J. DIFFERENTIAL GEOMETRY
77 (2007) 149-167

KILLING VECTOR FIELDS WITH TWISTOR DERIVATIVE

Andrei Moroianu

Abstract

Motivated by the possible characterization of Sasakian manifolds in terms of twistor forms, we give the complete classification of compact Riemannian manifolds carrying a Killing vector field whose covariant derivative (viewed as a 2–form) is a twistor form.

1. Introduction

The concept of *twistor forms* on Riemannian manifolds was introduced and intensively studied by the Japanese geometers in the '50s. Some decades later, theoretical physicists became interested in these objects, which can be used to define quadratic first integrals of the geodesic equation (cf. Penrose and Walker [**9**]), or to obtain symmetries of field equations (cf. [**2**], [**3**]). More recently, a new impetus in this direction of research was given by the work of Uwe Semmelmann [**10**] (see also [**1**], [**5**], [**6**]).

Roughly speaking, a twistor form on a Riemannian manifold M is a differential p–form u such that one of the three components of its covariant derivative ∇u with respect to the Levi–Civita connection vanishes (the two other components can be identified respectively with the differential du and codifferential δu). If moreover the codifferential δu vanishes, u is called a *Killing form*. For $p = 1$, twistor forms correspond to conformal vector fields and Killing forms correspond to Killing vector fields via the isomorphism between T^*M and TM induced by the metric.

Two basic examples of manifolds carrying twistor forms are the round spheres and Sasakian manifolds, cf. [**10**], Prop. 3.2 and Prop. 3.4. A common feature of these examples is the existence of Killing 1–forms whose exterior derivatives are twistor 2–forms.

Conversely, if ξ is a Killing 1–form *of constant length* with twistor derivative, then it defines a Sasakian structure (see Proposition 2.3 below). It is therefore natural to drop the assumption on the length, and to address the question of classifying all Riemannian manifolds with this property.

Received 11/10/2005.

After some preliminaries on twistor forms in Section 2, we study the behavior of closed twistor 2–forms with respect to the curvature tensor in Section 3. This is used to obtain the following dichotomy in Section 4: if ξ is a Killing 1–form with twistor exterior derivative, then either ξ satisfies a Sasaki–type equation, or its kernel is an integrable distribution on M. The two possibilities are then studied in the last four sections, where in particular new examples of Riemannian manifolds carrying twistor 2–forms are exhibited. A complete classification is obtained in the compact case, cf. Theorem 8.9.

Acknowledgments. It is a pleasure to thank Paul Gauduchon and Christophe Margerin for many enlightening discussions.

2. Preliminaries

Let (M^n, g) be a Riemannian manifold. Throughout this paper vectors and 1–forms as well as endomorphisms of TM and two times covariant tensors are identified via the metric. In the sequel, $\{e_i\}$ will denote a local orthonormal basis of the tangent bundle, parallel at some point. We use Einstein's summation convention whenever subscripts appear twice.

We refer the reader to [**10**] for an extensive introduction to twistor forms. We only recall here their definition and a few basic properties.

Definition 2.1. A p–form u is a *twistor form* if and only if it satisfies the equation

$$\nabla_X u = \frac{1}{p+1} X \lrcorner\, du - \frac{1}{n-p+1} X \wedge \delta u, \tag{1}$$

for all vector fields X, where du denotes the exterior derivative of u and δu its codifferential. If, in addition, u is co–closed ($\delta u = 0$) then u is said to be a *Killing form.*

By taking one more covariant derivative in (1) and summing over an orthonormal basis $X = e_i$ we see that every twistor p–form satisfies

$$\nabla^* \nabla u = \frac{1}{p+1} \delta d u + \frac{1}{n-p+1} d\delta u.$$

Taking $p = 1$ and $\delta u = 0$ in this formula shows that

$$\nabla^* \nabla u = \frac{1}{2} \Delta u, \tag{2}$$

for every Killing 1–form u. For later use we also recall here the usual Bochner formula holding for every 1–form u:

$$\Delta u = \nabla^* \nabla u + \mathrm{Ric}(u). \tag{3}$$

Definition 2.2. A Sasakian structure on M is a Killing vector field ξ of constant length, such that

$$\nabla^2_{X,Y}\xi = k\big(\langle\xi,Y\rangle X - \langle X,Y\rangle\xi\big), \qquad \forall\ X,Y \in TM, \tag{4}$$

for some positive constant k.

Notice that we have extended the usual definition (which assumes $k = 1$ and ξ of unit length) in order to obtain a class of manifolds invariant through constant rescaling.

If we denote by u the 2–form corresponding to the skew–symmetric endomorphism $\nabla\xi$, then (4) is equivalent to

$$\nabla_X u = k\xi \wedge X, \qquad k > 0. \tag{5}$$

In particular, if ξ defines a Sasakian structure, then $d\xi$ is a closed twistor 2–form, a fact which was noticed by U. Semmelmann (cf. [**10**], Prop. 3.4). As a partial converse, we have the following characterization of Sasakian manifolds:

Proposition 2.3. *Let ξ be a Killing vector field of constant length on some Riemannian manifold such that $d\xi$ is a twistor 2–form. Then ξ is either parallel or defines a Sasakian structure on M.*

Proof. We may assume that ξ has unit length. Let us denote by u the covariant derivative of ξ

$$\nabla_X\xi =: u(X), \qquad \forall\ X \in TM. \tag{6}$$

It is a direct consequence of the Kostant formula that u is parallel in the direction of ξ (see Section 4 for details). Since u is a closed twistor form, we have

$$\nabla_X u = \frac{1}{n-1} X \wedge \delta u, \qquad \forall\ X \in TM,$$

whereas for $X = \xi$ we get that δu is collinear to ξ. Since ξ never vanishes, there exists some function f on M such that

$$\nabla_X u = fX \wedge \xi, \qquad \forall\ X \in TM. \tag{7}$$

On the other hand, ξ has unit length so $u(\xi) = 0$. Differentiating this last relation with respect to some arbitrary vector X and using (6) and (7) yields

$$u^2(X) = fX - f\langle X,\xi\rangle\xi, \qquad \forall\ X \in TM, \tag{8}$$

and in particular the square norm of u (as tensor) is

$$\langle u,u\rangle := \langle u(e_i), u(e_i)\rangle = -\langle u^2(e_i), e_i\rangle = (1-n)f.$$

On the other hand, (6) yields for every $X \in TM$

$$\nabla_X(\langle u,u\rangle) = 2f\langle X\wedge\xi, u\rangle = 4fu(X,\xi) = 0.$$

Thus f is a constant, non–positive by (8). If $f = 0$, ξ is parallel, otherwise ξ defines a Sasakian structure by (7). q.e.d.

3. Closed twistor 2–forms

In this section (M^n, g) is a (not necessarily compact) Riemannian manifold of dimension $n > 3$. We start with the following technical result:

Proposition 3.1. *Let u be a closed twistor 2–form, identified with a skew–symmetric endomorphism of TM. Then, for every other skew–symmetric endomorphism ω of TM, one has*

$$(n-2)(R_\omega\circ u-u\circ R_\omega) = (R_u\circ\omega-\omega\circ R_u)+(u\circ\mathrm{Ric}\circ\omega-\omega\circ\mathrm{Ric}\circ u), \tag{9}$$

where R_ω is the skew–symmetric endomorphism of TM defined by

$$R_\omega(X) := \frac{1}{2}R_{e_j,\omega(e_j)}X.$$

Proof. The identification between 2–forms and skew–symmetric endomorphisms is given by the formula

$$u = \frac{1}{2}e_i\wedge u(e_i). \tag{10}$$

Depending on whether u is viewed as a 2–form or as an endomorphism, the induced action of the curvature on it reads

$$R_\omega(u) = R_\omega e_k\wedge u(e_k) \qquad \text{and} \qquad R_\omega(u) = R_\omega\circ u - u\circ R_\omega. \tag{11}$$

Let X and Y be vector fields on M parallel at some point. Differentiating the twistor equation satisfied by u

$$\nabla_Y u = \frac{1}{1-n}Y\wedge\delta u \qquad \forall\ Y\in TM \tag{12}$$

in the direction of X yields

$$\begin{aligned}
\nabla^2_{X,Y}u &= \frac{1}{1-n}Y\wedge\nabla_X\delta u = \frac{1}{n-1}Y\wedge e_j\lrcorner\nabla^2_{X,e_j}u\\
&= \frac{1}{n-1}Y\wedge e_j\lrcorner R_{X,e_j}u + \frac{1}{n-1}Y\wedge e_j\lrcorner\nabla^2_{e_j,X}u\\
&= \frac{1}{n-1}Y\wedge e_j\lrcorner R_{X,e_j}u + \frac{1}{n-1}Y\wedge e_j\lrcorner\nabla_{e_j}\left(\frac{1}{1-n}X\wedge\delta u\right)\\
&= \frac{1}{n-1}Y\wedge e_j\lrcorner R_{X,e_j}u - \frac{1}{(n-1)^2}Y\wedge\nabla_X\delta u\\
&= \frac{1}{n-1}Y\wedge e_j\lrcorner R_{X,e_j}u + \frac{1}{n-1}\nabla^2_{X,Y}u,
\end{aligned}$$

whence

$$\nabla^2_{X,Y}u = \frac{1}{n-2}Y\wedge e_j\lrcorner R_{X,e_j}u. \tag{13}$$

Using the first Bianchi identity we get

$$R_u(X) = \frac{1}{2}R_{e_j,u(e_j)}X = \frac{1}{2}(R_{X,u(e_j)}e_j + R_{e_j,X}u(e_j)) = R_{X,u(e_j)}e_j.$$

This, together with (11) and (13), yields

$$\begin{aligned}(n-2)\nabla^2_{X,Y}u &= Y\wedge e_j \lrcorner R_{X,e_j}e_k\wedge u(e_k)\\ &= g(e_j, R_{X,e_j}e_k)Y\wedge u(e_k) - Y\wedge R_{X,u(e_k)}e_k\\ &= -Y\wedge u(\mathrm{Ric}(X)) - Y\wedge R_u(X).\end{aligned}$$

After skew–symmetrizing in X and Y we get

$$(n-2)R_{X,Y}u = X\wedge u(\mathrm{Ric}(Y)) + X\wedge R_u(Y) - Y\wedge u(\mathrm{Ric}(X)) - Y\wedge R_u(X).$$

Let now ω be some skew–symmetric endomorphism of TM. We take $X = e_i$, $Y = \omega(e_i)$ in the previous equation and sum over i to obtain:

$$\begin{aligned}(n-2)R_\omega(u) &= \frac{1}{2}\big(e_i\wedge u(\mathrm{Ric}(\omega(e_i))) + e_i\wedge R_u(\omega(e_i))\\ &\quad - \omega(e_i)\wedge u(\mathrm{Ric}(e_i)) - \omega(e_i)\wedge R_u(e_i)\big)\\ &= e_i\wedge u(\mathrm{Ric}(\omega(e_i))) + e_i\wedge R_u(\omega(e_i))\\ &= (u\circ\mathrm{Ric}\circ\omega - \omega\circ\mathrm{Ric}\circ u) + (R_u\circ\omega - \omega\circ R_u),\end{aligned}$$

taking into account that for every endomorphism A of TM, the 2–form $e_i\wedge A(e_i)$ corresponds to the skew–symmetric endomorphism $A - {}^tA$ of TM. q.e.d.

Corollary 3.2. *If u is a closed twistor 2–form, the square of the endomorphism corresponding to u commutes with the Ricci tensor:*

$$u^2\circ\mathrm{Ric} = \mathrm{Ric}\circ u^2.$$

Proof. Taking $\omega = u$ in (9) yields

$$(n-3)(R_u\circ u - u\circ R_u) = 0, \tag{14}$$

so u and R_u commute (as we assumed $n > 3$). We then have

$$\begin{aligned}0 &= (n-2)\mathrm{tr}\big(u\circ(R_\omega\circ u - u\circ R_\omega)\big)\\ &\overset{(9)}{=} \mathrm{tr}\big(u\circ R_u\circ\omega - u\circ\omega\circ R_u + u^2\circ\mathrm{Ric}\circ\omega - u\circ\omega\circ\mathrm{Ric}\circ u\big)\\ &\overset{(14)}{=} \mathrm{tr}(u^2\circ\mathrm{Ric}\circ\omega - \mathrm{Ric}\circ u^2\circ\omega)\\ &= -\langle\omega, u^2\circ\mathrm{Ric} - \mathrm{Ric}\circ u^2\rangle.\end{aligned}$$

Since $u^2\circ\mathrm{Ric} - \mathrm{Ric}\circ u^2$ is skew–symmetric and the equality above holds for every skew–symmetric endomorphism ω, the corollary follows. q.e.d.

4. Killing vector fields with twistor derivative

We will use the general results above in the particular setting which interests us. No compactness assumption will be needed in this section.

Let ξ be a Killing vector field on M, and denote by u its covariant derivative:

$$\nabla_X\xi =: u(X). \tag{15}$$

By definition, u is a skew–symmetric tensor, which can be identified with $\frac{1}{2}d\xi$. Taking the covariant derivative in (15) yields

$$\nabla^2_{X,Y}\xi = (\nabla_X u)(Y). \tag{16}$$

This equation, together with the Kostant formula

$$\nabla^2_{X,Y}\xi = R_{X,\xi}Y \tag{17}$$

(which holds for every Killing vector field ξ) shows that

$$\nabla_\xi u = 0. \tag{18}$$

Suppose now, and throughout the remaining part of this article, that the covariant derivative u of ξ is a twistor 2–form. Notice that in contrast to Proposition 2.3, we no longer assume the length of ξ to be constant. Taking $Y = \xi$ in (12) and using (18) yields

$$\xi \wedge \delta u = 0, \tag{19}$$

so δu and ξ are collinear. We denote by f the function defined on the support of ξ satisfying $(1-n)\delta u = f\xi$ (this normalization turns out to be the most convenient one in the computations below). On the support of ξ the twistor equation (12) then reads

$$\nabla_X u = fX \wedge \xi, \qquad \forall\ X \in TM. \tag{20}$$

Recall now the formula

$$(n-2)\nabla^2_{X,Y}u = -Y \wedge u(\mathrm{Ric}(X)) - Y \wedge R_u(X)$$

obtained in the previous section. We take the inner product with Y in this formula and sum over an orthonormal basis $Y = e_i$ to obtain:

$$-(n-2)\nabla_X \delta u = -(n-1)(u(\mathrm{Ric}(X)) + R_u(X)).$$

Taking the scalar product with some vector Y in this equation and symmetrizing the result yields

$$-\frac{n-2}{n-1}(\langle \nabla_X \delta u, Y\rangle + \langle \nabla_Y \delta u, X\rangle) = \langle \mathrm{Ric}(u(X)), Y\rangle + \langle \mathrm{Ric}(u(Y)), X\rangle.$$

If we replace Y by $u(Y)$ in this last equation and use Corollary 3.2, we see that the expression

$$\langle \nabla_X \delta u, u(Y)\rangle + \langle \nabla_{u(Y)}\delta u, X\rangle$$

is symmetric in X and Y, i.e.,

$$\langle \nabla_X \delta u, u(Y)\rangle + \langle \nabla_{u(Y)}\delta u, X\rangle = \langle \nabla_Y \delta u, u(X)\rangle + \langle \nabla_{u(X)}\delta u, Y\rangle. \tag{21}$$

A straightforward calculation taking (20) and (21) into account yields

$$u(\xi) \wedge df + u(df) \wedge \xi = 0. \tag{22}$$

On the other hand we have $u(\xi) = \nabla_\xi \xi = -\frac{1}{2}d(|\xi|^2)$ and

$$X(|u|^2) = 2\langle \nabla_X u, u\rangle = 2f\langle X \wedge \xi, u\rangle = -2f\langle X, u(\xi)\rangle,$$

whence

$$d(|u|^2) = -2fu(\xi) = fd(|\xi|^2). \tag{23}$$

Notice that the norm $|u|$ used here is the norm of u as 2–form, and differs by a factor $\sqrt{2}$ from the norm of u as tensor. More explicitly, $|u|^2 = \frac{1}{2}\langle u(e_i), u(e_i)\rangle$. Taking the exterior derivative in (23) yields

$$0 = df \wedge d(|\xi|^2) = -2df \wedge u(\xi), \tag{24}$$

which, together with (22), leads to

$$u(df) \wedge \xi = 0. \tag{25}$$

The main goal of this section is to show the following:

Proposition 4.1. *Either f is constant on M, or u has rank 2 on M and $\xi \wedge u = 0$.*

Proof. Suppose that f is non–constant. Since the support of ξ (say M_0) is a dense open subset of M, there exists a non–empty connected open subset U of M_0 where df does not vanish. We restrict to U for the computations below. First, (24) shows that $u(\xi)$ is collinear to df, which, together with (25), implies that

$$u^2(\xi) = \alpha\xi, \tag{26}$$

for some function α defined on U.

Differentiating this relation with respect to some vector X and using (15) and (20) yields

$$(X \wedge f\xi)(u(\xi)) + u((X \wedge f\xi)(\xi)) + u^3(X) = \alpha u(X) + X(\alpha)\xi,$$

or equivalently

$$u^3(X) - (f|\xi|^2 + \alpha)u(X) = (X(\alpha) - f\langle X, u(\xi)\rangle)\xi - f\langle X, \xi\rangle u(\xi). \tag{27}$$

In terms of endomorphisms of TM, identified with $(2,0)$–tensors, (27) becomes

$$u^3 - (f|\xi|^2 + \alpha)u = (d\alpha - fu(\xi)) \otimes \xi - f\xi \otimes u(\xi).$$

The left hand side of this relation is clearly skew–symmetric. The symmetric part of the right hand side thus vanishes: $(d\alpha - 2fu(\xi)) \odot \xi = 0$, whence $d\alpha = 2fu(\xi))$ on U. Using (23) we get $d\alpha = -d(|u|^2)$, so

$$\alpha = -|u|^2 + c \tag{28}$$

for some constant c. We now use (27) in order to compute the trace of the symmetric endomorphism u^2 on T_xM for some $x \in U$. It is clear that ξ and $u(\xi)$ are linearly independent eigenvectors of u^2 with eigenvalue α. Let V denote the orthogonal complement of $\{\xi, u(\xi)\}$ in T_xM. For $X \in V$, (27) becomes $u^3(X) - (f|\xi|^2 + \alpha)u(X) = 0$, so the minimal polynomial of the endomorphism $u|_V$ divides the degree 2 polynomial $\lambda(\lambda-(f|\xi|^2+\alpha))$. Thus u^2 has at most 2 different eigenvalues on V: $f|\xi|^2+\alpha$ and 0, with multiplicities denoted by k and $n-k-2$ respectively. We obtain:

$$-2|u|^2 = \mathrm{tr}(u^2) = 2\alpha + k(f|\xi|^2+\alpha) \overset{(28)}{=} 2c - 2|u|^2 + k(f|\xi|^2+\alpha),$$

showing that either $f|\xi|^2+\alpha$ is constant or $k=0$. In the first case we obtain by taking the exterior derivative

$$\begin{aligned} 0 &= d(f|\xi|^2+\alpha) = |\xi|^2 df + f d(|\xi|^2) + d\alpha \\ &\overset{(28)}{=} |\xi|^2 df + f d(|\xi|^2) - d(|u|^2) \overset{(23)}{=} |\xi|^2 df. \end{aligned}$$

This shows that f is constant on U, contradicting the definition of U.

We therefore get $k=0$. This means that the restriction of u to the distribution V vanishes, so

$$\frac{1}{2}d\xi = u = \frac{\xi\wedge u(\xi)}{|\xi|^2} \tag{29}$$

on U. In particular we get

$$\xi\wedge u = 0 \qquad \text{and} \qquad u\wedge u = 0 \ \text{ on } \ U. \tag{30}$$

It remains to show that the equation $\xi\wedge u = 0$ holds on the entire manifold M, not only on the (possibly small) open set U. This is a consequence of the following remark. The covariant derivatives of the 3–form $\xi\wedge u$ and of the 4–form $u\wedge u$ can be computed at every point of M_0 using (15) and (20):

$$\nabla_X(\xi\wedge u) = u(X)\wedge u + \xi\wedge(fX\wedge\xi) = \frac{1}{2}X\lrcorner(u\wedge u)$$

$$\nabla_X(u\wedge u) = 2fX\wedge\xi\wedge u.$$

This can be interpreted by saying that the section $(\xi\wedge u, u\wedge u)$ of $\Lambda^3M_0\oplus\Lambda^4M_0$ is parallel with respect to the covariant derivative D on this bundle defined by

$$D_X(\sigma,\tau) = \left(\nabla_X\sigma - \frac{1}{2}X\lrcorner\tau, \nabla_X\tau - 2fX\wedge\sigma\right).$$

Since a parallel section which vanishes at some point is identically zero, (30) implies that $\xi\wedge u$ vanishes identically on M_0, thus on M because M_0 is dense in M. q.e.d.

Most of the remaining part of this paper is devoted to the study of the two possibilities given by the above proposition.

5. The case where f is constant

In this section we consider the case where the function f defined on the support of ξ is constant, and we assume that M is compact. We then have

Theorem 5.1. *If the covariant derivative $u := \nabla\xi$ of a non–parallel Killing vector field ξ on M satisfies*

$$\nabla_X u = c\xi \wedge X \tag{31}$$

for some constant c, then either ξ defines a Sasakian structure on M, or M is a space form.

Proof. For the reader's convenience we provide here a proof of this rather standard fact. We start by determining the sign of the constant c. From (16), (17) and (31) we obtain

$$R_{X,\xi}Y = \nabla^2_{X,Y}\xi = (\nabla_X u)(Y) = (c\xi \wedge X)(Y) = c(\langle \xi, Y\rangle X - \langle X, Y\rangle \xi).$$

Taking the trace over X and Y in this formula yields

$$\mathrm{Ric}(\xi) = -R_{e_i,\xi}e_i = (n-1)c\xi.$$

Now, the two Weitzenböck formulas (2) and (3) applied to the Killing 1–form ξ read

$$\nabla^*\nabla\xi = \frac{1}{2}\delta d\xi = \frac{1}{2}\Delta\xi \qquad \text{and} \qquad \Delta\xi = \nabla^*\nabla\xi + \mathrm{Ric}(\xi).$$

Thus $\mathrm{Ric}(\xi) = \nabla^*\nabla\xi$ so taking the scalar product with ξ and integrating over M yields

$$(n-1)c|\xi|^2_{L^2} = |\nabla\xi|^2_{L^2}.$$

This shows that c is non–negative, and $c = 0$ if and only if ξ is parallel, a case which is not of interest for us. By rescaling the metric on M if necessary, we can therefore assume that $c = 1$, i.e., ξ satisfies the Sasakian condition (4)

$$\nabla^2_{X,Y}\xi = \langle \xi, Y\rangle X - \langle X, Y\rangle \xi.$$

If the norm of ξ is constant, we are in the presence of a Sasakian structure by Definition 2.2.

Suppose that $\lambda := |\xi|^2$ is non–constant. Then the function λ is a characteristic function of the round sphere. More precisely, the second covariant derivative of the 1–form $d\lambda$ can be computed as follows. Using the relation $\nabla_X\xi = u(X)$ we first get $d\lambda = -2u(\xi)$; therefore (31) gives

$$\nabla_Y d\lambda = -2(\xi \wedge Y)(\xi) - 2u^2(Y).$$

By taking another covariant derivative with respect to some vector X (at a point where Y is assumed to be parallel) we obtain after a straightforward calculation

$$\nabla^2_{X,Y} d\lambda + 2X(\lambda)Y + Y(\lambda)X + d\lambda\langle X, Y\rangle = 0.$$

By a classical theorem of Tanno (cf. [**11**]), if $d\lambda$ does not vanish identically, the sectional curvature of M has to be constant, so M is a finite quotient of the round sphere. q.e.d.

We end up this section by remarking that conversely, every Killing vector field on the round sphere (and all the more on its quotients) satisfies (14). This follows for instance from [**10**], Prop. 3.2. The main idea is that the space of Killing 1–forms (respectively of closed twistor 2–forms) on the sphere coincides with the eigenspace for the least eigenvalue of the Laplace operator on co–closed 1–forms (respectively on closed 2–forms), and the exterior differential defines an isomorphism between these two spaces.

6. The case where $\xi \wedge u = 0$

From now on we suppose that the function f defined by (20) is non–constant. By Proposition 4.1 the 3–form $\xi \wedge d\xi$ vanishes on M, thus the distribution orthogonal to ξ (defined on the support of ξ) is integrable. We start by a local study of the metric, at points where ξ does not vanish.

Proposition 6.1. *Around every point in the support of ξ, the manifold M is locally isometric to a warped product $I \times_\lambda N$ of an open interval I and a $(n-1)$–dimensional manifold N such that the differential of the warping function λ is a twistor 1–form on N.*

Proof. By the integrability theorem of Frobenius, M can be written locally as a product $I \times N$ where $\xi = \frac{\partial}{\partial t}$ and N is a local leaf tangent to the distribution $\xi^\perp$. The metric g can be written

$$g = \lambda^2 dt^2 + h_t$$

for some positive function λ on $I \times N$ and some family of Riemannian metrics h_t on N. Of course, the fact that $\xi = \frac{\partial}{\partial t}$ is Killing just means that λ and h_t do not depend on t, i.e., $g = \lambda^2 dt^2 + h$ is a warped product. The 1–form ζ, metric dual to ξ, is just $\lambda^2 dt$, so $u = \frac{1}{2} d\zeta = \lambda d\lambda \wedge dt$. We now express the fact that u is a twistor form on M in terms of the new data (λ, h). Let X denote a generic vector field on N, identified with the vector field on M projecting over it. Similarly, we will identify 1–forms on N with their pull–back on M. Since the projection $M \to N$ is a Riemannian submersion, these identifications are compatible with the metric isomorphisms between vectors and 1–forms.

The O'Neill formulas (cf. [**8**], p. 206) followed by a straightforward computation give

$$\nabla_{\frac{\partial}{\partial t}} u = 0 \qquad \text{and} \qquad \nabla_X u = \lambda \nabla_X d\lambda \wedge dt, \qquad \forall\ X \in TN,$$

where we denoted by the same symbol ∇ the covariant derivative of the Levi–Civita connection of h on N. Taking the inner product with X

in the second equation and summing over an orthonormal basis of N yields $\delta^M u = \lambda\Delta^N\lambda dt$, so u is a twistor form if and only if

$$\nabla_X d\lambda = -\frac{1}{n-1} X\Delta^N\lambda, \qquad \forall\ X \in TN$$

which just means that $d\lambda$ is a twistor 1–form on N. q.e.d.

We can express the above property of $d\lambda$ by the fact that its metric dual is a *gradient conformal vector field* on N. These objects were intensively studied in the '70s by several authors. In particular Bourguignon [**4**] has shown that a compact manifold carrying a gradient conformal vector field is conformally equivalent to the round sphere. The converse of this result does not hold (i.e., not every conformally flat metric on the sphere carries gradient conformal vector fields, cf. Remark 8.3 below). We study this notion in greater detail in the next section.

7. Gradient conformal vector fields

Definition 7.1. A *gradient conformal vector field* (denoted for convenience GCVF in the remaining part of this paper) on a connected Riemannian manifold (M^n, g) is a conformal vector field X whose dual 1–form is exact: $X = d\lambda$. The function λ (defined up to a constant) is called the *primitive* of X.

Let X be a GCVF. Since X is a gradient vector field, its covariant derivative is a symmetric endomorphism, and the fact that X is conformal just means that the trace–free symmetric part of ∇X vanishes. Thus X satisfies the equation

$$\nabla_Y X = \alpha Y, \qquad \forall\ Y \in TM \tag{32}$$

where $\alpha = -\frac{\delta X}{n}$. In particular, $\mathcal{L}_X g = 2\alpha g$.

In the neighbourhood of every point where X is non–zero, the metric g can be written

$$g = \psi(t)(dt^2 + h) \tag{33}$$

for some positive function ψ. Conversely, if g can be written in this form, then $\frac{\partial}{\partial t}$ is a GCVF whose primitive is Ψ (the primitive of ψ in the usual sense).

We thus see that the existence of a GCVF does not impose hard restrictions on the metric in general. Remarkably, if the GCVF has zeros, the situation is much more rigid:

Proposition 7.2. *Let X be a GCVF on a Riemannian manifold (M^n, g) vanishing at some $x \in M$. Then there exists an open neighbourhood of x in M on which the metric can be expressed in polar coordinates*

$$g = ds^2 + \gamma^2(s) g_{S^{n-1}}, \tag{34}$$

where $g_{S^{n-1}}$ denotes the canonical round metric on S^{n-1} and γ is some positive function $\gamma : (0, \varepsilon) \to \mathbb{R}^+$. The norm of X in these coordinates is a scalar multiple of γ:

$$|X| = c\gamma. \tag{35}$$

Notice that the metric defined by (34) is in particular of type (33), as shown by the change of variable $s(t) := \int_0^t \sqrt{\psi(r)}dr$.

Proof. Let τ be the unit tangent vector field along geodesics passing trough x. From ([**4**], Lemme 4) we have that X is everywhere collinear to τ. Using the Gauss Lemma, we know that the metric g can be expressed as $g = ds^2 + h_s$ in geodesic coordinates on some neighbourhood U of x, where h_s is a family of metrics on S^{n-1} (of course, $\tau = \frac{\partial}{\partial s}$ in these coordinates). Since x is an isolated zero of X (cf. [**4**], Corollaire 1), the norm of X is a smooth function $|X| = \beta$ defined on $U - \{x\}$, and $X = \beta\tau$. We then compute

$$\begin{aligned} \dot{h_s} &= \mathcal{L}_\tau g = \beta^{-1}\mathcal{L}_X g + 2d(\beta^{-1}) \odot X^\flat = 2\alpha\beta^{-1}g - 2\frac{d\beta}{\beta} \odot ds \\ &= 2\alpha\beta^{-1}h_s + 2\alpha\beta^{-1}ds^2 - 2\frac{d\beta}{\beta} \odot ds. \end{aligned}$$

By identification of the corresponding terms in the above equality we obtain the differential system

$$\begin{cases} d\beta = \alpha ds \\ \dot{h_s} = 2\alpha\beta^{-1}h_s. \end{cases}$$

The first equation shows that β only depends on s: $\beta = \beta(s) = \int_0^s \alpha(t)dt$. The second equation yields

$$h_s = \beta^2(s)h \tag{36}$$

for some metric h on S^{n-1}.

We claim that h is (up to a scalar multiple) the canonical round metric on the sphere. To see this, we need to understand the family of metrics h_s on S^{n-1}. We identify (T_xM, g) with $(\mathbb{R}^n, eucl)$ and S^{n-1} is viewed as the unit sphere in T_xM. If V is a tangent vector to S^{n-1} at some $v \in T_xM$, then $h_s(V, V)$ is the square norm with respect to g of the image of V by the homothety of ratio s followed by the differential at v of the exponential map $\exp_x$. In other words,

$$h_s = s^2(\exp_x)^*(g)\Big|_{T_vS^{n-1}}.$$

Since the differential at the origin of the exponential map is the identity, we get

$$\lim_{s\to 0}\frac{h_s}{s^2} = g_{S^{n-1}}.$$

Using this together with (36) shows that $\lim_{s\to 0}\frac{\beta(s)}{s}$ is a positive real number denoted by c and $c^2h = g_{S^{n-1}}$.

We thus have proved that $g = ds^2 + \gamma^2(s)g_{S^{n-1}}$, where $\gamma = \frac{\beta}{c}$. q.e.d.

8. The classification

We turn our attention back to the original question. Recall that ξ is a non–parallel Killing vector field on (M^n, g) such that $\xi \wedge d\xi = 0$ and $d\xi$ is a twistor form. We distinguish two cases, depending on whether ξ vanishes or not on M.

Case I. The vector field ξ has no zero on M. The distribution orthogonal to ξ is then globally well–defined and integrable, its maximal leaves turn out to be compact and can be used in order to obtain a dimensional reduction of our problem.

Definition 8.1. Let N be a Riemannian manifold, let λ be a positive smooth function on N, and let φ be an isometry of N preserving λ (that is, $\lambda \circ \varphi = \lambda$). The quotient of the warped product $\mathbb{R} \times_\lambda N$ by the free $\mathbb{Z}$–action generated by $(t, x) \mapsto (t+1, \varphi(x))$ is called the *warped mapping torus* of φ with respect to λ and is denoted by $N_{\lambda,\varphi}$.

Proposition 8.2. *A compact Riemannian manifold (M^n, g) carries a nowhere vanishing Killing vector field ξ as above if and only if it is isometric to a warped mapping torus $N_{\lambda,\varphi}$ where (N^{n-1}, h) is a compact Riemannian manifold carrying a GCVF with primitive λ and φ is an isometry of N preserving λ.*

Proof. The "if" part follows directly from the local statement given by Proposition 6.1. Suppose, conversely, that (M, ξ) satisfy the conditions above. We denote by φ_t the flow of ξ and by N_x the maximal leaf of of the integrable distribution $\xi^\perp$. Clearly φ_t maps N_x isometrically over $N_{\varphi_t(x)}$. We claim that this action of $\mathbb{R}$ on the space of leaves of $\xi^\perp$ is transitive. Let $x \in M$ be an arbitrary point of M and denote

$$M_x := \bigcup_{t\in\mathbb{R}} N_{\varphi_t(x)}.$$

For every $y \in M_x$ we define a map $\psi : (-\varepsilon, \varepsilon) \times N_y \to M$ by

$$\psi(t, z) := \varphi_t(z).$$

The differential of ψ at $(0, y)$ is clearly invertible, thus the inverse function theorem ensures that the image of ψ contains an open neighbourhood of y in M. On the other hand M_x contains the image of ψ by construction; therefore M_x contains an open neighbourhood of y. Thus M_x is open. For any $x, y \in M$ one either has $M_x = M_y$ or $M_x \cap M_y = \emptyset$. Thus M is a disjoint union of open sets

$$M = \bigcup_{x\in M} M_x$$

so by connectedness we get $M_x = M$ for all x.

Since the norm of ξ is constant along its flow, we deduce that $|\xi|$ attains its maximum and its minimum on each integral leaf N_x. By the main theorem in [**4**], each leaf is conformally diffeomorphic to the round sphere, so in particular it is compact. Reeb's stability theorem then ensures that the space of leaves is a compact 1–dimensional manifold S and the natural projection $M \to S$ is a fibration. Hence S is connected, i.e., $S \cong S^1$. On the other hand we have a group action of $\mathbb{R}$ on S given by $t(N_x) := N_{\varphi_t(x)}$ and S is the quotient of $\mathbb{R}$ by the isotropy group of some point. Since S is a manifold, this isotropy group has to be discrete, therefore is generated by some $t_0 \in \mathbb{R}$. Then clearly M can be identified with the warped mapping torus $N_{\lambda,\varphi}$, where $N := N_x$, $\varphi := \varphi_{t_0}$ and the warping function λ is the restriction to N of $|\xi|$. q.e.d.

Remark 8.3. A compact Riemannian manifold admitting gradient conformal vector fields is completely classified by one single smooth function defined on some closed interval and satisfying some boundary conditions. More precisely, such a manifold is isometric to the Riemannian completion of a cylinder $(0,l) \times S^{n-2}$ with the metric $dt^2 + f(t) g_{S^{n-2}}$, where $f : (0,l) \to \mathbb{R}^+$ is smooth and satisfies the boundary conditions

$$f(t) = t^2(1+t^2a(t^2)) \quad \text{and} \quad f(l-t) = t^2(1+t^2b(t^2)), \quad \forall\, |t| < \varepsilon, \tag{37}$$

for some smooth functions $a, b : (-\varepsilon, \varepsilon) \to \mathbb{R}^+$.

The proof is very similar to that of Theorem 8.6 below and will thus be omitted.

Case II. The vector field ξ has zeros on M. The study of this situation is more involved since the distribution orthogonal to ξ is no longer globally defined. On the other hand one can prove that the orbits of ξ are always closed in this case, which turns out to be crucial for the classification. This follows from a more general statement:

Proposition 8.4. *Let M be a compact Riemannian manifold and let ξ be a Killing vector field on M. If the covariant derivative of ξ has rank* 2 (*as skew–symmetric endomorphism*) *at some point $x \in M$ where ξ vanishes, then ξ is induced by an isometric S^1–action on M, and in particular its orbits are closed.*

Proof. Let Z denote the set of points where ξ vanishes, and let Z_0 be the connected component of Z containing x. It is well–known that Z_0 is a totally geodesic submanifold of M of codimension 2 (equal to the rank of $\nabla\xi$). Moreover, at each point of Z_0, $\nabla\xi$ vanishes on all vectors tangent to Z_0.

Since M is compact, its isometry group G is also compact. The Killing vector field ξ defines an element X of the Lie algebra $\mathfrak{g}$ of G. The exponential map of G sends the line $\mathbb{R}X$ onto a (not necessarily closed) Abelian subgroup of G. Let T be the closure of this subgroup and denote by $\mathfrak{t}$ its Lie algebra. T is clearly a compact torus. We claim that T is actually a circle. If this were not the case, one could find an element $Y \in \mathfrak{t}$ defining a Killing vector field ζ on M non–collinear to ξ. Let y be some point in Z_0. Since by definition $\xi_y = 0$ we get $\exp(tX) \cdot y = y$ for all $t \in \mathbb{R}$, whence $g \cdot y = y$ for all $g \in T$, thus showing that ζ vanishes on Z_0. Since the space of skew–symmetric endomorphisms of T_xM vanishing on T_xZ_0 is one–dimensional, we deduce that $(\nabla\zeta)_x$ is proportional to $(\nabla\xi)_x$. Finally, since a Killing vector field is determined by its 1–jet at some point, and $\xi_x = \zeta_x = 0$, we deduce that ζ is collinear to ξ, a contradiction.

Therefore T is a circle acting isometrically on M and ξ is the Killing vector field induced by this action. q.e.d.

Let M_0 denote as before the set of points where ξ does not vanish. The integrable distribution $\xi^\perp$ is well–defined along M_0 and T acts freely and transitively on its maximal integral leaves. If (N, h) denotes such a maximal integral leaf, Proposition 6.1 shows that M_0 is isometric to the warped product $S^1 \times_\lambda N$, $g = \lambda^2 d\theta^2 + h$, where λ is a positive function on N whose gradient is a conformal vector field X. Since λ is the restriction of the continuous function $|\xi|$ on M, it attains its maximum at some $x \in N$. Of course, X vanishes at x.

We thus may apply Proposition 7.2 to the gradient conformal vector field X on N. The metric on N can be written $h = ds^2 + \gamma^2(s) g_{S^{n-2}}$ on some neighbourhood of x. The length of X, which by (34) is equal to $c\gamma(s)$, only depends on the distance to x. Assume that X vanishes at some point $y := \exp_x(tV)$ (where V is a unit vector in T_xN). Then it vanishes on the whole geodesic sphere of radius t. On the other hand X has only isolated zeros, so the geodesic sphere $S(x,t)$ is reduced to y. This would imply that N is compact, homeomorphic to S^{n-1}, so $M_0 = S^1 \times N$ is compact, too. On the other hand M_0 is open, so by connectedness $M_0 = M$, contradicting the fact that ξ has zeros on M.

This proves that x is the unique zero of X on N. In fact we can now say much more about the global geometry of M. Recall that M is the disjoint union of M_0 and Z, where Z, the nodal set of ξ, is a codimension 2 submanifold and $M_0 = N \times S^1$ is endowed with a warped product metric. In order to state the global result we need the following

Definition 8.5. Let $l > 0$ be a positive real number and let $\gamma, \lambda : (0, l) \to \mathbb{R}^+$ be two smooth functions satisfying the following boundary conditions:

$$\begin{cases} \lim_{s\to 0} \gamma(s) = 0, & \lim_{s\to l} \gamma(s) > 0 \\ \lim_{s\to 0} \lambda(s) > 0, & \lim_{s\to l} \lambda(s) = 0 \end{cases} \tag{38}$$

We view the sphere S^n as the *topological join* of S^{n-2} and S^1, obtained from $[0,l]\times S^{n-2}\times S^1$ by shrinking $\{0\}\times S^{n-2}\times S^1$ to $\{\text{point}\}\times S^1$ and by shrinking $\{l\}\times S^{n-2}\times S^1$ to $\{\text{point}\}\times S^{n-2}$.

Then S^n, endowed with the Riemannian metric

$$g = ds^2+\gamma^2(s)g_{S^{n-2}}+\lambda^2(s)d\theta^2$$

defined on its open submanifold $(0,l)\times S^{n-2}\times S^1$ is called the *Riemannian join* of S^{n-2} and S^1 with respect to γ and λ and is denoted by $S^{n-2}*_{\gamma,\lambda}S^1$.

Notice that the metric g extends to a continuous metric on S^n. We will see below under which circumstances this extension is *smooth*.

Theorem 8.6. *Let N be a maximal leaf of the distribution $\xi^\perp$ of M_0 and let $x\in N$ be the unique zero of the gradient conformal vector field $X=\nabla(|\xi|)$ on N. We then have*

(i) *There exists some positive number l, not depending on N, such that the exponential map at x maps diffeomorphically the open ball $B(0,l)$ in T_xN onto N.*
(ii) *The submanifold Z is connected, isometric to a round sphere S^{n-2}. The closure of each integral leaf N defined above is $\overline{N}=N\cup Z$.*
(iii) *M is isometric to a Riemannian join $S^{n-2}*_{\gamma,\lambda}S^1$, where γ is* (*up to a constant*) *equal to the derivative of λ.*

Proof. (i) Consider the isometric action of S^1 on M induced by ξ. For $\theta\in S^1$ denote by N_θ the image of N through the action of θ on M. Of course, N_θ is itself a maximal integral leaf of $\xi^\perp$. For every unit vector $V\in T_xN$, we define

$$l(x,V):=\sup\{t>0\mid \exp_x(rV)\in N,\ \forall\, r\le t\}.$$

Clearly, $l(x,V)$ is the distance along the geodesic $\exp_x(tV)$ from x to the first point on this geodesic where X vanishes. Of course, the exponential map on N_θ coincides (as long as it is defined) with the exponential map on M since each N_θ is totally geodesic. As noticed before, the norm of X along geodesics issued from x only depends on the parameter along the geodesic, therefore $l(x,V)$ is independent of V and can be denoted by $l(x)$. Since we have a transitive isometric action on the N_θ's, $l(x)$ actually does not depend on x neither, and will be denoted by l. This proves that each N_θ is equal to the image of the open ball $B(0,l)$ in $T_{\theta(x)}N_\theta$ via the exponential $\exp_{\theta(x)}$.

(ii) Let us denote by Z_θ the set $\overline{N_\theta}\setminus N_\theta$. By the above, Z_θ is the image of the round sphere $S(0,l)$ in $T_{\theta(x)}N_\theta\subset T_{\theta(x)}M$ via the exponential map (on M) $\exp_{\theta(x)}$. In particular, each Z_θ is a connected subset of Z, diffeomorphic to S^{n-2}. Every element $\theta'\in S^1$ maps (by continuity) Z_θ to $Z_{\theta'\theta}$ and on the other hand, it preserves Z. We deduce that $Z_\theta=Z_{\theta'}$

for all $\theta, \theta' \in S^1$, and since Z is the union of all Z_θ, we obtain $Z = Z_\theta$. The other assertions are now clear.

(iii) This point is an implicit consequence of the local statements from the previous sections. First, by Proposition 6.1 M_0 is diffeomorphic to $N \times S^1$ with the warped product metric $g = g_N + \lambda^2 d\theta^2$, where λ is a function on N whose gradient X is a GCVF vanishing at x. From Proposition 7.2 and (i) above we see that $N \setminus \{x\}$ is diffeomorphic to $(0, l) \times S^{n-2}$ with the metric $g_N = ds^2 + \gamma^2(s) g_{S^{n-2}}$. If we denote by S the orbit of x under the S^1–action on M defined by ξ, this shows that $M_0 \setminus S$ is diffeomorphic to $(0, l) \times S^{n-2} \times S^1$ with the metric

$$g = ds^2 + \gamma^2(s) g_{S^{n-2}} + \lambda^2(s) d\theta^2,$$

where λ represents the norm of ξ and $X = \nabla\lambda = \lambda'(s)\frac{\partial}{\partial s}$. From (35) we get $|\lambda'| = |X| = c\gamma$. Taking into account that X does not vanish on $M_0 \setminus S$, we see that λ' does not change sign on $(0, l)$, so γ equals the derivative of λ up to some non–zero constant. Finally, the boundary conditions (38) are easy to check: $\lim_{s\to 0} \gamma(s) = \frac{1}{c}|X_x| = 0$, $\lim_{s\to l} \gamma(s)$ is equal to the radius of the round $(n-2)$–sphere Z and is thus positive, $\lim_{s\to 0} \lambda(s) = |\xi_x| > 0$ and $\lim_{s\to l} \lambda(s) = 0$ because ξ vanishes on Z. q.e.d.

In order to obtain the classification we have to understand which of the above Riemannian join metrics are actually smooth on the entire manifold. For this we will use the following folkloric result:

Lemma 8.7. *Let $f : (0, \varepsilon) \to \mathbb{R}^+$ be a smooth function such that $\lim_{t\to 0} f(t) = 0$. The Riemannian metric $dt^2 + f(t) g_{S^{n-1}}$ extends to a smooth metric at the singularity $t = 0$ if and only if the function $\tilde{f}(t) := f(t^{\frac{1}{2}})$ has a smooth extension at $t = 0$ and $\tilde{f}'(0) = 1$.*

Notice that the above condition on f amounts to saying that $f(t) = t^2 + t^4 h(t^2)$ for some smooth germ h around 0.

Corollary 8.8. *Let $\gamma : (0, l) \to \mathbb{R}^+$ be a smooth function satisfying $\lim_{s\to 0} \gamma(s) = 0$ and $\lim_{s\to l} \gamma(s) > 0$. For $c > 0$ consider the function*

$$\lambda(s) := c \int_s^l \gamma(t) dt. \tag{39}$$

The Riemannian join metric

$$g = ds^2 + \gamma^2(s) g_{S^{n-2}} + \lambda^2(s) d\theta^2$$

defined on $(0, l) \times S^{n-2} \times S^1$ extends to a smooth metric on S^n if and only if there exist two smooth functions a and b defined on some interval $(-\varepsilon, \varepsilon)$ such that

$$\gamma(t) = t(1 + t^2 a(t^2)) \quad \textit{and} \quad \gamma(l - t) = \frac{1}{c} + t^2 b(t^2), \quad \forall\ |t| < \varepsilon. \tag{40}$$

Proof. Since $\lambda(0) > 0$, g extends to a smooth metric at $s = 0$ if and only if $ds^2 + \gamma^2(s)g_{S^{n-2}}$ extends smoothly at $s = 0$. By Lemma 8.7, this is equivalent to $\gamma^2(t) = t^2 + t^4h(t^2)$ for some smooth h, so $\gamma(t) = t\sqrt{1 + t^2h(t^2)} = t(1 + t^2a(t^2))$ for some smooth function a. Similarly, g extends smoothly at $s = l$ if and only if the same holds for $ds^2+\lambda^2(s)d\theta^2$, which, by Lemma 8.7 is equivalent to the existence of some smooth function d defined around 0 such that $\lambda(l-t) = t+t^3d(t^2)$. Taking (39) into account, this is of course equivalent to the second part of (40). q.e.d.

Summarizing, we have

Theorem 8.9. *Let (M^n, g) be a compact Riemannian manifold carrying a non–parallel Killing vector field ξ whose covariant derivative is a twistor 2–form. Then one of the following possibilities occurs:*

1. *M is a space form of positive curvature and ξ is any Killing vector field on M.*
2. *M is a Sasakian manifold and ξ is the Sasakian vector field.*
3. *M is a warped mapping torus $N_{\lambda,\varphi}$*

   $$M = (\mathbb{R} \times N)/_{(t,x)\sim(t+1,\varphi(x))}, \qquad g = \lambda^2 d\theta^2 + g_N,$$

   where N is a compact $(n-1)$–dimensional Riemannian manifold carrying a GCVF with primitive λ (cf. Remark 8.3), and φ is an isometry of N preserving λ and $\xi = \frac{\partial}{\partial\theta}$.
4. *M is a Riemannian join $S^{n-2} *_{\gamma,\lambda} S^1$ with the metric $g = ds^2 + \gamma^2(s)g_{S^{n-2}} + \lambda^2(s)d\theta^2$ where $\gamma : (0, l) \to \mathbb{R}^+$ is a smooth function satisfying the boundary conditions (40), λ is given by formula (39), and $\xi = \frac{\partial}{\partial\theta}$.*

We end these notes with some open problems related to the classification above. One natural question is the following : which compact Riemannian manifolds carry *twistor* 1–forms ξ with twistor exterior derivative? To the author's knowledge, in all known examples ξ is either closed or co–closed. In the first case, the metric dual of ξ is a GCVF, so the manifold is described by Proposition 7.2. The second case just means that ξ is Killing, and the possible manifolds are described by Theorem 8.9.

More generally, one can address the question of classifying all compact Riemannian manifolds M^n carrying a Killing or twistor p–form whose exterior derivative is a non–zero twistor form $(2 \le p \le n-2)$. Besides the round spheres, the only known examples are Sasakian manifolds (for odd p), nearly Kähler 6–manifolds (for $p = 2$ and $p = 3$) and nearly parallel G_2–manifolds (for $p = 3$).

References

[1] F. Belgun, A. Moroianu, & U. Semmelmann, *Killing Forms on Symmetric Spaces,* Diff. Geom. Appl. **24** (2006) 215–222, MR 2216936.

[2] I.M. Benn, Ph. Charlton, & J. Kress, *Debye potentials for Maxwell and Dirac fields from a generalization of the Killing–Yano equation,* J. Math. Phys. **38** (1997) 4504–4527, MR 1468648, Zbl 0885.53077.

[3] I.M. Benn & Ph. Charlton, *Dirac symmetry operators from conformal Killing-Yano tensors,* Classical Quantum Gravity **14** (1997) 1037–1042, MR 1448285, Zbl 0879.58079.

[4] J.P. Bourguignon, *Transformations infinitésimales conformes fermées des variétés riemanniennes connexes complètes,* C.R. Acad. Sci. Paris **270** (1970) 1593–1596, MR 0270302, Zbl 0194.53001.

[5] A. Moroianu & U. Semmelmann, *Twistor Forms on Kähler Manifolds,* Ann Scuola Norm. Sup Pisa **II** (2003) 823–845, MR 2040645.

[6] ———, *Killing Forms on Quaternion–Kähler Manifolds,* Ann. Global Anal. Geom. **28** (2005) 319–335, MR 2199996.

[7] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds,* J. Differential Geom. **6** (1972) 247–258, MR 0303464, Zbl 0236.53042.

[8] B. O'Neill, *Semi-Riemannian geometry,* Academic Press, New York, 1983, MR 0719023, Zbl 0531.53051.

[9] R. Penrose & M. Walker, *On quadratic first integrals of the geodesic equations for type* {22} *spacetimes,* Commun. Math. Phys. **18** (1970) 265–274, MR 0272351, Zbl 0197.26404.

[10] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds,* Math. Z. **243** (2003) 503–527, MR 2021568, Zbl 1061.53033.

[11] S. Tanno, *Some differential equations on Riemannian manifolds*, J. Math. Soc. Japan **30** (1978) 509–531, MR 0500721, Zbl 0387.53015.

CMLS – École Polytechnique
UMR 7640 du CNRS
91128 Palaiseau, France
E-mail address: am@math.polytechnique.fr