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RIGIDITY OF AMALGAMATED PRODUCTS IN NEGATIVE CURVATURE

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Abstract

Let Γ be the fundamental group of a compact riemannian manifold X of sectional curvature $K \leq -1$ and dimension $n \geq 3$. We suppose that $\Gamma = A *_C B$ is the free product of its subgroups A and B amalgamated over the subgroup C. We prove that the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n-2$. The equality occurs if and only if there exist an embedded compact hypersurface $Y \subset X$, totally geodesic, of constant sectional curvature -1, whose fundamental group is C and which separates X in two connected components whose fundamental groups are A and B respectively. Similar results hold if Γ is an HNN extension, or more generally if Γ acts on a simplicial tree without fixed point.

1. Introduction

In [**17**], Y. Shalom proved the following theorem that says that for every lattice Γ in the isometry group of the hyperbolic space and for any decomposition of Γ as an amalgamated product $\Gamma = A *_C B$, the group C has to be "big". In order to measure how "big" C is, let us define the critical exponent of a discrete group C acting on a Cartan Hadamard manifold by

$$\delta(C) = \inf\{s > 0 \quad | \quad \Sigma_{\gamma\in\Gamma} e^{-sd(\gamma x,x)} < +\infty\}.$$

Theorem 1.1 ([**17**]). *Let Γ be a lattice in $PO(n,1)$. Assume that Γ is an amalgamated product of its subgroups A and B over C. Then, the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n-2$.*

An example is given by any n-dimensional hyperbolic manifold X which contains a compact separating connected totally geodesic hypersurface Y. The Van Kampen theorem then says that the fundamental group Γ of X is isomorphic to the free product of the fundamental groups of the two halves of $X - Y$ amalgamated over the fundamental group C of the incompressible hypersurface Y. Such examples do exist; this is the case in dimension 3 thanks to the W.Thurston's hyperbolization theorem. In these cases there is equality in Theorem 1.1, i.e., $\delta(C) = n-2$

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where C is the fundamental group of Y, and Y. Shalom suggested in [**17**] that the equality case in the Theorem 1.1 happens only in that case. In any dimension, A. Lubotsky showed that any standard arithmetic lattice of $PO(n,1)$ has a finite cover whose fundamental group is an amalgamated product, cf. [**12**]. In fact, A. Lubotsky proved that any standard arithmetic lattice Γ has a finite index subgroup Γ_0 which is mapped onto a nonabelian free group. A nonabelian free group can be written in infinitely many ways as an amalgamated product, so one gets infinitely many decompositions of Γ_0 as an amalgamated product by pulling back the amalgamated decomposition of the nonabelian free group.

The aim of this paper is to show that Theorem 1.1 still holds when Γ is the fundamental group of a compact riemannian manifold of variable sectional curvature less than or equal to -1, and characterize the equality case.

Theorem 1.2. *Let X be an n-dimensional compact riemannian manifold of sectional curvature $K \leq -1$. We assume that the fundamental group Γ of X is an amalgamated product of its subgroups A and B over C, and that neither A nor B equals Γ. Then, the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n-2$. Equality $\delta(C) = n-2$ happens if and only if C cocompactly preserves a totally geodesic isometrically embedded copy $\mathbb{H}^{n-1}$ of the hyperbolic space of dimension $n-1$. Moreover, in the equality case, the hypersurface $Y^{n-1} := \mathbb{H}^{n-1}/C$ is embedded in X and separates X in two connected components whose fundamental groups are respectively A and B.*

Remark 1.3.

(i) By the assumption on A or B not being equal to Γ we exclude the trivial decomposition $\Gamma = \Gamma *_C C$ where $A = \Gamma$ and $C = B$ can be an arbitrary subgroup of Γ, for example any cyclic subgroup, in which case the conclusion of Theorem 1.2 fails. Also note that because of this assumption on A and B, we have $A \neq C$ and $B \neq C$.

(ii) Let us recall that standard arithmetic lattices in $PO(n,1)$ have finite index subgroups with infinitely many non-equivalent decompositions as amalgamated products, cf. [**12**]. In fact, among these decompositions, all but finitely many of them are such that $\delta(C) > n-2$. Indeed, by Theorem 1.2, if $\delta(C) = n-2$ then C is the fundamental group of an embedded totally geodesic hypersurface in X, but there are only finitely many totally geodesic hypersurfaces by [**23**].

When a group is an amalgamated product, it acts on a simplicial tree without fixed point and Theorem 1.1 is a particular case of

Theorem 1.4 ([**17**], Theorem 1.6)**.** *Let $\Gamma \subset SO_0(n,1)$, $n \geq 3$, be a lattice. Suppose that Γ acts on a simplicial tree T without fixed vertex. Then there is an edge of T whose stabilizer C satisfies $\delta(C) \geq n-2$.*

In the case Γ is cocompact, the conclusion of Theorem 1.4 holds for the stabilizer of any edge which separates the tree T in two unbounded components, and the proof of this is exactly the same as the proof of Theorem 1.2. In particular, when the action of Γ on T is minimal, (i.e., there is no proper subtree of T invariant by Γ), the conclusion of Theorem 1.4 holds for every edge of T, in the variable curvature setting, and we are able to handle the equality case.

Theorem 1.5. *Let Γ be the fundamental group of an n-dimensional compact riemannian manifold X of sectional curvature less than or equal to -1. Suppose that Γ acts minimally on a simplicial tree T without fixed point. Then, the stabilizer C of every edge of T satisfies $\delta(C) \geq n-2$. The equality $\delta(C) = n-2$ happens if and only if there exists a compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, Y with its induced metric has constant sectional curvature -1.*

Another interesting case contained in Theorem 1.5 is the case of an HNN extension. Let us recall the definition of an HNN extension. Let A and C be groups and $f_1 : C \to A$, $f_2 : C \to A$ two injective morphisms of C into A. The HNN extension $A*_C$ is the group generated by A and an element t with the relations $tf_1(\gamma)t^{-1} = f_2(\gamma)$. For example, let X be a compact manifold containing a non-separating compact incompressible hypersurface $Y \subset X$. Let A be the fundamental group of the manifold with boundary $X - Y$ obtained by cutting X along Y and let C be the fundamental group of Y. The boundary of $X - Y$ consists in two connected components $Y_1 \subset X - Y$ and $Y_2 \subset X - Y$ homeomorphic to Y. By the incompressibility assumption, these inclusions give rise to two embeddings of C into A, and the fundamental group of X is the associated HNN extension $A*_C$. Such examples do exist, cf. [**13**].

Theorem 1.6. *Let Γ be the fundamental group of a compact riemannian manifold X of dimension n and sectional curvature less than or equal to -1. Suppose that $\Gamma = A*_C$ where A is a proper subgroup of Γ. Then, we have $\delta(C) \geq n-2$ and equality $\delta(C) = n-2$ if and only if there exists a non separating compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, Y with its induced metric is of constant sectional curvature -1, and the HNN decomposition arising from Y is the one we started with.*

Let us summarize the ideas of the proof of Theorem 1.2. We work on $\tilde{X}/C$. The amalgamation assumption provides an essential hypersurface Z in $\tilde{X}/C$, namely Z is homologically nontrivial in $\tilde{X}/C$. The volume of all hypersurfaces homologous to Z is bounded below by a positive constant because their systoles are bounded away from zero. We then construct a smooth map $F : \tilde{X}/C \to \tilde{X}/C$, homotopic to the identity which contracts the volume of all compact hypersurfaces Y by the factor $\left(\frac{\delta(C)}{n-2}\right)^{n-1}$, namely $\mathrm{vol}_{n-1}F(Y) \leq \left(\frac{\delta(C)}{n-2}\right)^{n-1}\mathrm{vol}_{n-1}Y$. This contracting property together with the lower bound on the volume of hypersurfaces in the homology class of Z gives the inequality $\delta(C) \geq n-2$. This map is different from the map constructed in [**3**]; in particular, it can be defined under the single condition that the limit set of C is not reduced to one point. Moreover, its derivative has an upper bound depending only on the critical exponent of C.

The equality case goes as follows. When $\delta(C) = n-2$, the map $F : \tilde{X}/C \to \tilde{X}/C$ contracts the $(n-1)$-dimensional volumes, i.e., $|\mathrm{Jac}_{n-1}F| \leq 1$. This contracting property is infinitesimally rigid in the following sense. Let us consider a lift $\tilde{F}$ of F. If $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$ at some point $x \in \tilde{X}$, then $\tilde{F}(x) = x$, there exists a tangent hyperplane $E \subset T_x\tilde{X}$ such that $D\tilde{F}(x)$ is the orthogonal projection of $T_x\tilde{X}$ onto E, and the limit set Λ_C is contained in the *topological equator* $E(\infty) \subset \partial\tilde{X}$ associated to E. By topological equator $E(\infty) \subset \partial\tilde{X}$ associated to E, we mean the set of end points of those geodesic rays starting at x tangent to E.

We then prove the existence of a point $x \in \tilde{X}$ such that

$$|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1. \tag{1.1}$$

If there would exist a minimizing cycle in the homology class of Z in $\tilde{X}/C$, any point of such a cycle would satisfy 1.1. As no such minimizing cycle a priori exists because of non-compactness of $\tilde{X}/C$, we prove instead the existence of an L^2 harmonic $(n-1)$-form dual to Z, which is enough to prove existence of a point x such that 1.1 holds.

At this stage of the proof, there is a big difference between the constant curvature case and the variable curvature case.

In the constant curvature case, any topological equator bounds a totally geodesic hyperbolic hypersurface $\mathbb{H}^{n-1}$, and therefore, as the group C preserves $\Lambda_C \subset E(\infty) = \partial\mathbb{H}^{n-1}$, it is not hard to see that C also preserves $\mathbb{H}^{n-1}$ and acts cocompactly on it, and the hypersurface of the equality case in Theorem 1.2 is $\mathbb{H}^{n-1}/C$, [**2**].

In the variable curvature case, we first show the existence of a C-invariant totally geodesic hypersurface $\tilde{Z}_\infty \subset \tilde{X}$ whose boundary at infinity coincides with $\Lambda(C)$, and then we show that $\tilde{Z}_\infty$ is isometric to the real hyperbolic space. We then show that $Y =: \mathbb{H}^{n-1}/C$, which is compact, injects in $X = \tilde{X}/\Gamma$ and separates X in two connected components whose fundamental groups are A and B respectively.

In order to show the existence of such a totally geodesic hypersurface $\tilde{Z}_\infty$, we first prove that C is a convex cocompact group, i.e., the convex hull of the limit set of C in $\tilde{X}$ has a compact quotient under the action of C, and that the limit set of C is homeomorphic to an $(n-2)$-dimensional sphere.

The convex cocompactness property of C and the fact that the limit set $\Lambda(C)$ of C is homeomorphic to an $(n-2)$-dimensional topological sphere are the two key points in the equality case.

This compactness property then allows us to prove the existence of a mimimizing current in the homology class of the essential hypersurface $Z \subset \tilde{X}/C$. By a regularity theorem this minimizing current $\tilde{Z}_\infty$ is a smooth manifold except at a singular set of codimension at least 8. By the contracting properties of our map F, $\tilde{Z}_\infty$ is fixed by F and the geometric properties of F at fixed points where the $(n-1)$-jacobian of F equals 1 allow us to prove that $\tilde{Z}_\infty$ is totally geodesic and isometric to the hyperbolic space.

Let us now briefly describe the proof of the convex cocompactness property of C in the equality case.

The group C (or a finite index subgroup of it) actually globally preserves a smooth cocompact hypersurface $\tilde{Z} \subset \tilde{X}$ which separates $\tilde{X}$ into two connected components and whose boundary $\partial\tilde{Z} \subset \partial\tilde{X}$ coincides with $\Lambda_C \subset E(\infty)$. In the case where C would not be convex cocompact, we are able to find a horoball $HB(\theta_0)$ centered at some point $\theta_0 \in \Lambda_C$ in the complement of the hypersurface $\tilde{Z}$.

The contradiction then comes from the following.

Consider a sequence of points $\theta_i \in \partial\tilde{X}$ converging to θ_0 and geodesic rays α_i starting from the point $x \in \tilde{X}$ at which $|\mathrm{Jac}_{n-1}(x)| = 1$ and ending up at θ_i. These geodesic rays have to cross $\tilde{Z}$ at points z_i which are at bounded distance from the orbit Cx of x, and therefore the shadows $\mathcal{O}_i$ of balls centered at these z_i lit from x must contain points of Λ_C by the shadow lemma of D. Sullivan. On the other hand, we show that it is possible to choose the sequence θ_i in such a way that these shadows $\mathcal{O}_i$ do not meet Λ_C. This property $\mathcal{O}_i \cap \Lambda_C = \emptyset$ comes from a choice of θ_i such that the distance between z_i and the set H of all geodesics rays at x tangent to $E \subset T_x\tilde{X}$ tends to ∞. Intuitively, in order to chose z_i as far as possible from H, the points θ_i have to be chosen tranversally to Λ_C. This transversality condition is not well-defined because the limit set Λ_C might be highly non-regular. Thus, in order to prove that such a choice is possible, we argue again by contradiction. If for any choice of a sequence θ_i converging to θ_0, the distance between z_i and H stays bounded, then the Gromov distances $d(\theta_i, \theta_0)$ between θ_i and θ_0 satisfy $d(\theta_i, \Lambda_C) = o(d(\theta_i, \theta_0))$, and therefore any tangent cone of Λ_C at θ_0 would coincide with a tangent cone of $\partial\tilde{X}$ at θ_0, which is known to be topologically $\mathbb{R}^{n-1}$. But on the other hand, the existence of a point x such that $|\mathrm{Jac}_{n-1}(x)| = 1$ and the fact that C acts uniformly quasi-conformally with respect to the Gromov distance on $\partial\tilde{X}$ imply that the

Alexandroff compactification of the above tangent cone of Λ_C at θ_0 is homeomorphic to Λ_C, which is contained in a topological sphere S^{n-2}, leading to a contradiction.

From convex cocompactness of C and the fact that the limit set of C is a topological $(n-2)$-dimensional sphere, there is an alternative proof of the existence of a totally geodesic C-invariant copy of the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}} \subset \tilde{X}$ that consists of observing that the topological dimension and the Hausdorff dimension of the limit set $\Lambda(C)$ are equal to $n-2$, and then using the following result of M. Bonk and B. Kleiner (which we quote in the riemannian manifold setting although it remains true for $CAT(-1)$ spaces) instead of the (simpler) minimal current argument.

Theorem 1.7 ([**4**]). *Let X be a Cartan Hadamard n-dimensional manifold whose sectional curvature satisfies $K \leq -1$, and C a convex cocompact discrete subgroup of isometries of X with limit set Λ_C. Let us assume that the topological dimension and the Hausdorff dimension (with respect to the Gromov distance on $\partial\tilde{X}$) of Λ_C coincide and are equal to an integer p. Then, C preseves a totally geodesic embedded copy of the real hyperbolic space $\mathbb{H}^{p+1}$, with $\partial\mathbb{H}^{p+1} = \Lambda_C$.*

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2. Essential hypersurfaces

Let Γ be a discrete cocompact group of isometries of a n-dimensional Cartan-Hadamard manifold $(\tilde{X}, \tilde{g})$ whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$. Let us assume that the compact manifold $X = \tilde{X}/\Gamma$ is orientable. Let us also assume that $\Gamma = A *_C B$ is an amalgamated product of its subgroups A and B over C.

The goal of this section is to construct a C-equivariant hypersurface $\tilde{Z} \subset \tilde{X}$ such that $\tilde{Z}/C$ is a compact essential hypersurface in $\tilde{X}/C$. By essential hypersurface we mean the following:

Definition 2.1. A compact smooth orientable hypersurface Z of an n-dimensional manifold Y is essential in Y if $i_*([Z]) \neq 0$ where $[Z] \in H_{n-1}(Z, \mathbb{R})$ denotes the fundamental class of Z and $i_* : H_{n-1}(Z, \mathbb{R}) \to H_{n-1}(Y, \mathbb{R})$ the morphism induced by the inclusion $i : Z \hookrightarrow Y$.

Before finding the essential hypersurface $Z \subset \tilde{X}/C$, let us show that under the assumption of Theorem 1.2 we have $H_{n-1}(\tilde{X}/C, \mathbb{Z}) \neq 0$.

We first reduce to the case where $[\Gamma : C]$ is infinite.

Namely, if $[\Gamma : C] < \infty$, then the critical exponent $\delta(C) = \delta(\Gamma) \geq n-1$, and the inequality in Theorem 1.2 holds.

We then can assume that $[\Gamma : C] = \infty$.

Lemma 2.2. *Let $\Gamma = A *_C B$ be as above, with $[\Gamma : C] = \infty$. If neither A nor B equals Γ, then $H_{n-1}(\tilde{X}/C, \mathbb{Z}) \neq 0$.*

Proof. The Mayer-Vietoris sequence coming from the decomposition $\Gamma = A *_C B$ writes (cf. [**6**], Corollary 7.7),

$$H_n(\tilde{X}/C, \mathbb{Z}) \to H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z}) \to H_n(\tilde{X}/\Gamma, \mathbb{Z}) \to ...$$

$$... \to H_{n-1}(\tilde{X}/C, \mathbb{Z}) \to ...$$

As $[\Gamma : C] = \infty$, $H_n(\tilde{X}/C, \mathbb{Z}) = 0$ thus, if furthermore $H_{n-1}(\tilde{X}/C, \mathbb{Z}) = 0$, we deduce from the Mayer-Vietoris sequence that $H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z})$ is isomorphic to $H_n(\tilde{X}/\Gamma, \mathbb{Z})$. As $H_n(\tilde{X}/\Gamma, \mathbb{Z}) = \mathbb{Z}$, we then deduce that either $[\Gamma : A] = \infty$ and $B = \Gamma$, or $[\Gamma : B] = \infty$ and $A = \Gamma$. q.e.d.

In fact, in the sequel we will make use of a smooth essential hypersurface Z in $\tilde{X}/C$ and the end of this section is devoted to finding such an hypersurface.

Let us recall a few facts about amalgamated products and their actions on trees, following the Bass-Serre theory, [**16**]. Let $\Gamma = A *_C B$ be an amalgamated product of its subgroups A and B over C. Then, Γ acts on a simplicial tree $\tilde{T}$, called the Bass-Serre tree with a fundamental domain $T \subset \tilde{T}$ being a segment, i.e., an edge joining two vertices. Let us describe this tree $\tilde{T}$. There are two orbits of vertices Γv_A and Γv_B, the stabilizer of the vertex v_A (resp. v_B) being A (resp. B). There is one orbit of edges Γe_C, the stabilizer of the edge e_C being C. The fundamental domain T can be chosen as the edge e_C joining the two vertices v_A and v_B. The set of vertices adjacent to v_A (resp. v_B), is in one to one correspondance with A/C (resp. B/C). Note that as neither A nor B are equal to Γ, $[A : C] \neq 1$ and $[B : C] \neq 1$, and therefore for an arbitrary point t_0 on the edge e_C we see that $\tilde{T} - t_0$ is a disjoint union of two unbounded connected components. This fact will be used later on.

Let us consider a continuous Γ-equivariant map $\tilde{f} : \tilde{X} \to T$ where T is the Bass-Serre tree associated to the amalgamation $\Gamma = A *_C B$. One regularizes $\tilde{f}$ such that it is smooth in restriction to the complement of the inverse image of the set of vertices of T. Let t_0 a regular value of $\tilde{f}$ contained in the edge of T which is fixed by the subgroup C, and define $\tilde{Z} = \tilde{f}^{-1}(t_0)$. $\tilde{Z}$ is a smooth orientable possibly not connected hypersurface in $\tilde{X}$, globally C-invariant.

Let us write $Z = \tilde{Z}/C$. We will show $Z \subset \tilde{X}/C$ is compact and that one of the connected components of Z is essential.

Lemma 2.3. *$Z \subset \tilde{X}/C$ is compact.*

Proof. Let us show that for any sequence $z_n \in \tilde{Z}$, there exists a subsequence z_{n_k} and $\gamma_k \in C$ such that $\gamma_k z_{n_k}$ converges. As Γ is cocompact, there exists $g_n \in \Gamma$ such that the set $(g_n z_n)$ is relatively compact. Let $g_{n_k} z_{n_k}$ be a subsequence which converges to a point $z \in \tilde{X}$. By continuity, the sequence $\tilde{f}(g_{n_k} z_{n_k})$ converges to $\tilde{f}(z)$, and by equivariance we get

$$g_{n_k}\tilde{f}(z_{n_k}) = g_{n_k} t_0 \to \tilde{f}(z)$$

when k tends to ∞. As Γ acts simplicially on the tree T and transitively on the set of edges, the sequence $g_{n_k} t_0$ is stationary, i.e., $g_{n_k} t_0 = t_0' = g t_0$ for k large enough. Thus, $g^{-1} g_{n_k} = \gamma_k \in C$ for k large enough since it fixes t_0 and $\gamma_k z_{n_k} = g^{-1} g_{n_k} z_{n_k}$ converges to $g^{-1} z$ q.e.d.

The smooth compact hypersurface Z we constructed might not be connected. Let us write $Z = Z_1 \cup Z_2 \cup \ldots \cup Z_k$ where the Z_j's are the connected components of Z. Each Z_j is a compact smooth oriented hypersurface of $\tilde{X}/C$.

The aim of what follows is to prove that at least one component Z_i of Z is essential.

Lemma 2.4. *There exists $i \in [1, k]$ such that Z_i is essential in $\tilde{X}/C$.*

Proof. If there exists a Z_i which does not separate $\tilde{X}/C$ in two connected components one can easily show that Z_i is then essential in $\tilde{X}/C$. So we can assume that every Z_j, $j = 1, \ldots, k$, does separate $\tilde{X}/C$ in two connected components. In that case we will show that there exists a Z_i that separates $\tilde{X}/C$ in two unbounded connected components, which easily implies using the Mayer-Vietoris sequence that Z_i is essential.

Let us denote U_l, $l = 1, 2, \ldots, p$, the connected components of $\tilde{X}/C - \cup_{j=1}^{k} Z_j$.

Claim 2.5. At least two components U_m, $U_{m'}$ are unbounded.

Assuming the claim, let us finish the proof of the lemma. For each Z_j we denote V_j, V_j' the two connected components of $\tilde{X}/C \setminus Z_j$. Then $U_m = W_1 \cap W_2 \cap \ldots \cap W_k$ where for each j, $W_j = V_j$ or $W_j = V_j'$. In the same way, $U_{m'} = W_1' \cap W_2' \cap \ldots \cap W_k'$. As $U_m \cap U_{m'} = \emptyset$, there exists $i \in [1, k]$ such that $W_i \cap W_i' = \emptyset$; thus $U_m \subset V_i$ and $U_{m'} \subset V_i'$ or $U_m \subset V_i'$ and $U_{m'} \subset V_i$, so Z_i separates $\tilde{X}/C$ into two unbounded components. This proves the lemma. q.e.d.

Proof of the claim. We have already noticed that $T - \{t_0\}$ is the disjoint union of two unbounded connected components T_1 and T_2. As C acts on T isometrically and simplicially, then $T/C - \{t_0\} = T_1/C \cup T_2/C$ is the disjoint union of two unbounded connected components. Let $\bar{f} : \tilde{X}/C \to T/C$ be the quotient map of $\tilde{f}$. For each component U_i, we have $\bar{f}(U_i) \subset T_1/C$ or $\bar{f}(U_i) \subset T_2/C$, which thus proves the claim since $\bar{f}$ is onto. q.e.d.

In the sequel it will be convenient to have Z_i of Lemma 2.4 as a quotient of some connected hypersurface $\tilde{Z} \subset \tilde{X}$ by C. This may be not possible so we have to replace C by a subgroup, as we shall now see.

Let $\pi : \tilde{X} \to \tilde{X}/C$ be the natural projection. For any $i = 1, 2, \ldots, k$, let us denote $\{\tilde{Z}_i^j\}_{j\in J}$ the set of connected components of $\tilde{Z}_i =: \pi^{-1}(Z_i)$. For each $i \in [1, k]$, we claim that C acts transitively on the set $\{\tilde{Z}_i^j\}_{j\in J}$. Namely, let us consider $\tilde{z} \in \tilde{Z}_i^j$, $\tilde{z}' \in \tilde{Z}_i^{j'}$, and write $z = \pi\tilde{z} \in Z_i$ and $z' = \pi\tilde{z}' \in Z_i$. Let α be a continuous path on Z_i such that $\alpha(0) = z$ and $\alpha(1) = z'$, and $\tilde{\alpha}$ the lift of α such that $\tilde{\alpha}(0) = \tilde{z}$. We have $\pi\tilde{\alpha}(1) = z'$ and $\tilde{\alpha}(1) \in \tilde{Z}_i^j$; thus, there exists $c \in C$ such that $c(\tilde{\alpha}(1)) = \tilde{z}'$, and therefore $c\tilde{Z}_i^j = \tilde{Z}_i^{j'}$ Let us denote C_i^j the stabilizer of $\tilde{Z}_i^j$, and $Z_i^j = \tilde{Z}_i^j/C_i^j \subset \tilde{X}/C_i^j$. Let us write $p : \tilde{X}/C_i^j \to \tilde{X}/C$ the natural projection.

Lemma 2.6. *The restriction of p to Z_i^j is a diffeomorphism onto Z_i. In particular, Z_i^j is compact.*

Proof. Let z and z' be the two points in Z_i^j such that $p(z) = p(z')$. Let $\tilde{z}$ and $\tilde{z}'$ be lifts of z and z' in $\tilde{X}$. These two points $\tilde{z}$ and $\tilde{z}'$ which are in $\tilde{Z}_i$ actually belong to the same connected component $\tilde{Z}_i^j$ because for $j \neq j'$, $(\bigcup_{j'\neq j} \tilde{Z}_i^{j'})/C_i^j \cap \tilde{Z}_i^j/C_i^j = \emptyset$. As $p(z) = p(z')$, there exists $c \in C$ such that $\tilde{z}' = c\tilde{z}$; thus $c \in C_i^j$, and $z = z'$, and therefore the restriction of p to Z_i^j is injective. The surjectivity comes from the fact that $\pi^{-1}Z_i = \bigcup_{j\in J} \tilde{Z}_i^j$ and C acts transitively on the set $\{\tilde{Z}_i^j\}_{j\in J}$.
q.e.d.

Let us consider the integer $i \in [1, k]$ as in Lemma 2.4, i.e., such that $Z_i \hookrightarrow \tilde{X}/C$ is essential, and choose $\tilde{Z}_i^l$ one component of $\pi^{-1}(Z_i)$. After possibly replacing C_i^l by an index two subgroup, we may assume that C_i^l globally preserves each of the two connected components U_i^l and V_i^l of $\tilde{X} - \tilde{Z}_i^l$.

Lemma 2.7. *Let i, l and C_i^l be chosen as above. The compact hypersurface $Z_i^l = \tilde{Z}_i^l/C_i^l$ is essential in $\tilde{X}/C_i^l$. Moreover, the two connected components U_i^l/C_i^l and V_i^l/C_i^l of $\tilde{X}/C_i^l - Z_i^l$ are unbounded.*

Proof. Let us consider $p : \tilde{X}/C_i^l \to \tilde{X}/C$. By Lemma 2.6, the restriction of p to Z_i^l is a diffeomorphism onto Z_i, and therefore Z_i^l is essential in $\tilde{X}/C_i^l$ because Z_i is essential in $\tilde{X}/C$. As C_i^l preseves U_i^l and V_i^l, Z_i^l separates $\tilde{X}/C_i^l$ into two connected components U_i^l/C_i^l and V_i^l/C_i^l, and as Z_i^l is essential in $\tilde{X}/C_i^l$, U_i^l/C_i^l and V_i^l/C_i^l are unbounded. q.e.d.

In the sequel we will denote $\tilde{Z}' = \tilde{Z}_i^l$, $C' = C_i^l$, and $Z' = Z_i^l =$ $\tilde{Z}_i^l/C_i^l$. Notice that C_i^l may not be of finite index in C.

3. Isosystolic inequality

In this section we summarize facts and results due to M. Gromov, [**10**]. These results will be used later on to give a uniform lower bound of the volume of hypersurfaces of $\tilde{X}/C$ homotopic to the hypersurface $Z' = \tilde{Z}'/C'$ constructed in Section 2.

Let Z be a p-dimensional compact orientable manifold and $i : Z \hookrightarrow Y$ an embedding of Z into Y where Y is an aspherical space. We suppose that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \to H_p(Y, \mathbb{R})$ is the morphism induced by the embedding $Z \hookrightarrow Y$. Let us fix a riemanniann metric g on Z. For each $z \in Z$ we consider the set $\mathcal{C}_z$ of the loops α at z such that $i \circ \alpha$ is homotopically nontrivial in Y.

Let us define the systole of (Z, g, i) at the point z by

Definition 3.1. $\mathrm{sys}_i(Z, g, z) = \inf\{\mathrm{length}(\alpha);\ \alpha \in \mathcal{C}_z\}$,

where the length is taken with respect to the metric g and the systole of (Z, g, i) by

Definition 3.2. $\mathrm{sys}_i(Z, g) = \inf\{\mathrm{sys}_i(Z, g, z);\ z \in Z\}$.

The following isosystolic inequality, due to M. Gromov says that the volume of any essential submanifold Z of an aspherical space Y relative to any riemanniann metric on Z is universally bounded below by its systole.

Theorem 3.3 ([**10**]). *There exists a constant C_p such that for each p-dimensional riemanniann manifold (Z, g) and each embedding $Z \hookrightarrow Y$ into an aspherical space Y such that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \to H_p(Y, \mathbb{R})$ is the induced morphism in homology,*

$$\mathrm{vol}_p(Z, g) \geq C_p(\mathrm{sys}_i(Z, g))^p.$$

In the above theorem, $\mathrm{vol}_p Z$ stands for the riemannian volume of the p-dimensional manifold (Z, g). We will apply this volume estimate to the essential hypersurface $i : Z' \hookrightarrow \tilde{X}/C'$ that we constructed in Lemma 2.7.

The following lemma is immediate.

Lemma 3.4. *Let C' be a discrete group acting on a simply connected manifold $\tilde{X}$, $\tilde{Z}'$ a C'-invariant hypersurface of $\tilde{X}$ and $i : Z' = \tilde{Z}'/C' \hookrightarrow \tilde{X}/C'$ the natural inclusion. Let g any riemanniann metric on Z' and $\tilde{g}$ the lift of g to $\tilde{Z}'$. Then for any $z \in Z'$, we have*

$$\mathrm{sys}_i(Z', g, z) = \inf\{d_{\tilde{g}}(\tilde{z}, \gamma\tilde{z});\ \gamma \in C'\},$$

where $\tilde{z} \in \tilde{Z}'$ is a lift of $z \in Z'$ and $d_{\tilde{g}}$ is the distance induced by $\tilde{g}$ on $\tilde{Z}'$.

Proof. Let α a loop based at $z \in Z'$. As $i \circ \alpha$ is an homotopically non trivial loop at $i(z) = z$ in $\tilde{X}/C'$, its lift $\widetilde{i \circ \alpha}$ at some $\tilde{z} \in \tilde{Z}'$ ends up at $\gamma\tilde{z}$ for some $\gamma \in C'$ q.e.d.

4. Volume of hypersurfaces in $\tilde{X}/C$

Let $(\tilde{X}, \tilde{g})$ be a n-dimensional Cartan-Hadamard manifold whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$, and C a discrete group of isometries of $(\tilde{X}, \tilde{g})$. We assume that the group C is non-elementary, namely C fixes neither one nor two points in the geometric boundary $\partial\tilde{X}$ of $(\tilde{X}, \tilde{g})$.

The aim of this section is to give a general construction of a C-equivariant map $\tilde{F} : \tilde{X} \to \tilde{X}$ whose p-jacobian is related to the critical exponent $\delta(C)$ of C, namely such that for any p, $2 \leq p \leq n$ and any $x \in \tilde{X}$, we have

$$|\mathrm{Jac}_p(\tilde{F}(x)| \leq \Big(\frac{\delta(C)+1}{p}\Big)^p.$$

This map gives rise to a quotient map $F : \tilde{X}/C \to \tilde{X}/C$ which dilates the volume of compact hypersurfaces Z of $\tilde{X}/C$ by the above factor, namely,

$$\mathrm{vol}_{n-1}(F(Z)) \leq \Big(\frac{\delta(C)+1}{n-1}\Big)^{n-1} \mathrm{vol}_{n-1}(Z)$$

where $\mathrm{vol}_{n-1}(Z)$ stands for the $(n-1)$-dimensional volume of the metric on Z induced from g.

For every subgroup $C' \subset C$ and any hypersurface Z' of $\tilde{X}/C'$ the lift $F' : \tilde{X}/C' \to \tilde{X}/C'$ of F will also clearly verify

$$\mathrm{vol}_{n-1}(F'(Z')) \leq \Big(\frac{\delta(C)+1}{n-1}\Big)^{n-1} \mathrm{vol}_{n-1}(Z').$$

In order to construct the map $\tilde{F}$, we need a few preliminaries. We consider a finite positive Borel measure μ on the boundary $\partial\tilde{X}$ whose support contains at least two points. Let us fix an origin $o \in \tilde{X}$ and denote $B(x,\theta)$ the Busemann function defined for each $x \in \tilde{X}$ and $\theta \in \partial\tilde{X}$ by

$$B(x,\theta) = \lim_{t\to\infty} \mathrm{dist}(x, c(t)) - t$$

where $c(t)$ is the geodesic ray such that $c(0) = o$ and $c(+\infty) = \theta$.

Let us recall that $B(x,\theta) \to -\infty$ when x tends to θ and that $B(x,\theta) \to +\infty$ when x tends to a point $\theta' \neq \theta$ in $\partial\tilde{X}$.

Let $\mathcal{D}_\mu : \tilde{X} \to \mathbb{R}$ be the function defined by

$$\mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} e^{B(y,\theta)} d\mu(\theta). \tag{4.1}$$

A computation shows that

$$Dd\mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} (DdB_{(y,\theta)} + dB_{(y,\theta)} \otimes dB_{(y,\theta)}) e^{B(y,\theta)} d\mu(\theta). \tag{4.2}$$

When $K_{\tilde{g}} \leq -1$, the Rauch comparison theorem says that for every $y \in \tilde{X}$, and $\theta \in \partial\tilde{X}$,

$$DdB(y,\theta) + dB_{(y,\theta)} \otimes dB_{(y,\theta)} \geq \tilde{g}. \tag{4.3}$$

We then get

$$Dd\mathcal{D}_\mu(y) \geq \mathcal{D}_\mu(y)\tilde{g}, \tag{4.4}$$

thus $Dd\mathcal{D}_\mu(y)$ is positive definite and $\mathcal{D}_\mu$ is strictly convex.

Lemma 4.1. *If the support of μ contains at least two points, we have* $\lim_{y \to \partial\tilde{X}} \mathcal{D}_\mu(y) = +\infty$.

Proof. Let $y_k \in \tilde{X}$ be a sequence such that

$$\lim_{k\to\infty} y_k = \theta_0 \in \partial\tilde{X}. \tag{4.5}$$

As the support of μ contains at least two points, we have $\mathrm{supp}(\mu) \cap (\partial\tilde{X} \setminus \{\theta_0\}) \neq \emptyset$, thus there exists a compact subset $K \subset \partial\tilde{X} \setminus \{\theta_0\}$ such that $\mu(K) > 0$. Therefore,

$$\int_{\partial\tilde{X}} e^{B(y_k,\theta)} d\mu \geq \int_K e^{B(y_k,\theta)} d\mu \to +\infty. \tag{4.6}$$

q.e.d.

Corollary 4.2. *Let μ a finite borel measure on $\partial\tilde{X}$ whose support contains at least two points. The function $\mathcal{D}_\mu$ has a unique minimum. This minimum will be denoted by $\mathcal{C}(\mu)$.*

Let us now consider some discrete subgroup $C \subset \mathrm{Isom}(\tilde{X}, \tilde{g})$. Recall that a family of Patterson measures $(\mu_x)_{x\in\tilde{X}}$ associated to C is a set of positive finite measures μ_x on $\partial\tilde{X}$, $x \in \tilde{X}$, such that the following holds for all $x \in \tilde{X}$, $\gamma \in C$,

$$\mu_{\gamma x} = \gamma_* \mu_x \tag{4.7}$$

$$\mu_x = e^{-\delta B(x,\theta)} \mu_o, \tag{4.8}$$

where $o \in \tilde{X}$ is a fixed origin, B the Busemann function associated to o and δ the critical exponent of C. We now assume that $\mathrm{supp}(\mu_o)$ contains at least two points and define the map $\tilde{F} : \tilde{X} \to \tilde{X}$ for $x \in \tilde{X}$ by

$$\tilde{F}(x) = \mathcal{C}(e^{-B(x,\theta)} \mu_x). \tag{4.9}$$

Remark 4.3.

The construction of this map is close to the one worked out in [**1**], [**3**]. In these papers we used the functional $\mathcal{D}'_\mu(y) = \int_{\partial\tilde{X}} B(y,\theta) d\mu(\theta)$ instead of $\mathcal{D}_\mu$ and defined the barycenter of μ as the unique critical point of $\mathcal{D}_\mu$. This construction required properties of the measures μ_x that we cannot ensure in our setup. This is the reason why we modify the construction as above. Furthermore, the new construction turns out to be technically easier to handle and provides a map with better properties.

Here are a few notations. For a subspace E of $T_x\tilde{X}$, we will write $\mathrm{Jac}_E\tilde{F}(x)$ the determinant of the matrix of the restriction of $D\tilde{F}(x)$ to E with respect to orthonormal bases of E and $D\tilde{F}(x)E$. For an integer p, we denote by $\mathrm{Jac}_p\tilde{F}(x)$ the supremum of $|\mathrm{Jac}_E\tilde{F}(x)|$ as E runs through the set of p-dimensional subspaces of $T_x\tilde{X}$.

Lemma 4.4. *The map $\tilde{F}$ is smooth, homotopic to the Identity, and verifies for all $x \in \tilde{X}$, $\gamma \in C$ and $p \in [2, n = dim(X)]$,*

i) $\tilde{F}(\gamma x) = \gamma \tilde{F}(x)$
ii) $|\mathrm{Jac}_p\tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p$.

Proof. The map

$$(x, y) \to \int_{\partial\tilde{X}} e^{B(y,\theta)-B(x,\theta)} d\mu_x(\theta) = \int_{\partial\tilde{X}} e^{B(y,\theta)-(\delta+1)B(x,\theta)} d\mu_o(\theta)$$

is smooth because $y \to B(y, \theta)$ is smooth. For all x the map $y \to \int_{\partial\tilde{X}} e^{B(y,\theta)-(\delta+1)B(x,\theta)} d\mu_o(\theta)$ is strictly convex by 4.4 and tends to infinity when y tends to $\partial\tilde{X}$ (cf. Lemma 4.1); thus the unique minimum $\tilde{F}(x)$ is a smooth function of x. The equivariance of $\tilde{F}$ comes from the cocycle relation $B(\gamma y, \gamma\theta) - B(\gamma x, \gamma\theta) = B(y, \theta) - B(x, \theta)$.

For each $x \in \tilde{X}$ let c_x be the geodesic in $\tilde{X}$ such that $c_x(0) = x$, $c_x(1) = \tilde{F}(x)$ and which is parametrized with constant speed. The map $\tilde{F}_t : \tilde{X} \to \tilde{X}$ defined by $\tilde{F}_t(x) = c_x(t)$ is a C-equivariant homotopy between $Id_{\tilde{X}}$ and $\tilde{F}$.

It remains to prove (ii).

The point $\tilde{F}(x)$ is characterized by the implicit equation

$$\int_{\partial\tilde{X}} dB_{(\tilde{F}(x),\theta)} e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0. \tag{4.10}$$

In order to simplify the notation we will denote by ν_x the measure $e^{B(\tilde{F}(x),\theta)-B(x,\theta)}\mu_x$ and δ instead of $\delta(C)$. We will also write $D\tilde{F}(u)$ instead of $D\tilde{F}(x)(u)$. By differentiating 4.10 we get the following characterization of the differential of $\tilde{F}$: for $u \in T_x\tilde{X}$ and $v \in T_{\tilde{F}(x)}\tilde{X}$, one has

$$\int_{\partial\tilde{X}} [DdB_{(\tilde{F}(x),\theta)}(D\tilde{F}(u), v) + dB_{(\tilde{F}(x),\theta)}(v) dB_{(\tilde{F}(x),\theta)}(D\tilde{F}(u))] d\nu_x(\theta)$$

$$= (\delta + 1) \int_{\partial\tilde{X}} dB_{(\tilde{F}(x),\theta)}(v) dB_{(x,\theta)}(u) d\nu_x(\theta). \tag{4.11}$$

We define the quadratic forms k and h for $v \in T_{\tilde{F}(x)}\tilde{X}$ by

$$k(v,v) = \int_{\partial\tilde{X}} [DdB_{(\tilde{F}(x),\theta)}(v,v) + (dB_{(\tilde{F}(x),\theta)}(v))^2] d\nu_x(\theta) \tag{4.12}$$

and

$$h(v,v) = \int_{\partial\tilde{X}} dB_{(\tilde{F}(x),\theta)}(v)^2 d\nu_x(\theta). \tag{4.13}$$

The relation 4.11 becomes, for $u \in T_x\tilde{X}$ and $v \in T_{\tilde{F}(x)}\tilde{X}$:

$$k(D\tilde{F}(u),v) = (\delta+1)\int_{\partial\tilde{X}} dB_{(\tilde{F}(x),\theta)}(v) dB_{(x,\theta)}(u) d\nu_x(\theta). \tag{4.14}$$

We define the quadratic form h' on $T_x\tilde{X}$ for $u \in T_x\tilde{X}$ by

$$h'(u,u) = \int_{\partial\tilde{X}} dB_{(x,\theta)}(u)^2 d\nu_x(\theta), \tag{4.15}$$

and one derives from 4.14 by applying the Cauchy-Schwarz inequality

$$|k(D\tilde{F}(x)(u),v)| \leq (\delta+1) h(v,v)^{1/2} h'(u,u)^{1/2}. \tag{4.16}$$

One can now estimate $\mathrm{Jac}_p\tilde{F}(x)$. Let P be a subspace of $T_x\tilde{X}$, with $\dim P = p$. If $D\tilde{F}(P)$ has dimension lower than p, then $\mathrm{Jac}_p\tilde{F}(x) = 0$. Let us assume that $\dim D\tilde{F}(P) = p$. Denote by the same letters H' [resp. H and K] the selfadjoint endomorphisms (with respect to $\tilde{g}$) associated to the quadratic forms h' [resp. h, k] restricted to P [resp. $D\tilde{F}(P)$].

Let $(v_i)_{i=1}^p$ be an orthonormal basis of $D\tilde{F}(P)$ which diagonalizes H and $(u_i)_{i=1}^p$ an orthonormal basis of P such that the matrix of $K \circ D\tilde{F}(x) : P \to D\tilde{F}(P)$ is triangular. Then,

$$\det K \cdot |\mathrm{Jac}_P\tilde{F}(x)| \leq (\delta+1)^p (\Pi_{i=1}^p h(v_i,v_i)^{1/2})(\Pi_{i=1}^p h'(u_i,u_i)^{1/2}), \tag{4.17}$$

and thus,

$$\det K \cdot |\mathrm{Jac}_P\tilde{F}(x)| \leq (\delta+1)^p \Big(\frac{\mathrm{trace} H}{p}\Big)^{p/2} \Big(\frac{\mathrm{trace} H'}{p}\Big)^{p/2}. \tag{4.18}$$

In these inequalities one can normalize at each point x the measure

$$\nu_x = e^{B(\tilde{F}(x),\theta) - B(x,\theta)} \mu_x$$

such that its total mass equals one, which gives

$$\mathrm{trace} H = \Sigma_{i=1}^p h(v_i,v_i) \leq 1, \tag{4.19}$$

the last inequality coming from the fact that for all $\theta \in \partial\tilde{X}$

$$\Sigma_{i=1}^p dB_{(\tilde{F}(x),\theta)}(v_i)^2 \leq ||dB_{(\tilde{F}(x),\theta)}||^2 = 1, \tag{4.20}$$

and from the previous normalization. Similarly,

$$\mathrm{trace} H' = \Sigma_{i=1}^p h'(u_i,u_i) \leq 1. \tag{4.21}$$

We then obtain from 4.18

$$\det K \cdot |\mathrm{Jac}_P \tilde{F}(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p. \tag{4.22}$$

Thanks to 4.3, we have $\det K \geq 1$, so that

$$|\mathrm{Jac}_P \tilde{F}(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p. \tag{4.23}$$

We get (ii) by taking the supremum in P. q.e.d.

As the map $\tilde{F} : \tilde{X} \to \tilde{X}$ is C-equivariant, then for every subgroup $C' \subset C$, $\tilde{F}$ gives rise to a map $F' : \tilde{X}/C' \to \tilde{X}/C'$ and so does the homotopy $\tilde{F}_t$ between $\tilde{F}$ and $Id_{\tilde{X}}$. We easily get the

Corollary 4.5. *The map $F' : \tilde{X}/C' \to \tilde{X}/C'$ is homotopic to the Identity map and verifies for all $x \in \tilde{X}/C'$ and $p \in [2, n = \dim X]$*

$$|\mathrm{Jac} F'_p(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p.$$

The C-equivariant map $\tilde{F} : \tilde{X} \to \tilde{X}$ constructed in Lemma 4.4 satisfies

$$|\mathrm{Jac}\tilde{F}_p(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p,$$

for $2 \leq p \leq n = \dim \tilde{X}$. When $\delta = p-1$, we then have $|\mathrm{Jac}\tilde{F}_p(x)| \leq 1$. In particular, when $\delta = n-2$ we get $|\mathrm{Jac}\tilde{F}_{n-1}(x)| \leq 1$.

Let us now analyze the equality case, $|\mathrm{Jac}\tilde{F}_{n-1}(x)| = 1$ when $\delta = n-2$. Let $x \in \tilde{X}$ and $E \subset T_x\tilde{X}$ be a codimension one subspace. For each $u \in T_x\tilde{X}$ with $\tilde{g}(u,u) = 1$, one considers the geodesic c_u defined by $c_u(0) = x$ and $\dot{c_u}(0) = u$. We define the equator $E(\infty)$ associated to E as the subset of $\partial\tilde{X}$

$$E(\infty) = \{c_u(+\infty) / u \in E\}. \tag{4.24}$$

Proposition 4.6. *We suppose that $\delta = n-2$. Let $x \in \tilde{X}$ such that $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$. Then there exists $E \subset T_x\tilde{X}$ such that the limit set Λ_C satisfies $\Lambda_C \subset E(\infty)$. Moreover, $\tilde{F}(x) = x$ and $D\tilde{F}(x)$ is the orthogonal projector onto E.*

Proof. Let $x \in \tilde{X}$ such that $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$. By compactness there exist a subspace $E \subset T_x\tilde{X}$ such that $|\mathrm{Jac}_E\tilde{F}(x)| = 1$. By (4.18) and $\det K \geq 1$, we have

$$\begin{aligned} |\mathrm{Jac}_E\tilde{F}(x)| &\leq (n-1)^{n-1}\Big(\frac{\mathrm{trace} H}{n-1}\Big)^{\frac{n-1}{2}}\Big(\frac{\mathrm{trace} H'}{n-1}\Big)^{\frac{n-1}{2}} \\ &\leq (n-1)^{n-1}\Big(\frac{1}{n-1}\Big)^{n-1}. \end{aligned} \tag{4.25}$$

In particular, as $|\mathrm{Jac}_E \tilde{F}(x)| = 1$, we have equality in the inequalities 4.17 and 4.18; thus, trace$H$ = trace$(h) = 1$, which implies

$$H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}. \tag{4.26}$$

Let us recall that the quadratic form h is defined by

$$h(v,v) = \int_{\partial\tilde{X}} dB_{(\tilde{F}(x),\theta)}(v)^2 e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta)$$

where μ_x is the Patterson-Sullivan measure of C normalized by

$$\int_{\partial\tilde{X}} e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 1. \tag{4.27}$$

We then have

$$\begin{aligned}1 = \mathrm{trace}(h) = \mathrm{trace}H &= \Sigma_{i=1}^{n-1} h(v_i,v_i)\\ &= \int_{\partial\tilde{X}} \Sigma_{i=1}^{n-1} dB_{(\tilde{F}(x),\theta)}(v_i)^2 e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta)\\ &\leq \int_{\partial\tilde{X}} ||dB_{(\tilde{F}(x),\theta)}||^2 e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) \leq 1,\end{aligned}$$

because $\Sigma_{i=1}^{n-1} dB_{(\tilde{F}(x),\theta)}(v_i)^2 \leq ||dB_{(\tilde{F}(x),\theta)}||^2 = 1$ for all $\theta \in \partial\tilde{X}$. Therefore for μ_x-almost all $\theta \in \mathrm{supp}(e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta)) = \mathrm{supp}(\mu_x)$, we have

$$\Sigma_{i=1}^{n-1} dB_{(\tilde{F}(x),\theta)}(v_i)^2 = ||dB_{(\tilde{F}(x),\theta)}||^2 = 1. \tag{4.28}$$

In 4.28, $\Sigma_{i=1}^{n-1} dB_{(\tilde{F}(x),\theta)}(v_i)^2$ represents the square of the norm of the projection of the gradient $\nabla B(\tilde{F}(x),\theta)$ on E. By continuity of $B(x,\theta)$ in θ, one then gets $\Lambda_C = \mathrm{supp}(\mu_x) \subset E(\infty)$.

Let us now prove that $\tilde{F}(x) = x$. When $\mathrm{Jac}\tilde{F}_E(x) = 1$, we have equality in the Cauchy-Schwarz inequality 4.16, and therefore for each $i = 1, \ldots, n-1$ and $\theta \in \Lambda_C$ we get $dB_{(\tilde{F}(x),\theta)}(v_i) = dB_{(x,\theta)}(u_i)$. Therefore we deduce from 4.28 that $\nabla B_{(x,\theta)} = \Sigma_{i=1}^{n-1} dB_{(x,\theta)}(u_i) u_i$, which implies with 4.10 that

$$\int_{\partial\tilde{X}} dB_{(x,\theta)} e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0. \tag{4.29}$$

On the other hand, as $\int_{\partial\tilde{X}} dB_{(\tilde{F}(x),\theta)} e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0$ and $H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}$, the support of μ_x cannot be just a pair of points; therefore the barycenter of the measure $e^{B(\tilde{F}(x),\theta)-B(x,\theta)}\mu_x$ defined in [**3**], see the remark in Section 4, is well defined and characterized as the unique point $z \in \tilde{X}$ such that

$$\int_{\partial\tilde{X}} dB_{(z,\theta)} e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0.$$

Thus 4.29 and 4.10 imply $x = \tilde{F}(x)$.

It is now easy to check that $D\tilde{F}(x)$ is the orthogonal projector onto E. q.e.d.

Let $C \subset \mathrm{Isom}(\tilde{X}, \tilde{g})$ be as above, i.e., such that the support of the Patterson- Sullivan measures contains at least two points and with critical exponent δ. Let $C' \subset C$ be a subgroup. We now explain how we can prove the inequality $\delta(C') \geq n-2$ from Corollary 4.5.

Let us consider a compact hypersurface $Z' \subset \tilde{X}/C'$. Denote $F'^k = F' \circ F' \circ \ldots \circ F'$ the composition of F' k-times. If we knew that $F'^k(Z')$ were a submanifold of $\tilde{X}/C'$ with volume bounded below by a positive constant independent of k, then the inequality $\delta(C') \geq n-2$ would follow from the remark that if $\delta(C') < n-2$, by Corollary 4.5 one would get

$$\mathrm{vol}_{n-1} F'^k(Z') \leq \Big(\frac{\delta+1}{n-1}\Big)^{k(n-1)} \mathrm{vol}_{n-1} Z' \to 0$$

when k tend to ∞.

As $F'^k(Z')$ may not be a submanifold of $\tilde{X}/C'$, we cannot apply directly Theorem 3.3 to get a lower bound on $\mathrm{vol}_{n-1} F'^k(Z')$. Thus, below in Section 5 we shall work with a perturbation of the symmetric two tensor $g_k = (F'^k)^* g$.

For $\epsilon > 0$, the following symmetric 2-tensor $g_{\epsilon,k}$ is a riemanniann metric on $\tilde{X}/C'$

$$g_{\epsilon,k} = g_k + \epsilon^2 g. \tag{4.30}$$

Let us denote $h_{\epsilon,k}$ its restriction to Z'. The following shows that if $\delta < n-2$ then there exist a sequence ϵ_k tending to 0 as k tends to ∞ such that

$$\lim_{k\to\infty} \mathrm{vol}_{n-1}(Z', h_{\epsilon_k,k}) = 0.$$

Lemma 4.7. *Let $h_{\epsilon,k}$ be the restriction of $g_{\epsilon,k}$ to the hypersurface Z' and $g_{Z'}$ the restriction of g to Z'. Let $\Phi_{\epsilon,k} : Z' \to \mathbb{R}$ the density defined for all $x \in Z'$ by $dv_{h_{\epsilon,k}}(x) = \Phi_{\epsilon,k}(x) dv_{g_{Z'}}(x)$. For any sequence ϵ_k such that $\lim_{k\to\infty} \epsilon_k = 0$, there exists a sequence ϵ'_k, $\lim_{k\to\infty} \epsilon'_k = 0$, such that for all $x \in Z'$,*

$$0 < \Phi_{\epsilon'_k,k}(x) \leq |\mathrm{Jac}_{n-1} F'^k(x)| + \epsilon_k.$$

In particular,

$$\Phi_{\epsilon'_k,k}(x) \leq \Big(\frac{\delta+1}{n-1}\Big)^{k(n-1)} + \epsilon_k$$

and

$$\mathrm{vol}(Z', h_{\epsilon'_k,k}) \leq \Big[\Big(\frac{\delta+1}{n-1}\Big)^{k(n-1)} + \epsilon_k\Big] \mathrm{vol}(Z', g_{Z'}).$$

Corollary 4.8. *Under the above asumptions, if* $\delta < n-2$ *there exists a sequence* ϵ'_k *such that* $\lim_{k\to\infty} \epsilon'_k = 0$*, and* $\lim_{k\to\infty} \mathrm{vol}(Z', h_{\epsilon'_k,k}) = 0$.

Proof of Lemma 4.6. Let us fix k an integer. Let $x \in Z'$ and $u \in T_xZ'$. We have $g_{\epsilon,k}(u,u) = h_{\epsilon,k}(u,u) = g(DF'^k(x)(u), DF'^k(x)(u)) + \epsilon^2 g(u,u)$, thus $h_{\epsilon,k}(u,u) = g(A_{x,\epsilon}u, u)$ where $A_{x,\epsilon} \in \mathrm{End}(T_xZ')$ is the self adjoint endomorphism $A_{x,\epsilon} = DF'^k(x)^* \circ DF'^k(x) + \epsilon^2 Id$, where we have written $DF'^k(x)^*$ the adjoint of the endomorphism $DF'^k(x) : (T_xZ', g(x)) \to (DF'^k(x)(T_xZ'), g(F'^k(x)))$.

By compactness of Z' and continuity of $A_{x,\epsilon}$, there exist ϵ'_k such that

$$\Phi_{\epsilon'_k,k}(x) = \det A^{1/2}_{x,\epsilon'_k} \le \det A_{x,0} + \epsilon_k.$$

Thus

$$\Phi_{\epsilon'_k,k}(x) \le |\mathrm{Jac}_{n-1} F'^k(x)| + \epsilon_k.$$

The lemma then follows from Corollary 4.5. q.e.d.

5. Proof of the inequality of Theorem 1.2

This section is devoted to the proof of the inequality of Theorem 1.2. Let Γ be a discrete cocompact group of isometries of a n-dimensional Cartan-Hadamard manifold $(\tilde{X}, \tilde{g})$ whose sectional curvature satisfies $K_{\tilde{g}} \le -1$. We assume that $\Gamma = A *_C B$. At the end of Section 2 we constructed a subgroup $C' \subset C$ and an orientable hypersurface $\tilde{Z}' \subset \tilde{X}$ such that $C'.\tilde{Z}' = \tilde{Z}'$ and $Z' = \tilde{Z}'/C'$ is compact in $\tilde{X}/C'$. Moreover, Z' is essential in $\tilde{X}/C'$ i.e., $i_*([Z']) \ne 0$ where $i_* : H_{n-1}(Z', \mathbb{R}) \to H_{n-1}(\tilde{X}/C', \mathbb{R})$ is the morphism induced on homology groups by the inclusion $i : Z' \to \tilde{X}/C'$ and $[Z']$ the fundamental class of Z'.

Proof of the inequality in Theorem 1.2. In order to prove the inequality in Theorem 1.2, let us assume that $\delta < n-2$ and derive a contradiction. Let $h_{\epsilon'_k,k}$ be the sequence of metrics defined on Z' in Lemma 4.7; then by Corollary 4.8 we have

$$\lim_{k\to\infty} \mathrm{vol}(Z', h_{\epsilon'_k,k}) = 0. \tag{5.1}$$

We now show that the systole of the metric $h_{\epsilon'_k,k}$ on Z' is bounded below independently of k, which will give the desired contradiction. Recall that the systole of $i : Z' \to \tilde{X}/C'$ at a point $z \in Z'$ with respect to a metric $h_{\epsilon,k}$ can be defined by

$$\mathrm{sys_i(Z', h_{\epsilon,k}, z) = inf_{\gamma\in C'} dist_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z})}, \tag{5.2}$$

where $\tilde{z}$ is any lift of z and $\tilde{h}_{\epsilon,k}$ the lift on $\tilde{Z}'$ of $h_{\epsilon,k}$ (cf. Lemma 3.4). Let $\alpha(t)$ be a minimizing geodesic between $\tilde{z}$ and $\gamma\tilde{z}$ on $(Z', \tilde{h}_{\epsilon,k})$. By definition of $\tilde{h}_{\epsilon,k}$ we have

$$\mathrm{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \ge l_{\tilde{g}}(\tilde{F}^k \circ \alpha) \tag{5.3}$$

where $l_{\tilde{g}}$ stands for the length with respect to $\tilde{g}$ on $\tilde{X}$. We get

$$\mathrm{dist}_{(\tilde{Z}',\tilde{h}_{\epsilon,k})}(\tilde{z},\gamma\tilde{z}) \geq \mathrm{dist}_{(\tilde{X},\tilde{g})}(\tilde{F}^k(\tilde{z}),\gamma\tilde{F}^k(\tilde{z})) \geq \mathrm{inj}(\tilde{X}/C'), \tag{5.4}$$

where $\mathrm{inj}(\tilde{X}/C')$ is the injectivity radius of $\tilde{X}/C'$. We then have

$$\mathrm{sys_i(Z', h_{\epsilon,k}) \geq inj(\tilde{X}/C')}, \tag{5.5}$$

and from Theorem 3.3 we obtain

$$\mathrm{vol(Z', h_{\epsilon'_k,k}) \geq C_n\Big(inj(\tilde{X}/C')\Big)^{n-1}}, \tag{5.6}$$

which contradicts 5.1. q.e.d.

The next sections 6–11 are devoted to proving the equality case of Theorem 1.2. We will assume from now on that $\delta(C) = n-2$.

6. The limit set Λ_C of C is contained in a topological equator.

The goal of this section is to prove, under the assumption $\delta = \delta(C) = n-2$, that the limit set Λ_C of C is contained in some topological equator, namely:

Proposition 6.1. *Let us assume that $\delta = \delta(C) = n-2$; then there exists an equator $E(\infty)$ such that $\Lambda_C \subset E(\infty)$.*

Recall that for $x \in \tilde{X}$ and $E \subset T_x\tilde{X}$ a codimension one subspace, the equator $E(\infty)$ associated to E is the subset of $\partial\tilde{X}$

$$E(\infty) = \{c_u(+\infty)/u \in E\}, \tag{6.1}$$

where c_u is the geodesic such that $c_u(0) = x$ and $\dot{c_u}(0) = u$.

Recall that $C' \subset C$ globally preserves an hypersurface $\tilde{Z}'$ such that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact and essential.

Let us also recall that we have constructed a C-equivariant map $\tilde{F} : \tilde{X} \to \tilde{X}$ such that, for all $x \in \tilde{X}$,

$$|\mathrm{Jac}_{n-1}\tilde{F}(x)| \leq \Big(\frac{\delta+1}{n-1}\Big)^{n-1} \tag{6.2}$$

where the critical exponent δ of C satisfies $\delta = n-2$, and we thus have

$$|\mathrm{Jac}_{n-1}\tilde{F}(x)| \leq 1. \tag{6.3}$$

Proposition 6.1 is a direct consequence of Proposition 4.6 and the following

Proposition 6.2. *Let us assume that $\delta = \delta(C) = n-2$. Then there exists a point x in $\tilde{X}$ such that $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$.*

Proof. If we knew that there exists a minimizing hypersurface Z_0 in the homology class of Z, then all points $x \in Z_0$ would verify $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$. We unfortunately do not know if there exists such a minimizing hypersurface or a minimizing current in the homology class of Z by lack of compactness. Instead, we will consider an $L^2(\tilde{X}/C')$ harmonic $(n-1)$-form dual to the homology class of Z.

We need the following lemmas in order to prove the existence of such a dual form.

Let $\lambda_1(\tilde{X}/C')$ be the bottom of the spectrum of the Laplacian on $(\tilde{X}/C', \tilde{g})$, i.e.,

$$\lambda_1(\tilde{X}/C') = \inf_{u \in C_0^\infty(\tilde{X}/C')} \left\{ \frac{\int_{\tilde{X}/C'} |du|^2}{\int_{\tilde{X}/C'} u^2} \right\}. \tag{6.4}$$

Lemma 6.3. *Let $C \subset Isom(\tilde{X}, \tilde{g})$ be a discrete group of isometries with critical exponent $\delta = n-2$ where $(\tilde{X}, \tilde{g})$ is an n-dimensional Cartan-Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$. Then for any subgroup $C' \subset C$ we have $\lambda_1(\tilde{X}/C') \geq n-2$.*

Proof. Thanks to a theorem of Barta, cf. [**19**] Theorem 2.1, the lemma reduces to finding a positive function $c : \tilde{X}/C' \to \mathbb{R}_+$ such that $\Delta c(x) \geq (n-2)c(x)$. Here, the Laplacian Δ is the positive operator i.e., $\Delta c = -\text{trace} Ddc$. We consider the smooth function $\tilde{c} : \tilde{X} \to \mathbb{R}_+$ defined by $\tilde{c}(x) = \mu_x(\partial\tilde{X})$ where $\{\mu_x\}_{x \in \tilde{X}}$ is a family of Patterson-Sullivan measures of C. The function $\tilde{c}$ is C-equivariant, therefore it defines a map $c : \tilde{X}/C' \to \mathbb{R}_+$ for any subgroup $C' \subset C$. Let us show that

$$\Delta c(x) \geq \delta(n-1-\delta)c(x) = (n-2)c(x). \tag{6.5}$$

We have

$$\tilde{c}(x) = \int_{\partial\tilde{X}} e^{-\delta B(x,\theta)} d\mu_o(\theta);$$

therefore

$$\Delta\tilde{c}(x) = \int_{\partial\tilde{X}} [-\delta\Delta B(x,\theta) - \delta^2] d\mu_x(\theta).$$

The sectional curvature $K_{\tilde{g}}$ of $(\tilde{X}, \tilde{g})$ satisfies $K_{\tilde{g}} \leq -1$. We thus have $-\Delta B(x,\theta) \geq n-1$, and as $\delta = n-2$ we get

$$\Delta\tilde{c}(x) \geq [\delta(n-1) - \delta^2]\tilde{c}(x) = (n-2)\tilde{c}(x).$$

q.e.d.

The following lemma is due to G. Carron and E. Pedon, [**9**]. For a complete riemannian manifold Y, we denote $H_c^1(Y, \mathbb{R})$ the first cohomology group generated by differential forms with compact support.

Lemma 6.4 ([**9**], Lemme 5.1)**.** *Let Y be a complete riemannian manifold with $\lambda_1(Y) > 0$ and such that each end of Y has infinite volume, then the natural morphism*

$$H_c^1(Y, \mathbb{R}) \to H_{L^2}^1(Y, \mathbb{R})$$

is injective. In particular, any $\alpha \in H^1_c(Y,\mathbb{R})$ *admits a representative* $\bar{\alpha}$ *which is in* $L^2(Y,\mathbb{R})$ *and harmonic.*

Corollary 6.5. *Let* C' *be as above and assume that there exists a compact essential hypersurface* $Z' \subset \tilde{X}/C'$*. Then there exists an harmonic* $(n-1)$*-form* ω *in* $L^2(\tilde{X}/C')$ *such that* $\int_{Z'}\omega \neq 0$*.*

Proof.
Let $\alpha \in H^1_c(\tilde{X}/C',\mathbb{R})$ be a Poincaré dual of $[Z'] \in H_{n-1}(\tilde{X}/C',\mathbb{R})$. By definition of α, for any $\beta \in H^{n-1}(\tilde{X}/C',\mathbb{R})$, one has

$$\int_{Z'}\beta = \int_{\tilde{X}/C'}\beta\wedge\alpha, \tag{6.6}$$

([**5**] p. 51, note that $\tilde{X}/C'$ has a "finite good cover"). Notice that the above equality makes sense also when α and β are L^2. After Lemma 6.4, α admits a non trivial harmonic representative $\bar{\alpha}$ in $L^2(\tilde{X}/C')$. In order to apply the Lemma 6.4, one has to check that all ends of $\tilde{X}/C'$ have infinite volume, i.e., for a compact $K \subset \tilde{X}/C'$ each unbounded connected component of $\tilde{X}/C' - K$ has infinite volume. This comes from the fact that the injectivity radius of $\tilde{X}/C'$ is bounded below by the injectivity radius of $X = \tilde{X}/\Gamma$ and the sectional curvature is bounded above by -1. The $(n-1)$-harmonic form $\omega = (-1)^{n-1} * \bar{\alpha}$, where $*$ is the Hodge operator, is in $L^2(\tilde{X}/C')$ and verifies after 6.6

$$\int_{Z'}\omega = \int_{\tilde{X}/C'}\omega\wedge\bar{\alpha} = \int_{\tilde{X}/C'}\omega\wedge *\omega = ||\omega||^2_{L^2(\tilde{X}/C')} \neq 0. \tag{6.7}$$

q.e.d.

We now can prove Proposition 6.2. Let us briefly describe the idea. We consider the iterates F'^k of $F' : \tilde{X}/C' \to \tilde{X}/C'$. As F' is homotopic to the identity map, $F'^k(Z')$ is homologous to Z', and if ω is the harmonic form of corollary 6.5 we have

$$\int_{Z'}(F'^k)^*(\omega) = \int_{Z'}\omega = a \neq 0. \tag{6.8}$$

We do not know if $F'^k(Z')$ converges or stays in a compact subset of $\tilde{X}/C'$, but we will show that $F'^k(Z')$ cannot entirely diverge in $\tilde{X}/C'$ and that there exists a $z' \in Z'$ such that $F'^k(z')$ subconverges to a point $x \in \tilde{X}/C'$ with $|\mathrm{Jac}_{n-1}F'(x)| = 1$.

Let us now give the details. From 6.8 one gets

$$0 < |a| = |\int_{Z'}(F'^k)^*(\omega)| \leq \int_{Z'}|\mathrm{Jac}_{TZ'}F'^k(z)|.||\omega(F'^k(z))||dz \tag{6.9}$$

where

$$|\mathrm{Jac}_{TZ'}F'^k(z)| = ||DF'^k(z)(u_1)\wedge DF'^k(z)(u_2)\wedge\ldots\wedge DF'^k(z)(u_{n-1})||$$

with $(u_1, \dots, u_{n-1})$ is an orthonormal basis of $T_z(Z')$.

Let us define

$$\mathcal{B} = \{z \in Z', |\mathrm{Jac} F'^k_{TZ'}(z)| \text{ does not converge to } 0\}.$$

For $z \in Z'$ we define the sequence z_k by $z_0 = z$ and $z_k = F'(z_{k-1}) = F'^k(z) \in \tilde{X}/C'$.

Lemma 6.6. *There exists $z \in \mathcal{B}$ and a subsequence z_{k_j} such that z_{k_j} converges to a point $x \in \tilde{X}/C'$ with $|\mathrm{Jac}_{n-1}(x)| = 1$.*

Proof. We first remark that $\lim_{x\to\infty} ||\omega(x)|| = 0$. This comes from the following facts: ω is harmonic, $\omega \in L^2(\tilde{X}/C')$, and the injectivity radius of $\tilde{X}/C'$ is bounded below by a positive constant. Let us assume that for all $z \in \mathcal{B}$ the sequence z_k diverges in $\tilde{X}/C'$. We then have, for all $z \in \mathcal{B}$,

$$||\omega(z_k)|| = ||\omega(F'^k(z))|| \to 0 \tag{6.10}$$

whenever k tends to ∞ because of the previous remark. On the other hand, as $||\omega(F'^k(z))|| \le C$ and $|\mathrm{Jac}_{n-1} F'^k| \le 1$, it follows from 6.9 that

$$\begin{aligned} &\lim_{k\to\infty} \Big| \int_{Z'} (F'^k)^*(\omega) \Big| \\ &\le \lim_{k\to\infty} \Big[\int_{\mathcal{B}} ||\omega(F'^k(z))|| dz + C \int_{Z'-\mathcal{B}} |\mathrm{Jac}_{Z'} F'^k(z)| dz \Big] = 0, \end{aligned}$$

which contradicts our assumption.

Thus, there exists a point $z \in \mathcal{B}$ such that $z_k = F'^k(z)$ does not diverge, and therefore there exists a subsequence $z_{k_j} = F'^{k_j}(z)$ with

$$\lim_{j\to\infty} z_{k_j} = x \in \tilde{X}/C' \tag{6.11}$$

and

$$|\mathrm{Jac}_{Z'} F'^k(z)| \to \alpha \neq 0. \tag{6.12}$$

Let us define

$$E_0 = T_z Z', E_1 = DF'(z)(E_0)$$

and

$$E_k = DF'(z_{k-1})(E_{k-1}) \subset T_{z_k}(\tilde{X}/C').$$

As $z_{k_j} \to x$ we can assume, after extracting again a subsequence, that $E_{k_j} \to E \subset T_x(\tilde{X}/C')$. On the other hand, we also have

$$|\mathrm{Jac}_{Z'} F'^k(z)| = |\mathrm{Jac}_{E_{k-1}} F'(z_{k-1})| \cdot |\mathrm{Jac}_{E_{k-2}} F'(z_{k-2})| \cdots |\mathrm{Jac}_{E_0} F'(z)|. \tag{6.13}$$

We know that $|\mathrm{Jac}_{E_k} F'(z_k)| = 1 - \epsilon_k$ where $0 \le \epsilon_k < 1$. As $z \in \mathcal{B}$, we have

$$\lim_{k\to\infty} \prod_{j=1}^{k} (1 - \epsilon_j) = \alpha > 0;$$

therefore $\lim_{k\to\infty} \epsilon_k = 0$, and by continuity we have $|\mathrm{Jac}_E F'(x)| = 1$.
q.e.d.

This finishes the proof of Proposition 6.2. q.e.d.

We now can finish the proof of Proposition 6.1. Let us consider a lift of x in $\tilde{X}$ and a subspace E in $T_x\tilde{X}$ that we again call x and E. We then have $|\mathrm{Jac}_E \tilde{F}'(x)| = 1$, and by Proposition 4.6, we get $\Lambda_{C'} \subset E(\infty)$.
q.e.d.

Let us remark that Corollary 4.5 and 6.8 give another proof of the inequality $\delta \geq n-2$, which does not use the isosystolic inequality, i.e., Theorem 3.3.

7. The weak tangent of $\partial\tilde{X}$ and Λ_C

The goal of this section is to introduce some tools which we will use later.

We first recall the definition of the Gromov-distance on $\partial\tilde{X}$. For two arbitrary points θ and θ' in $\partial\tilde{X}$ let us define

$$l(\theta,\theta') = \inf\{t > 0/\mathrm{dist}(\alpha_\theta(t), \alpha_{\theta'}(t)) = 1\} \tag{7.1}$$

where α_θ denotes the geodesic ray between a fixed origin o and θ and

$$d(\theta,\theta') = e^{-l(\theta,\theta')}; \tag{7.2}$$

then d is a distance on $\partial\tilde{X}$.

We now recall a few definitions following [**4**]. A complete metric space $(S,\bar{d})$ is a weak tangent of a metric space (Z,d) if there exist a point $0 \in S$, a sequence of points $z_k \in Z$ and a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(Z, \lambda_k d, z_k)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(Z, \lambda_k d)$ stands for the set Z endowed with the rescaled metric $\lambda_k d$.

In the sequel, we shall denote $B(x,R)$ a ball of radius R centered at x without mentioning in which metric space it lies if there is no ambiguity. Let us recall that the sequence of metric spaces (Z_k, d_k, z_k) converges to $(S,\bar{d},0)$ in the pointed Gromov-Hausdorff topology if the following conditions hold (cf. [**7**], Definition 8.1.1).

Definition 7.1. A sequence of metric spaces (Z_k, d_k, z_k) converges to $(S,\bar{d},0)$ if for any $R > 0$, $\epsilon > 0$ there exists k_0 such that for any $k \geq k_0$ there exists a (not necessarily continuous) map $f : B(z_k, R) \to S$ such that

(i) $f(z_k) = 0$, for any two points x and y in $B(z_k, R)$,
(ii) $|\bar{d}(f(x), f(y)) - d_k(x,y)| \leq \epsilon$, and
(iii) the ϵ-neighborhood of the set $f(B(z_k,R))$ contains $B(0, R-\epsilon)$.

For a metric space (Z,d) we will denote $WT(Z,d)$ the set of weak tangents of (Z,d).

Let Γ be a cocompact group of isometries of $(\tilde{X},\tilde{g})$, an n-dimensional Cartan-Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$, and C' be a subgroup of Γ. The limit set Λ_Γ of Γ is the full boundary $\partial\tilde{X}$, namely a topological $(n-1)-$dimensional sphere S^{n-1}. We endow $\partial\tilde{X}$ with the Gromov distance d defined in 7.2. In [**4**] Lemma 5.2, M. Bonk and B. Kleiner show among other properties the following:

Lemma 7.2. *For any weak tangent space* $(S,\bar{d})$ *in* $WT(\partial\tilde{X},d)$*,* S *is homeomorphic to* $\partial\tilde{X}$ *less a point, thus to* $\mathbb{R}^{n-1}$*.*

In fact, the crucial asumption in the above lemma, coming from the cocompactness of Γ, is the property that any triple of points in $\partial\tilde{X}$ can be uniformly separated by an element of Γ, i.e., there is $\nu > 0$ such that for any three points $\theta_1,\theta_2,\theta_3 \in \partial\tilde{X}$ there exists a $\gamma \in \Gamma$ such that $d(\gamma\theta_i,\gamma\theta_j) \geq \nu$ for all $1 \leq i \neq j \leq 3$. Following the argument of M. Bonk and B. Kleiner one can show that if C' is a subgroup of Γ and $\mathcal{L}$ a C'-invariant subset of $\partial\tilde{X}$ such that one weak tangent $(S,\bar{d})$ of $(\mathcal{L},d)$ is in $WT(\partial\tilde{X},d)$, and enough triples of points of $(\mathcal{L},d)$ can be uniformly separated by elements of C', then S is homeomorphic to $(\mathcal{L},d)$ less a point. In particular, $(\mathcal{L},d)$ is homeomorphic to $\partial\tilde{X}$. We will apply later the following lemma to the case of $\mathcal{L}$ being either the limit set $\Lambda_{C'}$ or a hemisphere of $\partial\tilde{X}$.

Lemma 7.3. *Let* $\mathcal{L} \subset \partial\tilde{X}$ *be a closed* C'*-invariant set and* $\theta_0 \in \mathcal{L}$*. We assume that there exists a sequence of positive real numbers* $\lambda_k \to \infty$ *such that the sequence of pointed metric spaces* $(\mathcal{L},\lambda_k d,\theta_0)$ *converges in the pointed Gromov-Hausdorff topology to* $(S,\bar{d},0)$ *where* $(S,\bar{d},0)$ *is a weak tangent of* $(\partial\tilde{X},d)$*. We also assume that there exist positive constants* C *and* ν*, a sequence of points* $\theta_0^k = \theta_0,\theta_1^k,\theta_2^k \in \mathcal{L}$ *and a sequence of elements* $\gamma_k \in C'$ *such that* $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k,\theta_j^k) \leq C\lambda_k^{-1}$ *and* $d(\gamma_k\theta_i^k,\gamma_k\theta_j^k) \geq \nu$ *for all* $0 \leq i \neq j \leq 2$*. Then,* S *is homeomorphic to* $\mathcal{L}$ *less a point. In particular,* $\mathcal{L}$ *is homeomorphic to* $\partial\tilde{X}$*.*

The proof of this lemma is postponed to the Appendix. In the sequel it will be applyed twice; a first time in the proof of proposition 8.8 with $\mathcal{L}$ being the limit set of C', and a second time in the proof of proposition 9.3 where $\mathcal{L}$ is a topological hemisphere.

Let C' a discrete group of isometries of $\tilde{X}$ and $\Lambda_{C'} \subset \partial\tilde{X}$ the limit set of C'. The following lemma amounts to saying that if $\Lambda_{C'}$ is very "irregular" near a point $\theta_0 \in \Lambda_{C'}$, then weak tangents of $\partial\tilde{X}$ are weak tangents of $\Lambda_{C'}$.

Lemma 7.4. *Let us assume that for every sequence* $\theta_i \neq \theta_0$ *of points in* $\partial\tilde{X}$ *converging to* θ_0*,* $\lim_{i\to\infty} \frac{d(\theta_i,\Lambda_{C'})}{d(\theta_i,\theta_0)} = 0$*. Let* $\lambda_k \to \infty$ *be such* *that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to a space $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology; then the sequence of spaces $(\Lambda_{C'}, \lambda_k d, \theta_0)$ also converges to $(S, \bar{d}, 0)$.*

Proof. Let us define

$$r(\epsilon) =: \sup\{\frac{d(\theta, \Lambda_{C'})}{d(\theta, \theta_0)}, \theta \neq \theta_0, d(\theta, \theta_0) \leq \epsilon\}.$$

The assumption says that

$$\lim_{\epsilon \to 0} r(\epsilon) = 0. \tag{7.3}$$

For an arbitrary metric space (Y, d) and Y' a subset of Y, let us denote $B_{(Y,d)}(y, R)$ the closed ball of (Y, d) of radius R centered at $y \in Y$, and $\mathcal{U}_\epsilon^{(Y,d)}(Y')$ the ϵ-neighborhood of Y' in (Y, d). For a metric space (Y, d) and a positive number λ, let us denote, for sake of simplicity, by λY the rescaled space $(Y, \lambda d)$. By definition of the function r, we have for any R,

$$\begin{aligned} B_{\lambda_k \partial\tilde{X}}(\theta_0, R) &\subset \mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k) \\ &\subset B_{\lambda_k \partial\tilde{X}}(\theta_0, R + 2\epsilon_k) \end{aligned} \tag{7.4}$$

where $\epsilon_k =: Rr(R/\lambda_k)$.

Let us fix $\alpha > 0$. By Definition 7.1 of Hausdorff-Gromov convergence, for any $R > 0$, $\epsilon > 0$, and any $k \geq k_0$ there exists a map $f : B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha) \to S$ such that for $k \geq k_0$,

(i) $f(\theta_0) = 0$,
(ii) $|\bar{d}(f(x), f(y)) - \lambda_k d(x, y)| \leq \epsilon$ for any two points x and y in $B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha)$ and,
(iii) $B_{(S,\bar{d})}(0, R + \alpha - \epsilon) \subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha))$.

Moreover, let us prove that

(iv) $B_{(S,\bar{d})}(0, R - 2\epsilon) \subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(B_{\lambda_k \partial\tilde{X}}(\theta_0, R))$.

Indeed, let $z \in B_{(S,\bar{d})}(0, R-2\epsilon)$. By (iii), there exists $\theta \in B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha)$ such that $\bar{d}(z, f(\theta)) \leq \epsilon$. Since $\bar{d}(z, 0) \leq R - 2\epsilon$, we thus deduce from the triangle inequality that $\bar{d}(f(\theta), 0) \leq R - \epsilon$, and therefore we get, using (i) and (ii), that $\lambda_k d(\theta, \theta_0) \leq R$.

By 7.3, for ϵ small enough, there exists $k_1 \geq k_0$ such that for any $k \geq k_1$, $2\epsilon_k \leq \epsilon$ and

$$B_{\lambda_k \partial\tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha).$$

Therefore, by 7.4 and the above properties (i), (ii), (iii) and (iv) of the map f, and the triangle inequality, we get

$$\begin{aligned} B_{(S,\bar{d})}(0, R - 2\epsilon) &\subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(B_{\lambda_k \partial\tilde{X}}(\theta_0, R)) \\ &\subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(\mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)) \end{aligned}$$

$$\subset \mathcal{U}^{(S,\bar{d})}_{2\epsilon+\epsilon_k} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R+\epsilon_k)).$$

About the second inclusion above, let us recall that f is not everywhere defined. However, we just have to remark that the $\mathcal{U}^{\lambda_k \partial \tilde{X}}_{\epsilon_k} B_{\lambda_k \Lambda_{C'}}(\theta_0, R+\epsilon_k)$ is contained in $B_{\lambda_k \partial \tilde{X}}(\theta_0, R+2\epsilon_k) \subset B_{\lambda \partial \tilde{X}}(\theta_0, R+\alpha)$, so that we can apply f to this set. From the above inclusions we obtain

$$B_{(S,\bar{d})}(0, R-2\epsilon) \subset \mathcal{U}^{(S,\bar{d})}_{3\epsilon+2\epsilon_k} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R)),$$

which implies the convergence of $(\Lambda_{C'}, \lambda_k d, \theta_0)$ to $(S, \bar{d}, 0)$. q.e.d.

Remark 7.5. Note that Lemma 7.4 is a purely metric property and remains valid for any metric space Z instead of $\partial \tilde{X}$ and any subset $Y \subset Z$ instead of $\Lambda_{C'} \subset \partial \tilde{X}$. Namely, the proof does not make use of special properties of $\partial \tilde{X}$ or $\Lambda_{C'}$.

8. The limit set $\Lambda_{C'}$ of C' and the limit set Λ_C of C are equal to a topological equator

The goal of this section is to prove that if $\delta = \delta(C) = n-2$, then the limit set Λ_C of C and the limit set $\Lambda_{C'}$ of C' are equal to some topological equator.

Proposition 8.1. *Let us assume that $\delta(C) = n-2$; then there exists a topological equator $E(\infty)$ such that $\Lambda_{C'} = E(\infty)$ and $\Lambda_C = E(\infty)$.*

We have shown in Section 6 the existence of a point $o \in \tilde{X}$ and a subspace $E \subset T_o\tilde{X}$ such that $\tilde{F}(o) = o$, $|\mathrm{Jac}_E \tilde{F}(o)| = 1$ and $\Lambda_C \subset E(\infty)$ where $E(\infty)$ is the equator associated to E.

Recall that there exists a subgroup C' of C which globally preserves a connected hypersurface $\tilde{Z}' \subset \tilde{X}$, and that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact. Furthermore, $\tilde{Z}'$ separates $\tilde{X}$ into two connected components $\tilde{U}$ et $\tilde{V}$. We can assume that $\tilde{U}$ and $\tilde{V}$ are globally invariant by C' after having replaced C' by an index 2 subgroup.

The limit set $\Lambda_{C'}$ of C' is contained in Λ_C; therefore $\Lambda_{C'} \subset E(\infty)$. In this section we will show that $\Lambda_{C'} = E(\infty)$.

For any subset $W \in \tilde{X}$ we define the boundary at infinity ∂W of W by

$$\partial W = Cl(W) \cap \partial \tilde{X} \tag{8.1}$$

where $Cl(W)$ stands for the closure of W in $\tilde{X} \cup \partial \tilde{X}$.

Our first goal is to show that if $\Lambda_{C'} \neq E(\infty)$, then $\partial \tilde{U} = \Lambda_{C'}$ or $\partial \tilde{V} = \Lambda_{C'}$. As $\tilde{Z}'/C'$ is compact, $\tilde{Z}'$ is at bounded distance of the orbit $C'z \subset \tilde{Z}'$ of some point z in $\tilde{Z}'$, thus

$$Cl(\tilde{Z}') \cap \partial \tilde{X} = \Lambda_{C'} \tag{8.2}$$

By definition we have $\Lambda_{C'} \subset \partial \tilde{U}$ and $\Lambda_{C'} \subset \partial \tilde{V}$.

Lemma 8.2. *Let us assume that $\Lambda_{C'} \neq E(\infty)$; then either $\partial\tilde{U} = \Lambda_{C'}$ or $\partial\tilde{V} = \Lambda_{C'}$.*

Proof. We know that $\Lambda_{C'} \subset \partial\tilde{U}$ and $\Lambda_{C'} \subset \partial\tilde{V}$.

Let us assume that the conclusion of the lemma is not true, so there is $\zeta \in \partial\tilde{U} - \Lambda_{C'}$ and $\theta \in \partial\tilde{V} - \Lambda_{C'}$.

As $\Lambda_{C'} \subset E(\infty)$, $\Lambda_{C'} \neq E(\infty)$ and $\partial\tilde{X}$ is a sphere, any two points of $\partial\tilde{X} - \Lambda_{C'}$ can be joined by a continuous path contained in $\partial\tilde{X} - \Lambda_{C'}$. Let α be a path joining ζ and θ.

The set $\tilde{Z}' \cup \Lambda_{C'}$ is a closed subset of $\tilde{X} \cup \partial\tilde{X}$, thus there is an open connected neighborhood W of α in $\tilde{X} \cup \partial\tilde{X}$ contained in the complement of $Z' \cup \Lambda_{C'}$.

As ζ and θ can be approximated by points in $\tilde{U}$ and $\tilde{V}$, respectively, there exist points $x \in \tilde{U} \cap W$ and $y \in \tilde{V} \cap W$ that can be joined by a continuous path by connectedness of W, which leads to a contradiction. q.e.d.

Remark 8.3. In fact, the end of the section is devoted to showing that under the assumption $\delta(C') = n-2$, it is impossible to have $\partial\tilde{U} = \Lambda_{C'}$ or $\partial\tilde{V} = \Lambda_{C'}$.

For any $x \in \tilde{X}$ and $\theta \in \partial\tilde{X}$ let us denote $HB(x, \theta)$ the open horoball centered at θ and passing through x.

Lemma 8.4. *Let us assume that $\partial\tilde{U} = \Lambda_{C'}$. Then there exist $\theta_0 \in \Lambda_{C'}$ and $z' \in \tilde{Z}'$ such that $HB(z', \theta_0) \subset \tilde{U}$.*

Proof. Let us recall that $\tilde{X}/C' - Z' = U \cup V$ where $U = \pi(\tilde{U})$ $V = \pi(\tilde{V})$ and $\pi : \tilde{X} \to \tilde{X}/C'$ is the projection. By Lemma 2.7, we know that U and V are unbounded. Let x_n be a sequence of points in U such that $\mathrm{dist}(x_n, Z') \to \infty$. Let $z_n \in Z'$ such that $\mathrm{dist}(x_n, Z') = \mathrm{dist}(x_n, z_n)$. We consider a compact fundamental domain $D \subset \tilde{Z}'$ of C'. There exist lifts $\tilde{z}_n \in D$ and $\tilde{x}_n \in \tilde{U}$ such that $\mathrm{dist}(\tilde{x}_n, \tilde{Z}') = \mathrm{dist}(\tilde{x}_n, \tilde{z}_n)$ tends to infinity. By compactness we can assume that a subsequence $\tilde{x}_{n_j}$ converges to a point $\theta_0 \in \partial\tilde{X}$ and $\tilde{z}_{n_j}$ also converges to a point $\tilde{z} \in \bar{D}$. Furthermore, the sequence of open balls $B(\tilde{x}_{n_j}, \mathrm{dist}(\tilde{x}_{n_j}, \tilde{z}_{n_j})) \subset \tilde{U}$ converges to the open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$. q.e.d.

Let us now describe the idea of the proof of Proposition 8.1 and next state some facts that we will need.

As $\Lambda_{C'} \subset \Lambda_C \subset E(\infty)$ the proposition reduces to proving that $\Lambda_{C'} = E(\infty)$. We will assume that $\Lambda_{C'} \neq E(\infty)$ and find a contradiction. More precisely, we will show that for any sequence θ_i converging to θ_0, the geodesic starting at a point o such that $|\mathrm{Jac}_{n-1}\tilde{F}(o)| = 1$ and ending at θ_i crosses the hypersurface $\tilde{Z}'$ in a point z_i. For an **appropriate choice** of such a sequence θ_i (roughly speaking, the sequence

θ_i is chosen to be converging to θ_0 "transversely to $\Lambda_{C'}$"), the shadow (defined below) projected from o through some geodesic ball $B(z_i, r)$ will not intersect $\Lambda_{C'}$. On the other hand, this shadow has to meet the limit set $\Lambda_{C'}$ because of the shadow lemma of D. Sullivan, which leads to a contradiction.

Precisely, by Lemma 8.2 and Lemma 8.4 we know that $\partial\tilde{U} = \Lambda_{C'}$ and that there exists an open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ centered at a point $\theta \in \Lambda_{C'}$, and whose closure contains a point $\tilde{z} \in \tilde{Z}'$.

Let $o \in \tilde{X}$ and $E \in T_o\tilde{X}$ a hyperplane such that $\tilde{F}(o) = o$, $|\mathrm{Jac}_E\tilde{F}(o)| = 1$, and $\Lambda_{C'} \subset E(\infty)$ where $E(\infty)$ is the topological equator associated to E. For each $\theta \in \partial\tilde{X}$ we denote by α_θ the geodesic starting from o and such that $\alpha_\theta(+\infty) = \theta$.

Let $\theta_i \in \partial\tilde{X} \setminus \Lambda_{C'} \subsetneqq \partial\tilde{V} \setminus \partial\tilde{U} = \partial\tilde{V} \setminus \Lambda_{C'}$ be a sequence converging to θ_0. By continuity, for each i large enough, the geodesic α_{θ_i} spends some time inside the horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up inside $\tilde{V}$ because θ_i converges to θ_0 and θ_i belongs to $\partial\tilde{V} - \partial\tilde{U}$.

Thus α_{θ_i} eventually crosses $\tilde{Z}'$. Let $z_i \in \alpha_{\theta_i} \cap \tilde{Z}'$ such that z_i tends to θ_0. As $\tilde{Z}'/C'$ is compact, there is an element $\gamma_i \in C'$ such that $z_i = \gamma_i(x_i)$ where x_i is a point in the closure $\bar{D}$ of a fundamental domain D for the action of C' on $\tilde{Z}'$. The points $\gamma_i(x_i)$ and $\gamma_i(o)$ stay at bounded distance because $\mathrm{dist}(\gamma_i(x_i), \gamma_i(o)) = \mathrm{dist}(x_i, o) \leq \mathrm{dist}(o, D) + \mathrm{diam} D$. In particular, $\lim_{i\to\infty} \gamma_i(o) = \theta_0$.

We have proved

Lemma 8.5. *Let us assume $\Lambda_{C'} = \partial\tilde{U}$. Let $\theta_i \in \partial\tilde{X} \setminus E(\infty)$ be a sequence which converges to $\theta_0 \in \Lambda_{C'}$. There exists a constant A such that for i large enough, there exists $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ and $\gamma_i \in C'$ such that* $\mathrm{dist}(z_i, \gamma_i(o)) \leq A$, *and both z_i and $\gamma_i(o)$ converge to θ_0.*

Let x and y be two points in $\tilde{X}$. We define the shadow $\mathcal{O}(x, y, R) \subset \partial\tilde{X}$ of the ball $B(y, R)$ lit from the point x by

$$\mathcal{O}(x, y, R) = \{\alpha(+\infty)\} \tag{8.3}$$

where α runs through the set of geodesic rays starting from x and meeting $B(y, R)$. Let $\{\mu_x\}_x$ be a family of Patterson measures associated to the discrete group C' with critical exponent $\delta' = \delta(C')$.

The following lemma, called the shadow lemma, is essentially due to D. Sullivan.

Lemma 8.6 ([**20**], [**15**], [**22**]). *There exist positive constants C and R such that for any y in $\tilde{X}$,*

$$\mu_y(\mathcal{O}(y, \gamma(y), R)) \geq Ce^{-\delta' d(y, \gamma(y))}.$$

In particular, by Lemma 8.6, the shadow $\mathcal{O}(y, \gamma(y), R)$ intersects the support of μ_y and hence $\Lambda_{C'}$.

Corollary 8.7. *Let z_i be defined in Lemma* 8.5*; then we have $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} \neq \emptyset$ for i large enough.*

We now prove that for a good choice of θ_i, the shadow $\mathcal{O}(o, z_i, R+A)$ (with z_i associated to θ_i as in Lemma 8.5) never meets $\Lambda_{C'}$ for all sufficiently large i, i.e., for any $\theta \in \Lambda_{C'}$ the geodesic α_θ does not cross $B(z_i, R + A)$ which will lead to a contradiction. We have no control on the radius R coming from the shadow lemma, nor on the constant A, but we will show

Proposition 8.8. *There exists a sequence $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$ such that θ_i converges to θ_0 and*

$$\lim_{i\to\infty} \inf_{\theta\in\Lambda_{C'}} \operatorname{dist}(z_i, \alpha_\theta) = +\infty,$$

where $z_i = \tilde{Z}' \cap \alpha_{\theta_i}$ has been constructed in Lemma 8.5*.*

Corollary 8.9. *There exists a sequence $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$ such that for i large enough, $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} = \emptyset$, where $z_i = \tilde{Z}' \cap \alpha_{\theta_i}$ has been constructed in Lemma* 8.5*.*

Corollary 8.7 and Corollary 8.9 lead to a contradiction, which will complete the proof of Proposition 8.1. The remainder of this section is devoted to proving Proposition 8.8.

Lemma 8.10. *Let $\theta_i \neq \theta_0$ be a sequence of points in $\partial\tilde{X}$ converging to θ_0 and z_i constructed in Lemma* 8.5*. If*

$$\liminf_{i\to\infty} inf_{\theta\in\Lambda_{C'}} \operatorname{dist}(z_i, \alpha_\theta) = C < +\infty,$$

then $\lim_{i\to\infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$.

Proof. We first show that

$$\lim_{i\to\infty} \operatorname{dist}(z_i, \alpha_{\theta_0}) = \infty. \tag{8.4}$$

Recall that, for any $\tilde{z} \in \tilde{X}$ and $\theta \in \partial\tilde{X}$, $B(\tilde{z}, \theta)$ equals the decreasing limit as t tends to infinity of $\operatorname{dist}(\tilde{z}, \alpha_\theta(t)) - \operatorname{dist}(o, \alpha_\theta(t))$ where $\alpha_\theta(t)$ is the geodesic ray joigning o to θ. Therefore, as the points $z_i \in \tilde{Z}$ belong to the complement of the fixed horoball $HB(\tilde{z}, \theta_0)$, we have

$$\operatorname{dist}(z_i, \alpha_{\theta_0}(T_i) \geq T_i + B(\tilde{z}, \theta_0) \tag{8.5}$$

where T_i is chosen so that $\operatorname{dist}(z_i, \alpha_{\theta_0}(T_i)) = \operatorname{dist}(z_i, \alpha_{\theta_0})$. On the other hand, as z_i tends to θ_0, T_i tends to infinity, so 8.4 is proven.

Let t_i be such that $z_i = \alpha_{\theta_i}(t_i)$. By 8.4, we have

$$\lim_{i\to\infty} \operatorname{dist}(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) = \infty. \tag{8.6}$$

Let u_i be such that

$$\operatorname{dist}(\alpha_{\theta_i}(u_i), \alpha_{\theta_0}(u_i)) = 1; \tag{8.7}$$

then in particular $u_i \leq t_i$ for i large enough, and by the triangle inequality we have

$$\mathrm{dist}(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) \leq 2(t_i - u_i) + 1. \tag{8.8}$$

By 8.6, we get

$$\lim_{i\to\infty} (t_i - u_i) = +\infty. \tag{8.9}$$

Let us assume there exists a sequence $\theta'_i \in \Lambda_{C'}$ and a constant C such that

$$\mathrm{dist}(z_i, \alpha_{\theta'_i}) \leq C < +\infty. \tag{8.10}$$

We can assume that $C \geq 1$. Let v_i be such that

$$\mathrm{dist}(z_i, \alpha_{\theta'_i}) = \mathrm{dist}(z_i, \alpha_{\theta'_i}(v_i)). \tag{8.11}$$

By the triangle inequality

$$|t_i - v_i| \leq C \tag{8.12}$$

and

$$\mathrm{dist}(\alpha_{\theta'_i}(t_i), \alpha_{\theta_i}(t_i)) \leq 2C. \tag{8.13}$$

On the other hand, as the curvature of $\tilde{X}$ is bounded above by -1, a classical comparison theorem gives for any $t \in [0, t_i]$

$$\sinh\left(\frac{\mathrm{dist}(\alpha_{\theta'_i}(t), \alpha_{\theta_i}(t))}{2}\right) \leq \sinh C . \frac{\sinh t}{\sinh t_i}. \tag{8.14}$$

Let s_i be such that

$$\mathrm{dist}(\alpha_{\theta'_i}(s_i), \alpha_{\theta_i}(s_i)) = 1. \tag{8.15}$$

If $s_i < t_i$, we get from 8.13 and 8.14 the existence of a constant A such that for any i,

$$s_i \geq t_i - A, \tag{8.16}$$

and this inequality also trivially holds when $s_i \geq t_i$. From 8.16, we get

$$\frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = e^{-s_i + u_i} \leq e^A e^{-t_i + u_i}; \tag{8.17}$$

therefore, thanks to 8.9, we obtain

$$\lim_{i\to\infty} \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = 0, \tag{8.18}$$

which ends the proof of Lemma 8.10. q.e.d.

The two lemmas 8.10 and 7.4 imply the following:

Corollary 8.11. *Let us assume that for every sequence θ_i of points in $\partial\tilde{X} - \Lambda_{C'}$ converging to θ_0 and z_i the sequence of points constructed in Lemma* 8.5, $\liminf_{i\to\infty} \inf_{\theta\in\Lambda_{C'}} \mathrm{dist}(z_i, \alpha_\theta) < +\infty$. *Let $\lambda_k \to \infty$ be such that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to the space $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology; then the sequence of spaces $(\Lambda_{C'}, \lambda_k d, \theta_0)$ also converges to $(S, \bar{d}, 0)$.*

We will show now that there exists a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ converging to θ_0, such that the mutual distances $d(\theta_1^k, \theta_2^k)$, $d(\theta_1^k, \theta_0)$, $d(\theta_2^k, \theta_0)$ are tending to zero at the same rate, and the triple $\theta_1^k, \theta_2^k, \theta_0$ can be uniformly separated by elements $\gamma_k \in C'$.

Lemma 8.12. *Assume that every weak tangent of $(\partial\tilde{X}, d)$ at θ_0 belongs to $WT(\Lambda_{C'}, d)$; then there exist positive constants c, ν, a sequence ϵ_k tending to* 0 *when k tends to ∞, a sequence $\gamma_k \in C'$, and a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ such that for $i = 1, 2$,*
$c^{-1}\epsilon_k \leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k$,
$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_0) \leq c\epsilon_k$ *and*
$d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) \geq \nu$, $d(\gamma_k\theta_i^k, \gamma_k\theta_0)) \geq \nu$.

Proof. For any $x \in \tilde{X} \cup \partial\tilde{X}$ and $y \in \tilde{X} \cup \partial\tilde{X}$ let us define $\alpha_{x,y}$ as the geodesic ray joining x and y. Let $o \in \tilde{X}$ and $E \in T_o\tilde{X}$ be such that $|\mathrm{Jac}_E \tilde{F}(o)| = 1$ and $E(\infty)$ the equator associated to E. Let $\gamma_k \in C'$ be a sequence such that $\gamma_k(o)$ converges to the point $\theta_0 \in \Lambda_{C'}$ which is the point coming from Lemma 8.4. In particular, according to that lemma, there exists a point $z' \in \tilde{Z}'$ such that the hypersurface $\tilde{Z}'$ is contained in the complement of the open horoball $HB(z', \theta_0)$. We define $D := \mathrm{dist}(\tilde{z}, o)$. As $\tilde{Z}'$ lies outside the open horoball $HB(z', \theta_0)$, the points $\gamma_k(o)$ belong to the complement of the open horoball $HB(\alpha_{\tilde{z},\theta_0}(D), \theta_0)$. By standard triangle comparison argument (comparison with the hyperbolic case) the angle $\angle(\alpha_{\gamma_k(o),\theta_0}, \alpha_{\gamma_k(o),o})$ between the two geodesic rays $\alpha_{\gamma_k(o),\theta_0}$ and $\alpha_{\gamma_k(o),o}$ satisfies :

$$\lim_{k\to\infty} \angle(\alpha_{\gamma_k(o),\theta_0}, \alpha_{\gamma_k(o),o}) = 0. \tag{8.19}$$

By equivariance we have $\Lambda_{C'} \subset (\gamma_k E)(\infty)$ where $\gamma_k E \subset T_{\gamma_k(o)}\tilde{X}$. For any $v \in T\tilde{X}$ let α_v be the geodesic ray such that $\dot{\alpha}_v(0) = v$. Let us denote by u_k the unit vector in $\gamma_k E$ such that $\alpha_{u_k}(+\infty) = \theta_0$ and let us choose some $w_k \in \gamma_k E$ such that $< u_k, w_k >= 0$ (this is possible because $n - 1 \geq 2$).

We claim now that there exist unit vectors $v_k \in \gamma_k E$ such that the angle between v_k and w_k is not too far from 0 or π, namely

$$|\langle v_k, w_k\rangle| \geq \frac{1}{(n-1)^{1/2}}, \tag{8.20}$$

and $\alpha_{v_k}(+\infty) \in \Lambda_{C'}$ or $\alpha_{v_k}(-\infty) \in \Lambda_{C'}$.

Let us prove this claim. According to Proposition 4.6 and to 4.26, the restriction $h_{\gamma_k E}$ to $\gamma_k E$ of the quadratic form

$$h(u) = \int dB_{(\gamma_k(o),\theta)}(u)^2 d\mu_{\gamma_k(o)}(\theta)$$

verifies

$$h_{\gamma_k E}(u) = \frac{||u||^2}{n-1}. \tag{8.21}$$

Therefore, if for all $u \in \gamma_k(E)$ such that $\alpha_u(+\infty) = \theta \in \Lambda_{C'}$ we had $| < u, w_k > | < \frac{1}{(n-1)^{1/2}}$, then one would get $h(w_k) < \frac{1}{n-1}$, which contradicts 8.21 and proves the claim.

In particular, the angle between u_k and v_k is not too far from $\pi/2$ for k large enough, i.e.,

$$| < u_k, v_k > | \leq \left(\frac{n-2}{n-1}\right)^{1/2}, \tag{8.22}$$

and thanks to 8.19, we have for k large enough

$$| < \dot\alpha_{\gamma_k(o),o}(0), v_k > | \leq \left(\frac{n-\frac{3}{2}}{n-1}\right)^{1/2}. \tag{8.23}$$

Let us now define $\theta_k = \alpha_{v_k}(+\infty) \in \Lambda_{C'}$. Let us show that

$$\lim_{k\to\infty} d(\theta_0, \theta_k) = 0. \tag{8.24}$$

We proceed by contradiction and assume that 8.24 is not true. Then, one can assume after extracting a subsequence that θ_k converges to $\theta \neq \theta_0$. Therefore, the geodesic rays $\alpha_{\gamma_k(o),o}$ and α_{v_k} would converge to the geodesics $\alpha_{\theta_0,o}$ and $\alpha_{\theta_0,\theta}$ and thus the angle $\angle(\alpha_{\gamma_k(0),o}, \alpha_{v_k})$ would converge to 0. But this would contradict 8.23.

Let us now denote $\epsilon_k =: d(\theta_k, \theta_0)$. According to 8.24, we have $\lim_{k\to\infty} \epsilon_k = 0$. We now consider the following sequence of pointed metric space $(\partial\tilde X, \epsilon_k^{-1} d, \theta_0)$, a subsequence of which converges to some metric space $(S, \bar d)$, cf. [**4**]. For convenience we still denote by the same index k the subsequence. By Corollary 8.11, the sequence $(\Lambda_{C'}, \epsilon_k^{-1} d, \theta_0)$ also converges to $(S, \bar d)$. According to Lemma 7.2, the space S is homeomorphic to $\mathbb{R}^{n-1}$. In particular, there exist a sequence of points $\theta'_k \in \Lambda_{C'}$ and a constant c such that

$$c^{-1}\epsilon_k \leq d(\theta_k, \theta'_k) \leq c\epsilon_k, \tag{8.25}$$

$$c^{-1}\epsilon_k \leq d(\theta'_k, \theta_0)) \leq c\epsilon_k. \tag{8.26}$$

The points $\theta_1^k = \theta_k$ and $\theta_2^k = \theta'_k$ satisfy the two first properties of Lemma 8.12.

In order to complete the proof of Lemma 8.12, we will show that the elements $\eta_k =: \gamma_k^{-1}$ uniformly separate θ_0, θ_1^k and θ_2^k.

The inequality 8.22 shows that the angle at $\gamma_k(o)$ between θ_1^k and θ_0 is uniformly bounded away from 0 and π, and so is the angle at o between $\gamma_k^{-1}(\theta_1^k)$ and $\gamma_k^{-1}(\theta_0)$. Therefore, as the angle is Hölder-equivalent to the distance d, cf. [**11**], there is a constant c such that

$$d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_0)) \geq c. \tag{8.27}$$

Now, the cocompact group Γ acts uniformly quasi-conformally on $(\partial\tilde{X}, d)$ ([**4**] and [**21**] Theorem 5.2), and so does $C' \subset \Gamma$; therefore

$$d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_2^k)) \geq c, \tag{8.28}$$

and

$$d(\gamma_k^{-1}(\theta_2^k), \gamma_k^{-1}(\theta_0)) \geq c, \tag{8.29}$$

which ends the proof of Lemma 8.12. q.e.d.

Proof of Proposition 8.8. Let us assume that for every sequence θ_i of points in $\partial\tilde{X}$ converging to θ_0,

$$\liminf_{i\to\infty} \inf_{\theta\in\Lambda_{C'}} \mathrm{dist}(z_i, \alpha_\theta) < +\infty.$$

Then by Corollary 8.11 and Lemma 8.12 there exists a positive constant c, a sequence ϵ_k tending to 0 when k tends to ∞, a sequence $\gamma_k \in C'$, a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ such that

$$c^{-1}\epsilon_k \leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k\,, \quad c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_0) \leq c\epsilon_k,$$

and

$$d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) \geq \nu, \quad d(\gamma_k\theta_i^k, \gamma_k\theta_0) \geq \nu.$$

Applying Lemma 7.3 for $\mathcal{L} = \Lambda_{C'}$ and $\lambda_k = \epsilon_k^{-1}$ we conclude that $\Lambda_{C'}$ is homeomorphic to $\partial\tilde{X}$, which is impossible because $\Lambda_{C'}$ is contained in a topological equator $E(\infty)$. q.e.d.

This completes the proof of Proposition 8.1 as a consequence of Corollary 8.7 and Corollary 8.9.

9. C' and C are convex cocompact

We first define convex cocompactness. For a discrete group C of isometries acting on a Cartan Hadamard manifold of negative sectional curvature with limit set Λ_C, one defines the geodesic hull $\mathcal{G}(\Lambda_C)$ of Λ_C as the set of all geodesics both ends of which belong to Λ_C.

The geodesic hull of Λ_C is a C invariant set. One says that C is convex cocompact if $\mathcal{G}(\Lambda_C)/C$ is compact.

The goal of this section is to prove the following:

Lemma 9.1. *Let us assume that $\delta(C) = n - 2$; then C' is convex cocompact.*

Proof. Let $\pi : \tilde{X} \to \tilde{X}/C'$ denote the isometric covering map. Assume that C' is not convex cocompact. Then, there exists a sequence $x_n \in \pi\mathcal{G}(C')$ such that x_n tends to infinity. In particular, $\mathrm{dist}(x_n, Z') \to +\infty$, where $Z' = \tilde{Z}'/C'$ is the compact hypersurface which separates $\tilde{X}/C'$ in two unbounded connected components. There exist lifts $\tilde{x}_n$ of x_n such that

$$\tilde{x}_n \to \theta_0 \in \Lambda_{C'} \tag{9.1}$$

$$\mathrm{dist}(\tilde{x}_n, \tilde{Z}') = \mathrm{dist}(\tilde{x}_n, \tilde{z}_n) \tag{9.2}$$

where $\tilde{z}_n \in \tilde{Z}'$ is bounded. Therefore, by taking the limit of a subsequence of balls centered at $\tilde{x}_n$ and of radius equal to $\mathrm{dist}(\tilde{x}_n, \tilde{z}_n)$ we see that there exists $\tilde{z} \in \tilde{Z}'$ such that $HB(\tilde{z}, \theta_0) \subset \tilde{U}$, where $\tilde{U}$ is one of the two connected components of $\tilde{X} \setminus \tilde{Z}'$, the other being $\tilde{V}$.

We recall that $\mathcal{M}$, $\mathcal{N}$ are the two connected components of $\partial\tilde{X} \setminus \Lambda_{C'}$. By Proposition 8.1 we also have $\partial\tilde{Z}' = \Lambda_{C'} = E(\infty)$, and after possibly replacing C' by an index two subgroup, we can assume that C' preserves $\tilde{U}$ and $\tilde{V}$.

Claim 9.2. There are the two following cases. Either one of the two boundaries $\partial\tilde{U}$ or $\partial\tilde{V}$ is equal to $\Lambda_{C'}$ (in this case the other boundary is equal to $\partial\tilde{X}$), or $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$, where $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ are the closure of $\mathcal{M}$ and $\mathcal{N}$ respectively.

Proof of the claim. We first remark that if there exist $\theta \in \partial\tilde{U} \cap \mathcal{M}$, then $\mathcal{M} \subset \partial\tilde{U}$. Indeed, let ξ be any other point in $\mathcal{M}$ and α a continuous path in $\mathcal{M}$ joining θ and ξ. Since the set $\tilde{Z}' \cup \Lambda_{C'}$ is a closed subset in $\tilde{X} \cup \partial\tilde{X}$, there exists an open connected neighborhood W of α in $\tilde{X} \cup \partial\tilde{X}$ contained in the complementary of $\tilde{Z}' \cup \Lambda_{C'}$. Therefore, as $W \cap \tilde{U} \neq \emptyset$, we have $W \cap \tilde{X} \subset \tilde{U}$ and $\xi \in \partial\tilde{U}$. Let us assume that neither $\partial\tilde{U}$ nor $\partial\tilde{V}$ is equal to $\Lambda_{C'}$. Then, each boundary $\partial\tilde{U}$ and $\partial\tilde{V}$ contains $\mathcal{M}$ or $\mathcal{N}$. But on the other hand, since the set $\tilde{Z}' \cup \Lambda_{C'}$ is closed, $(\partial\tilde{U} \setminus \Lambda_{C'}) \cap (\partial\tilde{V} \setminus \Lambda_{C'}) = \emptyset$, and thus we have $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$ or the other way around, and the claim is proved. q.e.d.

Case 1: $\partial\tilde{U} = \Lambda_{C'}$ and $\partial\tilde{V} = \partial\tilde{X}$ or the other way around.

In this case, we are in the situation of the Section 8, which leads to a contradiction, cf. Remark 8.3.

Case 2: $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$.

In that case, assuming C' is not convex-cocompact, there exists an open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ where $\theta_0 \in \Lambda_{C'}$, $\tilde{z} \in \tilde{Z}'$. This intuitively means that $\tilde{Z}'$ touches $\partial\tilde{X}$ tangentially. We will find a contradiction in a similar way as in case 1, i.e., Section 8. By Proposition 8.1 there exists a point $o \in \tilde{X}$ and a hyperplane $E \subset T_o\tilde{X}$ such that $|\mathrm{Jac}_E\tilde{F}(o)| = 1$ and $\Lambda_{C'} = E(\infty)$.

Let $\theta_i \in \mathcal{N}$ be a sequence which converges to θ_0. By continuity, for i large enough, the geodesic ray α_{o,θ_i} spends some time in $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up in $\tilde{V}$ because θ_i converges to θ_0 and θ_i belongs to $\mathcal{N} = \partial\tilde{V} \setminus \Lambda_{C'}$. Therefore, α_{o,θ_i} eventually crosses $\tilde{Z}'$. Let z_i be some point in $\tilde{Z}' \cap \alpha_{o,\theta_i}$.

We will prove the following proposition, similar to Proposition 8.8:

Proposition 9.3. *There exists a sequence $\theta_i \in \mathcal{N}$ such that θ_i converges to θ_0 and*

$$\lim_{i\to\infty} \inf_{\theta\in\Lambda_{C'}} \operatorname{dist}(z_i, \alpha_\theta) = +\infty$$

where $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$.

Remark 9.4. The difference between Propositions 9.3 and 8.8 is that we are looking for a sequence $\theta_i \in \mathcal{N}$ instead of $\theta_i \in \partial\tilde{X} \setminus \Lambda_{C'}$.

Assuming Proposition 9.3, we find a contradiction in the same way as in Section 8. Namely, as $\tilde{Z}'/C'$ is compact, the points $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ stay at bounded distance from the C'-orbit of a fixed point, say, o; thus there exist a constant $A > 0$ and elements $\gamma_i \in C'$ such that for any i,

$$\operatorname{dist}(z_i, \gamma_i o) \leq A. \tag{9.3}$$

From 9.3 and the shadow Lemma 8.6, we obtain $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} \neq \emptyset$, and on the other hand, from Proposition 9.3, we have $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} = \emptyset$, which gives the contradiction. This ends the proof of Lemma 9.1 assuming Proposition 9.3. q.e.d.

It remains to prove Proposition 9.3.

Proof of Proposition 9.3. We argue by contradiction, as in the proof of Proposition 8.8. Let us assume that there exists a constant $C > 0$ such that for any sequence of points $\theta_i \in \mathcal{N}$ converging to θ_0, $\lim_{i\to\infty} \inf_{\theta\in\Lambda_{C'}} \operatorname{dist}(z_i, \alpha_\theta) \leq C$, then by Lemma 8.10, we have for any such sequence $\theta_i \in \mathcal{N}$

$$\lim_{i\to\infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0. \tag{9.4}$$

The proof of the following lemma is the same as the proof of Lemma 7.4.

Lemma 9.5. *Let us assume that for any sequence $\theta_i \in \mathcal{N}$, $\theta_i \neq \theta_0$, converging to θ_0, $\lim_{i\to\infty} \frac{d(\theta_i,\Lambda_{C'})}{d(\theta_i,\theta_0)} = 0$. Let $\{\lambda_k\}$ be a sequence of positive numbers tending to $+\infty$ such that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to a space $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology; then $(\mathcal{M}, \lambda_k d, \theta_0)$ also converges to $(S, \bar{d}, 0)$.*

Proof. Since $\Lambda_{C'} \subset \bar{\mathcal{M}}$, the assumption implies that $\lim_{\epsilon\to 0} r(\epsilon) = 0$ where

$$r(\epsilon) = \sup\left\{\frac{d(\theta,\mathcal{M})}{d(\theta,\theta_0)}, \theta \neq \theta_0, \theta \in \mathcal{N}, d(\theta,\theta_0) \leq \epsilon\right\}$$

and the proof goes the same way as in Lemma 7.4 replacing $\Lambda_{C'}$ by $\mathcal{M}$. q.e.d.

Similarly to Lemma 8.12, we have

Lemma 9.6. *Let us assume that every weak tangent of $(\partial\tilde{X}, d)$ at θ_0 belongs to $WT(\mathcal{M}, d)$. There exist positive constants c, ν, a sequence ϵ_k tending to 0 when k tends to $+\infty$, a sequence of $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0,1,2\}$, $c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c\epsilon_k$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_i^k) \geq \nu$.*

We can now end the proof of Proposition 9.3. Let us assume that there exists a constant $C > 0$ such that for every sequence $\theta_i \neq \theta_0$ of points in $\mathcal{N}$ converging to θ_0, $\lim_{i\to\infty} \inf_{\theta\in\Lambda_{C'}} \mathrm{dist}(\theta_i, \alpha_\theta) \leq C$, then by 9.4, Lemma 9.5 and Lemma 9.6, there exist a sequence ϵ_k tending to 0 when k tends to ∞, a sequence $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0,1,2\}$, $c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c\epsilon_k$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_i^k) \geq \nu$.

Applying Lemma 7.3 for $\mathcal{L} = \mathcal{M}$ and $\lambda_k = \epsilon_k^{-1}$, we conclude that $\mathcal{M}$ is homeomorphic to $\partial\tilde{X}$, which is impossible because $\partial\tilde{X}$ is a sphere, and $\mathcal{M}$ is homeomorphic to an hemisphere. This ends the proof of Proposition 9.3. q.e.d.

Corollary 9.7. *C is convex cocompact.*

Proof. The subgroup C' of C is convex cocompact and the limit sets of C' and C coincide by Section 8; therefore C is convex cocompact. q.e.d.

10. C preserves a copy of the $(n-1)$-dimensional hyperbolic space $\mathbb{H}^{n-1}$ totally geodesically embedded in $\tilde{X}$

From Sections 6-9, we know that the groups C and C' are convex cocompact, and that their limit set Λ_C and $\Lambda_{C'}$ are equal to a topological equator $E(\infty)$. Our goal is now to prove the existence of a C-invariant hypersurface $\tilde{Z}_\infty$ in $\tilde{X}$, such that $\tilde{Z}_\infty$ is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$ and totally geodesic in $\tilde{X}$.

Proposition 10.1. *Let us assume that $\delta(C) = n-2$; then there exists a C-invariant hypersurface $\tilde{Z}_\infty$ in $\tilde{X}$ such that $\tilde{Z}_\infty$ is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$ and totally geodesic in $\tilde{X}$.*

Proof. Let us first sketch the idea of proof. We consider the essential hypersurface $Z' \subset \tilde{X}/C'$. We will show that there exists a minimizing current representing the class of Z' in $H_{n-1}(\tilde{X}/C', \mathbb{R})$ and that this minimizing current lifts to a C'-invariant totally geodesic hypersurface $\tilde{Z}_\infty$ embedded in $\tilde{X}$. We will then eventually show that this totally geodesic C'-invariant hypersurface $\tilde{Z}_\infty$ is hyperbolic. Then, we will show that $\tilde{Z}_\infty$ is C-invariant since C and C' have the same limit set $\Lambda_C = \Lambda_{C'} = E(\infty)$.

We work in $\tilde{X}/C'$ and consider the essential hypersurface $Z' \subset \tilde{X}/C'$. We will now prove that there exists a minimizing current representing the class of Z' in $H_{n-1}(\tilde{X}/C', \mathbb{R})$. Let $\{Z_k\}$ be a minimizing sequence of currents homologous to Z'. The orthogonal projection onto the convex core of $\tilde{X}/C'$ is distance nonincreasing and thus volume nonincreasing. Therefore, we can assume that the Z_k's are in the the convex core of $\tilde{X}/C'$, which is compact. By [**14**] (5.5), the sequence $\{Z_k\}$ subconverges to a minimal current Z_∞ in $\tilde{X}/C'$. By [**14**] (8.2), Z_∞ is a manifold with singularities of codimension greater than or equal to 7. By Corollary 4.5 and minimality, we get that $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$ at every regular point $x \in Z_\infty$. We will use the fact that $|\mathrm{Jac}_{n-1}\tilde{F}(x)| = 1$ at every regular point $x \in Z_\infty$ in order to prove that Z_∞ is a totally geodesic hypersurface.

Lemma 10.2. *Let x and y two distinct points in $\tilde{X}$ and $E_x \subset T_x\tilde{X}$, $E_y \subset T_y\tilde{X}$ be such that* $\mathrm{Jac}_{n-1}\tilde{F}(x) = \mathrm{Jac}_{E_x}\tilde{F}(x) = 1$ *and* $\mathrm{Jac}_{n-1}\tilde{F}(y) = \mathrm{Jac}_{E_y}\tilde{F}(y) = 1$. *Then, the geodesic $\alpha_{x,y}$ (resp. $\alpha_{y,x}$) joining x and y (resp. y and x) satisfies $\dot{\alpha}_{x,y}(0) \in E_x$ (resp. $\dot{\alpha}_{y,x}(0) \in E_y$). In particular, $\alpha_{x,y}(+\infty)$ and $\alpha_{y,x}(+\infty)$ belong to $\Lambda_{C'}$.*

Proof. Let S_x and S_y be the unit spheres of E_x and E_y. For any unit tangent vector $u \in T_z\tilde{X}$ at some point z, we define $\theta_u \in \partial\tilde{X}$ by $\dot{\alpha}_{z,\theta_u}(0) = u$. By Section 8, $\Lambda_{C'} = E_x(\infty) = E_y(\infty)$; therefore for every $u \in S_x$, $\theta_u \in \Lambda_{C'}$ and there exist $v \in E_y$ such that $\theta_u = \theta_v$. As E_y is a vector space, θ_{-v} belongs to $\Lambda_{C'}$; therefore there exist $w \in E_x$ such that $\theta_w = \theta_{-v}$. The map $f : S_x \to S_x$ defined by $f(u) = w$ is a continuous map. The lemma then reduces to proving that there exist $u \in S_x$ such that $f(u) = -u$ because the negative curvature forces x, y and θ_u are on the same geodesic α_{x,θ_u}. The following properties of f are obvious.

(i) For every $u \in S_x$, $f(u) \neq u$.
(ii) $f \circ f = Id$.

So f is an involution of the sphere without fixed point, and for any such map, we claim that there exist u in the sphere such that $f(u) = -u$. In order to prove the claim, we follow a very similar argument as in [**18**], Theorem 1. We argue by contradiction. Let us assume that for every $u \in S_x$, $f(u) \neq -u$. The map $g : S_x \to S_x$ defined by $g(u) = \frac{f(u)+u}{\|f(u)+u\|}$, is then well defined and continuous. Let us remark that as for every $u \in S_x$, $f(u) \neq -u$, then f is homotopic to the Identity, and so is g. Moreover by (ii) we clearly have $g \circ f = g$, thus the map

g factorizes through S_x/G_f where G_f is the group generated by the involution f. By (i), f has no fixed point; thus S_x/G_f is a manifold and the projection $p : S_x \to S_x/G_f$ is a degree 2 map. Therefore, the induced endomorphism g_* on $H_{n-1}(S_x, \mathbb{Z}_2)$ is trivial, which contradicts the fact that g is homotopic to the Identity. q.e.d.

Corollary 10.3. *Let $\mathcal{H}^{n-1} \subset \tilde{X}$ be a hypersurface with possibly nonempty boundary $\partial\mathcal{H}^{n-1}$, such that for any $x \in \mathcal{H}^{n-1}$, $\mathrm{Jac}_{n-1}\tilde{F}'(x) = \mathrm{Jac}_{E_x}\tilde{F}'(x) = 1$, where E_x is the tangent space of $\mathcal{H}^{n-1}$ at x. Let us consider $x \in \mathcal{H}^{n-1}$ such that $\mathrm{dist}_{\tilde{X}}(x, \partial\mathcal{H}^{n-1}) = r > 0$. Then, for any $x' \in \mathcal{H}^{n-1}$ with $\mathrm{dist}_{\tilde{X}}(x, x') < r$, the geodesic $\alpha_{x,x'}$ joining x and x' is contained in $\mathcal{H}^{n-1}$. In particular, $\mathcal{H}^{n-1}$ is locally convex.*

Proof. Let us fix $\theta \in \Lambda_{C'}$ and consider the vector field $\nabla B(y, \theta)$ for y in $\tilde{X}$. Let $x \in \mathcal{H}^{n-1}$. As $\mathrm{Jac}_{E_x}\tilde{F}'(x) = 1$, we have $\Lambda_{C'} = E_x(\infty)$ by Proposition 4.6. Then, for any $x \in \mathcal{H}^{n-1}$, $\nabla B(x, \theta)$ is tangent to $\mathcal{H}^{n-1}$, and therefore the geodesic $\alpha_{x,\theta}$ satisfies $\alpha_{x,\theta}(t) \in \mathcal{H}^{n-1}$ for all $t \in [0, r)$. Let $x' \in \mathcal{H}^{n-1}$. By Lemma 10.2, $\dot{\alpha}_{x,x'}(0) \in E_x$, therefore $\alpha_{x,x'} = \alpha_{x,\theta}$ for some $\theta \in \Lambda_{C'}$ and $\alpha_{x,x'}(t) \in \mathcal{H}^{n-1}$ for all $t \in [0, r)$. q.e.d.

We now prove that Z_∞ is a totally geodesic hypersurface in $\tilde{X}/C'$. Let us recall that Z_∞ is a manifold which is smooth except at a singular subset of codimension at least 7. Let us consider a lift $\tilde{Z}_\infty \subset \tilde{X}$ of Z_∞ and denote $\tilde{Z}_\infty^{\mathrm{reg}}$ (resp. $\tilde{Z}_\infty^{\mathrm{sing}}$) the set of regular (resp. singular points) of $\tilde{Z}_\infty$.

Lemma 10.4. *$\tilde{Z}_\infty$ is a totally geodesic hypersurface in $\tilde{X}$.*

Proof. Let us consider a regular point $x \in \tilde{Z}_\infty^{\mathrm{reg}}$. We shall show that for every point $x' \in \tilde{Z}_\infty^{\mathrm{reg}}$ the geodesic segment joining x and x' is contained in $\tilde{Z}_\infty$, and as the set of regular points is dense in $\tilde{Z}_\infty$ (as the complement of a subset of codimension at least 8), this will show that $\tilde{Z}_\infty$ is totally geodesic.

We claim that there exists a sequence $y_k \in \tilde{Z}_\infty^{\mathrm{reg}}$ such that $\lim_{k\to\infty} y_k = x'$ and the geodesic segment joining x and y_k is contained in $\tilde{Z}_\infty$. This claim immediately implies that the geodesic segment joining x and x' is contained in $\tilde{Z}_\infty$. Let us prove the claim.

For $y \in \tilde{Z}_\infty^{\mathrm{reg}}$ we consider $\alpha_{x,y}$ the geodesic joining x and y and define

$$t_y = \inf\{t > 0, \alpha_{x,y}(t) \notin \tilde{Z}_\infty\}. \tag{10.1}$$

As x is a regular point, by Corollary 10.3, there exist $\epsilon > 0$ such that $t_y > \epsilon$.

In order to prove the claim, we argue by contradiction. Let us assume that there exist $r > 0$ such that for any $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{\mathrm{reg}}$, $t_y < \mathrm{dist}(x, y)$. By Corollary 10.3 applied to $\tilde{Z}_\infty^{\mathrm{reg}}$, we have $\alpha_{x,y}(t_y) \in \tilde{Z}_\infty^{sing}$. As the set of regular points is an open subset of $\tilde{Z}_\infty$, if r is small enough we have $B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{\mathrm{reg}} = B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$. We choose such an r and we consider the set S of all singular points contained in the union of all geodesic segments joining x to a point $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$. Let us consider the map defined on S by

$$p(y) = \alpha_{x,y}(\epsilon).$$

As we already saw, for any $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$, we have $t_y > \epsilon$; therefore the map p is distance decreasing, and by assumption p is surjective onto an open subset of the sphere and $p(S)$ is homeomorphic to an open subset of $\mathbb{R}^{n-1}$. Therefore, the Hausdorff dimension of S is greater than or equal to $n-1$, which contradicts the fact that the singular set has codimension at least 7 in $\tilde{Z}_\infty$. q.e.d.

The totally geodesic hypersurface $\tilde{Z}_\infty \subset \tilde{X}$ is preserved by C', and $\tilde{Z}_\infty / C'$ is of minimal volume in its homology class. Let us prove that $\tilde{Z}_\infty$ is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$.

Lemma 10.5. *$\tilde{Z}_\infty$ is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$.*

Proof. As $\tilde{Z}_\infty / C'$ is of minimal volume in its homology class, we have by Proposition 4.6, for all $x \in \tilde{Z}_\infty$, $\mathrm{Jac}_{E_x} \tilde{F}(x) = 1$ and $\tilde{F}(x) = x$, where E_x is the tangent space of $\tilde{Z}_\infty$ at x. Moreover, we saw in the proof of Proposition 4.6 that

$$H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E_x)} = \frac{1}{n-1} Id_{E_x};$$

therefore, we get from 4.11 and $\tilde{F}(x) = x$ that for all $u, v \in T_x \tilde{Z}_\infty$,

$$\begin{aligned} &\int_{\partial \tilde{X}} [DdB_{(x,\theta)}(u,v) + dB_{(x,\theta)}(u) dB_{(x,\theta)}(v)] d\nu_x(\theta) \\ &= \tilde{g}(u,v), \end{aligned} \tag{10.2}$$

where $\tilde{g}$ is the metric on $\tilde{X}$. As $\tilde{Z}_\infty$ is totally geodesic, the relation 10.2 remains true with the Busemann function $B^{\tilde{Z}_\infty}$ of $\tilde{Z}_\infty$ instead of the Busemann function B of $\tilde{X}$:

$$\begin{aligned} &\int_{\partial \tilde{X}} [DdB^{\tilde{Z}_\infty}_{(x,\theta)}(u,v) + dB^{\tilde{Z}_\infty}_{(x,\theta)}(u) dB^{\tilde{Z}_\infty}_{(x,\theta)}(v) \\ &= \tilde{g}(u,v). \end{aligned} \tag{10.3}$$

On the other hand, as $\tilde{Z}_\infty$ is totally geodesic, its sectional curvature is less than or equal to -1, and thus by Rauch comparison theorem, we have

$$DdB^{\tilde{Z}_\infty}_{(x,\theta)} + dB^{\tilde{Z}_\infty}_{(x,\theta)} \otimes dB^{\tilde{Z}_\infty}_{(x,\theta)} \geq \tilde{g}_{\tilde{Z}_\infty} \tag{10.4}$$

for all $\theta \in \partial \tilde{Z}_\infty = \Lambda_{C'}$, where $\tilde{g}_{\tilde{Z}_\infty}$ is the restriction of $\tilde{g}$ to $\tilde{Z}_\infty$. As the support of the measure ν_x is $\partial \tilde{Z}_\infty = \Lambda_{C'}$ (by convex cocompactness of C') and the Busemann function is continuous, we get from 10.3 and 10.4 that for all $x \in \tilde{Z}_\infty$ and all $\theta \in \tilde{Z}_\infty$

$$DdB^{\tilde{Z}_\infty}(x,\theta) + dB^{\tilde{Z}_\infty}_{(x,\theta)} \otimes dB^{\tilde{Z}_\infty}_{(x,\theta)} = \tilde{g}_{\tilde{Z}_\infty}(x). \tag{10.5}$$

This last relation is characteristic of the hyperbolic space, cf. [**8**]. q.e.d.

We now can end the proof of Proposition 10.1. By Lemma 10.4 and Lemma 10.5, there exists a C'-invariant totally geodesic hypersurface $\tilde{Z}_\infty$ in $\tilde{X}$ which is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$. But since $\Lambda_{C'} = \Lambda_C = \partial \tilde{Z}_\infty$ we conclude that $\tilde{Z}_\infty$ is also C-invariant, which concludes the proof of Proposition 10.1. q.e.d.

11. Conclusion

We are now ready to conclude the proof of the equality case of Theorem 1.2. So by Proposition 10.1, we know that whenever $\delta(C) = n-2$, C preserves a totally geodesic copy of the real hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}} \subset \tilde{X}$ such that $\mathbb{H}^{n-1}_{\mathbb{R}}/C$ is compact.

Our goal now is to show that $Y =: \mathbb{H}^{n-1}/C$ injects diffeomorphically in $X = \tilde{X}/\Gamma$ and separates X in two connected components R and S such that $A = \pi_1(R)$ and $B = \pi_1(S)$.

Let us define $\bar{C} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} = \mathbb{H}^{n-1}\}$. We have $C \subset \bar{C}$ and as $\mathbb{H}^{n-1}/C$ is compact, so is $\mathbb{H}^{n-1}/\bar{C}$ and thus $[\bar{C} : C] < \infty$.

Let $p : \tilde{X}/C \to X = \tilde{X}/\Gamma$ and $\bar{p} : \tilde{X}/\bar{C} \to X = \tilde{X}/\Gamma$ be the natural projections. We now show that the restriction of p to $\mathbb{H}^{n-1}/C$ is an embedding, thus $Y := p(\mathbb{H}^{n-1}/C)$ is a compact totally geodesic hypersurface of X.

In Section 2, we constructed a C-invariant hypersurface $\tilde{Z} \subset \tilde{X}$ such that $Z = \tilde{Z}/C \subset \tilde{X}/C$ is compact. The hypersurface is defined as $\tilde{Z} = \tilde{f}^{-1}(t_0)$ where $\tilde{f} : \tilde{X} \to T$ is an equivariant map onto the Bass-Serre tree associated to the amalgamation $A *_C B$ and t_0 belongs to the edge of T which is fixed by C. Let us first show two lemmas.

Lemma 11.1. *The restriction of p to $\tilde{Z}/C$ is an embedding into $X = \tilde{X}/\Gamma$.*

Proof. Let $\gamma \in \Gamma$, z, z' in $\tilde{Z}$ such that $z' = \gamma z$. By equivariance,

$$t_0 = \tilde{f}(z') = \tilde{f}(\gamma z) = \gamma \tilde{f}(z) = \gamma t_0\,,$$

thus $\gamma \in C$. q.e.d.

Lemma 11.2. *The restriction of $\bar{p}$ to $\mathbb{H}^{n-1}/\bar{C}$ is an embedding into $X = \tilde{X}/\Gamma$.*

Proof. Let us assume that there is a $\gamma \in \Gamma - \bar{C}$ such that $\gamma\mathbb{H}^{n-1} \cap \mathbb{H}^{n-1} \neq \emptyset$ and choose an $x \in \gamma\mathbb{H}^{n-1} \cap \mathbb{H}^{n-1}$. As $\gamma \notin \bar{C}$, there exist $u \in T_x\gamma\mathbb{H}^{n-1} \setminus T_x\mathbb{H}^{n-1}$. We consider c_u the geodesic ray such that

$\dot{c}_u(0) = u$. We know that $\tilde{Z}$ is contained in an ϵ-neighbourhood $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ of $\mathbb{H}^{n-1}$ which separates $\tilde{X}$ in two connected components $\mathcal{V}_\epsilon$ and $\mathcal{W}_\epsilon$, and for $t > 0$ large enough, we have, say, $c_u(t) \in \mathcal{V}_\epsilon$ and $c_u(-t) \in \mathcal{W}_\epsilon$.

Let $\tilde{Z}'$ be the connected component of $\tilde{Z}$ that we constructed at the end of Section 2, whose stabilizer (or an index two subgroup of it) C' is such that $\tilde{Z}'/C'$ separates $\tilde{X}/C'$ in two unbounded connected components U'/C' and V'/C' where U' and V' are the two connected components of $\tilde{X} \setminus \tilde{Z}'$.

We claim that $\mathcal{V}_\epsilon \subset U'$ and $\mathcal{W}_\epsilon \subset V'$ or the other way around. Indeed, if not, $\mathcal{V}_\epsilon$ and $\mathcal{W}_\epsilon$ would be both contained in, say, U'. But in that case, V' would be contained in $\mathcal{U}_\epsilon H^{n-1}$ and therefore V'/C' would be bounded, which is a contradiction.

As $\gamma\tilde{Z}$ lies in the ϵ neighborhood of $\gamma\mathbb{H}^{n-1}$, there exist sequences z_k, z'_k in $\gamma\tilde{Z}$ such that $\mathrm{dist}(z_k, c_u(k)) \leq \epsilon$ and $\mathrm{dist}(z'_k, c_u(-k)) \leq \epsilon$. By Proposition 8.1 and Lemma 9.1, C' also acts cocompactly on $\mathbb{H}^{n-1}$; thus C' is of finite index in C, and therefore there are finitely many connected components of $\tilde{Z}$ and the same holds for $\gamma\tilde{Z}$. We thus can assume that the z_k's and z'_k's belong to a single connected component of $\gamma\tilde{Z}$. Let us consider a continuous path $\alpha \subset \gamma\tilde{Z}$ joining z_k and z'_k.

By construction, the distance between $c_u(k)$ [resp. $c_u(-k)$] and $\mathbb{H}^{n-1}$ tends to infinity and thus, for k large enough, $z_k \in \mathcal{V}_\epsilon$ and $z'_k \in \mathcal{W}_\epsilon$ or the other way around. By the claim, we then have $z_k \in U'$ and $z'_k \in V'$; therefore the path α has to cross $\tilde{Z}'$, which contradicts Lemma 11.1 and ends the proof of Lemma 11.2. q.e.d.

As we already saw, $\tilde{Z}$ has finitely many connected components, and so does $\tilde{X} \setminus \tilde{Z}$. Let us write $\{W_j\}_{j=1,..,m}$ the connected components of $\tilde{X} \setminus \tilde{Z}$. As C acts cocompactly on $\tilde{Z}$ and $\mathbb{H}^{n-1}$ there exist $\epsilon > 0$ such that $\mathbb{H}^{n-1} \subset \mathcal{U}_\epsilon \tilde{Z}$ and $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. Moreover, let us recall that $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ separates $\tilde{X}$ into two connected components $\mathcal{V}_\epsilon$ and $\mathcal{W}_\epsilon$.

Lemma 11.3. *Let us consider ϵ such that $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ and $\mathbb{H}^{n-1} \subset \mathcal{U}_\epsilon \tilde{Z}$. Let $\mathcal{V}_\epsilon$ and $\mathcal{W}_\epsilon$ be the two connected components of $\tilde{X} \setminus \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. Then there are two distinct connected components W_1 and W_2 of $\tilde{X} \setminus \tilde{Z}$ such that $\mathcal{V}_\epsilon \subset W_1$ and $\mathcal{W}_\epsilon \subset W_2$. Moreover, $\tilde{f}(W_1) \subset \tilde{T}_1$ and $\tilde{f}(W_2) \subset \tilde{T}_2$, where $\tilde{T}_1$ and $\tilde{T}_2$ are the two connected components of $\tilde{T} \setminus \{t_0\}$.*

Proof. We argue by contradiction. Let us assume that $\mathcal{V}_\epsilon$ and $\mathcal{W}_\epsilon$ are contained in the same connected component W_1 of $\tilde{X} \setminus \tilde{Z}$. Then, all other components W_j, $j \neq 1$, satisfy $W_j \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1} \subset \mathcal{U}_{2\epsilon} \tilde{Z}$. Therefore, as C acts cocompactly on $\mathcal{U}_{2\epsilon} \tilde{Z}$, there exist a constant D such that for any $j \neq 1$, $\max_{w \in W_j} \mathrm{dist}_{\tilde{T}}(\tilde{f}(w), t_0) \leq D$. Thus, $\tilde{f}(W_1)$ is contained in one connected component of $\tilde{T} - \{t_0\}$ and $\tilde{f}(\cup_{j \neq 1} W_j)$, contained in the ball $B_{\tilde{T}}(t_0, D)$ of $\tilde{T}$ of radius D centered at t_0, is bounded. This is clearly impossible because $\tilde{T} - \{t_0\}$ has two unbounded connected components and $\tilde{f}$ is onto. q.e.d.

Let us denote $\mathcal{A} = A\mathbb{H}^{n-1}$ the A-orbit of the C-invariant totally geodesic copy of the real hyperbolic space $\mathbb{H}^{n-1}$, and $\bar{A}$ the stabilizer of $\mathcal{A}$, i.e., $\bar{A} = \{\gamma \in \Gamma; \gamma\mathcal{A} = \mathcal{A}\}$. We define in a similar way $\mathcal{B} = B\mathbb{H}^{n-1}$ and $\bar{B} = \{\gamma \in \Gamma; \gamma\mathcal{B} = \mathcal{B}\}$. Note that due to Lemma 11.2, $\mathcal{A}$ is a collection of disjoint translates $\gamma\mathbb{H}^n$ and so is $\mathcal{B}$. Let us recall that $\bar{C}$ is the stabilizer of $\mathbb{H}^{n-1}$ in Γ. We now prove the following lemma:

Lemma 11.4. *We have $\bar{A} \subset \{\gamma \in \Gamma; \gamma\mathbb{H}^{n-1} \in \mathcal{A}\} \subset A\bar{C}$ and $\bar{B} \subset \{\gamma \in \Gamma; \gamma\mathbb{H}^{n-1} \in \mathcal{B}\} \subset B\bar{C}$.*

Proof. The first inclusion $\bar{A} \subset \{\gamma \in \Gamma; \gamma\mathbb{H}^{n-1} \in \mathcal{A}\}$ is obvious. Let us prove the second inclusion, $\{\gamma \in \Gamma; \gamma\mathbb{H}^{n-1} \in \mathcal{A}\} \subset A\bar{C}$. Let $\gamma' \in \Gamma$ be such that $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$. Then there exists $\gamma \in A$ such that $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$, thus $\gamma^{-1}\gamma' \in \bar{C}$ and therefore $\gamma' \in \gamma\bar{C} \subset A\bar{C}$. This proves the second inclusion. The inclusions $\bar{B} \subset \{\gamma \in \Gamma; \gamma\mathbb{H}^{n-1} \in \mathcal{B}\} \subset B\bar{C}$ are proved similarly. q.e.d.

For each $\gamma \in \Gamma$, $\gamma\mathbb{H}^{n-1}$ separates $\tilde{X}$ in two connected components U_γ and V_γ. Let us now prove the following lemma.

Lemma 11.5.

(i) *Let $\gamma \in A$, [resp. $\gamma \in B$]. Then, we have $\mathcal{A} \setminus \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$ or $\mathcal{A}\setminus\{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$, [resp. $\mathcal{B}\setminus\{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$ or $\mathcal{B}\setminus\{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$]. Moreover, if $\mathcal{A} \setminus \mathbb{H}^{n-1} \subset U_\gamma$, then $\mathcal{B} \setminus \mathbb{H}^{n-1} \subset V_\gamma$ or the other way around.*

(ii) *Let γ be an element of $\Gamma \setminus \bar{A}$, [resp. $\Gamma \setminus \bar{B}$]. Then $\mathcal{A} \subset U_\gamma$ or $\mathcal{A} \subset V_\gamma$, [resp. $\mathcal{B} \subset U_\gamma$ or $\mathcal{B} \subset V_\gamma$].*

Proof. (i) For the first part we argue by contradiction. Let us consider $\gamma\mathbb{H}^{n-1}$, $\gamma'\mathbb{H}^{n-1}$ and $\gamma''\mathbb{H}^{n-1}$ three distinct elements in $\mathcal{A}$ such that $\gamma'\mathbb{H}^{n-1} \subset U_\gamma$ and $\gamma''\mathbb{H}^{n-1} \subset V_\gamma$. By equivariance we can assume that γ is the identity. Let us recall that $\mathcal{V}_\epsilon$ and $\mathcal{W}_\epsilon$ are the two connected components of $\tilde{X} \setminus \mathcal{U}_\epsilon\mathbb{H}^{n-1}$. We then have $\gamma'\mathbb{H}^{n-1} \cap \mathcal{V}_\epsilon \neq \emptyset$ and $\gamma''\mathbb{H}^{n-1} \cap \mathcal{W}_\epsilon \neq \emptyset$, which implies $\gamma'\tilde{Z} \cap \mathcal{V}_\epsilon \neq \emptyset$ and $\gamma''\tilde{Z} \cap \mathcal{W}_\epsilon \neq \emptyset$. By Lemma 11.3, $\mathcal{V}_\epsilon \subset W_1$ and $\mathcal{W}_\epsilon \subset W_2$ where W_1 and W_2 are two connected components of $\tilde{X}\setminus\tilde{Z}$ and $\tilde{f}(\mathcal{V}_\epsilon) \subset \tilde{T}_1$ and $\tilde{f}(\mathcal{W}_\epsilon) \subset \tilde{T}_2$. Therefore, $\tilde{f}(\mathcal{V}_\epsilon)$ contains $\gamma' t_0 \in \tilde{T}_1$ and $\tilde{f}(\mathcal{W}_\epsilon)$ contains $\gamma'' t_0 \in \tilde{T}_2$. This is impossible because for all elements γ' and γ'' in A, $\gamma' t_0$ and $\gamma'' t_0$ belong to the same connected component of $\tilde{T} \setminus \{t_0\}$. This proves the first part of (i).

Let us now prove the second part of (i). We can assume that $\mathcal{A} \setminus \mathbb{H}^{n-1} \subset U_\gamma$ and we have to prove that $\mathcal{B} \setminus \mathbb{H}^{n-1} \subset V_\gamma$. From our assumption, for each $\gamma\mathbb{H}^{n-1} \in \mathcal{A}$ distinct from $\mathbb{H}^{n-1}$, we get $\gamma\mathbb{H}^{n-1} \cap \mathcal{V}_\epsilon \neq \emptyset$ and thus $\gamma\tilde{Z} \cap \mathcal{V}_\epsilon \neq \emptyset$. Therefore, as above we deduce $\gamma t_0 \in \tilde{T}_1$ for each $\gamma \in A$ such that $\gamma\mathbb{H}^{n-1} \in \mathcal{A}$ distinct from $\mathbb{H}^{n-1}$. This implies that $\gamma t_0 \in \tilde{T}_1$ for each $\gamma \in A$ such that $\gamma t_0 \neq t_0$. This argument would similarly prove that $\gamma' t_0 \in \tilde{T}_1$ for each $\gamma' \in B$ such that $\gamma' t_0 \neq t_0$, if we had assumed that $\mathcal{B} \setminus \mathbb{H}^{n-1} \subset U_\gamma$. But this is clearly impossible since if $\gamma t_0 \in \tilde{T}_1$ for each $\gamma \in A$ such that $\gamma t_0 \neq t_0$, then $\gamma' t_0 \in \tilde{T}_2$ for each $\gamma' \in B$ such that $\gamma' t_0 \neq t_0$ by definition of the Bass-Serre tree.

(ii) Let us consider $\gamma \in \Gamma \setminus \bar{A}$. For any $\gamma' \in A$, then $\gamma' \mathbb{H}^{n-1} \subset U_\gamma$ or $\gamma' \mathbb{H}^{n-1} \subset V_\gamma$. We again argue by contradiction. Let us assume there exist γ', γ'' in A such that

$$\begin{aligned} \gamma' \mathbb{H}^{n-1} &\subset U_\gamma \\ \gamma'' \mathbb{H}^{n-1} &\subset V_\gamma. \end{aligned} \tag{11.1}$$

By Lemma 11.3, we have $\tilde{f}(\gamma \mathcal{V}_\epsilon) \subset \gamma \tilde{T}_1$ and $\tilde{f}(\gamma \mathcal{W}_\epsilon) \subset \gamma \tilde{T}_2$, where $\gamma \tilde{T}_1$ and $\gamma \tilde{T}_2$ are the two connected components of $\tilde{T} \setminus \{\gamma t_0\}$. By Lemma 11.1, we have $\gamma' \mathbb{H}^{n-1} \cap \gamma \mathcal{V}_\epsilon \neq \emptyset$ and $\gamma'' \mathbb{H}^{n-1} \cap \gamma \mathcal{W}_\epsilon \neq \emptyset$, which implies $\gamma' \tilde{Z} \cap \gamma \mathcal{V}_\epsilon \neq \emptyset$ and $\gamma'' \tilde{Z} \cap \gamma \mathcal{W}_\epsilon \neq \emptyset$. Therefore $\gamma' t_0 \in \gamma \tilde{T}_1$ and $\gamma'' t_0 \in \gamma \tilde{T}_2$, which is impossible because in the tree $\tilde{T}$, the points $\gamma' t_0$ and $\gamma'' t_0$ belong to two adjacent edges since γ' and γ'' are contained in A. q.e.d.

By Lemma 11.5 (i), for every γ in $\bar{A}$, either U_γ or V_γ is the connected component of $\tilde{X} - \gamma \mathbb{H}^{n-1}$ which contains all $\gamma' \mathbb{H}^{n-1}$ with γ' in $\bar{A}$ such that $\gamma' \mathbb{H}^{n-1} \neq \gamma \mathbb{H}^{n-1}$. For all $\gamma \in \bar{A}$ we can therefore denote by U_γ^A this component. For all $\gamma \in \bar{B}$ we similarly denote U_γ^B as the connected component of $\tilde{X} - \gamma \mathbb{H}^{n-1}$ which contains all $\gamma' \mathbb{H}^{n-1}$ with γ' in $\bar{B}$ such that $\gamma' \mathbb{H}^{n-1} \neq \gamma \mathbb{H}^{n-1}$. Let us define

$$U_A := \cap_{\gamma \in A} U_\gamma^A \text{ and } U_B := \cap_{\gamma \in B} U_\gamma^B. \tag{11.2}$$

Note that by definition of $\bar{A}$ and $\bar{B}$, we can write $U_A := \cap_{\gamma \in \bar{A}} U_\gamma^A$ and $U_B := \cap_{\gamma \in \bar{B}} U_\gamma^B$.

Lemma 11.6. *The sets U_A and U_B are nonempty, open and convex. The boundary of U_A [resp. U_B] is $\mathcal{A}$ [resp. $\mathcal{B}$].*

Proof. We argue for U_A. By definition, U_A is a convex set in $\tilde{X}$ whose boundary is the collection $\mathcal{A}$, of disjoint translates of $\gamma \mathbb{H}^{n-1}$, γ in $\bar{A}$. Moreover, we claim that for each $\gamma \in A$,

$$\operatorname{dist}(\gamma \mathbb{H}^{n-1}, \mathcal{A} \setminus \gamma \mathbb{H}^{n-1}) > 0.$$

By equivariance the proof reduces to the case of $\gamma \mathbb{H}^{n-1} = \mathbb{H}^{n-1}$, and the fact that $\operatorname{dist}(\mathbb{H}^{n-1}, \mathcal{A} - \mathbb{H}^{n-1}) > 0$ follows from the fact that $\bar{C}$ acts cocompactly on $\mathbb{H}^{n-1}$ and the discretness of Γ. From the claim we deduce that U_A contains a nonempty open neighbourhood of $\mathbb{H}^{n-1}$ in U_{Id}^A, which proves that U_A is nonempty. The fact that U_A is an open set follows from the discreteness of Γ. q.e.d.

Let us now show that U_A and U_B are disjoint.

Lemma 11.7. *The sets U_A and U_B are disjoint.*

Proof. It is sufficient to prove that U_{Id}^A and U_{Id}^B are disjoint. We already know that $U_{Id}^A = U_{Id}$ or $U_{Id}^A = V_{Id}$ and the same is true for U_{Id}^B. We can assume that $U_{Id}^A = U_{Id}$ and we then have to show that $U_{Id}^B = V_{Id}$. But this follows from Lemma 11.5, second part of (i). q.e.d.

Corollary 11.8. *The sets U_A and U_B are two disjoint connected components of $\tilde{X} \setminus \Gamma\mathbb{H}^{n-1}$.*

Proof. The sets U_A and U_B are non-empty disjoint open convex sets in $\tilde{X}$ by Lemma 11.6 and Lemma 11.7. Moreover, they are contained in $\tilde{X} \setminus \Gamma\mathbb{H}^{n-1}$ as shown in Lemma 11.5 (ii) and their boundaries are contained in $\Gamma\mathbb{H}^{n-1}$. q.e.d.

Lemma 11.9. *The closures of U_A and U_B intersect along $\mathbb{H}^{n-1}$, $\bar{A} \cap \bar{B} = \bar{C}$, $\bar{A} = A\bar{C}$ and $\bar{B} = B\bar{C}$. Moreover, for any $\gamma \in \Gamma$, then $\gamma U_A \cap U_B = \emptyset$.*

Proof. The convex set U_A is the intersection of open half spaces U_γ, $\gamma \in \bar{A}$, and is delimited by the disjoint union of hyperplanes $\gamma\mathbb{H}^{n-1}$, for some $\gamma \in \bar{A}$. The same is true for U_B, and as $U_A \cap U_B = \emptyset$, the closures of U_A and U_B can intersect only along one of the connected components of their boundaries, thus along $\mathbb{H}^{n-1}$, which is obviously in both closures. This proves the first part of the lemma; let us prove the second part. By Lemma 11.5, (ii), $\bar{C} \subset \bar{A} \cap \bar{B}$. Conversely, let us take $\gamma \in \bar{A} \cap \bar{B}$. Then γ preserves the closures of U_A and U_B, thus it preserves their intersection $\mathbb{H}^{n-1}$, and therefore $\gamma \in \bar{C}$. The fact that $\bar{A} = A\bar{C}$ follows then from Lemma 11.4.

Let us prove the last part of the lemma. The group Γ acts on the set of connected components of $\tilde{X} \setminus \Gamma\mathbb{H}^{n-1}$, thus for any $\gamma \in \Gamma$ either $\gamma U_A = U_B$ or $\gamma U_A \cap U_B = \emptyset$. Let us argue by contradiction and suppose that $\gamma \in \Gamma$ is such that $\gamma U_A = U_B$. As $\mathbb{H}^{n-1}$ is one component of the boundary $\mathcal{B}$ of U_B, there exists one component $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$, γ' being in $\bar{A}$, such that $\gamma(\gamma'\mathbb{H}^{n-1}) = \mathbb{H}^{n-1}$. Therefore, $\gamma\gamma' \in \bar{C}$, thus $\gamma \in \bar{A}$. The same argument yields $\gamma^{-1} \in \bar{B}$, so $\gamma \in \bar{A} \cap \bar{B} = \bar{C}$ and γ preserves U_A and U_B, which contradicts our choice of γ. q.e.d.

Lemma 11.10. *The Γ-orbit of the closure of $U_A \cup U_B$ covers $\tilde{X}$.*

Proof. Let x be a point in $\tilde{X}$. If $x \in \gamma\mathbb{H}^{n-1}$, $\gamma \in \Gamma$, then x is in the closure of γU_A. If x is in $\tilde{X} \setminus \Gamma\mathbb{H}^{n-1}$, then x belongs to a connected component U of $\tilde{X} \setminus \Gamma\mathbb{H}^{n-1}$. Let $\gamma\mathbb{H}^{n-1}$ be a component of the boundary of U, then $\gamma^{-1}U$ is a connected component of $\tilde{X} \setminus \Gamma\mathbb{H}^{n-1}$ which contains $\mathbb{H}^{n-1}$ in its closure; that is $\gamma^{-1}U = U_A$ or $\gamma^{-1}U = U_B$ by Lemma 11.9. Therefore, $x \in \gamma U_A$ or $x \in \gamma U_B$. q.e.d.

Lemma 11.11. *The stabilizer of U_A [resp. U_B] in Γ is $\bar{A}$ [resp. $\bar{B}$].*

Proof. Let γ be an element of Γ such that $\gamma U_A = U_A$. Then γ globally preserves the boundary $\mathcal{A}$ of U_A, thus in particular $\gamma\mathbb{H}^{n-1} \in \mathcal{A}$ and therefore $\gamma \in \mathcal{A}$. Conversely, let us consider $\gamma \in \bar{A}$. Then, γU_A is a connected component of $\tilde{X}\setminus\Gamma\mathbb{H}^{n-1}$ whose boundary is $\mathcal{A}$ and so contains $\mathbb{H}^{n-1}$. The set γU_A is thus a component containing $\mathbb{H}^{n-1}$ that is either U_A or U_B. Therefore, by Lemma 11.9, $\gamma U_A = U_A$. Hence $\bar{A}$ coincides with the stabilizer of U_A. The same argument shows that $\bar{B}$ coincides with the stabilizer of U_B. q.e.d.

Let us construct a tree $\bar{T}$ embedded in $\tilde{X}$ in the following way: the set of vertices is the set of connected components of $\tilde{X}\setminus\Gamma\mathbb{H}^{n-1}$, and two vertices are joined by an edge if the boundaries of their corresponding connected components intersect nontrivially in $\tilde{X}$. By equivariance and Lemma 11.9 the boundaries of two connected components of $\tilde{X}\setminus\Gamma\mathbb{H}^{n-1}$ intersect along some $\gamma\mathbb{H}^{n-1}$ or are disjoint. Since any two $\gamma\mathbb{H}^n$ separate $\mathbb{H}^n$ in two connected components, one can go from one of these components to the other only across $\gamma\mathbb{H}^n$. Consequently, there are no cycles. By construction Γ acts on $\bar{T}$, the stabilizers of the vertices a and b corresponding to U_A and U_B are $\bar{A}$ and $\bar{B}$ by Lemma 11.11, the stabilizer of the edge between a and b is $\bar{C}$, and a fundamental domain for this action is the segment joining a and b. By [**16**], I, 4, Theorem 6, the group Γ is the amalgamated product of $\bar{A}$ and $\bar{B}$ over $\bar{C}$.

We now claim that $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$.

As A, B and C are subgroups of $\bar{A}$, $\bar{B}$, and $\bar{C}$, the corresponding Mayer-Vietoris sequences of $A *_C B$ and $\bar{A} *_{\bar{C}} \bar{B}$ are related by the following commutative diagram

$$\begin{array}{ccccc} H_n(A,\mathbb{R})\oplus H_n(B,\mathbb{R}) & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(C,\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(\bar{A},\mathbb{R})\oplus H_n(\bar{B},\mathbb{R}) & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C},\mathbb{R}). \end{array}$$

We know that the index $[\bar{C} : C]$ is finite. On the other hand the indices $[\Gamma : A]$ and $[\Gamma : B]$ are infinite by assumption, thus the previous diagram becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(C,\mathbb{R}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C},\mathbb{R}). \end{array}$$

Moreover, the map $H_n(\Gamma,\mathbb{R}) \to H_{n-1}(\bar{C},\mathbb{R})$ is bijective. Namely, the injectivity comes from the above diagram, and the surjectivity from the fact that the hypersurface $\mathbb{H}^{n-1}/\bar{C}$ bounds in $\mathbb{H}^n/\bar{A}$ and $\mathbb{H}^n/\bar{B}$ so that the map $H_{n-1}(\bar{C},\mathbb{R}) \to H_{n-1}(\bar{A},\mathbb{R})\oplus H_{n-1}(\bar{B},\mathbb{R})$ is trivial. Therefore the index $[\bar{C} : C] = 1$, so $\bar{C} = C$, and since $\bar{A} = A\bar{C}$, $\bar{B} = B\bar{C}$ by Lemma 11.9, we obtain $\bar{A} = A$ and $\bar{B} = B$. q.e.d.

12. Proof of the theorems 1.5 and 1.6

The proof of Theorem 1.5 is exactly the same as the proof of Theorem 1.2. The actions of Γ on $\tilde{X}$ and T give rise to a continuous Γ-equivariant map $\tilde{f} : \tilde{X} \to T$. Like in Section 2, we build a hypersurface $\tilde{f}^{-1}(t_0)$ where t_0 is a regular value of $\tilde{f}$ belonging to the interior of an edge. As the edge separates the tree in two unbounded components, Section 2 applies and we get a subgroup C' of C, and a hypersurface $\tilde{Z}' \subset \tilde{X}/C'$ which is essential. Now, if the action of Γ is minimal, every edge separates T into two unbounded components. $\square$

13. Appendix

The goal of this section is to give a proof of Lemma 7.3. This lemma is contained in Lemmas 2.1, 5.1 and 5.2 of [**4**], but our situation being not exactly the same, we reproduce it here for the sake of completeness.

Let us restate Lemma 7.3.

Lemma 13.1. *Let $\mathcal{L} \subset \partial\tilde{X}$ be a closed C'-invariant subset and $\theta_0 \in \mathcal{L}$. We assume that there exists a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(S, \bar{d}, 0)$ is a weak tangent of $(\partial\tilde{X}, d)$. We also assume that there exist positive constants C and ν_0, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \nu_0$ for all $0 \leq i \neq j \leq 2$. Then, $\mathcal{L}$ is homeomorphic to the one point compactification $\hat{S}$ of S. In particular, $\mathcal{L}$ is homeomorphic to $\partial\tilde{X}$.*

We first give a definition of pointed Gromov-Haussdorff convergence which is equivalent to the Definition 7.1. We follow [**4**], paragraph 4.

A sequence of metric spaces (Z_k, d_k, z_k) converges to the metric space $(S, \bar{d}, 0)$ if for every $R > 0$, and every $\epsilon > 0$, there exist an integer N, a subset $D \subset B_S(0, R)$, subsets $D_k \subset B_{Z_k}(z_k, R)$ and bijections $f_k : D_k \to D$ such that for $k \geq N$,

(i) $f_k(z_k) = 0$,
(ii) the set D is ϵ-dense in $B_S(0, R)$, and the sets D_k are ϵ-dense in $B_{Z_k}(z_k, R)$,
(iii) $|d_{Z_k}(x, y) - \bar{d}(f_k(x), f_k(y))| < \epsilon$, where x, y belong to D_k.

Let us describe now Lemmas 2.1 and 5.1 of [**4**] following [**4**]. For a metric space (Z, d) the cross ratio of four points $\{z_i\}$, $i = 1, \ldots, 4$, is the quantity

$$[z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}. \tag{13.1}$$

Given two metric spaces X and Y, a homeomorphism $\phi : [0,\infty) \to [0,\infty)$, and an injective map $f : X \to Y$, we say that f is a ϕ-quasi-Möbius map if for any four points $\{x_i\}$, $i = 1,..,4$, in X, we have

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \le \phi([x_1, x_2, x_3, x_4]). \tag{13.2}$$

For example, any discrete cocompact group of isometries of $\tilde{X}$, where $\tilde{X}$ is a Cartan-Hadamard manifold with sectional curvature $K \le -1$, acts on the ideal boundary $(\partial\tilde{X}, d)$ endowed with the Gromov distance by ϕ-quasi-Möbius transformations for some ϕ.

Lemma 13.2 ([**4**, Lemma 2.1])**.** *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and for any integer k, $g_k : \tilde{D}_k \to Y$ a ϕ-quasi-Möbius map defined on a subset $\tilde{D}_k$ of X. We assume that the Hausdorff distance between $\tilde{D}_k$ and X satisfies*

$$\lim_{k\to\infty} \mathrm{dist}_H(\tilde{D}_k, X) = 0,$$

and that for any integer k, there exist points (x_1^k, x_2^k, x_3^k) in $\tilde{D}_k$ and (y_1^k, y_2^k, y_3^k) in Y, such that $g_k(x_i^k) = y_i^k$ for $i \in \{1,2,3\}$, $d_X(x_i^k, x_j^k) \ge \nu$ and $d_Y(y_i^k, y_j^k) \ge \nu$ for $i, j \in \{1,2,3\}, i \ne j$, where ν is independant of k. Then a subsequence of g_k converges uniformly to a quasi-Möbius map $f : X \to Y$, i.e., $\lim_{k_j\to\infty} \mathrm{dist}_H(g_{k_j}, f|_{\tilde{D}_{k_j}}) = 0$. If, in addition, we suppose that

$$\lim_{k\to\infty} \mathrm{dist}_H(g_k(\tilde{D}_k), Y) = 0,$$

then the sequence $\{g_{k_j}\}$ converges uniformly to a quasi-Möbius homeomorphism $f : X \to Y$.

Before stating the second lemma, let us define a metric space Z to be uniformly perfect if there exists a constant $\lambda \ge 1$ such that for every $z \in Z$ and $0 < R < \mathrm{diam} Z$, we have $B(z, R) - B(z, \frac{R}{\lambda}) \ne \emptyset$. For example, all weak tangents of $(\partial\tilde{X}, d)$ and their one point compactifications are uniformly perfect.

Lemma 13.3 ([**4**, Lemma 5.1])**.** *Let Z be a compact uniformly perfect metric space and G a ϕ-quasi-Möbius action on Z. Suppose that for each integer k we are given a set D_k in a ball $B_k = B(z, R_k) \subset Z$ that is $(\epsilon_k R_k)$-dense in B_k, where $\epsilon_k > 0$, distinct points $x_1^k, x_2^k, x_3^k \in B(z, \lambda_k R_k)$, where $\lambda_k > 0$, with*

$$d_Z(x_i^k, x_j^k) \ge \nu_k R_k$$

for $i, j \in \{1,2,3\}, i \ne j$, where $\nu_k > 0$, and elements $\gamma_k \in G$ such that for $y_i^k := \gamma_k(x_i^k)$ we have,

$$d_Z(y_i^k, y_j^k) \ge \nu'$$

for $i, j \in \{1,2,3\}, i \neq j$, where ν' is independent of k. Let $D'_k = \gamma_k(D_k)$, and suppose that $\lambda_k \to \infty$ when $k \to \infty$, and the sequence $\frac{\epsilon_k}{\nu_k^2}$ is bounded. Then $\lim_{k\to\infty} \mathrm{dist}_H(D'_k, Z) = 0$.

Let us go back to the proof of Lemma 13.1. By definition of convergence, there exist a subsequence of $\{\lambda_k\}$, which we still denote by $\{\lambda_k\}$, subsets $\tilde{D}_k^1 \subset B_S(0,k)$, $D_k^1 \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$, where $\tilde{D}_k^1$ and D_k^1 are $1/2k$-dense subsets of $B_S(0,k)$ and $B_{\lambda_k\mathcal{L}}(\theta_0,k)$, and bijections $f_k : \tilde{D}_k^1 \to D_k^1$ such that for all $x, y \in \tilde{D}_k^1$,

$$|\lambda_k d(f_k(x), f_k(y)) - \bar{d}(x,y)| \leq \frac{1}{2k}\,. \tag{13.3}$$

We choose now $\tilde{D}_k$ a maximal $\frac{1}{k}$-separated subset of $\tilde{D}_k^1$ so that $\tilde{D}_k$ is $2/k$-dense in $B_S(0,k)$ and $D_k = f_k(\tilde{D}_k)$ is $1/2k$-separated and $2/k$-dense in $B_{\lambda_k\mathcal{L}}(\theta_0,k)$; therefore for all x, y in $\tilde{D}_k$ we have

$$\frac{1}{2}\bar{d}(x,y) \leq \lambda_k d(f_k(x), f_k(y)) \leq 2\bar{d}(x,y), \tag{13.4}$$

cf. [**4**], (5.4).

We can suppose that the points $\theta_0^k := \theta_0$, θ_1^k, and θ_2^k in Lemma 13.1 belong to the set D_k. By assumption there exist elements $\gamma_k \in C'$ and a constant ν_0 such that

$$d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \nu_0 \tag{13.5}$$

for all $i, j \in \{0,1,2\}$.

Lemma 13.1 is a direct consequence of Lemma 13.2 applied to $(X, d_X) = (\hat{S}, \hat{d})$ and $(Y, d_Y) = (\mathcal{L}, d)$ and to the sequence of maps $g_k := \gamma_k \circ f_k$, where $\hat{S} = S \cup \{\infty\}$ is the one point compactification of S and $\hat{d}$ the distance on $\hat{S}$ associated to $\bar{d}$ in the following way. Let $h_0 : \hat{S} \to [0, +\infty[$ be the function defined by $h_0(\infty) = 0$ and for all $x \in S$, $h_0(x) = \frac{1}{1+\bar{d}(0,x)}$, where 0 is a fixed point on S. Now for any $x, y \in S$ let $d_0(x,y) = h_0(x)h_0(y)\bar{d}(x,y)$, $d_0(x,\infty) = d_0(\infty,x) = h_0(x)$ and $d_0(\infty,\infty) = 0$. In [**4**], the authors define for all $x, y \in \hat{S}$, $\hat{d}(x,y) = \inf \Sigma_{i=0}^{k-1} d_0(x_i, x_{i+1})$ where the infimum is taken over all finite sequences of points $x_0, \ldots, x_k \in \hat{S}$ with $x_0 = x$ and $x_k = y$ and show that $\hat{d}$ is a distance on $\hat{S}$ inducing the topology of $\hat{S}$ and that the identity map $Id : (S, \bar{d}) \to (S, \hat{d})$ is a ϕ-quasi Möbius map, with $\phi(t) = 16t$, cf. [**4**], Lemma 2.2.

Let us denote x_0^k, x_1^k, x_2^k the points in S such that $f_k(x_i^k) = \theta_i^k$, for $i \in \{0,1,2\}$. We now check that the assumptions of Lemma 13.2 are verified.

The fact that $\lim_{k\to\infty} \mathrm{dist}_H(\tilde{D}_k, \hat{S}) = 0$ is a direct consequence of $\lim_{k\to\infty} \mathrm{dist}_H(\tilde{D}_k, S) = 0$ and the definition of $\hat{d}$. By 13.4, we have

$\bar{d}(x_i^k, x_j^k) \geq \frac{\lambda_k}{2} d(\theta_i^k, \theta_j^k)$, and by assumption we then get

$$\bar{d}(x_i^k, x_j^k) \geq \frac{1}{2C}. \tag{13.6}$$

By 13.5 we then get the separation assumption on triples of points in Lemma 13.2 by choosing $\nu := \ inf\{\nu_0, \frac{1}{2C}\}$.

It remains to check the assumption on $g_k(\tilde{D}_k) = \gamma_k \circ f_k(\tilde{D}_k) = \gamma_k(D_k)$, namely,

$$\lim_{k \to \infty} \mathrm{dist}_H(\gamma_k(D_k), \mathcal{L}) = 0. \tag{13.7}$$

In order to prove Property 13.7, we want to apply Lemma 13.3, but as the set $(\mathcal{L}, d)$ may be not uniformly perfect, we shall replace the uniform perfectness by the fact that $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to a space $(S, \bar{d}, 0)$, which is uniformly perfect, cf.[**4**]. We now show

Lemma 13.4. *We consider the subsets* $\tilde{D}_k \subset B_S(0, k)$ *and* $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$*, where* $\tilde{D}_k$ *and* D_k *are* $1/k$*-dense subsets of* $B_S(0, k)$ *and* $B_{(\mathcal{L}, \lambda_k d)}(\theta_0, k)$*, and the bijections* $f_k : \tilde{D}_k \to D_k$ *coming from the convergence of the sequence of pointed metric spaces* $(\mathcal{L}, \lambda_k d, \theta_0)$ *to* $(S, \bar{d}, 0)$ *where* $(S, \bar{d}, 0)$ *is a weak tangent of* $(\partial \tilde{X}, d)$*. We also assume that there exist positive constants* C *and* ν*, a sequence of points* $\theta_1^k, \theta_2^k \in \mathcal{L}$ *and a sequence of elements* $\gamma_k \in C'$ *such that* $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ *and* $d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \nu$ *for all* $0 \leq i \neq j \leq 2$*. Then, the Hausdorf distance* $\mathrm{dist}_H(\gamma_k D_k, \mathcal{L})$ *tends to* 0 *as* k *tends to infinity.*

Proof. The proof is word-by-word the same as the proof of Lemma 13.3, i.e., Lemma 5.1 (i) of [**4**] with a difference in case 2).

We have $B_{\lambda_k \mathcal{L}}(\theta_0, k) = B_{\mathcal{L}}(\theta_0, \frac{k}{\lambda_k})$ and $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$ a $\frac{1}{k}$-dense subset, for the metric $\lambda_k d$. In term of the distance d, the set D_k is $(\epsilon_k R_k)$-dense in $B_{\mathcal{L}}(\theta_0, R_k)$, where $R_k := \frac{k}{\lambda_k}$ and $\epsilon_k := \frac{1}{k^2}$. By assumption, the points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k$ belong to $B_{\mathcal{L}}(\theta_0, \mu_k R_k)$, and satisfy

$$d(\theta_i^k, \theta_j^k) \geq \nu_k R_k \tag{13.8}$$

where $\nu_k := \frac{1}{Ck}$, and $\mu_k := \frac{C}{k}$. The points $\gamma_k \theta_i^k$ satisfy

$$d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \nu, \tag{13.9}$$

and $\frac{\epsilon_k}{\nu_k^2} = C^2$ is bounded.

Let us consider a point $\theta \in \mathcal{L}$. We want to approximate it by a point of $\gamma_k D_k$. We can write $\theta = \gamma_k \theta_k$, for some $\theta_k \in \mathcal{L}$. There are two cases.

Case 1: For infinitely many indices k, $\theta_k \in B_{\mathcal{L}}(\theta_0, R_k)$. We work in that case with these indices k; thus there are points $\theta'_k \in D_k \cap B_{\mathcal{L}}(\theta_0, R_k)$, with $d(\theta_k, \theta'_k) \leq \epsilon_k R_k$.

Since the distance between the θ_i^k's is bounded below by $\nu_k R_k$, we can find at least two of them which we call a_k and b_k, such that

$$d(\theta_k, b_k) \geq \frac{\nu_k R_k}{2}$$

and

$$d(\theta'_k, a_k) \geq \frac{\nu_k R_k}{2}. \tag{13.10}$$

As C' is contained in the cocompact group Γ, it acts in a quasi-Möbius way on $(\partial\tilde{X}, d)$. Thus,

$$\frac{d(\gamma_k\theta'_k, \gamma_k\theta_k)d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k\theta'_k, \gamma_k b_k)d(\gamma_k a_k, \gamma_k\theta_k)} \leq \phi\left(\frac{d(\theta'_k, \theta_k)d(a_k, b_k)}{d(\theta'_k, b_k)d(\theta_k, a_k)}\right) \tag{13.11}$$

for some homeomorphism $\phi : [0, \infty) \to [0, \infty)$. This implies

$$d(\gamma_k\theta'_k, \gamma_k\theta_k) \leq \frac{(\mathrm{diam}\mathcal{L})^2\phi(8\epsilon_k\mu_k/\nu_k^2)}{\nu}; \tag{13.12}$$

therefore $d(\gamma_k\theta'_k, \gamma_k\theta_k)$ tends to zero as k tends to infinity.

Case 2: For all but finitely many indices k, $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$. We work with the indices k such that $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$. We know that ϵ_k/ν_k^2 is bounded above independently of k, and by assumption, $\nu_k \leq 2\mu_k$.

We claim that there exist $\xi_k \in D_k$ and a positive constant c_0 such that for all k,

$$\frac{d(\xi_k, \theta_0)}{R_k} \geq c_0. \tag{13.13}$$

Let us prove the claim.

On the one hand, as $(\partial\tilde{X}, d)$ is uniformly perfect, so is its weak tangent $(S, \bar{d})$ because the one point compactification $(\hat{S}, \hat{d})$ of $(S, \bar{d})$ is quasi-Möbius homeomorphic to $(\partial\tilde{X}, d)$. Therefore, there exists a constant $C_0 \in [0, 1)$ such that for every $x \in S$ and $R > 0$, we have

$$\bar{B}_{(S,\bar{d})}(0, R) - B_{(S,\bar{d})}(0, C_0 R) \neq \emptyset. \tag{13.14}$$

On the other hand, $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to $(S, \bar{d}, 0)$. After reindexing the sequence $\{\lambda_k\}$, we have for each $\epsilon > 0$ a map $g_k : B_{\lambda_k\mathcal{L}}(\theta_0, k) \to S$ such that

(i) $g_k(\theta_0) = 0$, for any two points θ and θ' in $B_{\lambda_k\mathcal{L}}(\theta_0, k)$,
(ii) $|\bar{d}(g_k(\theta), g_k(\theta')) - \lambda_k d(\theta, \theta')| \leq \epsilon$,
(iii) the ϵ-neighborhood of $g_k(B_{\lambda_k\mathcal{L}}(\theta_0, k))$ contains $B_{(S,\bar{d})}(0, k - \epsilon)$.

By (iii), we have

$$\bar{B}_{(S,\bar{d})}(0, k - \epsilon) \subset \mathcal{U}_\epsilon^{(S,\bar{d})} g_k(\bar{B}_{\lambda_k\mathcal{L}}(\theta_0, k)). \tag{13.15}$$

By 13.14 there exists $y_k \in \bar{B}_{(S,\bar{d})}(0, k-\epsilon) - B_{(S,\bar{d})}(0, C_0(k-\epsilon))$, and by 13.15 there exist $\xi'_k \in \bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)$ such that

$$\bar{d}(y_k, g_k(\xi'_k)) \leq \epsilon. \tag{13.16}$$

We now evaluate $d(\xi'_k, \theta_0)$. By the above properties (i), (ii), 13.16 and the triangle inequality we have

$$\begin{aligned} \lambda_k d(\xi'_k, \theta_0) &\geq \bar{d}(g_k(\xi'_k), 0) - \epsilon \\ &\geq \bar{d}(y_k, 0) - \bar{d}(y_k, g_k(\xi'_k)) - \epsilon \\ &\geq C_0(k-\epsilon) - 2\epsilon. \end{aligned} \tag{13.17}$$

As D_k is $\epsilon_k R_k$-dense in $B_{(\mathcal{L},d)}(\theta_0, k/\lambda_k)$, there exist $\xi_k \in D_k$ such that $d(\xi_k, \xi'_k) \leq \epsilon_k R_k = \frac{k\epsilon_k}{\lambda_k}$.

Let us denote $c_0 = C_0/2$. For k large enough we have

$$\frac{C_0(k-\epsilon) - 2\epsilon - k\epsilon_k}{\lambda_k} \geq \frac{c_0 k}{\lambda_k},$$

and therefore by 13.17 we get

$$d(\xi_k, \theta_0) \geq d(\xi'_k, \theta_0) - d(\xi'_k, \xi_k) \geq c_0 R_k, \tag{13.18}$$

which proves the claim.

We can assume that for k large enough, $\mu_k < c_0/2 < 1/2$. We choose $a_k = \theta_1^k$ and $b_k = \theta_2^k$, and we get

(13.19)

$$\begin{aligned} \frac{d(\gamma_k\xi_k, \gamma_k\theta_k)d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k\xi_k, \gamma_k b_k)d(\gamma_k a_k, \gamma_k\theta_k)} &\leq \phi\left(\frac{d(\xi_k, \theta_k)d(a_k, b_k)}{d(\xi_k, b_k)d(\theta_k, a_k)}\right) \\ &\leq \phi\left(\frac{(d(\xi_k, \theta_0) + d(\theta_0, \theta_k))\mu_k R_k}{R_k(c_0 - \mu_k)(d(\theta_k, \theta_0) - \mu_k R_k))}\right) \\ &\leq \phi\left(\frac{2\mu_k d(\theta_k, \theta_0)}{(c_0 - \mu_k)(d(\theta_k, \theta_0) - \mu_k R_k))}\right) \\ &\leq \phi(8\mu_k/c_0), \end{aligned}$$

where the third inequality follows from $d(\theta_0, \xi_k) \leq d(\theta_0, \theta_k)$ since $d(\theta_0, \theta_k) \geq R_k$ by assumption, and the last inequality from the fact that for k large enough, $\mu_k R_k \leq \frac{d(\theta_k, \theta_0)}{2}$.

We get

$$d(\gamma_k\theta_k, \gamma_k\xi_k) \leq (\mathrm{diam}\mathcal{L})^2 \phi(8\mu_k/c_0)/\nu.$$

q.e.d.

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