DIFFEOMORPHISM OF SIMPLY CONNECTED ALGEBRAIC SURFACES

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This article is dedicated to the memory of Boris Moishezon

Abstract

In this paper we show that even in the case of simply connected minimal algebraic surfaces of general type, deformation and differentiable equivalence do not coincide.

Exhibiting several simple families of surfaces which are not deformation equivalent, and proving their diffeomorphism, we give a counterexample to a weaker form of the speculation DEF = DIFF of R. Friedman and J. Morgan, i.e., in the case where (by M. Freedman's theorem) the topological type is completely determined by the numerical invariants of the surface.

We hope that the methods of proof may turn out to be quite useful to show diffeomorphism and indeed symplectic equivalence for many important classes of algebraic surfaces and symplectic 4-manifolds.

1. Introduction

One of the basic problems in the theory of algebraic surfaces is to understand the moduli spaces of surfaces of general type, in particular their connected components, which parametrize deformation equivalence classes of minimal surfaces of general type.

If two compact complex manifolds X, X' are deformation equivalent, there exists a diffeomorphism between them carrying the canonical class ($c_1(K_X) \in H^2(X, \mathbb{Z})$) of X to the one of X'.

Freedman's classification of topological simply connected 4-manifolds ([**Free**]) and the above consideration allowed to easily exhibit moduli spaces, for a fixed oriented topological type, having several connected

components ([**Cat3**]). The Def = Diff problem asks whether (orient-edly) diffeomorphic algebraic surfaces are equivalent by deformation. A negative answer was conjectured by the first author (see Problem part of the volume [**Katata**]). Later on, a positive answer was conjectured in 1987 by Friedman and Morgan in [**F-M1**], in view of the exciting new developments of gauge theory and 4-manifolds theory.

In fact, Donaldson's theory made clear that diffeomorphism and home-omorphism differ drastically for algebraic surfaces ([**Don1**], [**Don2**], [**Don3**], [**Don4**]) and the successes of gauge theory led Friedman and Morgan to "speculate" that the diffeomorphism type of algebraic sur-faces should determine the deformation type (Def = Diff).

Friedman and Morgan realized later (see the exercise left to the reader on page 208 of [**F-M3**]) that there are elliptic surfaces S with $\pi_1(S)$ infi-nite such that S is not deformation equivalent to the complex conjugate surface $\bar{S}$; moreover, by their results, for surfaces of special type the only possible exceptions to Def = Diff are produced in this way.

The first counterexamples to this 'speculation' for surfaces of general type were given by Marco Manetti ([**Man4**]), whose examples however are based on a somewhat complicated construction of Abelian coverings of rational surfaces.

Moreover, Manetti's surfaces are, like the examples of ([**Cat4**]) and the ones by Kharlamov-Kulikov ([**K-K**]), not simply connected.

For the latter ([**Cat4**] and [**K-K**]) examples, one would take pairs of complex conjugate surfaces, whence the condition of being orientedly diffeomorphic was tautologically fulfilled. However, the diffeomorphism would send the canonical class to its opposite.

These observations led the referee of the paper ([**Cat4**]) to ask about a weakening of the Friedman and Morgan conjecture; namely, to ask about the case of simply connected minimal surfaces (of course then of general type), and to require a diffeomorphism preserving the canonical class.

Indeed, by the results of Seiberg Witten theory, any diffeomorphism carries $c_1(K_X)$ either to $c_1(K_{X'})$ or to $-c_1(K_{X'})$ (cf. [**Witten**] or [**Mor**]), whence the second requirement is not difficult to fulfill once one can produce $h \geq 3$ such surfaces $S_1, \ldots S_h$ which are diffeomorphic but pairwise not deformation equivalent.

Our main result is that for each positive integer h, we can find h such surfaces $S_1, \ldots S_h$ which are simply connected, diffeomorphic to each other, but pairwise not deformation equivalent.

This follows directly from the two main results of the paper

Theorem 2.5. *Let S, S' be simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types $((2a, 2b), (2c, 2b))$, and $((2a + 2k, 2b), (2c - 2k, 2b))$, and assume*

(I) *a, b, c, k are strictly positive even integers with $a, b, c - k \geq 4$,*

(II) $a \geq 2c + 1$,
(III) $b \geq c + 2$ and either
   (IV1) $b \geq 2a + 2k - 1$ or
   (IV2) $a \geq b + 2$.

Then S and S' are not deformation equivalent.

Theorem 3.1. Let S, S' be simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types $((2a, 2b), (2c, 2b))$, and $((2a + 2, 2b), (2c - 2, 2b))$, and assume that $a, b, c - 1$ are integers with $a, b, c - 1 \geq 2$.
Then S and S' are diffeomorphic.

It suffices in fact to apply Theorem 3.1 k-times under the assumptions of Theorem 2.5, and then to consider the $h := (k/2 + 1)$- families of surfaces S_i, for $0 \leq i \leq h - 1 = k/2$, obtained by taking S_i as a simple bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ of type $((2a + 2i, 2b), (2c - 2i, 2b))$.

Apart from the numerology, these examples are surprisingly simple, obtained by taking two square roots of two polynomials $f(x, y)$, $g(x, y)$ in 2 variables, and are a special case of the original examples considered in [**Cat1**].

The fact that these examples provide distinct deformation classes appeared already in the preprint [**Cat6**] (part of which is subsumed in the present paper) and is a consequence of a series of results of the first author and of M.Manetti.

In the first section (Section 2) we also try to motivate the choice of our examples, as surfaces which are pairwise homeomorphic by a homeomorphism carrying the canonical class to the canonical class, but which are not deformation equivalent.

The bulk and the novel part of the paper is, however, dedicated to proving the diffeomorphism of the pair of surfaces S and S' considered in 3.1.

The guiding idea here is that the surfaces in question have a holomorphic map to $\mathbb{P}^1_{\mathbb{C}}$ provided by the first coordinate x, and a small perturbation of this map in the symplectic category realizes them as symplectic Lefschetz fibrations (cf. [**Don7**], [**G-S**]).

Actually, the construction easily shows that they are fibre sums of the same pair of symplectic Lefschetz fibrations over the disk, but under different glueing diffeomorphisms.

Ultimately, the proof involves comparing two different factorizations of the identity in the mapping class group, and, via a very useful lemma of [**Aur02**], one is reduced to show that

($**$) the class of the "glueing difference" diffeomorphism ψ, in the Mapping Class Group of the fibre Riemann surface, is a product of Dehn twists occurring in the monodromy factorization.

Our results allow to show diffeomorphism of huge classes of families of surfaces of general type, but for the sake of brevity and clarity we stick to our simple (a, b, c) -examples.

We postpone for the moment the investigation whether our surfaces are indeed symplectomorphic for the canonical symplectic structure which is uniquely associated to a surface of general type ([**Cat6**]). Symplectomorphism would follow if one could prove a similar statement to $(**)$ but "downstairs", i.e., in a certain coloured subgroup of the braid group.

This is much more complicated and we decided not to further postpone the publication of the present results.

Here is the organization of the paper.

In Section 2 we describe our (a,b,c) -surfaces, recall the basic properties of bidouble covers (Galois $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-covers), and explain in some more detail the proof of Theorem 2.5, which is an improvement of Theorem 2.6 of [**Cat6**] (there, only case (IV2) with $a \geq b + 2$ was contemplated).

In Section 3 we begin to prepare the proof of Theorem 3.1, namely, we show how our surfaces S, S' are obtained as fibre sums of two Lefschetz fibrations over the disk, according to different glueing maps. And we observe that their diffeomorphism is guaranteed (Cor. 3.8) once we show $(**)$.

In Section 4 we show how, via the concept of symmetric mapping class groups, we can reduce the determination of the Dehn twists appearing in the monodromy factorization to results already existing in the literature (even if these are standard for experts, we sketch however here the argument of proof, and we give an alternative self contained direct proof in the appendix).

In Section 5, after recalling the concepts of chain of curves on a Riemann surface and of the associated Coxeter homeomorphism, we finally show $(**)$, exhibiting the class of ψ as a product of six such Coxeter homeomorphisms, and hence we prove the diffeomorphism Theorem 3.1.

The proof, based on the useful Cor. 5.1, generalizing results of Epstein, requires next a long but easy verification about the isotopy of certain explicitly given curves. For this purpose we find it useful to provide the reader with some pictures.

In Section 6 we explain why one could conjecture that our surfaces are symplectomorphic for the canonical symplectic structure, and we comment on the difficulties one encounters in trying to prove this assertion.

Both authors are especially thankful to Denis Auroux for hinting at his lemma (Lemma 2 of [**Aur02**]), and to a referee for several useful remarks.

2. The (a,b,c) surfaces

This section describes certain series of families of surfaces, depending on 3 integer parameters (a,b,c).

Once we fix b and the sum $(a+c)$, it will be immediately clear to the experts that the surfaces in the respective families are homeomorphic by a homeomorphism carrying the canonical class to the canonical class. This fact is a consequence of the following rather well known proposition (which was the guiding principle for the construction):

Proposition 2.1. *Let S, S' be simply connected minimal surfaces of general type such that $p_g(S) = p_g(S') \geq 1$, $K_S^2 = K_{S'}^2$, and moreover such that the divisibility indices of K_S and $K_{S'}$ are the same.*

Then there exists a homeomorphism F between S and S', unique up to isotopy, carrying $K_{S'}$ to K_S.

Proof. By Freedman's theorem ([**Free**], cf. especially [**F-Q**], page 162) for each isometry $h : H_2(S, \mathbb{Z}) \to H_2(S', \mathbb{Z})$ there exists a homeomorphism F between S and S', unique up to isotopy, such that $F_* = h$. In fact, S and S' are smooth 4-manifolds, whence the Kirby-Siebenmann invariant vanishes.

Our hypotheses that $p_g(S) = p_g(S')$, $K_S^2 = K_{S'}^2$ and that $K_S, K_{S'}$ have the same divisibility imply that the two lattices $H_2(S, \mathbb{Z})$, $H_2(S', \mathbb{Z})$ have the same rank, signature, and parity, whence they are isometric (since S, S' are algebraic surfaces, cf. e.g., [**Cat1**]). Finally, by Wall's theorem ([**Wall**]) (cf. also [**Man2**], p. 93) such an isometry h exists since the vectors corresponding to the respective canonical classes have the same divisibility and by Wu's theorem they are characteristic: in fact, Wall's condition $b_2 - |\sigma| \geq 4$ (σ being the signature of the intersection form) is equivalent to $p_g \geq 1$. q.e.d.

As in [**Cat1**, Sections 2,3,4] we consider smooth bidouble covers S of $\mathbb{P}^1 \times \mathbb{P}^1$: these are smooth finite Galois covers of $\mathbb{P}^1 \times \mathbb{P}^1$ having Galois group $(\mathbb{Z}/2)^2$. Bidouble covers are divided into those of simple type, and those not of simple type.

Those of *simple type* (and type $(2a, 2b), (2c, 2d)$) are defined by 2 equations

$$
\begin{aligned}
(1) \qquad z^2 &= f(x,y) \\
w^2 &= g(x,y),
\end{aligned}
$$

where f and g are bihomogeneous polynomials, belonging to respective vector spaces of sections of line bundles: $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2a, 2b))$ and

$g \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2c, 2d))$.

Bidouble covers of simple type (cf.[**Cat1**]) are embedded in the total space of the direct sum of 2 line bundles L_i: in the above case L_1, L_2 are just $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c, d)$.

Notice moreover that the smoothness of S is ensured by the condition that the 2 branch divisors be smooth and intersect transversally.

We recall from [**Cat1**, Sections 2, 3, 4] that our surface S has the following invariants:

(2) $\qquad\qquad\qquad$ setting $\; n = 2a + 2c, m = 2b + 2d$,

(3) $\qquad\qquad \chi(\mathcal{O}_S) = \frac{1}{4}((n-4)(m-4) + 4(ab+cd))$

(4) $\qquad\qquad\qquad\qquad K_S^2 = 2(n-4)(m-4)$.

Moreover, (cf. [**Cat1**, Proposition 2.7]) our surface S is simply connected.

Definition 2.2. Example (a, b, c) consists of two simple bidouble covers S, S' of respective types $((2a, 2b), (2c, 2b))$, and $((2a+2, 2b), (2c-2, 2b))$. We shall moreover assume, for technical reasons, that $a \geq 2c+1$, $b \geq c+2$, and that a, b, c are even and ≥ 3.

By the previous formulae, these two surfaces have the same invariants $\chi(S) = 2(a+c-2)(b-1) + b(a+c), K_S^2 = 16(a+c-2)(b-1)$.

Remark 2.3. The divisibility index of the canonical divisor K for the above family of surfaces is easily calculated by Lemma 4 of [**Cat3**], asserting that the pull back of $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ is primitively embedded in $H^2(S, \mathbb{Z})$. Now, K_S is the pull back of a divisor of bidegree $(a+c-2, 2b-2)$ whence its divisibility equals simply $G.C.D.$ $(a+c-2, 2b-2)$. Therefore the divisibility index is the same for the several families (vary the integer k) of simple covers of types $((2a+2k, 2b), (2c-2k, 2b))$.

Crucial to the study of small deformations of our bidouble covers is the concept of *natural deformations of bidouble covers* (ibidem, Def. 2.8, p. 494) which will be also used in the forthcoming Theorem 2.6.

Natural deformations are parametrized by a 4-tuple of bihomogeneous polynomials f, g, ϕ, ψ which yield equations

(5) $\qquad\qquad\qquad \begin{aligned} z^2 &= f(x, y) + w\phi(x, y) \\ w^2 &= g(x, y) + z\psi(x, y), \end{aligned}$

(f, g are as above whereas ϕ has bidegree (2a-c,2b-d), respectively ψ has bidegree (2c-a,2d-b)).

Remark 2.4.

A) Theorem 3.8, ibidem, says in particular that the natural deformations yield all the local deformations, and give an irreducible

component of the moduli space. This component uniquely determines the numbers a, b, c, d up to the obvious permutations corresponding to the possibilities of exchanging f with g and x with y.

B) In our (a, b, c) case, to see that the irreducible component determines the numbers a, b, c, one can also notice that the dimension of this component equals $M := (b+1)(4a+c+3)+2b(a+c+1)-8$, while $K^2/16 = (a + c - 2)(b - 1)$, and $(8\chi - K^2)/8 = b(a + c)$. Set $\alpha = a+c, \beta = 2b$: α, β are then the roots of a quadratic equation, so they are determined up to exchange, and uniquely if we would restrict our numbers either to the inequality $a \geq 2b$ or to the inequality $b \geq a$.

Finally $M = (\frac{\beta}{2} + 1)(\alpha + 3) + \beta(\alpha + 1) - 8 + 3a(\frac{\beta}{2} + 1)$ would then determine a, whence the ordered triple (a, b, c).

In the case of example (a, b, c) the natural deformations of S do not preserve the action of the Galois group $(\mathbb{Z}/2)^2$ (this would be the case for a cover of type $((2a, 2b), (2c, 2d))$ with $a \geq 2c + 1$, $d \geq 2b + 1$, cf. [**Cat1**, **Cat2**]).

But, since $a \geq 2c + 1$, it follows that ψ must be identically zero and the natural deformations yield equations of type

$$(6) \qquad \begin{aligned} z^2 &= f(x, y) + w\phi(x, y) \\ w^2 &= g(x, y), \end{aligned}$$

whence there is preserved the $(\mathbb{Z}/2)$ action sending

$$(z, w) \to (-z, w)$$

and also the action of $(\mathbb{Z}/2)$ on the quotient of S by the above involution (sending $w \to -w$).

That is, every small deformation preserves the structure of *iterated double cover* ([**Man3**, **Man2**]).

We prove now the main result of this section, which is an improvement of Theorem 2.6 of [**Cat6**] (beyond correcting a misprint in the statement, namely replacing the assumption $c \geq b + 2$ by $b \geq c + 2$, we give an alternative to the further assumption $a \geq b + 2$):

Theorem 2.5. *Let S, S' be simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types $((2a, 2b), (2c, 2b))$, and $((2a + 2k, 2b), (2c$ -2k, 2b))$, and assume*

(I) *a, b, c, k are strictly positive even integers with $a, b, c - k \geq 4$,*

(II) *$a \geq 2c + 1$,*

(III) *$b \geq c + 2$ and either*

(IV1) *$b \geq 2a + 2k - 1$ or*

(IV2) *$a \geq b + 2$.*

Then S and S' are not deformation equivalent.

Proof. It suffices to construct a family $\mathcal{N}'_{a,b,c}$, containing all simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types $((2a, 2b), (2c, 2b)$, and to show that it yields a connected component of the moduli space under the conditions

(I′) a, b, c are strictly positive even integers ≥ 4,
(II′) $a \geq 2c + 1$,
(III′) $b \geq c + 2$ and either
(IV′1) $b \geq 2a - 1$ or
(IV′2) $a \geq b + 2$.

The family $(\mathcal{N}'_{a,b,c})$ consists of all the natural deformations of simple bidouble covers of the Segre-Hirzebruch surfaces $\mathbb{F}_{2h}$ which have only Rational Double Points as singularities and are of type $((2a, 2b), (2c, 2b))$.

In order to explain what this means, let us recall, as in [**Cat0**] pp. 105–111, that a basis of the Picard group of $\mathbb{F}_{2h}$ is provided, for $h \geq 1$, by the fibre F of the projection to $\mathbf{P}^1$, and by $F' := \sigma_\infty + hF$, where σ_∞ is the unique section with negative self-intersection $= -2h$. Observe that $F^2 = F'^2 = 0, FF' = 1$, and that F is nef, while $F' \cdot \sigma_\infty = -h$.

We set $\sigma_0 := \sigma_\infty + 2hF$, so that $\sigma_\infty \sigma_0 = 0$, and we observe (cf. Lemma 2.7 of [**Cat0**]) that $|m\sigma_0 + nF|$ has no base point if and only if $m, n \geq 0$. Moreover, $|m\sigma_0 + nF|$ contains σ_∞ with multiplicity ≥ 2 if $n < -2h$.

Then we say here (the present notation differs from the one of [**Cat0**]) that two divisors D, D' are of type $((2a, 2b), (2c, 2b))$ if either $D \equiv 2aF + 2bF'$, $D' \equiv 2cF + 2bF'$, or the roles of F, F' are reversed.

Assume that we have a natural deformation of such a simple bidouble cover: then, by (II′), there is no effective divisor in $|(2c - a)F' + bF|$, as we see by intersecting with the nef divisor F; also, any effective divisor $\operatorname{div}(\psi)$ in $|(2c-a)F+bF'| = |b\sigma_0+(2c-a-bh)F| = |b\sigma_\infty+(2c-a+bh)F|$ exists only for $h \geq 1$, since $h \geq \frac{a-2c}{b}$; in any case it contains σ_∞ with multiplicity at least 2 by our previous remark.

Since, however, the divisor $\operatorname{div}(z\psi + g)$ must be reduced, it is not possible that $\operatorname{div}(g)$ also contains σ_∞ with multiplicity at least 2. But this is precisely the case, since $|2cF + 2bF'| = |2b\sigma_0 + (2c - 2bh)F|$ and again by condition (III′) $2c - 2bh \leq -4h$.

We have thus shown two things:
1) that all such natural deformations are iterated double covers,
2) that $\operatorname{div}(g)$ is in the linear system $|2cF' + 2bF|$.

Therefore, under our assumptions, we are considering only surfaces X defined by equations of type

(7)
$$z^2 = f(x, y) + w\phi(x, y)$$
$$w^2 = g(x, y),$$

where $\mathrm{div}\,(f) \in |2aF' + 2bF|$, $\mathrm{div}\,(g) \in |2cF' + 2bF|$, $\mathrm{div}\,(\phi) \in |(2a - c)F' + bF|$.

In this case we see also that, since the divisor of g cannot contain σ_∞ with multiplicity at least 2, and neither can $\mathrm{div}\,(f)$ and $\mathrm{div}\,(\phi)$ simultaneously, we obtain the inequalities $h \leq \frac{b}{c-1}$, $h \leq \frac{b}{a-1}$ (the latter inequality is in fact $\iff h \leq \mathrm{Max}\,(\frac{b}{a-1}, \frac{b}{2a-c-2}))$, i.e., the single inequality $h \leq \frac{b}{a-1}$.

STEP I.

Let us prove that the family of canonical models $(\mathcal{N}'_{a,b,c})$ yields an open set in the moduli space: to this purpose it suffices to show that, for each such surface X, the Kodaira Spencer map is surjective, and by Theorem 1.8 of [**Cat1**] we might also assume $h \geq 1$.

In fact, exactly as in [**Cat0**], we see that the family is parametrized by a smooth variety.

Observe that the tangent space to the Deformations of X is provided by $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.

Denoting by $\pi : X \to \mathbb{F} := \mathbb{F}_{2h}$ the projection map and differentiating equations (7) we get an exact sequence for Ω^1_X

$$o \to \pi^*(\Omega^1_{\mathbb{F}}) \to \Omega^1_X \to \mathcal{O}_{R_z}(-R_z) \oplus \mathcal{O}_{R_w}(-R_w) \to 0$$

as in (1.7) of [**Man1**], where $R_z = \mathrm{div}\,(z), R_w = \mathrm{div}\,(w)$.

Applying the derived exact sequence for $\mathrm{Hom}_{\mathcal{O}_X}(\dots, \mathcal{O}_X)$ we obtain the same exact sequence as Theorem (2.7) of [**Cat0**], and (1.9) of [**Man1**], namely:

$(**)$
$$0 \to H^0(\Theta_X) \to H^0(\pi^*\Theta_{\mathbb{F}}) \to H^0(\mathcal{O}_{R_z}(2R_z)) \oplus H^0(\mathcal{O}_{R_w}(2R_w)) \to$$
$$\to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^1(\pi^*\Theta_{\mathbb{F}}).$$

The argument is now completely identical to the one given in [**Cat0**].

First, we claim that $H^1(\pi^*\Theta_{\mathbb{F}}) = H^1(\Theta_{\mathbb{F}} \otimes \pi_*(\mathcal{O}_X))$ equals $H^1(\Theta_{\mathbb{F}})$.

To conclude this we need a minor improvement to loc. cit. (2.12), namely concerning the vanishing $H^1(\Theta_{\mathbb{F}}(-d_1\sigma_0 - d_2F)$ on the Segre Hirzebruch surface $\mathbb{F}_{2h}$.

Lemma 2.6. *Let $\mathbb{F}$ be the Segre Hirzebruch surface $\mathbb{F}_{2h}$: then $H^1(\Theta_{\mathbb{F}}(-d_1\sigma_0 - d_2F) = 0$ as soon as $d_1 \geq 3, d_2 \geq -2h+3$.*
In other words, $H^1(\Theta_{\mathbb{F}}(-d'F'-dF) = 0$ if $d' \geq 3$ and $d \geq (d'-2)h+3$.

Proof of the lemma. In fact, by the relative tangent bundle exact sequence, $H^1(\Theta_{\mathbb{F}}(-d_1\sigma_0-d_2F))$ stays between $H^1(\mathcal{O}_{\mathbb{F}}(-d_1\sigma_0-(d_2-2)F))$ and $H^1(\mathcal{O}_{\mathbb{F}}(-(d_1 - 2)\sigma_0 - (d_2 + 2h)F))$. To get the vanishing of these, we notice as in loc. cit. that $H^1(\mathcal{O}_{\mathbb{F}}(-\Delta)) = 0$ if the divisor Δ is reduced and connected. This is the case for $\Delta \equiv e\sigma_0 + rF$ if $r \geq 0, e \geq 1$, or $r \geq -2h + 1, e \geq 1$, since in the last case we write $\Delta \equiv (e - 1)\sigma_0 + \sigma_\infty + (2h + r)F$. q.e.d.

We have now to verify that $H^1(\Theta_{\mathbb{F}} \otimes \pi_*(\mathcal{O}_X)) = H^1(\Theta_{\mathbb{F}})$, i.e., by virtue of the standard formula for $\pi_*(\mathcal{O}_X)$, the vanishing of $H^1(\Theta_{\mathbb{F}}(-d'F' - dF)) = 0$ for the three respective cases:

- $d' = a, d = b$: here we use $b \geq (a-1)h$, whence $b \geq (a-2)h + h \geq (a-2)h + 3$ for $h \geq 3$. For $h = 2$ we use assumption (IV'1) $b \geq 2a - 1$, which also implies $b \geq a + 1$, i.e., the case $h = 1$.
- $d' = c, d = b$: here we use the previous inequality plus $a \geq 2c + 1$.
- $d' = a + c, d = 2b$: just take the sum of the two previous inequalities.

Observe that under assumption (IV'2), $a \geq b + 2$, for $h \geq 1$ $div(f)$ and $div(\phi)$ contain σ_∞ with multiplicity at least 2. Whence, the case $h \geq 1$ does not occur.

To finish the proof of Step I we argue exactly as in [**Cat0**], observing that the smooth parameter space of our family surjects onto $H^1(\Theta_{\mathbb{F}})$, and its kernel, provided by the natural deformations, surjects onto $H^0(\mathcal{O}_{R_z}(2R_z)) \oplus H^0(\mathcal{O}_{R_w}(2R_w))$. Thus the Kodaira Spencer is onto and we get an open set in the moduli space.

STEP II.

We want now to show that our family yields a closed set in the moduli space.

It is clear at this moment that we obtained an irreducible component of the moduli space. Let us consider the surface over the generic point of the base space of our family: then it has $\mathbb{Z}/2$ in the automorphism group (sending $z \to -z$, as already mentioned).

As shown in [**Cat0**], this automorphism then acts biregularly on the canonical model X_0 of each surface corresponding to a point in the closure of our open set.

We use now the methods of [**Cat2**] and [**Man3**], and more specifically Theorem 4.1 of [**Man3**] to conclude that if X_0 is a canonical surface which is a limit of surfaces in our family, then the quotient of X_0 by the subgroup $\mathbb{Z}/2 \subset \mathrm{Aut}\,(X_0)$ mentioned above is a surface with Rational Double Points.

Again, the family of such quotients has a $\mathbb{Z}/2$-action over the generic point, and dividing by it we get (cf. [**Man3**, Theorem 4.10]) a Hirzebruch surface, and our surface X_0 is also an iterated double cover of some $\mathbb{F}_{2h}$; thus it belongs to the family we constructed.　　q.e.d.

Corollary 2.7. *If, as in* [**Cat6**], *we assume*
(IV2) $a \geq b + 2$,
the connected component of the moduli space contains only iterated double covers of $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. We already observed that under this assumption, for $h \geq 1$ $div(f)$ and $\mathrm{div}\,(\phi)$ contain σ_∞ with multiplicity at least 2. Hence, the case $h \geq 1$ does not occur.　　q.e.d.

Remark 2.8. We observe moreover that in Section 5 of [**Man3**] it is proven that the general surface in the family has $\mathbb{Z}/2$ as automorphism group: this follows also from the Noether-Lefschetz property that, for the general surface S, $\mathrm{Pic}\,(S)$ is the pull back of $\mathrm{Pic}\,(\mathbf{P}^1 \times \mathbf{P}^1)$.

3. (a, b, c)-surfaces as fibre sums

In the next three sections we shall establish the following main result:

Theorem 3.1. *Let S, S' be simple bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$ of respective types $((2a, 2b), (2c, 2b))$, and $((2a + 2, 2b), (2c$ - $2, 2b))$, and assume that $a, b, c - 1$ are integers with $a, b, c - 1 \geq 2$.*
Then S and S' are diffeomorphic.

Remark 3.2. Consider the two families of surfaces of example (a, b, c) (Def. 2.2), and choose respective surfaces S, S' each in one of the two respective families.

Then there exist two 4-manifolds with boundary M_1, M_2 such that S and S' are obtained (as oriented differentiable manifolds) by glueing M_1 and M_2 through two respective glueing maps $\Phi, \Phi' \in Diff(\partial M_1, \partial M_2)$.

Proof. In order to describe the manifolds M_1 and M_2 we make an explicit choice of branch curves.

We cut $\mathbb{P}^1$ into two disks

$\Delta_0 : \{|x| \leq 1\}, \Delta_\infty : \{|x| \geq 1\} \cup \{\infty\}$ and write
$\mathbb{P}^1 \times \mathbb{P}^1$ as $(\Delta_0 \times \mathbb{P}^1) \cup (\Delta_\infty \times \mathbb{P}^1)$.

We let

(3.2.1) $D_1' : (x - \zeta_1)(x - \zeta_6)A(x)F(y) = 0,$
(3.2.2) $D_2' : (x - \zeta_2)(x - \zeta_3)(x - \zeta_4)(x - \zeta_5)C(x)F(-y) = 0$

where

- $0 < \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 < \zeta_5 < \zeta_6 << 1$ are real numbers,
- $F(y) = \prod_{j=1}^{2b}(y - B_j)$, where the B_i's are real with $0 < B_1 < B_2 < \cdots < B_{2b}$ and
- $A(x)$ and $C(x)$ are polynomials of degree $2a-2$, respectively $2c-4$, with all roots real, pairwise distinct, and much bigger than 1.

Let D_1 and D_2 be respective nearby smoothings of D_1' and D_2' which will be made explicit later.

We then let M_1 be the simple $(\mathbb{Z}/2)^2$ cover of $(\Delta_0 \times \mathbb{P}^1)$ with branch curves

$D_1 \cap (\Delta_0 \times \mathbb{P}^1)$ and $D_2 \cap (\Delta_0 \times \mathbb{P}^1)$.

Instead, we let M_2 be the simple $(\mathbb{Z}/2)^2$ cover of $(\Delta_\infty \times \mathbb{P}^1)$ with branch curves

$D_1 \cap (\Delta_\infty \times \mathbb{P}^1)$ and $D_2 \cap (\Delta_\infty \times \mathbb{P}^1)$.

We let S be the simple $(\mathbb{Z}/2)^2$ cover of $(\mathbb{P}^1 \times \mathbb{P}^1)$ with branch curves D_1 and D_2.

In other words, if we denote by f the composition of the Galois cover with the first projection onto $\mathbb{P}^1$, we have $M_1 = S \cap f^{-1}(\Delta_0), M_2 = S \cap f^{-1}(\Delta_\infty)$.

To explain the definition of S', observe that the symmetry of the second $\mathbb{P}^1$ given by $y \to -y$ allows to interchange the roles of D_1' and D_2'.

We choose now the branch curves for the surface S' setting
$E_1' : (x - \zeta_2)(x - \zeta_3)(x - \zeta_4)(x - \zeta_5)A(x)F(y) = 0$,
$E_2' : (x - \zeta_1)(x - \zeta_6)C(x)F(-y) = 0$
and letting E_1 and E_2 be respective nearby smoothings of E_1' and E_2'.

We let then S' be the simple $(\mathbb{Z}/2)^2$ cover of $(\mathbb{P}^1 \times \mathbb{P}^1))$ with branch curves E_1 and E_2. The restrictions of S' to Δ_0 and to Δ_∞ are very similar to M_1 and M_2. We shall describe respective diffeomorphisms between them.

Since $|\zeta_i| \ll 1$ there is a well defined branch
$h(x) = (((x - \zeta_2)(x - \zeta_3)(x - \zeta_4)(x - \zeta_5))/((x - \zeta_1)(x - \zeta_6)))^{1/2} = x(1 + \text{small})^{1/2}$ for $|x| \geq 1$.

Before the smoothing of the branch curves the coverings are defined by
$z^2 = (x - \zeta_1)(x - \zeta_6)A(x)F(y)$,
$w^2 = (x - \zeta_2)(x - \zeta_3)(x - \zeta_4)(x - \zeta_5)C(x)F(-y)$
and
$z'^2 = (x - \zeta_2)(x - \zeta_3)(x - \zeta_4)(x - \zeta_5)A(x)F(y)$,
$w'^2 = (x - \zeta_1)(x - \zeta_6)C(x)F(-y)$.

Thus, the transformation $z' = h(x)z$, $w' = h(x)^{-1}w$ defines a diffeomorphism ϕ between the coverings over Δ_∞ and then, once the equations of the branch loci are slightly perturbed, we choose diffeomorphisms provided by Tjurina's Theorem on simultaneous resolution ([**Tju**]).

For $|x| < 1$, we set
$\psi(x, y, z, w) = (x, -y, w \ (A(x)/C(x))^{1/2}, z \ (C(x)/A(x))^{1/2})$
and perturb as above the diffeomorphisms of the minimal resolutions. In fact, since the roots of $A(x)$ and $C(x)$ are real, positive and very big, a branch of $(A(x)/C(x))^{1/2}$ is well defined over Δ_0.   q.e.d.

The importance of the previous decomposition is its compatibility with the map f given by the composition of the Galois covering map with the first projection of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

This is almost a Lefschetz fibration, if the branch curves D_1 and D_2 are general. I.e., the singularities of the fibres are only nodes, but, because of the Galois structure, the fibres which are the inverse image of a vertical line simply tangent to one of the two branch curves D_i at one point possess two nodal singularities instead of only one. We can easily find (cf. [**G-S**], page 287 for a more general result) a differentiable

perturbation on the respective pieces M_1, M_2 of the given projections, leaving them pointwise fixed on the boundary, so that we obtain two differentiable Lefschetz fibrations over a disk (Δ_∞, resp. Δ_0).

We shall then use a very useful criterion stated in [**Kas**] as folklore, and explicitly proven by Auroux ([**Aur02**]) for the equivalence of the fibre sum of two Lefschetz pencils.

To this purpose, let us first recall the standard

Definition 3.3. Let (M, ω) be a differentiable (resp.: symplectic) compact 4-manifold.

Then a Lefschetz fibration is a differentiable map $f : M \to \mathbb{P}^1_{\mathbb{C}}$ which is a submersion except for a finite set of critical values $b_1, \ldots b_m \in \mathbb{P}^1_{\mathbb{C}}$, where $f^{-1}(b_i)$ is smooth except at a single point p_i where $f = x^2 + y^2$ for local compatible complex coordinates (x, y) (i.e., in the symplectic case, coordinates for which the form ω corresponds to the standard symplectic structure on $\mathbb{C}^2$).

Remark 3.4.

i) A similar definition can be given for the case where M has boundary and we replace $\mathbb{P}^1_{\mathbb{C}}$ by a disk $D \subset \mathbb{C}$.

ii) By a theorem of Donaldson ([**Don6**]) every compact symplectic 4-manifold M' admits a Lefschetz pencil, i.e., a symplectic blow up (cf. [**MD-S**]) $\pi : M \to M'$ and a Lefschetz fibration $f : M \to \mathbb{P}^1_{\mathbb{C}}$.

Recall that a genus g Lefschetz fibration (i.e., the fibres have genus g) with critical values $b_1, \ldots b_m \in \mathbb{P}^1_{\mathbb{C}}$ determines a Hurwitz equivalence class of a factorization of the identity

$$\tau_1 \circ \tau_2 \circ \ldots \tau_m = Id$$

as a product of Dehn twists in the mapping class group

$$\mathcal{M}ap_g := \pi_0(\mathcal{D}iff^+(C_g)).$$

If the Lefschetz fibration is over a disk $D \subset \mathbb{C}$, then we get instead a factorization

$$\tau_1 \circ \tau_2 \circ \ldots \tau_m = \phi$$

of the monodromy of the fibration restricted on the boundary of D.

The above follows, as well known, by considering a geometric quasi-basis $\gamma_1, \gamma_2, \ldots \gamma_m$ of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{b_1, \ldots b_m\})$, which means as usual that there is a disk $D \subset \mathbb{P}^1_{\mathbb{C}}$ containing $\{b_1, \ldots b_m\}$, and that the γ_i's are of the form $L_i \beta_i L_i^{-1}$ where the L_i's are non intersecting segments stemming from a base point p_0 on the boundary of D, and following each other counterclockwise, and β_i is a counterclockwise oriented circle with centre b_i and containing the end point of L_i.

The monodromy of the restriction of M to $\mathbb{P}^1_{\mathbb{C}} - \{b_1, \ldots b_m\}$ is determined, once we fix a geometric quasi-basis $\gamma_1, \gamma_2, \ldots \gamma_m$ of $\pi_1(\mathbb{P}^1_{\mathbb{C}} -$

$\{b_1, \ldots b_m\}$), by the classes of the diffeomorphisms $\tau_i := \mu(\gamma_i)$, which satisfy $\tau_1 \tau_2 \ldots \tau_m = Id$.

The condition that f is a symplectic Lefschetz pencil (e.g., in the complex case) yields that τ_i is a positive Dehn twist around the vanishing cycle of the fibre over b_i.

For the reader's convenience we recall that Hurwitz equivalence of factorizations is generated by the so-called Hurwitz moves

$$\tau_1 \circ \tau_2 \circ \ldots \tau_i \circ \tau_{i+1} \circ \cdots \circ \tau_m \cong \tau_1 \circ \tau_2 \circ \ldots \tau_{i+1} \circ (\tau_i)_{\tau_{i+1}} \circ \cdots \circ \tau_m$$

where we use the notation

$$(a)_b := (b^{-1}ab),$$

and by their inverses

$$\tau_1 \circ \tau_2 \circ \ldots \tau_i \circ \tau_{i+1} \circ \cdots \circ \tau_m \cong \tau_1 \circ \tau_2 \circ \ldots (\tau_{i+1})_{\tau_i^{-1}} \circ \tau_i \circ \cdots \circ \tau_m.$$

The Hurwitz moves simply reflect the dependence of the given factorization upon the choice of a geometric quasi-basis of the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{b_1, \ldots b_m\})$.

Define (simultaneous) conjugation equivalence for two factorizations as the equivalence given by $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m \equiv (\tau_1)_b \circ (\tau_2)_b \circ \cdots \circ (\tau_m)_b$.

One says that two Lefschetz fibrations (M, f), (M', f') are equivalent if there are two diffeomorphisms $u : M \to M', v : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f \circ u = v \circ f$.

We have then the following fundamental result established by Kas ([**Kas**]) in 1980 (actually in a greater generality) and reproven by Matsumoto in [**Mat**]:

Proposition 3.5 (Kas). *The equivalence class of a differentiable Lefschetz fibration (M, f) of genus $g \geq 2$ is completely determined by the equivalence class of its factorization of the identity in the Mapping class group, for the equivalence relation generated by Hurwitz equivalence and by conjugation equivalence.*

More generally, the same result holds if the basis of the fibration is a disk.

Very useful also is the concept of the fibre sum of two symplectic Lefschetz fibrations (cf. [**G-S**] Def. 7.1.11, p. 245, Theorem 10.2.1, p. 394 and references therein).

This is just the glueing of two symplectic Lefschetz pencils over the disk (the case where the basis is $\mathbb{P}^1_{\mathbb{C}}$ reduces to the previous by removing a small disk over which the fibration is trivial).

Definition 3.6.

1) Let f, f' be Lefschetz fibrations over a disk D, and let $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m = \phi$, $\tau_1' \circ \tau_2' \circ \cdots \circ \tau_r' = \phi'$ be their corresponding factorizations: then their fibre sum is the fibration corresponding to the factorization $\tau_1 \circ \tau_2 \circ \ldots \tau_m \circ \tau_1' \circ \tau_2' \circ \cdots \circ \tau_r' = \phi\phi'$.

If moreover $\phi\phi' = Id$, we obtain also a corresponding Lefschetz fibration over $\mathbb{P}^1_{\mathbb{C}}$.

2) More generally, for each $\psi \in \mathcal{D}iff^+(C_g)$, one can define a twisted fibre sum by considering instead the factorization

$$\tau_1 \circ \tau_2 \circ \ldots \tau_m \circ (\tau'_1)_\psi \circ (\tau'_2)_\psi \circ \cdots \circ (\tau'_r)_\psi = \phi(\phi')_\psi.$$

If ψ commutes with ϕ' we obtain a new factorization of $\phi\phi'$.

In the case where moreover ϕ' is trivial, the new factorization is associated to the Lefschetz fibration obtained by glueing the two pieces in a different way via the diffeomorphism of the boundary $C \times S^1$ provided by $\psi \times Id_{S^1}$.

Thus, the problem of finding a diffeomorphism between two differentiable manifolds (M, ω), (M', ω'), endowed with respective Lefschetz pencils f, f' over $\mathbb{P}^1_{\mathbb{C}}$ can be solved by showing the equivalence of the respective factorizations of the Identity in the mapping class group.

Very useful in this context is the Lemma of Auroux, for which we also reproduce the very simple proof.

Lemma 3.7 (Auroux). *Let τ be a Dehn twist and let F be a factorization of a central element $\phi \in \mathcal{M}ap_g$, $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m = \phi$.*

If there is a factorization F' such that F is Hurwitz equivalent to $\tau \circ F'$, then $(F)_\tau$ is Hurwitz equivalent to F.

In particular, if F is a factorization of the identity, $\psi = \Pi_h \tau'_h$, and $\forall h \exists F'_h$ such that $F \cong \tau'_h \circ F'_h$, then the fibre sum with the Lefschetz pencil associated with F yields the same Lefschetz pencil as the fibre sum twisted by ψ.

Proof. If $\cong$ denotes Hurwitz equivalence, then

$$(F)_\tau \cong \tau \circ (F')_\tau \cong F' \circ \tau \cong (\tau)_{(F')^{-1}} \circ F' = \tau \circ F' \cong F.$$

q.e.d.

Corollary 3.8. *Notation as above, assume that $F : \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m = \phi$, is a factorization of the Identity and that ψ is a product of the Dehn twists τ_i appearing in F. Then a fibre sum with the Lefschetz pencil associated with F yields the same result as the same fibre sum twisted by ψ.*

Proof. We need only to verify that for each h , there is F'_h such that $F \cong \tau_h \circ F'_h$.

But this is immediately obtained by applying $h - 1$ Hurwitz moves, the first one between τ_{h-1} and τ_h, and proceeding further to the left till we obtain τ_h as first factor. q.e.d.

In the rest of this section we want to show how the two surfaces S, S' appearing in Theorem 3.1 are indeed obtained as the respective fibre sums of the same pair of Lefschetz fibrations over the disk. We shall

later show the more delicate point: viewing the second as the twisted fibre sum of the two first by a diffeomorphism ψ, then the hypothesis of Auroux's lemma applies, namely, ψ is generated by Dehn twists appearing in one factorization.

In order to understand which map ψ is in our example, recall that we have previously defined diffeomorphisms of the coverings S and S' when restricted over Δ_0, respectively over Δ_∞. Before the perturbations the diffeomorphism over Δ_∞ covers the identity map of $\Delta_\infty \times \mathbb{P}^1$ while the diffeomorphism Ψ over Δ_0 covers the map $(x, y) \to (x, -y)$ of $\Delta_0 \times \mathbb{P}^1$. The situation remains similar after we perturb slightly the coverings and the diffeomorphisms.

We shall describe a fibre over a point $x \in \Delta_0$ with $|x| = 1$.

Letting C, ψ be the pair of the curve given by the bidouble cover of $\mathbb{P}^1_{\mathbb{C}}$ of equation

- $z^2 = F(y), w^2 = F(-y)$

   and of the automorphism given by

- $(*)$ $y \to -y, z \to w, w \to z$

we see immediately that

Proposition 3.9. *The monodromy of S over the unit circle $\{x \mid |x| = 1\}$ is trivial, and the pair C, ψ of the fibre over $x = 1$, considered as a differentiable 2-manifold, together with the isotopy class of the attaching map ψ, is given by $(*)$ above.*

Proof. For both the first assertion and the second assertion it suffices to let the roots ζ_i tend to 0 and let the roots of A, resp. C tend to ∞.

Then we see that the bundle is trivial over the circle $\{x \mid |x| = 1\}$, and then ψ becomes the map in $(*)$.   q.e.d.

In order to use the Lemma of Auroux we need to compute, at least partially, a monodromy factorization of the Lefschetz fibration over Δ_0.

We shall first describe certain cycles (simple closed curves) on the fibre C of the trivial fibration described by the above proposition. The fiber C is a four-tuple cover of $\mathbb{P}^1$ and is branched over the roots $B_1, B_2, ..., B_{2b}$ of $F(y)$ and over the roots $-B_1, -B_2, ..., -B_{2b}$ of $F(-y)$.

Over each branch point there are two simple ramification points.

Observe moreover that, if we cut the projective line $\mathbb{P}^1_{\mathbb{C}}$ along the closed subinterval of the real axis $[-B_{2b}, B_{2b}]$, we get a domain covered by four disjoint sheets. The ramification points over B_i connect two pairs of sheets (z changes sign) while the ramification points over $-B_i$ connect two different pairs of sheets (w changes sign). The sheets will be numbered as follows: we declare that the points over the B_i's connect sheets (1,2) and (3,4), whereas the points over the $-B_i$'s connect the sheets (1,3) and (2,4). Each segment $[B_i, B_{i+1}]$ and $[-B_{i+1}, -B_i]$ is covered by 4 arcs, two pairs with common end points, which form

two cycles - (α_i between sheets (1,2) and γ_i between sheets (3,4)) over $[B_i, B_{i+1}]$ and (β_i between sheets (1,3) and δ_i between sheets (2,4)) over $[-B_{i+1}, -B_i]$. The segment $[-B_1, B_1]$ is covered by one long cycle called σ, composed of four arcs. Each pair (α_i, α_{i+1}), (β_i, β_{i+1}), (γ_i, γ_{i+1}), (δ_i, δ_{i+1}) has one intersection point. Also σ meets $\alpha_1, \beta_1, \gamma_1, \delta_1$ at one point each, in this cyclic order for a suitable orientation of σ. No other pair of these cycles meet. Thus the configuration of the cycles α_i, β_i, γ_i, δ_i and σ on C, including their regular neighbourhood, is as in Figure 1. The regular neighbourhood of the union of all these cycles has genus $g = 4b - 3$ and 4 boundary components. By the Riemann-Hurwitz formula this is also the genus of C (with four punctures), hence each boundary component bounds a disk on C and the picture on Figure 1 represents the configuration of the cycles α_i, β_i, γ_i, δ_i and σ on C.

We now describe the action of the diffeomorphism ψ on these cycles. It covers the automorphism $y \to -y$ of $\mathbb{P}^1$. In particular, it interchanges the segments $[B_i, B_{i+1}]$ and $[-B_{i+1}, -B_i]$, thus ψ takes α_i onto β_i or onto δ_i. The situation is symmetric and ψ has order two so we may assume that ψ interchanges α_i and δ_i. It also takes σ onto itself. The intersection points of σ with α_1 and β_1 are consecutive along the cycle σ and ψ interchanges them, so it must reverse the orientation of σ. The diffeomorphism ψ preserves the intersection number of cycles.

Taking under consideration the orientation of the cycles described in Figure 1 (the Riemann Surface C) we conclude that $\psi(\sigma) = -\sigma$ and
$\psi(\alpha_i) = -\delta_i$, $\psi(\delta_i) = -\alpha_i$,
$\psi(\gamma_i) = -\beta_i$, $\psi(\beta_i) = -\gamma_i$ for all i.

Since the complement of the regular neighbourhood of the union of these cycles consists of disks the above formulae determine ψ up to isotopy (see Section 5).

We shall prove in Section 5 that ψ is isotopic to a product of Dehn twists with respect to the cycles α_i, β_i, γ_i, δ_i and σ on C (we shall then use the same symbol for an oriented cycle and for the Dehn twist determined by it).

We shall now prove in the next section 4 that all of these Dehn twists appear in a monodromy factorization of $S|_{\Delta_0}$.

We shall do it via a new conceptual approach, and by referring to a well known result. An alternative computational approach is contained in the appendix.

4. Coloured mapping class groups

In the first part of this section we want to point out a simple generalization of the concept (cf. [**Sieb-Tian**] and also [**Aur02**]) of the hyperelliptic mapping class group.

Recall that the braid group $\mathcal{B}^S_{2g+2}$ of the Riemann sphere (i.e., $\mathcal{B}^S_n$ is the fundamental group of the configuration space of n distinct points on $\mathbb{P}^1_{\mathbb{C}}$) has the presentation

$$\langle \sigma_1, \ldots \sigma_{2g+1} | \sigma_1 \ldots \sigma_{2g+1}\sigma_{2g+1} \ldots \sigma_1 = 1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \;\; \text{for} \;\; |i-j| \geq 2, \;\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

It is a quotient of Artin's braid group $\mathcal{B}_{2g+2}$, presented as follows:

$$\langle \sigma_1, \ldots \sigma_{2g+1} | \sigma_i\sigma_j = \sigma_j\sigma_i \;\; \text{for} \;\; |i-j| \geq 2, \;\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

A further quotient of the braid group $\mathcal{B}^S_{2g+2}$ of the Riemann sphere is the mapping class group $\mathcal{M}ap_{0,2g+2}$ of $(\mathbb{P}^1 - \{1, \ldots 2g+2\})$, which admits the following presentation:

$$\langle \sigma_1, \ldots \sigma_{2g+1} | \sigma_1 \ldots \sigma_{2g+1}\sigma_{2g+1} \ldots \sigma_1 = 1, (\sigma_1 \ldots \sigma_{2g+1})^{2g+2} = 1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \;\; \text{for} \;\; |i-j| \geq 2, \;\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle,$$

i.e., one further mods out the centre of the braid group, which is generated by $\Delta^2 := (\sigma_1 \ldots \sigma_{2g+1})^{2g+2}$.

Example 4.1. Let $C = C_g$ be a compact curve of genus g endowed with a hyperelliptic involution H. Then, for $g \geq 1$, the hyperelliptic mapping class group, denoted by $\mathcal{M}ap^h_g$, is the subgroup of the mapping class group defined by the following central extension of the cyclic subgroup $\mathbb{Z}/2$ generated by H:

$$1 \to \mathbb{Z}/2 = \langle H \rangle \to \mathcal{M}ap^h_g \to \mathcal{M}ap_{0,2g+2} \to 1.$$

It is a proper subgroup for $g \geq 3$, and it measures the obstruction to having a lifting map $\mathcal{M}ap_{0,2g+2} \to \mathcal{M}ap_g$; we have in fact only an injection:

$$\rho : \mathcal{M}ap_{0,2g+2} \to \mathcal{M}ap_g/\langle H \rangle.$$

The hyperelliptic mapping class group $\mathcal{M}ap^h_g$ has the presentation

$$\langle \xi_1, \ldots \xi_{2g+1}, H | \xi_1 \ldots \xi_{2g+1}\xi_{2g+1} \ldots \xi_1 = H,$$

$$H^2 = 1, (\xi_1 \ldots \xi_{2g+1})^{2g+2} = 1,$$

$$H\xi_i = \xi_i H \;\forall i, \xi_i\xi_j = \xi_j\xi_i \;\; for |i-j| \geq 2, \;\; \xi_i\xi_{i+1}\xi_i = \xi_{i+1}\xi_i\xi_{i+1} \rangle.$$

Definition 4.2.

1) Let C be a (compact) Riemann surface of genus $g \geq 2$, and let $p : C \to C'$ be a non constant holomorphic map to another Riemann surface of genus g'. Then we consider the subgroup $\mathcal{M}ap^p_g$ of $\mathcal{M}ap_g$ consisting of those isotopy classes of diffeomorphisms $F : C \to C$ such that there is a diffeomorphism $F' : C' \to C'$ such that $p \circ F = F' \circ p$. We shall call this subgroup the p-symmetric mapping class group (cf. explanation later).

2) A special case we consider is the one where p is the quotient map for an effective action of a finite group G (thus, $C' = C/G$). In this case we shall also use the notation $\mathcal{M}ap_g^G$ for the above subgroup and we shall also call it the G-symmetric mapping class group.

We have that, B being the branch locus of p, and $\mu : \pi_1(C' - B) \to \mathcal{S}_d$ being the monodromy of p, then clearly such an F' induces an isotopy class in the Mapping class group of $(C' - B)$. However, the branch points cannot be permuted arbitrarily, since a point $b \in B$ gets a "colouring" given by the conjugacy class of a local monodromy element $\mu(\gamma_b)$.

In general, elementary covering space theory guarantees that the class of a diffeomorphism F' corresponding to a diffeomorphism F in the p-symmetric mapping class group belongs to the p-coloured subgroup $\mathcal{M}ap(C' - B)^p$ of the Mapping class group $\mathcal{M}ap(C' - B)$ consisting of those diffeomorphisms F' whose action F'_* on $\pi_1(C' - B)$ satisfies the property

$(* * *)$ μ and $\mu' := \mu \circ F'_*$ are conjugated by an inner automorphism of $\mathcal{S}_d$.

Conversely, every element F' in the p-coloured subgroup $\mathcal{M}ap(C' - B)^p$ has some lift F'' in the punctured mapping class group of $C - f^{-1}(B)$, which uniquely determines the class of F'.

But the image F of F'' in the mapping class group of C may be trivial even if F' is non trivial.

Notice that, for Galois coverings with group G, condition $(* * *)$ means that $\mu_G : \pi_1(C' - B) \to G \subset \mathcal{S}(G)$ and $\mu'_G : \pi_1(C' - B) \to G$ are related by an automorphism of G (not necessarily inner!).

If $\mathcal{H}$ is the kernel subgroup of the monodromy, $N_\mathcal{H}$ its normalizer, and G' is the quotient group $N_\mathcal{H}/\mathcal{H}$, we obtain the exact sequence

$$1 \to G' \to \mathcal{M}ap_g^p(C - f^{-1}(B)) \to \mathcal{M}ap(C' - B)^p \to 1.$$

A general investigation of these coloured (resp.: symmetric) mapping class groups, and especially their presentations, seem to us to be of considerable interest, yet they could present some difficulties.

For our present much more limited purposes, it suffices to restrict ourselves to the case where C' has genus zero. Then we may assume that $C' = \mathbb{P}^1$ and that ∞ is not in B. In this case, the datum of p is determined by giving a factorization of the identity in the group $\mathcal{S}_d$, and we get the subgroup of the Mapping class group of the punctured sphere stabilizing the (simultaneous) conjugacy class of the factorization.

If the covering p is an Abelian Galois covering with group G, we have a relation with groups of coloured braids: in fact, since G is abelian and the local monodromy at a point b is determined up to conjugacy, the local monodromy $\mu(\gamma_b) \in G$ is an element in G which is independent of the path (γ_b) chosen. Thus we can talk of the G- labelling of the point b.

Proposition 4.3. *Let G be a finite Abelian group, and assume that G operates on a Riemann surface C with quotient $\mathbb{P}^1$. Assume that m is the cardinality of the branch locus B, and let P be the partition of B given by the G-labelling of B (i.e., to b corresponds the label $\mu(\gamma_b) \in G$). Then we have an exact sequence*

$$1 \to G \to \overline{\mathcal{M}ap_g^G(C - f^{-1}(B))} \to \mathcal{M}ap_{0,m,P} \to 1$$

where $\mathcal{M}ap_{0,m,P}$ is the Mapping class group of the diffeomorphisms of the punctured sphere which leave each set of the partition P invariant, and $\overline{\mathcal{M}ap_g^G}$ is the kernel of the natural homomorphism $\mathcal{M}ap_g^G \to \mathrm{Aut}\,(G)$ described in $(\ast\ast\ast)$.

The coloured Mapping class groups $\mathcal{M}ap_{0,m,P}$ are obviously quotients of the coloured braid groups $\mathcal{B}_{m,P}$ which were investigated by Manfredini in [**Manf**], where an explicit presentation was given.

As a warm up the author considers there the particular case where the partition consists of the two sets $\{1, 2, \ldots n-k\}$ and $\{n-k+1, \ldots n\}$ and Theorem 1.6 gives a presentation of the group $\mathcal{B}_{n,k}$ showing in particular

Theorem 4.4 (Manfredini)**.** *The two coloured braid group $\mathcal{B}_{n,k}$ is generated by σ_i, for $i \neq n-k, 1 \leq i \leq n-1$, and by $B := \sigma_{n-k}^2$, subject to the standard braid relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \ \ for |i-j| \geq 2, \ \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

and to the following relations

$$ABAB = BABA, BCBC = CBCB, ABA^{-1}CBC^{-1} = CBC^{-1}ABA^{-1}$$

where $A := \sigma_{n-k-1}$, $C := \sigma_{n-k+1}$.

The two coloured braid group is particularly relevant to our situation. In fact, in the case $G = (\mathbb{Z}/2)^2$ there are exactly three non zero elements in the group.

The restriction however that the bidouble cover be simple is precisely the restriction that a local monodromy transformation be only $(1,0)$ or $(0,1)$: whence we get only two labels.

For the bidouble cover considered in proposition 3.9 we get therefore the following non central exact sequence

$$1 \to (\mathbb{Z}/2)^2 \to \overline{\mathcal{M}ap_g^{(\mathbb{Z}/2)^2}(C - f^{-1}(B))} \to \mathcal{M}ap_{0,4b,2b} \to 1.$$

Let us look at the extension, and especially at the Manfredini generators. For σ_i, $i \leq 2b-1$, it lifts to the product of a couple of Dehn twists on disjoint curves α_i, γ_i. We denote here the corresponding Dehn twists by the same symbols. Hence we may assume that $e_1 := (1,0)$ leaves these curves invariant and thus centralizes the two Dehn twists, while $e_2 := (0,1)$ permutes the two curves, thus conjugation by e_2 exchanges α_i, γ_i. Hence, $e_1\alpha_i = \alpha_i e_1$, $e_1\gamma_i = \gamma_i e_1$, while $e_2\gamma_i = \alpha_i e_2$, $e_2\alpha_i = \gamma_i e_2$.

Furthermore, the generator B lifts to a Dehn twist along the curve denoted by σ in Figure 1, which is left invariant by e_1, e_2, and thus σ commutes with e_1, e_2.

Finally, for $i \geq 2b+1$, σ_i lifts to the product of a couple of Dehn twists β_i, δ_i on disjoint curves, and we have formulae similar to the above ones.

All the calculations of monodromies for the algebraic surfaces obtained as bidouble covers therefore take place in our bi-coloured subgroups (recall that our branch locus was bipartite into the set of positive roots, i.e., the roots of the polynomial $F(y)$, and the set of negative roots (roots of $F(-y)$)).

Remark 4.5. Our given diffeomorphism ψ is a lift of the involution ι such that $\iota(y) = -y$.

ι and the Galois group $(\mathbb{Z}/2)^2$ generate a dihedral symmetry on our Riemann surface, with cyclic symmetry of order 4 given by the transformation
$$y \to -y, \ w \to -z, \ z \to w.$$
ι generates the extension $\mathcal{M}ap_{0,4b}{}^{(\mathbb{Z}/2)^2}/\mathcal{M}ap_{0,4b,2b}$ while ψ generates the quotient $\mathcal{M}ap_g^{(\mathbb{Z}/2)^2}(C - f^{-1}(B))/\mathcal{M}ap_g^{(\mathbb{Z}/2)^2}(C - f^{-1}(B))$ (since the bidouble cover is simple).

We want now to compute the braid monodromy factorization corresponding to the branch curves for our branched coverings.

In order to write it down, let us change notation for the Manfredini generators of the bicoloured braid group in the case where $n = 2m, k = m$, and let us call them

$$x_1 := \sigma_1, \ldots x_{m-1} := \sigma_{m-1}, z^2 := \sigma_m^2, \ y_1 := \sigma_{m+1}, \ldots y_{m-1} := \sigma_{2m-1}.$$

Proposition 4.6. *Consider the fibration $M_1 \to \Delta_0$ given by the composition $p_1 \circ \pi$, where $p_1 : \Delta_0 \times \mathbb{P}^1 \to \Delta_0$ is the first projection, and $\pi : M_1 \to \Delta_0 \times \mathbb{P}^1$ is the Galois degree 4 covering corresponding to the two curves $D_1 \cap (\Delta_0 \times \mathbb{P}^1)$ and $D_2 \cap (\Delta_0 \times \mathbb{P}^1)$. Then*

1) *The braid monodromy factorization in the bicoloured braid group $\mathcal{B}_{2m,m}$ (where $m = 2b$) equals the product of four equal factors of the form*

$$x_1 \circ x_1 \circ (x_2)_{x_1} \circ (x_2)_{x_1} \circ (x_3)_{x_2 x_1}$$
$$\circ (x_3)_{x_2 x_1} \cdots \circ (x_{m-1})_{x_{m-2} \ldots x_2 x_1}$$
$$\circ (z^2)_{x_{m-1} \ldots x_2 x_1} \circ (y_1^2)_{z x_{m-1} \ldots x_2 x_1}$$
$$\circ (y_2^2)_{y_1 z x_{m-1} \ldots x_2 x_1} \cdots \circ (y_{m-1}^2)_{y_{m-2} \ldots y_1 z x_{m-1} \ldots x_2 x_1}$$

with two symmetrical factors

$$y_{m-1} \circ y_{m-1} \circ (y_{m-2})_{y_{m-1}} \circ (y_{m-2})_{y_{m-1}} \cdots \circ (y_1)_{y_2 \ldots y_{h-1}} \circ$$
$$\circ (z^2)_{y_1 \ldots y_{m-1}} \circ (x_{m-1}^2)_{z y_1 \ldots y_{m-1}} \cdots \circ (x_1^2)_{x_2 \ldots x_{m-1} z y_1 \ldots y_{m-1}}.$$

2) *The mapping class group factorization of the perturbed Lefschetz fibration is obtained via the following replacement:*
  - *each factor which is a half twist on a segment whose end points are either both positive or both negative roots is replaced by the product of the two Dehn twists on the two circles lying above it,*
  - *each factor which is a full twist on a segment whose end points are roots of opposite sign is replaced by the Dehn twist on the circle lying over the segment.*

Proof. As a first preliminary remark we observe that if we take a general k-tuple of curves belonging to k respective linear systems on a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, the braid monodromy factorization will be unique (up to Hurwitz equivalence), since the parameter space for these k-tuples is then connected.

In the special case where our ruled surface is $\mathbb{P}^1 \times \mathbb{P}^1$ we observe that the braid monodromy factorization corresponding to a curve of bidegree (d, m) is the fibre sum of d factorizations corresponding to curves of bidegree $(1, m)$.

This last is well known (cf. [**Aur02**]) to be

$$\sigma_1 \circ \cdots \circ \sigma_{m-1} \circ \sigma_{m-1} \circ \cdots \circ \sigma_1$$

and is Hurwitz equivalent to the following factorization (ibidem, Lemma 5)

$$(\bullet) \; \sigma_1 \circ \sigma_1 \circ (\sigma_2)_{\sigma_1} \circ (\sigma_2)_{\sigma_1} \cdots \circ (\sigma_{m-1})_{\sigma_{m-2}\ldots\sigma_2\sigma_1} \circ (\sigma_{m-1})_{\sigma_{m-2}\ldots\sigma_2\sigma_1}.$$

This last factorization of a curve of bidegree $(2, m)$ corresponds to the following regeneration of the union of two vertical lines and of m horizontal lines: first regenerate partially to a curve of bidegree $(2, 1)$ and $m - 1$ horizontal lines, then regenerate so that each of the $2(m - 1)$ nodes splits into two vertical tangencies.

Let us now consider the case of $k = 2$ curves of bidegree $(2, m)$: the first curve will be coloured red and will consist of the regeneration of two red vertical lines (with respective abscissae $0 < \zeta_1' < \zeta_2' \ll \epsilon$) plus m red horizontal lines (of equation $F(y) = 0$); the other will be coloured blue and will consist of the regeneration of two blue vertical lines (with respective abscissae $0 < \zeta_5' := 2\epsilon - \zeta_2' < \zeta_6' := 2\epsilon - \zeta_1'$) plus m blue horizontal lines (of equation $F(-y) = 0$).

We use the symmetry in $\Delta_0 \times \mathbb{P}^1$ given by $(x, y) \to (2\epsilon - x, -y)$ and assume that the blue curve is the symmetrical of the red curve. Thus we couple the factorization $(\bullet)$ with its symmetrical by ι. This is a factorization for the union of the regeneration of the two curves. But now we are not allowed a complete regeneration, because only the double points where two curves of the same colour meet are allowed to be smoothed (and correspondingly we have simple factors corresponding

to a pair of vertical tangencies). Meanwhile the other square factors are to be considered as elements of $\mathcal{B}_{2m.m}$ (where they are not squares).

In this way we obtain factorization 1).

2) is a direct consequence of the exact sequence in Proposition 4.3, and of the explicit description of the lifts of half twists and full twists, which is an easy exercise to obtain.   q.e.d.

Corollary 4.7. *The factors of the braid monodromy factorization generate the bicoloured braid group* $\mathcal{B}_{2m.m}$.

Proof. We see immediately that the factors generate a subgroup containing all the x_i's, all the y_j's , and z^2. But these elements generate $\mathcal{B}_{2m,m}$ by Manfredini's theorem.   q.e.d.

The involution ι is not a product of those, but we shall prove in the next section that the diffeomorphism ψ is a product of the Dehn twists appearing in the mapping class group factorization of the Lefschetz fibration on M_1.

Since ψ is a lift of ι one may hope that ι be a product of the braid monodromy factors that we obtain when we perturb the branch locus, but we fix the colouring given by the symmetric group $\mathcal{S}_4$ (containing the abelian group $G = (\mathbb{Z}/2)^2$ as the subgroup of left translations on G).

We will describe elsewhere this "perturbed" braid monodromy factorization.

5. Factorization of the diffeomorphism ψ

In this section we shall give a geometric proof that the diffeomorphism ψ of the (Riemann) surface C can be expressed as a product of Dehn twists with respect to the curves α_i, β_i, γ_i, δ_i and σ (see Figure 1).

It follows from the definition of the curves α_i, β_i, γ_i, δ_i and σ and from Proposition 4.6 (using Hurwitz moves) that each of these Dehn twists appears in some monodromy factorization of the fibration $M_1 \to \Delta_0$. Therefore, by Lemma 3.7, the above claim implies that the surfaces S and S' are diffeomorphic, i.e., we conclude then the proof of Theorem 3.1.

We shall first recall some known and some less known facts about Dehn twists.

Definition 5.1. A positive Dehn twist with respect to a simple closed curve α on C is an isotopy class of a diffeomorphism h of C which is equal to the identity outside a neighbourhood of α orientedly homeomorphic to an annulus while inside the annulus h rotates the inner boundary of the annulus by 360 degrees to the right and damps the rotation down to the identity at the outer boundary.
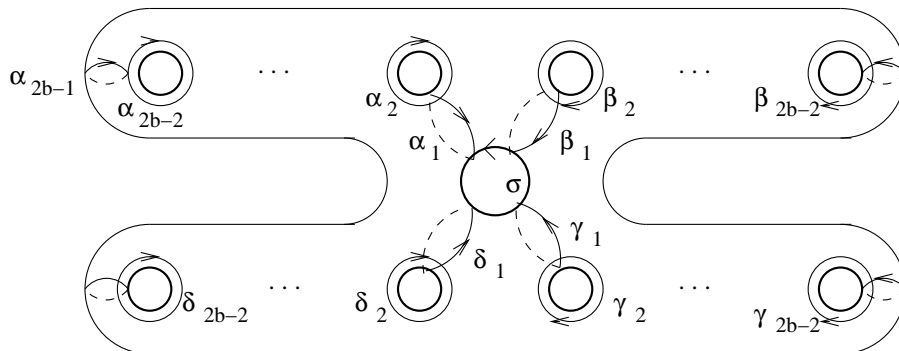
Figure 1. Surface C.

The Dehn twist with respect to α will be denoted by T_α.

Observe that the twist does not depend on the orientation of the curve α (it depends only on the orientation of C) but if we apply T_α to an oriented curve β then $T_\alpha(\beta)$ inherits the orientation of β.

Our notation will in the sequel be as follows: an equality sign between diffeomorphisms and between curves shall mean that they are isotopic on C.

One can easily check the following

Lemma 5.2. *Let α and β be oriented simple closed curves on C which intersect transversally at one point with a positive intersection index $i(\alpha, \beta) = 1$ (i.e.,the angle from the positive direction of α to the positive direction of β is positive on the oriented surface S).*
  Then $T_\alpha T_\beta(\alpha) = -\beta$ and $T_\beta T_\alpha(\beta) = \alpha$.

Definition 5.3. We say that simple closed curves $\alpha_1, \alpha_2, \ldots, \alpha_n$ on S form a chain of curves if α_i intersects α_{i+1} in one point, transversally for $i = 1, \ldots, n-1$, and α_i is disjoint from α_j for $|i - j| > 1$. We shall say that the chain is oriented if also each curve α_i is oriented and $i(\alpha_i, \alpha_{i+1}) = 1$ for $i = 1, \ldots, n-1$.

Chains of curves were introduced by Dennis Johnson in 1979. He also proved the following

Lemma 5.4. *Let $\alpha_1, \ldots, \alpha_n$ form a chain of curves on C. Let N be a regular neighborhood of the union $\bigcup \alpha_i$. Then N has one boundary component if n is even and N has two boundary components if n is odd.*

Dennis Johnson associated to each chain a *chain map* which acts trivially on the homology of C.

We shall associate another diffeomorphism of C to a chain of curves.

Definition 5.5. Let $\alpha_1, \ldots, \alpha_n$ form a chain of curves on S. Then the Coxeter diffeomorphism $\Delta = \Delta(\alpha_1, \ldots, \alpha_n)$ of the chain is defined by

$$\Delta = (T_{\alpha_1})(T_{\alpha_2}T_{\alpha_1})\ldots(T_{\alpha_{n-1}}T_{\alpha_{n-2}}\ldots T_{\alpha_1})(T_{\alpha_n}T_{\alpha_{n-1}}\ldots T_{\alpha_1}).$$

In order to understand the Coxeter diffeomorphism we need the following

Lemma 5.6. *Let N be an oriented connected Riemann surface, possibly with boundary, and let $\tau_1, \tau_2, \ldots, \tau_n$ be a family of oriented proper arcs and s.c.c. on N such that any two members of the family are either disjoint or intersect transversally in one point. Suppose that the graph which has a vertex for each τ_i and an edge for each intersecting pair is a tree. Suppose also that the complement of the union $\bigcup \tau_i$ on N is a union of disks.*

Let h be a diffeomorphism of N such that h is the identity on the boundary of N and $h(\tau_i)$ is isotopic to τ_i for each i. Then h is isotopic to the identity.

Proof. We can modify h by an isotopy in the neighbourhood of consecutive τ_i's and make it pointwise fixed on each τ_i. We can choose the order in such a way that the next τ_i intersects precisely one of the previous curves and arcs, on which h was already modified.

The rest is then easy.

When we modify h on the last τ_i it is the identity on the boundary of each complementary disk and thus it is isotopic to the identity by Alexander's lemma. q.e.d.

Corollary 5.7. *Let $\tau_1, \tau_2, \ldots, \tau_n$ be as in the previous lemma. Let h and g be two diffeomorphisms of N which are the identity on the boundary of N and such that $h(\tau_i)$ is isotopic to $g(\tau_i)$ for all i. Then g and h are isotopic.*

Proof. The diffeomorphism gh^{-1} satisfies the assumptions of the previuos lemma. Thus gh^{-1} is isotopic to the identity. q.e.d.

Remark 5.8. The assumption in the above lemma, that the intersection pattern of the τ_i's yields a tree, may not be necessary.

However, in some other case it may be more difficult (or impossible) to change the image of one curve by an isotopy and keep fixed the curves where h was already modified.

Proposition 5.9. *Let $\alpha_1, \ldots, \alpha_n$ form an oriented chain of curves on C and let Δ be the Coxeter diffeomorphism of the chain. Let N be a regular neighborhood of the union $\bigcup_{i=1}^n \alpha_i$ (see Figure 2 and Figure 3). We may assume that Δ takes N onto itself and leaves the boundary of N pointwise fixed. Then*

1. *If n is odd then Δ is isotopic to the rotation of N around the axis by 180 degrees followed by two positive half Dehn twists on the respective boundary components (they rotate each boundary component by 180 degrees).*

> *It is characterized by the properties:*
> $\Delta(\alpha_i) = \alpha_{n+1-i}$ *for i odd,*
> $\Delta(\alpha_i) = -\alpha_{n+1-i}$ *for i even,*
> *and* $\Delta(\gamma) = \delta$
> (*see Figure* 3).
> 2. *If n is even then* Δ^2 *rotates the neighborhood by* 180 *degrees keeping the boundary fixed* (*see Figure* 2).
> *It is characterized by the properties:*
> $\Delta^2(\alpha_i) = -\alpha_i$ *for all i and*
> $\Delta^2(\beta) = \delta$ (*see Figure* 2).

Proof. The isotopy class $\Delta(\alpha_i)$ is determined by Lemma 5.2. The isotopy classes $\Delta^2(\beta)$ and $\Delta(\gamma)$ can be very easily determined by drawing pictures because most of the curves α_i are disjoint from the consecutive images of β and γ and the corresponding factors of Δ have no effect. The Theorem follows then by Corollary 5.7, see Figure 2 and Figure 3.
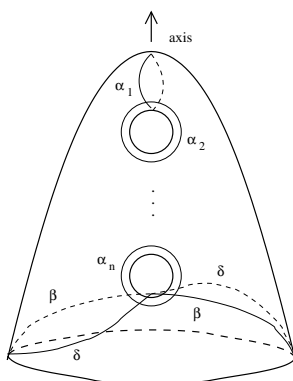
q.e.d.

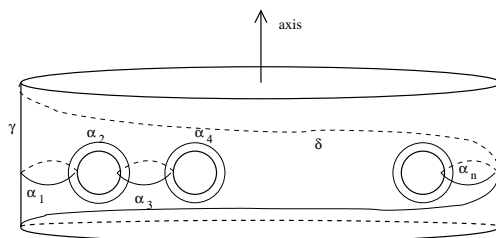

Figure 2. Coxeter homeomorphism for n-even.

Figure 3. Coxeter homeomorphism for n-odd.

Consider now the surface C and the curves represented on Figure 1.

We define the following diffeomorphisms of the surface, in the case where $n = 2b - 1$ is odd:

$A_1 = \Delta(\delta_n, \delta_{n-1}, \ldots, \delta_2, \delta_1, \sigma, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n)$,

$A_2 = \Delta(\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1, \sigma, \beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n)$,

$A_3 = \Delta(\beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1, \sigma, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n)$,

$A_4 = \Delta^{-2}(\alpha_n, \alpha_{n-1}, \ldots, \alpha_2)$,

$A_5 = \Delta^{-2}(\gamma_n, \gamma_{n-1}, \ldots, \gamma_2, \gamma_1, \sigma)$,

$A_6 = \Delta^{-1}(\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1, \sigma, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n)$.

Proposition 5.10. *The diffeomorphism ψ of the surface C is isotopic to the product*

$g = A_6 A_5 A_4 A_3 A_2 A_1$.

Proof. The complement of the union of all the curves α_i, β_i, γ_i, δ_i and σ is a union of four disjoint disks so ψ is determined by the following images of all the curves:

$\psi(\alpha_i) = -\delta_i$, $\psi(\delta_i) = -\alpha_i$,

$\psi(\gamma_i) = -\beta_i$, $\psi(\beta_i) = -\gamma_i$ for $i = 1, 2, \ldots, n$

and $\psi(\sigma) = -\sigma$.

It follows by the definition and the basic properties of Coxeter diffeomorphism that $g = \psi$ for all curves except possibly for α_1, β_1, γ_1 and δ_1. (The other curves either are completely contained in the domain of A_i and are interchanged by A_i or are disjoint from the domain of A_i and are not affected by A_i.)

We check through a series of pictures that the images of α_1, β_1, γ_1 and δ_1 by g are the same as by ψ. We shall not consider the orientation of the curves. It must be correct since the diffeomorphisms preserve the intersection index.

By Proposition 5.9 each A_i, for $i = 1, \ldots, 5$ rotates the neighborhood of the corresponding chain of curves by 180 degrees keeping the boundary of the neighborhood pointwise fixed. The boundary and the axis of each rotation is shown on figure 4. For $i = 1, 2, 3$ the neighborhood has two boundary components but one of the components bounds a disk on C so we can cap it with a disk and forget it. For $i = 6$ the neighborhood of the chain has two boundary components as shown on the last picture of Figure 4. In order to perform A_6 we cut the surface along the boundary components, rotate the middle part (last picture of Figure 4) by 180 degrees around the axis and then apply two negative half Dehn twists with respect to each boundary component and glue back to the remaining pieces of C.

We now describe the consecutive images of $\alpha_1, \beta_1, \gamma_1, \delta_1$.

Figure 5 shows the initial position.

By the description of A_i's we have $A_1(\delta_1) = \alpha_1$, $A_2(\alpha_1) = \beta_1$, $A_3(\beta_1) = \gamma_1$, A_4 and A_5 leave γ_1 invariant and $A_6(\gamma_1) = \alpha_1$ as required.

Figure 6 shows the consecutive images of the transforms of α_1, where A_4 leaves $A_3 A_2 A_1(\alpha_1)$ fixed.
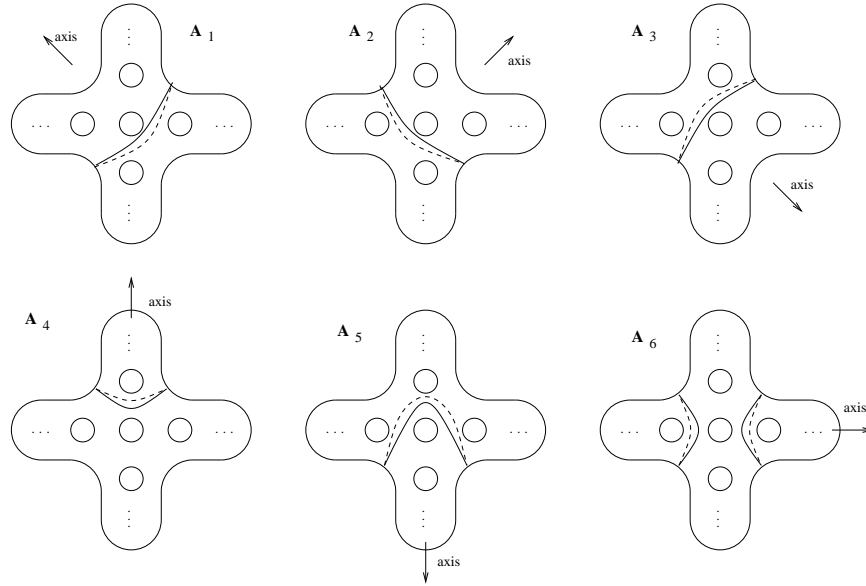
Figure 4. Axes and boundaries of the rotations A_i.
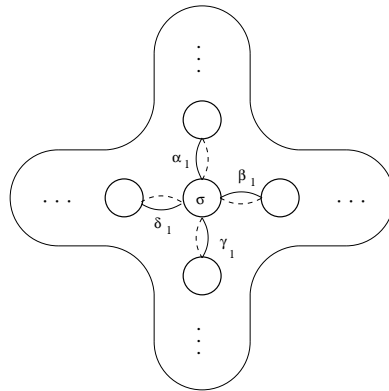

Figure 5. Initial positions of the curves α_1, β_1, γ_1, δ_1.

Figure 7 shows the consecutive images of β_1.

Figure 8 shows the consecutive images of γ_1, where A_4 leaves $A_3 A_2 A_1(\gamma_1)$ fixed.

When we apply A_6 to the last picture on Figures 6, 7 and 8 we get the curves δ_1, γ_1 and β_1 respectively, as required.

Thus the diffeomorphism $\psi = g$ of the (Riemann) surface C can be expressed as a product of Dehn twists with respect to the curves α_i, β_i, γ_i, δ_i and σ.   q.e.d.
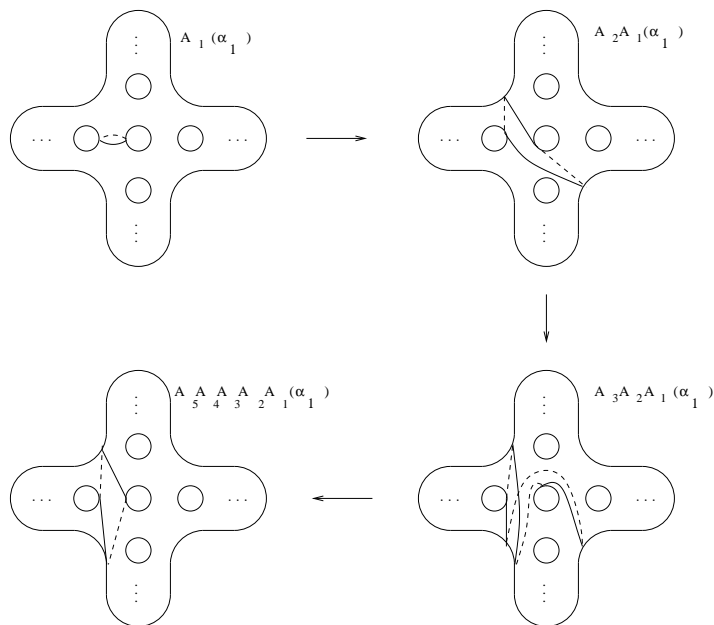
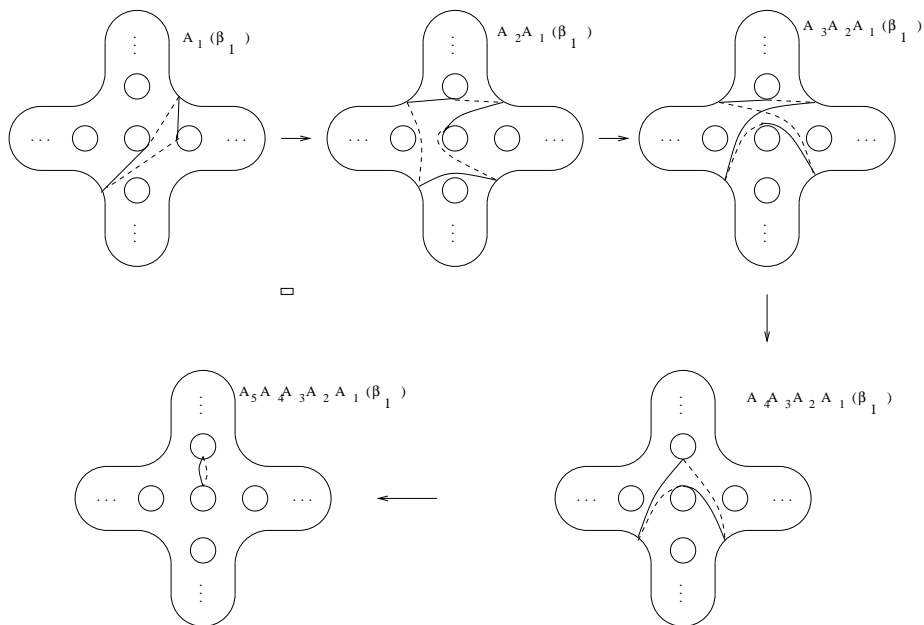**Figure 6.** Consecutive images of α_1.

Figure 7. Consecutive images of β_1.

6. Final comments: Symplectomorphism

In [**Cat6**] it was observed that each minimal algebraic surface of general type S has a canonical symplectic structure, namely, a symplectic
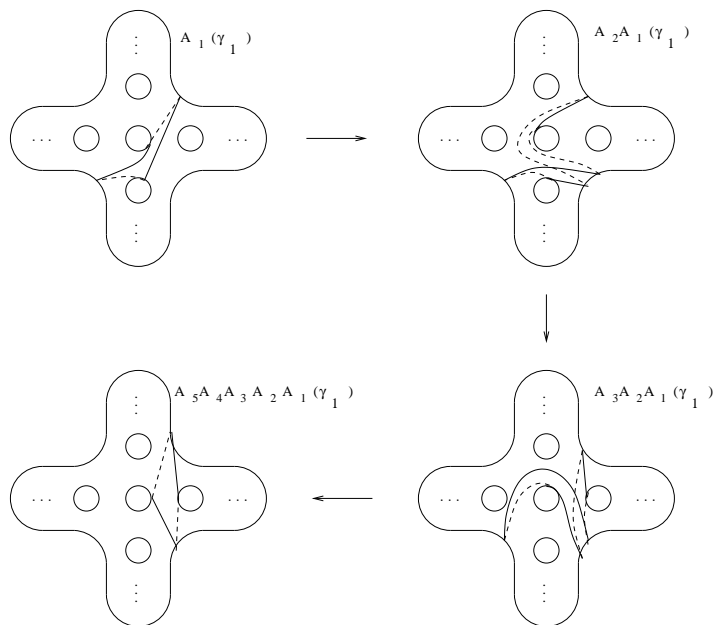
Figure 8. Consecutive images of γ_1.

structure (S, ω) uniquely determined up to symplectomorphism, and such that the class $[\omega]$ of the symplectic structure equals the canonical class K_S.

If K_S is ample, this ω is simply the pull back of the Fubini-Study Kähler class via an m-canonical embedding, divided by m.

Are then the (a,b,c) examples, endowed with this canonical symplectic structure, not only diffeomorphic but also symplectomorphic?

This question can be set in a much wider context.

Assume that $f : S \to C$ is a (holomorphic) fibration over a curve, with fibres curves of genus $g \geq 2$.

We may think of f as given by a classifying morphism ϕ to the compactified moduli space $\overline{\mathcal{M}_g}$ of curves of genus g.

Assume that ϕ admits a small symplectic perturbation ϕ' which yields a morsification f' of f, i.e., a Lefschetz type fibration. By a further perturbation we may assume that ϕ' is an embedding and it meets transversally the boundary divisor Δ_0 of irreducible curves with one node, but does not meet the other divisors $\Delta_i, i \geq 1$.

The differentiable type of the perturbed fibration f' is encoded in the monodromy factorization in the mapping class group $\mathcal{M}ap_g$. On the other hand $\mathcal{M}ap_g$ acts properly discontinuously on the Teichmüller domain $\mathcal{T}_g$ with quotient equal to $\mathcal{M}_g$. It is natural to interpret the equivalence class of this factorization as the isotopy class of the perturbed map ϕ'.

If two factorizations are equivalent, then we should get an isotopy for the natural symplectic structures which one should obtain pulling back from the relative canonical model of the universal curve (keeping in mind the positivity of the direct image of the relative canonical divisor) and so, again by Moser's lemma, one should get a natural symplectic structure corresponding to the relative canonical class $K_{S|C}$.

At this point, if C has genus at least two, we could add the pull back of a Kähler form in the canonical class of C, and get a symplectic structure.

If $C = \mathbb{P}^1$, then there is some difficulty in showing that the sum of the two closed 2- forms is a symplectic form, since we lift the negative of a Kähler form.

Adding the multiple of a sufficiently positive form on the base, one should get natural symplectic structures (with fixed cohomology class). Notice that Gompf ([**Gompf1**, Theorem 1.3]) proved that a Lefschetz fibration carries a natural symplectic form, but only up to deformations which do not keep the cohomology class fixed.

The general question is then: under the above assumptions does the equivalence of Lefschetz fibrations over $\mathbb{P}^1$ imply symplectomorphism of the respective canonical symplectic structures?

Let's observe here that, using a standard technique introduced in [**A-K**] in order to show symplectic equivalence of branched covers, it would suffice in our particular case to be able to mimic the calculation done for ψ in the mapping class group and obtain a similar result for a certain involution ι on the Riemann sphere (of which ψ is a lift).

Thus in our case it seems more convenient to try to show that the involution ι on $\mathbb{P}^1$ is a product of the diffeomorphisms appearing in the perturbed braid monodromy factorization. Whether this is possible is not completely clear to the authors at this stage.

The difficulty for such an attempt is provided by the existence of a large kernel for the 'lifting' homomorphism from the $\mathcal{S}_4$-coloured subgroup of the braid group to the mapping class group.

7. Appendix

In this appendix we give another and direct proof of the main corollary of Proposition 4.6, asserting that the Dehn twists with respect to the curves α_i, β_i, γ_i, δ_i and the curve σ appear in some monodromy factorization of the fibration $M_1 \to \Delta_0$.

We regenerate the equations of F and G in a few steps and choose a suitable set of paths from the base point $x = 1$ to the critical values in Δ_0. Eventually we get smooth curves F and G which intersect transversally and have some vertical tangents (parallel to the y-axis) for distinct values of x. To each noncritical value of x there correspond $2b$ branches of F and $2b$ branches of G as functions $y(x)$.

When x approaches a critical value x_i two branches approach the same value. A short arc between them is called *a vanishing arc*. If x_i corresponds to a vertical tangent then the vanishing arc is covered by two *vanishing cycles* on S, because S is a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$, and if x_i corresponds to the intersection of F and G then the vanishing arc is covered by one vanishing cycle. Recall the elements $\rho_i = L_i \theta_i L_i^{-1}$ of the geometric quasi-basis.

We move back along the path L_i to the base point $x = 1$ and the vanishing arc deforms along L_i in such a way that it always stays in a line $x = \text{const.}$ and never meets the points of F or G except at its end points.

This determines the final position of the vanishing arc for $x = 1$ up to isotopy which fixes the branch points. The vanishing arc is covered by one or two vanishing cycles on C - the fiber of S over $x = 1$. To the loop ρ_i corresponds a factor in the monodromy factorization of $S|_{\Delta_0}$ equal to the Dehn twist with respect to the vanishing cycle in the case of an intersection point of F and G and to the product of twists with respect to two disjoint vanishing cycles in the case of a point with vertical tangent. In the last case the singular fiber over x_i splits into two singular fibers with one double point each after a symplectic morsification of the fibration and the critical value x_i splits into two nearby critical values. We replace ρ_i by two loops.

To each of them corresponds a factor of the monodromy factorization consisting of one Dehn twist. The twists commute so their order and the order of the two loops is not important.

We shall choose arcs L_i which connect the base point $x = 1$ with the critical values. To get the corresponding path ρ_i we cut off a very short final piece of L_i and we replace it by a small circle around the critical point.

At the beginning the critical values are real and satisfy $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_6 < 1$. We begin with an arc L_0 which starts at $x = 1$, continues along the real axis to the left towards the greatest critical value ζ_6, turns around ζ_6 clockwise along the half circle of small radius ϵ_0 and continues towards the next critical value ζ_5.

When we move along L_0 the branches of F and G are constant, equal to $y_i(x) = B_i$ and $y_i'(x) = -B_i$ respectively. Therefore after a sufficiently small change in F and G the branches are still almost constant along the path L_0. We now regenerate the intersection of the fibre $x = \zeta_6$ with the horizontal components of F.

The first change produces $F : (x - \zeta_1)((x - \zeta_6)(y - B_1) - \epsilon_1) \prod_{i=1}^{2a-2}(x - A_i) \prod_{j>1}(y - B_j) = 0$, $0 < \epsilon_1 \ll \epsilon_0$.

We get a conic $(x - \zeta_6)(y - B_1) = \epsilon_1$ whose real part is a hyperbola with a negative slope, which intersects the horizontal lines $y = B_j$ at

points (x_j, B_j) with $x_2 > x_3 > \cdots > x_{2b} > \zeta_6$ and intersects the lines $y = -B_j$ at points x'_j with $x'_1 < x'_2 < \cdots < x'_{2b} < \zeta_6$ (see Figure 9).
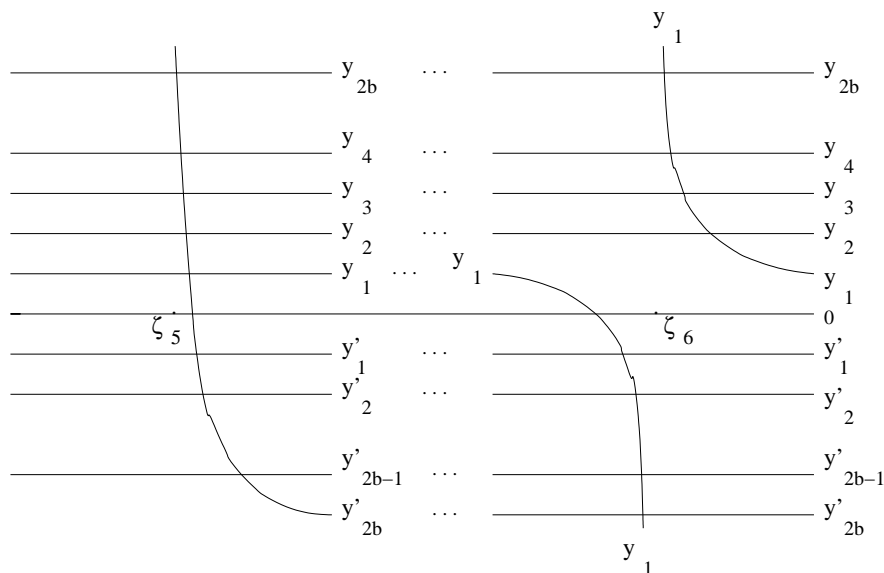

Figure 9. The real part of D_1 and D_2 after the first regeneration at ζ_5 and ζ_6.

Consider an arc L_j which starts at $x = 1$, moves to the left along the real axis, makes half a turn clockwise along a small circle around each consecutive critical value and continues along the real axis until it gets to x_j where it ends.

The branches of G are constant along these paths and the branches of F are constant except for $y_1(x)$, which starts at B_1, goes up along the real axis, makes half a turn clockwise around consecutive points B_i and ends at $B_j = y_j(x_j)$. The values of $y_1(x)$ trace a vanishing arc corresponding to L_j and isotopic to a simple arc which lies in the upper half-plane except for its end points at B_1 and B_j.

We now make a further regeneration of F getting

$F : (x-\zeta_1) \prod_{i=1}^{2a-2}(x-A_i)[((x-\zeta_6)(y-B_1)-\epsilon_1) \prod_{j>1}(y-B_j)+\epsilon_2] = 0$, $0 < \epsilon_2 \ll \epsilon_1$.

Each intersection point (x_j, B_j) splits into two vertical tangents, both real or both complex, lying near (x_j, B_j). We split L_j into two paths which end at the new critical points (and we split each of the new paths again after morsification).

The branches of F remain almost constant along each of these new paths except for $y_1(x)$ and except for $y_j(x)$ which changes a little near the end of the path and coincides with the value of $y_1(x)$ for x equal to the critical value.

To each of the two paths obtained from L_j corresponds the same, up to isotopy, vanishing arc described above. The arc is obtained from an interval $[B_{j-1}, B_j]$ by a sequence of half-twists.

It is covered by two vanishing cycles and the corresponding Dehn twists are equal to $\mu_j = \alpha_1 \alpha_2 \ldots \alpha_{j-2} \alpha_{j-1} \alpha_{j-2}^{-1} \ldots \alpha_2^{-1} \alpha_1^{-1}$ and $\nu_j = \gamma_1 \gamma_2 \ldots \gamma_{j-2} \gamma_{j-1} \gamma_{j-2}^{-1} \ldots \gamma_2^{-1} \gamma_1^{-1}$, for $j = 2, 3, \ldots, 2b$ where α_i and γ_i denote both the vanishing cycles on C and the corresponding Dehn twists. Each of these twists appears twice, consecutively in the monodromy factorization of $S|_{\Delta_0}$. We compose the twists from the right to the left.

It is easy to see that the product
$\mu_{2b}^2 \nu_{2b}^2 \mu_{2b-1}^2 \nu_{2b-1}^2 \ldots \mu_3^2 \nu_3^2 \mu_2^2 \nu_2^2$ is Hurwitz equivalent to
$\alpha_1 \alpha_2 \ldots \alpha_{2b-2} \alpha_{2b-1}^2 \alpha_{2b-2} \ldots \alpha_2 \alpha_1 \gamma_1 \gamma_2 \ldots \gamma_{2b-2} \gamma_{2b-1}^2 \gamma_{2b-2} \ldots \gamma_2 \gamma_1$.

In particular each of the cycles α_i and γ_i appears in the product.

Consider now an arc which starts as L_0, turns clockwise around ζ_6 along a half-circle of radius ϵ_0, comes back to the real axis and then moves right, up the real axis, in the direction of the smallest critical value x_1'. All values of y_i and y_i' are almost constant along this arc except for the last piece of it where y_1 moves to the left along the real axis and reaches y_1' at $-B_1$ for $x = x_1$ (see Figure 9). The corresponding vanishing arc is just the interval $[-B_1, B_1]$ covered by the cycle σ and the corresponding factor in the monodromy is σ.

We regenerate next the curve G in two steps, in a way similar to the regeneration of F. We let
$G : (x - \zeta_2)(x - \zeta_3)(x - \zeta_4) \prod_{i=1}^{2c-4}(x - C_i)[((x - \zeta_5)(y + B_{2b}) - \epsilon_3) \prod_{j=2}^{2b}(y + B_j) + \epsilon_4] = 0$
where $0 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2$.

We choose paths which start as L_0. All horizontal branches of F and G are almost constant along L_0. The path approaches ζ_5. After the regeneration we get critical values of x derived from ζ_5 and we continue as in the regeneration of F at ζ_6 getting new loops around new critical values.

The order of the branches of G is now from $-B_{2b}$ to $-B_1$ (see left part of Figure 9) so the corresponding monodromy factorization is Hurwitz equivalent to

$\beta_{2b-1} \beta_{2b-2} \ldots \beta_2 \beta_1^2 \beta_2 \ldots \beta_{2b-2} \beta_{2b-1} \delta_{2b-1} \delta_{2b-2} \ldots \delta_2 \delta_1^2 \delta_2 \ldots \delta_{2b-2} \delta_{2b-1}$.

Since the regeneration of G is much smaller than the previous regeneration of F it does not change the monodromy along the loops ρ_i. Subsequently, the final regeneration of F and G is much smaller so it will not change the monodromy along these new loops. The set of loops constructed above can be completed to a geometric basis of $\pi_1(\Delta_0 - \{critical\ values\}, 1)$.

It follows that the twists along all curves α_i, β_i, γ_i, δ_i and the curve σ appear in the monodromy factorization of $S|_{\Delta_0}$.

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