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DELIGNE PAIRINGS AND THE KNUDSEN-MUMFORD EXPANSION

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Abstract

Let $X \to B$ be a proper flat morphism between smooth quasi-projective varieties of relative dimension n, and $L \to X$ a line bundle which is ample on the fibers. We establish formulas for the first two terms in the Knudsen-Mumford expansion for $\det(\pi_* L^k)$ in terms of Deligne pairings of L and the relative canonical bundle K. This generalizes the theorem of Deligne [**1**], which holds for families of relative dimension one. As a corollary, we show that when X is smooth, the line bundle η associated to $X \to B$, which was introduced in Phong-Sturm [**12**], coincides with the CM bundle defined by Paul-Tian [**10**, **11**]. In a second and third corollaries, we establish asymptotics for the K-energy along Bergman rays generalizing the formulas obtained in [**11**].

1. Introduction

Let $\pi : X \to B$ be a flat proper morphism of integral schemes with constant relative dimension n, and let $L \to X$ be a relatively ample line bundle. The theorem of Knudsen-Mumford [**6**] says that there exist functorially defined line bundles $\lambda_j = \lambda_j(X, L, B) \to B$ with the property:

$$\det\, \pi_*(L^k) \;\approx\; \lambda_{n+1}^{\binom{k}{n+1}} \otimes \lambda_n^{\binom{k}{n}} \otimes \cdots \otimes \lambda_0 \quad \text{for } k \gg 0. \tag{1.1}$$

In the case $n = 1$, Deligne [**1**] showed that $\lambda_2(L, X, B) = \langle L, L \rangle_{X/B}$, the Deligne pairing of L with itself. If in addition the varieties X and B are smooth, Deligne proved that $\lambda_1(L, X, B)^2 = \langle LK^{-1}, L \rangle_{X/B}$, where $K = K_{X/B} = K_X \otimes K_B^{-1}$ is the relative canonical line bundle. Our first result provides a generalization of these formulas to the case where $n \geq 0$:

Theorem 1. *Let $\pi : X \to B$ be a proper flat morphism of integral schemes of relative dimension $n \geq 0$ and let $L \to X$ be a line bundle which is very ample on the fibers.*

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1) *There is a canonical functorial isomorphism*

$$\lambda_{n+1}(L, X, B) = \langle L, \ldots, L\rangle_{X/B}. \tag{1.2}$$

2) *If X and B are smooth, and K is the relative canonical line bundle of $X \to B$, then there is a canonical functorial isomorphism*

$$\lambda_n^2(L, X, B) = \langle L^n K^{-1}, L, \ldots, L\rangle_{X/B}, \tag{1.3}$$

where the right sides of (1.2) *and* (1.3) *are Deligne pairings of $n+1$ line bundles.*

Remark 1. Knudsen-Mumford show that the leading term $\lambda_{n+1}(L, X, B)$ is equal to the Chow bundle, so (1.2) follows immediately by combining their result with the theorem of Zhang [**24**], which gives a formula for the Chow bundle in terms of the Deligne pairing $\langle L, \ldots, L\rangle$. Thus the main content of Theorem 1 is the isomorphism (1.3).

Remark 2. We need to make precise the meaning of "functorial" in statements (1.1), (1.2), and (1.3). This will be done in §4.

Remark 3. The proof of Theorem 1 generalizes to the case, where $X \to B$ is a relative complete intersection, but for simplicity of exposition, we restrict to the smooth setting.

Remark 4. If the base B is compact, then it is easy to see that both sides of (1.2) and (1.3) have the same Chern class. But in the statement of Theorem 1, the base B is not assumed to be compact. This means that we can not use a Chern class argument and that we must work directly with the sections of the relevant line bundles. Allowing B to be non-compact is important for applications to Kähler geometry, where cases of interest are test configurations $X \to B$ for which the base B is the complex plane.

We shall give proofs of (1.1), (1.2) and (1.3), which use Bertini's theorem and go by induction on the dimension n. The proofs are self-contained, and do not rely on the $n = 1$ result of Deligne, or the results of Knudsen-Mumford [**6**] and Zhang [**24**].

Next we describe two applications of Theorem 1. The first says that the line bundle η, which was introduced in [**12**], coincides with the CM line bundle η_{CM}, which was defined by Paul-Tian [**10**, **11**]. In order to state the precise result, we first recall the necessary definitions.

Let $\pi : X \to B$ be a flat proper map of smooth quasi-projective varieties of relative dimension n, and let $L \to X$ be a line bundle, which is relatively ample on the fibers. Let $p(k) = a_0 k^n + a_1 k^{n-1} + \cdots$ be the Hilbert polynomial of the fibers of π, and let $\mu = \frac{2a_1}{a_0} = \frac{nc_1(X)\cdot c_1(L)^{n-1}}{c_1(L)^n}$. Let $K = K_{X/B}$ be the relative canonical bundle of $X \to B$.

The line bundle $\eta \to B$ was introduced in [**12**] and it is defined as follows:

$$\eta(L, X) = \langle L, \dots, L\rangle^{\mu} \otimes \langle K, L, \dots, L\rangle^{(n+1)}, \tag{1.4}$$

where in each of the two Deligne pairings there is a total of $n+1$ line bundles. In [**12**] it is proved that the line bundle η has a metric given by the Mabuchi K-energy. In [**13**] this point of view was developed and applied to the calculation of the K-energy of a complete intersection, thus providing a non-linear generalization of the Futaki invariant formulas discovered by Lu [**8**] and Yotov [**22**] (see also the related work of Hou [**5**], Liu [**7**], and Yotov [**21**]).

Remark 5. We learned recently that the line bundle η had also been deduced from Riemann-Roch by Shou-Wu Zhang in a 1993 letter to P. Deligne [**23**].

The line bundle $\eta_{\mathrm{CM}} \to B$ was introduced in [**10**] and it is defined as follows:

$$\eta_{\mathrm{CM}}(L, X) = \lambda_{n+1}(L, X, B)^{\mu} \otimes \left(\frac{\lambda_{n+1}(L, X, B)^n}{\lambda_n(L, X, B)^2}\right)^{(n+1)}. \tag{1.5}$$

This extends to a bundle on the Hilbert scheme whose weights are the Donaldson-Futaki invariants, which were defined in [**3**].

Corollary 1. *Let $\pi : X \to B$ be a flat proper map of smooth quasi-projective varieties and $L \to X$ be a line bundle, which is very ample on the fibers. Then there is a canonical functorial isomorphism $\eta(L, X) \to \eta_{\mathrm{CM}}(L, X)$.*

Before stating the next corollary, we need to recall some background from Kähler geometry (full definitions will be provided in §6): Let X_1 be a compact complex manifold and $L_1 \to X_1$ be an ample line bundle. Then Donaldson [**3**] defines a test configuration T for (X_1, L_1) to be a triple $L \to X \to \mathbf{C}$ plus a homomorphism $\rho : \mathbf{C}^{\times} \to \mathrm{Aut}(L, X, \mathbf{C})$, where X is a scheme, $L \to X$ is a $\mathbf{C}^{\times}$ equivariant line bundle, $\pi : X \to \mathbf{C}$ is flat and $\mathbf{C}^{\times}$ equivariant, $\pi^{-1}(1) = X_1$ and $L|_{X_1} = L_1$. Donaldson associates to T a rational number $F(T)$ which is called the Futaki invariant of T, and which is defined by

$$\rho(\tau)|_{\eta_{\mathrm{CM}}(L,X)_0} = \tau^{-F(T)},$$

where $\eta_{\mathrm{CM}}(L, X)_0$ is the fiber of $\eta_{\mathrm{CM}}(L, X) \to \mathbf{C}$ at the origin. This invariant generalizes the invariant defined by Tian [**18**] in the case where the central fiber is normal. We say that (X_1, L_1) is K-stable if $F(T) \leq 0$ for all test configurations T, with equality if and only if T is a product. The conjecture of Yau [**19**], Tian [**18**] and Donaldson [**3**] says that X_1 has a metric of constant scalar curvature in $c_1(L_1)$ if and only if the pair (X_1, L_1) is K-stable.

Now assume that $L \to X \to \mathbf{C}$ is a test configuration with X smooth. Let ω be a Kähler metric on X, and let $\omega_t = \omega|_{X_t}$ where for $t \in \mathbf{C}^\times$, we put $X_t = \pi^{-1}(t)$. Let $d = \int_{X_1} \omega_1^n$ and define, as in [**18**], the function

$$\psi(t) = \frac{1}{d}\int_{X_t} \log\left[\frac{\omega^n \wedge (d\pi \wedge d\bar{\pi})}{\omega^{n+1}}\right] \omega_t^n. \tag{1.6}$$

Here we view $\pi : X \to \mathbf{C}$ as a holomorphic function, so that $d\pi$ is a 1-form on X. The expression $f = \frac{\omega^n \wedge (d\pi\wedge d\bar{\pi})}{\omega^{n+1}}$ is the ratio of two $(n+1, n+1)$ forms on X whose denominator is strictly positive, and thus f is a non-negative smooth function on X. Then $\psi : \mathbf{C}^\times \to \mathbf{R}$ is smooth and bounded above, as $t \to 0$. Let $\nu(t) = \nu(\omega_1, \rho(t)^*\omega_t)$ be the K-energy of $\rho(t)^*\omega_t$ with respect to the base point ω_1. Then we have the following generalization of the formula proved in [**11**]:

Corollary 2. *Let $L \to X \to \mathbf{C}$ be a test configuration with X smooth, and let $\omega \in c_1(L)$ be a* Kähler *metric on X. Then*

$$\nu(t) - \psi(t) = \frac{F(T)}{(n+1)d}\log|t|^2 + O(1). \tag{1.7}$$

Hence, if the central fiber of X has no component of multiplicity greater than one, then

$$\lim_{t\to 0} \frac{\nu(t)}{\log|t|^2} = \frac{F(T)}{(n+1)d}. \tag{1.8}$$

This result was obtained in [**11**] under the following additional assumption: There is a triple $(\mathcal{L}, \mathcal{X}, \mathcal{B})$ with $\mathcal{X} \to \mathcal{B}$ a flat map between smooth projective varieties, $\mathcal{L} \to \mathcal{X}$ relatively very ample, $\mathbf{P}(\pi_*\mathcal{L}) \approx B \times \mathbf{P}^N$, $(X, L) \approx (\mathcal{X}_b, \mathcal{O}_{\mathbf{P}^N}(1))$ for some $b \in \mathcal{B}$, with the property: There is an action of $SL(N+1, \mathbf{C})$ on the data commuting with all the projections such that ρ is the restriction of a one parameter subgroup of $SL(N+1, \mathbf{C})$. The purpose of Corollary 2 is to remove this assumption.

More generally, suppose $L \to X \to \mathbf{C}$ is a test configuration T and $L' \to X' \to B$ a flat family satisfying the hypothesis of Corollary 1. Suppose that there is an imbedding $\mathbf{C} \subseteq B$ such that T is the restriction of the flat family to $\mathbf{C}$. Thus X' is smooth, but X need not be smooth. Then we have the following generalization of Theorem 1 in [**11**] :

Corollary 3. *Let $\omega \in c_1(L')$ be a* Kähler *metric on X' and η be a* Kähler *metric on B. Define*

$$\psi(t) = \frac{1}{d}\int_{X_t} \log\left[\frac{\omega^n \wedge \pi^*\eta^m}{\omega^{n+m}}\right] \omega_t^n, \tag{1.9}$$

where $n + m$ is the dimension of X. Then

$$\nu(t) - \psi(t) = \frac{F(T)}{(n+1)d}\log|t|^2 + O(1). \tag{1.10}$$

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2. Review of the Deligne pairing

We recall some of the results of Deligne [**1**] and Zhang [**24**]: Let $\pi : X \to B$ be a flat projective morphism of integral schemes of pure relative dimension n. If $L_0, \ldots, L_n$ are line bundles on X, then the Deligne pairing $\langle L_0, \ldots, L_n \rangle(X/B)$ is a line bundle on B. It is locally generated by symbols $\langle s_0, \ldots, s_n \rangle$, where the s_j are rational sections of the L_j whose divisors, (s_j), have empty intersection. The transition functions are determined by the following relation:

$$\langle s_0, \ldots, fs_i, \ldots s_n \rangle = f[Y] \cdot \langle s_0, \ldots, s_n \rangle, \tag{2.1}$$

where f is a rational function, $Y = \cap_{j \neq i}(s_j)$ is flat over B, and $f[Y] = \mathrm{Norm}_{Y/B}(f)$. The fact that (2.1) determines a well defined line bundle follows from the Weil reciprocity formula, which says that if f and g are rational functions on a projective curve such that $(f) \cap (g) = \emptyset$, then

$$f[(g)] = g[(f)], \tag{2.2}$$

where $g[(f)] = \prod_j g(p_j)^{\mu_j}$ if $(f) = \sum_j \mu_j p_j$.

The Deligne pairing satisfies a useful induction formula, which is described as follows: Suppose s_j is a rational section of L_j such that $Y \to B$ is flat, where $Y = (s_j)$. Then the map $\langle s_0|_Y, \ldots, \hat{s}_j|_Y, \ldots, s_n|_Y \rangle(Y/B) \to \langle s_0, \ldots, s_n \rangle(X/B)$ defines an isomorphism

$$\langle L_0|_Y, \ldots, \hat{L}_j|_Y, \ldots, L_n|_Y \rangle(Y/B) \to \langle L_0, \ldots, L_n \rangle(X/B). \tag{2.3}$$

Next let us suppose that for some $i < j$ that $L_i = L_j$, and let $\langle s_0, \ldots, s_n \rangle$ and $\langle t_0, \ldots, t_n \rangle$ be generating sections for $\langle L_0, \ldots, L_n \rangle$ as above. Assume that $s_k = t_k$ for all $k \neq i, j$. Assume also that $s_i = t_j$ and $s_j = t_i$. Then we have the following formula from [**1**]:

$$\langle s_0, \ldots, s_n \rangle = (-1)^d \langle t_0, \ldots, t_n \rangle, \tag{2.4}$$

where $d = \prod_{k \neq i} c_1(L_k)$. In [**1**], the relation (2.4) is only stated for $n = 1$, but the general case follows from this and from (2.3).

Finally, we recall that if h_j is a smooth metric on L_j, then there is an induced metric $\langle h_0, \ldots, h_n \rangle$ on the line bundle $\langle L_0, \ldots, L_n \rangle$. This metric has the following property: Let ϕ be a smooth function on X. Then $h_0 e^{-\phi}$ is a metric on L_0 and

$$\langle h_0 e^{-\phi}, \ldots, h_n \rangle = \langle h_0, \ldots, h_n \rangle \cdot e^{-\psi}, \tag{2.5}$$

where $\psi : B \to \mathbf{C}$ is the function

$$\psi = \int_{X/B} \phi \cdot \omega_1 \wedge \cdots \wedge \omega_n \tag{2.6}$$

and $\omega_j = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_j$ is the curvature of h_j.

3. Bertini's Theorem

Our proofs require a variant of Bertini's theorem to cut down the dimension of our family $X \to B$ so that we obtain a smooth family of smaller relative dimension. We state the version of the theorem that we need and, for the sake of completeness, we supply a proof.

Proposition 1. *Let $X \subseteq \mathbf{P}^N$ be a smooth quasi-projective subvariety and let $Y \subseteq X$ be any subvariety with $Y \neq X$. Let $H_0 \subseteq \mathbf{P}^N$ be a hyperplane such that $H_0 \cap X$ is smooth. Let ℓ be a generic pencil of hyperplanes containing H_0, that is, ℓ is a generic line in $\mathbf{P}^N_* = \{H \subseteq \mathbf{P}^N : H \text{ is a hyperplane}\}$, which passes through $H_0 \in \mathbf{P}^N_*$. Then there exists an open set $U \subseteq X$ such that $Y \subseteq U$ and $H \cap U$ is smooth for all $H \in \ell$.*

Proof. We first recall the proof of the usual Bertini theorem: Let $X \subseteq \mathbf{P}^N$ be a smooth variety of dimension n and let ℓ be a generic pencil of hyperplanes. We wish to show that $H \cap X$ is smooth for all but finitely many $H \in \ell$. To see this, let $F : X \to X \times \mathbf{Gr}(n, N)$ (the set of n planes in $\mathbf{P}^N$) be the map which sends x to $(x, T_x(X))$. Let $p : X \times \mathbf{Gr}(n, N) \to \mathbf{Gr}(n, N)$ be the projection map. Let $Z \subseteq \mathbf{Gr}(n, N) \times \mathbf{P}^N_*$ be the set of pairs (λ, H) such that $\lambda \subseteq H$. Then $\pi_1 : Z \to \mathbf{Gr}(n, N)$ has fibers of dimension $N - n - 1$. Let $\pi_2 : Z \to \mathbf{P}^N_*$ be the projection onto the second factor. Thus $\mathcal{B} = (\pi_2 \pi_1^{-1} pF)(X) \subseteq \mathbf{P}^N_*$ is a constructible set of dimension at most $N - 1$. Moreover, $H \in \mathcal{B} \iff H \cap X$ is not smooth. Thus most hyperplanes (a non-empty Zariski open set) will intersect X along a smooth divisor. On the other hand, a line in $\mathbf{P}^N_*$ (i.e., a pencil of hyperplanes) will, in general, contain a finite number of hyperplanes H for which $H \cap X$ is singular.

Now we prove the proposition: Let $\mathcal{B}_Y = (\pi_2 \pi_1^{-1} pF)(Y) \subseteq \mathbf{P}^N_*$. Then $\mathcal{B}_Y$ is constructible and has dimension at most $N - 2$. Thus, a generic pencil ℓ misses $\mathcal{B}_Y$. Let $\{H_1, \ldots, H_k\} = \{H \in \ell : H \cap X \text{is not smooth}\}$. Let $U_j \subseteq H_j \cap X$ be the set of smooth points. Then $Y \subseteq U_j$. Then $U = \cap U_j$ is the desired open set, proving the proposition.

Next we let $f : X \to B$ be a projective flat morphism between smooth varieties and $L \to X$ a line bundle which is very ample on fibers. Assume that B is affine and fix $b_0 \in B$. Assume as well that $X \subseteq B \times \mathbf{P}^N$ is the imbedding given by the complete linear series of L.

Proposition 2. *There exists $s \in H^0(X,L)$ such that $\{s=0\} \subseteq X$ is smooth and flat over B' where $b_0 \in B' \subseteq B$ is open. Moreover, if $s_1, s_2 \in H^0(X,L)$ are two such sections, then there exists $s' \in H^0(X,L)$ such that $\{ts_1 + (1-t)s' = 0\}$ and $\{ts_2 + (1-t)s' = 0\}$ are smooth and flat over B'' for all $t \in \mathbf{C}$ (where $b_0 \subseteq B'' \subseteq B'$ is open).*

Proof. Let $f : B \to \mathbf{C}^m$ be an imbedding where $f = (f_0, \ldots, f_m)$ and $f_j : B \to \mathbf{C}$ a regular function, and $f_0 = 1$. Let $x_0, \ldots, x_N$ be the homogeneous coordinates on $\mathbf{P}^N$. Then the map $\mathbf{P}^N \times B \to \mathbf{P}^M$ given by $(x,b) \to (f_i x_j)$ is an imbedding which restricts to an imbedding of $X \hookrightarrow \mathbf{P}^M$. Here $M = (m+1)(N+1) - 1$. Thus, if c_{ij} are generic constants, the usual Bertini theorem says that $s = \sum_{ij} c_{ij} f_i x_j = 0$ is a smooth subvariety of X. And this subvariety is flat over B (possibly after shrinking B a little).

Now let $s_1, s_2 \in H^0(X,L)$. Then there exist regular functions $f_0, \ldots, f_m$ with $f_0 = 1$ such that s_i is a $\mathbf{C}$ linear combination of the sections $f_i x_j$ and such that $(x,b) \to (f_i x_j)$ is an imbedding of $\mathbf{P}^N \times B$. The existence of s' now follows from Proposition 1.

4. Formula for the leading term

In this section we prove (1.1) and (1.2). Although the results are not new, a methodology will be developed which will also yield, with suitable adaptations, a proof of (1.3).

Before constructing the isomorphisms of (1.1) and (1.2), we must first make precise the meaning of "functorial": Let B be a scheme and suppose we are given, for all sufficiently large positive integers k, a line bundle $M_k \to B$. Then associated to such a sequence $M^\bullet = (M_k)$, we define, as in [**6**], a new sequence $\Delta M^\bullet$ as follows: $(\Delta M^\bullet)_k = M_k \otimes M_{k-1}^{-1}$.

Now let $\pi : X \to B$ be a flat proper morphism of integral schemes of relative dimension n, and $L \to X$ a line bundle which is relatively ample on the fibers. Then a precise formulation of the Knudsen-Mumford theorem (1.1) is the following:

Theorem 2. *There exists, for $k \gg 0$, an isomorphism*

$$\sigma_k(L,X,B) : \Delta^{(n+2)}\det(\pi_* L^k) \;\to\; \mathcal{O}_B \tag{4.1}$$

with the functorial property: If $\phi : (L', X', B') \to (L, X, B)$ is a cartesian morphism then

$$\sigma_k(L',X',B') \circ \phi^* = \phi^* \circ \sigma_k(L,X,B), \tag{4.2}$$

where ϕ^* denotes the maps $\mathcal{O}_B \to \mathcal{O}_{B'}$ and

$$\Delta^{(n+2)}\det(\pi_* L^\bullet)_k \to \Delta^{(n+2)}\det(\pi'_* {L'}^\bullet)_k$$

induced by ϕ.

Here we say that $\phi = (\phi_2, \phi_1, \phi_0)$ is cartesian if $\phi_0 : B' \to B$ and $\phi_1 : X' \to X$ are morphisms such that $\pi\phi_1 = \phi_0\pi'$, the induced map $X' \to X \times_B B'$ is an isomorphism, and $\phi_2 : L' \to \phi_1^* L$ is an isomorphism of line bundles over B'.

Let us spell out the equivalence between (4.1) and (1.1), which is the usual formulation of the Knudsen-Mumford theorem: Fix $n \geq 0$ and assume that (4.1) holds. For $k \gg 0$, we define

$$\lambda_{n+1}(X, L, k) = \Delta^{(n+1)}\det(\pi_* L^\bullet)_k. \tag{4.3}$$

Then (4.1) defines a family of functorial isomorphisms $\sigma_{k,k'}(n+1) : \lambda_{n+1}(X, L, k) \to \lambda_{n+1}(X, L, k')$. Now fix $k_0 \gg 0$ and define $\lambda_{n+1}(X, L)$ to be $\lambda_{n+1}(X, L, k_0)$. Let

$$\lambda_n(X, L, k) = \Delta^{(n)} \left\{ \det(\pi_* L^\bullet) \otimes \left[\lambda_{n+1}(X, L)^{\binom{\bullet}{n+1}} \right]^{-1} \right\}_k . \tag{4.4}$$

Then $\sigma_{k,k'}(n+1)$ defines functorial isomorphisms

$$\sigma_{k,k'}(n) : \lambda_n(X, L, k) \to \lambda_n(X, L, k').$$

Let $\lambda_n(X, L) = \lambda_n(X, L, k_0)$ and define

$$\lambda_{n-1}(X, L, k) = \Delta^{(n-1)} \left\{ \det(\pi_* L^\bullet) \otimes \left[\lambda_{n+1}(X, L)^{\binom{\bullet}{n+1}} \right]^{-1} \otimes \left[\lambda_n(X, L)^{\binom{\bullet}{n}} \right]^{-1} \right\}_k .$$

Continuing in this fashion, we construct line bundles $\lambda_j(X, L)$ for $0 \leq j \leq n+1$ which satisfy (1.1). Moreover, if $\phi : B' \to B$ is a base change, then the construction of the λ_j provides an isomorphism $\phi^* \lambda_j(X, L) \to \lambda_j(X', L')$ which is compatible with the isomorphism $\phi^* \det(\pi_* L^k) \to \det(\pi'_* L'^k)$.

Next we give a precise formulation of the statement (1.2) which gives the formula for $\lambda_{n+1}(X, L)$ in terms of a Deligne pairing:

Theorem 3. *Let $\pi : X \to B$ be a flat projective morphism of quasi-projective schemes of relative dimension n, and $L \to X$ be a line bundle, which is very ample on the fibers. There exists for $k \gg 0$ an isomorphism*

$$\tau_k(L, X, B) : \Delta^{(n+1)}\det(\pi_* L^\bullet)_k \ \to \ \langle L, \ldots, L \rangle \tag{4.5}$$

with the functorial property: If $\phi : (L', X', B') \to (L, X, B)$ is a cartesian morphism, then

$$\tau_k(L', X', B') \circ \phi^* = \phi^* \circ \tau_k(L, X, B), \tag{4.6}$$

where ϕ^* denotes the isomorphism $\phi^* \langle L, \ldots, L \rangle \to \langle L', \ldots, L' \rangle$ as well as the isomorphism $\phi^* \Delta^{(n+2)}\det(\pi_* L^k) \to \Delta^{(n+2)}\det(\pi'_* L'^k)$ induced by ϕ. In particular, there is a functorial isomorphism

$$\lambda_{n+1}(L, X, B) \ \to \ \langle L, \ldots, L \rangle. \tag{4.7}$$

Note that Theorem 3 implies Theorem 2 upon defining $\sigma_\bullet(L, X, B) = \Delta\tau_\bullet(L, X, B)$.

Finally we give the precise formulation of (1.3):

Theorem 4. *Let $\pi : X \to B$ be a flat projective morphism of smooth quasi-projective varieties of relative dimension n, and $L \to X$ be a line bundle, which is very ample on the fibers. There exists for $k \gg 0$ an isomorphism*

(4.8)
$$\mu_k = \mu_k(L, X, B) : [\Delta^{(n)}\det\pi_*(L^k)]^2 \ \to \ \langle L, \dots L\rangle^{2k}\langle K_{X/B}L^n, \dots, L\rangle^{-1}$$

with the functorial property: If $\phi : (L', X', B') \to (L, X, B)$ is a cartesian morphism, then

$$\mu_k(L', X', B') \circ \phi^* = \phi^* \circ \mu_k(L, X, B). \tag{4.9}$$

In particular, there is a functorial isomorphism

$$\lambda_n^2(L, X, B) = \langle L^n K^{-1}, L, \dots L\rangle_{X/B}. \tag{4.10}$$

Proof of Theorem 3. We will define $\tau_k(L, B)$ first on the level of stalks. Thus, we fix $b_0 \in B$, and we define $\tau(L, B')$, where $b_0 \in B' \subseteq B$ is some small open neighborhood. In order to avoid cumbersome notation, we shall write B instead of B' with the understanding that B has possibly been replaced by a smaller open neighborhood of b_0. The definition we give will depend on some choices, so the main task will be to show that after shrinking B even further, that different sets of choices define the same $\tau_k(L, X, B)$.

We start with the case of relative dimension zero: Fix a section s which generates L (shrinking the base B if necessary). Then multiplication by s defines an isomorphism between L^{k-1} and L^k and thus an isomorphism $\det(\pi_*L^{k-1}) \to \det(\pi_*L^k)$. This provides a nowhere vanishing section ξ of $\Delta\det(\pi_*L^k)$ and the map $\langle s\rangle \mapsto \xi$ defines $\tau_k(L, B)^{-1}$.

Now let n, the relative dimension, be arbitrary. Choose generic sections $s_1, \dots, s_n$ of π_*L and for $0 \le k \le n$, let $X_k \subseteq X$ be the subscheme defined by $s_{k+1} = s_{k+2} = \cdots = s_n = 0$. Thus, $X_n = X$ and applying Proposition 2, we conclude that $X_j \to B$ is a projective flat map between smooth quasi-projective varieties (again, with the understanding that B has been replaced by a smaller open neighborhood of b_0).

Multiplication by s_j defines an exact sequence

$$0 \ \to \ \pi_*L^{k-1}_{X_j} \ \to \ \pi_*L^k_{X_j} \ \to \ \pi_*L^k_{X_{j-1}} \ \to \ 0. \tag{4.11}$$

Taking determinants defines an isomorphism

$$\kappa_{s_j} : \Delta\det(\pi_*L^k_{X_j}) \ \to \ \det(\pi_*L|^k_{X_{j-1}}). \tag{4.12}$$

More precisely, if $t_1, \dots, t_a$ is a basis of $H^0(B, \pi_* L^{k-1}_{X_j})$, choose $u_1, \dots, u_b \in H^0(B, \pi_* L^k_{X_j})$ so that $\{s_j t_1, \dots, s_j t_a, u_1, \dots, u_b\}$ is a basis of $H^0(B, \pi_* L^k_{X_j})$. Then we define

$$\kappa_{s_j}\left((s_j t_1 \wedge \cdots \wedge s_j t_a \wedge u_1 \cdots \wedge u_b) \otimes (t_1 \wedge \cdots \wedge t_a)^{-1}\right) = \tilde{u}_1 \wedge \cdots \wedge \tilde{u}_b,$$

where $\tilde{u}_i = u_i | X_{j-1}$.

Define the isomorphism $\kappa(s_1, \dots, s_n) = \Delta\kappa_{s_1} \circ \Delta^{(2)}\kappa_{s_2} \circ \cdots \circ \Delta^{(n)}\kappa_{s_n}$. Then

$$\kappa(s_1, \dots, s_n) : \Delta^{(n+1)}\det(\pi_* L^k) \ \to \ \Delta\det(\pi_* L^k_{X_0}). \tag{4.13}$$

Now let $\iota_{s_j} : \langle L|_{X_j}, \dots, L|_{X_j}\rangle_{X_j/B} \to \langle L|_{X_{j-1}}, \dots, L|_{X_{j-1}}\rangle_{X_{j-1}/B}$ be the induction isomorphism defined by (2.3). Define the isomorphism

$$\iota(s_1, \dots, s_n) = \iota_{s_1} \circ \cdots \circ \iota_{s_n}.$$

Then

$$\iota(s_1, \dots, s_n) : \langle L, \dots, L\rangle_{X/B} \ \to \ \langle L|_{X_0}\rangle_{X_0/B}. \tag{4.14}$$

Finally, define

$$\tau_k(L, X, B)(s_1, \dots, s_n) = \iota(s_1, \dots, s_n)^{-1} \circ \tau_k(L|_{X_0}, X_0, B) \circ \kappa(s_1, \dots, s_n). \tag{4.15}$$

Lemma 1. *The isomorphism $\tau_k(L, X, B)(s_1, \dots, s_n)$ is independent of the choice of generic sections $s_1, \dots, s_n$.*

Once Lemma 1 is proved, we can define $\tau_k(L, X, B)$ to be given by $\tau_k(L, X, B)(s_1, \dots, s_n)$ for any choice of defining sections. This is a local definition, since B has been replaced by a small open subset of b_0. Now Lemma 1 implies that the local isomorphisms glue together to give a unique isomorphism, which is globally defined and easily seen to satisfy the functorial properties: If $s_1, \dots, s_n$ are general elements of $H^0(L)$ used to define $\tau_k(X, L, B)$, then we can use the pullbacks of these to $H^0(L')$, which are also general, to define $\tau(X', L', B')$. Thus, to prove Theorem 3, it suffices to prove the lemma.

Proof of Lemma 1. First we show that τ_k is independent of the ordering of the sections $s_1, \dots, s_n$. To see this, it suffices to show that τ_k is invariant under a permutation which switches s_{j-1} and s_j for some $j < n$, and fixes all the other sections. Let's verify this for $j = n$ (the general case is similar): Let $\tilde{\kappa}(s_{n-1}, s_n) = \kappa_{s_{n-1}} \circ \Delta\kappa_{s_n}$. Thus we have $\tilde{\kappa}(s_{n-1}, s_n) : \Delta^{(2)}\det(\pi_* L^k) \to \det(\pi_* L^k|_{X_{n-2}})$. From the definition of κ_{s_j}, we easily see that

$$\tilde{\kappa}(s_n, s_{n-1}) = (-1)^{p_{n-1}(k-1)}\tilde{\kappa}(s_{n-1}, s_n), \tag{4.16}$$

where $p = p_n$ is the Hilbert polynomial for $L \to X \to B$ and $p_{j-1}(k) = \Delta p_j(k) = p_j(k) - p_j(k-1)$. Indeed, let E_{k-1} be an ordered basis of

$H^0(X, L^{k-1})$. Let $F_k \subseteq H^0(X, L^k)$ be an ordered set of sections whose restrictions to X_{n-1} form a basis of $H^0(X_{n-1}, L^k)$. Then $(s_{n-1}E_{k-1}, F_k)$ is an ordered basis of $H^0(X, L^k)$ and

$$\kappa_{s_n}\left(\frac{\det(s_n E_{k-1}, F_k)}{\det(E_{k-1})}\right) = \det(F_k|_{X_{n-1}}).$$

We repeat this process replacing X with X_{n-1}: Let $F_{k-1} \subseteq H^0(X, L^{k-1})$ be an ordered set such that $F_{k-1}|_{X_{n-1}} \subseteq H^0(X_{n-1}, L^{k-1})$ is a basis. Let $H_k \subseteq H^0(X, L^k)$ an ordered set such that $(s_{n-1}F_{k-1}|_{X_{n-1}}, H_k|_{X_{n-1}})$ is a basis of $H^0(X_{n-1}, L^k)$. So, starting with F_{k-1} and H_k we define $F_k = (s_{n-1}F_{k-1}, H_k)$. If, in addition, we fix a basis $E_{k-2} \subseteq H^0(X, L^{k-2})$, we can define $E_{k-1} = (s_{n-1}E_{k-2}, F_{k-1})$. Then we obtain

$$\kappa_{s_n}\left(\frac{\det(s_n s_{n-1}E_{k-2}, s_n F_{k-1}, s_{n-1}F_{k-1}, H_k)}{\det(s_{n-1}E_{k-2}, F_{k-1})}\right) = \det(s_{n-1}F_{k-1}|_{X_{n-1}}, H_k|_{X_{n-1}}),$$

$$\kappa_{s_n}\left(\frac{\det(s_n E_{k-2}, F_{k-1})}{\det(E_{k-2})}\right) = \det(F_{k-1}|_{X_{n-1}}).$$

Thus,

$$\Delta\kappa_{s_n}\left(\frac{\det(s_n s_{n-1}E_{k-2}, s_n F_{k-1}, s_{n-1}F_{k-1}, H_k)\det(E_{k-2})}{\det(s_{n-1}E_{k-2}, F_{k-1})\det(s_n E_{k-2}, F_{k-1})}\right) = \frac{\det(s_{n-1}F_{k-1}, H_k)}{\det(F_{k-1})},$$

where to ease the notation, we've omitted the restrictions to X_{n-1}. Applying $\kappa_{s_{n-1}}$ to both sides we obtain that $\det(H_k)$ equals

$$\tilde{\kappa}(s_{n-1}, s_n)\left(\frac{\det(s_n s_{n-1}E_{k-2}, s_n F_{k-1}, s_{n-1}F_{k-1}, H_k)\det(E_{k-2})}{\det(s_{n-1}E_{k-2}, F_{k-1})\det(s_n E_{k-2}, F_{k-1})}\right).$$

If we interchange s_n and s_{n-1} in the ratio

$$\left(\frac{\det(s_n s_{n-1}E_{k-2}, s_n F_{k-1}, s_{n-1}F_{k-1}, H_k)\det(E_{k-2})}{\det(s_{n-1}E_{k-2}, F_{k-1})\det(s_n E_{k-2}, F_{k-1})}\right),$$

the sign changes by a factor of $(-1)^{|F_{k-1}|}$. On the other hand, (4.11) implies that $|F_{k-1}| = p_{n-1}(k-1)$. This proves (4.16).

Next, applying Δ successively, we have

$$\Delta^{(n-1)}\tilde{\kappa}(s_n, s_{n-1}) = (-1)^d \Delta^{(n-1)}\tilde{\kappa}(s_{n-1}, s_n),$$

where $d = \Delta^{(n-1)}p_{n-1}(k-n) = p_0(k-n)$. But $p_0 = c_1(L)^n$ is a constant. Thus $d = c_1(L)^n$. On the other hand, (2.4) implies that permuting s_n and s_{n-1} in $\iota(s_1, \ldots, s_n)$ introduces the same factor of $(-1)^d$. Thus the two factors cancel, and we see that τ_k is independent of the ordering of the s_j.

Now let us fix two choices of sections, $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$. We must show that $\tau_k(L, X, B)(s_1, \ldots, s_n) = \tau_k(L, X, B)(s'_1, \ldots, s'_n)$. Clearly we may assume that $s_i = s'_i$ for all but one index i. Since τ_k is independent of the ordering of the sections, we may assume that $s_j = s'_j$ for all $j \geq 2$. In other words, in the proof of Lemma 1, we may assume that $n = 1$.

If $n = 1$, then

$$\tau_k^{-1}(s_1)(\langle s_1, s_0\rangle) = \frac{\det(E)\det(s_0 s_1 E, s_1 F, s_0 F)}{\det(s_0 E, F)\det(s_1 E, F)}. \tag{4.17}$$

Here E is any basis of $H^0(\pi_* L^{k-2})$ and $F \subseteq H^0(\pi_* L^{k-1})$ is any set of linearly independent elements such that $(s_0 E, F)$, $(s_1 E, F)$ and $(s'_1 E, F)$ each form a basis of $H^0(\pi_* L^{k-1})$. We must show that $\tau_k(s_1)(\langle s_1, s_0\rangle) = \tau_k(s'_1)(\langle s_1, s_0\rangle)$. Since $\langle s'_1, s_0\rangle = f((s_0))\langle s_1, s_0\rangle$ where, $f = s'_1/s_1$ and (s_0) is the divisor of s_0, it suffices to show that $\tau_k(s'_1)(\langle s'_1, s_0\rangle) = f((s_0))\tau_k(s_1)(\langle s_1, s_0\rangle)$. But this follows immediately from (4.17). This completes the proof of Lemma 1 as well as the proofs of Theorem 3 and Theorem 2.

5. Relative dimension zero

In this section we prove Theorem 4 in the case $n = 0$.

Lemma 2. *Let $\pi : Y \to B$ be a finite flat morphism of integral schemes and let $L \to Y$ be a line bundle. Then there is a canonical isomorphism*

$$\det(\pi_* L)^2 = \langle L\rangle^2 \otimes \delta_{Y/B}, \tag{5.1}$$

where $\delta_{Y/B} \subseteq \mathcal{O}_B$ is the discriminant of the extension $Y \to B$.

Proof. This question is local on the base, so we may assume $Y \to B$ is a finite morphism of degree r between affine varieties. Thus $Y = \mathrm{spec}(T)$ and $B = \mathrm{spec}(S)$ where $S \subseteq T$ is a extension of rings such that T is a free S module of rank r.

1) We must prove

$$\det(\pi_* \mathcal{O})^2 = \delta_{Y/B}. \tag{5.2}$$

2) We must also show

$$(\det\, \pi_* L) \otimes (\det\, \pi_* \mathcal{O})^{-1} = \langle L\rangle. \tag{5.3}$$

To see (5.3), let s be a section of L. Then multiplication by s defines a map $\mathcal{O}_Y \to L$ and thus a map $\pi_* \mathcal{O} \to \pi_* L$ and hence a section of $(\det\, \pi_* L) \otimes (\det\, \pi_* \mathcal{O}_Y)^{-1}$. One easily checks that this defines the isomorphism asserted by (5.3).

As for (5.2), let $\theta_1, \ldots, \theta_r \in T$, where r is the degree of $S \subseteq T$. Let $\sigma_1, \ldots, \sigma_r$ be the imbeddings of T into the algebraic closure of the fraction field of S. Then $\det(\sigma_i(\theta_j))^2 \in \delta_{Y/B}$. In fact, $\delta_{Y/B}$ is generated by all such elements, so this map gives the isomorphism (5.2) and Lemma 2 is proved.

Lemma 3. *Let $\pi : Y \to B$ be a finite flat separable morphism of quasiprojective varieties of relative dimension zero, and let $\mathcal{D}_{Y/B} \subseteq \mathcal{O}_Y$ be the sheaf of ideals which annihilates $\Omega_{Y/\mathcal{B}}$, the module of relative differentials. Then $\mathcal{D}_{Y/B}$ is locally free of rank one and*

$$\langle \mathcal{D}_{Y/B} \rangle_{Y/B} = \delta_{Y/B}. \tag{5.4}$$

Proof. We may assume that B and Y are affine: $Y = \mathrm{spec}(T)$ and $\mathcal{B} = \mathrm{spec}(S)$. Then $\mathcal{D} \subseteq T$ is an ideal. Choose $\alpha \in T$ such that $T = S[\alpha]$. The existence of such an α is guaranteed by Nakayama's lemma (perhaps after shrinking the base). Let f be the minimal monic polynomial for α, so that $f(\alpha) = 0$ and $\deg f = d$. Then, as is well known,

$$\Omega_{Y/B} \approx T/(f'(\alpha)). \tag{5.5}$$

Let us briefly recall the proof. Since $S[\alpha] = S[X]/(f)$ we have an exact sequence [**4**]:

$$(f)/(f)^2 \;\to\; \Omega_{S[X]/S} \otimes T \;\to\; \Omega_{T/S} \;\to 0,$$

where the first arrow is the map $u \mapsto du \otimes 1$. The module in the middle is the free T module generated by $dX \otimes 1$. The image of the first map is the free T module generated by $df \otimes 1 = f'(X)dX \otimes 1 = dX \otimes f'(\alpha)$. This proves (5.5).

Now we can define the isomorphism of (5.4): If $f'(\alpha)$ is a generator of $\mathcal{D}$ then we associate to it the basis $\{1, \alpha, \ldots, \alpha^{d-1}\}$ of $p_*\mathcal{O}$ and thus an element of $\det(p_*\mathcal{O})^2$. To see that this is well defined, let $f'(\gamma)$ be another generator. Then, from the definition of the Deligne pairing,

$$\langle f'(\alpha) \rangle = \mathrm{Norm}_{T/S}\left(\frac{f'(\alpha)}{f'(\gamma)}\right) \langle f'(\gamma) \rangle.$$

On the other hand, if M is the matrix defined by

$$(1 \; \alpha \; \cdots \; \alpha^{d-1})M = (1 \; \gamma \; \cdots \gamma^{d-1}),$$

then

$$(\alpha_j^k)M = (\gamma_j^k),$$

where α_j ranges over all the conjugates of α ($j = 1, \ldots, d$) and $k = 0, 1, \ldots, d-1$. Taking the determinant of both sides, and using the Vandermonde determinant formula, we see that

$$\det(M)^2 = \frac{\prod_{i \neq j}(\gamma_i - \gamma_j)}{\prod_{i \neq j}(\alpha_i - \alpha_j)} = \mathrm{Norm}_{T/S}\left(\frac{f'(\gamma)}{f'(\alpha)}\right).$$

Thus the map $\langle f'(\alpha)\rangle \mapsto [\det(\alpha_j^k)]^2$ is a well defined isomorphism from $\langle\mathcal{D}\rangle$ to δ, and this proves Lemma 3.

Lemma 4. *Let $\pi : Y \to B$ be a finite flat separable morphism of smooth quasi-projective varieties. Then $\mathcal{D}_{Y/B} = K_{Y/B}^{-1}$. Thus, if $L \to Y$ is a line bundle, we have a canonical isomorphism:*

$$\det(\pi_* L)^2 \ \to\ \langle L\rangle^2 \otimes \langle K_{Y/B}^{-1}\rangle. \tag{5.6}$$

Replacing L by L^k yields an isomorphism

$$\mu_k(L,Y,B) : \det(\pi_* L^k)^2 \ \to\ \langle L\rangle^{2k} \otimes \langle K_{Y/B}^{-1}\rangle. \tag{5.7}$$

Proof. As before, we may assume $B = \mathrm{spec}(S)$ and $Y = \mathrm{spec}(T)$. We have an exact sequence [**4**]

$$0 \ \to\ \pi^*\Omega_{B/\mathbf{C}} \ \to\ \Omega_{Y/\mathbf{C}} \ \to\ \Omega_{Y/B} \ \to 0,$$

where injectivity of the second arrow follows from the fact that it is a full rank morphism of two vector bundles. According to (5.5), $\Omega_{Y/B} = T/(f'(\alpha))$. Thus $H^0(Y, \Omega_{Y/\mathbf{C}})$ has a basis $\omega_1, \ldots, \omega_m$ with the property: $f'(\alpha)\omega_1, \omega_2, \ldots, \omega_m$ is a basis of $H^0(Y, \pi^*\Omega_{B/\mathbf{C}})$. This shows that $K_{Y/B}^{-1}$ is principal, and generated by $f'(\alpha)$. Since $f'(\alpha)$ also generates $\mathcal{D}_{Y/B}$, Lemma 4 is proved.

6. Arbitrary relative dimension

In this section we complete the proof of Theorem 4. First we define the isomorphism μ_k of (4.8). To do this, we choose sections $s_1, \ldots, s_N$ of L which are in general position.

The adjunction formula implies $(K_{X/B}L^n)|_{X_{n-1}} = K_{X_{n-1}/B}(L|_{X_{n-1}})^{n-1}$. Thus we have an isomorphism:

$$\begin{aligned}\tilde\iota_{s_N} :&\langle L, \ldots L\rangle_{X/B}^{2k}\langle K_{X/B}L^n, \ldots, L\rangle_{X/B}^{-1} \ \to \\ &\langle L, \ldots L\rangle_{X_{n-1}/B}^{2k}\langle K_{X_{n-1}/B}L^{n-1}, \ldots, L\rangle_{X_{n-1}/B}^{-1},\end{aligned} \tag{6.1}$$

where on the left side, there are $n+1$ terms in each pairing and on the right side there are n terms. Continuing in this fashion, we obtain an isomorphism:

(6.2)
$$\begin{aligned}\tilde\iota(s_1, \ldots, s_N) = \tilde\iota_{s_1} \circ \cdots \circ \tilde\iota_{s_N} :&\langle L, \ldots L\rangle_{X/B}^{2k}\langle K_{X/B}L^n, \ldots, L\rangle_{X/B}^{-1} \to \\ &\langle L_Y^k\rangle_{Y/B}^2\langle K_{Y/B}^{-1}\rangle_{Y/B},\end{aligned}$$

where $Y = X_0$.

Define the isomorphism $\tilde\kappa(s_1, \ldots, s_n) = \kappa_{s_1} \circ \Delta\kappa_{s_2} \circ \cdots \circ \Delta^{(n-1)}\kappa_{s_n}$. Then

$$\tilde\kappa(s_1, \ldots, s_n) : \Delta^{(n)}\det(\pi_* L^k) \ \to\ \det(\pi_* L_{X_0}^k). \tag{6.3}$$

Finally, define

$$\mu_k(L,X,B)(s_1,\dots,s_n) = \tilde{\iota}(s_1,\dots,s_n)^{-1}\circ \mu_k(L|_{X_0},X_0,B)\circ \tilde{\kappa}^2(s_1,\dots,s_n). \tag{6.4}$$

As in the proof of Theorem 3, the proof of Theorem 4 follows from the following:

Lemma 5. *The isomorphism $\mu_k(L,X,B)(s_1,\dots,s_n)$ is independent of the choice of generic sections $s_1,\dots,s_n$.*

The first step is to show that μ_k does not change if the sections $s_1,\dots,s_n$ are permuted. The proof is similar to the first step in the proof of Lemma 1, so we omit it.

As before, we are reduced to proving Lemma 5 in the case $n=1$: We shorten the notation by writing $\mu(s)=\mu_k(L,X,B)(s)$. Thus

$$\mu(s):[\Delta\det\, \pi_*(L^k)]^2 \;\to\; \langle L,L\rangle^{2k}\langle K_{X/B}L,L\rangle^{-1} = \langle L^{2k-1}K_{X/B}^{-1},L\rangle.$$

Step 1. If s is a generic section, then for every $t\in \mathbf{C}^\times$ we have

$$\mu(s)=\mu(ts). \tag{6.5}$$

To prove this, we first recall the definition of μ:

$$\mu(s)=\tilde{\iota}(s)^{-1}\circ\mu(L|_{X_0},X_0,B)\circ\tilde{\kappa}(s)^2. \tag{6.6}$$

Here $X_0=\{s=0\}$. We claim that there is an integer p with the property

$$\tilde{\kappa}^2(ts)=t^p\tilde{\kappa}^2(s) \quad\text{and}\quad \tilde{\iota}(ts)=t^p\tilde{\iota}(s). \tag{6.7}$$

If we prove (6.7) then (6.6) implies $\mu(s)=\mu(ts)$. Thus we need only prove (6.7).

To ease the notation, we shall write ι for $\tilde{\iota}$ and κ for $\tilde{\kappa}$. First we examine $\kappa(ts)$. Recall that $\kappa(s):\Delta\det(\pi_*L^k)\to\det(\pi_*L^k_{X_0})$ is defined by

$$\kappa(s)\left(\frac{\det(sF,H)}{\det(F)}\right)=\det(H),$$

where F is an ordered basis of $H^0(X,L^{k-1})$ and $H\subseteq H^0(X,L^k)$ is an ordered set such that (sF,H) is a basis of $H^0(X,L^k)$. Replacing s by ts, we have

$$\kappa(s)\left(\frac{\det(sF,H)}{\det(F)}\right)=\kappa(ts)\left(\frac{\det(tsF,H)}{\det(F)}\right)=\kappa(ts)t^{|F|}\left(\frac{\det(sF,H)}{\det(F)}\right).$$

Thus, $\kappa(s)=t^{|F|}\kappa(ts)$.

Next we calculate $\iota(ts)$. Recall that

$$\iota(s):\langle L^{2k-1}K_{X/B}^{-1},L\rangle \;\to\; \langle L^{2k}K_{X_0/B}^{-1}\rangle.$$

In other words,

$$\iota(s) : \langle L^{1-2k} K_{X/B}, L\rangle^{-1} \;\to\; \langle L^{-2k} K_{X_0/B}\rangle^{-1}.$$

The definition is given as follows:

$$\iota(s)(\langle \omega, s\rangle^{-1}) = \langle \mathrm{Ad}(s)(\omega)\rangle^{-1}, \tag{6.8}$$

where $\mathrm{Ad}(s) : L^{1-2k}K^{-1}_{X/B}|_{X_0} \to L^{-2k}K_{X_0/B}$ is the isomorphism given by the adjunction formula, and is characterized by the formula:

$$\frac{\omega}{s} = \frac{df}{f} \wedge \mathrm{Ad}(s)(\omega),$$

where f is any local defining equation of $X_0 \subseteq X$.

Thus replacing s by ts, we see that

$$\mathrm{Ad}(ts)(\omega) = t^{-1}\mathrm{Ad}(s)(\omega);$$

so, if $\deg(L)$ is the degree of L on a fiber of $X \to B$, we have

$$\langle \mathrm{Ad}(ts)(\omega)\rangle = t^{-\deg(L)}\langle \mathrm{Ad}(s)(\omega)\rangle,$$

using properties of the Deligne pairing. Replacing s by ts in (6.8), we get

$$\iota(ts)(\langle \omega, ts\rangle^{-1}) = \langle \mathrm{Ad}(ts)(\omega)\rangle^{-1} = t^{\deg(L)}\langle \mathrm{Ad}(s)(\omega)\rangle^{-1}. \tag{6.9}$$

On the other hand, the properties of the Deligne pairing imply

$$\langle \omega, ts\rangle^{-1} = t^m \langle \omega, s\rangle^{-1},$$

where m is the degree of $L^{2k-1}K^{-1}$ on a fiber of $X \to B$. Thus we conclude

$$\kappa^2(s) = t^{2|F|}\kappa^2(ts) \quad \text{and} \quad \iota(s) = t^{m-\deg(L)}\iota(ts).$$

To prove (6.7), we must show that $2|F| = m - 1$, that is, we must show

(6.10)

$$2\dim(H^0(X, L^{k-1}) = \deg(L^{2k-1}K^{-1}) - \deg(L) = \deg(L^{2(k-1)}K^{-1}).$$

The Riemann-Roch formula implies $2\dim(H^0(X, L^k)) = \deg(L^{2k}K^{-1})$ for k sufficiently large. This now proves (6.5) and completes Step 1.

To describe Step 2, we first recall that

$$\mu^{-1}(s) : \langle L^{2k-1}K^{-1}_{X/B}, L\rangle \;\to\; [\Delta\det\, \pi_*(L^k)]^2.$$

To prove Lemma 5, we must show that if s, s' are generic sections, then $\mu(s) = \mu(s')$. To do this, we connect s to s' by a line $s_t = (u + t)s' = s + ts'$, and study $\mu(s_t)$ as t varies. But $\mu(s_t)$ is only defined if $Y_t = \{s_t = 0\}$ is smooth and flat over B. Bertini's theorem says that Y_t is smooth and flat over B for *all but finitely many* t. On the other hand, by Proposition 2, if s'' is a generic section, then $\{s + ts'' = 0\}$ and $\{s' + ts'' = 0\}$ are smooth and flat for all $t \in \mathbf{C}$ (possibly after shrinking the base B). Thus we may assume that Y_t is smooth and flat over B for all $t \in \mathbf{C}$ so that $\mu(s_t)$ is well defined for *all* $t \in \mathbf{C}$.

Thus we let $s, s' \in H^0(X, \pi_* L)$ be generic sections with the property: $Y_t = \{s + ts' = 0\}$ is smooth and flat over B for all $t \in \mathbf{C}$. Let $\eta \in H^0(X, \pi_*(L^{2k-1}K^{-1}))$ be such that $(\eta) \cap (s) = (\eta) \cap (s') = \emptyset$. To prove Lemma 5, we must show that

$$\mu^{-1}(s)(\langle \eta, s\rangle) = \mu^{-1}(s')(\langle \eta, s\rangle). \tag{6.11}$$

We write $s = us'$, where u is a rational function such that $(u) \cap (\eta) = \emptyset$. Thus,

$$\langle \eta, s\rangle = u\big[(\eta)\big]\langle \eta, s'\rangle, \tag{6.12}$$

where for $b \in B$,

$$u\big[(\eta)\big](b) = \prod_{q \in (\eta)\cap X_b} u(q). \tag{6.13}$$

Step 2. Let m be the degree of $L^{2k-1}K^{-1}$ on a fiber of $X \to B$. Then we claim that

$$\mu^{-1}(s')(\langle \eta, s'\rangle) = \lim_{t\to\infty} t^{-m}\mu^{-1}(s_t)(\langle \eta, s_t\rangle). \tag{6.14}$$

To prove (6.14), first recall that

$$\mu^{-1}(s) : \langle L^{2k-1}K_{X/B}^{-1}, L\rangle \;\to\; [\Delta\mathrm{det}\ \pi_*(L^k)]^2.$$

We must show that if s, s' are generic, then $\mu(s) = \mu(s')$, which is equivalent to showing that $\mu^{-1}(s) = \mu^{-1}(s')$.

We now return to the proof of (6.14). Recall that $s_t = ts' + s$. Relation (6.5) implies $\mu(s_t) = \mu(s' + \frac{1}{t}s)$. We can thus rewrite (6.14) as follows:

$$\lim_{t\to 0} \mu^{-1}(s' + ts)\langle(\eta, s' + ts\rangle) = \mu^{-1}(s')(\langle \eta, s'\rangle).$$

Thus, to prove (6.14), we must show that for fixed η, the map

$$F(t) = \mu^{-1}(s' + ts)(\langle \eta, s' + ts\rangle) \tag{6.15}$$

is a continuous function of t and a neighborhood of $t = 0$. Here $s, s' \in H^0(B, \pi_* L)$ and $\eta \in H^0(B, \pi_*(L^{2k-1}K_{X/B}^{-1}))$. To do this, we let $\tilde{X} = X \times \mathbf{C}$, $\tilde{B} = X \times \mathbf{C}$ and we let $\tilde{\pi} : \tilde{X} \to \tilde{B}$ be the map $\tilde{\pi}(x, t) = (\pi(x), t)$. We let $p : \tilde{X} \to X$ be the projection map and $\tilde{L} = p^* L$ so that $\tilde{L} \to \tilde{X}$ is a line bundle.

If $\tilde{s} \in H^0(\tilde{B}, \tilde{\pi}_* \tilde{L})$ and $\tilde{\eta} \in H^0(\tilde{B}, \tilde{\pi}_*(\tilde{L}^{2k-1}K_{\tilde{X}/\tilde{B}}^{-1}))$ then $\tilde{\mu}^{-1}(\tilde{s})(\langle \tilde{\eta}, \tilde{s}\rangle)$ is a section of $[\Delta\mathrm{det}\tilde{\pi}_*(\tilde{L}^k)]^2$, that is,

$$\tilde{\mu}^{-1}(\tilde{s})(\langle \tilde{\eta}, \tilde{s}\rangle) : B \times \mathbf{C} \to [\Delta\mathrm{det}\tilde{\pi}_*(\tilde{L}^k)]^2.$$

Now fix $t_0 \in \mathbf{C}$ and let $\sigma_{t_0} : B \to B \times \mathbf{C}$ be the map $\sigma_{t_0}(b) = (b, t_0)$. Then

$$\sigma_{t_0}^*[\Delta\det\tilde{\pi}_*(\tilde{L}^k)]^2 = [\Delta\det\pi_*(L^k)]^2.$$

With these preliminaries, we return to the proof of the continuity of F. Fix s, s' as before and let $\tilde{s} : X \times \mathbf{C} \to L$ be the map $\tilde{s}(x) = s'(x) + ts(x)$. Thus $\tilde{s} \in H^0(\tilde{B}, \tilde{\pi}_*\tilde{L})$, in other words, $\tilde{s}$ is a section of $\tilde{L} \to \tilde{X}$.

It follows from the definition of μ and $\tilde{\mu}$ that for each $t_0 \in \mathbf{C}$ and $b \in B$ we have

$$\mu^{-1}(s' + t_0 s)(\langle \eta, s' + ts \rangle)(b) = \sigma_{t_0}^*[\tilde{\mu}^{-1}(\tilde{s})(\langle \tilde{\eta}, \tilde{s} \rangle)(b, t_0)].$$

Now the right side of this equation is manifestly continuous (in fact, analytic) in the t_0 variable. This shows that $F(t)$ is an analytic function of t, and thus continuous at $t = 0$. This completes Step 2.

Now we can finish the proof of Lemma 5: for $t \in \mathbf{C}$ with $(\eta) \cap (s_t) = \emptyset$, we define $\rho_\eta(b; t) \in \mathbf{C}^\times$ by the formula

$$\mu(s)\mu(s_t)^{-1}(\langle \eta, s_t \rangle) = \rho_\eta(b; t)\langle \eta, s \rangle. \tag{6.16}$$

We define $\rho_\eta(b; t)$ for an arbitrary number $t \in \mathbf{C}$ as follows: Suppose $t_0 \in \mathbf{C}$ is such that $(\eta) \cap (s_{t_0}) \neq \emptyset$. Then choose η^* a generic section of $H^0(B, \pi_*(L^{2k-1}K^{-1}))$. Then

$$\rho_{\eta^*}(b; t)\frac{\eta^*}{\eta}((s)) \cdot \frac{\eta}{\eta^*}((s_t)) = \rho_\eta(b; t).$$

The left side is well defined in a neighborhood of $t = t_0$ and this shows that $\rho_\eta(b; t)$ extends to an analytic function on all of $t \in \mathbf{C}$ which vanishes precisely when $(\eta) \cap (s_t) \neq \emptyset$, that is, precisely when $u(q) = -t$ for some $q \in (\eta)$. Moreover, according to (6.14), $t^{-m}\rho_\eta(b; t)$ has a finite limit as $t \to \infty$. This shows that $\rho_\eta(b; t)$ is a polynomial of degree m. Thus we have

$$\rho_\eta(b; t) = \alpha(b) \prod_{q \in (\eta)} (t + u(q))$$

with $\alpha(b) \neq 0$. Since $\rho_\eta(b; 0) = 1$, we must have $\alpha(b) = u\big[(\eta)\big]^{-1}$. Now (6.16) implies

$$t^{-m}\mu(s_t)^{-1}(\langle \eta, s_t \rangle) = t^{-m}\rho_\eta(b; t)\mu^{-1}(s)\langle \eta, s \rangle. \tag{6.17}$$

Taking the limit as $t \to \infty$ and applying (6.14), we obtain

$$\mu^{-1}(s')(\langle \eta, s' \rangle) = u[(\eta)]^{-1}\mu^{-1}(s)\langle \eta, s \rangle. \tag{6.18}$$

Thus (6.11) follows from (6.12).

7. Asymptotics of the Mabuchi K-energy

In this section we prove Corollary 2 (the proof of Corollary 3 is exactly the same, so we omit it). First, we recall some properties of the Deligne metric: Let $\pi : X \to B$ be a flat projective morphism between smooth quasi-projective varieties. Let n be the relative dimension of π, and for $0 \le j \le n$, let h_j be a smooth metric on L_j. Let $\langle h_0, \dots, h_n \rangle$ be the Deligne metric on $\langle L_0, \dots, L_n \rangle$ as defined by Deligne [**1**] and Zhang [**24**]. This metric is continuous by the result of Moriwaki [**9**] and it satisfies the following change of metric property [**13**]:

$$\langle h_0, \dots, h_{n-1}, h_n e^{-\phi} \rangle = \langle h_0, \dots, h_n \rangle e^{-\Phi}, \tag{7.1}$$

where

$$\Phi = \int_{X/B} \phi \cdot \prod_{k<n} c_1'(h_k) \tag{7.2}$$

and $c_1'(h_k) = -\frac{i}{2\pi}\partial\bar\partial \log h_k$.

Next we recall the formula for η:

$$\eta(L, X) = \langle L, \dots, L \rangle^{\mu} \otimes \langle K, L, \dots, L \rangle^{(n+1)}. \tag{7.3}$$

If h is a metric on L with positive curvature ω, then $\frac{1}{\omega^n}$ is a smooth metric on K and thus we obtain a Deligne metric

$$\eta(h) = \langle h, \dots, h \rangle^{\mu} \otimes \langle \frac{1}{\omega^n}, h, \dots, h \rangle^{(n+1)}, \tag{7.4}$$

which is smooth on $X' \subseteq X$, the union of all the smooth fibers.

The key property of the metric $\eta(h)$ is the following transformation rule [**12**, **13**]: Let $\omega' = \omega + \frac{\sqrt{-1}}{2\pi}\phi$ be a Kähler metric in the same Kähler class as ω. Then

$$\eta(he^{-\phi}) = \eta(h) e^{-\tilde\nu(\omega,\omega')}, \tag{7.5}$$

where $\tilde\nu(\omega,\omega') = d(n+1)\nu(\omega,\omega')$, d is the degree of L on a smooth fiber, and $\nu(\omega,\omega')$ is the K-energy of ω' with respect to ω. In fact, one may use (7.5) to define $\nu(\omega,\omega')$. Since the map $\rho(t) : (L_1, h_1 e^{-\phi}) \to (L_t, h_t)$ is an isometry (this is the definition of ϕ) we conclude that for any non-zero section s_1 of $\eta(L, X)$,

$$\nu(t) = \nu(\omega_1, \rho(t)^*\omega_t) = \frac{1}{d(n+1)} \log \frac{\|\rho(t)s_1\|^2}{\|s_1\|^2}, \tag{7.6}$$

where $\|\cdot\|$ is the metric defined by $\eta(h)$.

Proof of Corollary 2. Let

$$\eta^*(h) = \langle h, \dots, h \rangle^{\mu} \otimes \langle \frac{\pi^*(dt \wedge d\bar t)}{\omega^{n+1}}, h, \dots, h \rangle^{(n+1)}. \tag{7.7}$$

Then $\eta^*(h)$ is a continuous metric on all of $\eta(L, X)$. Moreover, (7.1) implies

$$\eta^*(h) = \eta(h) e^{-d(n+1)\psi}. \tag{7.8}$$

Combining this with (7.5), we see that if s_1 is a non-zero element of L_1, then

$$\frac{1}{d(n+1)} \cdot \log \frac{\|\rho(t) s_1\|_*^2}{\|s_1\|_*^2} = \nu(t) - \psi(t) + \psi(1), \tag{7.9}$$

where $\| \cdot \|_*$ is the metric defined by $\eta^*(h)$.

To complete the proof of Corollary 2, we need the following:

Lemma 6.

$$\log \frac{\|\rho(t) s_1\|_*}{\|s_1\|_*} = F(T) \log |t| + \epsilon(t), \tag{7.10}$$

where $\epsilon(t)$ is a continuous function in a neighborhood of $t = 0$.

Proof. Let $s : \mathbf{C} \to \eta(L, X)$ be a nowhere vanishing section. Define for $t \neq 0$ and $z \in \mathbf{C}$ the function $f(t, z)$ by the formula

$$\rho(t)(s(z)) = f(t, z) s(tz). \tag{7.11}$$

Then Theorem 1 together with [**10**] implies $f(t, 0) = t^{F(T)}$. Applying $\rho(t')$ to both sides of (7.11), we get

$$\rho(t')\rho(t)(s(z)) = f(t', tz) f(t, z) s(t'tz) = f(t't, z) s(t'tz). \tag{7.12}$$

Thus,

$$f(t, z) = \frac{f(tz, 1)}{f(z, 1)} = \frac{g(tz)}{g(z)} \tag{7.13}$$

for all $z \neq 0$, where $g(t) = f(t, 1)$. Since the right side of (7.13) approaches $t^{F(T)}$ as z approaches zero, we see that g does not have an essential singularity at the origin. Define an integer

$$q \in \mathbf{Z} \tag{7.14}$$

such that $h(t) = t^{-q} g(t)$ is non-zero and holomorphic in a neighborhood of $t = 0$. Then $f(t, z) = t^q h(tz)/h(z)$. This shows that

$$q = F(T) \tag{7.15}$$

and

$$\log \frac{\|\rho(t) s_1\|_*}{\|s_1\|_*} \ = \ F(T) \log |t| \ + \epsilon(t), \tag{7.16}$$

where

$$\epsilon(t) = \log |h(t)/h(1)| + \log \|s(t)\|_* - \log \|s_1\|_*. \tag{7.17}$$

By Moriwaki [**9**], the term $\log \|s(t)\|_*$ is continuous. Since $h(t)$ is holomorphic and non-vanishing near $t = 0$, we conclude that $\epsilon(t)$ is continuous.

Remark 6. It has been pointed out to us by Shou-Wu Zhang that q can be viewed as a non-archimedian Mabuchi functional on the space of test configurations.

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