J. DIFFERENTIAL GEOMETRY
76 (2007) 485-493

SIU'S INVARIANCE OF PLURIGENERA: A ONE–TOWER PROOF

Mihai Păun

Abstract

In this article our main goal is to establish an L^2 version of the twisted invariance of plurigenera. We equally simplify the original "two towers" approach of Yum-Tong Siu.

1. Introduction

Let $\mathcal{X} \to \Delta$ be a smooth projective family of manifolds, and let $L \to \mathcal{X}$ be a line bundle, endowed with a (possibly singular) metric h_L, such that:

1) $\Theta_{h_L}(L) \geq 0$ as a current on $\mathcal{X}$;
2) The restriction of h_L to the central fiber $\mathcal{X}_0$ is well defined.

In a series of recent and remarkable papers, ([**8**], [**9**] and [**10**]; see also [**3**] for an algebraic–geometric version), Y.-T. Siu proves that the pluricanonical sections twisted with L on $\mathcal{X}_0$, which are *bounded* with respect to the metric h_L, extend to $\mathcal{X}$.

The main purpose of the following lines is to establish a more general version of the extension theorem, and also to simplify the original proof in [**9**].

Theorem 1. *Let $\pi : \mathcal{X} \to \Delta$ be a projective family over the unit disk, and let (L, h) be a hermitian line bundle, which satisfy the properties* (1) *and* (2) *above. Then any section of $(mK_{\mathcal{X}_0} + L) \otimes \mathcal{I}(h_{L|\mathcal{X}_0})$ extends over $\mathcal{X}$.*

Thus, we are able to replace the L^∞ hypothesis in the theorem of Siu by a L^2 condition.

In the argument presented in [**9**], there are two main technical tools, namely the global generation statement, and the Ohsawa-Takegoshi extension theorem. Here we will present a proof which only makes use of the second technique. Although our approach is very similar to the "classical" one, we point out now very briefly the main differences.

Let $\sigma \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$ be a pluricanonical section defined on the central fiber (we assume that L is trivial, to simplify the discussion).

Received 11/03/2005.

There exists $A \to \mathcal{X}$ an ample line bundle on $\mathcal{X}$, such that $pK_{\mathcal{X}} + A$ is generated by its global sections, say $(s_j^{(p)})$, for $0 \le p \le m-1$, $1 \le j \le N_p$, and such that any section of the bundle $mK_{\mathcal{X}_0} + A$ extends over a neighbourhood of the origin. Consider now the family of sections $(\sigma \otimes s_j^{(0)})_{j=1,\dots,N_0}$ over the central fiber (so we are at the bottom of the tower). By the properties of A, all these sections extend to $\mathcal{X}$, and denote by $h^{(0)}$ the metric induced on $mK_{\mathcal{X}} + A$. Observe that the singularities of $h^{(0)}$ restricted to $\mathcal{X}_0$ are precisely the zeroes of σ. We take now the adjoint bundle $K_{\mathcal{X}} + mK_{\mathcal{X}} + A$; over $\mathcal{X}_0$ each of the sections $\sigma \otimes s_j^{(1)}$, $j = 1, \dots, N_1$ are integrable with respect to $h^{(0)}$ and the Ohsawa-Takegoshi theorem [**5**] (more precisely, the version given by Siu in [**9**]) will imply that we can extend this family of sections of the adjoint bundle over $\mathcal{X}$. Then we iterate this procedure, and we obtain metrics for the $(km+p)K_{\mathcal{X}} + A$, such that their restriction to $\mathcal{X}_0$ is the metric given by the family of sections $(\sigma^{\otimes k} \otimes s_j^{(p)})_{j=1,\dots,N_p}$. An important point is that we can perform each step in a very effective manner; so by extracting roots and passing to the limit, we produce a metric for $mK_{\mathcal{X}}$, with all the properties required by the extension theorem to end the proof.

The original argument of Siu goes as follows: he starts (on the top of the first tower) with the section $\sigma^{\otimes k} \otimes s_A$ (where $k \gg 0$, and s_A is a non-zero section of A), which he contracts with a local section of the anti-canonical bundle. The result is a local section of the bundle $(km-1)K_{\mathcal{X}_0} + A$, but the (effective) global generation property will show that it is a combination of elements $\sigma_j^{(km-1)} \in H^0(\mathcal{X}_0, (km-1)K_{\mathcal{X}_0} + A)$, with holomorphic functions as coefficients, which are divisible by large powers of σ (remark that in our case, the corresponding sections are just $\sigma^{\otimes(k-1)} \otimes s_j^{(m-1)}$). At this point, the L^∞ hypothesis is needed, in order to obtain effective bounds for the coefficients. Next he applies this procedure inductively, and at the bottom of the tower the Ohsawa-Takegoshi extension theorem is used. Finally, a "second tower" is needed, to extend step by step the sections $\sigma_j^{(p)}$ on $\mathcal{X}$. Again, at the end the metric on $mK_{\mathcal{X}}$ is produced by extracting roots.

Acknowledgments. We would like to thank Yum-Tong Siu for many discussions about the invariance of plurigenera, and also Bo Berndtsson, whose results on the Bergman kernels (very much related to the topic presented here) helped us to understand Siu's proof from a different point of view.

2. Proof of the theorem

To start with, we recall some basic constructions and results needed later on.

Let $E \to X$ be a line bundle over a complex manifold X and let $s_1, \dots, s_l \in H^0(X, E)$ be a collection of homolorphic sections of E. One can use the sections (s_j) to define in a canonical way a metric on E, as we recall here (see [**2**] for more examples and properties concerning this notion).

Via a local trivialization of the bundle E on an open set $\Omega \subset X$, the sections (s_j) correspond respectively to the holomorphic functions (f_j), and then the local weight of the metric will be

$$\varphi_\Omega := 1/2 \log \left(\sum_{j=1}^{l} |f_j|^2 \right).$$

A more global way of expressing this is the following: we consider h a hermitian metric on E, and then the metric associated to the family of sections (s_j) is

$$\widetilde{h} := \frac{1}{\sum_j \|s_j\|_h^2} h. \tag{1}$$

Remark that $\widetilde{h}$ is singular along the common zeroes of the sections (s_j), but nevertheless we can define the notion of curvature, which in this case will be a closed positive current.

We come now to the main technical tool of the proof, namely the Ohsawa–Takegoshi theorem (see [**5**], and the further generalizations by T. Ohsawa); the next version was established by Siu in [**9**].

Theorem 2.1 ([**5**], [**9**]). *Let $\pi : \mathcal{X} \to \Delta$ be a projective family of smooth manifolds. Let $E \to \mathcal{X}$ be a line bundle, endowed with a (possibly singular) metric h, with semi-positive curvature current. If $\sigma_0 \in H^0(\mathcal{X}_0, K_{\mathcal{X}_0} + E)$ is a section of the adjoint bundle of E restricted to the central fiber, such that*

$$\int_{\mathcal{X}_0} \|\sigma_0\|_h^2 < \infty,$$

then there exist a section $\sigma \in H^0(\mathcal{X}, K_{\mathcal{X}} + E)$ such that $\sigma_{|\mathcal{X}_0} = \sigma_0$ and moreover

$$\int_{\mathcal{X}} \|\sigma\|_h^2 < C_0 \int_{\mathcal{X}_0} \|\sigma_0\|_h^2.$$

The result stated (and proved) in [**9**] is in fact more general, but the previous version is enough for our purposes. An essential point is that the constant C_0 above is universal (about 200 will do).

Consider a section $\sigma \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0} + L)$, as in the hypothesis of Theorem 1. In order to extend it over $\mathcal{X}$, we first fix an ample line bundle $A \to \mathcal{X}$ such that:

(A_1) For each $p = 0, \dots, m-1$, the bundle $pK_{\mathcal{X}} + A$ is generated by its global sections, say $(s_j^{(p)})$, with $1 \leq j \leq N_p$.

(A_2) Every section of the line bundle $mK_{\mathcal{X}} + L + A_{|\mathcal{X}_0}$ extends to $\mathcal{X}$.

Let ω be a hermitian metric on $\mathcal{X}$; we denote by h_ω the metric induced on the canonical bundle. We also consider h_L a smooth, hermitian metric on L, and let h_A be a smooth, positively curved metric on A. Then for each integers $q, r \in \mathbb{Z}_+$, we denote by $h^{(q,r)}$ the metric on the bundle $qK_{\mathcal{X}} + rL + A$ induced by h_ω, h_L and h_A.

By hypothesis, there exists another metric $\widetilde{h}_L := \exp(-2\varphi_L)h_L$ on L such that

$$\Theta_{\widetilde{h}_L}(L) = \Theta_{h_L}(L) + i/\pi\partial\overline{\partial}\varphi_L \geq 0$$

as currents on $\mathcal{X}$; thus, the weight function φ_L is bounded from above.

Then we have the next statement (compare with [**9**, Proposition 4.1]).

Proposition 2.2. *There exists a positive constant $C_\infty > 0$ such that for all $k \in \mathbb{Z}_+$, and $0 \leq p \leq m-1$, we have a family of sections $(\widetilde{s}_j^{(km+p)})_{j=1,\dots,N_p}$ of the bundle $(km+p)K_{\mathcal{X}} + kL + A$ such that:*

(a) *For each integer k, p as above, $\widetilde{s}_{j|\mathcal{X}_0}^{(km+p)} = \sigma^{\otimes k} \otimes s_j^{(p)}$.*

(b) *For each $1 \leq p \leq m-1$, the next inductive estimate holds*

$$\int_{\mathcal{X}} \frac{\sum_{j=1}^{N_p} \|\widetilde{s}_j^{(km+p)}\|^2_{h^{(km+p,k)}}}{\sum_{j=1}^{N_{p-1}} \|\widetilde{s}_j^{(km+p-1)}\|^2_{h^{(km+p-1,k)}}} dV_\omega \leq C_\infty.$$

(c) *For $p = 0$, we have*

$$\int_{\mathcal{X}} \frac{\sum_{j=1}^{N_0} \|\widetilde{s}_j^{(km)}\|^2_{h^{(km,k)}}}{\sum_{j=1}^{N_{m-1}} \|\widetilde{s}_j^{((k-1)m+m-1)}\|^2_{h^{(km-1,k-1)}}} dV_\omega \leq C_\infty.$$

Proof. We will construct inductively the sections $(\widetilde{s}_j^{(km+p)})_{j,k,p}$, and the uniformity properties (b), (c) will be a consequence of the Ohsawa–Takegoshi theorem.

To this end, let us start with the family of sections $\sigma \otimes s_j^{(0)} \in mK_{\mathcal{X}_0} + L + A$, where $j = 1, \dots, N_0$. By the property (A_2) of the bundle A, each of the previous sections extend over $\mathcal{X}$, thus we get $\widetilde{s}_j^{(m)} \in mK_{\mathcal{X}} + L + A$ such that $\widetilde{s}_{j|\mathcal{X}_0}^{(m)} = \sigma \otimes s_j^{(0)}$.

As recalled above, the family of sections $(\widetilde{s}_j^{(m)})_{j=1,\dots,N_0}$ define in a canonical way a metric on the bundle $mK_{\mathcal{X}} + L + A$, with semi-positive curvature current. Consider the adjoint bundle $K_{\mathcal{X}_0} + mK_{\mathcal{X}_0} + L + A$; we have

$$\sigma \otimes s_j^{(1)} \in H^0\big(\mathcal{X}_0, (m+1)K_{\mathcal{X}_0} + L + A\big)$$

for each index $j = 1, \dots, N_1$. Moreover, remark that by the global generation property (A_1), there exist a constant $C_1 > 0$, such that

$$\max_{r,q} \sup_{\mathcal{X}} \left(\frac{\sum_{j=1}^{N_r} \|s_j{}^{(r)}\|^2_{h^{(r,0)}}}{\sum_{j=1}^{N_q} \|s_j{}^{(q)}\|^2_{h^{(q,0)}}} \right) = C_1. \tag{2}$$

Thus we have a constant $C_2 > 0$ such that

$$\frac{\|\sigma \otimes s_j^{(1)}\|^2_{h^{(m+1,1)}}}{\sum_{k=1}^{N_0} \|\sigma \otimes s_k^{(0)}\|^2_{h^{(m,1)}}} < C_2 \tag{3}$$

pointwise on the central fiber $\mathcal{X}_0$. We apply now the Ohsawa–Takegoshi theorem, for the line bundle $E := mK_{\mathcal{X}} + L + A$, endowed with the metric given by the family of sections $(s_j^{(m)})$. Then for each $j = 1, \dots, N_1$, we obtain a holomorphic section

$$\widetilde{s}_j{}^{(m+1)} \in H^0(\mathcal{X}, (m+1)K_{\mathcal{X}} + L + A)$$

such that $\widetilde{s}_{j|\mathcal{X}_0}^{(m+1)} = \sigma \otimes s_j^{(1)}$, together with the estimate

$$\int_{\mathcal{X}} \frac{\|\widetilde{s}_j{}^{(m+1)}\|^2_{h^{(m+1,1)}}}{\sum_{k=1}^{N_0} \|\widetilde{s}_k{}^{(m)}\|^2_{h^{(m,1)}}} dV_\omega \leq C_0 \int_{\mathcal{X}_0} \frac{\|\sigma \otimes s_j^{(1)}\|^2_{h^{(m+1,1)}}}{\sum_{k=1}^{N_0} \|\sigma \otimes s_k^{(0)}\|^2_{h^{(m,1)}}} dV_\omega \tag{4}$$

(in the above relation, we denote by dV_ω both volume elements on $\mathcal{X}$, respectively $\mathcal{X}_0$, by an abuse of notation). The above relation may look odd, since it involves smooth metrics with no curvature assumption, but recall the expression (1) of the metric defined by a family of sections.

The relations (3) and (4) imply

$$\int_{\mathcal{X}} \frac{\|\widetilde{s}_j{}^{(m+1)}\|^2_{h^{(m+1,1)}}}{\sum_{k=1}^{N_0} \|\widetilde{s}_k{}^{(m)}\|^2_{h^{(m,1)}}} dV_\omega \leq C_0 \cdot C_2 \cdot \mathrm{Vol}(\mathcal{X}_0, \omega) := C_3. \tag{5}$$

We apply this procedure inductively; thus for $p = 1, \dots, m-1$ and for each $j = 1, \dots, N_p$, we get the sections

$$\widetilde{s}_j{}^{(m+p)} \in H^0(\mathcal{X}, (m+p)K_{\mathcal{X}} + L + A)$$

such that $\widetilde{s}_{j|\mathcal{X}_0}^{(m+p)} = \sigma \otimes s_j^{(p)}$, and such that

$$\int_{\mathcal{X}} \frac{\|\widetilde{s}_j{}^{(m+p)}\|^2_{h^{(m+p,1)}}}{\sum_{k=1}^{N_{p-1}} \|\widetilde{s}_k{}^{(m+p-1)}\|^2_{h^{(m+p-1,1)}}} dV_\omega \leq C_3. \tag{6}$$

In order to "climb at the m'th floor", we will consider the adjoint bundle of $(2m-1)K_{\mathcal{X}} + L + A$, *twisted with* L; we endow the bundle $(2m-1)K_{\mathcal{X}} + 2L + A$, with the metric canonically defined by the family of sections $(\widetilde{s}_j{}^{(2m-1)})_{j=1,\dots,N_{m-1}}$, twisted with the metric $\widetilde{h}_L$. We observe that by hypothesis, this metric has semi-positive curvature.

On the central fiber $\mathcal{X}_0$, consider the sections

$$\sigma^{\otimes 2} \otimes s_j^{(0)} \in H^0(\mathcal{X}_0, 2mK_{\mathcal{X}} + 2L + A).$$

The L^2 hypothesis on σ, and the previous estimates show that we have
(7)

$$\int_{\mathcal{X}_0} \frac{\|\sigma^{\otimes 2} \otimes s_j^{(0)}\|^2_{h^{(2m,1)}\otimes \widetilde{h}_L}}{\sum_{k=1}^{N_{m-1}} \|\sigma \otimes s_k^{(m-1)}\|^2_{h^{(2m-1,1)}}} dV_\omega \leq C_1 \int_{\mathcal{X}_0} \|\sigma\|^2_{h_\omega^{\otimes m}\otimes \widetilde{h}_L} dV_\omega := C_4$$

By the Ohsawa-Takegoshi theorem, there exists a section

$$\widetilde{s}_j^{(2m)} \in H^0(\mathcal{X}, 2mK_{\mathcal{X}} + 2L + A)$$

such that $\widetilde{s}_{j|\mathcal{X}_0}^{(2m)} = \sigma^{\otimes 2} \otimes s_j^{(0)}$, and such that

$$\int_{\mathcal{X}} \frac{\|\widetilde{s}_j^{(2m)}\|^2_{h^{(2m,1)}\otimes \widetilde{h}_L}}{\sum_{k=1}^{N_{m-1}} \|\widetilde{s}_k^{(2m-1)}\|^2_{h^{(2m-1,1)}}} dV_\omega \leq C_0 \cdot C_4 := C_5. \tag{8}$$

Now recall that we have $\widetilde{h}_L = \exp(-\varphi_L) h_L$, and the curvature condition implies that the function φ_L is bounded from above by a constant; thus we get

$$\int_{\mathcal{X}} \frac{\|\widetilde{s}_j^{(2m)}\|^2_{h^{(2m,2)}}}{\sum_{k=1}^{N_{m-1}} \|\widetilde{s}_k^{(2m-1)}\|^2_{h^{(2m-1,1)}}} dV_\omega \leq C_6 \tag{9}$$

for each $j = 1, \ldots, N_0$.

It becomes now very clear how to proceed: assume that we already have the sections $(\widetilde{s}_j^{(km+p)})_{j=1,\ldots,N_p}$ of the bundle $(km+p)K_{\mathcal{X}} + kL + A$; if $p = m-1$, then we will consider the adjoint bundle twisted with L, and if not, we can use simply the adjoint bundle. We can take the constant

$$C_\infty := \max(C_3, C_6) \cdot \max(N_1, \ldots, N_{m-1})$$

and the uniformity conditions (2) and (3) of the proposition will be satisfied. Indeed, at each step of the previous construction, the quantity we need to estimate in order to apply the extension theorem of Ohsawa–Takegoshi is either

$$\int_{\mathcal{X}_0} \frac{\|\sigma^{\otimes k} \otimes s_j^{(p)}\|^2_{h^{(km+p,k)}}}{\sum_{r=1}^{N_{p-1}} \|\sigma^{\otimes k} \otimes s_r^{(p-1)}\|^2_{h^{(km+p-1,k)}}} dV_\omega$$

or

$$\int_{\mathcal{X}_0} \frac{\|\sigma^{\otimes k} \otimes s_j^{(0)}\|^2_{h^{(km,k-1)}\otimes \widetilde{h}_L}}{\sum_{r=1}^{N_{m-1}} \|\sigma^{\otimes k-1} \otimes s_r^{(m-1)}\|^2_{h^{(mk-1,k-1)}}} dV_\omega$$

and these quantities are clearly bounded by a uniform constant, as indicated along the previous lines. The proposition is proved. q.e.d.

The rest of the argument follows clearly from the [**9**, pp. 259–263], but for the convenience of the reader (and because we give a more intrinsic presentation) we reproduce it now.

For each $k \in \mathbb{Z}_+$, and $0 \leq p \leq m-1$, consider the following quasi-psh function on the manifold $\mathcal{X}$

$$f_{k,p} := \log \left(\sum_{j=1}^{N_p} \| \widetilde{s_j}^{(km+p)} \|^2_{h^{km+p,k}} \right).$$

The properties (b), (c) in the proposition and the concavity property of the logarithm function show that for some constant $C'_\infty > 0$ we have

$$\int_{\mathcal{X}} (f_{k,p} - f_{k,p-1}) dV_\omega \leq C'_\infty \tag{10}$$

with the convention that for $p = 0$, the function $f_{k,-1}$ equals $f_{k-1,m-1}$. We add up the inequalities (10), and thus we get

$$\int_{\mathcal{X}} f_{k,0} dV_\omega \leq mkC'_\infty + C_7$$

for some constant $C_7 > 0$, and for any positive integer k.

In conclusion, we have the following properties:

(i) There exists a constant $C > 0$ such that $\frac{1}{k} \int_{\mathcal{X}} f_{k,0} dV_\omega \leq C$.

(ii) The inequality

$$\Theta_{h_{\omega,L}}(mK_{\mathcal{X}} + L) + i/k \partial\overline{\partial} f_{k,0} \geq -\frac{1}{k} \Theta_{h_A}(A)$$

holds true in the sense of currents on $\mathcal{X}$, where $h_{\omega,L} := h_\omega^{\otimes m} \otimes h_L$.

(iii) On the central fiber the following equality is satisfied

$$\frac{1}{k} f_{k,0|\mathcal{X}_0} = \log \|\sigma\|^2 + \frac{1}{k} \log \big(\sum_{j=1}^{N_0} \| s_j{}^{(0)} \|^2_{h^{(0,0)}} \big).$$

As a consequence of the properties (i) and (ii), one can show the existence of a uniform upper bound for the functions $(\frac{1}{k} f_{k,0})$ on $\pi^{-1}(\Delta_{1-\eta})$, for any positive η (this is a consequence of the mean value inequality for the psh functions). But then (see e.g., [**4**]) we can consider the limit

$$f_\infty := \lim_{reg} \sup_k \frac{1}{k} f_{k,0}$$

(the *reg* in the above limit means that we take the upper semi-continuous envelope of the limit). Remark that the function f_∞ cannot be identically $-\infty$, because of (iii). Moreover, the limit f_∞ inherits from the sequence $f_{k,0}$ the following properties:

$$\Theta_{h_{\omega,L}}(mK_{\mathcal{X}} + L) + i\partial\overline{\partial} f_\infty \geq 0 \tag{11}$$

as a current on $\mathcal{X}$, as well as

$$f_\infty(z) \geq \log \|\sigma_z\|^2 + \mathcal{O}(1) \tag{12}$$

pointwise on the central fiber $\mathcal{X}_0$.

We consider the metric $h_\infty := \exp(-f_\infty)h_{\omega,L}$ on the line bundle $mK_{\mathcal{X}} + L$. It has semi-positive curvature, thanks to the relation (11); then we can endow the bundle $(m-1)K_{\mathcal{X}} + L$ with the metric $h_\infty^{\frac{m-1}{m}} \otimes \widetilde{h_L}^{1/m}$. The curvature of this metric is still semi-positive, and moreover the section σ is integrable with respect to its restriction to $\mathcal{X}_0$; indeed, if we denote by φ_∞, respectively φ_L the local weights of the metrics h_∞, $\widetilde{h}_L$ on a set Ω, then we have

$$\int_\Omega |\sigma|^2 \exp\left(-\frac{m-1}{m}\varphi_\infty - \frac{1}{m}\varphi_L\right) d\lambda$$
$$\leq C \int_\Omega |\sigma|^{\frac{2}{m}} \exp\left(-\frac{1}{m}\varphi_L\right) d\lambda \leq C$$

where the last inequality is a consequence of the hypothesis and of the Hölder inequality. Thus, a final application of the Ohsawa–Takegoshi will end the proof.

Remark 2.3. In the paper [**1**], Bo Berndtsson proved a result which could be very useful for the questions investigated above. The general framework is the following. Let Y, X be compact Kähler manifolds, and consider a fibration $\pi_1 : X \to Y$. Let $L \to X$ be a line bundle, endowed with a hermitian (eventually singular) metric h, such that $\Theta_h(L) \geq 0$.

Theorem 2.4 ([**1**])**.** *Consider the metric on the adjoint bundle $K_X + L$ whose restriction to the fiber X_y is given by an orthonormal basis of L^2-sections of $K_X + L$. Then this metric is either identically $+\infty$, or it has semi–positive curvature.*

Our remark in that this result could be used to obtain a "canonical" metric on the bundle $mK_{\mathcal{X}} + L$; observe that the metric constructed in our paper (as well as the one in [**9**]) depends on the particular section σ we want to extend (compare with [**12**]).

References

[1] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, math.CV/0511225.

[2] J.-P. Demailly, *Singular hermitian metrics on positive line bundles*, Proceedings of the Bayreuth conference "Complex algebraic varieties", April 2-6, 1990, edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math., **1507**, Springer-Verlag, 1992, MR 1178721.

[3] J. Kawamata, *Deformations of canonical singularities*, J. Amer. Math. Soc. **12**(**1**) (1999), MR 1631527, Zbl 0906.14001.

[4] P. Lelong & L. Gruman, *Entire functions of several complex variables*, Springer, 1986, MR 0837659, Zbl 0583.32001.

[5] T. Ohsawa & K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987) 197–204, MR 0892051, Zbl 0625.32011.

[6] S. Takayama, *Pluricanonical Systems on Algebraic Varieties of General Type*, Invent. Math. **165**(**3**) (2006) 551–587, MR 2242627.

[7] ———, *On the Invariance and Lower Semi–Continuity of Plurigenera of Algebraic Varieties*, J. Algebraic Geom. **16**(**1**) (2007) 1–18, MR 2257317.

[8] Y.-T. Siu, *Invariance of Plurigenera*, Inv. Math. **134** (1998) 661–673, MR 1660941, Zbl 0955.32017.

[9] ———, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type*, Complex geometry (Göttingen, 2000), 223–277, Springer, Berlin, 2002, Zbl 1007.32010.

[10] ———, *Multiplier ideal sheaves in complex and algebraic geometry*, Sci. China Ser. **A 48** (2005), MR 2156488, Zbl 1044.32016.

[11] H. Tsuji, *Pluricanonical Systems of Projective Varieties of General Type*, preprint CV/0409318, 2004.

[12] ———, *Invariance of Plurigenera of Varieties with Nonnegative Kodaira Dimension*, AG/0011257, 2000.

Institut Élie Cartan
Université Henri Poincaré Nancy 1
B.P. 239, F-54506 Vandoeuvre-les-Nancy Cédex
France

E-mail address: Mihai.Paun@iecn.u-nancy.fr