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ON THE CLASSIFICATION OF LORENTZIAN HOLONOMY GROUPS

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Abstract

If an $(n+2)$-dimensional Lorentzian manifold is indecomposable, but non-irreducible, then its holonomy algebra is contained in the parabolic algebra $(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n$. We show that its projection onto $\mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold. This leads to a classification of Lorentzian holonomy groups and implies that the holonomy group of an indecomposable Lorentzian spin manifold with parallel spinor equals to $G\ltimes\mathbb{R}^n$ where G is a product of $SU(p)$, $Sp(q)$, G_2 or $Spin(7)$.

1. Introduction

Holonomy groups. An important tool to study the geometric structure of a smooth manifold M equipped with a linear connection ∇ is its *holonomy group*. Parallel sections in geometric vector bundles, such as tensor products of the tangent bundle or the spin bundle, correspond to invariant objects under the holonomy representation. By a result of J. Hano and H. Ozeki [**22**] any closed subgroup of $Gl(m,\mathbb{R})$ can be obtained as a holonomy group of a connection, but possibly a connection with torsion. By imposing conditions on the torsion there arises a classification problem of possible holonomy groups. In order to tackle such a classification problem one usually assumes that the connection is torsion free and that holonomy group acts irreducibly. If the connection is torsion free, its curvature satisfies the Bianchi-identity imposing algebraic constraints to the holonomy algebra via the Ambrose-Singer holonomy theorem [**3**]. By evaluating these constraints M. Berger classified the irreducible semi-Riemannian holonomy groups (see [**6**] for not locally symmetric semi-Riemannian manifolds, [**7**] for symmetric spaces, and [**35**], [**2**], [**10**] and [**11**] for simplifications, corrections and existence results in the Riemannian case), while L. Schwachhöfer and S. Merkulov ([**31**], [**32**], [**33**]) classified irreducible holonomy groups of torsion free connections which are not necessarily compatible with a metric.

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Due to the de Rham splitting theorem, which asserts that any simply connected, complete Riemannian manifold is isometric to a product of simply connected, complete Riemannian manifolds with trivial or irreducible holonomy representation, the Berger-classification leads to a classification of possible holonomy groups of simply connected complete Riemannian manifolds. The generalisation of the de Rham theorem to semi-Riemannian manifolds is due to H. Wu [**38**] assuring a decomposition into manifolds with indecomposable holonomy groups, instead of irreducible ones. Thus, in order to classify holonomy groups of pseudo-Riemannian manifolds, one has to determine the holonomy groups which act indecomposably, but not necessarily irreducibly.

In one of his articles about open problems in geometry S.-T. Yau referred to this situation and related it to the problem of the existence of parallel spinors [**39**]: "Berger has classified holonomy groups for Riemannian manifolds. If the metric is not Riemannian but allows different signature, the corresponding theorem of Berger should exist. More importantly, we would like to find a *complete* connection whose holonomy group is the given Lie group. In particular, classify complete Lorentzian manifolds with parallel spinors." In the present article we shall deal with two questions which are subsidiary to the questions posed by Yau. Firstly, we restrict ourselves to Lorentzian manifolds and study the problem, which Lie groups might occur as their holonomy groups. We shall show that the screen holonomy group of an indecomposable, non-irreducible Lorentzian manifold is always a Riemannian holonomy group. Secondly, we show which of the groups we found are holonomy groups of Lorentzian spin manifolds with a parallel spinor field. Both questions should reasonably be answered before tackling the completeness problem in Yau's quote.

Lorentzian holonomy groups and results of the paper. For a simply connected and complete semi-Riemannian manifold (M,h) of Lorentzian signature $(-+\cdots+)$ the de Rham/Wu decomposition leads to the following isometry:

$$(M,h) \overset{\text{isometric}}{\simeq} (M',h') \times (N_1,g_1) \times \ldots (N_k,g_g),$$

where (N_i,g_i) are Riemannian manifolds which are flat or with irreducible holonomy representation and (M',h') is a simply-connected, complete Lorentzian manifold. Then three cases arise:

1) (M',h') *is flat*, i.e., the holonomy representation is trivial.
2) (M',h') *is irreducible.* In this case its holonomy group is the full $SO_0(1,m-1)$ where $m = dim\ M'$. This follows from Berger's classification, but it was also proven directly in [**17**].
3) (M',h') *is indecomposable, but non-irreducible.* We will treat this case in the present article.

Since we shall work on the level of Lie algebras, *in the following we always assume that our manifold is simply connected or that the results are true only for the restricted holonomy group, i.e., the connected component of the holonomy group.* We consider an $(n+2)$-dimensional Lorentzian manifold. If the holonomy representation is indecomposable and non-irreducible it admits a one-dimensional, light-like holonomy invariant subspace. Thus the holonomy group is contained in the isotropy group of this subspace, which is equal to the parabolic group $(\mathbb{R}^+ \times SO(n)) \ltimes \mathbb{R}^n$. Its projection onto $SO(n)$ is called *screen holonomy*, since it is the holonomy of the so-called *screen bundle* (see [**29**] and [**30**]). If the screen holonomy and the projection onto $\mathbb{R}$ are trivial, i.e., if the holonomy is Abelian, then the manifold is a pp-wave [**26**]. L. Berard-Bergery and A. Ikemakhen proved in [**5**] a Borel-Lichnerowicz-type decomposition property for the screen holonomy. This property is satisfied for holonomy groups of Riemannian manifolds and these were the only examples of such a projection. In their paper they posed the question whether the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian holonomy algebra is always the holonomy algebra of a Riemannian manifold. We show that this is the case:

Theorem 1.1. *Let (M,h) be a Lorentzian manifold of dimension $n+2>2$ with an indecomposable, non-irreducible restricted holonomy group H. Then the $SO(n)$-projection of H is the holonomy group of an n-dimensional Riemannian manifold.*

We will explain shortly how this result can be used for a classification of indecomposable, non-irreducible Lorentzian holonomy groups. First of all, there is no further obstruction on a Lie group to be the screen holonomy of a Lorentzian manifold, i.e., any Riemannian holonomy group can be realised as screen holonomy of a certain indecomposable, non-irreducible Lorentzian manifold. This is due to a simple construction method which we shall not prove in the present article (see [**27**] for a proof and [**29**] for a slight generalisation).

Secondly one has to recall that in [**5**] four different types of indecomposable subalgebras of the parabolic algebra are distinguished. For two of these types the algebra has the shape $(\mathbb{R}\oplus\mathfrak{g})\ltimes\mathbb{R}^n$ resp. $\mathfrak{g}\ltimes\mathbb{R}^n$ with a subalgebra $\mathfrak{g}\subset\mathfrak{so}(n)$ — we call these *uncoupled* types — while two further types show a coupling between the centre of the $\mathfrak{so}(n)$-projection and the $\mathbb{R}$- resp. the $\mathbb{R}^n$-projection. As the mentioned construction method leads to Lorentzian manifolds with holonomy of uncoupled type, there are no further obstrucions on a Riemannian holonomy group to be the screen holonomy of a Lorentzian holonomy group of uncoupled type. For the coupled types A. Galaev showed recently [**20**] the following: If $\mathfrak{h}$ is an indecomposable subalgebra of the parabolic algebra (possibly of coupled type) and with a Riemannian holonomy algebra as $\mathfrak{so}(n)$-projection, then there is a Lorentzian metric with restricted holonomy $\mathfrak{h}$. Using this we can summarise the results as follows.

Corollary 1.2. *Let $H \subset (\mathbb{R}^+ \times SO(n)) \ltimes \mathbb{R}^n \subset SO(1, n+1)$ be an indecomposable subgroup and $G = pr_{SO(n)}H$. Then H is the restricted holonomy group of a Lorentzian manifold if and only if G is the holonomy group of a Riemannian manifold.*

Regarding the existence of parallel spinors fields on a semi-Riemannian spin manifold, one has to recall that a parallel spinor field on a simply-connected manifold corresponds to a spinor which is annihilated under the spin representation of the holonomy algebra. As soon as one knows the holonomy algebra one may calculate if the manifold admits a parallel spinor field. In the case of Riemannian manifolds this was done by M. Wang [**37**] and for irreducible pseudo-Riemannian manifolds by H. Baum and I. Kath [**4**]. Since for the Lorentzian manifolds the only irreducible holonomy group is the full $SO_0(1, m-1)$, the existence of a parallel spinor field immediately leads to the non-irreducible case. In other words: On a Lorentzian manifold, the existence of a parallel spinor field implies the existence of a parallel vector field. If this vector field is time-like, the Lorentzian part in the Wu-decomposition is flat; if it is light-like, we are in the case of an indecomposable, non-irreducible Lorentzian manifold. From Theorem 1.1 and the fact that the holonomy of a Lorentzian manifold with parallel spinor cannot be of coupled type one can deduce the following consequence.

Corollary 1.3. *Let (M, h) be an indecomposable Lorentzian spin manifold of dimension $(n+2) > 2$ with restricted holonomy group H admitting a parallel spinor field. Then it is $H = G \ltimes \mathbb{R}^n$ where G is the restricted holonomy group of an n-dimensional Riemannian manifold with parallel spinor, i.e., G is a product of $SU(p)$, $Sp(q)$, G_2 or $Spin(7)$.*

This generalises a result of R. L. Bryant in [**12**] (see also [**18**]) where it is shown up to $n \leq 9$ that the maximal subalgebras of the parabolic algebra admitting a trivial subrepresentation of the spin representation are of type (Riemannian holonomy)$\ltimes\mathbb{R}^n$. We conclude:

Theorem 1.4. *Let (M, h) be a simply connected Lorentzian spin manifold which admits a parallel spinor. Then (M, h) is isometric to a product $(M', h') \times (N_1, g_1) \times \ldots (N_k, g_g)$, where the (N_i, g_i) are flat or irreducible Riemannian manifolds with a parallel spinor and (M', h') is either $(\mathbb{R}, -dt)$ or it is an indecomposable, non-irreducible Lorentzian manifold of dimension $(n+2) > 2$ with holonomy $G \ltimes \mathbb{R}^n$ where G is the holonomy group of a Riemannian manifold with parallel spinor.*

Finally one should remark that the question for the general form of the metric having a prescribed holonomy is still open (up to $n \leq 9$ see

[**12**], for more detailed results about the local form of the metric in any dimension see [**8**]). Of course, the question of completeness, which was part of Yau's problem, is not touched yet. We should remark that the result of Theorem 1.1 up to $n \leq 9$ was obtained in [**19**], by using some of our results of [**26**]. Finally, it would be interesting to have a more conceptual proof of Theorem 1.1.

Methods of the proof and structure of the paper. In Section 2 we recall basic properties of indecomposable, non-irreducible Lorentzian holonomy groups, in particular the results of [**5**]. We introduce the notion of a *weak-Berger algebra*, in contrast to a *Berger algebra*, and show that the $\mathfrak{so}(n)$-projection $\mathfrak{g}$ of an indecomposable, non-irreducible Lorentzian holonomy algebra is a weak-Berger algebra, as well as all its irreducibly acting components. As a consequence, we have to classify all irreducibly acting, real weak-Berger algebras. A real Lie algebra is weak-Berger if and only if its complexification is weak-Berger, but in order to switch to the complex situation we have to distinguish two cases: the Lie algebra acts irreducibly on the complexified module (the representation is of *real type*) or it acts reducibly on the complexified module (the representation is of *non-real type*). Since the Lie algebra we start with acts completely reducibly it has to be reductive. Thus, in the first case the Lie algebra has to be semisimple. In the second case the complexified module splits into two irreducible submodules which are conjugate to each other.

In Section 3 we classify weak-Berger algebras of non-real type. Using a classification of complex Lie algebras with non-vanishing first prolongation by S. Kobayashi and T. Nagano [**25**] we show that any weak-Berger algebra which is unitary is a Berger algebra, i.e., the holonomy algebra of a Riemannian Kähler manifold.

In Section 4 we consider the case where the representation is of real type. Because of the semisimplicity we can use the methods of root space and weight space decomposition and transform the weak-Berger property in the language of roots and weights. In Section 5 we classify irreducible, complex, simple weak-Berger algebras and in Section 6 the semisimple, non-simple ones. We show for both that they are Berger algebras, i.e., that they are complexifications of Riemannian holonomy algebras.

2. Lorentzian holonomy and weak-Berger algebras

2.1. Indecomposable, non-irreducible Lorentzian holonomy algebras. Let (M, h) be an indecomposable, non-irreducible Lorentzian manifold with $dim\ M = n + 2 > 2$. The holonomy group at a point $p \in M$ acting on T_pM has a degenerate invariant subspace. Intersecting this subspace with its orthogonal complement yields a light-like, one-dimensional invariant subspace Ξ_p which is the fibre of a parallel distribution Ξ. This distribution corresponds to a light-like recurrent vector field X. The subspace $\Xi_p^\perp$ is holonomy invariant too and the fibre of a parallel distribution $\Xi^\perp$ of co-dimension 1.

The Lie algebra of the isotropy group of Ξ_p can be identified with the parabolic algebra $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n \subset \mathfrak{so}(\mathbb{R}^{n+2}, \eta)$, and by fixing a basis of T_pM of the form

$$\begin{array}{ll} (x, e_1, \ldots e_n, z) \text{ with } x \in \Xi_p, e_i \in \Xi_p^\perp \text{ such that } h(e_i, e_j) = \delta_{ij} \\ h(z,z) = h(z, e_i) = h(x, e_i) = 0 \text{ and } h(x,z) = 1, \end{array} \tag{1}$$

it can be written in matrices as follows:

$$(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n = \left\{ \left(\begin{array}{ccc} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{array} \right) \middle| \, a \in \mathbb{R}, v \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.$$

The holonomy algebra of an indecomposable, non-irreducible Lorentzian manifold, denoted by $\mathfrak{hol}_p(M,h)$, is contained in this Lie algebra. If the manifold admits not only a recurrent but also a parallel light-like vector field, then the projection of $\mathfrak{hol}_p(M,h)$ on $\mathbb{R}$ vanishes.

We set $E := span(e_1, \ldots, e_n)$ and write $\mathfrak{so}(n) = \mathfrak{so}(E, (h_p)_{|E})$. The main ingredient of such a holonomy algebra is the $\mathfrak{so}(n)$-projection. We call this projection $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{hol}_p(M,h)$ the *screen holonomy* of an indecomposable, non-irreducible Lorentzian manifold, because it is equal to the holonomy of the so-called *screen bundle* $\Xi^\perp/\Xi \to M$ (see [**29**] and [**30**]). Since $\mathfrak{g} \subset \mathfrak{so}(n)$ acts completely reducibly, $\mathfrak{g}$ is reductive, i.e., its Levi decomposition is $\mathfrak{g} = \mathfrak{z} + \mathfrak{d}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ the derived Lie algebra, which is semisimple [**15**]. Choosing a different basis of type (1) corresponds to conjugation in the parabolic group. Hence, the $\mathfrak{so}(n)$-component is uniquely defined up to conjugation in $O(n)$. For the screen holonomy L. Berard-Bergery and A. Ikemakhen proved the following Borel-Lichnerowicz-type property.

Theorem 2.1 ([**5**]). *Let $\mathfrak{h} \subset (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ be an indecomposably acting Lie algebra, $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{h})$, and $E = E_0 \oplus E_1 \oplus \cdots \oplus E_r$ the complete decomposition of E, i.e., $\mathfrak{g}$ acts trivially on E_0 and irreducibly on each E_i. If $\mathfrak{h}$ is the holonomy algebra of an indecomposable, non-irreducible Lorentzian manifold, then also $\mathfrak{g}$ decomposes into ideals*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

such that each $\mathfrak{g}_i$ acts irreducibly on E_i and trivially on E_j for $i \neq j$.

This theorem has two important consequences making a further algebraic investigation of $\mathfrak{g}$ possible. Irreducibly acting, connected subgroups of $SO(n)$ are closed and therefore compact. Now, by the theorem the group $G := pr_{SO(n)} Hol_p^0(M,h)$ decomposes into irreducibly acting subgroups. Thus we have as a first consequence that G is compact, although the whole holonomy group must not be compact (for such examples see also [**5**]). The second consequence is, that it suffices to study irreducibly acting groups or algebras $\mathfrak{g}$, a fact which is necessary for trying a classification. We shall see this in detail in the following section.

A further result of [**5**] is the distinction of indecomposable subalgebras of the parabolic algebra into four types.

Theorem 2.2 ([**5**]). *Let $\mathfrak{h}$ be a subalgebra of the parabolic algebra which acts indecomposably on $\mathbb{R}^{n+2}$, $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{h}) = \mathfrak{z} \oplus \mathfrak{d}$ as above. Then $\mathfrak{h}$ belongs to one of the following types.*

1) *If $\mathfrak{h}$ contains $\mathbb{R}^n$, then there are three types:*
   Type 1: *$\mathfrak{h}$ contains $\mathbb{R}$. Then $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$.*
   Type 2: *$pr_{\mathbb{R}}(\mathfrak{h}) = 0$, i.e., $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$.*
   Type 3: *Neither Type 1 nor Type 2. In this case there exists a epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, such that*
$$\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{d}) \ltimes \mathbb{R}^n,$$
   where $\mathfrak{l} := graph\ \varphi = \{(\varphi(T), T) | T \in \mathfrak{z}\} \subset \mathbb{R} \oplus \mathfrak{z}$. Or written in matrix form:
$$\mathfrak{h} = \left\{ \left(\begin{array}{ccc} \varphi(A) & v^t & 0 \\ 0 & A+B & -v \\ 0 & 0 & -\varphi(A) \end{array} \right) \middle| A \in \mathfrak{z}, B \in \mathfrak{d}, v \in \mathbb{R}^n \right\}.$$
2) *In the case where $\mathfrak{h}$ does not contain $\mathbb{R}^n$ we have* **Type 4:** *There exists a non-trivial decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$, $0 < k, l < n$ and a a epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}^l$, such that $\mathfrak{g} \subset \mathfrak{so}(k)$ and $\mathfrak{h} = (\mathfrak{d} \oplus \mathfrak{l}) \ltimes \mathbb{R}^k \subset \mathfrak{p}$ where $\mathfrak{l} := \{(\varphi(T), T)\, | T \in \mathfrak{z}\} = graph\ \varphi \subset \mathbb{R}^l \oplus \mathfrak{z}$. Or written in matrix form:*
$$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} 0 & \varphi(A)^t & v^t & 0 \\ 0 & 0 & 0 & -\varphi(A) \\ 0 & 0 & A+B & -v \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{z}, B \in \mathfrak{d}, v \in \mathbb{R}^k \right\}.$$

These four algebraic types give four types of indecomposable, non-irreducible Lorentzian holonomy groups with the same properties. These types are independent of conjugation within the parabolic group.

2.2. Berger algebras and weak-Berger algebras. Here we will recall the notion of weak-Berger algebras which we have introduced in [**26**]. We derive some basic properties, in particular a decomposition property and the behaviour under complexification. Let E be a vector space over the field $\mathbb{K}$. For a subalgebra $\mathfrak{g} \subset \mathfrak{gl}(E)$ we set

$$\begin{aligned} \mathcal{K}(\mathfrak{g}) &:= \{R \in \Lambda^2 E^* \otimes \mathfrak{g} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0\}, \\ \underline{\mathfrak{g}} &:= \operatorname{span}\{R(x,y) \mid x, y \in E, R \in \mathcal{K}(\mathfrak{g})\}, \end{aligned}$$

and for $\mathfrak{g} \subset \mathfrak{so}(E,h)$, where h is a quadratic form, not necessarily positive definite, we set

$$\mathcal{B}_h(\mathfrak{g}) := \{Q \in E^* \otimes \mathfrak{g} \mid h(Q(x)y,z) + h(Q(y)z,x) + h(Q(z)x,y) = 0\},$$
$$\mathfrak{g}_h := \operatorname{span}\{Q(x) \mid x \in E, Q \in \mathcal{B}_h(\mathfrak{g})\}.$$

Both, $\mathcal{K}(\mathfrak{g})$ and $\mathcal{B}_h(\mathfrak{g})$ are spaces of *curvature endomorphisms*. For distinction, one may call $\mathcal{B}_h(\mathfrak{g})$ the space of *weak curvature endomorphisms*. Both are $\mathfrak{g}$-modules. $\underline{\mathfrak{g}}$ and $\mathfrak{g}_h$ are ideals in $\mathfrak{g}$ (for the weak curvature endomorphisms see [**26**]).

Definition 2.3. Let $\mathfrak{g} \subset \mathfrak{gl}(E)$ be a subalgebra. Then $\mathfrak{g}$ is is called *Berger algebra* if $\underline{\mathfrak{g}} = \mathfrak{g}$. If $\mathfrak{g} \subset \mathfrak{so}(E,h)$ with $\mathfrak{g}_h = \mathfrak{g}$, then we call it *weak-Berger algebra*.

Equivalent to the (weak-)Berger property is the fact that there is no proper ideal $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathcal{K}(\mathfrak{h}) = \mathcal{K}(\mathfrak{g})$ (resp. $\mathcal{B}_h(\mathfrak{h}) = \mathcal{B}_h(\mathfrak{g})$). The following lemma relates both curvature endomorphism modules to each other. Its proof is straightforward and can be found in [**26**].

Lemma 2.4. *The vector space $\mathcal{R}(\mathfrak{g})$ spanned by $\{R(x,.) \in \mathcal{B}(\mathfrak{g}) \mid R \in \mathcal{K}(\mathfrak{g}),\ x \in E\}$ is a $\mathfrak{g}$-submodule of $\mathcal{B}_h(\mathfrak{g})$.*

This implies $\underline{\mathfrak{g}} \subset \mathfrak{g}_h$, so we get a justification of the nomenclature:

Proposition 2.5. *A Berger algebra in $\mathfrak{so}(E,h)$ is a weak-Berger algebra.*

The lemma also implies the following property which we will need later on:

$$\operatorname{span}\{R(x,y) + Q(z) | R \in \mathcal{K}(\mathfrak{g}), Q \in \mathcal{B}_h(\mathfrak{g}), x,y,z \in E\} \subset \mathfrak{g}_h \tag{2}$$

for $\mathfrak{g} \subset \mathfrak{so}(E,h)$. For a weak Berger algebra we obtain a statement similar to the Borel-Lichnerowicz property of Theorem 2.1.

Proposition 2.6. *Let $\mathfrak{g} \subset \mathfrak{so}(E,h)$ be a subalgebra, h being positive definite, and $E = E_0 \oplus E_1 \oplus \cdots \oplus E_r$ the complete decomposition of E, i.e., E_0 is a trivial submodule and the E_i are irreducible for $i = 1,\ldots,n$. If $\mathfrak{g}$ is a weak-Berger algebra, then it decomposes into commuting ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ such that $\mathfrak{g}_i$ acts irreducibly on E_i and trivially on E_j. Each of the $\mathfrak{g}_i \subset \mathfrak{so}(E_i, h_{|E_i})$ is a weak Berger algebra.*

Proof. Since h is positive definite, E is completely reducible and decomposes into irreducible and a trivial subspace as in the proposition. Then we consider for $i = 0,\ldots,r$ the subalgebras $\mathfrak{g}_i := \operatorname{span}\{Q(x_i) | Q \in \mathcal{B}_h(\mathfrak{g}),\ x_i \in E_i\}$. $\mathfrak{g}_i$ is by definition a weak Berger algebra, and since $\mathfrak{g}$ is a weak-Berger algebra it is $\mathfrak{g} = \sum_{i=0}^r \mathfrak{g}_i$. Now we show that for $x_i \in E_i$ and $x_j \in E_j$ for $i \neq j$ it holds that $Q(x_i)x_j = 0$ and $Q(x_0) = 0$ for any

$Q \in \mathcal{B}_h(\mathfrak{g})$. By the Bianchi identity and by the invariance of E_j we get for any $y, z \in E$:

$$h(Q(x_i)y, z) \;=\; h(Q(y)x_i, z) - h(Q(z)x_i, y).$$

This gives $Q(x_0) = 0$. Setting $z = x_j$ and $y = y_j$ also shows that $Q(x_j)x_i = 0$. Hence $\mathfrak{g}_i$ annihilates E_j, which implies $\mathfrak{g}_i \cap \mathfrak{g}_j = \{0\}$ for $i \neq j$, and acts irreducibly on E_i, for $i > 0$. q.e.d.

By the Ambrose-Singer holonomy theorem holonomy algebras of torsion free connections — in particular of a Levi-Civita-connection — are Berger algebras. The list of all irreducible Berger algebras is known ([**6**] for orthogonal, non-symmetric Berger algebras, [**7**] for orthogonal symmetric ones, and [**31**] in the general affine case). The $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian manifold *a priori* is no holonomy algebra, and therefore not necessarily a Berger algebra. But we can show that it is a weak-Berger algebra.

Theorem 2.7. *Let $\mathfrak{h} \subset (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ be an indecomposable, non-irreducible subalgebra with $\mathfrak{g} = pr_{\mathfrak{so}(n)}(\mathfrak{h})$. Then $\mathfrak{h}$ is a Berger algebra if and only if $\mathfrak{g}$ is a weak-Berger algebra. In particular, if $\mathfrak{h}$ is the holonomy algebra of an indecomposable, non irreducible Lorentzian manifold, then $\mathfrak{g}$ is a weak-Berger algebra and it decomposes into irreducibly acting ideals which are weak-Berger algebras.*

Proof. First we suppose that $\mathfrak{h}$ is a Berger algebra. Hence it is generated by endomorphisms of the form $A = \mathcal{R}(U,V)$, with $U, V \in \mathbb{R}^{n+2}$ and $\mathcal{R} \in \mathcal{K}(\mathfrak{h})$. Fixing a basis $(X, E_1, \ldots E_n, Z)$ in $\mathbb{R}^{n+2}$ of the form (1) and setting $E := \operatorname{span}(E_1, \ldots, E_n)$ we obtain for an $A \in \mathfrak{h}$ that

$$pr_{\mathfrak{so}(n)}A = pr_E \circ \mathcal{R}(U,V)_{|E} \quad \text{for arbitrary } U, V \in \mathbb{R}^{n+2},$$

i.e., for an $Y \in E$ it is

$$\left(pr_{\mathfrak{so}(n)}A\right)Y = pr_E\left(\mathcal{R}(U,V)Y\right) = \sum_{k=1}^{n} h(\mathcal{R}(U,V)Y, E_k)E_k.$$

With respect to the above basis we can write $U = \xi_1 X + Y_1 + \zeta_1 Z$ and $V = \xi_2 X + Y_2 + \zeta_2 Z$ with $Y_i \in E$, ξ_i, $\zeta_i \in \mathbb{R}$, $i = 1, 2$. Using the symmetries of the $\mathcal{R}$ we obtain for U, V as above and $Y \in E$:

$$\begin{aligned} &h(\mathcal{R}(U,V)Y, E_k) \\ &\quad = h\left(\mathcal{R}\left(Y_1, Y_2\right)Y, E_k\right) + h\left(\mathcal{R}\left(Z, \zeta_1 Y_2 - \zeta_2 Y_1\right)Y, E_k\right). \end{aligned}$$

Finally it is

$$
\begin{aligned}
\left(pr_{\mathfrak{so}(n)}A\right)Y &= \sum_{k=1}^{n} h([\mathcal{R}(Y_1,Y_2)+\mathcal{R}(Z,\zeta_1Y_2-\zeta_2Y_1)]Y,E_k)E_k \\
&= pr_E\left(\mathcal{R}(Y_1,Y_2)Y\right)+pr_E\left(\mathcal{R}(Z,\zeta_1Y_2-\zeta_2Y_1)Y\right) \\
&= R(Y_1,Y_2)Y+Q(\zeta_1Y_2-\zeta_2Y_1)Y,
\end{aligned}
$$

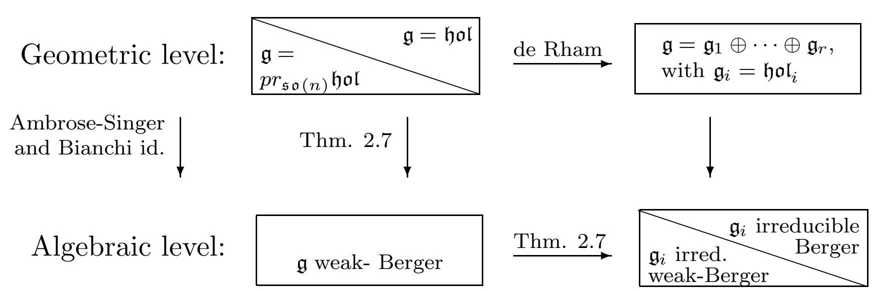
with

$$
\begin{aligned}
R(.,.) &:= pr_E\circ\mathcal{R}(.,.)_{|E\times E\times E} &\in \wedge^2E^*\otimes\mathfrak{g} \\
Q(.) &:= pr_E\circ\mathcal{R}(Z,.)_{|E\times E} &\in E^*\otimes\mathfrak{g}.
\end{aligned}
$$

The Bianchi identity implies $R\in\mathcal{K}(\mathfrak{g})$ and $Q\in\mathcal{B}_h(\mathfrak{g})$. Hence, the generators of $\mathfrak{g}$, which are $\mathfrak{so}(n)$-projections of the generators of $\mathfrak{hol}_p(M,h)$, are of the form $R(Y_1,Y_2)+Q(Y_3)$ with $Y_i\in E$, $R\in\mathcal{K}(\mathfrak{g})$, $Q\in\mathcal{B}_h(\mathfrak{g})$. By (2) we obtain $\mathfrak{g}\subset\mathfrak{g}_h$, i.e., $\mathfrak{g}$ is a weak-Berger algebra.

For the other direction we refer to the result of [**19**] who proved that $\mathfrak{h}\subset(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n$ is a Berger algebra if $\mathfrak{g}:=pr_{\mathfrak{so}(n)}(\mathfrak{h})$ is weak Berger algebra and satisfies the Borel-Lichnerowicz decomposition property of Theorem 2.1. But Proposition 2.6 shows that this property is satisfied for weak-Berger algebras. Hence, this is no additional condition and we get the other direction. The remaining statements follow from the Ambrose-Singer holonomy theorem [**3**] and Proposition 2.6. q.e.d.

The 'only if'-direction of this theorem we proved in [**26**]. It ensures that we are at a similar point as in the Riemannian situation, but reaching it in a different way. This is shown schematically in the following diagram:



2.3. Real and complex weak-Berger algebras. Because of the above result we have to classify real weak-Berger algebras. Since we will use representation theory of complex semisimple Lie algebras we have to describe the transition of a real weak-Berger algebra to its complexification. First we note that the spaces $\mathcal{K}(\mathfrak{g})$ and $\mathcal{B}_h(\mathfrak{g})$ for $\mathfrak{g}\subset\mathfrak{so}(E,h)$

can be described by the following exact sequences:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{K}(\mathfrak{g}) & \hookrightarrow & \wedge^2 E^* \otimes \mathfrak{g} & \stackrel{\lambda}{\rightarrow} & \wedge^3 E^* \otimes E \\
0 & \rightarrow & \mathcal{B}_h(\mathfrak{g}) & \hookrightarrow & E^* \otimes \mathfrak{g} & \stackrel{\lambda_h}{\rightarrow} & \wedge^3 E^*,
\end{array}
$$

where the map λ is the skew-symmetrisation and λ_h the dualisation by h and the skew-symmetrisation. If we consider a real Lie algebra $\mathfrak{g}$ acting orthogonally on a real vector space E, then h extends by complexification to a non-degenerate complex-bilinear form $h^{\mathbb{C}}$ which is invariant under $\mathfrak{g}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$. The complexification of the above exact sequences gives

$$
\mathcal{K}(\mathfrak{g})^{\mathbb{C}} = \mathcal{K}(\mathfrak{g}^{\mathbb{C}}) \tag{3}
$$

$$
(\mathcal{B}_h(\mathfrak{g}))^{\mathbb{C}} = \mathcal{B}_{h^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}) \tag{4}
$$

and leads to the following statement.

Proposition 2.8. *$\mathfrak{g} \subset \mathfrak{so}(E, h)$ is a (weak-) Berger algebra if and only if $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$ is a (weak-) Berger algebra.*

Thus complexification preserves the weak-Berger as well as the Berger property. But irreducibility is a property which is *not* preserved under complexification. In order to deal with this problem one recalls the following definition, distinguishing two cases for an irreducible module of a real Lie algebra.

Definition 2.9. Let $\mathfrak{g}$ be a real Lie algebra. Irreducible real $\mathfrak{g}$-modules E for which $E^{\mathbb{C}}$ is irreducible and irreducible complex modules V for which $V_{\mathbb{R}}$ is reducible are called of *real type*. Irreducible real $\mathfrak{g}$-modules E for which $E^{\mathbb{C}}$ is reducible and irreducible complex modules V for which $V_{\mathbb{R}}$ is irreducible are called of *non-real type*.

In the original papers of Cartan [**14**] and Iwahori [**23**], see also [**21**], where these distinction is introduced, a representation of real type is called "representation of first type" and a representation of non-real type is called of "second type". The above notation makes sense because the complexification of a real module of real type is of real type — recall that $(E^{\mathbb{C}})_{\mathbb{R}}$ is a reducible $\mathfrak{g}$-module — and the reellification of a complex module of non-real type is of non-real type. The notation corresponds to the distinction of complex irreducible $\mathfrak{g}$-modules into real, complex and quaternionic ones: A complex $\mathfrak{g}$ module V is of real type if and only if it is self-conjugate and the invariant anti-linear bijection J satisfies $J^2 = id$. V is called of *quaternionic type* if and only if it is self-conjugate with $J^2 = -id$, and it is called of *complex type* if V is not self conjugate. Obviously, a complex $\mathfrak{g}$-module is of non-real type if and only if it is of complex or quaternionic type. For unitary $\mathfrak{g}$ we recall the following equivalences [**9**].

Proposition 2.10. *If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is unitary w.r.t. a positive definite form, then V is*

1) *of real type if and only if it is orthogonal,*
2) *of complex type if and only if it is not self-dual,*
3) *of quaternionic type if and only if it is symplectic.*

Before we make further remarks on real and non-real type modules, we consider the transition from the real Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(E)$ to the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(E^{\mathbb{C}})$ represented on the complex module $W := E^{\mathbb{C}}$. Obviously, this transition preserves irreducibility:

Lemma 2.11. *Let $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(W)$ be the complexification of $\mathfrak{g} \subset \mathfrak{gl}(W)$ with a complex $\mathfrak{g}$-module W. Then it holds that:*

1) *$\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}^{\mathbb{C}}$ is irreducible.*
2) *$\mathfrak{g} \subset \mathfrak{so}(W,H)$ if and only if $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(W,H)$, where H is a symmetric bilinear form.*

2.3.1. Representations of non-real type. (See again [**14**], [**23**], also [**21**] and the appendix of [**28**] or [**29**] for more details.) Suppose that $\mathfrak{g} \subset \mathfrak{so}(E)$ is of non-real type. In this case the complexified module $E^{\mathbb{C}}$ splits into two submodules $E^{\mathbb{C}} = V \oplus \overline{V}$, where $\overline{V}$ is the $\mathfrak{g}$-module which is conjugated to V according to the conjugation induced by the real subspace $E \subset E^{\mathbb{C}}$. Of course, E is isomorphic to the $\mathfrak{g}$-module $V_{\mathbb{R}}$ which is the reellification of V or of $\overline{V}$. If we start with $\mathfrak{g} \subset \mathfrak{so}(E,h)$, where h is positive definite, the fact that V is a $\mathfrak{g}$-module which is of non-real type implies by Proposition 2.10 the following result, which will be of importance later on.

Proposition 2.12. *Let $\mathfrak{g} \subset \mathfrak{so}(E,h)$ be a real Lie algebra, E a $\mathfrak{g}$-module of non-real type, i.e., $E^{\mathbb{C}} = V \oplus \overline{V}$, and suppose that h is positive definite. Then $\mathfrak{g} \subset \mathfrak{gl}(V)$ is unitary with respect to a positive definite Hermitian form and not orthogonal.*

This means that the complexified symmetric form $h^{\mathbb{C}}$ gives a pairing between V and $\overline{V}$ and vanishes on $V \times V$. Since we shall switch to the complex Lie algebra we should remark that the decomposition $E^{\mathbb{C}} = V \oplus \overline{V}$ is also $\mathfrak{g}^{\mathbb{C}}$-invariant.

2.3.2. Weak-Berger algebras of real type. Before we start to classify weak-Berger algebras of real type in Section 4, 5 and 6 we have to make some observation and remarks.

Let $\mathfrak{g}_0$ be a real Lie algebra and E a real irreducible module of real type. Furthermore we suppose $\mathfrak{g}_0 \subset \mathfrak{so}(E,h)$, now h not necessarily positive definite. Then $E^{\mathbb{C}}$ is an irreducible $\mathfrak{g}_0$-module (also of real type) and $\mathfrak{g}_0 \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$. Now we may extend h to a Hermitian form θ^h on $E^{\mathbb{C}}$, which is invariant under $\mathfrak{g}_0$. Thus, $\mathfrak{g}_0 \subset \mathfrak{u}(V, \theta^h)$ and θ^h has the same signature as h.

Subalgebras of $\mathfrak{so}(E,h)$ which act completely reducibly are reductive [**15**]. If h is positive definite, then $\mathfrak{g}_0$ is even compact. In any case its Levi-decomposition is $\mathfrak{g}_0 = \mathfrak{z}_0 \oplus \mathfrak{d}_0$, with centre $\mathfrak{z}_0$ and semisimple derived algebra $\mathfrak{d}_0$. Thus $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{z} \oplus \mathfrak{d}$ is also reductive. But since it is irreducible by assumption, the Schur lemma implies that the centre $\mathfrak{z}$ is $\mathbb{C}\, Id$ or zero, and by $\mathfrak{g}_0^{\mathbb{C}} \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$ it is zero. Hence $\mathfrak{g}$ is semisimple. Resuming all this, Proposition 2.8 can be reformulated as follows.

Proposition 2.13. *If $\mathfrak{g}_0 \subset \mathfrak{so}(E,h)$ is a weak-Berger algebra of real type, then $\mathfrak{g}_0^{\mathbb{C}} \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$ is an irreducible complex weak-Berger algebra. $E^{\mathbb{C}}$ is a $\mathfrak{g}_0$-module of real type and $\mathfrak{g}_0^{\mathbb{C}}$ is semisimple.*

Let $\mathfrak{g} \subset \mathfrak{so}(V,H)$ be a semisimple complex Lie algebra with a complex $\mathfrak{g}$-module V of real type. Then $\mathfrak{g}$ has a real form $\mathfrak{g}_0$, $V = E^{\mathbb{C}}$ with a real $\mathfrak{g}_0$-module E of real type, $\mathfrak{g}_0$ is unitary with respect to a Hermitian form θ and $\mathfrak{g}_0 \subset \mathfrak{so}(E,h)$ where the signatures of h and θ are equal. For the compact real form of $\mathfrak{g}$ the quadratic form h is positive definite. If $\mathfrak{g}$ is in addition a weak-Berger algebra, then $\mathfrak{g}_0$ is a weak-Berger too.

Proof. The first direction follows directly from Proposition 2.8 and the above definitions. Since $\mathfrak{g}$ is semisimple it has a compact real form $\mathfrak{g}_0$. If V is a $\mathfrak{g}_0$-module of real type then it is $V = E^{\mathbb{C}}$, and $\mathfrak{g}_0 \subset \mathfrak{gl}(V)$ is unitary since it is orthogonal, both w.r.t. the same signature. Then the proposition follows by Proposition 2.8. (For details of the proof see appendix of [**29**] or [**28**].) q.e.d.

Considering the four different types of indecomposable, non-irreducible holonomy algebras from Theorem 2.2, the above proposition already yields the following observation.

Corollary 2.14. *Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian holonomy algebra, which is supposed to be of coupled type* 3 *or* 4*. Then at least one of the irreducibly acting ideals of $\mathfrak{g} \subset \mathfrak{so}(n)$ is of non-real type.*

Remark 2.15. We have to make a remark about the definition of holonomy up to conjugation. The $SO(n)$-component of an indecomposable, non-irreducible Lorentzian manifold was defined modulo conjugation in $O(n)$. Hence we shall not distinguish between subalgebras of $\mathfrak{gl}(n,\mathbb{C})$ which are isomorphic under Ad_φ where φ is an element from $O(n,\mathbb{C})$ and Ad the adjoint action in of $Gl(n,\mathbb{C})$ on $\mathfrak{gl}(n,\mathbb{C})$. We say that an orthogonal representation κ_1 of a complex semisimple Lie algebra $\mathfrak{g}$ is *congruent* to an orthogonal representation κ_2 if there is an element $\varphi \in O(n,\mathbb{C})$ such that the following equivalence of $\mathfrak{g}$–representations is valid: $\kappa_1 \sim Ad_\varphi \circ \kappa_2$. Hence we have to classify semisimple, orthogonal, irreducibly acting, complex weak-Berger algebras of real type up to this congruence of representations.

If the automorphism Ad_φ is inner, then the representations are equivalent; if it is outer then only congruent. For semisimple Lie algebras it holds that $Out(\mathfrak{g}) := Aut(\mathfrak{g})/Inn(\mathfrak{g})$ counts the connection components of $Aut(\mathfrak{g})$, and $Out(\mathfrak{g})$ is isomorphic to the automorphism of the fundamental system, i.e., symmetries of the Dynkin diagram (see for example [**36**]). This will become relevant in case of $\mathfrak{so}(8,\mathbb{C})$ where the symmetries of the Dynkin diagram generate the symmetric group $\mathcal{S}_3$, i.e., $Out(\mathfrak{so}(8,\mathbb{C})) = \mathcal{S}_3$ and it contains the so-called "triality automorphism" which interchanges vector and spin representations of $\mathfrak{so}(8,\mathbb{C})$ without fixing one. We shall use that the automorphism which interchanges the vector representation with one spinor representation and fixes the second spinor representation resp. interchanges the spinor representations and fixes the vector representation comes from Ad_φ with $\varphi \in O(n,\mathbb{C})$. Hence the vector and the spinor representations of $\mathfrak{so}(8,\mathbb{C})$ are congruent to one another.

Finally, we should remark that compact real forms equivalent to a given one correspond to inner automorphism of $\mathfrak{g}$. Hence the corresponding representations are equivalent.

3. Weak-Berger algebras of non-real type

In this section we shall classify real weak-Berger algebras (w.r.t. a positive definite quadratic form h) of non-real type, by showing that they are Berger algebras. We will use the classification of first prolongations of irreducible complex Lie algebras and shall show that the complexification of the space $\mathcal{B}_h(\mathfrak{g}_0)$ is isomorphic to the first prolongation of the complexified Lie algebra.

Throughout this section $\mathfrak{g}_0$ is a real Lie algebra and E a real $\mathfrak{g}_0$-module of non-real type, i.e., $W := E^{\mathbb{C}} = V \oplus \overline{V}$ is not irreducible. Furthermore we assume $\mathfrak{g}_0 \subset \mathfrak{so}(E,h)$ with h positive definite and set $H := h^{\mathbb{C}}$.

We define the following complex Lie algebra:

$$\mathfrak{g} := \left\{ A_{|V} \;\middle|\; A \in \mathfrak{g}_0^{\mathbb{C}} \subset \mathfrak{so}(V\oplus\overline{V}, h^{\mathbb{C}}) \right\} \subset \mathfrak{gl}(V). \tag{5}$$

Since the symmetric bilinear form we start with is positive definite, $\mathfrak{g} \subset \mathfrak{gl}(V)$ is unitary but not orthogonal by Proposition 2.12.

In $\mathfrak{g}_0^{\mathbb{C}}$ as well as in $\mathfrak{g}$ we have a conjugation $^{-}$ with respect to $\mathfrak{g}_0$ and $(\mathfrak{g}_0)_{|V}$ respectively. Since $A \in \mathfrak{g}_0$ acts on $V \oplus \overline{V}$ by $A(v+\overline{w}) = Av + \overline{Aw}$ we have for $iA \in \mathfrak{g}_0^{\mathbb{C}}$ that $iA(v+\overline{w}) = i(Av + \overline{Aw}) = (iAv + \overline{-iAw})$. So we write the action of $A \in \mathfrak{g}_0^{\mathbb{C}}$ with the help of the conjugation in $\mathfrak{g}$:

$$A(v+\overline{w}) = Av + \overline{\overline{A}w}. \tag{6}$$

This gives the following Lie algebra isomorphism

$$\begin{array}{rcccc} \varphi & : & \mathfrak{g}_0^{\mathbb{C}} & \simeq & \mathfrak{g} \\ & & A & \mapsto & A_{|V}. \end{array}$$

It is injective because for $A_{|V} = B_{|V}$ it holds that $A(v+\overline{w}) = Av+\overline{\overline{A}w} = Bv + \overline{\overline{B}w} = B(v+\overline{w})$ for all $v, w \in V$, i.e., $A = B$. By definition it is surjective and φ^{-1} is given by

$$\varphi^{-1}(A) \ : \ v + \overline{w} \ \longmapsto \ Av + \overline{\overline{A}w} \quad \text{for all} \quad A \in \mathfrak{g}. \tag{7}$$

These notations are needed to show the relation to the first prolongation.

3.1. The first prolongation of a weak-Berger algebra of non-real type. Given a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, the $\mathfrak{g}$-module

$$\mathfrak{g}^{(1)} \ := \ \{Q \in V^* \otimes \mathfrak{g} \mid Q(u)v = Q(v)u\}. \tag{8}$$

is called *first prolongation* of $\mathfrak{g} \subset \mathfrak{gl}(V)$. Furthermore we set

$$\tilde{\mathfrak{g}} \ := \ span\{Q(u) \in \mathfrak{g} \mid Q \in \mathfrak{g}^{(1)}, u \in V\} \subset \mathfrak{g}.$$

Now we describe the space $\mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$ with the help of $\mathfrak{g}^{(1)}$.

Proposition 3.1. *Let E be a $\mathfrak{g}_0$-module of non-real type, orthogonal with respect to a positive definite scalar product h, $E^{\mathbb{C}} = V \oplus \overline{V}$ the corresponding $\mathfrak{g}_0^{\mathbb{C}}$ invariant decomposition, and $\mathfrak{g}$ defined as in* (5)*. Then there is an isomorphism*

$$\begin{array}{rcccc} \phi & : & \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}}) & \simeq & \mathfrak{g}^{(1)} \\ & & Q & \mapsto & Q_{|V\times V}. \end{array}$$

Proof. For the proof we will use the $\mathfrak{g}_0$–invariant Hermitian form θ on V which is given by $\theta(u,v) = h^{\mathbb{C}}(u,\overline{v})$, where $^-$ is the conjugation in $E^{\mathbb{C}} = V \oplus \overline{V}$ with respect to E. The linearity of ϕ is clear. We have to show the following:

ϕ is well-defined, i.e., for $Q \in \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$ it is $Q_{|V\times V} \in \mathfrak{g}^{(1)}$: For every $u, v, w \in V$ and $H = h^{\mathbb{C}}$ it holds

$$\begin{array}{rcl} \theta(Q(u)v, w) & = & h^{\mathbb{C}}(Q(u)v, \overline{w}) \\ & = & -h^{\mathbb{C}}(Q(v)\overline{w}, u) \ - \ \underbrace{h^{\mathbb{C}}(Q(\overline{w})u, v)}_{\substack{= 0 \\ \text{since } h^{\mathbb{C}}_{V\times V} = 0 \text{ (Proposition 2.12)}}} \\ & = & h^{\mathbb{C}}(Q(v)u, \overline{w}) \\ & = & \theta(Q(v)u, w), \end{array}$$

i.e., $Q(u)v = Q(v)u$ which means that $Q_{|V\times V} \in \mathfrak{g}^{(1)}$.

The homomorphism ϕ is injective: Let Q_1 and Q_2 be in $\mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$ with $(Q_1)_{|V\times V} = (Q_2)_{|V\times V}$. Then

a) $(Q_1)_{|\overline{V}\times\overline{V}} = (Q_2)_{|\overline{V}\times\overline{V}}$, since $Q_1(\overline{u})\overline{v} = \overline{Q_1(u)v} = \overline{Q_2(u)v} = Q_2(\overline{u})\overline{v}$,

b) $(Q_1)_{|\overline{V}\times V} = (Q_2)_{|\overline{V}\times V}$, since

$$\begin{array}{rcccl} \theta(Q_1(\overline{u})v,w) & = & h^{\mathbb{C}}(Q_1(\overline{u})v,\overline{w}) & = & -h^{\mathbb{C}}(v,Q_1(\overline{u})\overline{w}) = \\ = h^{\mathbb{C}}(v,Q_2(\overline{u})\overline{w}) & = & h^{\mathbb{C}}(Q_2(\overline{u})v,\overline{w}) & = & \theta(Q_2(\overline{u})v,w). \end{array}$$

c) $(Q_1)_{|V\times\overline{V}} = (Q_2)_{|V\times\overline{V}}$, by b) with the same argument as in a).

The homomorphism ϕ is surjective: For $Q \in \mathfrak{g}^{(1)}$ we define ϕ^{-1} using φ,

$$(\phi^{-1}Q)(u) := \varphi^{-1}(Q(u)) \quad \text{and} \quad (\phi^{-1}Q)(\overline{u}) := \varphi^{-1}\left(\overline{Q(u)}\right) \in \mathfrak{gl}(E^{\mathbb{C}}).$$

It is $(\phi^{-1}Q)(\overline{u},\overline{v}) = \overline{(\phi^{-1}Q)(u,v)}$. Then obviously $\phi\circ\phi^{-1} = id$, since $\phi\left(\phi^{-1}(Q)\right) = \phi^{-1}(Q)_{|V\times V} = Q$. By the symmetry of Q and recalling that $H|_{V\times V} = 0$, a direct calculation gives $(\phi^{-1}Q) \in \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$. q.e.d.

The proof of this proposition relies on the fact that V is not orthogonal, which was due to Proposition 2.12. An analogous result can be proven for the space $\mathcal{K}(\mathfrak{g})$ (see [**26**]). We obtain two corollaries.

Corollary 3.2. *Let $\mathfrak{h}_0 \subset \mathfrak{g}_0 \subset \mathfrak{so}(E^{\mathbb{C}},h^{\mathbb{C}})$ be subalgebras of non-real type, $\mathfrak{h}$ and $\mathfrak{g}$ defined as above. If $\mathfrak{h}^{(1)} = \mathfrak{g}^{(1)}$, then $(\mathfrak{h}_0^{\mathbb{C}})_H = (\mathfrak{g}_0^{\mathbb{C}})_H$. i.e., if in $\mathfrak{g}$ there exists a proper subalgebra which has the same first prolongation and a compact real form in $\mathfrak{g}_0$ of non-real type, then $\mathfrak{g}_0^{\mathbb{C}}$ and therefore $\mathfrak{g}_0$ cannot be weak-Berger algebras.*

Proof. Because of $Q \in \mathcal{B}_H(\mathfrak{h}_0^{\mathbb{C}}) \simeq \mathfrak{h}^{(1)} = \mathfrak{g}^{(1)} \simeq \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$ we have $Q(u) \in (\mathfrak{g}_0^{\mathbb{C}})_H$ if and only if $Q(u) \in (\mathfrak{h}_0^{\mathbb{C}})_H$. q.e.d.

Corollary 3.3. *Let $\mathfrak{g}_0 \subset \mathfrak{so}(E^{\mathbb{C}},H)$ be a Lie algebra of non-real type, and $\mathfrak{g}$ defined as above. Then $(\mathfrak{g}_0^{\mathbb{C}})_H = \mathfrak{g}_0^{\mathbb{C}}$ (i.e., $\mathfrak{g}_0^{\mathbb{C}}$ is a weak-Berger-algebra) if and only if $\mathfrak{g} = \tilde{\mathfrak{g}}$.*

Proof. First we show the sufficiency: Let $A \in \mathfrak{g}_0^{\mathbb{C}}$ be arbitrary. $\mathfrak{g} = \tilde{\mathfrak{g}}$ gives w.l.o.g. that $\varphi(A) = Q(u)$ with $Q \in \mathfrak{g}^{(1)}$ and $u \in V$. But then:

$$(\phi^{-1}Q)(u) \overset{\text{per def.}}{=} \varphi^{-1}(Q(u)) = \varphi^{-1}(\varphi(A)) = A,$$

with $(\phi^{-1}Q) \in \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$, i.e., $A \in (\mathfrak{g}_0^{\mathbb{C}})_H$.

Now we show the necessity: If $A \in \mathfrak{g}$, the assumption $\mathfrak{g}_0^{\mathbb{C}} = (\mathfrak{g}_0^{\mathbb{C}})_H$ gives w.l.o.g. that $\varphi^{-1}(A) = \hat{Q}(u+\overline{v})$ with $\hat{Q} \in \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}})$, $u \in V$ and $\overline{v} \in \overline{V}$. By Proposition 3.1 there is a $Q \in \mathfrak{g}^{(1)}$ such that

$$\varphi^{-1}(A) = \hat{Q}(u+\overline{v}) = (\phi^{-1}Q)(u+\overline{v}) = \varphi^{-1}(Q(u)) + \varphi^{-1}(\overline{Q(v)}).$$

But this means that $A = \underbrace{Q(u)}_{\in\tilde{\mathfrak{g}}} + \underbrace{\overline{Q(v)}}_{\in\tilde{\mathfrak{g}}} \in \tilde{\mathfrak{g}}$, i.e., $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. q.e.d.

Analogous results can be obtained for Berger algebras leading — with the same reasoning as below — to a classification of irreducible Berger algebras of non-real type [**26**].

As a result of the previous and this section we have to investigate complex irreducible representations of complex Lie algebras with non-vanishing first prolongation. Only their real forms of non-real type may be candidates for weak-Berger algebras.

3.2. Lie algebras with non-trivial first prolongation. There are only a few complex Lie algebras irreducibly contained in $\mathfrak{gl}(V)$which have non vanishing first prolongation. The classification can be found in [**13**] and [**25**]. We collect them in two tables, following [**33**].

Table 1. Complex Lie-groups and algebras with $\mathfrak{g}^{(1)} \neq 0$ and $\mathfrak{g}^{(1)} \neq V^*$:

	G	$\mathfrak{g}$	V	$\mathfrak{g}^{(1)}$
1.	$Sl(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n, \quad n \geq 2$	$(V \otimes \odot^2 V^*)_0$
2.	$Gl(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathbb{C}^n, \quad n \geq 1$	$V \otimes \odot^2 V^*$
3.	$Sp(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}, \quad n \geq 2$	$\odot^3 V^*$
4.	$\mathbb{C}^* \times Sp(n,\mathbb{C})$	$\mathbb{C} \oplus \mathfrak{sp}(n,\mathbb{C})$,	$\mathbb{C}^{2n}, \quad n \geq 2$	$\odot^3 V^*$

Table 2. Complex Lie-groups and algebras with first prolongation $\mathfrak{g}^{(1)} = V^*$:

	G	$\mathfrak{g}$	V
1.	$CO(n,\mathbb{C})$	$\mathfrak{co}(n,\mathbb{C})$	$\mathbb{C}^n$, $n \geq 3$
2.	$Gl(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$\odot^2\mathbb{C}^n$, $n \geq 2$
3.	$Gl(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$\wedge^2\mathbb{C}^n$, $n \geq 5$
4.	$Gl(n,\mathbb{C}) \cdot Gl(m,\mathbb{C})$	$\mathfrak{sl}(\mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(m,\mathbb{C}))$	$\mathbb{C}^n \otimes \mathbb{C}^m$, $m,n \geq 2$
5.	$\mathbb{C}^* \cdot Spin(10,\mathbb{C})$	$\mathbb{C} \oplus \mathfrak{spin}(10,\mathbb{C})$	$\Delta_{10}^+ \simeq \mathbb{C}^{16}$
6.	$\mathbb{C}^* \cdot E_6$	$\mathbb{C} \oplus \mathfrak{e}_6$	$\mathbb{C}^{27}$

For a details of these representations see [**1**].

3.2.1. The algebras of Table 1. Regarding Table 1, its first three entries are complexifications of the Riemannian holonomy algebras $\mathfrak{su}(n)$, $\mathfrak{u}(n)$ acting on $\mathbb{R}^{2n}$ and $\mathfrak{sp}(n)$ acting on $\mathbb{R}^{4n}$, which are Berger algebras. The fourth has the compact real form $\mathfrak{so}(2) \oplus \mathfrak{sp}(n)$ acting irreducibly on $\mathbb{R}^{4n}$. Since the representation of $\mathfrak{sp}(n)$ on $\mathbb{R}^{4n}$ is of non-real type, we are in the situation of Corollary 3.2, because $(\mathbb{C}Id \oplus \mathfrak{sp}(n,\mathbb{C}))^{(1)} = \mathfrak{sp}(n,\mathbb{C})^{(1)}$. Hence $\mathfrak{so}(2) \oplus \mathfrak{sp}(2n)$ is not a weak-Berger algebra.

3.2.2. The algebras of Table 2. Looking at the unique compact real form of the Lie algebras and the reellification of the representations in

Table 2, one sees that they correspond to the holonomy representations of Riemannian symmetric spaces which are Kählerian (for a detailed proof see the appendix of [**26**]).

Proposition 3.4. *The compact real forms of the algebras in Table* 2 *and the reellification of the representations are equivalent to the holonomy representations of the Riemannian, Kählerian symmetric spaces, i.e., the symmetric spaces of type BDI, CI, DIII, AIII, EIII and EVII.*

3.3. The result for weak-Berger algebras of non-real type. The conclusion from the previous section is: every real Lie algebra $\mathfrak{g}_0$ of non-real type, i.e., contained in $\mathfrak{u}(n)$, which is a weak-Berger algebra is a Berger algebra. Furthermore, each of these Lie algebras is the holonomy algebra of a Riemannian manifold, the first three entries of Table 1 of non-symmetric ones, and the entries of Table 2 of symmetric ones.

Theorem 3.5. *Let $\mathfrak{g}$ be a Lie algebra and E an irreducible $\mathfrak{g}$–module of non-real type. If $\mathfrak{g} \subset \mathfrak{so}(E,h)$ is a weak-Berger algebra, where h is positive definite, then it is a Berger algebra, in particular a Riemannian holonomy algebra.*

Before we we apply this to the irreducible components of the screen holonomy algebra of an indecomposable, non-irreducible Lorentzian manifold, we prove a lemma to get the result in full generality.

Lemma 3.6. *Let $\mathfrak{g} \subset \mathfrak{u}(n) \subset \mathfrak{so}(2n)$ be a Lie algebra with the decomposition property of Theorem* 2.1*, i.e., there exist decompositions of $\mathbb{R}^{2n}$ into orthogonal subspaces and of $\mathfrak{g}$ into ideals*

$$\mathbb{R}^{2n} = E_0 \oplus E_1 \oplus \cdots \oplus E_r \quad \textit{and} \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

where $\mathfrak{g}$ acts trivially on E_0, $\mathfrak{g}_i$ acts irreducibly on E_i and $\mathfrak{g}_i(E_j) = \{0\}$ for $i \neq j$. Then $\mathfrak{g} \subset \mathfrak{u}(n)$ implies dim $E_i = 2k_i$ and $\mathfrak{g}_i \subset \mathfrak{u}(k_i)$ for $i = 1, \ldots, r$.

Proof. Let $\mathbb{R}^{2n} = \mathbb{C}^n$ and θ be the positive definite Hermitian form on $\mathbb{C}^n$. Let E_i be an invariant subspace on which $\mathfrak{g}$ acts irreducibly. If $E_i = V^i_{\mathbb{R}}$ for a complex vector space V^i, then we can restrict θ to V^i. Because θ is positive definite it is non-degenerate on V^i — since $\theta(v,v) > 0$ for $v \neq 0$ — we get that $\mathfrak{g}_i \subset \mathfrak{u}(V^i, \theta)$, i.e., $\mathfrak{g} \subset \mathfrak{u}(k_i)$. Hence, we have to consider a subspace E_i which is not the reellification of a complex vector space. Let J be the complex structure on $\mathbb{R}^{2n}$. We consider the real vector space JE_i, which is invariant under $\mathfrak{g}$, since J commutes with $\mathfrak{g}$. Then the space $JE_i \cap E_i$ is contained in E_i as well as in JE_i and invariant under $\mathfrak{g}$. Because $\mathfrak{g}$ acts irreducibly on E_i we get two cases. The first is $E_i \cap JE_i = E_i = JE_i$, but this was excluded since E_i was not a reellification. The second is $E_i \cap JE_i = \{0\}$. So we have two invariant irreducible subspaces on which $\mathfrak{g}$ acts simultaneously, i.e.,

$A(x, Jy) = (Ax, AJy)$, but this is not possible because of the Borel-Lichnerowicz decomposition property from Theorem 2.1. q.e.d.

By this lemma and by Theorem 3.5 we get the conclusion.

Theorem 3.7. *Let (M, h) be an indecomposable, non-irreducible $(n+2)$-dimensional Lorentzian manifold and set $\mathfrak{g} := pr_{\mathfrak{so}(n)}\mathfrak{hol}_p(M, h)$. Then every irreducible component $\mathfrak{g}_i$ of $\mathfrak{g}$ (due to Theorem 2.1) which is unitary, i.e., $\mathfrak{g}_i \subset \mathfrak{u}(d_i/2)$, for d_i the dimension of E_i, is the holonomy algebra of a Riemannian manifold. In particular, if $\mathfrak{g} \subset \mathfrak{u}(n) \subset \mathfrak{so}(2n)$, then $\mathfrak{g}$ is the holonomy algebra of a Riemannian manifold.*

4. Semisimple complex weak-Berger algebras

Due to the arguments of Section 2.3.2, in order to classify real weak-Berger algebras of real type we have to classify irreducible, complex, semisimple weak-Berger algebras. The tools for doing this we will explain in this section. The argumentation here and in the following sections is analogous to the reasoning in [**33**], [**32**] and [**31**].

4.1. Irreducible, complex, orthogonal, semisimple Lie algebras. In the following, V will be a complex vector space equipped with a non-degenerate symmetric bilinear 2–form H. $\mathfrak{g}$ is an irreducibly acting, complex, semisimple subalgebra of $\mathfrak{so}(V, H)$. Thus, we may use all the tools provided by roots and weights. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{t}^*$ be the roots of $\mathfrak{g}$, and set $\Delta_0 := \Delta \cup \{0\}$. $\mathfrak{g}$ decomposes into its root spaces $\mathfrak{g}_\alpha := \{A \in \mathfrak{g} | [T, A] = \alpha(T) \cdot A \text{ for all } T \in \mathfrak{t}\} \neq \{0\}$:

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_0 = \mathfrak{t}.$$

Let $\Omega \subset \mathfrak{t}^*$ be the weights of $\mathfrak{g} \subset \mathfrak{so}(V, H)$. Then V decomposes into weight spaces $V_\mu := \{v \in V | T(v) = \mu(T) \cdot v \text{ for all } T \in \mathfrak{t}\} \neq \{0\}$,

$$V = \bigoplus_{\mu \in \Omega} V_\mu.$$

As $\mathfrak{g} \subset \mathfrak{so}(V, H)$, the weight spaces are related as follows.

Proposition 4.1. *Let $\mathfrak{g} \subset \mathfrak{so}(V, H)$ be a complex, semisimple Lie algebra with weight space decomposition. Then $V_\mu \perp V_\lambda$ if and only if $\lambda \neq -\mu$. In particular, if μ is a weight, then $-\mu$ too.*

Proof. For any $T \in \mathfrak{t}$, $u \in V_\mu$ and $v \in V_\lambda$ we have

$$0 = H(Tu, v) + H(u, Tv) = (\mu(T) + \lambda(T))\, H(u, v).$$

If $\lambda \neq -\mu$, there is a T such that $\mu(T) + \lambda(T) \neq 0$. But this implies $V_\lambda \perp V_\mu$. On the other hand $V_\mu \perp V_{-\mu}$ would imply $V_\mu \perp V$, which contradicts the non-degeneracy of H. This also implies that $\mu \in \mathfrak{t}^*$ is a weight if and only if $-\mu$ is a weight. q.e.d.

4.2. Irreducible complex weak-Berger algebras. If $\mathfrak{g}$ is a weak-Berger algebra, then $\mathcal{B}_h(\mathfrak{g})$ is a non-zero $\mathfrak{g}$–module. If we denote by Π its weights, it decomposes into weight spaces

$$\mathcal{B}_H(\mathfrak{g}) = \bigoplus_{\phi\in\Pi} \mathcal{B}_\phi.$$

Now we define a subset of $\mathfrak{t}^*$,

$$\Gamma := \left\{ \mu+\phi \;\middle|\; \begin{array}{l} \mu\in\Omega,\ \phi\in\Pi \ \text{ and there is an } u\in V_\mu \\ \text{and a } Q\in\mathcal{B}_\phi \text{ such that } Q(u)\neq 0 \end{array} \right\} \subset \mathfrak{t}^*,$$

which is contained in Δ_0:

Lemma 4.2. $\Gamma \subset \Delta_0$.

Proof. For $\mu\in\Omega$ and $\phi\in\Pi$ we consider weight elements $Q_\phi\in\mathcal{B}_\phi$ and $u_\mu\in V_\mu$ with $0\neq Q_\phi(u_\mu)$. Then, by recalling how $\mathfrak{g}$ acts on $\mathcal{B}_H(\mathfrak{g})$,

$$\begin{aligned}[][T, Q_\phi(u_\mu)] &= (TQ_\phi)(u_\mu) + Q_\phi(T(u_\mu)) \\ &= (\phi(T)+\mu(T))\, Q_\phi(u_\mu)\end{aligned}$$

for every $T\in\mathfrak{t}$. i.e., $\phi+\mu$ is a root or zero. q.e.d.

For weak-Berger algebras the other inclusion is true.

Proposition 4.3. *If $\mathfrak{g}\subset\mathfrak{so}(V,h)$ is an irreducible, semisimple Lie algebra which is weak-Berger, then $\Gamma=\Delta_0$.*

Proof. The decomposition of $\mathcal{B}_H(\mathfrak{g})$ and V into weight spaces and the fact that $Q_\phi(u_\mu)\in\mathfrak{g}_{\phi+\mu}$ imply the following inclusion:

$$\mathfrak{g}_H \;=\; span\{Q_\phi(u_\mu) \mid \phi+\mu\in\Gamma\} \subset \bigoplus_{\beta\in\Gamma}\mathfrak{g}_\beta.$$

But if $\mathfrak{g}=\bigoplus_{\alpha\in\Delta_0}\mathfrak{g}_\alpha$ is weak-Berger it follows that $\mathfrak{g}\subset\mathfrak{g}_H$ and thus

$$\bigoplus_{\alpha\in\Delta_0}\mathfrak{g}_\alpha \subset \bigoplus_{\beta\in\Gamma}\mathfrak{g}_\beta \subset \bigoplus_{\alpha\in\Delta_0}\mathfrak{g}_\alpha.$$

This implies $\Gamma=\Delta_0$. q.e.d.

For a root $\alpha\in\Delta$ we denote by Ω_α the following subset of Ω:

$$\Omega_\alpha := \{\lambda\in\Omega \mid \lambda+\alpha\in\Omega\}.$$

Then $\alpha+\Omega_\alpha$ are the weights of $\mathfrak{g}_\alpha V$.

Proposition 4.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra with roots Δ and $\Delta_0=\Delta\cup\{0\}$. Let $\mathfrak{g}\subset\mathfrak{so}(V,H)$ be irreducible, weak-Berger with weights Ω. Then the following properties are satisfied:*

(PI): *There is a $\mu\in\Omega$ and a hyperplane $U\subset\mathfrak{t}^*$ such that*

$$\Omega \subset \{\mu+\beta \mid \beta\in\Delta_0\} \cup U \cup \{-\mu+\beta \mid \beta\in\Delta_0\}. \tag{9}$$

(PII): *For every $\alpha \in \Delta$ there is a $\mu_\alpha \in \Omega$ such that*

$$\Omega_\alpha \subset \{\mu_\alpha - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\mu_\alpha + \beta \mid \beta \in \Delta_0\}\,. \tag{10}$$

Proof. If $\mathfrak{g}$ is a weak-Berger algebra, then $\Gamma = \Delta_0$. We use this property for $0 \in \Delta_0$ as well as for every $\alpha \in \Delta$.

(PI): By $\Gamma = \Delta_0$ there are $\phi \in \Pi$ and $\mu \in \Omega$ such that $0 = \phi + \mu$ with $Q \in \mathcal{B}_\phi$ and $u \in V_\mu$ such that $0 \neq Q(u) \in \mathfrak{t}$, i.e., $\phi = -\mu \in \Pi$. We fix such u, Q and μ. For arbitrary $\lambda \in \Omega$ the following case may occur:

Case 1. There is a $v_+ \in V_\lambda$ such that $Q(v_+) \neq 0$ or a $v_- \in V_{-\lambda}$ such that $Q(v_-) \neq 0$. This implies $-\mu + \lambda \in \Delta_0$ or $-\mu - \lambda \in \Delta_0$, i.e., $\lambda \in \{\mu + \beta \mid \beta \in \Delta_0\} \cup \{-\mu + \beta \mid \beta \in \Delta_0\}$.

Case 2. For all $v \in V_\lambda \oplus V_{-\lambda}$ it holds $Q(v) = 0$. Then the Bianchi identity implies for $v_+ \in V_\lambda$ and $v_- \in V_{-\lambda}$ that $0 = \lambda(Q(u))H(v_+, v_-)$. Now one can choose v_+ and v_- such that $H(v_+, v_-) \neq 0$. This implies $\lambda \in Q(u)^\perp =: U$ and we get (PI).

(PII): Let $\alpha \in \Delta$. $\Gamma = \Delta_0$ implies the existence of $\phi \in \Pi$ and $\mu_\alpha \in \Omega$ such that $\alpha = \phi + \mu_\alpha$ with $Q \in \mathcal{B}_\phi$ and $u \in V_{\mu_\alpha}$ such that $0 \neq Q(u) \in \mathfrak{g}_\alpha$. We fix Q and u for α. Hence $\alpha - \mu_\alpha = \phi \in \Pi$ is a weight of $\mathcal{B}_H$.

Now, let λ be a weight in Ω_α, i.e., $\lambda + \alpha$ is also a weight. Hence $-\lambda - \alpha$ is a weight. If $v \in V_\lambda$ then $Q(u)v \in V_{\lambda+\alpha}$. Since H is non-degenerate, there is a $w \in V_{-\lambda-\alpha}$ such that $H(Q(u)v, w) \neq 0$. Since $Q \in \mathcal{B}_H(\mathfrak{g})$, the Bianchi identity gives

$$0 = H(Q(u)v, w) + H(Q(v)w, u) + H(Q(w)u, v),$$

i.e., at least one of $Q(v)$ or $Q(w)$ has to be non-zero. Hence we have two cases for $\lambda \in \Omega_\alpha$:

Case 1. $Q(v) \neq 0$. This implies $-\mu_\alpha + \alpha + \lambda \in \Delta_0$, and thus $\lambda \in \{\mu_\alpha - \alpha + \beta \mid \beta \in \Delta_0\}$.

Case 2. $Q(w) \neq 0$. This implies $-\mu_\alpha + \alpha - \lambda - \alpha = -\mu_\alpha - \lambda \in \Delta_0$, i.e., $\lambda \in \{-\mu_\alpha + \beta \mid \beta \in \Delta_0\}$. q.e.d.

Of course, it is desirable to find weights μ and μ_α which are extremal in order to handle criteria (PI) and (PII).

Lemma 4.5. *Let $\mathfrak{g} \subset \mathfrak{so}(V, H)$ be an irreducible, complex semisimple Lie algebra with $\mathcal{B}_H(\mathfrak{g}) \neq 0$. Then for any extremal weight vector $u \in V_\Lambda$ there is a weight element $Q \in \mathcal{B}_H(\mathfrak{g})$ such that $Q(u) \neq 0$.*

Proof. Let $u \in V_\Lambda$ be extremal with $Q(u) = 0$ for every weight element Q. Since $\mathcal{B}_H(\mathfrak{g}) = \bigoplus_{\phi \in \Pi} \mathcal{B}_\phi$, the assumption implies $Q(u) = 0$ for all $Q \in \mathcal{B}_H(\mathfrak{g})$. This gives for every $A \in \mathfrak{g}$ and every weight element Q that

$$Q(Au) = [A, Q(u)] - \underbrace{(A \cdot Q)}_{\in \mathcal{B}_H(\mathfrak{g})}(u) = 0.$$

On the other hand, V is irreducible and thus generated as vector space by elements of the form $A_1 \cdot \ldots \cdot A_k \cdot u$ with $A_i \in \mathfrak{g}$ and $k \in \mathbb{N}$ (see for example [**34**]). Successive application of $\mathfrak{g}$ to u yields $Q(v) = 0$ for every weight element Q and every weight vector v. Hence $Q(v) = 0$ for all $Q \in \mathcal{B}_H(\mathfrak{g})$ and every $v \in V$, i.e., $\mathcal{B}_H(\mathfrak{g}) = 0$. q.e.d.

Proposition 4.6. *Let $\mathfrak{g} \subset \mathfrak{so}(V, H)$ be an irreducibly acting, semisimple weak-Berger algebra with roots Δ, $\Delta_0 = \Delta \cup \{0\}$ and weights Ω. Then there is a partial order of Δ (i.e., a set of simple roots) such that the following holds: If Λ is the highest weight of $\mathfrak{g} \subset \mathfrak{so}(V, H)$ with respect to this partial order, then*

(QI): *There is a $\delta \in \Delta_+ \cup \{0\}$ and a hyperplane $U \subset \mathfrak{t}^*$ such that*

$$\Omega \subset \{\Lambda - \delta + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\Lambda + \delta + \beta \mid \beta \in \Delta_0\}. \tag{11}$$

If δ cannot be chosen to be zero, then

(QII): *There is an $\alpha \in \Delta$ such that*

$$\Omega_\alpha \subset \{\Lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}. \tag{12}$$

Proof. We consider the extremal weights of the representation. Since these cannot lie in the same hyper plane, by (PI) of Proposition 4.4, there is a $\mu \in \Omega$ and an extremal weight Λ with $\Lambda + \mu \in \Delta_0$ or $\Lambda - \mu \in \Delta_0$. We fix Λ and choose a fundamental root system, i.e., a partial order on the roots, such that Λ is the highest weight. With respect to this fundamental root system the roots split into positive and negative roots $\Delta = \Delta_+ \cup \Delta_-$. This implies

$$\mu = \varepsilon(\Lambda - \delta) \quad \text{with } \delta \in \Delta_+ \text{ and } \varepsilon = \pm 1. \tag{13}$$

Then for arbitrary $\lambda \in \Omega$ it holds $\lambda \in U = Q(u)^\perp$ or $\lambda + \mu \in \Delta_0$ or $\lambda - \mu \in \Delta_0$. But with (13) this implies that we find a $\beta \in \Delta_0$ such that $\lambda = \pm(\Lambda - \delta) + \beta$ with $\beta \in \Delta_0$. This is (QI). Note that we are still free to choose Λ or $-\Lambda$ as highest weight.

Now we suppose that δ cannot be chosen to be zero. Let $v \in V_\Lambda$ or $v \in V_{-\Lambda}$ be a highest weight vector. By the proof of Proposition 4.4, for all weight elements $Q \in \mathcal{B}_h(\mathfrak{g})$ it holds $Q(v) \in \mathfrak{g}_\alpha$ for an $\alpha \in \Delta$. Since $\mathfrak{g}$ is weak-Berger $\mathcal{B}_H(\mathfrak{g})$ is non-zero. Thus we get by Lemma 4.5 that there is a weight element Q such that $0 \neq Q(v) \in \mathfrak{g}_\alpha$ and we are done (possibly by making $-\Lambda$ to the highest weight). q.e.d.

Example 4.7. *Representations of* $\mathfrak{sl}(2, \mathbb{C})$. To illustrate how these criteria will work we apply them to irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$.

Proposition 4.8. *Let V be an irreducible, complex, orthogonal $\mathfrak{sl}(2, \mathbb{C})$–module of highest weight Λ. If it is weak-Berger, then $\Lambda \in \{2, 4\}$.*

Proof. Let $\mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{so}(N,\mathbb{C})$ be an irreducible representation of highest weight Λ. I.e., $\Lambda(H) = l \in \mathbb{N}$ for $\mathfrak{sl}(2,\mathbb{C}) = span(H,X,Y)$ where X has the root α. Since the representation is orthogonal, l must be even (see for example [**36**]) and 0 is a weight. The hypersurface U is the point 0. Now property (9) ensures that $l \in \{2,4,6\}$. If $\mu = \Lambda$ we obtain $l \in \{2,4\}$. If $\mu \neq \Lambda$ we can apply (QII): We have that $\Omega_\alpha = \Omega \setminus \{\Lambda\}$ and $\Omega_{-\alpha} = \Omega \setminus \{-\Lambda\}$. Then (QII) implies $l \in \{2,4\}$. q.e.d.

So we get the first result:

Corollary 4.9. *Let $\mathfrak{su}(2) \subset \mathfrak{so}(E,h)$ be a real irreducible weak-Berger algebra of real type. Then it is a Berger algebra. In particular it is equivalent to the Riemannian holonomy representations of $\mathfrak{so}(3,\mathbb{R})$ on $\mathbb{R}^3$ or of the symmetric space of type AI, i.e., $\mathfrak{su}(3)/\mathfrak{so}(3,\mathbb{R})$ in the compact case or $\mathfrak{sl}(3,\mathbb{R})/\mathfrak{so}(3,\mathbb{R})$ in the non-compact case.*

4.3. Berger algebras, weak-Berger algebras, and spanning triples. In this section we shall describe a result of [**32**], [**33**], where holonomy groups of torsion free connections are classified. Sometimes we shall we refer to the unpublished [**32**], since some of its results are not contained in the published [**33**]. We shall describe our results in their language such that we can use a partial result of [**33**].

For a Berger algebra it holds that for every $\alpha \in \Delta_0$ there is a weight element $R \in \mathcal{K}(\mathfrak{g})$ and weight vectors $u_1 \in V_{\mu_1}$ and $u_2 \in V_{\mu_2}$ such that $0 \neq R(u_1,u_2) \in \mathfrak{g}_\alpha$. Choosing u_1,u_2 such that $0 \neq R(u_1,u_2) \in \mathfrak{t}$, by the Bianchi identity one gets for any $\lambda \in \Omega$ and $v \in V_\lambda$ that

$$\lambda(R(u_1,u_2))v \;=\; R(v,u_2)u_1 + R(u_1,v)u_2.$$

This implies $\lambda \in (R(u_1,u_2))^\perp \subset \mathfrak{t}^*$ or $V_\lambda \subset \mathfrak{g}V_{\mu_1} \oplus \mathfrak{g}V_{\mu_2}$, and hence

(RI): There are weights $\mu_1,\mu_2 \in \Omega$ such that

$$\Omega \subset \{\mu_1 + \beta \mid \beta \in \Delta_0\} \cup U \cup \{\mu_2 + \beta \mid \beta \in \Delta_0\}.$$

If one chooses u_1,u_2 such that $0 \neq R(u_1,u_2) = A_\alpha \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta$, for $\lambda \in \Omega$ we get that $A_\alpha V_\lambda \subset \mathfrak{g}V_{\mu_1} \oplus \mathfrak{g}V_{\mu_2}$. Hence, the weights of $A_\alpha V_\lambda$ are contained in $\{\mu_1 + \beta | \beta \in \Delta_0\} \cup \{\mu_2 + \beta | \beta \in \Delta_0\}$:

(RII): For every $\alpha \in \Delta$ there are weights $\mu_1,\mu_2 \in \Omega$ such that

$$\Omega_\alpha \subset \{\mu_1 - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{\mu_2 - \alpha + \beta \mid \beta \in \Delta_0\}.$$

Of course (PI) is a special case of (RI) with $\mu_1 = -\mu_2$. (PII) is not a special case of (RII) since $\mu_\alpha + \alpha$ is not a weight, *a priori*.

To describe this situation further, in [**32**] and [**33**] the following definitions are made. We point out that here Ω_α does not denote the weights of $\mathfrak{g}_\alpha V$ but the weights λ of V such that $\lambda + \alpha$ is a weight.

Definition 4.10. Let $\mathfrak{g} \subset End(V)$ be an irreducibly acting complex Lie algebra, Δ_0 be the roots and zero of the semisimple part of $\mathfrak{g}$, Ω the weights of $\mathfrak{g}$ and Ω_α as above.

1) A triple $(\mu_1, \mu_2, \alpha) \in \Omega \times \Omega \times \Delta$ is called *spanning triple* if
$$\Omega_\alpha \subset \{\mu_1 - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{\mu_2 - \alpha + \beta \mid \beta \in \Delta_0\}\,.$$
2) A spanning triple (μ_1, μ_2, α) is called *extremal* if μ_1 and μ_2 are extremal.
3) A triple extremal weights μ_1 and μ_2, and an affine hyperplane $U \subset \mathfrak{t}^*$ is called *planar spanning triple* if every extremal weight different from μ_1 and μ_2 is contained in U and
$$\Omega \subset \{\mu_1 + \beta \mid \beta \in \Delta_0\} \cup U \cup \{\mu_2 + \beta \mid \beta \in \Delta_0\}\,.$$

In [**32**] the following conclusion is deduced from (RI) and (RII).

Proposition 4.11 ([**32**, Proposition 3.13])**.** *Let $\mathfrak{g} \subset End(V)$ be an irreducible complex Berger algebra. Then, for every root $\alpha \in \Delta$ there is a spanning triple. Furthermore there is an extremal spanning triple or a planar spanning triple.*

Returning to weak-Berger algebras we reformulate Proposition 4.6:

Proposition 4.12. *Let $\mathfrak{g} \subset \mathfrak{so}(V, H)$ be an irreducible complex weak-Berger algebra. Then there is an extremal weight Λ such that one of the following properties is satisfied.*

(SI): *There is a planar spanning triple of the form $(\Lambda, -\Lambda, U)$.*
(SII): *There is an $\alpha \in \Delta$ such that $\Omega_\alpha \subset \{\Lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}\,.$*

There is a fundamental system such that the extremal weight in (SI) *and* (SII) *is the highest weight.*

Proof. The proof is analogous the one of Proposition 4.6. If there is an $\alpha \in \Delta$ such that the corresponding μ_α is extremal we are done. Otherwise it is $Q(u) \in \mathfrak{t}^*$ for every extremal weight vector $u \in V_\Lambda$ and every weight element $Q \in \mathcal{B}_\phi$. By Lemma 4.5 there is a Q such that $0 \neq Q(u) \in \mathfrak{t}^*$. As before this implies
$$\Omega \subset \{\Lambda + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}.$$
To ensure that $(\Lambda, -\Lambda, U)$ is a planar spanning triple we have to verify that every extremal weight λ different from Λ and $-\Lambda$ is contained in $U = Q(u)^\perp$. Let λ be extremal and different from Λ and $-\Lambda$, $v_\pm \in V_{\pm\lambda}$ and $u \in V_\Lambda$. Since $Q(v_\pm) \in \mathfrak{t}^*$, the Bianchi identity gives
$$\begin{aligned} 0 &= H(Q(u)v_+, v_-) + H(Q(v_+)v_-, u) + H(Q(v_-)u, v_+) \\ &= \lambda\,(Q(u))\,\underbrace{H(v_+, v_-)}_{\neq 0} - \underbrace{\lambda\,(Q(v_+))\,H(v_-, u) + \Lambda\,((Q(v_-))\,H(u, v_+)}_{=\,0 \text{ since } u \text{ is neither in } V_\lambda \text{ nor in } V_{-\lambda}}. \end{aligned}$$
Hence, $\lambda \in U$. q.e.d.

Obviously we are in a slightly different situation than in the Berger case since $-\Lambda + \alpha$ is not necessarily a weight and in case it is a weight, it is not necessarily extremal.

5. Classification of simple complex weak-Berger algebras

In this section we apply the result of Proposition 4.12 to simple complex irreducibly acting Lie algebras. We shall do this step by step under the following special conditions:

1) The highest weight of the representation is a root.
2) The representation satisfies (SI), i.e., admits a planar spanning triple $(\Lambda, -\Lambda, U)$.
3) The representation satisfies (SII) and has weight zero.
4) The representation satisfies (SII) and does not have weight zero.

Throughout this section the considered Lie algebra is supposed to be different from $\mathfrak{sl}(2,\mathbb{C})$. Before we start with this approach we have to recall some basic properties of root systems. Let Δ be a root system in the Euclidean vector space $(E, \langle .,. \rangle)$, i.e., Δ spans E, for any $\alpha, \beta \in \Delta$, the number $\frac{2\langle \alpha,\beta\rangle}{\|\alpha\|^2}$ is an integer, and the reflection $s_\alpha : \varphi \mapsto \varphi - \frac{2\langle \alpha,\varphi\rangle}{\|\alpha\|^2}$ maps Δ onto itself. For root systems, the following properties hold true (for a proof, see [**24**, pp. 149]).

Proposition 5.1. *Let Δ be a reduced root system in $(E, \langle .,. \rangle)$.*

1) *If $\alpha \in \Delta$, then the only root which is proportional to α is $-\alpha$.*
2) *If $\alpha, \beta \in \Delta$, then $\frac{2\langle \beta,\alpha\rangle}{\|\alpha\|^2} \in \{0, \pm 1, \pm 2, \pm 3\}$. If Δ is one of the indecomposable root systems, ± 3 occurs only for the root system G_2. If both roots are non proportional, then ± 2 only occurs for B_n, C_n, F_4 or G_2.*
3) *If α and β are non proportional in Δ and $\|\beta\| \leq \|\alpha\|$, then $\frac{2\langle \beta,\alpha\rangle}{\|\alpha\|^2} \in \{0, \pm 1\}$.*
4) *Let be $\alpha, \beta \in \Delta$. If $\langle \alpha, \beta\rangle > 0$, then $\alpha - \beta \in \Delta$. If $\langle \alpha, \beta\rangle < 0$, then $\alpha + \beta \in \Delta$. I.e., if neither $\alpha - \beta \in \Delta$ nor $\alpha + \beta \in \Delta$, then $\langle \alpha, \beta\rangle = 0$.*
5) *The subset of Δ defined by $\{\beta + k\alpha \in \Delta \cup \{0\} | k \in \mathbb{Z}\}$ is called α–string through β. It has no gaps, i.e., $\beta + k\alpha \in \Delta$ for $-p \leq k \leq q$ with $p, q \geq 0$ and it holds $p - q = \frac{2\langle \beta,\alpha\rangle}{\|\alpha\|^2}$. The maximal length of such a string is given by $\max_{\alpha,\beta\in\Delta} \frac{2\langle \beta,\alpha\rangle}{\|\alpha\|^2} + 1$, i.e., it contains at most four roots.*

5.1. Representations with roots as highest weight.

Proposition 5.2. *Let $\mathfrak{g} \subset \mathfrak{so}(N,\mathbb{C})$ be an irreducibly acting, complex simple Lie algebra, different from $\mathfrak{sl}(2,\mathbb{C})$ and satisfying* (SI) *or* (SII). *If we suppose in addition that there is an extremal weight Λ with $\Lambda = a\eta$ for a root $\eta \in \Delta$ and $a > 0$, then the following holds true:*

1) *If η is a long root, then $a = 1$, i.e., the representation is the adjoint one.*
2) *If Δ has roots of different length and η is a short root, then:*

a) *If* $\Delta = B_n$ *or* G_2 *then* $a = 1, 2$.
b) *If* $\Delta = C_n$ *or* F_4 *then* $a = 1$.

Proof. Let $\Lambda = a\eta$ with $\eta \in \Delta$, $a \in \mathbb{R}$. W.l.o.g. we may suppose that Λ is the extremal weight in the properties (SI) and (SII). First we show that $a \in \mathbb{N}$. If we chose an fundamental system $(\pi_1, \ldots, \pi_n)$ such that $\Lambda = a\eta$ is the highest weight we get that $\langle \Lambda, \pi_i \rangle = a\langle \eta, \pi_i \rangle \in \mathbb{N}$ for all i. $a \notin \mathbb{N}$ would imply that $\langle \eta, \pi_i \rangle \geq 2$ for all i with $\langle \eta, \pi_i \rangle \neq 0$. This holds only for the root system C_n where $\Lambda = \omega_1 = \frac{1}{2}\eta$. But this representation is symplectic, not orthogonal. (For an explicit formulation of this criterion see [**36**].) So we get $a \in \mathbb{N}$. Now we consider two cases.

Case 1: *η is a long root.* In this case the root system of long roots, denoted by Δ_l is the orbit of η under the Weyl group. Hence $a \cdot \Delta_l$ are the extremal weights and $\Delta \subset \Omega$. This implies $0 \in \Omega_\alpha$ for every $\alpha \in \Delta$.

Furthermore, it holds that $a \cdot \Delta \subset \Omega$. This is true because we can find a short root such that $\eta - \beta \in \Delta_s$. On the other hand it is $\frac{2\langle a\eta, \beta \rangle}{\|\beta\|^2} \geq a$, i.e., $a(\eta - \beta) \in \Omega$. Applying the Weyl group to this weight we get the property for all short roots. Now we check (SI) and (SII).

(SI) Let Λ satisfy (SI), i.e., Λ and $-\Lambda$ define a planar spanning triple $(\Lambda, -\Lambda, U)$. This would imply that every long root different from η lies in the hyperplane U. This is only possible for the the root system C_n, because all other root systems have an indecomposable system of long roots. For C_n holds that $\Delta_l = A_1 \times \cdots \times A_1$. But we have still a root β — possibly a short one — such that $\beta \notin U$ and β not proportional to η. This implies $\Omega \ni a\beta = \Lambda + \gamma = a\eta + \gamma$ or $\Omega \ni a\beta = -\Lambda + \gamma = -a\eta + \gamma$ with $\gamma \in \Delta_0$. Then Proposition 5.1 implies $a = 1$.

(SII) Let us suppose that Λ satisfies (SII), i.e., there is an $\alpha \in \Delta$ such that $\Omega_\alpha \subset \{\Lambda - \alpha + \beta | \beta \in \Delta_0\} \cup \{-\Lambda + \beta | \beta \in \Delta_0\}$. $0 \in \Omega_\alpha$ implies $0 = \Lambda - \alpha + \beta = a\eta - \alpha + \beta$ or $0 = -\Lambda + \beta = -a\eta + \beta$ with $\beta \in \Delta_0$. The second is not possible and the first implies by Proposition 5.1 that $a = 1$ or $a = 2$ and $\eta = \alpha$. In the second case we find a root $\gamma \not\sim \alpha$ such that $\langle \gamma, \alpha \rangle < 0$, hence $2\gamma \in \Omega_\alpha$. Since $2\gamma - 2\alpha \notin \Delta$ it has to be $2\gamma = \alpha + \beta$, but this is prevented by $\langle \gamma, \alpha \rangle < 0$ and Proposition 5.1.

Of course, if η is a long root the representation is the adjoint one.

Case 2: *η is a short root.* Let us denote by Δ_s the root system of short roots. Clearly, $\Delta_s \subset \Omega$ and $a \cdot \Delta_s$ are the extremal weights in Ω. For the root system B_n the root system of short roots Δ_s equals $A_1 \times \cdots \times A_1$, otherwise it is indecomposable. Furthermore, the following holds: If $a \geq 2$ then $\Delta \subset \Omega$. To verify this, we consider a long root $\beta \in \Delta_l$ with the property that $\langle \beta, \eta \rangle > 0$. Such a β always exists. Then we have $\frac{2\langle \eta, \beta \rangle}{\|\eta\|^2} > \frac{2\langle \eta, \beta \rangle}{\|\beta\|^2} \geq 1$. This implies $2\eta - \beta \in \Delta$ (see Proposition 5.1). On the other hand, $a \geq 2$ ensures that $\Omega \ni s_\beta(2\eta) = 2\left(\eta - \frac{2\langle \eta, \beta \rangle}{\|\beta\|^2}\beta\right)$.

Hence the long root $2\eta - \beta$ is a weight. Applying the Weyl group to β shows that every long root is a weight.

(SI) We suppose that there is a planar spanning triple $(\Lambda, -\Lambda, U)$, i.e., $a\beta$ lies in the hyperplane U if β is a short root. This is only possible for B_n because all other systems of short roots are indecomposable. In case of B_n we can at least find a long root α which is not in U. Since the long roots are weights, we have $\alpha = a\eta + \gamma$ or $\alpha = -a\eta + \gamma$ with $\gamma \in \Delta_0$. But this implies for B_n that $a \leq 2$.

(SII) Since $\Delta \subset \Omega$, it is $0 \in \Omega_\alpha$ for all α. $0 = -a\eta + \gamma$ with $\gamma \in \Delta_0$ would give $a = 1$. Hence, $a \geq 2$ implies

$$0 = a\eta - \alpha + \gamma. \tag{14}$$

Thus we have to deal with the following cases:

(a) $\alpha = \eta$ and $a = 2$.

(b) $\alpha \not\sim \eta$ and by 5 of Proposition 5.1 $a \leq \frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} \leq 3$. I.e., if $a \geq 2$, α is a long root.

We exclude the first case for any root system different from B_n. Set $a = 2$ and $\alpha = \eta$. If $\Delta \neq B_n$ the short roots are indecomposable, i.e., there is a short root β such that $\beta \not\sim \eta$ and $\langle \beta, \eta \rangle < 0$. Hence, $2\beta \in \Omega_\eta$ and $\beta + \eta \in \Delta$.

The existence of a spanning triple implies $2\beta = \eta + \gamma$ or $2\beta = -2\eta + \gamma$ with $\gamma \in \Delta_0$. The second case is impossible because of Proposition 5.1. The first implies $2\beta - \eta \in \Delta$. Again, this is not possible by Proposition 5.1 and $\langle \beta, \eta \rangle < 0$. Hence case (a) is excluded.

Now we consider case (b). First we show that $a = 3$ is not possible. Set $a = 3$. We notice that $\langle \eta, \alpha \rangle > 0$ implies $\frac{2\langle \eta, \alpha \rangle}{\|\alpha\|^2} \geq 1$ and hence $3\eta - 3\alpha \in \Omega_\alpha$. Thus we have the alternative $3\eta - 3\alpha = 3\eta - \alpha + \gamma$ or $3\eta - 3\alpha = -3\eta + \gamma$ with $\gamma \in \Delta_0$. The first implies $2\alpha \in \Delta$ and the second $6\eta - 3\alpha \in \Delta$. Both are not true, hence $a = 3$ is impossible.

We continue with case (b) and have that α is a long root with

$$\frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} \geq 2, \quad \text{i.e., } 2\eta - \alpha \in \Delta.$$

From now on we suppose that the root system is different from G_2. Then we have

$$\frac{2\langle \eta, \alpha \rangle}{\|\eta\|^2} = 2. \tag{15}$$

In a next step we show that under these conditions there is no short root β with

$$\beta \in \Delta_s \text{ with } \langle \alpha, \beta \rangle < 0\ ,\ \langle \beta, \eta \rangle < 0 \text{ and } \beta \not\sim \eta. \tag{16}$$

Suppose that there is such a β. Then the first condition implies that $2\beta \in \Omega_\alpha$ and hence $2\beta = 2\eta - \alpha + \gamma$ or $2\beta = -2\eta + \gamma$ with $\gamma \in \Delta_0$. The latter is not possible. The second implies the following using (15):

$$-2 \geq 2 \cdot \frac{2\langle \beta, \eta \rangle}{\|\eta\|^2} = \frac{2\langle 2\eta - \alpha, \eta \rangle}{\|\eta\|^2} + \frac{2\langle \gamma, \eta \rangle}{\|\eta\|^2} = 2 + \frac{2\langle \gamma, \eta \rangle}{\|\eta\|^2}.$$

Hence, $-4 \geq \frac{2\langle \gamma, \eta \rangle}{\|\eta\|^2}$ which is impossible.

Now by the Proposition 5.1 there is such a β. Hence, for every remaining root system different from G_2 and different from B_n we have that $a = 1$.

All in all we have shown, that for a long root it holds $a = 1$ and that for a short root $a = 2$ implies $\Delta = B_n$ or G_2. q.e.d.

Corollary 5.3. *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducible complex simple weak-Berger algebra different from $\mathfrak{sl}(2, \mathbb{C})$ and with the additional property that the highest weight is of the form $\Lambda = a\eta$ for a root $\eta \in \Delta$. Then $\mathfrak{g}$ is complexification of a holonomy algebra of a Riemannian manifold or the representation with highest weight $2\omega_1$ of G_2.*

Proof. If η is a long root the representation is the adjoint one, i.e., the complexification of a holonomy representation of a Lie group with positive definite bi-invariant metric. For a short root η we get the following:

$B_n,\ a = 1$: This is the standard representation of $\mathfrak{so}(2n+1, \mathbb{C})$ on $\mathbb{C}^{2n+1}$, and hence the complexification of the generic Riemannian holonomy representation.

$B_n,\ a = 2$: This is the representation of highest weight $2\omega_1$. This is the complexified isotropy representation of the Riemannian symmetric space of type AI, i.e., of the symmetric spaces $SU(2n+1)/SO(2n+1, \mathbb{R})$, respectively $SL(2n+1, \mathbb{R})/SO(2n+1, \mathbb{R})$.

$C_n,\ a = 1$: (for $n \geq 3$) This is the representation of highest weight ω_2. It is the complexified isotropy representation of the Riemannian symmetric space of type AII, i.e., of the symmetric spaces $SU(2n)/Sp(n)$, respectively $SL(2n, \mathbb{R})/Sp(n)$.

$F_4,\ a = 1$: This is the representation of highest weight ω_1. It is the complexified isotropy representation of the Riemannian symmetric space of type EIV, i.e., of the symmetric spaces E_6/F_4, respectively $E_{6(-26)}/F_4$.

$G_2,\ a = 1$: This is the representation of highest weight ω_1. It is the representation of G_2 on $\mathbb{C}^7$, i.e., the complexification of the holonomy representation of a Riemannian G_2–manifold.

$G_2,\ a = 2$: This is the representation $2\omega_1$ of G_2. It is a 27-dimensional representation of G_2 isomorphic to $Sym^2_0\mathbb{C}^7$, where $\mathbb{C}^7$ denotes the standard module of G_2 and $Sym^2_0\mathbb{C}^7$ its symmetric, trace free $(2, 0)$–tensors. This is the exception, because there is no Riemannian manifold with this complexified holonomy representation. q.e.d.

5.2. Representations with planar spanning triples. Now we consider representations of a simple Lie algebra under the condition that there is a planar spanning triple. For these we get the following proposition, its proof follows the proof of a similar proposition in [**32**, Proposition 3.20] under usage of the additional properties of our planar spanning triple.

Proposition 5.4. *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducibly acting, complex simple Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$ and satisfying* (SI), *i.e., with a planar spanning triple of the form $(\Lambda, -\Lambda, U)$. If there is no root α with $\Lambda = a\alpha$, then $\mathfrak{g}$ is of type D_n with $n \geq 3$ and the representation is congruent to the one with highest weight ω_1 or $2\omega_1$.*

Proof. Since $\Lambda \neq a\alpha$ there is no root such that $-\Lambda = s_\alpha(\Lambda)$. Hence, the existence of a planar spanning triple implies $s_\alpha(\Lambda) \in U$, for any $\alpha \in \Delta$ such that $\langle \Lambda, \alpha \rangle \neq 0$. If we set $U = T^\perp$, then

$$\text{for } \alpha \in \Delta \text{ with } \langle \alpha, \Lambda \rangle \neq 0 \text{ it holds } \langle \alpha, T \rangle = \frac{\|\alpha\|^2}{2\langle \Lambda, \alpha \rangle} \langle \Lambda, T \rangle \neq 0. \tag{17}$$

Following the lines of reasoning in [**32**], we prove various claims to get the wanted result.

Claim 1. For any non-proportional $\alpha, \beta \in \Delta$ with $\langle \Lambda, \alpha \rangle \neq 0$ and $\langle \Lambda, \beta \rangle \neq 0$ it holds that $\langle \alpha, \beta \rangle = 0$ or both have the same length.

To show this we prove that two such roots are orthogonal or $\langle \Lambda, s_\alpha \beta \rangle = \langle \Lambda, s_\beta \alpha \rangle = 0$. Suppose $\langle \Lambda, s_\alpha \beta \rangle \neq 0$. Then by (17):

$$\begin{aligned}
\|\beta\|^2 &= \|s_\alpha \beta\|^2 \\
&= \frac{2}{\langle \Lambda, T \rangle} \cdot \langle \Lambda, s_\alpha \beta \rangle \cdot \langle s_\alpha \beta, T \rangle \\
&= \frac{2}{\langle \Lambda, T \rangle} \cdot \left(\langle \Lambda, \beta \rangle - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \langle \Lambda, \alpha \rangle \right) \cdot \left(\langle \beta, T \rangle - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \langle \alpha, T \rangle \right) \\
&= 2 \cdot \left(\langle \Lambda, \beta \rangle - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \langle \Lambda, \alpha \rangle \right) \cdot \left(\frac{\|\beta\|^2}{2\langle \Lambda, \beta \rangle} - \frac{\langle \alpha, \beta \rangle}{\langle \Lambda, \alpha \rangle} \right) \\
&= 2 \cdot \left(\frac{\|\beta\|^2}{2} - \langle \alpha, \beta \rangle \, \frac{\langle \Lambda, \beta \rangle}{\langle \Lambda, \alpha \rangle} - \langle \alpha, \beta \rangle \, \frac{\langle \Lambda, \alpha \rangle}{\langle \Lambda, \beta \rangle} \frac{\|\beta\|^2}{\|\alpha\|^2} + \frac{2\langle \alpha, \beta \rangle^2}{\|\alpha\|^2} \right).
\end{aligned}$$

Subtracting $\|\beta\|^2$ and multiplying by the denominators gives

$$0 = \langle \alpha, \beta \rangle \left(\|\beta\|^2 \langle \Lambda, \alpha \rangle^2 + \|\alpha\|^2 \langle \Lambda, \beta \rangle^2 - 2\langle \beta, \alpha \rangle \langle \Lambda, \alpha \rangle \langle \Lambda, \beta \rangle \right).$$

This gives the following pair of equations

$$0 = \langle \alpha, \beta \rangle \Big(\underbrace{(\|\beta\| \langle \Lambda, \alpha \rangle + \|\alpha\| \langle \Lambda, \beta \rangle)^2}_{\geq 0} - 2 \underbrace{(\|\alpha\| \|\beta\| + \langle \beta, \alpha \rangle)}_{>0} \langle \Lambda, \alpha \rangle \langle \Lambda, \beta \rangle \Big)$$

$$0 = \langle\alpha,\beta\rangle\Big(\underbrace{(\|\beta\|\langle\Lambda,\alpha\rangle - \|\alpha\|\langle\Lambda,\beta\rangle)^2}_{\geq 0} + 2\underbrace{(\|\alpha\|\|\beta\| - \langle\beta,\alpha\rangle)}_{>0}\langle\Lambda,\alpha\rangle\langle\Lambda,\beta\rangle\Big).$$

This implies $\langle\alpha,\beta\rangle = 0$ or $\langle\Lambda,\alpha\rangle\langle\Lambda,\beta\rangle = 0$, but this was excluded. This argument is symmetric in α and β hence we get the same result for $s_\beta\alpha$. Thus we have proved that $\langle\Lambda, s_\alpha\beta\rangle = \langle\Lambda, s_\beta\alpha\rangle = 0$ or $\langle\alpha,\beta\rangle = 0$.

Now $\langle\Lambda, s_\alpha\beta\rangle = \langle\Lambda, s_\beta\alpha\rangle = 0$ implies $\langle\Lambda,\alpha\rangle = \frac{2\langle\alpha,\beta\rangle}{\|\alpha\|^2} \cdot \frac{2\langle\alpha,\beta\rangle}{\|\beta\|^2} \cdot \langle\Lambda,\alpha\rangle$. Since $\langle\Lambda,\alpha\rangle$ was supposed to be non zero we have that $\frac{2\langle\alpha,\beta\rangle}{\|\alpha\|^2} \cdot \frac{2\langle\alpha,\beta\rangle}{\|\beta\|^2} = 1$ which implies — since both factors are in $\mathbb{Z}$ — that $\|\alpha\|^2 = \|\beta\|^2$. This holds if $\langle\alpha,\beta\rangle \neq 0$.

Claim 2. All roots in Δ have the same length.

Suppose we have short and long roots. Then we can write a long root α as the sum of two short ones, let's say $\alpha = \beta + \gamma$. This implies $\langle\alpha,\beta\rangle \neq 0$ and $\langle\alpha,\gamma\rangle \neq 0$. Since α is long and β and γ are short, we have by the first claim that $\langle\Lambda,\alpha\rangle \cdot \langle\Lambda,\beta\rangle = 0$ and $\langle\Lambda,\alpha\rangle \cdot \langle\Lambda,\gamma\rangle = 0$. Now $\langle\Lambda,\alpha\rangle = \langle\Lambda,\beta\rangle + \langle\Lambda,\gamma\rangle$ gives that $\langle\Lambda,\alpha\rangle = 0$ for every long root. But this is impossible. Hence all roots have the same length and w.l.o.g. we can suppose for non-proportional roots that

$$\frac{2\langle\alpha,\beta\rangle}{\|\alpha\|^2} \in \{0, 1, -1\}. \tag{18}$$

Claim 3. There is an $a \in \mathbb{R}$ such that for any root α it holds $\langle\Lambda,\alpha\rangle \in \{0, \pm a\}$. a is less or equal than the square of the length of the roots.

We consider $\alpha \in \Delta$ with $\langle\Lambda,\alpha\rangle \neq 0$ and set $a := \langle\Lambda,\alpha\rangle$. Then we define the vector space $A := \mathit{span}\{\beta \in \Delta \mid \langle\Lambda,\beta\rangle = \pm a\} \subset \mathfrak{t}^*$. We show that every root γ with $\langle\Lambda,\gamma\rangle \notin \{0,\pm a\}$ is orthogonal to A and hence that $A = \mathfrak{t}^*$. To verify $A = \mathfrak{t}^*$ we show that every root is either in A or in $A^\perp$. First consider $\gamma \in \Delta$ with $\langle\Lambda,\gamma\rangle = 0$. If it is not in $A^\perp$ then there are roots $\beta \in A$ and $\delta \notin A$ such that $\gamma = \beta + \delta$. But this implies $0 = \langle\Lambda,\gamma\rangle = \langle\Lambda,\beta\rangle + \langle\Lambda,\delta\rangle = \pm a + \langle\Lambda,\delta\rangle$. Hence $\delta \in A$ and therefore $\gamma \in A$ which is a contradiction. Thus $\gamma \in A^\perp$. Now we consider a root γ with $\langle\Lambda,\gamma\rangle \notin \{0,\pm a\}$. Then for any β with $\langle\Lambda,\beta\rangle = \pm a$ we have because of (18) that $\langle\Lambda, s_\beta\gamma\rangle = \langle\Lambda,\gamma\rangle \pm a \neq 0$. Because of the proof of Claim 1 this gives $\langle\beta,\gamma\rangle = 0$. Hence $\gamma \in A^\perp$. Since the root system is indecomposable we have that $A = \mathfrak{t}^*$. Furthermore we have shown that any root with $\langle\Lambda,\gamma\rangle \notin \{0,\pm a\}$ is orthogonal to $A = \mathfrak{t}^*$. Thus, the first part of Claim 3 is proved.

Now we suppose that $a > c$ where c denotes the square of the length of the roots. We consider an $\alpha \in \Delta$ with $\langle\Lambda,\alpha\rangle = a$. $s_\alpha(\Lambda) = \Lambda - \frac{2a}{c}\alpha$ is an extremal weight in U. Then $a > c$ implies $\Lambda - 2\alpha \in \Omega$ but not in U. Then the existence of the planar spanning triple $(\Lambda, -\Lambda, U)$ implies

$\Lambda - 2\alpha = -\Lambda + \beta$ for a $\beta \in \Delta$. Hence,

$$\frac{2\langle \Lambda, \beta\rangle}{c} \;=\; 1 + \frac{2\langle \alpha, \beta\rangle}{c} = 2$$

and therefore $\langle \Lambda, \beta\rangle = a$ and $a = c$ which is a contradiction.

Now we consider for any $\alpha \in \Delta$ the set $\Delta_\alpha^\perp := \{\beta \in \Delta \mid \langle \alpha, \beta\rangle = 0\} \subset \Delta$. This set is a root system, reduced but not necessarily indecomposable. But we can make the following claim.

Claim 4. Let $\alpha \in \Delta$ with $\langle \Lambda, \alpha\rangle \neq 0$. Then one of the following cases holds:

1) $\Delta_\alpha^\perp$ is orthogonal to Λ or
2) there is a unique $\beta \in \Delta_\alpha^\perp$ with $\langle \Lambda, \beta\rangle \neq 0$ such that
   a) $\Lambda = \pm\frac{a}{c}(\alpha + \beta)$ where c is the lengths of the roots, and
   b) $\Delta_\alpha^\perp$ is decomposable with a direct summand $A_1 = \{\pm\beta\}$.

Suppose that the first alternative is false, i.e., there is a $\beta \in \Delta_\alpha^\perp$ with $\langle \Lambda, \beta\rangle \neq 0$. W.l.o.g. we can suppose that $\langle \Lambda, \beta\rangle = \langle \Lambda, \alpha\rangle = \pm a$. $\langle \alpha, \beta\rangle = 0$ then implies

$$s_\alpha s_\beta(\Lambda) \;=\; \Lambda \mp \frac{2a}{c}(\alpha + \beta).$$

Now we show with the help of (17) that $s_\alpha s_\beta(\Lambda)$ is not in U:

$$\begin{aligned}\langle s_\alpha s_\beta(\Lambda), T\rangle &= \langle \Lambda, T\rangle \mp \frac{2\langle \Lambda, \alpha\rangle}{\|\alpha\|^2}\langle \alpha, T\rangle \mp \frac{2\langle \Lambda, \beta\rangle}{\|\beta\|^2}\langle \beta, T\rangle \\ &= \mp\langle \Lambda, T\rangle \neq 0.\end{aligned}$$

But this implies $-\Lambda = s_\alpha s_\beta(\Lambda) = \Lambda \pm \frac{2a}{c}(\alpha + \beta)$. By this equation α determines β uniquely.

We still have to show that such β is orthogonal to all other roots in $\Delta_\alpha^\perp$. For $\gamma \not\sim \beta$ in $\Delta_\alpha^\perp$ uniqueness of β implies $\langle \Lambda, \gamma\rangle = 0$, and hence

$$\langle \Lambda, s_\beta\gamma\rangle = \underbrace{\langle \Lambda, \gamma\rangle}_{=0} - \frac{2\langle \beta, \gamma\rangle}{\|\beta\|^2}\langle \Lambda, \beta\rangle.$$

Again the uniqueness of β implies that β is orthogonal to $\Delta_\alpha^\perp$.

Claim 5. The root system of $\mathfrak{g}$ is of type A_n or D_n.

The only root system with roots of equal length where the root system $\Delta_\alpha^\perp$ is decomposable for a root α is D_n. Hence for every root system different from D_n we have that $\Delta_\alpha^\perp \perp \Lambda$ by Claim 4. Any root system different from A_n satisfies that $span(\Delta_\alpha^\perp) = \alpha^\perp$. Both together imply that for any root system different from D_n and A_n we have that $\alpha = \Lambda$ but this was excluded.

To find the representations of A_n and D_n which obey the above claims we introduce a fundamental system $\Pi = (\pi_1, \ldots, \pi_n)$ which makes Λ

to the highest weight of the representation. Then we have that $\Lambda = \sum_{k=1}^{n} m_k \omega_k$ with $m_k \in \mathbb{N} \cup \{0\}$ and ω_k the fundamental representations. $\langle \omega_i, \pi_j \rangle = \delta_{ij}$ implies $m_i = \langle \Lambda, \pi_i \rangle \in \{0, a\}$. Then we get

Claim 6. The root system is of type D_n and the representation has highest weight $\Lambda = a\omega_i$.

Applying Λ to the root $\sum_{k=1}^{n} \pi_k$ gives $\sum_{k=1}^{n} m_k = a$. Applying Λ to any of the π_i gives that $\sum_{k=1}^{n} m_k = m_i$ for any i.

Now we consider the root system A_n. $n = 1$ was excluded from the beginning. Recalling $A_3 \simeq D_3$ we can also exclude A_3. Now we impose the condition that the representation is orthogonal. This forces n to be odd and $\Lambda = a\omega_{\frac{n+1}{2}}$ where a has to be 2 when $\frac{n+1}{2}$ is odd. Thus we can suppose that $n > 3$. Now we consider the root $\sum_{k=1}^{n} \pi_k = e_1 - e_{n+1}$ for which holds that $\langle \Lambda, \eta \rangle = a$. Hence, by Claim 4 we have that $\Delta_\eta^\perp$ is orthogonal to Λ. On the other hand $\Delta_\eta^\perp = \{\pm(e_i - e_j) \mid 2 \le i < j \le n\}$ with $n > 3$ is not orthogonal to $a\omega_{\frac{n+1}{2}} = a\left(e_1 + \cdots + e_{\frac{n+1}{2}}\right)$. This yields a contradiction.

Finally, we show that only the representations of D_n given in the proposition satisfy the derived properties. The fundamental representations of D_n are given by $\omega_i = e_1 + \cdots + e_i$ for $i = 1 \dots n-2$ and $\omega_i = \frac{1}{2}(e_1 + \cdots + e_{n-1} \pm e_n)$ for $i = n-1, n$. Then $\langle a\omega_i, \pi_i \rangle = a$. On the other hand, for the largest root $\eta = e_1 + e_2$ it holds

$$\langle a\omega_i, \eta \rangle = \begin{cases} a & : \quad i = 1, n-1, n \\ 2a & : \quad 2 \le i \le n-2. \end{cases}$$

Hence, the representation of $a\omega_i$ with $2 \le i \le n-2$ does not satisfy Claim 3. Now we consider for $n > 4$ the representations $\Lambda = \frac{1}{2}(e_1 + \cdots + e_{n-1} \pm e_n)$. For the root $\alpha = e_{n-1} \pm e_n$ it holds that $\langle \Lambda, \alpha \rangle = a \neq 0$. The roots $\beta := e_1 - e_2$ and $\gamma := e_1 + e_3$ both satisfy $\langle \Lambda, \beta \rangle = \langle \Lambda, \gamma \rangle = a$ and $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = 0$. But this is a violation of the uniqueness property in Claim 4. Hence $n = 4$. For D_4 it holds that ω_3 and w_4 are congruent to ω_1, i.e., there is an involutive automorphism of the Dynkin diagram which interchanges ω_1 with ω_3 respectively ω_1 with ω_4. For $D_3 \simeq A_3$ only the representations ω_2 and $2\omega_2$ are orthogonal. q.e.d.

Corollary 5.5. *Every representation of a Lie algebra which satisfies the conditions of Proposition* 5.4 *is the complexification of a Riemannian holonomy representation.*

Proof. The representation with highest weight ω_1 of D_n is the standard representation of $\mathfrak{so}(2n, \mathbb{C})$ on $\mathbb{C}^{2n}$. Hence it is the holonomy representation of a generic Riemannian manifold. The representation with highest weight $2\omega_1$ is the complexified holonomy representation of a symmetric space of type AI for even dimensions, i.e., of $SU(2n)/SO(2n, \mathbb{R})$ respectively $Sl(2n, \mathbb{R})/SO(2n, \mathbb{R})$. q.e.d.

5.3. Representations with the property (SII) and weight zero. First we have to find out which representations admit zero as a weight.

Lemma 5.6. *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be a irreducible representation of a simple Lie algebra with weights Ω. If $0 \in \Omega$ then either*

1) *$\Delta \subset \Omega$, or*
2) *the extremal weights are short roots, or*
3) *$\Delta = C_n$ and the representation has highest weight ω_{2k} for $k \geq 2$.*

Proof. $0 \in \Omega$ implies that there is a $\lambda \in \Omega$ and an $\eta \in \Delta$ such that $0 = \lambda - \eta$, i.e $\lambda = \eta$. If η is a long root, the long and the short roots are weights. Thus, let us suppose that η is a short root. In this case we have to show that one long root is a weight if η is not extremal or that we are in the case of the C_n with the above representations. If η is not extremal then there exists an $\alpha \in \Delta$ such that $\eta + \alpha \in \Omega$ and $\eta - \alpha \in \Omega$. We fix this α and consider the following cases.

Case A: *$\alpha = \eta$, i.e., $2\eta \in \Omega$.* If $\Delta \neq G_2$ we find a long root β such that $\frac{2\langle \eta, \beta \rangle}{\|\eta\|^2} = -2$. This implies that $\beta + 2\eta$ is a long root but also a weight. In case of G_2 we find a short root β with $\langle \eta, \beta \rangle < 0$ and such that $2\eta + \beta \in \Delta$ a long root. This long root is also in Ω since $\langle \eta, \beta \rangle < 0$.

Case B: *$\alpha \not\sim \eta$ and $\langle \alpha, \eta \rangle \neq 0$.* W.l.o.g. we suppose that $\langle \alpha, \eta \rangle < 0$.

First we consider the case where α is a long root, i.e., $\frac{2\langle \alpha, \eta \rangle}{\|\eta\|^2} = -2$. Then $\alpha + \eta$ is a short root and by assumption a weight. One easily verifies that $\alpha + 2\eta$ is a long root, but also a weight: Since $\eta - \alpha$ is a weight and

$$\frac{2\langle \eta - \alpha, \eta + \alpha \rangle}{\underbrace{\|\eta + \alpha\|^2}_{=\|\eta\|^2}} = 2 - 2\frac{\|\alpha\|^2}{\|\eta\|^2} \leq -1,$$

we get that $\eta - \alpha + \eta + \alpha = 2\eta$ is a weight. Hence $2\eta + \alpha$ is a weight.

Now we consider the case where α is a short root too. $\alpha + \eta$ is a root. If it is long, we are done, because its a weight. If it is short we are left with the cases where the root system is C_n, F_4 or G_2 (see the appendix of [**24**]). For G_2 this implies that $\eta - \alpha$ is a root which is long, and we are done in this case. For C_n and F_4 one easily sees that there is a root γ such that $\eta - \alpha + \gamma$ is a weight and a long root.

Case C: *$\langle \alpha, \eta \rangle = 0$ and $\Delta \neq C_n$.* For G_2 this implies that α is a long root and that $\eta + \alpha$ is two times a short root. Then we can proceed as above to get the result.

If Δ is different from G_2 we consider the root system $\Delta_\eta^\perp$ of roots orthogonal to η which contains α. In case of C_n this root system is equal to $A_1 \times C_{n-2}$ and in the cases we are considering — B_n and F_4 — it is equal to B_{n-1} resp. B_3. Now we show that there is a short root α_1 in $\Delta_\eta^\perp$ such that $\eta + \alpha_1 \in \Omega$. If α is short this is trivial and if α is long we write $\alpha = \alpha_1 + \alpha_2$ with two orthogonal short roots from $\Delta_\eta^\perp$. Then $\langle \eta + \alpha, \alpha_2 \rangle > 0$ and thus $\eta + \alpha_1 \in \Omega$. On the other hand there is a short root $\gamma \in \Delta_\eta^\perp$ such that $\eta + \gamma$ is a long root. Applying the Weyl group of $\Delta_\eta^\perp$ on $\eta + \gamma$ we get that $\eta + \alpha_1$ is a long root. In case of C_n this argument does not apply since γ spans the A_1 factor of $\Delta_\eta^\perp$.

Hence we have verified $\Delta \subset \Omega$ in the cases A, B and C. For $\Delta = C_n$ and $\langle \alpha, \eta \rangle = 0$ one verifies directly that, if η is not an extremal weight, it holds that either $\Delta \subset \Omega$, or the representation of C_n is the one with highest weight ω_{2k} with $k \geq 2$. q.e.d.

Proposition 5.7. *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducibly acting, complex simple Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$ and satisfying* (SII). *If $0 \in \Omega$, then there is a root α such that for the extremal weight from property* (SII) *it holds $\Lambda = a\alpha$ or the representation is congruent to one of the following:*

1) *$\Delta = C_4$ with highest weight ω_4.*
2) *$\Delta = D_n$ with highest weight $2\omega_1$.*

Proof. Let Λ and α be the extremal weight and the root from property (SII). We suppose that Λ is not the multiple of a root, i.e., case (2) of the previous lemma, where the extremal weight were short roots is excluded. Hence, in this situation the only case where $0 \notin \Omega_\alpha$ occurs when $\Delta = C_n$, $\Lambda = \omega_{2k}$ with $k \geq 2$ and α is a long root. We shall treat this case at the end of the proof and suppose now that $0 \in \Omega_\alpha$. (SII) gives that $0 = -\Lambda + \beta$ — which was excluded — or $0 = \Lambda - \alpha - \beta$. The latter gives that $\alpha + \beta$ is not a root which implies that $\langle \alpha, \beta \rangle \geq 0$. We consider three cases.

Case 1: $\Delta = G_2$. In this case the fact that Λ is not proportional to a root implies $\langle \alpha, \beta \rangle > 0$ and α and β must have different length. Thus, we can chose a long root γ not proportional neither to α nor to β and such that $\langle \alpha, \gamma \rangle < 0$ and $\langle \beta, \gamma \rangle < 0$ which implies $\gamma \in \Omega_\alpha$ as well as $\gamma \in \Omega_\beta$. (SII) implies $\gamma - \beta \in \Delta$ or $\gamma - \alpha \in \Delta$ or $\gamma + \alpha + \beta \in \Delta$. The first two cases are not possible because of Proposition 5.1. For the third case we suppose that α is the long root and get that $\frac{2\langle \gamma + \beta, \alpha \rangle}{\|\alpha\|^2} = 0$ by analyzing the root diagram of G_2. Hence, $\gamma + \alpha + \beta$ cannot be a root.

Case 2: $\Delta \neq G_2$ and $\langle \alpha, \beta \rangle > 0$. This implies $\alpha - \beta \in \Delta$. We consider the number $k := \frac{2\langle \alpha, \alpha + \beta \rangle}{\|\alpha\|^2} = 2 + \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} \geq 3$. Since G_2 was excluded we have that $k \in \{3, 4\}$. Hence $\alpha + \beta - k\alpha = \beta - (k-1)\alpha \in \Omega_\alpha$. Then property (SII) implies $\beta - (k-1)\alpha = -\alpha - \beta + \gamma$ with $\gamma \in \Delta_0$, i.e., $2\beta - (k-2)\alpha \in \Delta$. At first this implies $k = 3$ and thus $\frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} = 1$. Secondly we must have $\frac{2\langle \alpha, \beta \rangle}{\|\beta\|^2} = 2$, therefore $\|\alpha\|^2 = 2\|\beta\|^2$, i.e., α as well as $2\beta - \alpha$ are long roots while β and $\beta - \alpha$ are short ones.

This implies $\frac{2\langle\beta-\alpha,\alpha+\beta\rangle}{\|\beta-\alpha\|^2} = \frac{2(\|\beta\|^2-\|\alpha\|^2)}{\|\beta\|^2} = -2$. Hence $\alpha+\beta+2(\beta-\alpha) = 3\beta-\alpha \in \Omega$ and since $\frac{2\langle\alpha,\alpha-3\beta\rangle}{\|\alpha\|^2} = 2-3 = -1$ it holds $\alpha - 3\beta \in \Omega_\alpha$. Then (SII) gives $\alpha - 3\beta = \beta - \gamma$ or $\alpha - 3\beta = -\beta - \alpha + \gamma$ with $\gamma \in \Delta_0$. But none of these equations can be true.

Case 3: $\langle\alpha,\beta\rangle = 0$ and $\Delta \neq G_2$. Since $\alpha+\beta$ is not the multiple of a root, the rank of Δ has to be greater than 3 or it is $\Delta = D_n$ and $\Lambda = 2e_i$, i.e., $\Lambda = 2w_1$. In the second case we are done and we exclude this representation in the following. We can suppose $rk\Delta \geq 4$. Recall that the weight Λ and the root α were defined by the property (SII). In this situation we prove the following lemma.

Lemma 5.8. *Let $rk\Delta \geq 4$ and let $\Lambda = \alpha+\beta$ be an extremal weight of a representation satisfying property* (SII) *with Λ and α, and let be a root such that $\langle\alpha,\beta\rangle = 0$ and $\alpha+\beta$ not the multiple of a root. Then Δ is a root system with roots of the same length or $\Delta = C_n$ and α and β are two short roots.*

Proof. Suppose that Δ has roots of different length. First we assume that β is a long root. We consider the root system $\Delta_\alpha^\perp$, which contains β. We note that β lies not in the A_1 factor of $\Delta_\alpha^\perp$ because otherwise $\alpha+\beta$ would be the multiple of a root. Since β is long we find a short root $\gamma \in \Delta_\alpha^\perp$ such that $\frac{2\langle\beta,\gamma\rangle}{\|\gamma\|^2} = -2$. Hence $\alpha+\beta+2\gamma \in \Omega$ and — since $\frac{2\langle\alpha,\alpha+\beta+2\gamma\rangle}{\|\alpha\|^2} = 2$ — it is $-\alpha-\beta-2\gamma \in \Omega_\alpha$. But this contradicts property (SII).

Now we suppose that α is a long root. Here we consider the root system $\Delta_\beta^\perp$ which contains α. Again α lies not in the A_1 factor of $\Delta_\beta^\perp$ because otherwise $\alpha+\beta$ would be the multiple of a root. Since α is long we find a short root $\gamma \in \Delta_\beta^\perp$ such that $\frac{2\langle\alpha,\gamma\rangle}{\|\gamma\|^2} = -2$. Hence, $\alpha+\beta+2\gamma \in \Omega$. Now we have that $\frac{2\langle\alpha,\gamma\rangle}{\|\alpha\|^2} = -1$ and therefore $\frac{2\langle\alpha,\alpha+\beta+2\gamma\rangle}{\|\alpha\|^2} = 2-1 = 1$. Thus $-\alpha-\beta-2\gamma \in \Omega_\alpha$. Again this contradicts (SII).

If α and β are short and orthogonal and the root system is not C_n, i.e., it is B_n or F_4, then the sum of two orthogonal short roots is the multiple of a root. q.e.d.

Now we prove a second claim.

Lemma 5.9. *The assumptions of the previous lemma imply that there is no $\gamma \in \Delta$ such that*

$$\langle\alpha,\gamma\rangle = 0 \quad \textit{and} \quad \frac{2\langle\beta,\gamma\rangle}{\|\gamma\|^2} = 1. \tag{19}$$

Proof. Let us suppose that there is a $\gamma \in \Delta$ such that $\langle\alpha,\gamma\rangle = 0$ and $\frac{2\langle\beta,\gamma\rangle}{\|\gamma\|^2} = 1$. In case of C_n γ is a short root. We note that both together imply that neither $\alpha+\gamma$ nor $\alpha-\gamma$ is a root. But $\gamma-\beta$ is a root, in case of C_n a short one. Furthermore $\Lambda-\gamma\in\Omega$. Hence,

$$\frac{2\langle\Lambda-\gamma,\gamma-\beta\rangle}{\|\gamma-\beta\|^2}=\frac{2\langle\alpha+\beta-\gamma,\gamma-\beta\rangle}{\|\gamma-\beta\|^2}=-2,$$

and thus $\Lambda-\gamma+2(\gamma-\beta)=\alpha-\beta+\gamma\in\Omega$. Now $\frac{2\langle\alpha-\beta+\gamma,\alpha\rangle}{\|\alpha\|^2}=2$, i.e., $-\alpha+\beta-\gamma\in\Omega_\alpha$. (SII) implies that $-\alpha+\beta-\gamma=\beta+\delta$ or $-\alpha+\beta-\gamma=-\alpha-\beta+\delta$ for $\delta\in\Delta_0$. But both options are not possible since $\alpha+\gamma$ is not a root and because γ is short. q.e.d.

We conclude that Lemma 5.8 leaves us with representations of A_n, D_n, E_6, E_7, E_8 or C_n where Λ is the sum of two orthogonal (short) roots but not a root.

Now one easily verifies that Lemma 5.9 implies $n\leq 4$ and $\Delta\neq A_4$. Hence the remaining representations are $2\omega_1$, $2\omega_3$ and $2\omega_4$ of D_4, which are congruent to each other, and w_4 of C_4.

To finish the proof we have to consider the representation of highest weight ω_{2k} (with $k\geq 2$) of C_n supposing α is a long root. $0\in\Omega$ implies that the short roots are weights. Let β be a short root with $\langle\alpha,\beta\rangle<0$, i.e., $\beta\in\Omega_\alpha$. (SII) then gives $\beta=\omega_{2k}-\alpha+\delta$ or $\beta=\omega_{2k}-\delta$ for a $\delta\in\Delta_0$. Analysing roots and fundamental weights of C_n we get that (SII) implies $k=2$ and $\alpha=2e_i$ for $1\leq i\leq 4$. But for $n>4$ Lemma 5.9 applies analogously. The remaining representation is ω_4 of C_4. q.e.d.

Corollary 5.10. *Let $\mathfrak{g}\subset\mathfrak{so}(N,\mathbb{C})$ be an orthogonal algebra of real type different from $\mathfrak{sl}(2,\mathbb{C})$ and satisfying* (SII). *If $0\in\Omega$, in particular if $\Delta=G_2,F_4$ or E_8, then it is the complexification of a Riemannian holonomy representation with the exception of G_2 in Corollary* 5.3.

Proof. If Λ is the multiple of a root then we are in the situation of Corollary 5.3. For D_n the remaining representations are those which appear in Corollary 5.5. The representation of highest weight ω_4 of C_4 is the complexification of the holonomy representation of the Riemannian symmetric space of type EI, i.e., of $E_6/Sp(4)$ resp. $E_{6(6)}/Sp(4)$. Analysing the roots and fundamental representations of the exceptional algebras we notice that every representation of G_2, F_4 and E_8 contains zero as weight. q.e.d.

5.4. Representations with the property (SII) where zero is no weight. Again we start with a lemma.

Lemma 5.11. *If $0\notin\Omega$, then there is a weight $\lambda\neq 0$, such that $\left|\frac{2\langle\lambda,\alpha\rangle}{\|\alpha\|^2}\right|\leq 1$ for every root α.*

Proof. In order to prove this indirectly, we fix a $0\neq\lambda\in\Omega$ with minimal length, i.e., $\|\lambda\|\leq\|\mu\|$ for all $0\neq\mu\in\Omega$. If the proposition were not true we could find a root $\alpha \in \Delta$ such that $\frac{2\langle \lambda, \alpha\rangle}{\|\alpha\|^2} \geq 2$. Since $0 \notin \Omega$, it is $\lambda \neq \alpha$, and we can consider the non-zero weight $\mu := \lambda - \alpha \neq 0$. For the square of its length we get

$$0 < \|\mu\|^2 = \|\lambda\|^2 - 2\langle \lambda, \alpha\rangle + \|\alpha\|^2 \leq \|\lambda\|^2 - \|\alpha\|^2 < \|\lambda\|^2.$$

This is a contradiction to the minimality of λ. q.e.d.

Proposition 5.12. *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducibly acting complex simple Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$, with $0 \notin \Omega$ and satisfying* (SII)*. Then $\left|\frac{2\langle \Lambda, \beta\rangle}{\|\beta\|^2}\right| \leq 3$ for all roots $\beta \in \Delta$.*

Proof. Let α be in Δ with the property (SII). By the previous lemma there is a $\lambda \in \Omega$ such that $\left|\frac{2\langle \lambda, \beta\rangle}{\|\beta\|^2}\right| \leq 1$ for all roots $\beta \in \Delta$. Applying the Weyl group one can choose λ such that $\langle \lambda, \alpha\rangle < 0$, i.e., $\lambda \in \Omega_\alpha$. Hence (SII) gives $\lambda = \Lambda - \alpha - \gamma$ or $\lambda = -\Lambda + \gamma$ with $\gamma \in \Delta_0$. Since we have excluded G_2, the second case gives for every $\beta \in \Delta$

$$\left|\frac{2\langle \Lambda, \beta\rangle}{\|\beta\|^2}\right| \leq \left|\frac{2\langle \lambda, \beta\rangle}{\|\beta\|^2}\right| + \left|\frac{2\langle \gamma, \beta\rangle}{\|\beta\|^2}\right| \leq 3.$$

Thus we have to consider the first case $\Lambda = \lambda + \alpha + \gamma$ with $\gamma \in \Delta_0$ and it is to verify that

$$\left|\frac{2\langle \Lambda, \beta\rangle}{\|\beta\|^2}\right| = \left|\frac{2\langle \lambda, \beta\rangle}{\|\beta\|^2} + \frac{2\langle \alpha, \beta\rangle}{\|\beta\|^2} + \frac{2\langle \gamma, \beta\rangle}{\|\beta\|^2}\right| \leq 3 \tag{20}$$

for all roots $\beta \in \Delta$.

For $\beta = \pm\alpha$ this is satisfied:

$$\frac{2\langle \Lambda, \beta\rangle}{\|\alpha\|^2} = \pm\frac{2\langle \lambda, \alpha\rangle}{\|\alpha\|^2} \pm 2 + \frac{2\langle \gamma, \alpha\rangle}{\|\alpha\|^2} = \mp 1 \pm 2 + \frac{2\langle \gamma, \alpha\rangle}{\|\alpha\|^2} \leq 3.$$

Hence, we have to show (20) for all $\beta \in \Delta$ with $\beta \not\sim \alpha$. For this we consider three cases.

Case 1: *All roots have the same length.* This implies $\left|\frac{2\langle \gamma, \beta\rangle}{\|\beta\|^2}\right| \leq 1$ for all roots which are not proportional to each other. Thus we get (20) for all $\beta \not\sim \gamma$:

$$\left|\frac{2\langle \Lambda, \beta\rangle}{\|\beta\|^2}\right| \leq \left|\frac{2\langle \lambda, \beta\rangle}{\|\beta\|^2}\right| + \left|\frac{2\langle \alpha, \beta\rangle}{\|\beta\|^2}\right| + \left|\frac{2\langle \gamma, \beta\rangle}{\|\beta\|^2}\right| \leq 3.$$

For $\beta = \pm\gamma$ we have

$$\frac{2\langle \Lambda, \beta\rangle}{\|\gamma\|^2} = \pm\frac{2\langle \lambda, \gamma\rangle}{\|\gamma\|^2} \pm \frac{2\langle \alpha, \gamma\rangle}{\|\gamma\|^2} \pm 2.$$

This has absolute value ≥ 4 only if $\langle \lambda, \gamma\rangle > 0$ and $\langle \alpha, \gamma\rangle > 0$. This implies that $\alpha - \gamma$ is a root. But for this root it holds $\frac{2\langle \lambda, \gamma - \alpha\rangle}{\|\gamma - \alpha\|^2} = \frac{2\langle \lambda, \gamma\rangle}{\|\gamma - \alpha\|^2} - \frac{2\langle \lambda, \alpha\rangle}{\|\gamma - \alpha\|^2} = 2$ since all roots have the same length. This is a contradiction to the choice of λ.

Case 2: *There are long and short roots and β is a long root.* This implies again $\left|\frac{2\langle\gamma,\beta\rangle}{\|\beta\|^2}\right| \leq 1$ for all β which are not proportional to γ. This implies (20) in this case.

For $\beta = \pm\gamma$ we argue as above, taking into account that $\|\gamma\| \geq \|\gamma-\alpha\| \geq \|\alpha\|$. We obtain $\frac{2\langle\lambda,\gamma-\alpha\rangle}{\|\gamma-\alpha\|^2} = \frac{2\langle\lambda,\gamma\rangle}{\|\gamma-\alpha\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\gamma-\alpha\|^2} \geq \frac{2\langle\lambda,\gamma\rangle}{\|\gamma\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\alpha\|^2} \geq 2$ which is a contradiction to the choice of λ

Case 3: *There are long and short roots and β is a short root.* First we consider the case where $\beta = \pm\gamma$. Again (20) is not satisfied only if $\langle\lambda,\gamma\rangle$ and $\langle\alpha,\gamma\rangle$ are non zero and have the same sign, lets say $+$. If α is a short root too, then, because of $\langle\alpha,\gamma\rangle \neq 0$,1, Proposition 5.1 gives that $\alpha-\gamma$ is also a short root. Hence, $\frac{2\langle\lambda,\gamma-\alpha\rangle}{\|\gamma-\alpha\|^2} = \frac{2\langle\lambda,\gamma\rangle}{\|\gamma-\alpha\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\gamma-\alpha\|^2} = \frac{2\langle\lambda,\gamma\rangle}{\|\gamma\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\alpha\|^2} = 2$ yields a contradiction.

If α is a long root, then $\gamma-\alpha$ has to be a short one and we get again a contradiction: $\frac{2\langle\lambda,\gamma-\alpha\rangle}{\|\gamma-\alpha\|^2} = \frac{2\langle\lambda,\gamma\rangle}{\|\gamma-\alpha\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\gamma-\alpha\|^2} \geq \frac{2\langle\lambda,\gamma\rangle}{\|\gamma\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\alpha\|^2} \geq 2$.

Now suppose that $\beta \not\sim \gamma$. Then $\frac{2\langle\Lambda,\beta\rangle}{\|\beta\|^2} = \frac{2\langle\lambda,\beta\rangle}{\|\beta\|^2} + \frac{2\langle\alpha,\beta\rangle}{\|\beta\|^2} + \frac{2\langle\gamma,\beta\rangle}{\|\beta\|^2}$ has absolute value ≥ 4 only if all three right hand side terms are ≥ 0 or ≤ 0 — lets say they are ≥ 0 — and at least one of the last two terms has absolute value greater than one, i.e., γ or α is a long root. If α is a long root then $\alpha-\beta$ is a short one and arguing as above gives the contradiction. If α is a short root then $\frac{2\langle\lambda,\beta\rangle}{\|\beta\|^2} > 0$ and $\langle\alpha,\beta\rangle > 0$ implies by Proposition 5.1 that $\beta-\alpha$ is a short root. Again, we get a contradiction: $\frac{2\langle\lambda,\beta-\alpha\rangle}{\|\beta-\alpha\|^2} = \frac{2\langle\lambda,\beta\rangle}{\|\beta-\alpha\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\beta-\alpha\|^2} = \frac{2\langle\lambda,\beta\rangle}{\|\beta\|^2} - \frac{2\langle\lambda,\alpha\rangle}{\|\alpha\|^2} = 2$. q.e.d.

Proposition 5.13. *Under the same assumptions as in the previous proposition it holds that* $\left|\frac{2\langle\Lambda,\eta\rangle}{\|\eta\|^2}\right| \leq 2$ *for all long roots* η.

Proof. Let Λ and α be the extremal weight and the root from property (SII). We suppose that there is a long root η with

$$\frac{2\langle\Lambda,\eta\rangle}{\|\eta\|^2} = -3 \tag{21}$$

and derive a contradiction considering different cases.

Case 1: *All roots have the same length.* By applying the Weyl group we find an extremal weight Λ' such that $a := \frac{2\langle\Lambda',\alpha\rangle}{\|\alpha\|^2} = -3$. Firstly, we find a root β with

$$\frac{2\langle\alpha,\beta\rangle}{\|\beta\|^2} = 1 \text{ and } \frac{2\langle\Lambda',\beta\rangle}{\|\beta\|^2} \leq -2.$$

This is obvious: We find a β such that $\frac{2\langle\alpha,\beta\rangle}{\|\beta\|^2} = 1$. If $\frac{2\langle\Lambda',\beta\rangle}{\|\beta\|^2} \geq -1$ then we consider the root $\alpha-\beta$. It satisfies $\frac{2\langle\alpha,\alpha-\beta\rangle}{\|\alpha-\beta\|^2} = 1$ and we have

$$\frac{2\langle\Lambda',\alpha-\beta\rangle}{\|\alpha-\beta\|^2} = -3 - \frac{2\langle\Lambda',\beta\rangle}{\|\alpha-\beta\|^2} \leq -2.$$

Hence we have $\Lambda' + k\beta \in \Omega$ for $0 \leq k \leq 2$ and $\Lambda' + k\alpha \in \Omega$ for $0 \leq k \leq 3$. Furthermore,

$$\frac{2\langle \Lambda' + l\beta, \alpha\rangle}{\|\alpha\|^2} = -3 - \frac{2\langle \Lambda', \alpha\rangle}{\|\alpha\|^2} = -3 + l.$$

But this gives

$$\Lambda' + k\alpha + l\beta \in \Omega_\alpha \text{ for } 0 \leq k \leq 2, 0 \leq k + l \leq 2.$$

Among others (SII) implies the existence of γ_i and δ_i from Δ_0 for $i = 0, 1, 2$ such that the following alternatives must hold:

$$\begin{array}{llcllcl}
(22) & \Lambda' + \alpha & = & \Lambda + \gamma_0 & \text{or} & \Lambda' & = & -\Lambda + \delta_0 \\
(23) & \Lambda' + 3\alpha & = & \Lambda + \gamma_1 & \text{or} & \Lambda' + 2\alpha & = & -\Lambda + \delta_1 \\
(24) & \Lambda' + \alpha + 2\beta & = & \Lambda + \gamma_2 & \text{or} & \Lambda' + 2\beta & = & -\Lambda + \delta_2.
\end{array}$$

First we suppose that the first alternative of (22) holds, i.e $\Lambda' + \alpha = \Lambda + \gamma_0$. Since $a = -3$ and both Λ and Λ' are extremal we have that $\alpha \neq -\gamma_0$. Hence the first case of (23) cannot be true and we have $\Lambda' + 2\alpha = -\Lambda + \delta_1$. We consider now (24): The left side of (22) gives that $\Lambda' + 2\beta + \alpha = \Lambda + \gamma_0 + 2\beta$. If the left side of (24) were true, we would have $\gamma_0 = -\beta$. Hence $\Lambda + \beta \in \Omega$ and on the other hand $\Omega \ni \Lambda' + \alpha = \Lambda - \beta$ which contradicts the extremality of Λ. Thus the right hand side of (24) must be satisfied. From $\Lambda' + 2\alpha = -\Lambda + \delta_1$ follows $\Lambda' + 2\beta = -\Lambda + \delta_1 + 2(\beta - \alpha)$ and therefore $\delta_1 = -(\beta - \alpha)$. Again we have $-\Lambda + (\beta - \alpha) \in \Omega$ and $-\Lambda - (\beta - \alpha) \in \Omega$ which contradicts the extremality of Λ.

If one starts with the right hand side of (22) one may proceed analogously and get a contradiction in the case where all roots have the same length.

Case 2. *The roots have different length and α is a short root.* On one hand we find a short root β which is orthogonal to α and $\alpha + \beta$ is a long root, and on the other we can find an extremal weight Λ' such that

$$\frac{2\langle \Lambda', \alpha + \beta\rangle}{\|\alpha + \beta\|^2} = -3.$$

Since $\alpha \perp \beta$ we have

$$-3 = \frac{2(\langle \Lambda', \alpha\rangle + \langle \Lambda', \beta\rangle)}{\|\alpha\|^2 + \|\beta\|^2} = \frac{1}{2}\left(\frac{2\langle \Lambda', \alpha\rangle}{\|\alpha\|^2} + \frac{2\langle \Lambda', \beta\rangle}{\|\beta\|^2}\right).$$

Because of the previous proposition we get

$$\frac{2(\langle \Lambda', \alpha\rangle}{\|\alpha\|^2} = \frac{2(\langle \Lambda', \beta\rangle}{\|\beta\|^2} = -3.$$

Hence $\Lambda' + k\alpha + l\beta \in \Omega$ for $0 \leq k, l \leq 3$ and therefore $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \leq k \leq 2$ and $0 \leq l \leq 3$. (SII) implies the following alternatives:

$$\begin{array}{lrclcrcl}
(25) & \Lambda' + \alpha & = & \Lambda + \gamma_0 & \text{or} & \Lambda' & = & -\Lambda + \delta_0 \\
(26) & \Lambda' + \alpha + 3\beta & = & \Lambda + \gamma_1 & \text{or} & \Lambda' + 3\beta & = & -\Lambda + \delta_1 \\
(27) & \Lambda' + 2\alpha + 3\beta & = & \Lambda + \gamma_2 & \text{or} & \Lambda' + \alpha + 3\beta & = & -\Lambda + \delta_2 \\
(28) & \Lambda' + 3\alpha + 2\beta & = & \Lambda + \gamma_3 & \text{or} & \Lambda' + 2(\alpha + \beta) & = & -\Lambda + \delta_3 \\
(29) & \Lambda' + 3\alpha + 3\beta & = & \Lambda + \gamma_4 & \text{or} & \Lambda' + 2\alpha + 3\beta & = & -\Lambda + \delta_4.
\end{array}$$

If the left hand side of the first alternative is valid then the left hand sides of the remaining four cannot be satisfied: For (26) we would have $3\beta = \gamma_1 - \gamma_0$ which is not possible. (27) would imply $3\beta + \alpha = \gamma_2 - \gamma_0$ which is by Proposition 5.1 a contradiction since $\alpha \neq -\beta$ and $\gamma_0 \neq -\alpha$. (28) would imply $2(\alpha+\beta) = \gamma_3 - \gamma_0$. Since $\alpha+\beta$ is a long root this would give $\gamma_0 = -(\alpha+\beta)$ and $\gamma_3 = \alpha+\beta$ and thus $\Lambda - (\alpha+\beta)$ and $\Lambda + \alpha + \beta$ would be weights. But this is a contradiction to the extremality of Λ, (29) would give $2\alpha + 3\beta = \gamma_4 - \gamma_0$ which also is not possible.

Thus for the last four equations the right hand side must hold. Taking everything together we would get $\alpha = \delta_2 - \delta_1 = \delta_4 - \delta_2$ and $\beta = \delta_4 - \delta_3$. This gives $2\alpha = \delta_4 - \delta_1$ and thus

$$\frac{2\langle\delta_4, \alpha\rangle}{\|\alpha\|^2} - \frac{2\langle\delta_1, \alpha\rangle}{\|\alpha\|^2} = \frac{4\|\alpha\|^2}{\|\alpha\|^2} = 4.$$

The extremality of Λ prevents that $\alpha = \delta_4 = -\delta_1$. Hence δ_1 and δ_4 are long roots, in particular

$$\frac{2\langle\delta_4, \alpha\rangle}{\|\alpha\|^2} = -\frac{2\langle\delta_1, \alpha\rangle}{\|\alpha\|^2} = 2.$$

For β again $\beta = \delta_4 = -\delta_3$ cannot hold by the extremality of Λ and we have

$$0 = \frac{2\langle\beta, \alpha\rangle}{\|\alpha\|^2} = \frac{2\langle\delta_4, \alpha\rangle}{\|\alpha\|^2} - \frac{2\langle\delta_3, \alpha\rangle}{\|\alpha\|^2} = 2 - \frac{2\langle\delta_3, \alpha\rangle}{\|\alpha\|^2}$$

which forces δ_3 to be a long root too. Now we have a contradiction because the short root β is the sum of two long roots. This is impossible.

If we start with the right hand side of the first alternative one proceeds analogously.

Case 3. *The roots have different length and α is a long root.* In this case we find an extremal weight Λ' such that $\frac{2\langle\Lambda', \alpha\rangle}{\|\alpha\|^2} = -3$. Now we can write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \perp \alpha_2$ two short roots. As above we get

$$\frac{2\langle\Lambda', \alpha_1\rangle}{\|\alpha_1\|^2} = \frac{2\langle\Lambda', \alpha_2\rangle}{\|\alpha_2\|^2} = -3. \tag{30}$$

Again this implies $\Lambda' + k\alpha_1 + l\alpha_2 \in \Omega$ for $0 \leq k, l \leq 3$ and therefore $\Lambda' + k\alpha_1 + l\alpha_2 \in \Omega_\alpha$ for $0 \leq k, l \leq 2$. Now (SII) implies the existence of

γ_i and δ_i from Δ_0 for $i = 0, \ldots, 8$ such that the following alternatives have to be true:

$$
\begin{array}{lrcr}
 & (L) & & (R) \\
(31) & \Lambda' + \alpha_1 + \alpha_2 = \Lambda + \gamma_0 & \text{or} & \Lambda' = -\Lambda + \delta_0 \\
(32) & \Lambda' + 2\alpha_1 + \alpha_2 = \Lambda + \gamma_1 & \text{or} & \Lambda' + \alpha_1 = -\Lambda + \delta_1 \\
(33) & \Lambda' + 3\alpha_1 + \alpha_2 = \Lambda + \gamma_2 & \text{or} & \Lambda' + 2\alpha_1 = -\Lambda + \delta_2 \\
(34) & \Lambda' + \alpha_1 + 2\alpha_2 = \Lambda + \gamma_3 & \text{or} & \Lambda' + \alpha_2 = -\Lambda + \delta_3 \\
(35) & \Lambda' + \alpha_1 + 3\alpha_2 = \Lambda + \gamma_4 & \text{or} & \Lambda' + 2\alpha_2 = -\Lambda + \delta_4 \\
(36) & \Lambda' + 2\alpha_1 + 2\alpha_2 = \Lambda + \gamma_5 & \text{or} & \Lambda' + \alpha_1 + \alpha_2 = -\Lambda + \delta_5 \\
(37) & \Lambda' + 2\alpha_1 + 3\alpha_2 = \Lambda + \gamma_6 & \text{or} & \Lambda' + \alpha_1 + 2\alpha_2 = -\Lambda + \delta_6 \\
(38) & \Lambda' + 3\alpha_1 + 2\alpha_2 = \Lambda + \gamma_7 & \text{or} & \Lambda' + 2\alpha_1 + \alpha_2 = -\Lambda + \delta_7 \\
(39) & \Lambda' + 3\alpha_1 + 3\alpha_2 = \Lambda + \gamma_8 & \text{or} & \Lambda' + 2\alpha_1 + 2\alpha_2 = -\Lambda + \delta_8.
\end{array}
$$

In the following we denote the left hand side formulas with an .L and the right hand side formulas with an .R. Again we suppose that (31.L) is satisfied, i.e., $\Lambda' + \alpha_1 + \alpha_2 = \Lambda + \gamma_0$. Under this assumption at first (39.L) can be excluded: It would imply $2(\alpha_1 + \alpha_2) = 2\alpha = \gamma_8 - \gamma_0$. Since α is a long root this is not possible for $\gamma_8 \neq \alpha$. But if $\gamma_8 = \alpha$ then (39.L) is equivalent to $\Lambda = \Lambda' + 2\alpha_1 + 2\alpha_2$. Hence $\Lambda \pm \alpha$ would be a weight, i.e., Λ would not be extremal. Secondly, (38.L) can not be true: It implies $2\alpha_1 + \alpha_2 = \gamma_7 - \gamma_0$. Recalling that we are dealing with the root system B_n or C_n this can only be true if γ_7 is equal to α or equal to α_1. Again one deduces a contradiction to the extremality of Λ. Analogously (37.L) can be excluded.

Hence (31.L) implies (39.R),(38.R) and (37.R). Using these three relations one can exclude (32.R), (33.R), (34.R) and (35.R), e.g., (35.R) implies $2\alpha_1 - \alpha_2 = \delta_7 - \delta_4$, thus $\delta_7 = \alpha_1$ or $\delta_7 = \alpha_1 - \alpha_2$, which contradicts to the extremality of Λ. Hence we get that (32.L), (33.L), (34.L) and (35.L) must hold true. But in this case (36.L) gives $2\alpha_1 + 2\alpha_2 = \gamma_2 - \gamma_4$ which is again impossible because Λ is extremal.

Hence we conclude that (36.R) must be true implying $\alpha_1 = \delta_7 - \delta_5 = \delta_8 - \delta_6$ and $\alpha_2 = \delta_6 - \delta_5 = \delta_8 - \delta_7$. But analysing the root systems B_n and C_n one concludes that this is not possible.

If we start with the right hand side of the first alternative one proceeds analogously.

All in all we have shown that the assumption of a long root with (21) leads to a contradiction. q.e.d.

Now we are in a position that we can use results of [**33**] explicitly. First we cite them.

Proposition 5.14. [**33**, Propositions 3.16 and 3.17] *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducible representation of real type of a complex simple Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$. Then it holds:*

1) *If there is an extremal spanning triple $(\Lambda_1, \Lambda_2, \alpha)$, then there is no weight λ for which exists a pair of orthogonal long roots η_1 and η_2 such that $\left|\frac{2\langle\lambda,\eta_i\rangle}{\|\eta_i\|^2}\right| = 2$.*
2) *If furthermore all roots have the same length, then there is no weight λ for which exists a triple of orthogonal roots $\eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1$ such that $\left|\frac{2\langle\lambda,\eta_1\rangle}{\|\eta_1\|^2}\right| = 2$ and $\left|\frac{2\langle\lambda,\eta_2\rangle}{\|\eta_2\|^2}\right| = \left|\frac{2\langle\lambda,\eta_3\rangle}{\|\eta_3\|^2}\right| = 1$.*

Now we shall show that the existence of such a pair or triple of roots implies that (SII) defines an extremal spanning pair.

Proposition 5.15. *Let $\mathfrak{g} \subset \mathfrak{so}(N, \mathbb{C})$ be an irreducible representation of real type of a complex simple Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$, with $0 \notin \Omega$ and satisfying* (SII)*. Then it holds: If there is a pair of orthogonal long roots η_1 and η_2 such that $\left|\frac{2\langle\Lambda,\eta_i\rangle}{\|\eta_i\|^2}\right| = 2$ for the extremal weight Λ from the property* (SII)*, then $\Lambda - \alpha$ is an extremal weight, i.e.,* (SII) *defines an extremal spanning triple.*

Proof. Again we argue indirectly considering three different cases for the root α from the property (SII)

Case 1: *All roots have the same length or α is a long root.* Again by applying the Weyl group the indirect assumption implies that there is an extremal weight Λ' and a root long β orthogonal to α such that $\frac{2\langle\Lambda',\alpha\rangle}{\|\alpha\|^2} = \frac{2\langle\Lambda',\beta\rangle}{\|\beta\|^2} = -2$. This gives that $\Lambda' + k\alpha + l\beta \in \Omega$ for $0 \le k, l \le 2$ and hence $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \le k \le 1, 0 \le l \le 2$. Among others (SII) implies the existence of γ_i and δ_i from Δ_0 for $i = 0, \ldots, 3$ such that the following alternatives must hold true:

$$
\begin{array}{lrcllrcl}
 & & (L) & & & & (R) & \\
(40) & \Lambda' + \alpha & = & \Lambda + \gamma_0 & \text{or} & \Lambda' & = & -\Lambda + \delta_0 \\
(41) & \Lambda' + 2\alpha & = & \Lambda + \gamma_1 & \text{or} & \Lambda' + \alpha & = & -\Lambda + \delta_1 \\
(42) & \Lambda' + \alpha + 2\beta & = & \Lambda + \gamma_2 & \text{or} & \Lambda' + 2\beta & = & -\Lambda + \delta_2 \\
(43) & \Lambda' + 2\alpha + 2\beta & = & \Lambda + \gamma_3 & \text{or} & \Lambda' + \alpha + 2\beta & = & -\Lambda + \delta_3.
\end{array}
$$

Supposing again (40.L) we conclude that (42.L) and (43.L) cannot hold because β is long and the extremality of Λ. Hence (42.R) and (43.R) must be satisfied. Again the extremality of Λ prevents that (41.R) can be valid. Hence we have (41.L). Now (40.L) gives that

$$\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} = \frac{2\langle\Lambda',\alpha\rangle}{\|\alpha\|^2} + 2 - \frac{2\langle\gamma_0,\alpha\rangle}{\|\alpha\|^2} = -\frac{2\langle\gamma_0,\alpha\rangle}{\|\alpha\|^2}$$

by assumption. On the other hand (40.L) together with (41.L) and (42.R) and (43.R) implies $\alpha = \gamma_1 - \gamma_0 = \delta_3 - \delta_2$. We note that γ_0 cannot be equal to 0 and γ_1 not equal to α. If $\gamma_0 = -\alpha$ and $\gamma_1 = 0$ then $\Lambda = \Lambda' + 2\alpha$. Then (42.R) and (43.R) imply

$$\begin{array}{rcccl} \langle \delta_2, \alpha \rangle & = & 2\langle \Lambda', \alpha \rangle + 2\|\alpha\|^2 & = & 0 \text{ and} \\ \langle \delta_3, \alpha \rangle & = & 2\langle \Lambda', \alpha \rangle + 3\|\alpha\|^2 & = & \|\alpha\|^2. \end{array}$$

Since α is long this entails $\delta_2 = 0$ and $\delta_3 = \alpha$. Taking now (40.L) and (42.R) together we get $\Lambda = \alpha - \beta$. But this forces $0 \in \Omega$ which was excluded. Thus we have $\alpha = \gamma_1 - \gamma_0$ with non-proportionality. But this implies, since α is long, that $\frac{2\langle \gamma_0, \alpha \rangle}{\|\alpha\|^2} = -1$ and hence $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = 1$. But this means that $\Lambda - \alpha$ is an extremal weight.

Case 2: *There are roots with different length and α is a short root.* By assumption there is a short root γ such that $\gamma \perp \alpha$ and $\eta := \alpha + \gamma$ is a long root and an extremal weight Λ' and a long root β such that $\frac{2\langle \Lambda', \eta \rangle}{\|\eta\|^2} = \frac{2\langle \Lambda', \beta \rangle}{\|\beta\|^2} = -2$. Analogously to the previous theorem the orthogonality of α and γ gives

$$-2 = \frac{1}{2} \left(\frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} + \frac{2\langle \Lambda', \gamma \rangle}{\|\gamma\|^2} \right).$$

Hence we have to consider three cases:

(a) $\frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} = \frac{2\langle \Lambda', \gamma \rangle}{\|\gamma\|^2} - 2$,
(b) $\frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} = -3$ and $\frac{2\langle \Lambda', \gamma \rangle}{\|\gamma\|^2} - 1$,
(c) $\frac{2\langle \Lambda', \alpha \rangle}{\|\alpha\|^2} = -1$ and $\frac{2\langle \Lambda', \gamma \rangle}{\|\gamma\|^2} - 3$.

Then an easy calculation shows that $\langle \alpha, \beta \rangle = \langle \gamma, \beta \rangle = 0$ in each case. We shall treat the cases (a),(b) and (c) separately.

Case (a): Here we can proceed completely analogously to the Case 1. We have that $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \le k \le 1, 0 \le l \le 2$ leading to the same set of equations (40) — (43) and the same implications since β is long again. The proportional case is excluded as above and we get that $\alpha = \gamma_1 - \gamma_0$ non proportional. At least one has to be a short root and $\langle \gamma_0, \alpha \rangle < 0$ and $\langle \gamma_1, \alpha \rangle > 0$. On the other hand we have $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = -\frac{2\langle \gamma_0, \alpha \rangle}{\|\alpha\|^2}$ and $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = -\frac{2\langle \gamma_1, \alpha \rangle}{\|\alpha\|^2} + 2$ by (40.L) and (41.L). But this implies that both are short and $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = 1$ which is the proposition.

Case (b): $\frac{2\langle\Lambda',\alpha\rangle}{\|\alpha\|^2} = -3$ implies that $\Lambda' + k\alpha + l\beta \in \Omega_\alpha$ for $0 \leq k, l \leq 2$. (SII) then implies:

$$\begin{array}{lrclcrcl}
 & & (L) & & & & (R) & \\
(44) & \Lambda' + \alpha & = & \Lambda + \gamma_0 & \text{or} & \Lambda' & = & -\Lambda + \delta_0 \\
(45) & \Lambda' + 2\alpha & = & \Lambda + \gamma_1 & \text{or} & \Lambda' + \alpha & = & -\Lambda + \delta_1 \\
(46) & \Lambda' + 3\alpha & = & \Lambda + \gamma_2 & \text{or} & \Lambda' + 2\alpha & = & -\Lambda + \delta_2 \\
(47) & \Lambda' + 2\alpha + 2\beta & = & \Lambda + \gamma_3 & \text{or} & \Lambda' + \alpha + 2\beta & = & -\Lambda + \delta_3 \\
(48) & \Lambda' + 3\alpha + 2\beta & = & \Lambda + \gamma_4 & \text{or} & \Lambda' + 2\alpha + 2\beta & = & -\Lambda + \delta_3.
\end{array}$$

Supposing (44.L) excludes (47.L) and (48.L) because β is long. Hence, (47.R) and (48.R) are valid and exclude (45.R) and (46.L). Hence (45.L) and (46.L) are satisfied. This gives $\alpha = \gamma_2 - \gamma_1 = \gamma_1 - \gamma_0$ with γ_0 different from 0 and $-\alpha$, γ_1 different from 0 and α and γ_2 different from $\pm\alpha$. Hence $\alpha + \pm\delta_1$ is a long root with $\alpha \perp \delta_1$. But this gives $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} = \frac{2\langle\Lambda',\alpha\rangle}{\|\alpha\|^2} + 4 = 1$, i.e., $\Lambda - \alpha$ is an extremal weight.

Case (c): Here we have that $\Lambda' + k\gamma + l\beta \in \Omega_\alpha$ for $0 \leq k \leq 3$ and $0 \leq l \leq 2$ since $\frac{2\langle\Lambda'+k\gamma+l\beta,\alpha\rangle}{\|\alpha\|^2} = -1$. The equations implied by (SII) lead easily to a contradiction:

$$\begin{array}{lrcr}
 & (L) & & (R) \\
(49) & \Lambda' + \alpha = \Lambda + \gamma_0 & \text{or} & \Lambda' = -\Lambda + \delta_0 \\
(50) & \Lambda' + \alpha + 3\gamma = \Lambda + \gamma_1 & \text{or} & \Lambda' + 3\gamma = -\Lambda + \delta_1 \\
(51) & \Lambda' + \alpha + 2\beta + 3\gamma = \Lambda + \gamma_2 & \text{or} & \Lambda' + 2\beta + 3\gamma = -\Lambda + \delta_2.
\end{array}$$

Supposing (49.L) excludes (50.L) and (51.L). Hence (50.R) and (51.R) are valid but contradict to each other because β is long. q.e.d.

Proposition 5.16. *Let $\mathfrak{g} \subset \mathfrak{so}(N,\mathbb{C})$ be an irreducible representation of real type of a complex simple Lie algebra different from $\mathfrak{sl}(2,\mathbb{C})$, with $0 \notin \Omega$ and satisfying* (SII). *If furthermore all roots have the same length, and if there is a triple of orthogonal roots $\eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1$ such that $\left|\frac{2\langle\Lambda,\eta_1\rangle}{\|\eta_1\|^2}\right| = 2$ and $\left|\frac{2\langle\Lambda,\eta_2\rangle}{\|\eta_2\|^2}\right| = \left|\frac{2\langle\Lambda,\eta_3\rangle}{\|\eta_3\|^2}\right| = 1$, then either*

1) $\Lambda - \alpha$ *is an extremal weight, i.e.,* (SII) *defines an extremal spanning triple, or*
2) $\Lambda = \alpha + \frac{1}{2}(\beta + \gamma)$ *with roots $\alpha \perp \beta \perp \gamma \perp \alpha$.*

Proof. Let α be the root determined by (SII). The assumption implies that there is an extremal weight Λ' and roots β and γ such that

$$\frac{2\langle\Lambda',\alpha\rangle}{\|\alpha\|^2} = -2 \quad\text{and}\quad \frac{2\langle\Lambda,\beta\rangle}{\|\beta\|^2} = \frac{2\langle\Lambda,\gamma\rangle}{\|\gamma\|^2} = -1.$$

Then $\Lambda'+k\alpha+l\beta+m\gamma \in \Omega$ for $k,l,m=0,1$. Hence (SII) implies again:

$$
\begin{array}{lrcr}
 & (L) & & (R) \\
(52) & \Lambda'+\alpha=\Lambda+\gamma_0 & \text{or} & \Lambda'=-\Lambda+\delta_0 \\
(53) & \Lambda'+2\alpha=\Lambda+\gamma_1 & \text{or} & \Lambda'+\alpha=-\Lambda+\delta_1 \\
(54) & \Lambda'+\alpha+\beta=\Lambda+\gamma_2 & \text{or} & \Lambda'+\beta=-\Lambda+\delta_2 \\
(55) & \Lambda'+2\alpha+\beta=\Lambda+\gamma_3 & \text{or} & \Lambda'+\alpha+\beta=-\Lambda+\delta_3 \\
(56) & \Lambda'+\alpha+\gamma=\Lambda+\gamma_4 & \text{or} & \Lambda'+\gamma=-\Lambda+\delta_4 \\
(57) & \Lambda'+2\alpha+\gamma=\Lambda+\gamma_5 & \text{or} & \Lambda'+\alpha+\gamma=-\Lambda+\delta_5 \\
(58) & \Lambda'+\alpha+\beta+\gamma=\Lambda+\gamma_6 & \text{or} & \Lambda'+\beta+\gamma=-\Lambda+\delta_6 \\
(59) & \Lambda'+2\alpha+\beta+\gamma=\Lambda+\gamma_7 & \text{or} & \Lambda'+\alpha+\beta+\gamma=-\Lambda+\delta_7.
\end{array}
$$

Supposing (52.L) excludes (59.R) because the roots are orthogonal. Thus (59.R) holds true. Now we consider two cases:

Case 1: $\langle\gamma_0,\beta\rangle=\langle\gamma_0,\gamma\rangle=0$. This excludes (54.L), (56.L) and (58.L) and implies therefore (54.R), (56.R) and (58.R). The latter together with (59.R) gives $\alpha=\delta_7-\delta_6$. Since $\delta_7\neq 0$ this implies $\langle\alpha,\delta_7\rangle>0$. On the other hand (59.R) and the assumption gives that $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2}=\frac{2\langle\delta_7,\alpha\rangle}{\|\alpha\|^2}>0$. If $\alpha\neq\delta_7$ we are done. But $\delta_7=\alpha$ implies $\Lambda'+\beta+\gamma=-\Lambda=-\Lambda'-\alpha-\gamma_0$ and hence $-2=2-2-\frac{2\langle\alpha,\gamma_0\rangle}{\|\alpha\|^2}$, i.e., $\gamma_0=-\alpha$. Taking everything together we get $2\Lambda=2\alpha-(\beta+\gamma)$.

Case 2: $\langle\gamma_0,\beta\rangle$ or $\langle\gamma_0,\gamma\rangle$ not equal to zero. This implies $\gamma_0\neq\pm\alpha$ and thus $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2}=-\frac{2\langle\gamma_0,\alpha\rangle}{\|\alpha\|^2}=\pm1$ or zero. Now (53.L) would imply $\alpha=\gamma_1-\gamma_0$, i.e., $\langle\alpha,\gamma_0\rangle<0$ which is the proposition.

Hence we suppose (53.R). This together with the starting point (52.L) gives

$$
\begin{aligned}
\Lambda &= \frac{1}{2}(\delta_1-\gamma_0) \quad \text{and} \\
\Lambda' &= -\alpha+\frac{1}{2}(\delta_1+\gamma_0).
\end{aligned}
$$

Using the assumption, the second equation implies 1 $\langle\alpha,\delta_1+\gamma_0\rangle=0$. For the length of both extremal weights it holds then

$$
\begin{aligned}
\|\Lambda\|^2 &= \frac{1}{4}\left(\|\delta_1\|^2+\|\gamma_0\|^2-2\langle\delta_1,\gamma_0\rangle\right) \\
\|\Lambda'\|^2 &= \|\alpha\|^2-\underbrace{\langle\alpha,\delta_1+\gamma_0\rangle}_{=0}+\frac{1}{4}\left(\|\delta_1\|^2+\|\gamma_0\|^2+2\langle\delta_1,\gamma_0\rangle\right).
\end{aligned}
$$

This gives $0=\|\alpha\|^2+\langle\delta_1,\gamma_0\rangle$. Since all roots have the same length this implies $\delta_1=-\gamma_0$. Hence Λ is a root. But this was excluded. q.e.d.

Now using the Proposition 5.14 of Schwachhöfer we get a corollary.

Corollary 5.17. *Let $\mathfrak{g} \subset \mathfrak{so}(N,\mathbb{C})$ be an irreducible representation of real type of a complex simple Lie algebra different from $\mathfrak{sl}(2,\mathbb{C})$, with $0 \notin \Omega$ and satisfying* (SII)*. Then it holds:*

1) *There is no pair of orthogonal long roots η_1 and η_2 such that $\left|\frac{2\langle\Lambda,\eta_i\rangle}{\|\eta_i\|^2}\right| = 2$ for the extremal weight Λ from the property* (SII).
2) *If furthermore all roots have the same length, and if there is a triple of orthogonal roots $\eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1$ such that $\left|\frac{2\langle\Lambda,\eta_1\rangle}{\|\eta_1\|^2}\right| = 2$ and $\left|\frac{2\langle\Lambda,\eta_2\rangle}{\|\eta_2\|^2}\right| = \left|\frac{2\langle\Lambda,\eta_3\rangle}{\|\eta_3\|^2}\right| = 1$ then $\Lambda = \alpha + \frac{1}{2}(\beta+\gamma)$ with roots $\alpha\perp\beta\perp\gamma\perp\alpha$.*

Before we apply this corollary we have to deal with the remaining exception in the second point.

Lemma 5.18. *If the representation of a simple Lie algebra with roots of the same length has an extremal weight Λ such that $\Lambda = \alpha + \frac{1}{2}(\beta+\gamma)$ with roots $\alpha\perp\beta\perp\gamma\perp\alpha$, then it holds*

1) *There is no root δ such that $\langle\delta,\beta\rangle = 0$, $\langle\delta,\gamma\rangle \neq 0$ and $\delta \not\sim \gamma$, i.e., $\Delta_\beta^\perp$ has the direct summand $\{\pm\gamma\}$.*
2) *The root system is D_n and the representation has one of the following highest weights: ω_3 for $n > 3$, or $\omega_1 + \omega_3$ or $\omega_1 + \omega_4$ for $n = 4$.*

Proof. The first point is easy to see: If there is such a δ then we have

$$\frac{2\langle\Lambda,\delta\rangle}{\|\delta\|^2} = \frac{2\langle\alpha,\delta\rangle}{\|\delta\|^2} + \frac{1}{2}\frac{2\langle\gamma,\delta\rangle}{\|\delta\|^2} = \frac{2\langle\alpha,\delta\rangle}{\|\delta\|^2} \pm \frac{1}{2} \notin \mathbb{Z}.$$

This is a contradiction.

For the second point one verifies directly that the fact that $\Delta_\beta^\perp$ contains a direct A_1 summand leaves us with $\Delta = D_n$. Hence we have to deal with representations of D_n. Now we consider the different root systems with roots of constant length. If $\alpha = e_i \pm e_j$, $\beta = e_p \pm e_q$ and $\gamma = e_r \pm e_s$ with all indices different from each other, then $\frac{2\langle\Lambda,e_i-e_p\rangle}{\|e_i-e_p\|^2}$ is not an integer. Thus we are left with two cases. The first is $\beta+\gamma = e_p + e_q + e_p - e_q = 2e_p$ and hence $\Lambda = e_i \pm e_j + e_p$. This leads to $\Lambda = \omega_3$ for $n > 3$. The second is $\alpha = e_i + e_j$, $\beta = e_i - e_j$ and $\gamma = e_p \pm e_q$. For $n > 4$ we found a root $e_p + e_s$ which leads to a contradiction by applying the first point. For $n = 4$ we have $\Lambda = \frac{3}{2}e_i + \frac{1}{2}(e_j + e_p \pm e_q)$. But this yields the remaining representations. q.e.d.

Now using all these properties we find the representations without weight zero and satisfying (SII).

Proposition 5.19. *Let $\mathfrak{g} \subset \mathfrak{so}(N,\mathbb{C})$ be an irreducibly acting complex simple Lie algebra different from $\mathfrak{sl}(2,\mathbb{C})$, with $0 \notin \Omega$ and satisfying*

(SII). *Then the root system and the highest weight of the representation is one of the following (modulo congruence):*

1) A_n: ω_4 *for* $n = 7$.
2) B_n: ω_n *for* $n = 3, 4, 7$.
3) D_n: ω_1, $2\omega_1$ *for arbitrary* n *and* ω_8 *for* $n = 8$.

Proof. We apply Proposition 5.13 and Corollary 5.17 to the remaining representations with $0 \notin \Omega$, i.e., representations of A_n, B_n, C_n, D_n, E_6 and E_7. We use a fundamental system such that the extremal weight Λ determined by (SII) is the highest weight, i.e., $\Lambda = \sum_{k=1}^{n} m_k \omega_k$ with $m_k \in \mathbb{N} \cup \{0\}$.

A_n: Proposition 5.13 gives for the largest root

$$2 \;\geq\; \frac{2\langle \Lambda, e_1 - e_{n+1}\rangle}{\|e_1 - e_{n+1}\|^2} = \sum_{k=1}^{n} m_k \langle \omega_k, e_1 - e_{n+1}\rangle = \sum_{k=1}^{n} m_k.$$

Since the representation has to be self dual we have that $m_i = m_{n+1-i}$. First we consider the case that $\Lambda = \omega_i + \omega_{n+1-i}$. For $n > 2$ we get in case $i > 1$ that $\langle \Lambda, e_2 - e_n\rangle = 2$. But $(e_2 - e_n) \perp (e_1 - e_{n+1})$ gives a contradiction to 1 of Corollary 5.17. For $n \geq 2$ it has to be

$$\Lambda = \omega_1 + \omega_n = 2e_1 + e_2 + \cdots e_n = e_1 - e_{n+1}.$$

This is the adjoint representation with $0 \in \Omega$.

Now we consider the case that $n+1$ is even. The case $\Lambda = 2\omega_{\frac{n+1}{2}}$ is excluded because of point 1 of Corollary 5.17: it is $\langle \Lambda, e_2 - e_n\rangle = \langle \Lambda, e_1 - e_{n+1}\rangle = 2$ for $n > 2$. Thus we have to study the case $\Lambda = \omega_{\frac{n+1}{2}}$. This representation is orthogonal if $\frac{n+1}{2}$ is even. The weights of this representation are given by $\frac{1}{2}(\pm e_{k_1} \pm \cdots \pm e_{k_{\frac{n+1}{2}}})$ where the $\pm$'s are meant to be independent of each other. We show that (SII) implies $n \leq 7$. We have to consider two cases for α of (SII). The first is that $\alpha = e_i - e_j$ with $1 \leq i \leq \frac{n+1}{2} < j \leq n+1$. W.l.o.g. we take $\alpha = e_{\frac{n+1}{2}} - e_{\frac{n+1}{2}+1}$ and consider the weight

$$\lambda := e_1 + \cdots e_{\frac{n+1}{2}-3} + e_{\frac{n+1}{2}+1} + e_{\frac{n+1}{2}+2} + e_{\frac{n+1}{2}+3}.$$

$\langle \lambda, \alpha\rangle < 0$ implies $\lambda \in \Omega_\alpha$. Then $\lambda - (\Lambda - \alpha) \in \Delta_0$ or $\lambda + \Lambda \in \Delta_0$. We check the first alternative: $\Lambda - \alpha = e_1 + \cdots e_{\frac{n+1}{2}-1} + e_{\frac{n+1}{2}+1}$ implies

$$\lambda - (\Lambda - \alpha) = e_{\frac{n+1}{2}-3} + e_{\frac{n+1}{2}-2} + e_{\frac{n+1}{2}+2} + e_{\frac{n+1}{2}+3}.$$

But this is not a root. For the second alternative we get

$$\lambda + \Lambda = e_1 + \cdots + e_{\frac{n+1}{2}-3} - e_{\frac{n+1}{2}+4} - \cdots - e_{n+1}.$$

This is not a root if $\frac{n+1}{2} > 4$, i.e., $n > 7$.

For the second type of root $\alpha = e_i - e_j$ with $1 \leq i < j \leq \frac{n+1}{2}$ and $\frac{n+1}{2} < i < j \leq n+1$ one derives analogously that $n \leq 5$.

Hence for $\Lambda = \omega_{\frac{n+1}{2}}$ the property (SII) can only be fulfilled if $n \leq 7$. These representations are orthogonal for $n = 7$ and $n = 3$. A_3 is isomorphic to D_3 and the representation with highest weight ω_2 of A_3 is equivalent to the one with ω_1 of D_3.

B_n: Proposition 5.13 gives for the largest root

$$2 \geq \frac{2\langle \Lambda, e_1 + e_2 \rangle}{\|e_1 + e_2\|^2} = m_1 + 2m_2 + \cdots 2m_{n-1} + m_n.$$

The only representations with $0 \notin \Omega$ have $\Lambda = \omega_1 + \omega_n$ and the spin representation $\Lambda = \omega_n$. There is no possibility to apply the first point of Corollary 5.17. But we verify that for $\Lambda = \omega_1 + \omega_n$ (SII) implies $n \leq 2$, and for the spin representation $\Lambda = \omega_n$ (SII) implies $n \leq 7$.

The spin representations: We show that (SII) implies $n \leq 7$. The spin representation has weights $\Omega = \left\{ \frac{1}{2}(\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n) | \varepsilon_i = \pm 1 \right\}$. We have to consider three types for the root α: $\alpha = e_i$, $\alpha = e_i + e_j$ and $\alpha = e_i - e_j$.

For the first we can assume w.l.o.g. that $\alpha = e_1$. Then $\Omega_\alpha = \{\frac{1}{2}(-e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_n e_n) | \varepsilon = \pm 1\}$. It is $\Lambda - \alpha = \frac{1}{2}(-e_1 + e_2 + \cdots + e_n)$. Hence for $\lambda \in \Omega_\alpha$ we have

$$\begin{aligned} \Lambda - \alpha - \lambda &= \frac{1}{2}((1 - \varepsilon_2)e_2 + \cdots + (1 - \varepsilon_n)e_n) \text{ and} \\ \Lambda + \lambda &= \frac{1}{2}((1 + \varepsilon_2)e_2 + \cdots + (1 + \varepsilon_n)e_n). \end{aligned}$$

If (SII) is satisfied one of these expression has to be a root. But if $n \geq 7$ we can choose $(\varepsilon_2, \ldots \varepsilon_n)$ such that none of them is a root.

The second type of root is w.l.o.g. $\alpha = e_1 - e_2$. In this case $\Omega_\alpha = \{\frac{1}{2}(-e_1 + e_2 + \varepsilon_3 e_3 + \cdots + \varepsilon_n e_n) | \varepsilon_i = \pm 1\}$ and $\Lambda - \alpha = \frac{1}{2}(-e_1 + 2e_2 + e_3 + \cdots + e_n)$. If $n \geq 4$ we can choose $\lambda \in \Omega_\alpha$ such that neither $\Lambda - \alpha - \lambda$ nor $\Lambda + \lambda$ is a root.

Now we consider the last type of root, $\alpha = e_1 + e_2$. $\Omega_\alpha = \{\frac{1}{2}(-e_1 - e_2 + \varepsilon_3 e_3 + \varepsilon_n e_n) | \varepsilon_i = \pm 1\}$ and $\Lambda - \alpha = \frac{1}{2}(-e_1 - e_2 + e_3 + \cdots + \cdots + e_n)$. If $n \geq 8$ we can choose $\lambda \in \Omega_\alpha$ such that neither $\Lambda - \alpha - \lambda$ nor $\Lambda + \lambda$ is a root. Hence if (SII) is satisfied, it has to be $n \leq 7$ and for $n = 7$ the pair of property (SII) is of the shape $(\Lambda, e_1 + e_2)$. But for $n = 2$, $n = 5$ and $n = 6$ the spin representations are symplectic but not orthogonal.

The representations of $\Lambda = \omega_1 + \omega_n = \frac{3}{2}e_1 + \frac{1}{2}(e_2 + \cdots + e_n)$. The weights are given by $\frac{1}{2}(a e_{k_1} + \varepsilon_2 e_{k_2} + \cdots + \varepsilon_n e_{k_n})$ with $a \in \{\pm 1, \pm 3\}$ and $\varepsilon_i = \pm 1$. For these one shows analogously that (SII) implies $n \leq 2$. For $n = 2$ this representation is symplectic.

C_n: For the largest root we get

$$2 \geq \frac{2\langle \Lambda, 2e_1 \rangle}{\|2e_1\|^2} = \sum_{k=1}^{n} m_k \langle \omega_k, e_1 \rangle = \sum_{k=1}^{n} m_k.$$

If $m_i = 2$ and all others are zero then $0 \in \Omega$. Hence we suppose that $\Lambda = \omega_i + \omega_j$ for $i \neq j$. If $i > 1$ we get for the root $2e_2$ which is orthogonal to $2e_1$ that $\frac{2\langle \Lambda, 2e_2\rangle}{\|2e_2\|^2} = 2$. Thus by 1 of Corollary 5.17 we have $i = 1$. $\Lambda = \omega_1 + \omega_i$ is only orthogonal if i is odd, but if i is odd we have that $0 \in \Omega$. Hence $\Lambda = \omega_i$. This is orthogonal if i is even, but in this case it is again $0 \in \Omega$.

D_n: Here we get for the largest root

$$2 \geq \frac{2\langle \Lambda, e_1 + e_2\rangle}{\|e_1 + e_2\|^2} = m_1 + 2m_2 + \cdots + 2m_{n-2} + m_{n-1} + m_n.$$

First we consider the representation where this number is equal to 2. For the representations $2\omega_n$ and $2\omega_{n-1}$ it is $0 \in \Omega$. For the representations $\Lambda = \omega_1 + \omega_n$ and $\Lambda = \omega_1 + \omega_{n-1}$ we get that $n = 4$ or there is no triple as in the second point of Proposition 5.17. Thus suppose in this case $n > 4$. We have that$\langle \Lambda, e_1 + e_2\rangle = 2$ and for the orthogonal roots $\langle \Lambda, e_1 - e_2\rangle = \langle \Lambda, e_3 \pm e_4\rangle = 1$. But this contradicts Proposition 5.17,1.

For $\Lambda = \omega_{n-1} + \omega_n = e_1 + \cdots + e_{n-1}$, $0 \notin \Omega$ implies $n-1$ even. The first point of Corollary 5.17 gives for $n > 4$ that $2 = \langle \Lambda, e_3 + e_4\rangle$ which is impossible. Hence $n \leq 4$, and $1 = \langle \Lambda, e_3 \pm e_4\rangle$ together with the second point of Corollary 5.17 implies $n \leq 3$.

Now suppose that $\Lambda = \omega_i$ for $2 \leq i \leq n-2$. We apply the first point of Corollary 5.17. If $n \geq 4$ we get that $\langle \omega_i, e_3 + e_4\rangle = 2$ for $i \geq 4$ but this was excluded. Hence $i \leq 3$. In the case $n = 3$ only ω_2 is an orthogonal representation. But for this is $0 \in \Omega$.

Thus, to get the assertion of the proposition we have to show that

1) For the spin representations $\Lambda = \omega_{n-1}$ and $\Lambda = \omega_n$ (SII) implies $n \leq 8$
2) $\Lambda = \omega_3$ does not satisfy (SII),
3) $\Lambda = \omega_1 + \omega_3$ and $\omega_1 + \omega_4$ for $n = 4$ do not satisfy (SII).

The spin representations: Because we are interested in the representations modulo congruence it suffices to consider the spin representation of highest weight $\Lambda = \frac{1}{2}(e_1 + \cdots + e_n)$ with weights $\Omega = \left\{\frac{1}{2}(\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n) | \varepsilon_i = \pm 1 \text{ and } \varepsilon_i = -1 \text{ for an even number}\right\}$. Analogously as for B_n we get for two types of roots $\alpha = e_i + e_j$ and $\alpha = e_i - e_j$ that (SII) implies $n \leq 8$. (We have to admit one dimension higher because of the sign restriction of the weights.) Now for n odd the spin representation is not self dual, and for $n = 6$ not orthogonal. For $n = 4$ it is congruent to ω_1.

$\Lambda = \omega_3 = e_1 + e_2 + e_3$: Here it is $\Omega = \{(\varepsilon_1 e_{k_1} + \varepsilon_2 e_{k_2} + \varepsilon_3 e_{k_3} | \varepsilon_i = \pm 1\} \cup \{\pm e_i\}$. For $n = 3$ and $n = 4$ this is a spin representation. Hence suppose $n \geq 5$. For $\alpha = e_1 + e_2$ we get $\Lambda - \alpha = e_3$. Set $\lambda := -e_1 + e_4 + e_5 \in \Omega_\alpha$. Hence $\Lambda - \alpha - \lambda = e_3 + e_1 - e_4 - e_5$ and $\Lambda + \lambda = e_2 + e_3 + e_4 + e_5$. None is a root, i.e., ω_3 for $n \geq 5$ does not satisfy (SII). For $\alpha = e_1 - e_2$ we get the same.

$\Lambda = \omega_1 + \omega_3$ *and* $\omega_1 + \omega_4$ *for* $n = 4$. These are congruent to each other and as above it can be shown that they do not satisfy (SII).

E_6 and E_7: For these we refer to [**33**]. There is shown that under the conclusions of Proposition 5.13 and 5.14 — which is our situation because of Lemma 5.18 — the only remaining representations are the standard representations of E_6 and E_7. But the first is not self dual and the latter symplectic but not orthogonal. q.e.d.

5.5. Consequences for simple weak-Berger algebras of real type. Before we get the result we have to exclude both exceptions.

Lemma 5.20. *The spin representation of B_7 and the representation of G_2 with two times a short root as highest weight are not weak-Berger.*

Proof. 1) Suppose that the spin representation of B_7 is weak-Berger. We have shown that it does not satisfy the property (SI). Hence it obeys (SII). Let (Λ, α) be the pair of (SII). We choose a fundamental system such that $\Lambda = \omega_7$ is the highest weight. In the proof of Proposition 5.19 we have shown that in this case $\alpha = e_i + e_j$.

Let now Q_ϕ be the weight element from $\mathcal{B}_H(\mathfrak{g})$ and $u_\Lambda \in V_\Lambda$ such that $Q_\phi(u_\Lambda) = A_{e_i+e_j} \in \mathfrak{g}_{e_i+e_j}$. Since $Q_\phi(u_\Lambda) \in \mathfrak{g}_{\phi+\Lambda}$ this implies that $\phi = e_i + e_j - \Lambda$ is a weight of $\mathcal{B}_H(\mathfrak{g})$. Hence $\phi = -\frac{1}{2}(e_1 + \cdots + e_{i-1} - e_i + e_{i+1} + \cdots + e_{j-1} - e_j + e_{j+1} + \cdots + e_7)$ is also an extremal weight of V and we can consider a weight vector $u_{-\phi} \in V_{-\phi}$. For this we get $Q_\phi(u_{-\phi}) \in \mathfrak{t}$. In case it does not vanish it would define a planar spanning triple $(\phi, -\phi, (Q_\phi(u_{-\phi}))^\perp)$, i.e., (SI) would be satisfied. But this was not possible, and thus $Q_\phi(u_{-\phi}) = 0$.

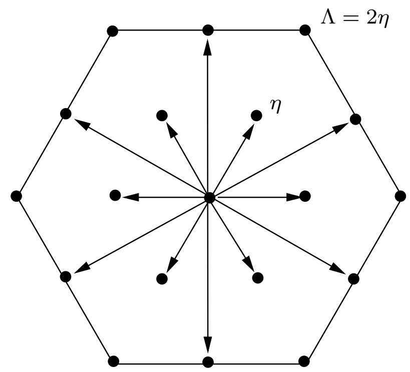
On the other hand we have that $0 \neq Q_\phi(u_\Lambda)u_{-\phi} \in V_\Lambda$ and thus there is a $v \in V_{-\Lambda}$ such that $H(Q_\phi(u_\Lambda)u_{-\phi}, v) \neq 0$. Now the Bianchi identity gives

$$0 = H(Q_\phi(u_\Lambda)u_{-\phi}, v) + \underbrace{H(Q_\phi(u_{-\phi})v, u_\Lambda))}_{=0} + H(Q_\phi(v)u_\Lambda, u_{-\phi}, v).$$

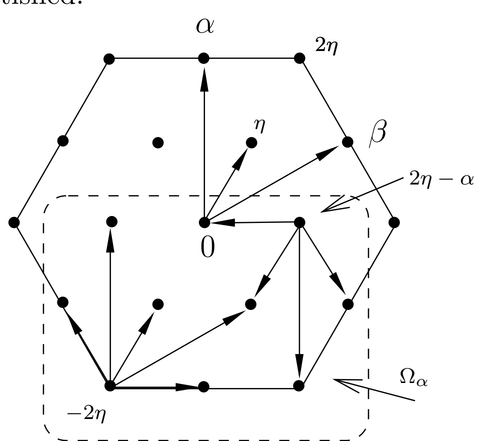
Hence $0 \neq Q_\phi(v) \in \mathfrak{g}_{\phi-\Lambda}$. But $\phi - \Lambda = -(e_1 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_{j-1} + e_{j+1} + \cdots + e_7)$ is not a root, hence $\mathfrak{g}_{\phi-\Lambda} = \{0\}$. This is a contradiction.

2) Suppose that the representation of G_2 with two times a short root as highest weight is weak-Berger. We shall argue analogously as for B_n.

In the picture we see the weight lattice of this representation. Obviously there is no planar spanning triple, because there is no hypersurface which contains all but two extremal weight (see also proof of Proposition 5.2). The weak-Berger property implies that there is a pair (Λ, α) such that (SII) is satisfied. We choose a fundamental system such that $\Lambda = 2\eta$ is the maximal weight.



Using the realisation of G_2 from the appendix of [**24**] we have that $\eta = e_3 - e_2$. Now we have to determine the roots for which (SII) is satisfied.



In the picture one can see that the long roots α and β satisfy (SII). (We illustrate the situation in detail only for α.) Contemplating the picture for a moment, one sees that there are no short roots and no other long root for which (SII) can be valid.
Now α and β are the only roots with $\langle \Lambda, \alpha \rangle > 0$ and $\langle \Lambda, \beta \rangle > 0$. Hence $\alpha = 2e_3 - e_1 - e_2$ and $\beta = -2e_2 + e_1 + e_3$.

We consider the case where (Λ, α) satisfies (SII). There is a weight element Q_ϕ from $\mathcal{B}_H(\mathfrak{g})$ such that $Q_\phi(u_\Lambda) = A_{2e_3 - e_1 - e_2}$, i.e., $\phi = 2e_3 - e_1 - e_2 - \Lambda = e_2 - e_1$. But this is a short root and therefore a weight. Thus we consider $u_{-\phi} \in = V_{-\phi}$. Then $Q_\phi(u_{-\phi}) \in \mathfrak{t}$. Since there is no planar spanning triple it has to be zero. As above, the Bianchi identity gives that $\phi - \Lambda$ has to be a root. But $\phi - \Lambda = e_2 - e_1 - 2e_3 + 2e_2 = 3e_2 - 2e_3 - e_1$ is no root. For β one proceeds analogously. q.e.d.

Now we can draw the conclusions from the previous sections. If a Lie algebra acts irreducible of real type then it is semisimple and obeys the properties (SI) or (SII). The simple Lie algebras with (SI) or (SII) we have listed above. Thus we have proven that an irreducibly acting, complex, simple weak-Berger algebra is the complexification of an irreducible holonomy representation of a Riemannian manifold, which implies Theorem 1.1 in the case that the irreducibly acting ideals are simple and of real type.

6. Classification of semisimple complex weak-Berger algebras

In this section we complete the proof of Theorem 1.1 by proving the following statement. We shall use several results of the previous section.

Proposition 6.1. *Any irreducibly acting, semisimple, non-simple complex weak-Berger algebra of real type is the complexification of an irreducible Riemannian holonomy algebra.*

6.1. Semisimple, non-simple weak-Berger algebras. From now on, let $\mathfrak{g}$ be a complex semisimple, non-simple Lie algebra, irreducibly represented on a complex vector space V. To a decomposition of $\mathfrak{g}$ into ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ corresponds a decomposition of the irreducible module V into factors $V = V_1 \otimes V_2$ which are irreducible $\mathfrak{g}_1$- resp. $\mathfrak{g}_2$-modules. $X = (X_1, X_2) \in \mathfrak{g}$ acts as follows: $X \cdot (v_1 \otimes v_2) = (X_1 \cdot v_1) \otimes v_2 + v_1 \otimes X_2 \otimes v_2$. The Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is the sum of the Cartan subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. If Δ are the roots of $\mathfrak{g}$ and Δ^i the roots of $\mathfrak{g}_i$ then $\Delta = \Delta^1 \cup \Delta^2$. For the weights it holds $\Omega = \Omega^1 + \Omega^2$. Analogously we denote for $\alpha \in \Delta^i$ the set Ω^i_α.

Lemma 6.2. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a semisimple Lie algebra with an irreducible module $V = V_1 \otimes V_2$. If $\alpha \in \Delta^1$, then*

$$\Omega_\alpha = \Omega^1_\alpha + \Omega^2. \tag{60}$$

Proof. For $\lambda \in \Omega_\alpha$ we have $\Omega \ni \lambda + \alpha = \lambda_1 + \alpha + \lambda_2$ with $\lambda_i \in \Omega_i$. Hence $\lambda_1 + \alpha \in \Omega^1$. If otherwise $\lambda_1 + \alpha \in \Omega^1$ then $\lambda_1 + \lambda_2 + \alpha \in \Omega$, i.e., $\lambda_1 + \lambda_2 \in \Omega_\alpha$. q.e.d.

Assuming the weak-Berger property this implies a second lemma.

Lemma 6.3. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a semisimple Lie algebra with irreducible module $V = V_1 \otimes V_2$ which is weak-Berger. If the dimensions of V_1 and V_2 are greater than 2, then for any $\alpha \in \Delta^i$ the set Ω^i_α contains at most 2 elements.*

Proof. Suppose that $\dim V_2 \geq 3$, i.e., $\#\Omega^2 \geq 3$. Let $\alpha \in \Delta^1$, $\lambda_1 \in \Omega^1_\alpha$ and $\lambda_2 \in \Omega^2$, i.e., $\lambda_1 + \lambda_2 \in \Omega^1_\alpha$. Now, from the property (PII) follows that there is a $\mu_\alpha =: \mu^1_\alpha + \mu^2_\alpha \in \Omega$ such that $\lambda_1 + \lambda_2 = \mu_\alpha - \alpha + \beta$ or $\lambda_1 + \lambda_2 = -\mu_\alpha + \beta$ with $\beta \in \Delta_0 = \Delta^1 \cup \Delta^2 \cup \{0\}$. If now $\#\Omega^2 \geq 3$ and $\#\Omega^1_\alpha \geq 3$, then we can choose $\lambda_1 \neq \mu^1_\alpha - \alpha$, $\lambda_1 \neq -\mu^1_\alpha$ and $\lambda_2 \neq \pm\mu^2_\alpha$. This gives a contradiction. q.e.d.

Now we can use a result of [**33**].

Proposition 6.4 ([**33**, Lemma 3.22])**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ be an irreducibly acting, semisimple Lie algebra. If for some α the set Ω_α contains at most two elements, then $\mathfrak{g}$ is conjugate to one of the following representations:*

1) $\mathfrak{sl}(n, \mathbb{C})$ *acting on $\mathbb{C}^n$; in this case Ω_α is a singleton for all $\alpha \in \Delta$.*

2) $\mathfrak{so}(n,\mathbb{C})$ *acting on* $\mathbb{C}^n$*; in this case* Ω_α *contains two elements for all* $\alpha \in \Delta$*, and their sum equals to* $-\alpha$*.*
3) $\mathfrak{sp}(n,\mathbb{C})$ *acting on* $\mathbb{C}^{2n}$*; in this case* Ω_α *contains two elements if* $\alpha \in \Delta$ *is short, and their sum equals to* $-\alpha$*, and* $\Omega_\alpha = \{-\frac{1}{2}\alpha\}$ *if* α *is long.*

From this result we obtain the following corollary, proving Proposition 6.1 if the dimensions of the factors of V are greater than 2.

Corollary 6.5. *Let* $\mathfrak{g} \subset \mathfrak{so}(V,h)$ *be a complex, semisimple, non-simple, irreducibly acting weak-Berger algebra. If* $\mathfrak{g}$ *decomposes into* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *such that for the corresponding decomposition of* $V = V_1 \otimes V_2$ *it holds that* $\dim V_i \geq 3$ *for* $i = 1,2$*, then it holds:* $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C}) \oplus \mathfrak{so}(m,\mathbb{C})$ *acting on* $\mathbb{C}^n \otimes \mathbb{C}^m$*, or* $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C}) \oplus \mathfrak{sp}(m,\mathbb{C})$ *acting on* $\mathbb{C}^{2n} \otimes \mathbb{C}^{2m}$*. In particular* $\mathfrak{g}$ *is the complexification of a Riemannian holonomy representation of a symmetric space of type BDI resp. CII.*

Proof. By Lemma 6.3 it must hold $\#\Omega^i_\alpha \leq 2$ for both summands. So we have to build sums of the Lie algebras of Proposition 6.4. But only the sum of two orthogonal Lie algebras, or a sum of two symplectic Lie algebras acts orthogonally. q.e.d.

By this result we are left with semisimple Lie algebras where the irreducible representation of one summand is two-dimensional, i.e., $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_2$ and $V = \mathbb{C}^2 \otimes V_2$. Since we are interested in $\mathfrak{g} \subset \mathfrak{so}(V,h)$ and $\mathfrak{sl}(2,\mathbb{C})$ acts symplectically on $\mathbb{C}^2$ the representation of $\mathfrak{g}_2$ on V_2 has to be symplectic too. In this situation we prove the following fact.

Proposition 6.6. *Let* $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_2$ *be a semisimple, complex weak-Berger algebra, acting irreducibly on* $\mathbb{C}^2 \otimes V_2$*. Then* $\mathfrak{g}_2 \subset \mathfrak{sp}(V_2)$ *satisfies the following properties:*

(PIII): *There is a* $\mu \in \Omega^2$ *and an affine hyperplane* $A \subset \mathfrak{t}_2^*$ *such that*

$$\Omega^2 \subset \left\{\mu + \beta \mid \beta \in \Delta_0^2\right\} \cup A \cup \{-\mu\}. \tag{61}$$

(PIV): *There is a* $\mu \in \Omega^2$ *such that*

$$\Omega^2 \subset \left\{\mu + \beta \mid \beta \in \Delta_0^2\right\} \cup \left\{-\mu + \beta \mid \beta \in \Delta_0^2\right\}. \tag{62}$$

Proof. Since $\mathfrak{g}$ is weak-Berger it satisfies the properties (PI) and (PII). We draw the consequences from both for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_2$. For the representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$ we have that $\Omega^1 = \{\Lambda, -\Lambda\}$. Let $\alpha \in \Delta^1 = \{\alpha, -\alpha\}$ be the positive root of $\mathfrak{sl}(2,\mathbb{C})$. Hence $\Omega^1_\alpha = \{-\Lambda\}$, since $-\Lambda = \Lambda - \alpha$.

(PIII) $\mathfrak{g}$ satisfies the property (PI) with a hyperplane $U := (T_1 + T_2)^\perp$ and a weight $\mu = \mu_1 + \mu_2$. Let $\lambda = \mu_1 + \lambda_2 \in \Omega$ be a weight of $\mathfrak{g}$. If λ lies in a hyperplane of $\mathfrak{t} = \mathfrak{t}_1 \oplus^\perp \mathfrak{t}_2$, then $0 = \langle \mu_1, T_1\rangle + \langle \lambda_2, T_2\rangle$, i.e., λ_2 lies in an affine hyperplane of $\mathfrak{t}$. If $\lambda = \mu + \beta = \mu_1 + \mu_2 + \beta$, then

$\lambda_2 = \mu_2 + \beta$ with $\beta \in \Delta^2$. If $\lambda = -\mu + \beta = -\mu_1 - \mu_2 + \beta$, then β has to be in Δ^1 and $\lambda_2 = \mu_2$. Hence, $\mathfrak{g}_2 \subset \mathfrak{sp}(V_2)$ satisfies (PIII).

(PIV) $\mathfrak{g}$ satisfies the property (PII). Suppose that α is the positive root of $\mathfrak{sl}(2, \mathbb{C})$. Then $\Omega^1_\alpha = \{-\Lambda\}$ and $\Omega_\alpha = \{-\Lambda\} \cup \Omega^2$. Now let $\lambda \in \Omega^2$, i.e., $-\Lambda + \lambda \in \Omega_\alpha$. By (PII) there is a $\mu_\alpha = \mu^1 + \mu^2$ such that $-\Lambda + \lambda = \mu_\alpha - \alpha + \beta$ or $-\Lambda + \lambda = -\mu_\alpha + \beta$ with $\beta \in \Delta_0 = \Delta^1 \cup \Delta^2 \cup \{0\}$. Since $\mu_1 = \pm\Lambda$ this implies $\lambda \in \{\mu_2 + \beta | \beta \in \Delta^2_0\} \cup \{-\mu_2 + \beta | \beta \in \Delta^2_0\}$, i.e., (PIV) is satisfied. q.e.d.

Example 6.7. We set $\mathfrak{g}_2 = \mathfrak{sl}(2, \mathbb{C})$ and check if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}^2 \otimes V_2$ is a weak-Berger algebra. This is to check whether $\mathfrak{sl}(2, \mathbb{C})$ acting on V_2 satisfies (PIII) and (PIV) and is symplectic. To be symplectic means that the representation has an even number of weights, (PIV) implies that V_2 has at most 6 weights but (PIII) implies that V_2 has at most 4 weights. Hence the only weak-Berger algebras with the structure of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ are those acting on $\mathbb{C}^4$ and on $\mathbb{C}^2 \otimes \mathrm{sym}^3_0 \mathbb{C}^2 = \mathbb{C}^8$. Both are of course complexifications of Riemannian holonomy representations, the first of $SO(4)$ on $\mathbb{R}^4$ and the second of the 8-dimensional symmetric space of type GI, i.e., $G_2/SU(2) \cdot SU(2)$.

Now we try to reduce the problem in a way that we only have to deal with simple Lie algebras.

Lemma 6.8. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple, complex Lie algebra acting irreducibly on V, satisfying the property* (PIV). *Then $\mathfrak{g}$ is simple or there is a $\mathfrak{g}_2$ acting on V_2 such that $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_2$ acting on $\mathbb{C}^2 \otimes V_2$.*

Proof. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and that $\#\Omega^1 \geq 3$. Let $\mu = \mu_1 + \mu_2$ be the weight from the property (PIV). We consider a weight $\lambda = \lambda_1 + \lambda_2 \in \Omega = \Omega^1 + \Omega^2$ with $\lambda_1 \neq \pm\mu_1$. Then (PIV) implies that $\lambda_2 = \mu_2$ or $\lambda_2 = -\mu_2$, i.e., $\#\Omega^2 \leq 2$. This implies the statement of the lemma. q.e.d.

Lemma 6.9. *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_3$ be a semisimple complex Lie algebra, acting irreducibly on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes V_3$ and satisfying the property* (PII). *Then for any root $\alpha \in \Delta^3$ of $\mathfrak{g}_3$ holds $\#\Omega^3_\alpha \leq 2$.*

Proof. Let $\alpha \in \Delta^3$ and $\mu^1_\alpha + \mu^2_\alpha + \mu^3_\alpha$ be the weight from the property (PII). Then $\Omega_\alpha = \Omega^1 + \Omega^2 + \Omega^3_\alpha \ni -\mu^1_\alpha + \mu^2_\alpha + \lambda$ with $\lambda \in \Omega^3_\alpha$ arbitrary. Again (PII) implies $\lambda = \mu^3_\alpha - \alpha$ or $\lambda = -\mu^3_\alpha$, i.e., $\#\Omega^3_\alpha \leq 2$. q.e.d.

Both lemmata give the following result.

Proposition 6.10. *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_2$ be a semisimple, complex Lie algebra, acting irreducibly on $\mathbb{C}^2 \otimes V_2$ which is supposed to be weak-Berger. Then $\mathfrak{g}_2$ is simple, acts irreducibly and symplectic on V_2 satisfying* (PIII) *and* (PIV), *or $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(4, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$ acting irreducibly on $\mathbb{C}^4 \otimes \mathbb{C}^n$.*

Proof. The proof is obvious by Lemma 6.8 and Lemma 6.9 and the result of Proposition 6.4 keeping in mind that $\mathfrak{g}$ is orthogonal, hence $\mathfrak{g}_2$ is symplectic and $\mathfrak{g}_3$ has to be orthogonal again. q.e.d.

Of course, the representation of $\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ on $\mathbb{C}^4 \otimes \mathbb{C}^n$ is the complexification of a Riemannian holonomy representation of the symmetric space of type BDI.

6.2. Simple Lie algebras satisfying (PIII) and (PIV). In this section we deal with the remaining problem to classify complex, simple irreducibly acting symplectic Lie algebras with the property (PIII) and (PIV).

Proposition 6.11. *Let* $\mathfrak{g} \subset \mathfrak{sp}(V)$ *be simple, irreducibly acting and satisfying* (PIV)*. Then it satisfies* (SII)*.*

Proof. First we note that the fact that the representation is symplectic leaves us with the simple Lie algebras with root systems A_n, B_n, C_n, D_n and E_7. In particular, the Lie algebra of type G_2 is excluded. This implies that for two roots α and β it holds that $\left|\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2}\right| \in \{\pm 1, \pm\frac{1}{2}, 0\}$, a fact which we shall use several times in the following proof.

Let μ be the weight from the property (PIV). We consider two cases.

Case 1: μ is not an extremal weight. In this case there is a root $\alpha \in \Delta$ such that $\mu + \alpha = \Lambda$ is extremal. We show indirectly that (SII) is satisfied with the triple $(\Lambda, -\Lambda, \alpha)$, i.e., we suppose that there is a $\lambda \in \Omega_\alpha \subset \Omega$ such that neither $\lambda = \Lambda - \alpha + \beta$ nor $\lambda = -\Lambda + \beta$ for a $\beta \in \Delta$. $\lambda \in \Omega$ and $\lambda + \alpha \in \Omega$ gives by (PIV) that $\lambda = -\Lambda + \alpha + \beta$ with $\beta \in \Delta$ and $\alpha + \beta \notin \Delta_0$, as well as $\lambda = \Lambda - 2\alpha + \gamma$ with $\gamma \in \Delta$ and $\alpha - \gamma \notin \Delta_0$. By properties of root systems this implies that $\langle\alpha,\beta\rangle \geq 0$ and $\langle\alpha,\gamma\rangle \leq 0$. Furthermore it is

$$2\Lambda = 3\alpha + \beta - \gamma. \tag{63}$$

Now it is $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} = 3 + \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2} - \frac{\langle\gamma,\alpha\rangle}{\|\alpha\|^2} \geq 3$, entailing $\Lambda - 3\alpha \in \Omega$. Since $\Lambda - 3\alpha \neq \Lambda - \alpha + \delta$ for a $\delta \in \Delta_0$ (PIV) gives $\Lambda - 3\alpha \neq -\Lambda + \alpha + \delta$, i.e.,

$$2\Lambda = 4\alpha + \delta, \tag{64}$$

with $\delta \neq -\alpha$. (63) and (64) give

$$0 = \alpha + \delta + \gamma - \beta. \tag{65}$$

Now suppose that $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} = 3$, i.e., $\langle\beta,\alpha\rangle = \langle\gamma,\alpha\rangle = 0$. In this case (64) gives $\frac{2\langle\delta,\alpha\rangle}{\|\delta\|^2} = -2$ and therefore $\frac{2\langle\Lambda,\delta\rangle}{\|\delta\|^2} = -3$. This implies that $\Lambda + 3\delta \in \Omega$, but this is together with (64) is a contradiction to (PIV).

Now suppose that $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} = 4$, i.e., $\frac{\langle\beta,\alpha\rangle - \langle\gamma,\alpha\rangle}{\|\alpha\|^2} = 1$. Then (65) implies $\langle\alpha,\delta\rangle = 0$. $\Lambda - 4\alpha \in \Omega$ implies by (PIV) and $3\alpha \notin \Delta$ that $2\Lambda = 5\alpha + \varepsilon$, i.e., $\alpha - \delta \in \delta$. Since $\langle\alpha,\delta\rangle = 0$ this implies that α and δ are short roots and $\alpha-\delta$ is a long one, i.e., $\frac{\|\delta\|^2}{\|\alpha-\delta\|^2}=\frac{\|\alpha\|^2}{\|\alpha-\delta\|^2}=\frac{1}{2}$. But this gives that $\frac{2\langle\Lambda,\alpha-\delta\rangle}{\|\alpha-\delta\|^2}=\frac{5}{2}$, which is a contradiction.

Finally suppose that $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2}\geq 5$. Hence, $\frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}-\frac{\langle\gamma,\alpha\rangle}{\|\alpha\|^2}\geq 2$. On the other hand, $\Lambda-5\alpha\in\Omega$ and by (PIV) $2\alpha-\delta\in\Delta$. This implies that $\frac{2\langle\alpha,\delta\rangle}{\|\alpha\|^2}\geq 2$. But both inequalities are a contradiction to (65).

Case 2. $\mu:=\Lambda$ is an extremal weight. To proceed analogously as in the first case we fix a root $\alpha\in\Delta$, which is supposed to be long in case of root systems with roots of different length, and we show again indirectly that (SII) is satisfied for the triple $(\Lambda,-\Lambda,\alpha)$. Suppose there is a $\lambda\in\Omega_\alpha\subset\Omega$ such that neither $\lambda=\Lambda-\alpha+\beta$ nor $\lambda=-\Lambda+\beta$ for a $\beta\in\Delta$. $\lambda\in\Omega$ and $\lambda+\alpha\in\Omega$ gives by (PIV) that $\lambda=\Lambda+\beta$ with $\beta\in\Delta$ and $\alpha+\beta\notin\Delta_0$, as well as $\lambda=-\Lambda-\alpha+\gamma$ with $\gamma\in\Delta$ and $\alpha-\gamma\notin\Delta_0$. By properties of root systems this implies that $\langle\alpha,\beta\rangle\geq 0$ and $\langle\alpha,\gamma\rangle\leq 0$. Since α is supposed to be a long root this the same as $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2}\in\{0,\frac{1}{2}\}$ and $\frac{\langle\alpha,\gamma\rangle}{\|\alpha\|^2}\in\{-\frac{1}{2},0\}$. Furthermore it is

$$2\Lambda=-\alpha-\beta+\gamma \tag{66}$$

and hence $\mathbb{Z}\ni\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2}=1-\frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}+\frac{\langle\gamma,\alpha\rangle}{\|\alpha\|^2}=:a\leq -1$. Then, of course $a\in\{-2,-1\}$.

First suppose that $a=-1$. In this case it is $\langle\alpha,\beta\rangle=\langle\alpha,\gamma\rangle=0$. Then, because of $\mathbb{Z}\ni\frac{2\langle\Lambda,\beta\rangle}{\|\beta\|^2}=-1+\frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}$ and $\mathbb{Z}\ni\frac{2\langle\Lambda,\gamma\rangle}{\|\gamma\|^2}=-1+\frac{\langle\beta,\gamma\rangle}{\|\gamma\|^2}$ it must hold that $\frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}$ and $\frac{\langle\beta,\gamma\rangle}{\|\gamma\|^2}$ are integers. But this can only be true if β and γ are both long and short. This is impossible.

Now suppose that $a=-2$, i.e., $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2}=\frac{1}{2}$ and $\frac{\langle\alpha,\gamma\rangle}{\|\alpha\|^2}=-\frac{1}{2}$. Then $\Lambda-2\alpha\in\Omega$, i.e., by (PIV) we get that

$$2\Lambda=-2\alpha+\delta \tag{67}$$

with $\delta\in\Delta_0$ with $\delta\neq\pm\alpha$ because otherwise we would get $a=-1$ or $a=-3$. Now, the existence of a root ε with the property $\|\delta\|\leq\|\varepsilon\|$ would give a contradiction since

$$\mathbb{Z}\ni\frac{2\langle\Lambda,\varepsilon\rangle}{\|\varepsilon\|^2}=-\underbrace{\frac{2\langle\alpha,\varepsilon\rangle}{\|\varepsilon\|^2}}_{\in\mathbb{Z}}+\underbrace{\frac{\langle\delta,\varepsilon\rangle}{\|\varepsilon\|^2}}_{\notin\mathbb{Z}}.$$

This implies that δ is a long root in the root system of type C_n. In C_n the system of long roots equals to $A_1\times\cdots\times A_1$. By this $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2}=\frac{1}{2}$ and $\frac{\langle\alpha,\gamma\rangle}{\|\alpha\|^2}=-\frac{1}{2}$ implies that β and γ are short roots recalling that α was supposed to be a long one. But then by (66) we get

$$\frac{2\langle\Lambda,\beta\rangle}{\|\beta\|^2}=-\frac{\langle\alpha,\beta\rangle}{\|\beta\|^2}+\frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}-1=-\frac{1}{2}\frac{\|\alpha\|^2}{\|\beta\|^2}+\frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}-1=-2+\frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}\notin\mathbb{Z}$$

since β and γ are short. But this is a contradiction. q.e.d.

As a consequence of this proposition we only have to check whether irreducible representations of simple Lie algebras satisfy (SII) — done in Section 5 — and then to add the condition that the representations are symplectic instead of orthogonal. We obtain the following result.

Proposition 6.12. *Let $\mathfrak{g} \subset \mathfrak{sp}(V)$ be a complex, simple, irreducibly and symplectic acting Lie algebra satisfying* (PIII) *and* (PIV) *and different from $\mathfrak{sl}(2,\mathbb{C})$. Then the root system and the highest weight of the representation are one of the following:*

1) *A_5: ω_3, i.e., $\mathfrak{g} = \mathfrak{sl}(6,\mathbb{C})$ acting on $\wedge^3\mathbb{C}^6$.*
2) *C_n: ω_1, i.e., $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$ acting on $\mathbb{C}^{2n}$.*
3) *C_3: ω_3, i.e., $\mathfrak{g} = \mathfrak{sp}(3,\mathbb{C})$ acting on $\mathbb{C}^{14}$.*
4) *D_6: ω_6, i.e., $\mathfrak{g} = \mathfrak{so}(12,\mathbb{C})$ acting on $\mathbb{C}^{32}$ as spin representation.*
5) *E_7: ω_1, i.e., the standard representation of E_7 of dimension* 56.

Proof. (PIV) implies (SII), so we use former results checking whether the Lie algebras satisfying (SII) are symplectic. For this we consider two cases. First we suppose that $0 \in \Omega$. In Proposition 5.7 and Corollary 5.10 of Section 5 it is proved that any such representation which satisfies (SII) and is self-dual is orthogonal. Hence if 0 is a weight, no symplectic representation satisfies (SII).

Now suppose that $0 \notin \Omega$. In the proof of Proposition 5.19 of Section 5 we have shown that the representations of the following Lie algebras with $0 \notin \Omega$ satisfy (SII). Now we check if these are symplectic and in some cases if they satisfy (PIII) and (PIV).

A_n with $n \leq 7$ odd, $\Lambda = \omega_{\frac{n+1}{2}}$. The only representation of these which is symplectic is the one for $n = 5$.

B_n: ω_n for $n \leq 7$ the spin representations, and $\omega_1 + \omega_2$ for $n = 2$. The latter is symplectic and the the former is symplectic for $n = 5, 6$. ($B_2 \simeq C_2$ we shall study in the next point.) Now we show that these remaining representations does not satisfy (PIII) or (PIV). Of course the representation of B_2 with highest weight $\Lambda = \omega_1 + \omega_2 = \frac{3}{2}e_1 + \frac{1}{2}e_2$ cannot satisfy (PIV) because it has 12 weights while B_2 has only 8 roots. The spin representation for $n = 6$ cannot obey (PIV): W.l.o.g we may assume that Λ from (PIV) is the highest weight $\Lambda = \frac{1}{2}(e_1 + \cdots + e_6)$. But then for the weight $\lambda = \frac{1}{2}(e_1+e_2+e_3-e_4-e_5-e_6)$ it holds neither $\Lambda - \lambda \in \Delta_0$ nor $\Lambda + \lambda \in \Delta_0$. The spin representation for $n = 5$ does satisfy (PIV) but not (PIII) since all the weights $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_5)$ with 3 minus signs can not lie on the same affine hyper plane. Hence, none of the symplectic representations satisfying (SII) satisfy (PIII) and (PIV).

C_n with $\Lambda = \omega_1 + \omega_i$ or $\Lambda = \omega_i$. These are symplectic for i even in the first case and for i odd in the second case. Again we have to put the condition (PIV) on both. First we consider the representation with highest weight $\Lambda = \omega_i = e_1 + \cdots + e_i$. Hence, $\Omega = \{\pm e_{k_1} \pm \cdots \pm e_{k_i}\} \cup \{\pm e_{k_1} \pm \cdots \pm e_{k_{i-2}}\} \cup \cdots \cup \{\pm e_k\}$. From this one sees that (PIV) can not be satisfied if $n \geq 5$. With analogous considerations we exclude the case where $\Lambda = \omega_1 + \omega_i$ with i even.

D_n with $\Lambda = \omega_n$ and $n \leq 8$. But these are only symplectic for $n = 6$ and $n = 2$. The latter is excluded since $D_2 = A_1 \times A_1$, a case which is handled in the previous subsection. For E_7 remains only the representation given in the proposition. q.e.d.

Combining the results of this and the previous subsection we get a final corollary, which — together with Corollary 6.5 — proves proposition 6.1 and therefore Theorem 1.1.

Corollary 6.13. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_2$ be a semisimple, complex weak-Berger algebra acting on $\mathbb{C}^2 \otimes V_2$. Then it is the complexification of a Riemannian holonomy representation, in particular the complexification of the holonomy representation of a non-symmetric $Sp(1)\cdot Sp(n)$-manifold or of the following Riemannian symmetric spaces (we list only the compact symmetric space):*

1) *Type EII: $E_6/SU(2) \cdot SU(6)$,*
2) *Type CII: $Sp(n+1)/Sp(1) \cdot Sp(n)$,*
3) *Type FI: $F_4/SU(2) \cdot Sp(3)$,*
4) *Type EVI: $E_7/SU(2) \cdot Spin(12)$,*
5) *Type EIX: $E_8/SU(2) \cdot E_7$,*
6) *Type GI, i.e., $G_2/SU(2) \cdot SU(2)$.*

7. Conclusions

7.1. Conclusions for the holonomy problem. By the previous sections we have proven Theorem 1.1 and the 'only if' direction of Corollary 1.2 from the Introduction. Concerning the four types of indecomposable, non-irreducible Lorentzian holonomy algebras due to Theorem 2.2, Theorem 1.1 gives the following consequence: If $\mathfrak{h}$ is the holonomy algebra of an indecomposable, non-irreducible $(n+2)$-dimensional Lorentzian manifold, then $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$ or $\mathfrak{g} \ltimes \mathbb{R}^n$, where $\mathfrak{g}$ is a Riemannian holonomy algebra, or $\mathfrak{h}$ is of coupled type type 3 or 4, in which case $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{h}$ is a Riemannian holonomy algebra with at least one irreducible factor equal to a Riemannian holonomy algebra with centre, i.e., equal to $\mathfrak{so}(2)$ acting on $\mathbb{R}^2$, $\mathfrak{so}(2) \oplus \mathfrak{so}(n)$ acting on $\mathbb{R}^{2n}$, $\mathfrak{so}(2) \oplus \mathfrak{so}(10)$ acting on $\mathbb{R}^{32}$ as the reellification of the complex spinor module of dimension 16, $\mathfrak{so}(2) \oplus \mathfrak{e}_6$ acting on $\mathbb{R}^{54}$, $\mathfrak{u}(n)$ acting on $\mathbb{R}^{2n}$ or on $\mathbb{R}^{n(n-1)}$.

In order to find out which of these algebras actually can be realised as holonomy algebras of Lorentzian manifolds, first we note that our result enables us to classify indecomposable, non-irreducible Lorentzian Berger algebras: An indecomposable $\mathfrak{h} \subset (\mathbb{R} \oplus \mathfrak{so}(n)) \times \mathbb{R}^n$ is a Berger algebra if and only if $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{h})$ is a Berger algebra. For realising the Berger algebras of *uncoupled type* as holonomy algebras we recall the following construction method, which we proved in [**27**], and in [**29**] in a slightly more general version.

Proposition 7.1. *Let (N, g) be an n-dimensional Riemannian manifold with holonomy algebra $\mathfrak{g}$, θ a closed form on N, and q a function on $N \times \mathbb{R}^2$, the latter sufficiently general. Then*

$$(M = N \times \mathbb{R}^2, h = 2dxdz + q\ dz^2 + \theta\ dz + g)$$

is a Lorentzian manifold with holonomy $(\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$ if q depends on x or $\mathfrak{g} \ltimes \mathbb{R}^n$ if q does not depend on x.

Obviously, this proposition gives the 'if' direction of Corollary 1.2 for the uncoupled types. For indecomposable algebras of the coupled types 3 and 4 we refer to the construction given recently in [**20**], but also to the results in [**8**, Théorème 3.IV.3 and Corollaire 3.IV.3].

7.2. Consequences for parallel spinors. Finally we want to draw the conclusions for the existence of parallel spinor fields on Lorentzian manifolds. The existence of a parallel spinor field on a Lorentzian spin manifold (M, h) implies the existence of a parallel vector field in the following way: To a spinor field φ, one may associate a vector field X_φ, defined by the the equation $h(V_\varphi, U) = \langle U \cdot \varphi, \varphi\rangle$ for any $U \in TM$, where $\langle ., .\rangle$ is the inner product on the spin bundle and $\cdot$ is the Clifford multiplication. Now, the vector field associated to a spinor in this way is light-like or time-like. If the spinor field is parallel, so is the vector field. In the case where it is time-like, the manifold splits by the de-Rham decomposition theorem into a factor $(\mathbb{R}, -dt^2)$ and Riemannian factors which are flat or irreducible with a parallel spinor, i.e., with holonomy $\{1\}$, G_2, $Spin(7)$, $Sp(k)$ or $SU(k)$.

In the case where the parallel vector field is light-like we have a Lorentzian factor which is indecomposable, but with parallel light-like vector field (and parallel spinor) and flat or irreducible Riemannian manifolds with parallel spinors. Hence, in this case one has to know which indecomposable Lorentzian manifolds admit a parallel spinor. The existence of the light-like parallel vector field forces the holonomy of such a manifold with parallel spinor to be contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ i.e., to be of type 2 or 4.

Furthermore, the spin representation of $\mathfrak{so}(n)$-projection $\mathfrak{g} \subset \mathfrak{so}(n)$ must admit a trivial subrepresentation. In fact, the dimension of the space of parallel spinor fields is equal to the dimension of the space of spinors which are annihilated by $\mathfrak{g} \subset \mathfrak{so}(n)$ [**27**]. But if this space is non-trivial a direct calculation shows that the screen holonomy cannot have a center, i.e., no $\mathfrak{so}(2)$-component (see also [**16**]). This excludes the coupled type 4 and yields Corollary 1.3 and consequently Theorem 1.4 of the Introduction.

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