_2^{(\infty)}$ such that $Iv_1^{(\infty)}$ generates the standard circle action.

Then just as before it follows from Guillemin's analysis that this data corresponds to a function satisfying Guillemin boundary conditions on a neighbourhood of q in $\overline{P}$.

4.3. Rescaling. It is standard practice in Riemannian geometry to rescale a metric in order to obtain a fixed bound on the curvature. We want to implement this idea in our special situation. Suppose u is a convex function on a polygon P which satisfies Guillemin boundary conditions defined by weights σ, with $u^{ij}_{ij} = -A$. Let λ be a positive real number. Define a function $\tilde{u}$ on the polygon $\tilde{P} = \lambda P$ by

$$\tilde{u}(x^1, x^2) = \lambda u(\lambda^{-1}x^1, \lambda^{-1}x^2).$$

Proposition 9.

- *The function $\tilde{u}$ satisfies Guillemin boundary conditions for the weights $\tilde{\sigma}(\lambda E) = \lambda\sigma(E)$.*
- *The curvature $\tilde{F}$ of $\tilde{u}$ satisfies*

$$|\tilde{F}|(\lambda p) = \lambda |F|(p).$$

- *The scalar curvature $\tilde{A} = \tilde{u}^{ij}_{ij}$ is*

$$\tilde{A}(\lambda p) = \lambda A(p).$$

- *If u satisfies an M-condition then so does $\tilde{u}$ (with the same value of M).*
- $\mu_{\tilde{P},\tilde{\sigma}} = \mu_{P,\sigma}$.

All of these are very easy to check. Notice that the second item implies that

$$\int_{\tilde{P}} |\tilde{F}|^2 d\mu_{\text{Euc}} = \int_P |F|^2 d\mu_{\text{Euc}}. \tag{5}$$

If (P,σ) is Delzant, so also is $(\tilde{P},\tilde{\sigma})$. There is then a canonical diffeomorphism from $X^c(P,\sigma)$ to $X^c(\tilde{P},\tilde{\sigma})$ and under this the Riemannian metric $\hat{g}$ is scaled by a factor λ. In this case (5) is just the standard fact that the L^2 norm of the curvature tensor is scale invariant in four real dimensions.

Using this rescaling we can transfer the results of Section 3, under the hypothesis that $|F| \leq 1$, to the general case. In fact we have the following refinement of Proposition 4.

Proposition 10. *Let (P,σ) be Delzant and let $\hat{g}$ be a metric on $X^c = X^c(P,\sigma)$ determined by a convex function u on P. Suppose u satisfies an M-condition. There is a $c > 0$, depending only on M and $\mu(P)$, with the following property. For any $\rho > 0$ and point $x \in X^c$:*

either *there is a point $x' \in X^c$ with* $\mathrm{Dist}_g(x,x') \leq \rho$ *and* $|F(x')| \geq \rho^{-2}$,

or $|F(x')| \leq \rho^{-2}$ *for all x' with* $\mathrm{Dist}_g(x,x') \leq \rho$ *and the exponential map at x is an embedding on the ball of radius $c\rho$*

This follows from Proposition 4 after rescaling and the observation that the hypothesis $|F| \leq 1$ in Proposition 4 is only used on points within a fixed distance of x.

4.4. Blow-up limits. Now suppose that, in our sequence $u^{(\alpha)}$ as considered in Theorem 1, the curvature $|F|$ does not satisfy a uniform bound. For each α choose a point p_α where the modulus of the curvature achieves its maximal value K_α and suppose that $K_\alpha \to \infty$. We want ultimately to derive a contradiction. By translation we can suppose that each p_α is the origin. We dilate by a factor K_α so we get a new sequence of data $(\tilde{P}^{(\alpha)},\tilde{\sigma}^{(\alpha)})$ and functions $\tilde{u}^{(\alpha)}$. It is clear that, perhaps after taking a subsequence, one of three cases must occur.

- The limit of the $\tilde{P}^{(\alpha)}$ is the whole of $\mathbf{R}^2$;
- The limit of the $\tilde{P}^{(\alpha)}$ is a half-plane;
- The limit of the $\tilde{P}^{(\alpha)}$ is a quarter-plane (i.e., a nontrivial intersection of two half-planes).

(Here by the statement that "the limit of $\tilde{P}^{(\alpha)}$ is G" we mean that point of G is contained in $\tilde{P}^{(\alpha)}$ for all large enough α and any point not in the closure of G is in the complement of $\tilde{P}^{(\alpha)}$ for all large enough α.)

The main result of this subsection is

Proposition 11. *If the limit of $\tilde{P}^{(\alpha)}$ is G, for one of the three cases above, then after taking a subsequence and adding suitable affine linear functions the $\tilde{u}^{(\alpha)}$ converge to a smooth convex function $\tilde{U}$ on G which satisfies the equation $\tilde{U}^{ij}_{ij} = 0$. The limit $\tilde{U}$ satisfies an M condition in G. In the case when G is a quarter plane, the limit satisfies Guillemin boundary conditions and defines a complete, non-flat, zero scalar curvature Kahler metric on $\mathbf{R}^4$ with curvature in L^2.*

We give the proof in the three cases.

Case 1 : *The limiting domain is the whole plane.*

We can apply the results from Section 3 to the functions $\tilde{u} = \tilde{u}^{(\alpha)}$. We want to show that on any compact subset $K \subset \mathbf{R}^2$ we have upper and lower bounds

$$C_K^{-1} \leq \tilde{u}_{ij} \leq C_K.$$

The upper bound follows immediately from Lemma 4 (since on compact sets the Euclidean distance to the boundary of $\tilde{P}^{(\alpha)}$ tends to infinity with α). Let $J = J^{(\alpha)}$ be the function $\det(\tilde{u}_{ij})$. The crucial thing is to get a lower bound on $J(0)$. Corollary 2 implies that the distance in the metrics $\tilde{g}^{\alpha}$ corresponding to $\tilde{u}^{\alpha}$ from the origin to the boundary of $\tilde{P}^{\alpha}$ tends to infinity. By construction, $|\tilde{F}^{(\alpha)}|$ is equal to 1 at the origin. We want to apply Proposition 5. Notice that when we rescale the derivatives of the scalar curvature function decrease, so are certainly uniformly bounded in the sequence. Thus by Proposition 5 we can find a fixed small number δ such that $|F^{(\alpha)}| \geq 1/2$ on the $\tilde{g}^{\alpha}$ ball of radius δ about the origin. On the other hand Lemma 8 implies that this ball contains a Euclidean ellipse of area at least $cJ(0)^{-1/2}\delta^2$, for some fixed c. Thus

$$\int_{\tilde{P}^{\alpha}} |\tilde{F}^{\alpha}|^2 d\mu_{\mathrm{Euc}} \geq c\delta^2 J(0)^{-1/2}.$$

Since, from (4) and (5), the integral on the left is bounded, we obtain a lower bound on $J(0)$, as required. Combined with the upper bound on $\tilde{u}_{ij}$ this lower bound on $J(0)$ yields an upper on $\tilde{u}^{ij}$ at the origin. Now Lemma 7 gives an upper bound on $\tilde{u}^{ij}$ at points of bounded $\tilde{g}$ distance from the origin. The upper bound on $\tilde{u}_{ij}$ implies that on compact subsets of the plane the $\tilde{g}$ distance to the origin is bounded. So we conclude that $\tilde{u}^{ij}$ is bounded above on compact subsets of the plane, which is the same as the lower bound on $\tilde{u}_{ij}$. Once we have these upper and lower bounds on $\tilde{u}_{ij}$ the convergence of a subsequence is straightforward, just as in the proof of Proposition 8, and the fact that the limit $\tilde{U}$ has $\tilde{U}^{ij}_{ij} = 0$ follows from the third item of Proposition 9.

Case 2 : *The limiting domain is a half-plane.*

The proof is similar to the first case. The upper bound on u_{ij} on compact subsets of the limiting half-plane is obtained just as before. Let d_α be the distance from the origin to the boundary of $\tilde{P}^{(\alpha)}$ if d_α is bounded below we can argue just as before. The only difficulty comes when $d_\alpha \to 0$, which is the same as saying that the origin is on the boundary of the limiting half-plane. Fix a parameter $\tau < \pi/2$. For each α we take a point q_α on the boundary of $\overline{P}^{\alpha}$ which minimises the g_α distance to the origin and let p_α be the point of $\tilde{P}^{\alpha}$ a distance τ from q_α along the geodesic emanating from q_α orthogonal to the boundary of $\tilde{P}^{\alpha}$. Then by Lemma 13 the distance from p_α to the boundary of

$\tilde{P}^\alpha$ is at least τ (once α is sufficiently large). Here we use the fact that the distance from the origin to all but one of the edges of $\tilde{P}^\alpha$ tends to infinity with α. Now by applying Proposition 5 to a geodesic ball centred at q_α we see that we can fix τ so that $|F| \geq 1/2$, say, on the ball of radius $\tau/2$ about p_α. Now the argument goes through just as before.

Case 3 : *The limiting domain is a quarter-plane.*

The proof in this case is much like that of Proposition 8. Let q_α be the vertex of $\tilde{P}^\alpha$ close to the vertex of the limiting quarter-plane and let E_1^α, E_2^α be the edges of $\tilde{P}^\alpha$ meeting in q_α with defining functions $\lambda_{i,\alpha} = \lambda_{E_i^\alpha}$. Observe that the definition of the $\tilde{\sigma}^\alpha$ implies that that these $\lambda_{i,\alpha}$ converge as α tends to infinity to defining functions for the edges of the quarter plane. We obtain the lower bounds on $\tilde{u}_{ij}$, or equivalently the upper bound on $\tilde{u}^{ij}$, by applying Lemma 6, using the geodesics emanating from q_α, and the upper bounds on u_{ij} using the M-condition and Lemma 4. Just as in the proof of Proposition 8 we show that the limit satisfies Guillemin boundary conditions along the edges of the quarter plane, and it is clear that the corresponding 4-manifold is diffeomorphic to $\mathbf{R}^4$. The completeness of the limiting metric follows from general principles or more directly from our estimate

$$\lambda_{i,\alpha} \leq \cosh(\mathrm{Dist}_g(q_\alpha, p)) - 1.$$

The fact that the curvature of the limiting metric is in L^2 follows from (4), (5) and Fatou's Lemma. Of course, the fact that the limiting metric is not flat follows from the normalisation that $|\tilde{F}^\alpha|$ is equal to 1 at the origin, and the C^∞ convergence.

With Proposition 11 in place the desired contradiction (to the hypothetical blow up of the curvature in the sequence) follows from the following two results.

Theorem 2. *There is no convex function U on a half-plane which satisfies an M-condition and the equation $U^{ij}_{ij} = 0$.*

Theorem 3. *If U is a convex function on a quarter plane which satisfies an M-condition and which defines a complete zero scalar curvature metric on $\mathbf{R}^4$ with curvature in L^2 then the metric is flat.*

We will give one proof of Theorem 3 now. This uses a result of Anderson [**2**], which we quote.

Theorem 4 (Anderson)**.** *Let g be a complete self-dual Riemannian metric on $\mathbf{R}^4$ with zero scalar curvature. Suppose that the curvature of g is in L^2 and that the volume $V(r)$ of the ball (in the metric g) of radius r about the origin satisfies $V(r) \geq cr^4$ for some $c > 0$. Then g is flat.*

To see that this applies to our case, recall first that scalar-flat Kahler metrics in two complex dimensions are self-dual. Thus the only thing we need to establish is the volume growth. This uses the M-condition. We can suppose the quarter plane in question is the standard one defined by $x^i \geq 0$ and that the boundary conditions correspond to the defining functions x^i. For $\tau > 0$ let $\Omega(\tau)$ be the triangle $\{(x^1, x^2) : x^i \geq 0, x^1 + x^2 \leq \tau\}$ and let $X^c(\tau)$ be the corresponding subset of X^c. By Corollary 1 we have $\mathrm{Dist}_g(p, 0) \leq C\sqrt{\tau}$ for $p \in \Omega(\tau)$, where C depends on M. So $X^c(\tau)$ is contained in the ball of radius $C\tau^2$. On the other hand the volume of $X^c(\tau)$ is equal to $(2\pi)^2$ times the Euclidean area of $\Omega(\tau)$ which is $\tau^2/2$. So we deduce that

$$V(C\sqrt{\tau}) \geq 2\pi^2\tau^2,$$

from which the statement follows.

5. Nonexistence of blow-up limits

5.1. The case of the half-plane. Throughout this section we will, contrary to our general convention, use lower indices for our coordinates (x_1, x_2) on the Euclidean plane.

We will first indicate the proof of Theorem 2 in the case of a convex function u satisfying the zero scalar curvature equation $u^{ij}_{ij} = 0$ and an M-condition on the whole plane. While this is subsumed in the harder case below the proof is substantially simpler. We suppose u is normalised to achieve its minimum at the origin. Then an easy elementary argument (see the proof of Lemma 14 below) shows that an M condition implies a uniform bound on the first derivative, $|\nabla u| \leq M'$, say, on $\mathbf{R}^2$. Now we apply Theorem 5 of the Appendix to the restriction of u to a large Euclidean disc of radius R centred at the origin. This yields $\det(u_{ij})(0) \leq CR^{-2}$, and we get a contradiction by letting R tend to infinity.

Now we give the proof for the case when the function is only defined on a half-plane.

Lemma 14. *Suppose (p_1, p_2) is fixed and u is a convex function on a neighbourhood of a rectangle $\{x_1, x_2 : |x_1 - p_1| \leq L_1, |x_2 - p_2| \leq L_2\}$ which satisfies the zero scalar curvature equation $u^{ij}_{ij} = 0$. Let*

$$V_1 = V((-L_1, 0), (L_1, 0)), V_2 = V((0, -L_2), (0, L_2))$$

and set $\Delta = \mathrm{Max}(V_1 L_1, V_2 L_2)$. Write J for the function $\det(u_{ij})$. Then there is a universal constant κ such that

$$J(0, t) \leq \frac{\kappa\Delta^2}{L_1^2 L_2^2},$$

for $|t| \leq L_1/4$,

(Recall that the function $V(p, q)$ is defined in (2) in Section 1.)

Obviously we can suppose $(p_1, p_2) = (0, 0)$. It is elementary to check that the statement is invariant under dilations of the co-ordinates, so we can reduce to the case when $L_1 = L_2 = 1$. We can also suppose that u is normalised so that it vanishes, together with its first derivatives, at the origin. So u is positive and, by convexity and the definition of V_1, the modulus of the partial derivative $\frac{\partial u}{\partial x_1}$ is bounded by Δ on the interval $\{(t,0) : -1 \leq t \leq 1\}$. Similarly for the x_2 variable. Thus $u \leq \Delta$ at the four points $(\pm 1, 0), (0, \pm 1)$. By convexity, $u \leq \Delta$ on the square K formed by the convex hull of these four points. Let D be the disc of radius $\frac{1}{2}$ about the origin. So D is contained in the interior of K and the distance from D to the boundary of K is $d = (1/\sqrt{2}) - \frac{1}{2}$. By an elementary property of convex functions we have $|\nabla u| \leq \rho = \Delta/d$ on D. Then by Theorem 5 of the Appendix there is a universal constant C such that $J \leq C\rho^2$ on the interior disc of radius $\frac{1}{4}$ centred on the origin. Thus we can take $\kappa = Cd^{-2}$.

Lemma 15. *Suppose u is a convex function on the half–plane $\{(x_1, x_2) : x_1 > 0\}$ which satisfies the M condition and the zero scalar curvature equation $u^{ij}_{ij} = 0$. Write $J = \det(u_{ij})$. Then*

1) *For any $\eta > 0$ there is an h_0 such that $J(x_1, x_2) \leq \eta x_1^{-1}$ if $x_1 \geq h_0$.*
2) *For any $\epsilon > 0$ and $0 < h_1 < h_2$ there is a $T > 0$ such that $J(x_1, x_2) \leq \epsilon$ if $|x_2| \geq T$ and $h_1 \leq x_1 \leq h_2$.*

To prove the first item we consider a point $p = (p_1, p_2)$ with $p_1 > 0$ and consider the rectangle $\{(x_1, x_2) : |x_1 - p_1| \leq p_1/2, |x_2 - p_2| \leq p_1\}$. We can apply Lemma 14, where $L_1 = p_1/2, L_2 = p_1$. The M condition implies that $V_1 \leq 2M, V_2 \leq M$, so $\Delta \leq Mp_1$. Then we obtain $J(p) \leq 4\kappa M^2 (p_1)^{-2}$ and the result follows (with $h_0 = 4\kappa M^2 \eta^{-1}$).

To prove the second item we first consider the case when $h_1 = \frac{3h}{4}, h_2 = \frac{5h}{4}$ for some h. It obviously suffices to show that the statement is true for $x^2 \geq T$, once T is suitable large. The M condition implies that

$$\int_{-\infty}^{\infty} u_{22}(h,t)dt \leq M < \infty,$$

so given any δ we can find a large S such that

$$\int_S^{\infty} u_{22}(h,t)dt \leq \delta. \tag{6}$$

Now, for $L_2 > 0$ consider a rectangle

$$\{(x^1, x^2) : \frac{h}{2} \leq x_1 \leq \frac{3h}{2}, |x_2 - p_2| \leq L_2\}$$

as before. Suppose that $p_2 - L_2 \geq S$. Then (6) implies that $V_2 \leq \delta$. Suppose that $\delta L_2 \leq V_1 L_1 = Mh$. Then $\Delta = Mh$ and Lemma 14 gives $J(x_1, p_2) \leq 4\kappa^2 M^2 / L_2^2$ for $3h/4 \leq x_1 \leq 5h/4$. So, given $\epsilon > 0$ we first choose a large L_2 such that $4\kappa^2 M^2 / L_2^2 \leq \epsilon$. Then we choose a small δ such that $\delta L_2 \leq Mh$. Then we choose S as above and set $T = S + L_2$.

Finally, for general $h_1 < h_2$, we cover the interval $[h_1, h_2]$ with a finite number of intervals of the form above and take the maximum value of the corresponding T's.

Now we can prove Theorem 2. Suppose that u satisfies the M-condition on the half-plane and $u^{ij}_{ij} = 0$. We consider the function $F = \det(u_{ij})^{-1}$ on the half-plane. This satisfies the equation

$$u^{ij} F_{ij} = 0.$$

(See [**8**], Sec. 2.1.) So, for any constant λ, the function $G = F - \lambda x_1$ satisfies $u^{ij} G_{ij} = 0$, hence can have no local minumum or maximum. Suppose, without loss of generality, that $F(1, 0) = 1$, and take $\lambda = 2$, so $G(1, 0) = -1$. Clearly $G(x_1, x_2) \geq -\frac{1}{2}$ if $x_1 \leq \frac{1}{4}$. By the first item in Lemma 15 we can choose h_0 so large that $F(x_1, x_2) \geq 10x_1$ if $x_1 \geq h_0$. This implies that $G(x_1, x_2) \geq 8x_1 > 0$ if $x_1 \geq h_0$. By the second item of Lemma 15 we can choose T so large that $F(x_1, x_2) \geq 10h_0$ if $|x_2| \geq T$ and $\frac{1}{4} \leq x^1 \leq h_0$. This implies that $G(x^1, x^2) \geq 8h_0 > 0$ if $|x^2| \geq T$ and $\frac{1}{4} \leq x^1 \leq h_0$. So $G(x^1, x^2) \geq -\frac{1}{2}$ if (x_1, x_2) lies on the boundary of the rectangle

$$Q = \{(x_1, x_2) : |x_2| \leq T, \frac{1}{4} \leq x_1 \leq h_0\}.$$

Since G takes the value -1 on the interior point $(1, 0)$ of Q it must have an interior minumum, which is the desired contradiction.

One point worth noting here is that in the case of a function defined on the whole plane the argument can be made entirely effective. There is no need to take the limit as α tends to infinity of the sequence $\tilde{u}^{(\alpha)}$ in Proposition 11. The same argument can be used to obtain an explicit *a priori* estimate of the form

$$|F| \leq C \mathrm{Dist}_g(\ , \partial P)^{-2}.$$

In the case of the half-plane it seems to be necessary to pass to the limit, and the proof does not yield an explicit *a priori* estimate in general. However if one considers the case when $A^{(\alpha)} \geq 0$ then this can be done, and one gets an explicit estimate of the form

$$|F| \leq C \mathrm{Dist}_g(\ , V)^{-2},$$

where V is the set of vertices.

5.2. The case of the quarter-plane. Here we give a second, self-contained, proof of Theorem 3. The general strategy of the proof is in part similar to Anderson's, in that we show that the curvature tensor vanishes by applying an integral formula for its L^2 norm, and the crux of the matter is to establish that the relevant boundary term vanishes in the limit. We will first state the relevant integral formula, in our special situation.

Let u be a convex function on $\mathbf{R}^+ \times \mathbf{R}^+$ with $u^{ij}_{ij} = 0$, as in the statement of the Theorem, and for $R > 0$ let $\Omega(R)$ be the triangle formed by the intersection of the quarter plane with the half-space $\{x_1 + x_2 \leq R\}$. Let $\partial\Omega(R)$ denote the ordinary boundary of the triangle, made up of three line segments and $\partial^0\Omega(R)$ be the single segment lying on the line $\{x_1 + x_2 = R\}$. (Recall that $\Omega(R)$ corresponds to a differentiably embedded ball in the 4-manifold X^c and $\partial^0\Omega(R)$ corresponds to the boundary of this ball. The other two segments in $\partial\Omega(R)$ correspond to fixed points for the two basic circle actions on X^c.) Now we have

$$\int_{\Omega(R)} |F|^2 d\mu = \int_{\partial^0\Omega(R} \nu^i \tag{7}$$

where

$$\nu^i = F^{ij}_{ab} u^{ab}_i - F^{ia}_{ib} u^{bj}_j . \tag{8}$$

The integrand on the right hand side of the formula (7) is written as a vector field but this can be viewed as a 1-form using the canonical identification furnished by the Euclidean area element $d\mu = dx_1 dx_2$. We leave the verification of this identity as an exercise for the reader (see also the similar discussion in [**8**], Sec. 5.2). The overall strategy of our proof is to show that the integral on the right hand side of (7) tends to zero as $R \to \infty$, which implies that F is identically zero.

We begin by establishing that the curvature decays as a function of the Riemannian distance from the origin. To fit in with the wider literature we will phrase this discussion in terms of the Riemannian 4-manifold X^c, although of course it can be translated into the two-dimensional language. The crucial thing is that this Riemannian manifold has the property stated in Proposition 10. (The discussion there assumed a compact manifold but it is easy to see that the proofs work equally well in the present situation.) Moreover if (in the notation of Proposition 10) $|F| \leq \rho^{-2}$ on the ball of radius ρ about a point x in X^c we have, by applying Proposition 5,

$$|F(x)|^2 \leq C\rho^{-4} \int_{B(c\rho,x)} |F|^2 dV$$

for some fixed C, where dV is the Riemannian volume element. Now we recall a general fact:

Lemma 16. *Let X be a complete, noncompact, Riemannian manifold with base point x_0. Let K be a continuous, non-negative, L^2 function on X with the following property. There are constants $c, C > 0$ such that for any $x \in X$ and $\rho > 0$, either there is a point $x' \in X$ with $d(x, x') \leq \rho$*

and $K(x') > \rho^{-2}$ *or*

$$K(x)^2 \leq C\rho^{-4} \int_{B(c\rho,x)} |K|^2.$$

Then $K(x)d(x,x_0)^2 \to 0$ *as* x *tends to infinity in* X.

To see this, let $E > 0$ and let $X_E \subset X$ be a compact set such that

$$\int_{X \setminus X_E} K^2 \leq C^{-1}E.$$

Define a function ρ_E, taking values in $(0,\infty]$, by

$$\rho_E(x)^{-4} = \frac{K(x)^2}{2E}.$$

It is convenient to work with this and one can check step-by-step in the argument below that, with the obvious interpretations, there are no problems from the zeros of K. The crucial thing is that ρ_E is bounded below by a strictly positive number on any compact set in X. Now suppose x is a point in X with $\rho_E(x) \leq \epsilon d(x, X_E)$ where $\epsilon < c$. Then the ball of radius $c\rho_E(x)$ about X does not meet X_E so the second alternative in the hypothesis (taking $\rho = \rho_E(x)$) would give $K(x)^2 \leq \frac{1}{2}K(x)^2$. Since $\rho_E(x)$ is finite $K(x)$ is nonzero and we conclude that the first alternative must hold; that is, there is a point x' with

$$d(x,x') \leq \rho_E, \quad K(x') > \rho_E(x)^{-2} = K(x)/\sqrt{2E}.$$

So now we have $\rho_E(x') \leq 2^{-1/4}\rho_E(x)$. We also have

$$\epsilon d(x', X_E) \geq \epsilon d(x, Z_E) - \epsilon d(x,x') \geq (1-\epsilon)\rho_E(x).$$

Thus

$$\rho_E(x') \leq \frac{\epsilon}{(1-\epsilon)2^{1/4}} d(x', Z_E).$$

Suppose ϵ is so small that $2^{1/4}(1-\epsilon) > 1$. Then $\rho_E(x') \leq \epsilon d(x', X_E)$. Thus x' satisfies the same hypothesis as x did. We continue in this way to generate a sequence x_n with

$$d(x_n, x_{n+1}) \leq \rho_E(x_n),$$

and

$$\rho_E(x_{n+1} \leq 2^{-1/4}\rho_E(x_n).$$

Thus x_n is a Cauchy sequence in X and $\rho_E(x_n)$ tends to zero, a contradiction. So we conclude that for $\epsilon < \mathrm{Min}(c, 1-2^{-1/4})$ we have

$$\rho_E(x) \geq \epsilon^{-1} d(x, X_E)$$

for all x in X. This says that

$$K(x)d(x,X_E)^2 \leq 2E/\epsilon,$$

and the result follows, since we can take E as small as we please.

So in our case we know that the function $|F|$ on the quarter-plane decays faster than than the inverse square of the Riemannian distance to the origin. The next step is to relate this distance to the Euclidean distance in the quarter-plane. For this we use another integral identity. Change Euclidean coordinates by setting $y = x_1 - x_2, z = x_1 + x_2$ and denote derivatives with respect to the new coordinates by u_{yy} *etc.*

Lemma 17. *If u satisfies $u^{ij}_{ij} = 0$ in the quarter plane and Guillemin boundary conditions then for any $R > 0$*

$$\int_{y=-R}^{R} u^{zz}(y, R) dy = R^2.$$

(The notation is slightly ambiguous here, so we should emphasise that in the formula above we are regarding $u = u(y, z)$ as a function of y, z. The region of integration is exactly $\partial^0 \Omega_R$, as considered above.)

To see this, let $f(y, z)$ be the function $f(y, z) = R - y$ on the region Ω_R. The zero scalar curvature condition takes the same form in the new coordinates, so we write it as $u^{\alpha\beta}_{\alpha\beta} = 0$, where α, β run over the labels y, z. So we have

$$\int_{\Omega_R} u^{\alpha\beta}_{\alpha\beta} f = 0.$$

Now we integrate by parts twice. Since f is linear we have $f_{\alpha\beta} = 0$ and there is no contribution from the interior so we get the identity

$$\int_{\partial\Omega_R} u^{\alpha\beta}_{\alpha} f = \int_{\partial\Omega_R} u^{\alpha\beta} f_\alpha.$$

The function f vanishes on $\partial^0\Omega(R)$ and the Guillemin boundary conditions imply that $u^{\alpha\beta}_\alpha$ has normal component 1 along the axes. Thus

$$\int_{\partial\Omega(R)} u^{\alpha\beta}_\alpha f = 2 \int_0^R R - t \; dt = R^2.$$

On the other hand the boundary conditions imply that the normal component of $u^{\alpha\beta} f_\alpha$ vanishes along the axes, so

$$\int_{\partial\Omega(R)} u^{\alpha\beta} f_\alpha = \int_{\partial^0\Omega(R)} u^{\alpha\beta} f_\alpha,$$

and the result follows.

Now for any fixed $z > 0$ let $s(z)$ be the Riemannian distance from the origin to the interval $\partial^0\Omega(z) = \{x^1 + x^2 = z\}$. Suppose for the moment that this distance is realised by a unique minimal geodesic γ and that there is no Jacobi field along γ which vanishes at the origin and is tangent to this interval at the other end point. Then s is smooth around this value of z and

$$\frac{ds}{dz} = |dz|_g^{-1} = \left(u^{zz}\right)^{-1/2},$$

where u^{zz} is evaluated at the distance-minimising point of the interval. In any case, $s(z)$ is a Lipschitz function and if we define

$$\phi(z) = \max u^{zz},$$

where the maximum is taken over this interval, then we have

$$\frac{ds}{dz} \geq \phi(z)^{-1/2}, \tag{9}$$

interpreted in an appropriate generalised sense.

We want to go from the integral identity of Lemma 17 to a pointwise bound on u^{zz}, and hence on $\phi(z)$. For this we use

Lemma 18. *Suppose f and σ are positive functions on an interval $[0, R]$, where $R \geq 1$, with*

$$|f''(t)| \leq f(t)\sigma(t),$$

and for any $\lambda > 0$ we have

$$\int_{\lambda/2}^{\lambda} \sigma(t)dt \leq 1.$$

Then for any t_0 in $[0, R]$ we have

$$f(t_0) \leq 18 \int_0^R f(t)dt.$$

To simplify notation we will give the proof in the case when f attains its maximum at $t_0 = 0$. It will be clear that this is the "worst" case and that the argument applies to all points. Set

$$I = \int_0^R f(t)dt.$$

For $h > 0$ use the formula

$$f(0) = f(h) - hf'(h) - \int_0^h tf''(t)dt.$$

Using the assumption that f attains its maximum at 0, and the given differential inequality $|f''| \leq f\sigma$, we have

$$\left|\int_{h/2}^h tf''(t)dt\right| \leq \frac{f(0)h}{2} \int_{h/2}^h \sigma(t)dt \leq \frac{f(0)h}{2}.$$

Summing over a geometric series, as in the proof of Lemma 2, we obtain

$$\left|\int_0^h tf''(t)dt\right| \leq hf(0).$$

So if $h \leq \frac{1}{2}$, say, we have

$$f(0) \leq 2(f(h) - hf'(h)). \tag{10}$$

Let $t_1 > 0$ be a point in the interval $[0, \frac{1}{3}]$ where f attains its minimum. Then

$$I \geq \int_0^{1/3} f(t)dt \geq \frac{f(t_1)}{3}.$$

If $t_1 < \frac{1}{3}$ the derivative $f'(t_1)$ vanishes and, taking $h = t_1$ in the inequality above, we have

$$f(0) \leq 2f(t_1) \leq 6I.$$

Suppose, on the other hand, that $t_1 = \frac{1}{3}$. Let g be the affine-linear function with $g(\frac{1}{3}) = f(\frac{1}{3})$ and $g(\frac{1}{2}) = f(\frac{1}{2})$. If $f'(\frac{1}{3}) < g'(\frac{1}{3})$ then there is a point h in the interval $(\frac{1}{3}, \frac{1}{2})$ where $f'(h) = g'(h)$ and $f(h) < g(h)$. If $f'(\frac{1}{3}) \geq g'(\frac{1}{3})$ we take $h = \frac{1}{3}$. In either case

$$f(h) - hf'(h) \leq g(0) = \frac{\frac{1}{2}f(\frac{1}{3}) - \frac{1}{3}f(\frac{1}{2})}{\frac{1}{2} - \frac{1}{3}} \leq 3f\left(\frac{1}{3}\right).$$

Then applying the inequality (10) above with this value of h we obtain

$$f(0) \leq 6f(1/3) \leq 18I.$$

Corollary 3. *Suppose that u satisfies $u^{ij}_{ij} = 0$ in $\mathbf{R}^+ \times \mathbf{R}^+$ and Guillemin boundary conditions. Suppose that u satisfies the M condition with $M = 1$ and that, for some $R > 1$, $|F| \leq 1$ on $\partial\Omega_R$. Then*

$$u^{zz} \leq 18R^2$$

on $\partial^0\Omega_R$.

To see this, observe that

$$|u^{zz}_{yy}| \leq |F| u^{zz}\, u_{yy}.$$

Then the result follows from Lemmas (17) and (18), taking $f(t) = u^{zz}(t + R, R)$ and $\sigma(t) = u_{yy}(t + R, R)$.

Lemma 19. *There is a constant C such that $z \leq Cs(z)^2$ for all z.*

For $z > 0$ define $\lambda(z) = \frac{z}{s(z)^2}$. We know, by Corollary 2, that $s(z) \to \infty$ as $z \to \infty$. Further, we know that $|F| = o(s^{-2})$, so it follows that for any $\epsilon > 0$ we can find an R_0 such that

$$|F| \leq \epsilon \frac{\lambda(R)}{R},$$

on $\partial^0\Omega_R$, once $R \geq R_0$. For a fixed $R \geq R_0$, suppose that $\lambda(R) \geq \epsilon^{-1}$ and set

$$\eta = \frac{\epsilon\lambda(R)}{R},$$

so by hypothesis $\eta \geq R^{-1}$. Now rescale using this factor η, so we define $\tilde{u}(x_1, x_2) = \eta u(x_1/\eta, x_2/\eta)$. Set $\tilde{R} = \eta R$ and consider the rescaled solution on the triangle $\Omega(\tilde{R})$, which corresponds to the original solution on the triangle $\Omega(R)$. The curvature tensor $\tilde{F}$ of $\tilde{u}$ satisfies

$|\tilde{F}| \leq \eta^{-1}|F| \leq 1$ on $\partial^0\Omega_{R'}$ and so we can apply Corollary 3 to $\tilde{u}$ to get

$$\tilde{u}^{zz} \leq 18\eta^2 R^2$$

on $\partial^0\Omega_{\tilde{R}}$. Transforming back, this becomes

$$u^{zz} \leq 18\eta^{-1}\eta^2 R^2 = 18\epsilon\lambda(R)R$$

on $\partial^0\Omega_R$. In other words we have the following: for $R \geq R_0$ if $\lambda(R) \geq \epsilon^{-1}$ then $\phi(R) \geq (18\epsilon\lambda(R)R)^{-1/2}$. Now consider the derivative of λ. Using (9) we have

$$\frac{d\lambda}{dz} = s(z)^{-2}\left(1 - \frac{2zs'(z)}{s}\right) \leq s(z)^{-2}\left(1 - \frac{2z\phi(z)}{s(z)}\right).$$

If $z \geq R_0$ and $\lambda(z) \geq \epsilon^{-1}$ then we have

$$\frac{d\lambda}{dz} \leq s(z)^{-2}\left(1 - \frac{2z}{s\sqrt{\epsilon\lambda z}}\right) = s(z)^{-2}\left(1 - \frac{2}{\sqrt{18\epsilon}}\right).$$

Now we fix $\epsilon < 2/9 = 4/18$ so that $(1 - 2/\sqrt{(18\epsilon})) < 0$ and we see that once $z \geq R_0$ and $\lambda > \epsilon^{-1}$ the function λ is decreasing. It follows then that $\lambda(z)$ is bounded.

Combining Lemma 16 and Lemma 19, we have

$$|F| = o(z^{-1}). \tag{11}$$

For $R > 1$ we now rescale by R, so we define $\tilde{u}^{(R)}$ (which we sometimes just denote by $\tilde{u}$) to be

$$\tilde{u}^{(R)}(x_1, x_2) = R^{-1}u(Rx_1, Rx_2) + L(x_1, x_2),$$

where L is an affine-linear function chosen so that $\tilde{u}^{(R)}$ and its first derivatives vanish at the point $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}$. We consider the restriction of $\tilde{u}^{(R)}$ to the fixed quadrilateral

$$Q = \left\{(x_1, x_2) : \frac{1}{2} < x_1 + x_2 < 2, \quad x_1, x_2 > 0\right\}.$$

We write $\tilde{F}^{(R)}$ for the curvature tensor corresponding to $\tilde{u}^{(R)}$. The decay condition (11) implies that $|\tilde{F}^{(R)}|$ tends to zero on Q, as $R \to \infty$. As usual, we obtain an upper bound on the Hessian $\tilde{u}_{ij}$ over compact subsets of the interior of Q. Now Corollary 3 gives an upper bound on $\tilde{u}^{zz}$ over Q. Lemma 3 gives

$$\frac{\partial^2}{\partial y^2}\left(\tilde{u}_{yy}^{-1}\right) \leq 1,$$

say over Q.The boundary conditions fix the values of $\frac{\partial}{\partial y}\left(\tilde{u}_{yy}^{-1}\right)$ on $y = \pm z$ and this gives lower bound on $\tilde{u}_{yy}$,

$$\tilde{u}_{yy}^{-1} \leq C(z - |y|).$$

Now

$$\tilde{u}^{zz} = \tilde{u}_{yy} / \det(\tilde{u}_{\alpha\beta}),$$

so we obtain a lower bound on the determinant

$$\det(\tilde{u}_{\alpha\beta}) \geq C^{-1}(z - |y|)^{-1}.$$

Combining with our upper bounds on the components of $u_{\alpha\beta}$ we obtain upper bounds $\tilde{u}^{yy} \leq C, |\tilde{u}^{yz}| \leq C$ on Q. Then, just as before, we can conclude that as $R \to \infty$ the $\tilde{u}^{(R)}$ converge on compact subsets of the interior of Q to a smooth limit $\tilde{u}^{(\infty)}$ with $\tilde{F}^{(\infty)} = 0$. Now the boundary term in (7) is scale invariant, so we get the same computing with $\tilde{u}^{(R)}$ and integrating over the fixed interval $\partial^0\Omega(1)$ in the interior of Q. It is then straightforward to check that this tends to zero with R.

6. Appendix: applications of the maximum principle

In this appendix we use the maximum principle to derive upper and lower bounds on the determinant of the Hessian of a solution to Abreu's equation. The results and their proofs are similar to those in [**8**], Sect. 4, but differ in being specific to the two-dimensional case. The inspiration for these results comes from the work of Trudinger and Wang in [**15**], [**16**] and, particularly, [**17**], Remark 4.1.

Theorem 5. *Suppose that u is a convex function on the closed disc of radius R in $\mathbf{R}^2$, smooth up to the boundary and with $\nabla u = 0$ at the origin. Let $A(x)$ be the function $A = -(\partial u^{ij})_{ij}$ and let*

$$A^+ = \max(\max_x(A(x), 0)), A^- = -\min(\min_x A(x), 0).$$

- *If the derivative ∇u maps the R-disc to the disc $|\xi| \leq \rho$ then on the interior disc $\{|x| \leq \frac{1}{4}\}$ we have*

$$\det(u_{ij}) \leq \left(\frac{\rho}{R}\right)^2 \left(c_1 + c_2 R^2 \rho^2 (A^-)^2\right).$$

- *If the derivative ∇u maps the R-disc onto the disc $|\xi| \leq \rho$ then on the set where $|\nabla u| \leq \frac{1}{4}$ we have*

$$\det(u_{ij}) \geq \left(\frac{\rho}{R}\right)^2 \left(c_3 + c_4 R^2 \rho^2 (A^+)\right)^{-2}$$

for universal constants c_1, c_2, c_3, c_4.

Rescaling the domain and multiplying u by a constant, we can assume that $R = \rho = 1$. We begin with the first item. Here we consider the function

$$f = -L + F - \alpha g^{ab} u_a u_b,$$

on the open disc, where $L = \log\det(u_{ij})$, F is a smooth function which tends to $+\infty$ on the boundary of the disc, to be specified shortly, α is an arbitrary strictly positive constant and g^{ij} denotes the standard Euclidean metric tensor. (Thus $g^{ab}u_a u_b$ is another notation for $|\nabla u|^2$.)

The function f attains its minimum in the disc and at this point we have $f_i = 0$ which gives

$$L_i = F_i - 2\alpha g^{ab} u_{ai} u_b. \tag{12}$$

We also have

$$f_{ij} = -L_{ij} + F_{ij} - 2\alpha g^{ab} u_{aij} u_b - 2\alpha g^{ab} u_{ai} u_{bj},$$

and at the minumum point $u^{ij} f_{ij} \geq 0$. Hence, at the minimum point,

$$2\alpha g^{ab} u_{ab} \leq -u^{ij} L_{ij} + u^{ij} F_{ij} - 2\alpha g^{ab} L_a u_q,$$

where we have used the identity

$$L_a = u^{ij} u_{aij}.$$

The defining equation $u^{ij}_{ij} = -A$ leads to the formula

$$u^{ij} L_{ij} = u^{ij} L_i L_j + A,$$

(see [**8**],Sect. 2.1) so we get

$$2\alpha g^{ab} u_{ab} \leq A^- - u^{ij} L_i L_j + u^{ij} F_{ij} - 2\alpha g^{ab} L_a u_b. \tag{13}$$

Now we use (12) to write

$$u^{ij} L_i L_j = u^{ij} (F_i - 2\alpha g^{pq} u_{pi} u_q)(F_j - 2\alpha g^{rs} u_{rj} u_s),$$

and expand this out to get

$$u^{ij} L_i L_j = u^{ij} F_i F_j - 4\alpha u^{ij} F_i g^{rs} u_{rj} u_s + 4\alpha^2 g^{pq} g^{rs} u^{ij} u_{pi} u_q u_{rj} u_s.$$

This simplifies to

$$u^{ij} L_i L_j = u^{ij} F_i F_j - 4\alpha F_r u_s g^{rs} + 4\alpha^2 g^{pq} g^{rs} u_q u_s u_{rp}. \tag{14}$$

Next we use (12) again to write

$$g^{pq} L_p u_q = g^{pq} (F_p - 2\alpha g^{rs} u_{rp} u_s) u_q,$$

so

$$4\alpha^2 g^{pq} g^{rs} u_q u_s u_{rp} = 2\alpha g^{pq} F_p u_q - 2\alpha g^{pq} L_p u_q. \tag{15}$$

Combining (13), (14) and (15) we obtain

$$2\alpha g^{ab} u_{ab} \leq A^- + u^{ij} (F_{ij} - F_i F_j) + 2\alpha F_r u_s g^{rs}.$$

Now take F to be the function $F(x) = -2\log(1 - |x|^2)$. If $E = e^{-F}$ we have

$$F_{ij} - F_i F_j = -E^{-1} E_{ij}$$

and $E = (1 - |x|^2)^2$, so the matrix (E_{ij}) is bounded. Using the formula for the inverse of a 2×2 matrix we get

$$u^{ij} (F_{ij} - F_i F_j) \leq c \frac{g^{ab} u_{ab}}{(|1 - |x|^2)^2 \det(u_{ij})},$$

for an easily-computable constant c. Similarly the derivative ∇F is bounded by a multiple of $(1-|x|^2)^{-1}$ so we obtain, at the minimum point of f,

$$2\alpha g^{ab}u_{ab} \leq A^+ + \frac{c}{\det(u_{ij})(1-|x|^2)^2} g^{ab}u_{ab} + \frac{\alpha c}{1-|x|^2}, \tag{16}$$

using the fact that $|\nabla u| \leq 1$.

Now suppose that, at this minimum point,

$$(1-|x|^2)^2 \det(u_{ij}) \geq \alpha^{-1}.$$

Then we can rearrange to obtain

$$\alpha g^{ab}u_{ab} \leq A^+ + \frac{\alpha c}{(1-|x|^2)}.$$

Since $4\det(u_{ij}) \leq (g^{ab}u_{ab})^2$ we have

$$(1-|x|^2)^2 \det(u_{ij}) \leq \frac{1}{4\alpha^2}\left(A^+(1-|x|^2)+\alpha c\right)^2 \leq \left(\frac{A^+ + \alpha c}{2\alpha}\right)^2.$$

So we conclude that, in any event, at the minimum point of f,

$$(1-|x|^2)^2 \det(u_{ij}) \leq C$$

where

$$C = \max\left(\alpha^{-1}, \left(\frac{A^+ + \alpha c}{2\alpha}\right)^2\right).$$

Taking logarithms, at the minumum point of f we have $-L + F \geq -\log C$, so $f \geq -\alpha - \log C$ since $g^{ab}u_a u_b = |\nabla u|^2 \leq 1$. So at any point of the disc $-L + F \geq -\alpha - \log C$ and in particular when $|x| \leq \frac{1}{4}$ we have $\det(u_{ij}) \leq \left(\frac{16}{15}\right)^2 Ce^{\alpha}$. This gives our first result, taking any fixed value of α.

The proof of the second item is very similar. Now we restrict attention to the open subset U of the unit disc on which $|\nabla u| < 1$ and consider the function on U

$$f = \log\det(u_{ij}) - \alpha|x|^2 + F(\nabla u)$$

where F is a function on the unit disc $|\xi| < 1$ which tends to infinity on the boundary. The easiest way to present the proof, in analogy with preceding case, is to take the Legendre transform ϕ of u, although it is not necessary to do so. The point is that the quantity $\det u_{ij}$we want to estimate can also be written as the inverse the determinant of the Hessian of ϕ. We calculate with respect to dual coordinates ξ^i. (There is a clash of notation here, in that we would often write these coordinates with lower indices, to fit in with the previous x^i, but that would not be convenient for the calculations we want to perform.) Our function becomes

$$f = -\log\det\phi_{ij} - \alpha g^{ab}\phi_a\phi_b + F(\xi),$$

thought of as a function on the unit disc, in ξ coordinates. We write $L = \log\det\phi_{ij}$, although we should keep in mind that this corresponds under the Legendre transform to the negative of the function we considered before. The defining equation for A yields

$$\phi^{ij}L_{ij} = -A.$$

With these preliminaries in place we can proceed with the argument. At the minimum we have $L_i = F_i - 2\alpha\phi_{ia}\phi_b g^{ab}$ just as before, and $\phi^{ab}f_{ab} \geq 0$. This leads to

$$2\alpha g^{ab}\phi_{ab} \leq A^+ + \phi^{ab}F_{ab} - 2\alpha g^{ab}F_a\phi_b + 4\alpha^2\phi_{ia}\phi_b\phi_j g^{ab}g^{ij}, \tag{17}$$

(at the minimum point). Now since $|\nabla\phi|^2 \leq 1$ we have

$$\phi_{ia}\phi_b\phi_j g^{ab}g^{ij} \leq \phi_{ab}g^{ab}.$$

(To see this, observe that, after rotating coordinates, we can suppose that $\phi_2 = 0$ at the point in question: then the left hand side of the expression above is ϕ_{11} and the right hand side is $\phi_{11} + \phi_{22}$.) So this time we choose $\alpha < 1/4$, in order that the last term in (17) is bounded by $\alpha g^{ab}\phi_{ab}$, and we obtain

$$\alpha g^{ab}\phi_{ab} \leq A^+ + \phi^{ab}F_{ab} - 2\alpha g^{ab}F_a\phi_b.$$

We use the same function F as before: $F(\xi) = -2\log(1 - |\xi|^2)$. The matrix F_{ab} is bounded by a multiple of $(1 - |\xi|^2)^{-2}$; the first derivative ∇F by a multiple of $(1 - |\xi|^2)^{-1}$ and the argument proceeds exactly as before.

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