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VANISHING OF THE TOP CHERN CLASSES OF THE MODULI OF VECTOR BUNDLES

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Abstract

We prove the vanishing of the top Chern classes of the moduli of rank three stable vector bundles on a smooth Riemann surface. More precisely, the Chern class c_i for $i > 6g-5$ of the moduli spaces of rank three vector bundles of degree one and two on a genus g smooth Riemann surface all vanish. This generalizes the rank two case, conjectured by Newstead and Ramanan and proved by Gieseker.

0. Introduction

Let Y be a smooth nonsingular curve of genus $g \geq 2$ and let $M_{r,d}(Y)$ be the moduli space of stable vector bundles of rank r and degree d on Y. In case d and r are relatively prime, $M_{r,d}(Y)$ is a smooth projective variety of dimension $r^2(g-1)+1$. A classical conjecture of Newstead and Ramanan states that

$$c_i(M_{2,1}(Y)) = 0 \quad \text{for } i > 2(g-1); \tag{0.1}$$

i.e., the top $2g-1$ Chern classes vanish. The purpose of this paper is to generalize this vanishing result to higher rank cases by generalizing Gieseker's degeneration method.

In the rank 2 case, there are two proofs of (0.1) due to Gieseker [**4**] and Zagier [**16**]. Zagier's proof is combinatorial based on the precise knowledge of the cohomology ring of the moduli space: by the Grothendieck-Riemann-Roch theorem, Zagier found an expression for the total Chern class $c(M_{2,1}(Y))$ and then used Thaddeus's formula on intersection pairing to show the desired vanishing. Because the computation is extremely complicated even in the rank 2 case, it seems almost impossible to generalize this approach to higher rank cases.

A more geometric proof of the vanishing (0.1) was provided by Gieseker via induction on the genus g. Let $W \to C$ be a flat family of projective curves over a pointed smooth curve $0 \in C$ such that

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(1) W is nonsingular,
(2) the fibers W_s over $s \neq 0$ are smooth projective curves of genus g,
(3) the central fiber W_0 is an irreducible stable curve X_0 with one node as its only singular point.

Gieseker constructed a flat family of projective varieties $\mathbf{M}_{2,1}(\mathfrak{W}) \to C$ such that

(1) the total space $\mathbf{M}_{2,1}(\mathfrak{W})$ is nonsingular,
(2) the fibers $\mathbf{M}_{2,1}(\mathfrak{W}_s)$ over $s \neq 0$ are the moduli spaces $M_{2,1}(W_s)$ of stable bundles over W_s,
(3) the central fiber $\mathbf{M}_{2,1}(\mathfrak{W}_0)$ over 0 has only normal crossing singularities.

Recently, this construction was generalized to the higher rank case by Nagaraj and Seshadri in [**11**] by geometric invariant theory, and the central fiber $\mathbf{M}_{r,d}(\mathfrak{W}_0)$ of their construction parameterizes certain vector bundles on semistable models of X_0. In this paper, we will provide a different construction, using the technique developed in [**8**].

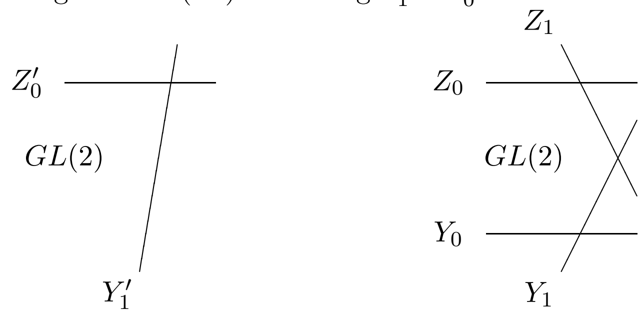
To prove the vanishing of Chern classes by induction on genus g, Gieseker relates the central fiber $\mathbf{M}_{2,1}(\mathfrak{W}_0)$ with the moduli space $M_{2,1}(X)$ where X is the normalization of the nodal curve $W_0 = X_0$. Let $\mathbf{M}^0$ be the normalization of $\mathbf{M}_{2,1}(\mathfrak{W}_0)$, which is a smooth projective variety. Its general points represent vector bundles on X_0 whose pull-back to X are stable bundles, and hence induces a rational map

$$\mathbf{M}^0 \dashrightarrow M_{2,1}(X). \tag{0.2}$$

Gieseker then proves that the indeterminacy locus of this rational map is precisely a projective bundle $\mathbb{P}E^+$ of a vector bundle E^+ over the product $B = Jac_0(X) \times Jac_1(X)$ of Jacobians, and the normal bundle to $\mathbb{P}E^+$ is the pull-back of a vector bundle $E^- \to B$ tensored with $\mathcal{O}_{\mathbb{P}E^+}(-1)$. This is a typical situation for flips: we blow up $\mathbf{M}^0$ along $\mathbb{P}E^+$ and then blow down along the $\mathbb{P}E^+$ direction in the exceptional divisor $\mathbb{P}E^+ \times_B \mathbb{P}E^-$. Let $\mathbf{M}^1$ be the result of this flip. Then the rational map becomes a morphism $\mathbf{M}^1 \to M_{2,1}(X)$, and is a fiber bundle with fiber $\overline{GL(2)}$ — the wonderful compactification of $GL(2)$ — that is constructed as follows. We first compactify $GL(2)$ by embedding it in

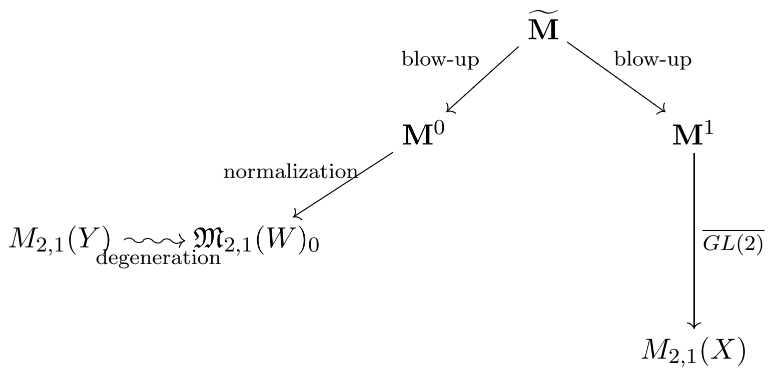
$$\mathbb{P}\left(\mathrm{End}(\mathbb{C}^2) \oplus \mathbb{C}\right).$$

Its complement consists of two divisors Z_0' (divisor at infinity) and Y_1' (zero locus of determinant). The wonderful compactification is by blowing it up along $0 \in \mathrm{End}(\mathbb{C}^2)$ and along $Y_1' \cap Z_0'$.



The wonderful compactification of $GL(r)$ for $r \geq 2$ is similarly defined by blowing up $\mathbb{P}(\mathrm{End}(\mathbb{C}^r) \oplus \mathbb{C})$ along smooth subvarieties $2(r-1)$ times and the complement of $GL(r)$ in it consists of $2r$ smooth normal crossing divisors. This was carefully studied by Kausz in [**7**]. In summary, Gieseker obtains the following diagram:

(0.3)



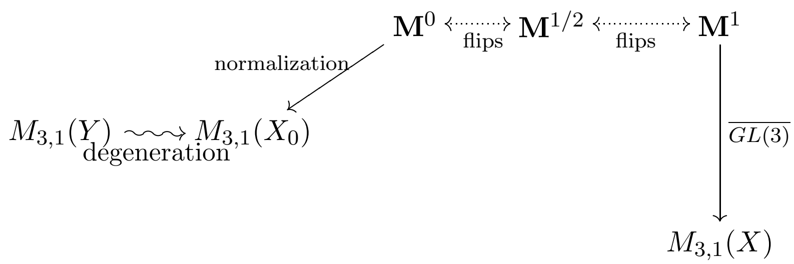
Afterwards, the proof of the vanishing result (0.1) is reduced to a series of very concrete Chern class computations.

To prove the vanishing of Chern classes for the higher rank case, we first construct a diagram similar to (0.3). In §1, we (re-)construct a degeneration $\mathbf{M}_{r,d}(\mathfrak{W})$ of Nagaraj and Seshadri, by using the stack of degeneration defined in [**8**]. We take $\mathbf{M}^0$ as the normalization of the central fiber of the family $\mathbf{M}_{r,d}(\mathfrak{W})$. Next, we define $\mathbf{M}^1$ as a fiber bundle over $M_{r,d}(X)$ whose fiber is $\overline{GL(r)}$ — the wonderful compactification of $GL(r)$. Explicitly, working with a universal bundle $\mathcal{U} \to M_{r,d}(X) \times X$, $\mathbf{M}^1$ is the blow-up of

$$\mathbb{P}\left(\mathrm{Hom}(\mathcal{U}|_{p_1}, \mathcal{U}|_{p_2}) \oplus \mathcal{O}\right)$$

along suitable smooth subvarieties exactly as in the construction of $\overline{GL(r)}$. The question is then how to relate $\mathbf{M}^0$ with $\mathbf{M}^1$. Our strategy is to construct a family of complex manifolds $\mathbf{M}^\alpha$ for $0 < \alpha < 1$ and study their variations as α moves from 1 to 0. We define a suitable stability condition for each α (Definition 1.1) and then show that the set of α-stable vector bundles on semistable models of X_0 admits the structure of a proper separated smooth algebraic space. In particular, $\mathbf{M}^\alpha$ is a compact complex manifold.

To prove the vanishing result of Chern classes, we need a very precise description of the variation of $\mathbf{M}^\alpha$. We achieved this for the case of rank 3. Since $M_{3,1}(Y) \cong M_{3,2}(Y)$ by the morphism $[E] \to [E^*]$ and tensoring a line bundle of degree 1, we only need to consider the case when $r = 3$ and $d = 1$. By the stability condition, the moduli spaces $\mathbf{M}^\alpha$ vary only at 1/3 and 2/3. We prove that $\mathbf{M}^{1/2}$ is obtained from $\mathbf{M}^1$ as the consequence of two flips and similarly $\mathbf{M}^0$ is the consequence of two flips from $\mathbf{M}^{1/2}$. The description is quite explicit and we have the following diagram.



Now it is a matter of explicit but very involved Chern class computations to verify the vanishing result by induction on genus g. The vanishing result we prove in the end is the following.

Theorem 0.1. *$c_i(M_{3,1}(Y)) = 0$ for $i > 6g - 5$.*

In other words, the top $3g - 3$ Chern classes vanish. It seems that c_{6g-5} should also vanish, but we haven't proved that. Notice that we also have $c_i(M_{3,2}(Y)) = 0$ for $i > 6g - 5$.

The paper is organized as follows. In Section 1, we will introduce and construct the moduli space of α-stable bundles on nodal curves; a special case of this construction is the Gieseker's degeneration for high rank cases. The next section is devoted to the study of the α-stable bundles and the generalized parabolic bundles on curves. The initial investigation of the variation of the moduli spaces is carried out in Section 3 and the detailed study of the flips is achieved in Section 4. The last section is about the Chern class calculation.

1. α-stable sheaves and Gieseker's degeration

In this section, we will introduce the notion of α-stable sheaves and prove their basic properties. We will then give an alternative construction of Gieseker's moduli of stable sheaves on nodal curves in high rank case. In the end, we will show that the normalization of such moduli spaces can be realized as the moduli spaces of α-stable bundles over marked nodal curves.

1.1. α-stable vector bundles. Let $g \geq 2$ be an integer and let X_0 be a reduced and irreducible curve of arithmetic genus g with exactly one node, $q \in X_0$. For $n \geq 0$, we denote by X_n the semistable model of X_0 that contains a chain of n-rational curves (i.e., $\mathbf{P}^1$). In this paper, we will fix such an X_0 once and for all. Let X be the normalization of X_0, with p_1 and $p_2 \in X$ the two liftings of the node of X_0 under the normalization morphism. For X_n, we denote by D the union of its rational curves and denote by $D_1, \ldots, D_n$ its n rational components. We order D_i so that $D_1 \cap X = p_1$, $D_i \cap D_{i+1} \neq \emptyset$ and $D_n \cap X = p_2$. We let $X^0 = X_0 - \{q\} = X - \{p_1, p_2\}$, which is an open subset of X_n. We define the based automorphisms of X_n to be

$$\mathfrak{Aut}_0(X_n) = \{\sigma : X_n \xrightarrow{\cong} X_n \mid \sigma|_{X^0} \equiv \mathrm{id}_{X^0}\} \cong (\mathbb{C}^\times)^n.$$

(Namely, they are automorphisms of X_n whose restrictions to $X^0 \subset X_n$ are the identity maps.)

Later, we need to study pairs $(X_n, q^\dagger)$, where $q^\dagger \in X_n$ are nodes of X_n. In this paper, we will call $(X_n, q^\dagger)$ based nodal curves, and denote them by $X_n^\dagger$ with $q^\dagger \in X_n$ implicitly understood. For $m \geq n$, we say $\pi : X_m \to X_n$ is a contraction if $\pi|_{X^0}$ is the identity and $\pi|_{D_k}$ is either an embedding or a constant map. A contraction of $X_m^\dagger \to X_n^\dagger$ is a contraction of the underlying spaces $X_m \to X_n$ that send the based node of $X_m^\dagger$ to the based node of $X_n^\dagger$.

We now fix a pair of positive integers $r \geq 2$ and χ. Let $X_n^\dagger = (X_n, q^\dagger)$ be a based nodal curve and let E be a rank r locally free sheaf of $\mathcal{O}_{X_n}$-modules with $\chi(E) = \chi$. We say E is admissible if the restriction $E|_{D_i}$ has no negative degree factor[1] for each i. (Here and later, for a closed subscheme $A \subset X_n$ we use $E|_A$ to mean $E \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_A$.) Next, we pick an n-tuple $\mathbf{d} = (d_i) \in \mathbb{Z}^{+n}$ and let ϵ be a sufficiently small positive rational. The pair $(\mathbf{d}, \epsilon)$ defines a $\mathbb{Q}$-polarization $\mathbf{d}(\epsilon)$ on X_n whose degree along the component $\overline{X^0}$ (resp. D_i) is $1 - \epsilon|\mathbf{d}|$ (resp. $d_i\epsilon$). (Here $|\mathbf{d}| = \sum d_i$.) Now let F be any subsheaf of E. We define the rank of $F \subset E$ at $q^\dagger \in X_n^\dagger$ to be

$$r^\dagger(F) = \dim \mathrm{Im}\{F|_{q^\dagger} \to E|_{q^\dagger}\}.$$

For real α we define the α-$\mathbf{d}$-slope (implicitly depending on the choice of ϵ) of $F \subset E$ to be

$$\mu_{\mathbf{d}}(F, \alpha) = (\chi(F) - \alpha r^\dagger(F))/\mathbf{rk}_{\mathbf{d}} F \in \mathbb{Q}. \tag{1.1}$$

[1] By Grothendieck's theorem, every vector bundle on $\mathbf{P}^1$ is a direct sum of line bundles. Each such line bundle is called a factor of this vector bundle.

Here the denominator is defined to be

$$\mathbf{rk_d}F = (1-\epsilon|\mathbf{d}|)\,\mathrm{rank}\,F|_{X^0} + \sum_{i=1}^{n} \epsilon d_i\,\mathrm{rank}\,F|_{D_i}.$$

We define the automorphism group $\mathfrak{Aut}_0(E)$ to be the group of pairs (σ, f) so that $\sigma \in \mathfrak{Aut}_0(X_n)$ and f is an isomorphism $E \cong \sigma^* E$.

Definition 1.1. Let E be a rank r locally free sheaf over $X_n^\dagger$. Let $\alpha \in [0,1)$ be any real number. We say E is α-$\mathbf{d}$-semistable (resp. weakly α-$\mathbf{d}$-stable) if for any proper subsheaf $F \subset E$ we have

$$\mu_{\mathbf{d}}(F,\alpha) \le \mu_{\mathbf{d}}(E,\alpha) \quad (\text{resp.} <)$$

for ϵ sufficiently small. We say E is α-$\mathbf{d}$-stable if E is weakly α-$\mathbf{d}$-stable and $\deg E|_{D_i} > 0$ for all i.

In case E is a vector bundle on X_n (without the marked node), we say E is $\mathbf{d}$-(semi)stable if the same condition holds with $\alpha = 0$.

We remark that when $\alpha = 0$ the α-$\mathbf{d}$-stability defined here coincides with the Simpson stability (compare also the stability used in [**5**]).

We now collect a few facts about α-$\mathbf{d}$-stable sheaves on $X_n^\dagger$. To avoid complications arising from strictly semistable sheaves, we will restrict ourselves to the case $(r,\chi) = 1$ and $\alpha \in [0,1) - \Lambda_r$:

(1.2)

$$\Lambda_r = \{\alpha \in [0,1) \mid \quad \alpha = \tfrac{r_0}{r_0 - r^\dagger}\big(\tfrac{\chi}{r} - \tfrac{\chi_0}{r_0}\big),\ \chi_0, r_0, r^\dagger \in \mathbb{Z},\ 0 < r_0 < r,\ 0 \le r^\dagger \le r,\ 2r_0 - r \le r^\dagger \le 2r_0\}.$$

Clearly, Λ_r is a discrete subset of $[0,1)$. When $(\chi, r) = 1$, $0 \notin \Lambda_r$.

Lemma 1.2. *Let $(r,\chi) = 1$ and $\chi > r$ be as before. Let $\alpha \in [0,1) - \Lambda_r$ be any real and let $\mathbf{d} \in \mathbb{Z}^{+n}$ be any weight. Then for any rank r α-$\mathbf{d}$-semistable sheaf E on $X_n^\dagger$ of Euler characteristic $\chi(E) = \chi$, we have*

(a) *The restriction $E|_{D_i}$ has no negative degree factors and there is no nontrivial section of E which vanishes on X.*
(b) *For any (partial) contraction $\pi : X_m^\dagger \to X_n^\dagger$ the pull back $\pi^* E$ is weakly α-$\mathbf{d}'$-stable for any weight $\mathbf{d}' \in \mathbb{Z}^{+m}$.*
(c) *Suppose E is α-$\mathbf{d}$-stable. Then $n \le r$ and $\mathfrak{Aut}_0(E) = \mathbb{C}^\times$.*

A similar statement holds for $\mathbf{d}$-(semi)stable sheaves E on X_n.

Proof. We first prove (a). Suppose $E|_{D_i}$ has a negative degree factor, say $\mathcal{O}_{D_i}(-t)$. Let F be the kernel of $E \to E|_{D_i} \to \mathcal{O}_{D_i}(-t)$. Then $\mathbf{rk_d}F < \mathbf{rk_d}E$, $r^\dagger(F) \le r$ and

$$\chi(F) = \chi(E) - \chi(\mathcal{O}_{D_i}(-t)) = \chi(E) + t - 1 \ge \chi(E).$$

Hence,

$$\mu_{\mathbf{d}}(F,\alpha) = \frac{\chi(F) - \alpha r^\dagger(F)}{\mathbf{rk_d}F} > \frac{\chi(E) - \alpha r}{r} = \mu_{\mathbf{d}}(E,\alpha).$$

This is a contradiction. Similarly, suppose there is a section $s \in H^0(E)$ so that its restriction to $X^0 \subset X_n$ is trivial. Let L be the subsheaf of E generated by this section. Then since $\mathbf{rk_d} L = c\epsilon$, $c > 0$ and $\chi(L) \geq 1$, $\mu_{\mathbf{d}}(L, \alpha) > (1-\alpha)/c\epsilon > \mu_{\mathbf{d}}(E, \alpha)$. This is a contradiction, which proves (a).

We now prove (b). Let F be any subsheaf of E. Since E is α-$\mathbf{d}$-semistable,

$$\mu_{\mathbf{d}}(F, \alpha) \leq \mu_{\mathbf{d}}(E, \alpha). \tag{1.3}$$

Let $r_0(F) = \operatorname{rank} F|_{X^0}$. If $r_0(F) = 0$, then $\mu_{\mathbf{d}}(F, \alpha) = c\epsilon^{-1}$, for some $c \in \mathbb{Q}$. Obviously $c > 0$ is impossible by (1.3) and $\chi > r$. In case $c = 0$, then $\chi(F) - \alpha r^\dagger(F) = 0$ and thus $\mu_{\mathbf{d}'}(F, \alpha) = 0$ for all $\mathbf{d}'$. Since $\chi(E) > r$, the strict inequality in (1.3) holds for all $\mathbf{d}'$. When $c < 0$, $\mu_{\mathbf{d}'}(F, \alpha) = c'\epsilon^{-1}$ for some $c' < 0$ as well. Hence the strict inequality in (1.3) holds for $\mathbf{d}'$ too.

We next consider the case $r > \operatorname{rank} E|_{X^0} > 0$. Then $\mu_{\mathbf{d}}(F, \alpha) - \mu_{\mathbf{d}}(E, \alpha)$ is a continuous function of ϵ whose value at $\epsilon = 0$ is

$$\frac{\chi(F) - \alpha r^\dagger(F)}{r_0(F)} - \frac{\chi}{r} + \alpha.$$

Because $\alpha \in [0, 1) - \Lambda_r$, this is never zero. Hence for sufficiently small ϵ,

$$\mu_{\mathbf{d}}(F, \alpha) \leq \mu_{\mathbf{d}}(E, \alpha) \Longleftrightarrow \mu_{\mathbf{d}'}(F, \alpha) < \mu_{\mathbf{d}'}(E, \alpha).$$

It remains to consider the case where $r_0(F) = r$. Obviously, we may consider only proper subsheaves F such that $F|_{X^0} \equiv E|_{X^0}$ and thus E/F is a nonzero sheaf of $\mathcal{O}_D$-modules. Because E/F is a quotient sheaf of $E|_D$ which is non-negative along each $D_i \subset D$, $\chi(E/F) \geq r - r^\dagger(F)$. Since each vector $v \in E|_{q^\dagger}$ that lies in a subspace complementary to $\operatorname{Im}\{F|_{q^\dagger} \to E|_{q^\dagger}\}$ extends to a section of E, $\chi(E) - \chi(F) > \alpha r - \alpha r^\dagger(F)$ and thus $\mu_{\mathbf{d}'}(F, \alpha) < \mu_{\mathbf{d}'}(E, \alpha)$ for any $\mathbf{d}'$. This proves that E is weakly $\mathbf{d}'$-stable for all $\mathbf{d}'$.

Now we prove that the pull-back of E to any X_m is weakly α-$\mathbf{d}'$-stable. We first consider the case $m = n + 1$. For simplicity we assume π is the contraction of the last rational component of X_{n+1}. We pick $\mathbf{d}'$ so that $d'_i = d_i$ for $i \leq n$ and $d'_{n+1} = 1$. Let F be a subsheaf of $\pi^* E$. Since $F|_{D_{n+1}}$ has no positive degree factor, $\pi_* F$ is torsion free and is a subsheaf of E. Further, since $R^1\pi_*(F)$ is a skyscraper sheaf, by Leray spectral sequence we have

$$\chi(F) = \chi(\pi_* F) - \chi(R^1 \pi_* F) \leq \chi(\pi_* F).$$

We now investigate the case $r_0(F) > 0$. (The other case can be treated as in the proof of (b), and will be omitted.) Because E is weakly α-$\mathbf{d}$-stable and $\alpha \notin \Lambda_r$, we must have

$$\frac{\chi(\pi_* F) - \alpha r^\dagger(\pi_* F)}{r_0(\pi_* F)} < \frac{\chi(E)}{r} - \alpha.$$

Combined with the above inequality and $r^\dagger(F) \geq r^\dagger(\pi_* F)$, we conclude

$$\mu_{\mathbf{d}'}(F, \alpha) < \mu_{\mathbf{d}'}(\pi^* E, \alpha).$$

This proves that $\pi^* E$ is weakly α-$\mathbf{d}'$-stable. The general case $m > n+1$ follows by induction. This completes the proof of (b).

Now assume E is α-$\mathbf{d}$-stable. Consider the vector space

$$V = \{s \in H^0(E|_D) \mid s(p_2) = 0\}.$$

We know that its dimension is no less than n by Riemann-Roch since degree$(E|_D) \geq n$. By part (b), the evaluation map

$$V \longrightarrow E|_{p_1}; \quad \text{via} \quad s \mapsto s(p_1) \in E|_{p_1}$$

is injective. Because $\dim E|_{p_1} = r$, we have $n \leq r$. This proves (c).

The proof of the second part of (d) is based on the notion of GPB, and will be proved in Section 2 (Corollary 2.7). q.e.d.

In the light of this lemma, the α-$\mathbf{d}$-stability is independent of the choice of $\mathbf{d}$ as long as $\alpha \notin \Lambda_r$. In the remainder of this paper, we will restrict ourselves to the case where the following is satisfied.

Basic Assumption 1.3. We assume $(r, \chi) = 1$, $\chi > r$ and $\alpha \in [0, 1) - \Lambda_r$

Henceforth, we can and will call α-$\mathbf{d}$-stable simply α-stable with some choice of $\mathbf{d}$ understood.

1.2. Gieseker's degeneration of moduli spaces.

In this subsection, we will give an alternative construction of Gieseker's degeneration of moduli of bundles in high rank case. The first such construction was obtained by Nagaraj and Seshadri [**11**].

Let $(r, \chi) = 1$ be as before and fix $\mathbf{d} = (1, \ldots, 1) \in \mathbb{Z}^n$ for any n. Let

$$\mathcal{V}_{r,\chi}(\mathfrak{X}) = \Big\{ E \mid E \text{ is } \mathbf{d}\text{-stable on } X_n \text{ for some } n, \quad \operatorname{rank} E = n \text{ and } \chi(E) = \chi \Big\} / \sim$$

Here two E and E' over X_n and X_m are isomorphic if there is a based isomorphism $\sigma : X_n \to X_m$ so that $\sigma^* E' \cong E$.

Lemma 1.4. *The set $\mathcal{V}_{r,\chi}(\mathfrak{X})$ is bounded.*

Proof. It follows immediately from the bound $n \leq r$ in Lemma 1.2. q.e.d.

Let $0 \in C$ be a pointed smooth curve and let $\pi : W \to C$ be a projective family of curves all of whose fibers W_s except W_0 are smooth and the central fiber W_0 is the nodal curve X_0 chosen before. Let $C^\circ = C - 0$ and $W^\circ = W - W_0$. For $s \in C^\circ$ we let $\mathfrak{M}_{r,\chi}(W_s)$ be the moduli space of rank r and Euler characteristic χ (namely $\chi(E) = \chi$) semistable vector bundles on W_s. We then let $\mathfrak{M}_{r,\chi}(W^\circ/C^\circ)$ be the associated relative moduli space; namely for $s \in C^\circ$ we have

$$\mathfrak{M}_{r,\chi}(W^\circ/C^\circ) \times_{C^\circ} s = \mathfrak{M}_{r,\chi}(W_s).$$

The goal of this section is to construct the degeneration of the family $\mathfrak{M}_{r,\chi}(W^\circ/C^\circ)$ by filling the central fiber of the family.

Our construction is aided by the construction of an Artin stack $\mathfrak{W}$ parameterizing all semi-stable models of W/C. Without loss of generality, we can assume $C \subset \mathbf{A}^1$ is a Zariski open subset with $0 \in C$ the origin of $\mathbf{A}^1$. Let $W[0] = W$ and let $W[1]$ be a small resolution of

$$W[0] \times_{\mathbf{A}^1} \mathbf{A}^2, \quad \text{where } \mathbf{A}^2 \to \mathbf{A}^1 \text{ is via } (t_1, t_2) \mapsto t_1 t_2.$$

The small resolution is chosen so that the fiber of $W[1]$ over $0 \in \mathbf{A}^2$ is X_1, and the fibers of $W[1]$ over the first (second) coordinate line $\mathbf{A}^1 \subset \mathbf{A}^2$ are a smoothing of the first (resp. second) node of X_1. Next $W[2]$ is constructed as a small resolution of

$$W[1] \times_{\mathbf{A}^2} \mathbf{A}^3, \quad \text{where } \mathbf{A}^3 \to \mathbf{A}^2 \text{ is via } (t_1, t_2, t_3) \mapsto (t_1, t_2 t_3).$$

The small resolution is chosen so that the fiber of $W[2]$ over $0 \in \mathbf{A}^3$ is X_2 and the fibers of $W[2]$ over the i-th coordinate line $\mathbf{A}^1 \subset \mathbf{A}^3$ are a smoothing of the i-th node of X_2. The degeneration $W[n]$ is defined inductively. For the details of this construction please see [**8**, §1].

Let $C[n]$ be $C \times_{\mathbf{A}^1} \mathbf{A}^{n+1}$. Then $W[n]$ is a projective family of curves over $C[n]$ whose fibers are isomorphic to one of $X_0, \ldots, X_n$. As shown in [**8**], there is a canonical $G[n] \equiv (\mathbb{C}^\times)^n$ action on $W[n]$ defined as follows. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a general element in $G[n]$. Then the $G[n]$ action on $\mathbf{A}^{n+1}$ is

$$(t_1, \ldots, t_{n+1})^\sigma = (\sigma_1 t_1, \sigma_1^{-1}\sigma_2 t_2, \ldots, \sigma_{n-1}^{-1}\sigma_n t_n, \sigma_n^{-1} t_{n+1}).$$

The $G[n]$ action on $W[n]$ is the unique lifting of the trivial action on $W[0]$ and the above action on $\mathbf{A}^{n+1}$. Consequently, we can view $W[n]/G[n]$ as an Artin stack. It is easy to see that $W[n]/G[n]$ is an open substack $W[n+1]/G[n+1]$ via the embedding $W[n]/\mathbf{A}^n \subset W[n+1]/\mathbf{A}^{n+1}$ with $\mathbf{A}^n \subset \mathbf{A}^{n+1}$ the embedding via $(t_i) \mapsto (t_i, 1)$.

We define the groupoid $\mathfrak{W}$ to be the category of pairs $(W_S/S, \pi)$, where S are C-schemes, W_S are families of projective curves over S and $\pi : W_S \to W$ are C-projections, such that there are open coverings U_α of S and $\rho_\alpha : U_\alpha \to C[n_\alpha]$ so that

$$W_S \times_S U_\alpha \cong W[n_\alpha] \times_{C[n_\alpha]} U_\alpha,$$

compatible with the projection $W_S \to W$. Two families W_S and W'_S are isomorphic if there is an S-isomorphism $f: W_S \to W'_S$ compatible to the tautological projections $W_S \to W$ and $W'_S \to W$. By our construction, $\mathfrak{W}$ is indeed a stack.

We next define the family of stable sheaves over $\mathfrak{W}$. Let S/C be any scheme over C. An S-family of locally free sheaves over $\mathfrak{W}/C$ consists of a member W_S in $\mathfrak{W}$ over S and a flat family of locally free sheaves E over W_S/S. We say E is admissible (resp. (semi)stable) if for each closed $s \in S$ the restriction of E to the fiber $W_s = W_S \times_S s$ is admissible (resp. (semi)stable). Let W_S and W'_S be two families in $\mathfrak{W}(S)$ and let E and E' be two families of sheaves over W_S and W'_S. We say $E \sim E'$ if there is an isomorphism $f: W_S \to W'_S$ in $\mathfrak{W}(S)$ and a line bundle L on S so that $f^*E' \cong E \otimes \mathrm{pr}_S^* L$.

We now define the groupoid of our moduli problem. For any C-scheme S, let $\mathfrak{F}_{r,\chi}(S)$ be the set of equivalence classes of all pairs (E, W_S), where W_S are members in $\mathfrak{W}(S)$ and E are flat S-families of rank r Euler characteristic χ stable vector bundles on W_S.

We continue to assume the basic assumption 1.3.

Proposition 1.5. *The functor $\mathfrak{F}_{r,\chi}(\mathfrak{W})$ is represented by an Artin stack $\mathfrak{M}_{r,\chi}(\mathfrak{W})$.*

Proof. The proof is straightforward and will be omitted. q.e.d.

Since all stable sheaves have automorphism groups isomorphic to $\mathbb{C}^\times$, the coarse moduli space $\mathbf{M}_{r,\chi}(\mathfrak{W})$ of $\mathfrak{M}_{r,\chi}(\mathfrak{W})$ exists as an algebraic space.

Theorem 1.6. *The coarse moduli space $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is separated and proper over C. Further, it is smooth and its central fiber (over $0 \in C$) has normal crossing singularities.*

We divide the proof into several lemmas.

We let $\mathfrak{F}_{r,\chi}(W[r])$ be the functor that associates to each $C[r]$-scheme $S/C[r]$ the set of equivalence classes[2] of stable sheaves E over $W[r]\times_{C[r]} S$ of rank r and Euler characteristic χ. Since $(r,\chi)=1$, the stability is independent of the choice of the polarizations $\mathbf{d}(\epsilon)$. By [**13**], $\mathfrak{F}_{r,\chi}(W[r])$ is represented by an Artin stack $\mathfrak{M}_{r,\chi}(W[r])$ and its coarse moduli space $\mathbf{M}_{r,\chi}(W[r])$ is quasi-projective over $C[r]$.

Lemma 1.7. *The moduli stack $\mathfrak{M}_{r,\chi}(W[r])$ and the moduli space $\mathbf{M}_{r,\chi}(W[r])$ are smooth over $C[r]$.*

Proof. This is a simple consequence of the deformation theory of sheaves on $W[r]$. Let $[E] \in \mathfrak{M}_{r,\chi}(W[r])$ be a closed point, represented by the sheaf E, as a sheaf of $\mathcal{O}_{W[r]}$-modules. Let Z be the support

[2]The equivalence relation for sheaves over $W[r]$ is the usual equivalence relation. Namely $E \sim E'$ if $E \cong E' \otimes \mathrm{pr}_S^* L$.

of E, which is a fiber of $W[r]$ over $\xi \in C[r]$. We denote by ι the inclusion $Z \to W[r]$. It follows from the deformation theory of sheaves that the first order deformation of E is given by the extension group $\mathrm{Ext}^1_{W[r]}(E, E)$ which fits into the exact sequence

$$0 \longrightarrow H^1(Z, \mathcal{E}nd(\iota^* E)) \longrightarrow \mathrm{Ext}^1_{W[r]}(E, E) \longrightarrow H^0(Z, \mathcal{E}nd(\iota^* E)) \otimes T_\xi C[r] \longrightarrow 0.$$

Since E is stable, $H^0(Z, \mathcal{E}nd(\iota^* E)) \equiv \mathbb{C}$. Further, it is a direct check that the homomorphism of the tangent spaces

$$T_{[E]}\mathfrak{M}_{r,\chi}(W[r]) \longrightarrow T_\xi C[r] \tag{1.4}$$

is the next to the last arrow in the above exact sequence.

To show that $\mathfrak{M}_{r,\chi}(W[r])$ is smooth, we need to show that there is no obstruction to deforming E. Since the support Z is a closed fiber of $W[r] \to C[r]$ and since E is a locally free sheaf of $\mathcal{O}_Z$-modules, the obstruction to deforming E lies in

$$H^2(Z, \mathcal{E}nd(\iota^* E)) \otimes T_\xi C[r] = 0.$$

This shows that $\mathfrak{M}_{r,\chi}(W[r])$ is smooth. Since (1.4) is surjective, $\mathfrak{M}_{r,\chi}(W[r])$ is smooth over $C[r]$. Finally, since the automorphism group of each $[E] \in \mathfrak{M}_{r,\chi}(W[r])$ is isomorphic to $\mathbb{C}^\times$, the coarse moduli space $\mathbf{M}_{r,\chi}(W[r])$ is also smooth over $C[r]$. This proves the lemma. q.e.d.

Corollary 1.8. *The coarse moduli space* $\mathbf{M}_{r,\chi}(\mathfrak{W})$ *is smooth and is flat over* C*.*

Proof. Clearly the $G[r]$-action on $W[r]$ naturally lifts to an action on $\mathfrak{M}_{r,\chi}(W[r])$ and $\mathbf{M}_{r,\chi}(W[r])$. By Lemma 1.2, the stabilizers of the $G[r]$-action at all points of $\mathbf{M}_{r,\chi}(W[r])$ are trivial. Hence the quotient $\mathbf{M}_{r,\chi}(W[r])/G[r]$ is an algebraic space. Further, because $\mathbf{M}_{r,\chi}(W[r])$ is smooth, $\mathbf{M}_{r,\chi}(W[r])/G[r]$ is also smooth.

Now let

$$\Phi : \mathbf{M}_{r,\chi}(W[r])/G[r] \longrightarrow \mathbf{M}_{r,\chi}(\mathfrak{W})$$

be the induced morphism. To prove that $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is smooth it suffices to show that Φ is surjective and is étale. Since $\mathbf{M}_{r,\chi}(W[r])$ is the coarse moduli space of the stack $\mathfrak{M}_{r,\chi}(W[r])$, its quotient $\mathbf{M}_{r,\chi}(W[r])/G[r]$ is the coarse moduli space of the stack $\mathfrak{M}_{r,\chi}(W[r])/G[r]$. Because the nature morphism

$$\mathfrak{M}_{r,\chi}(W[r])/G[r] \longrightarrow \mathfrak{M}_{r,\chi}(\mathfrak{W})$$

is étale, the morphism Φ is also étale. Further, Φ is surjective because of Lemma 1.2. This proves that $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is smooth. Finally, since $\mathbf{M}_{r,\chi}(W[r])$ is smooth over $C[r]$ and $C[r] \to C$ is flat, $\mathbf{M}_{r,\chi}(W[r]) \to C$ is flat. Hence $\mathbf{M}_{r,\chi}(W[r])/G[r] \to C$ is flat and so is $\mathbf{M}_{r,\chi}(\mathfrak{W}) \to C$. q.e.d.

Lemma 1.9. *The algebraic space* $\mathbf{M}_{r,\chi}(\mathfrak{W})$ *is separated and proper over* C.

Proof. We first check that $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is proper over C, using the valuation criterion. Let $\xi \in S$ be a closed point in a smooth curve over C. We let $S^\circ = S - \xi$ and let E° be a family of stable sheaves over W_{S° for a C-morphism $S^\circ \to C[n]$ for some n. We need to check that $S^\circ \to \mathbf{M}_{r,\chi}(\mathfrak{W})$ extends to $S \to \mathbf{M}_{r,\chi}(\mathfrak{W})$. Since $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is flat over C, it suffices to check those $S^\circ \to \mathbf{M}_{r,\chi}(\mathfrak{W})$ that are flat over C. In case ξ lies over C°, W_{S° extends to $W_S = W \times_C S$ with smooth special fiber W_ξ. Hence there is an extension of E° to a family of stable sheaves over W_S. We now assume ξ lies over $0 \in C$. Since S° is flat over C, $S \to C$ does not factor through $0 \in C$. Without loss of generality we can assume $S^\circ \to C$ factor through $C^\circ \subset C$. Then E° is a family of stable sheaves on $W \times_C S^\circ$. We consider $W_S = W \times_C S$, which possibly has singularity along the node of W_ξ. We let $\tilde{W}_S$ be the canonical desingularization of W_S. The central fiber $\tilde{W}_\xi$ (of $\tilde{W}_S$ over ξ) is isomorphic to X_m for some m. We next fix a polarization $\mathbf{d}(\epsilon)$ on $\tilde{W}_S$ so that its degrees along the irreducible components of $\tilde{W}_\xi$ are $1 - m\epsilon, \epsilon, \ldots, \epsilon$. Then by [**13**], there is an extension of E° to an $\mathcal{O}_S$-flat sheaf of $\mathcal{O}_{\tilde{W}_S}$-modules $\tilde{E}$ so that $\tilde{E}|_{\tilde{W}_\xi}$ is semistable with respect to a polarization $\mathbf{d}(\epsilon)$. Therefore $\tilde{E}$ is locally free since $\tilde{W}_S$ is smooth, $\tilde{E}$ is $\mathcal{O}_S$-flat and $\tilde{E}|_{\tilde{W}_\xi}$ has no torsion elements supported at points. By Lemma 1.2, for any rational curve $D_i \subset \tilde{W}_\xi$ the restriction $\tilde{E}|_{D_i}$ is either trivial ($\cong \mathcal{O}_{D_i}^{\oplus r}$) or is admissible along D_i. We let $\bar{W}_S$ be the contraction of $\tilde{W}_S$ along those D_i so that $\tilde{E}|_{D_i}$ is trivial. Let $\pi : \tilde{W}_S \to \bar{W}_S$ be the contraction morphism and let $\bar{E} = \pi_* \tilde{E}$. It is direct to check that $\bar{E}$ is locally free and its restriction to W_ξ is admissible and stable.

It remains to show that the family $\bar{W}_S$ can be derived from a C-morphism $S \to C[r]$. Let $t \in \Gamma(\mathcal{O}_{\mathbf{A}^1})$ be the standard coordinate function. Since the exceptional divisor of $\tilde{W}_S \to W_S$ has m irreducible components, $\rho^* t \in \mathfrak{m}_\xi^{m+1} - \mathfrak{m}_\xi^{m+2}$, where $\rho : S \to \mathbf{A}^1$ is the composite $S \to C \to \mathbf{A}^1$. Hence possibly after an étale base change, we can assume $\rho^*(t) = \tilde{t}_0 \cdots \tilde{t}_m$ for $\tilde{t}_i \in \Gamma(\mathfrak{m}_\xi)$. Let $S \to C[m] = C \times_{\mathbf{A}^1} \mathbf{A}^{m+1}$ be induced by $S \to C$ and $(\tilde{t}_0, \ldots, \tilde{t}_m) : S \to \mathbf{A}^{m+1}$. Then one checks directly that the fiber product $W[m] \times_{C[m]} S$ is isomorphic to $\tilde{W}_S$. As to $\bar{W}_S$, we consider the morphism $S \to C[m']$ defined as follows. Let $\tilde{\zeta}_0, \ldots, \tilde{\zeta}_m$ be the nodes of $\tilde{W}_\xi$ and let $\bar{\zeta}_0, \ldots, \bar{\zeta}_{m'}$ be the nodes of $\bar{W}_\xi$, ordered according to our convention. Then the restriction of the contraction $\pi_\xi : \tilde{W}_\xi \to \bar{W}_\xi$ induces a surjective map

$$\phi : \{\tilde{\zeta}_0, \ldots, \tilde{\zeta}_m\} \longrightarrow \{\bar{\zeta}_0, \ldots, \bar{\zeta}_{m'}\}.$$

We then define $\bar{t}_k = \prod\{\tilde{t}_i \mid \phi(\tilde{\zeta}_i) = \bar{\zeta}_k\}$. It is direct to check that $W[m'] \times_{C[m']} S$ is isomorphic to $\bar{W}_S$. Finally, we have $m' \le r$ by Lemma 1.2. Hence $W[m] \times_{C[m]} S$ can be realized as a product $W[r] \times_{C[r]} S$ for some $S \to C[r]$. This verifies the valuation criterion for properness.

We next show that $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is separated, using the valuation criterion. Let $\xi \in S$ be a closed point in a smooth curve over C, let W_{S° be a family associated to a C-morphism $S^\circ \to C[r]$ and let E° be a family of stable sheaves on W_{S°, as before. To verify the separatedness, we need to show that there is at most one extension of E° to families of stable sheaves over S. Suppose there are two extensions W'_S and W''_S of W_{S° and two extensions E' on W'_S and E'' on W''_S of E°. We need to show that there is a based isomorphism $\sigma : W'_S \to W''_S$ and an isomorphism $E' \cong \sigma^* E''$. Clearly, if ξ lies over a point in C°, this follows from the fact that the moduli of stable sheaves on smooth curves are separated. We now assume ξ lies over $0 \in C$. Again, because $\mathbf{M}_{r,\chi}(\mathfrak{W})$ is flat over C, we only need to check those so that S° is flat over C. As before, we let $W_S = W \times_C S$ and let $\tilde{W}_S$ be the canonical desingularization of W_S. Because both W'_S and W''_S are fiber products $W[n'] \times_{C[n']} S$ and $W[n''] \times_{C[n'']} S$ for some C-morphisms $S \to C[n']$ and $S \to C[n'']$, the canonical desingularizations of both W'_S and W''_S are isomorphic to $\tilde{W}_S$. Let

$$\pi' : \tilde{W}_S \to W'_S \quad \text{and} \quad \pi'' : \tilde{W}_S \to W''_S$$

be the projections. Then by Lemma 1.2, both $\pi'^* E'$ and $\pi''^* E''$ are families of locally free stable vector bundles, extending E°. Hence they are isomorphic. Further, we observe that a rational curve D is contracted by π' if and only if $\pi'^* E'|_D$ is trivial. Because $\pi'^* E' \cong \pi''^* E''$, both π' and π'' contract the same set of rational curves in $\tilde{W}_S$. Hence $W'_S \cong W''_S$ and under this isomorphism

$$E' \equiv \pi'_* \pi'^* E' \cong \pi''_* \pi''^* E'' \equiv E''.$$

This proves the separatedness. q.e.d.

We now complete the proof of the theorem.

Proof. It remains to show that the central fiber of $\mathbf{M}_{r,\chi}(\mathfrak{W})/C$ has normal crossing singularities. We consider the following Cartesian square

$$\begin{array}{ccc} \mathbf{M}_{r,\chi}(W[r]) & \longrightarrow & \mathbf{M}_{r,\chi}(W[r])/G[r] \\ \downarrow & & \downarrow \\ C[r] & \longrightarrow & C = C[r]/G[r] \end{array}$$

We know that the upper and left arrows are smooth morphisms while fibers of $C[r] \to C$ are smooth except the central fiber, which has normal crossing singularities. Thus the central fiber of the right arrow has normal crossing singularities. Finally, since $\mathbf{M}_{r,\chi}(W[r])/G[r] \to \mathbf{M}_{r,\chi}(\mathfrak{W})$

is étale and the arrow commutes with the two projections to C, the fibers of $\mathbf{M}_{r,\chi}(\mathfrak{W})$ over C are smooth except the central fiber which has normal crossing singularities. q.e.d.

We conclude this subsection by remarking that the set of closed points of $\mathbf{M}_{r,\chi}(\mathfrak{W})$ over $0 \in C$ is exactly the set $\mathcal{V}_{r,\chi}(\mathfrak{X})$ defined in the beginning of this subsection. This set is exactly the set used by Gieseker and Nagaraj - Seshadri in their construction degeneration of moduli spaces [**4**, **11**].

1.3. Normalization of the central fiber. We close this section by constructing the normalization of the central fiber

$$\mathbf{M}_{r,\chi}(\mathfrak{W}_0) = \mathbf{M}_{r,\chi}(\mathfrak{W}) \times_C 0.$$

Let $[E] \in \mathfrak{M}_{r,\chi}(\mathfrak{W}_0)$ be any closed point associated to a stable vector bundle E on X_n. We consider the coarse moduli space $\mathbf{M}_{r,\chi}(W[n])$ of stable sheaves on $W[n]/C[n]$. As we argued before, the induced morphism $\mathbf{M}_{r,\chi}(W[n])/G[n] \to \mathbf{M}_{r,\chi}(\mathfrak{W})$ is étale and its image contains $[E]$. For the same reason,

$$\mathbf{M}_{r,\chi}(W[n]) \times_{C[n]} (C[n] \times_C 0)/G[n] \longrightarrow \mathbf{M}_{r,\chi}(\mathfrak{W}_0)$$

is étale. Now let $\mathbf{H}^k \subset \mathbf{A}^{n+1}$ be the coordinate hyperplane transversal to the k-th coordinate axis (we agree that the coordinate axes of $\mathbf{A}^{n+1}$ are indexed from 0 to n) and let $\mathbf{H}^k \to C[n]$ be the induced immersion. Then

$$\coprod_{k=0}^{n} \mathbf{H}^k \longrightarrow C[n] \times_C 0$$

is the normalization morphism and the induced morphism

$$\coprod_{k=0}^{n} (\mathbf{M}_{r,\chi}(W[n]) \times_{C[n]} \mathbf{H}^k)/G[n] \longrightarrow \mathbf{M}_{r,\chi}(W[n]) \times_{C[n]} (C[n] \times_C 0)$$

is the normalization morphism. It follows that the closed points of the normalization of $\mathbf{M}_{r,\chi}(\mathfrak{W}_0)$ consist of the equivalence classes of triples $(E, X_n, q^\dagger)$ for some n where E are stable vector bundles over X_n and $q^\dagger$ are nodes of X_n.

The normalization of $\mathbf{M}_{r,\chi}(\mathfrak{W}_0)$ is indeed itself a moduli space. Let $\mathfrak{X}^\dagger$ be the Artin stack (groupoid) of pointed semistable models of X_0. Namely, $\mathfrak{X}^\dagger(S)$ consists of pairs $(W_S, q^\dagger)$ where W_S are members in $\mathfrak{W}(S)$ with S understood as C-schemes via the trivial morphisms $S \to 0$ with $0 \in C$, and $q^\dagger$ are sections of nodes of the fibers of W_S/S. An isomorphism between two families $(W_S, q^\dagger)$ and $(W'_S, q'^\dagger)$ is an isomorphism $\sigma : W_S \to W'_S$ in $\mathfrak{W}(S)$ that preserves the sections $q^\dagger$ and $q'^\dagger$. We denote the pair $(W_S, q^\dagger)$ by $W_S^\dagger$. We define the moduli groupoid $\mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ to be the category of all pairs $(E, W_S^\dagger)$, where $W_S \in \mathfrak{X}^\dagger(S)$ and

E is a family of α-stable sheaves on $W_S^\dagger$ whose members (namely the restriction to closed fibers of W_S/S) lie in the set $\mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ of isomorphism classes of α-stable rank r locally free sheaves E on $X_n^\dagger$ for some n with $\chi(E)=\chi$. Two families $(E,W_S^\dagger)$ and $(E',W_S'^\dagger)$ are equivalent if there is an isomorphism $\sigma: W_S^\dagger \to W_S'^\dagger$ in $\mathfrak{W}(S)$ and a line bundle L on S so that $f^*E' \cong E\otimes \mathrm{pr}_S^* L$. Following the proof of Theorem 1.5, one easily shows that the functor $\mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ is represented by a smooth Artin stack. Further, because of Corollary 2.7 its coarse moduli space $\mathbf{M}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ is a smooth algebraic space. Again, similar to the proof of Lemma 1.9, the coarse moduli space is proper and separated. Finally, the previous argument shows that $\mathbf{M}^0_{r,\chi}(\mathfrak{X}^\dagger)$ is a normalization of $\mathbf{M}_{r,\chi}(\mathfrak{W}_0)$. We summarize this in the following proposition.

Proposition 1.10. *The groupoid $\mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ forms a smooth Artin stack $\mathfrak{M}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ and its coarse moduli space $\mathbf{M}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ is a proper smooth and separated algebraic space. Further, the canonical morphism $\mathbf{M}^0_{r,\chi}(\mathfrak{X}^\dagger)\to \mathbf{M}_{r,\chi}(\mathfrak{W}_0)$ induced by forgetting the marked section of nodes is the normalization morphism.*

In case $r=2$, the moduli space $\mathbf{M}_{2,\chi}(\mathfrak{X}^\dagger)$ is exactly the normalization constructed in [**4**].

2. α-stable bundles and generalized parabolic bundles

In this section, we will first relate α-stable sheaves on $X^\dagger$ to Generalized-Parabolic-Bundle (in short GPB) on $X^+=(X,p_1+p_2)$. We will then show that the moduli of α-stable sheaves on $X^\dagger$ is a blow-up of the moduli of α-stable GPB on X^+. This will be used to study how $\mathbf{M}^0_{r,\chi}(\mathfrak{X}^\dagger)$ is related to $\mathbf{M}^{1^-}_{r,\chi}(\mathfrak{X}^\dagger)$ in the next section. We will continue to assume $(r,\chi)=1$, $\chi>r$ and $\alpha\notin\Lambda_r$ throughout this section.

2.1. GPB. We begin with the notion of GPB. Let X^+ be the pair (X,p_1+p_2). A rank r GPB on X^+ is a pair $V^G=(V,V^0)$ of a rank r vector bundle V on X and an r-dimensional subspace $V^0\subset V|_{p_1+p_2}$. In this paper, we will use the convention that for any sheaf F and closed $p\in X$ we denote by $F|_p$ the vector space $F\otimes \mathbf{k}(p)$ and denote by $V|_{p_1+p_2}$ the vector space $V|_{p_1}\oplus V|_{p_2}$. For any subsheaf $F\subset V$ we denote by $F|_{p_1+p_2}\cap V^0$ the subspace $\mathrm{Im}\{F|_{p_1+p_2}\to V|_{p_1+p_2}\}\cap V^0$ and define $r^+(F)=\dim F|_{p_1+p_2}\cap V^0$.

We begin with the investigation of locally free sheaves on a chain of rational curves. Let R be a chain of n $\mathbf{P}^1$'s coupled with two end points q_0 and q_n in the first and the last components of R. We order the n-rational curves into $R_1,\dots,R_n$ so that $R_i\cap R_{i+1}=\{q_i\}$ for $1\le i\le n-1$, $q_0\neq q_1\in R_1$ and $q_n\neq q_{n-1}\in R_n$. In the following we will call such R with q_0 and q_n understood an end-pointed chain of rational curves.

Now let F be any admissible locally free sheaf on R. Inductively, we define vector spaces $W_i \subset F|_{q_i}$ by $W_0 = \{0\}$ and

$$W_i = \{s(q_i) \mid s \in H^0(R_i, F), s(q_{i-1}) \in W_{i-1}\} \subset F|_{q_i}.$$

We call $T_\to = W_n$ the *transfer* of $0 \in F|_{q_0}$ along R. Note that

$$T_\to = \{s(q_n) \mid s(q_0) = 0 \text{ and } s \in H^0(R, F)\}. \tag{2.1}$$

If we reverse the order of R by putting $\tilde{R}_i = R_{n-i+1}$, we call the resulting transfer $T_\leftarrow \subset F|_{q_0}$ the *reverse transfer* of $0 \in F|_{q_n}$. Notice that we have a well-defined homomorphism $F_{q_n} \to F|_{q_0}/T_\leftarrow$ by assigning to each element $c \in F|_{q_n}$ the class of $[s(q_0)] \in F|_{q_0}/T_\leftarrow$ for some $s \in H^0(R, F)$ such that $s(q_n) = c$. The kernel of this homomorphism is precisely the transfer $T_\to$. Hence we have a canonical isomorphism

$$\xi : F|_{q_n}/T_\to \xrightarrow{\cong} F|_{q_0}/T_\leftarrow. \tag{2.2}$$

There is another way to see this isomorphism. Let $H^0(R, F^\vee) \otimes \mathcal{O}_R \to F^\vee$ and

$$\varphi : F \longrightarrow H^0(R, F^\vee)^\vee \otimes \mathcal{O}_R$$

be the canonical homomorphism. Then $\ker(\varphi(q_0)) = T_\leftarrow$, $\ker(\varphi(q_n)) = T_\to$ and the isomorphism (2.2) is induced by

$$F|_{q_0} \longrightarrow H^0(R, F^\vee)^\vee \longleftarrow F|_{q_n}.$$

Definition 2.1. We say a locally free sheaf F on R is regular if there are integers a_i so that to each $i \in [1, n]$,

$$F|_{R_i} = \mathcal{O}_{R_i}^{r-a_i} \oplus \mathcal{O}_{R_i}(1)^{a_i} \quad \text{and} \quad \dim W_i = \dim W_{i-1} + a_i.$$

Note that F is regular if and only if each restriction $F|_{R_i}$ has only degree 0 and 1 factors and

$$\{s \in H^0(R, F) \mid s(q_0) = s(q_n) = 0\} = 0. \tag{2.3}$$

Also, if F is regular then $\deg F = \dim T_\to = \dim T_\leftarrow$. We now prove a lemma concerning regular bundles on rational chains that will be useful later.

Lemma 2.2. *Let the notation be as before and let $\sigma_0 : F|_{q_0} \to F|_{q_0}/T_\leftarrow$ and $\sigma_n : F|_{q_n} \to F|_{q_n}/T_\to$ be the projections. Suppose $S_0 \subset F|_{q_0}$ and $S_n \subset F|_{q_n}$ are two subspaces so that $\xi(\sigma_n(S_n)) \subset \sigma_0(S_0)$. Then there is a subsheaf $\mathcal{F} \subset F$ so that $\mathrm{Im}\{\mathcal{F}|_{q_0} \to F|_{q_0}\} = S_0$, $\mathrm{Im}\{\mathcal{F}|_{q_n} \to F|_{q_n}\} = S_n$ and*

$$\chi(\mathcal{F}) \geq \dim S_n + \dim S_0 - \dim \sigma_0(S_0).$$

Proof. The proof is straightforward, is based on the following easy observation. Suppose $A \subset F|_{q_0}$ and $B \subset F|_{q_n}$ are one dimensional subspaces so that $\sigma_0(A) = \xi(\sigma_n(B)) \neq \{0\}$. Then there is a unique subsheaf $\mathcal{L} \cong \mathcal{O}_R \subset F$ so that $\mathrm{Im}\{\mathcal{L}|_{q_0} \to F|_{q_0}\} = A$ and $\mathrm{Im}\{\mathcal{L}|_{q_n} \to F|_{q_n}\} = B$.

We now construct the subsheaf $\mathcal{F}$. First, we write $S_n = A_1 \oplus A_2$ so that $A_2 = \ker\sigma_n \cap S_n$. Then by assumption, there is a subspace $A_1' \subset S_0$ so that A_1' is isomorphic to A_1 under $\sigma_n^{-1}\circ\xi^{-1}\circ\sigma_0$. We let $A_3 = \ker\sigma_0 \cap S_0$ and $A_4 \subset S_0$ be the compliment of $A_1' \oplus A_3$. Then by the observation just stated, there is a subsheaf $\mathcal{F}_1 \cong \mathcal{O}_R^{\oplus a_1} \subset F$, where $a_1 = \dim A_1$, so that $\mathcal{F}_1|_{q_0} \subset F|_{q_0}$ is A_1 and $\mathcal{F}_1|_{q_n} \subset F|_{q_n}$ is A_1'. We then pick $\mathcal{F}_4 \cong \mathcal{O}_R^{\oplus a_4} \subset F$, where $a_4 = \dim A_4$, so that $\mathcal{F}_4|_{q_0} \subset F|_{q_0}$ is A_4. We then define $\mathcal{F}_2 \subset F$ to be the subsheaf spanned by $\{s \in H^0(F(-q_n)) \mid s(q_0) \in A_2\}$. Similarly, we defined $\mathcal{F}_3 \subset F$ to be the subsheaf spanned by sections in $H^0(F(-q_0))$ whose values at q_n are in B_2. Clearly, $\chi(\mathcal{F}_i) = a_i$, where $a_i = \dim A_i$. Finally, let $\mathcal{F} \subset F$ be the image sheaf of

$$\varphi : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_4(-q_n) \to F.$$

By our construction, we certainly have $\mathrm{Im}\{\mathcal{F}|_{q_i} \to F|_{q_i}\} = S_i$ for $i = 0$ and n. As $\chi(\mathcal{F})$, it is equal to $\sum \chi(\mathcal{F}_i) - a_4 - \chi(\ker\varphi)$. Because the restriction of φ to q_0 and $q_n \in R$ are injective, the structures of $\mathcal{F}_i$ guarantee that $H^0(R, \ker\varphi) = 0$. Hence $\chi(\ker\varphi) \le 0$. This proves the inequality of $\chi(\mathcal{F})$. q.e.d.

Now we pick a pair of non-negative integers (n, m), and form the two end-pointed rational chains R and R' of length n and m, respectively. Let $q_0, q_n \in R$ and $q_0', q_m' \in R'$ be the respective end points with their nodes q_i and q_i', respectively. We then form the 2-pointed curve $X_{n,m}$ by gluing p_1 and p_2 in X with the $q_0 \in R$ and the $q_0' \in R'$, respectively, with q_n, q_m' its two marked points. Namely, $X_{n,m}$ has two tails of rational curves, the left tail $R \ni q_n$ and the right tail $R' \ni q_m'$. If we identify q_n and q_m', we obtain a pointed curve $X_{n+m}^\dagger$.

Now let E be a rank r vector bundle over $X_{n+m}^\dagger$ so that its restriction to the chain of rational curves $D \subset X_{n+m}$ is regular. Let

$$\rho : X_{n,m} \longrightarrow X_{n+m}^\dagger \quad \text{and} \quad \pi : X_{n,m} \longrightarrow X$$

be the tautological projections. First, $\tilde{E} \triangleq \rho^* E$ is a locally free sheaf on $X_{n,m}$ and E can be reconstructed from $\rho^* E$ and the isomorphism $\phi : \tilde{E}|_{q_n} \equiv E|_{q^\dagger} \equiv \tilde{E}|_{q_m'}$ via

$$0 \longrightarrow E \longrightarrow \rho_* \tilde{E} \longrightarrow (\tilde{E}|_{q_n} \oplus \tilde{E}|_{q_m'})/\Gamma_\phi \longrightarrow 0$$

where $\Gamma_\phi \subset \tilde{E}|_{q_n} \oplus \tilde{E}|_{q_m'}$ is the graph of ϕ. Here we view the last non-zero term in the sequence as a $\mathbf{k}(q^\dagger)$ vector space, which is naturally a sheaf of $\mathcal{O}_{X_{n+m}}$-modules. In this way the vector bundle E on $X_{n+m}^\dagger$ is equivalent to the GPB $(\tilde{E}, \Gamma_\phi)$ over $X_{n,m}$.

We now show how the GPB $(\tilde{E}, \Gamma_\phi)$ naturally associates to a GPB over X^+. First let $V = (\pi_* \tilde{E}^\vee)^\vee$. Since the restriction of $\tilde{E}$ to D is regular, $\rho_* \tilde{E}^\vee$ and hence V are locally free sheaves on X. Clearly, we have $\chi(V)=\chi(\tilde{E})=\chi(E)+r$. Next, by its construction we have canonical $\pi^*(\pi_*\tilde{E}^\vee)\to\tilde{E}^\vee$ and its dual

$$\tilde{E}\longrightarrow \pi^*V\equiv\pi^*(\pi_*\tilde{E}^\vee)^\vee. \tag{2.4}$$

Restricting to q_n and q'_m, we obtain

$$h_1:\tilde{E}|_{q_n}\longrightarrow\pi^*V|_{q_n}\equiv V|_{p_1}\quad\text{and}\quad h_2:\tilde{E}|_{q'_m}\longrightarrow\pi^*V|_{q'_m}\equiv V|_{p_2}.$$

We then define $V^0\subset V|_{p_1}\oplus V|_{p_2}$ to be the image of $\Gamma_\phi\subset\tilde{E}|_{q_n}\oplus\tilde{E}|_{q'_m}$ under the homomorphism $h_1\oplus h_2$. We claim that $\dim V^0=r$. Suppose $\dim V^0<r$. Then there is $v\in\Gamma_\phi$ so that $h_1(v)=h_2(v)=0$. Since $v\in\Gamma_\phi$, its image in $\tilde{E}|_{q_n}$ and in $\tilde{E}|_{q'_m}$ are identical, using $\tilde{E}|_{q_n}\equiv\tilde{E}|_{q'_m}$. By our previous discussion, $h_1(v)=0$ implies that v lies in the kernel $\tilde{E}|_{q_n}\to\tilde{E}|_{q_0}/T_\leftarrow$, which implies that there is a section $s\in H^0(\tilde{E}|_R(-q_0))$ so that $s(q_n)=v$. Similarly, $h_2(v)=0$ implies that there is a section $s'\in H^0(\tilde{E}|_{R'}(-q'_0))$ so that $s'(q'_m)=v$. The pair (s,s') then glues together to form a section of E that vanishes along X^0. Since $E|_D$ is regular, such section must be trivial, and hence $v=0$. This proves that $\dim V^0=r$ and hence the pair $V^G=(V,V^0)$ is a GPB on X.

2.2. α-stable bundles and α-stable GPB. In this subsection we will show that the correspondence constructed in the previous subsection relates α-stable bundles on $X^\dagger_{n+m}$ to α-stable GPBs on X^+.

Definition 2.3. Let $\alpha\in[0,1)$. A GPB $V^G=(V,V^0)$ is α-stable if for any proper subbundle $F\subset V$ we have $\mu^G(F,\alpha)<\mu^G(V,\alpha)$, where

$$\mu^G(F,\alpha)=(\chi(F)+(1-\alpha)r^+(F))/r(F).$$ [3]

Let $\mathcal{G}^\alpha_{r,\chi'}(X^+)$ be the set of all isomorphism classes of α-stable rank r GPBs on X^+ of Euler characteristics $\chi'=\chi+r$. In this subsection, we will show that the previous correspondence defines a map

$$\mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)\longrightarrow\mathcal{G}^\alpha_{r,\chi'}(X^+).$$

This map will be useful in studying the moduli of α-stable sheaves on $\mathfrak{X}^\dagger$.

We now prove the following equivalence of the stable GPB and stable vector bundles on $X^\dagger_{n+m}$.

Proposition 2.4. *Let E be a rank r vector bundle of Eular characteristic χ on $X^\dagger_{n+m}$. Suppose $E|_D$ is regular.*[4] *Then E is α-stable if and only if its associated GPB $V^G=(V,V^0)$ is α-stable.*

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[3] The α-stability is the $(1-\alpha)$-stability of GBP introduced in [**1**].

[4] By Lemma 1.2, $E|_D$ is regular whenever E is α-stable.

Proof. We first prove that E being α-stable implies that V^G is α-stable. Let $U \subset V$ be any proper subbundle. We need to show that $\mu^G(U,\alpha) < \mu^G(V,\alpha)$. We will prove this inequality by constructing a subsheaf $F \subset E$ so that $r_0(F) = r(U)$, $\chi(F) \geq \chi(U) - 2r(U) + r^+(U)$ and $r^\dagger(F) = r^+(U)$. Then we have the inequality because

$$\begin{aligned}\mu^G(U,\alpha) &= \frac{\chi(U) + (1-\alpha)r^+(U)}{r(U)} \\ &\leq \frac{(\chi(F) + 2r(U) - r^+(U)) + (1-\alpha)r^+(U)}{r(U)} \\ &= \frac{\chi(F) - \alpha r^\dagger(F)}{r_0(F)} + 2 < \frac{\chi(E) - \alpha r}{r} + 2 \\ &= \frac{\chi(V) + (1-\alpha)r}{r} = \mu^G(V,\alpha).\end{aligned}$$

We now construct such F. First let $\tilde{U} = \tilde{E}|_X \cap U$, which makes sense since by construction V is just the result of a elementary (Hecke) modification of $\tilde{E}|_X$ at p_1 and p_2 so that $\tilde{E}|_X \subset V$ is a subsheaf. Let $A_1 = \mathrm{Im}\{\tilde{U}|_{p_1} \to \tilde{E}|_{q_0}\}$ and $A_2 = \mathrm{Im}\{\tilde{U}|_{p_2} \to \tilde{E}|_{q'_0}\}$. We then let $\tilde{B} \subset \Gamma_\phi$ be the preimage of $U^0 = V^0 \cap (U|_{p_1} \oplus U|_{p_2})$ under the isomorphism $V^0 \cong \Gamma_\phi$, and let $B_1 \subset \tilde{E}|_{q_n}$ and $B_2 \subset \tilde{E}|_{q'_m}$ be the image of $\tilde{B}$ under the obvious projections. Let $\sigma_0 : \tilde{E}|_{q_0} \to \tilde{E}|_{q_0}/T_{\leftarrow}$ and $\sigma_n : \tilde{E}|_{q_n} \to \tilde{E}|_{q_n}/T_{\rightarrow}$ be the projections and $\xi : \tilde{E}|_{q_n}/T_{\rightarrow} \cong \tilde{E}|_{q_0}/T_{\leftarrow}$ be the tautological isomorphisms, constructed in (2.2). We claim that $\xi(\sigma_n(B_1)) \subset \sigma_0(A_1)$. Indeed, let $(u_1,u_2) \in \tilde{B}$ be any element with $(v_1,v_2) \in U^0$ its preimage. By definition, $\xi(\sigma_n(v_1)) = \sigma_0(u_1)$. Hence $\xi(\sigma_n(B_1)) \subset \sigma_0(A_1)$. Similarly, if we let σ'_0, σ'_m and ξ' be similar homomorphisms associated to q'_0, q'_m and p_2, we have $\xi'(\sigma'_m(B_2)) \subset \sigma'_0(A_2)$.

We now apply Lemma 2.2 to conclude that there is a subsheaf $F_1 \subset \tilde{E}|_R$ so that

$$\mathrm{Im}\{F_1|_{q_0} \to \tilde{E}|_{q_0}\} = A_1,\ \mathrm{Im}\{F_1|_{q_n} \to \tilde{E}|_{q_n}\} = B_1,\ \chi(F_1) \geq \dim B_1 + e_1,$$

where $e_1 = \dim\ker\{\tilde{U}|_{p_1} \to U|_{p_1}\}$. Similarly, we have a subsheaf $F_2 \subset \tilde{E}|_{R'}$ having

$$\mathrm{Im}\{F_2|_{q'_0} \to \tilde{E}|_{q'_0}\} = A_2,\ \mathrm{Im}\{F_2|_{q'_m} \to \tilde{E}|_{q'_m}\} = B_2,\ \chi(F_2) \geq \dim B_2 + e_2,$$

where $e_2 = \dim\ker\{\tilde{U}|_{p_2} \to U|_{p_2}\}$. Note that $B_1 = B_2$ under the identification $\tilde{E}|_{q_n} \equiv \tilde{E}|_{q'_m}$. Therefore, the subsheaves $\tilde{U} \subset \tilde{E}|_X$, $F_1 \subset \tilde{E}|_R$ and $F_2 \subset \tilde{E}|_{R'}$ glue together to form a subsheaf $F \subset E$ such that $r_0(F) = r(U)$, $r^\dagger(F) = \dim B_1 = \dim B_2 = \dim U^0 = r^+(U)$ and

$$\chi(F) = \chi(\tilde{U}) + \chi(F_1) + \chi(F_2) - 2r(U) - \dim C \geq \chi(U) - 2r(U) + \dim U^0.$$

Here we used the fact that $\chi(\tilde{U}) = \chi(U) - e_1 - e_2$. This F is the desired subsheaf. This prove the first part of the lemma.

We postpone the other part of the proof until we give a more precise description of the destabilizing subsheaf of E. q.e.d.

2.3. α-stable GPB and α-stable bundles. We will complete the other half of Proposition 2.4 in this subsection.

We begin with a characterization of the destabilizing subsheaf of E on $X^{\dagger}_{n+m}$. Let E be a vector bundle on $X^{\dagger}_{n+m}$ as in Proposition 2.4. Let $F \subset E$ be an α-destabilizing subsheaf. Namely, $\mu(F,\alpha) > \mu(E,\alpha)$ and is the largest possible among all subsheaves of E. Since $E|_D$ is regular, as mentioned before $r_0(F) = \operatorname{rank} F|_{X^0} > 0$. In the following, we say that a sheaf F' with $F \subset F' \subset E$ is a small extension of $F \subset E$ if F'/F is a sheaf of $\mathcal{O}_D$-modules. We say $F|_D = F \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_D$ is non-negative if the torsion free part of the restriction of F to each rational curves $D_i \subset D$ has no-negative factors.

Lemma 2.5. *Let E be as in Proposition 2.4 and $F \subset E$ be an α-destabilizing subsheaf. Then $r_0(F) > 0$, $F|_D$ is non-negative and there is no non-negative small extension $F \subset F' \subset E$ of $F \subset E$.*

Proof. First, $r_0(F) = 0$ is impossible because of our assumption $\chi > r$. Suppose there is an irreducible component $D_i \subset D$ so that $F|_{D_i}$ is not non-negative. Namely, there is a $t > 0$ so that $\mathcal{O}_{D_i}(-t)$ is a quotient sheaf of $F|_{D_i}$. We let F' be the kernel of the composite $F \to F|_{D_i} \to \mathcal{O}_{D_i}(-t)$. Clearly, $\chi(F') \geq \chi(F)$ and $r^{\dagger}(F') \leq r^{\dagger}(F)$ while $\mathbf{rk_d} F' < \mathbf{rk_d} F$. Hence $\mu_{\mathbf{d}}(F',\alpha) > \mu_{\mathbf{d}}(F,\alpha)$, violating that F is an α-destabilizing subsheaf of E.

Now suppose $F \subset F' \subset E$ is a small extension of F so that F' is also non-negative on D. We claim that $\mu_{\mathbf{d}}(F',\alpha) > \mu_{\mathbf{d}}(F,\alpha)$. We look at the quotient sheaf F'/F. Since $F'|_D$ is non-negative, F'/F is also non-negative. Further, since F'/F is a sheaf of $\mathcal{O}_D$-modules, it is easy to see that

$$\chi(F'/F) \geq r^{\dagger}(F') - r^{\dagger}(F).$$

Hence because $\alpha < 1$, the slope $\mu_{\mathbf{d}}(F',\alpha)$ is

$$\begin{aligned}&\frac{(\chi(F) - \alpha r^{\dagger}(F)) + (\chi(F'/F) - \alpha(r^{\dagger}(F') - r^{\dagger}(F)))}{\mathbf{rk_d} F + O(\epsilon)}\\ &> \frac{\chi(F) - \alpha r^{\dagger}(F)}{\mathbf{rk_d} F} = \mu_{\mathbf{d}}(F,\alpha).\end{aligned}$$

Here we used the fact that $r_0(F) > 0$ and that ϵ is sufficiently small. This violates the assumption that F is an α-destabilizing subsheaf of E. q.e.d.

We now give a more precise description of the destabilizing subsheaf $F \subset E$. We begin with more notation. Let $D_1, \ldots, D_{n+m}$ be the ordered rational curves of $D \subset X^{\dagger}_{n+m}$ with nodes $q_0, \ldots, q_{n+m}$ and the marked node $q^\dagger = q_n$. We let $D_{[i,j]} = \cup_{k=i+1}^{j} D_k$ be the subchain of D. There are several (possible) subsheaves of $E|_D$ that are important to our later study. The first is the subsheaf $\mathcal{O}_{[i,j]} \subset E|_D$, which as a sheaf is isomorphic to $\mathcal{O}_{D_{[i,j]}}$ and such that the induced homomorphisms $\sigma(q_i)\colon \mathcal{O}_{[i,j]} \otimes \mathbf{k}(q_i) \to E|_{q_i}$ and $\sigma(q_j)\colon \mathcal{O}_{[i,j]} \otimes \mathbf{k}(q_j) \to E|_{q_j}$ are non-zero. If we impose the condition $\sigma(q_i) \neq 0$ instead of $\sigma(q_i) = 0$, we denote the resulting subsheaf by $\mathcal{O}_{(i,j]}$. The sheaves $\mathcal{O}_{[i,j)}$ are defined similarly in the obvious way. The other subsheaf is $\mathcal{O}_{[0,n+m]}^{[k]} \subset E|_D$, which as a sheaf is an invertible sheaf of $\mathcal{O}_D$-modules. Its degree on D_k is 1 and its degrees on other components are all 0, and the induced homomorphisms $\sigma(q_0)$ and $\sigma(q_{n+m})$ are both non-zero.

Lemma 2.6. *Let E be as in Proposition* 2.4 *and $F \subset E$ be its α-destabilizing subsheaf. Then the image subsheaf of $F|_D \to E|_D$ is a direct sum of subsheaves from the list*

$$\mathcal{O}_{[0,i)},\ \mathcal{O}_{(i,n+m]},\ \mathcal{O}_{[0,n+m]},\ \mathcal{O}_{[0,n+m]}^{[i]}.$$

Proof. Since $E|_D$ is regular, all restrictions $F|_{D_k}$ have only degree 0 and 1 factors. We first assume there is a component D_k so that $F|_{D_k}$ has a factor $\mathcal{O}_{D_k}(1)$. Then there is a subchain $D_{[i,j]}$ containing D_k so that this factor $\mathcal{O}_{D_k}(1) \subset F|_{D_k}$ extends to a subsheaf $\mathcal{L} \subset F|_D$ so that $\mathcal{L}$ is an invertible sheaf of $\mathcal{O}_{D_{[i,j]}}$-modules and the degree of $\mathcal{L}$ along each $D_l \subset D_{[i,j]}$ is non-negative. However, if there is another $l \neq k$ so that the degree of $\mathcal{L}$ on D_l is 1, then we can find a section of $\mathcal{L}$ that vanishes at q_i and q_j. Using $\mathcal{L} \to E|_D$, this section induces a section of E that vanishes on X^0, violating the fact that $E|_D$ is regular. Hence $\deg \mathcal{L}|_{D_l} = 0$ for all $k \neq l \in [i+1, j]$. For the same reason we conclude that $\sigma(q_i)\colon \mathcal{L} \otimes \mathbf{k}(q_i) \to E|_{q_i}$ cannot be zero since otherwise we can find a section of E that vanishes on X^0. Since $E|_D$ is locally free, this is possible only if $i = 0$. For the same reason we have $j = n+m$ and $\sigma(q_{n+m}) \neq 0$. Hence $\mathcal{L} \subset E|_D$ is a subsheaf of the type $\mathcal{O}_{[0,n+m]}^{[k]} \subset E|_D$ described before.

Since $F|_D$ has only degree 0 and 1 factors, it is direct to see that $\mathcal{O}_{[0,n+m]}^{[k]} \subset F|_D$ must be a direct summand. Let $F|_D = \mathcal{O}_{[0,n+m]}^{[k]} \oplus \mathcal{F}'$ be a decomposition. By repeating this procedure to the sheaf $\mathcal{F}'$, we conclude that $F|_D$ is a direct sum of sheaves of type $\mathcal{O}_{[0,n+m]}^{[k]}$ with a sheaf $\mathcal{F}$ whose restriction to each D_i has only degree 0 factors. We now show that $\mathcal{F}$ must be a direct sum of a torsion sheaf and sheaves of the first three kinds in the list of the lemma: $\mathcal{O}_{[0,i]}$, $\mathcal{O}_{(i,n+m]}$ and $\mathcal{O}_{[0,n+m]}$. Suppose $\mathcal{F}$ is not supported at points. Then there is a subchain $D_{[i,j]} \subset D$ so that $\mathcal{O}_{D_{[i,j]}} \subset \mathcal{F}$ is a subsheaf. Let $\sigma(q_i)$ and $\sigma(q_j)$ be the induced homomorphisms $\mathcal{O}_{D_{[i,j]}} \otimes \mathbf{k}(q_i) \to E|_{q_i}$ and $\mathcal{O}_{D_{[i,j]}} \otimes \mathbf{k}(q_j) \to E|_{q_j}$. As before, we can show that $\sigma(q_i)$ and $\sigma(q_j)$ can not be simultaneously zero. Further, $\sigma(q_i) \neq 0$ is possible only if $i = 0$. Hence $\mathcal{F}$ has a factor from the first three kinds in the above list. By repeating this argument, we conclude that $F|_D$ is a direct sum of sheaves supported on q_0 and q_{n+m}, and sheaves from the list. q.e.d.

We now complete the proof of Proposition 2.4

Proof. We need to show that if E is α-unstable then V^G is α-unstable as well. Let $F \subset E$ be the α-destabilizing subsheaf of E. We let R and R' be the left and the right rational tails of $X_{n,m}$, let $\tilde{E} = \rho^* E$ be the pull back vector bundle on $X_{n,m}$ and let $T_{\leftarrow} \subset \tilde{E}|_{q_0}$ and $T_{\leftarrow} \subset \tilde{E}|_{q_0'}$ be the reverse transfer of $\tilde{E}|_R$ and $\tilde{E}|_{R'}$. We follow the notation introduced before (2.4) with q_i and q_j' the nodes of R and R' so that $q_0 = p_1$ and $q_0' = p_2$. Then $\tilde{E}|_X$ and V fits into the exact sequence

$$0 \longrightarrow \tilde{E}|_X \longrightarrow V \longrightarrow T_{\leftarrow} \otimes \mathbf{k}(p_1) \oplus T_{\leftarrow}' \otimes \mathbf{k}(p_2) \longrightarrow 0.$$

Let $\mathcal{F} \subset \tilde{E}|_X$ be the image subsheaf of $\rho^* F|_X \to \tilde{E}|_X$, let $\mathcal{F}' \supset \mathcal{F}$ be the largest subsheaf of $\tilde{E}|_X$ so that $\mathcal{F}'/\mathcal{F}$ is torsion and let $F' \subset V$ be the largest subsheaf so that $\mathcal{F} \subset F'$ and $F'/\mathcal{F}$ is torsion. Here the inclusion $\mathcal{F} \subset F'$ is understood in terms of the inclusion of sheaves $\tilde{E}|_X \subset V$. Since $\mathcal{F}'$ is the largest possible such subsheaf, it is a subbundle of $\tilde{E}|_X$ and $\mathcal{F}'/\mathcal{F}$ is contained in $\tilde{E}|_{p_1} \oplus \tilde{E}|_{p_2}$. We claim that $\mathcal{F}'/\mathcal{F}|_{p_1} \cap T_{\leftarrow} = \mathcal{F}'/\mathcal{F}|_{p_2} \cap T_{\leftarrow}' = \{0\}$, since otherwise we can find a small extension of $F \subset E$ so that its $\mu(\cdot, \alpha)$ degree is larger than $\mu(F, \alpha)$, violating the maximality of the latter. Combined with the maximality of $\mu(F, \alpha)$, we conclude that $F'/\mathcal{F}$ is a torsion sheaf supported at p_1 and p_2 and is indeed a direct sum of a $\mathbf{k}(p_1)$-module and a $\mathbf{k}(p_2)$-module.

We now write the torsion free part of $F|_D$ as a direct sum of sheaves in the list of Lemma 2.6. Let a_- (resp. a_+) be the number of summands of type $\mathcal{O}_{[0,i)}$ with $i \leq n$ (resp. $i > n$) in the decomposition; let b_- (resp. b_+) be the number of $\mathcal{O}_{(i,n+m]}$ in the summand with $i < n$ (resp. $i \geq n$); let c be the number of summand $\mathcal{O}_{[0,n+m]}$ and let d_- (resp. d_+) be the number of summands $\mathcal{O}_{[0,n+m]}^{[i]}$ with $i \leq n$ (resp. $i > n$). A direct check via the construction of V shows that

$$\dim F'/\mathcal{F}|_{p_1} = r_0(F) - (a_+ + c + d_+) \qquad \text{and}$$
$$\dim F'/\mathcal{F}|_{p_2} = r_0(F) - (b_- + c + d_-).$$

Hence

$$\chi(F') = \chi(\mathcal{F}) + 2r_0(F) - (a_+ + b_- + 2c + d_- + d_+).$$

Further, by a direct check we have $\chi(\mathcal{F}) = \chi(F) + c$ and $r^\dagger(F) = a_+ + b_- + c + d_- + d_+$. Hence

$$\chi(F') = \chi(F) + 2r_0(F) - r^\dagger(F).$$

A similar argument shows that $r^+(F') = a_+ + b_- + c + d_- + d_+$ and hence $r^\dagger(F) = r^+(F')$. This implies that

$$\frac{\chi(F') + (1-\alpha)r^+(F')}{r(F')} = \frac{\chi(F) - \alpha r^\dagger(F)}{r(F')} + 2 > \frac{\chi - \alpha r}{r} + 2 = \frac{\chi' + (1-\alpha)r}{r},$$

violating the α-stability of V^G. This completes the proof of Proposition 2.4. q.e.d.

Corollary 2.7. *Each* $E \in \mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ *has* $\mathfrak{Aut}_0(E) = \mathbb{C}^\times$.

Proof. Let $E \in \mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ be a sheaf over $X^\dagger_{n+m}$ and let (σ, f) be an automorphism of E. Namely, $\sigma : X_{n+m} \to X_{n+m}$ is a based automorphism and $f : \sigma^* E \cong E$ is an isomorphism. Let $\sigma' : X_{n,m} \cong X_{n,m}$ and $f' : \sigma'^* \tilde{E} \cong \tilde{E}$ be the induced isomorphisms. Since $\sigma|_X = \mathrm{id}$, f induces an isomorphism $\tilde{f} : \tilde{E}|_X \to \tilde{E}|_X$. Clearly, f' preserves the subspaces $T_\leftarrow \subset \tilde{E}|_{q_0}$ and $T'_\leftarrow \subset \tilde{E}|_{q'_0}$. Hence if we denote by $V^{\mathrm{G}} = (V, V^0)$ the associated GPB of E, $\tilde{f}$ extends to an isomorphism $\bar{f}$ as shown:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{E}|_X & \longrightarrow & V & \longrightarrow & T_\leftarrow \otimes \mathbf{k}(p_1) \oplus T'_\leftarrow \otimes \mathbf{k}(p_2) & \longrightarrow & 0 \\ & & \big\downarrow \tilde{f} & & \big\downarrow \bar{f} & & \big\downarrow & & \\ 0 & \longrightarrow & \tilde{E}|_X & \longrightarrow & V & \longrightarrow & T_\leftarrow \otimes \mathbf{k}(p_1) \oplus T'_\leftarrow \otimes \mathbf{k}(p_2) & \longrightarrow & 0 \end{array}$$

We claim that the GPB structure $V^0 \subset V|_{p_1} \oplus V|_{p_2}$ is preserved under $\bar{f}$. Indeed, because $\Gamma_\phi \subset \tilde{E}|_{q_n} \oplus \tilde{E}|_{q'_m}$ is the graph of the tautological isomorphism $\tilde{E}|_{q_n} \cong \tilde{E}|_{q'_m}$, and this isomorphism is preserved by f', Γ_ϕ is preserved by f'. Next, we look at the homomorphism

$$\tilde{E}|_{q_n} \longrightarrow \tilde{E}|_{q_n}/T_\rightarrow \xrightarrow{\cong} \tilde{E}|_{q_0}/T_\leftarrow.$$

Obviously, the first arrow is canonical. The second arrow is induced by $\mathcal{O}_R^{\oplus a} \subset \tilde{E}|_R$, which is also preserved by f'. Hence the composite of the above arrows is invariant under f'. Therefore, the image of Γ_ϕ in $V|_{p_1} \oplus V|_{p_2}$ will be preserved under f'. This shows that (f, σ) induces an isomorphism $\bar{f}$ of V^{G} whose restriction to $X^0 \subset X$ is exactly $f|_{X^0}$.

Since E is α-stable, V^G is α-stable by Proposition 2.4 and hence by the usual argument, the automorphism group of V^G is $\mathbb{C}^\times$. Hence $\bar{f}$ is a multiple of the identity map. In particular, after replacing f by a multiple of itself, we can assume $f|_{X^0} = \mathrm{id}$. We now show that $\sigma = \mathrm{id}$. As before, let $q_0, \ldots, q_{n+m}$ be the nodes of X_{n+m}. Since q_i are fixed points of σ, the isomorphism f induces automorphisms $f|_{q_i} : E|_{q_i} \to E|_{q_i}$. We first claim that all $f|_{q_i} = \mathrm{id}$. Suppose not; say $f|_{q_k} \neq \mathrm{id}$. Then $0 < k < n+m$ since $f|_{X^0} = \mathrm{id}$. Since $E|_D$ is non-negative, there is a section $s \in H^0(D, E)$ so that $s(q_k)$ is not fixed by $f|_{q_k}$. Hence $s - f^{-1*}\sigma^* s$ is a section of $H^0(D, E)$ that vanishes on q_0 and q_{n+m} but non-zero at q_k. This violates the α-stability of E. Hence all $f|_{q_i}$ are the identities.

Next, we claim that $\sigma_k = \sigma|_{D_k}$ are identities for all D_k. Indeed, since $E|_{D_k} \cong \mathcal{O}_{D_k}^{\oplus r-a} \oplus \mathcal{O}_{D_k}(1)^{\oplus a}$ for some $a > 0$, there is no isomorphism of $E|_{D_k}$ with $\sigma_k^*(E|_{D_k})$ whose restrictions to q_{k-1} and q_k are the identity maps unless $\sigma_k = id$. Finally, since $\sigma = \mathrm{id}$ and the restrictions of f to X^0 and all nodes are the identity maps, f must be an identity map since $E|_{D_k}$ has only degree 0 and 1 factors. This proves that $\mathfrak{Aut}_0(E) = \mathbb{C}^\times$. q.e.d.

The association from $E \in \mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ to $V^G \in \mathcal{G}^\alpha_{r,\chi'}(X^+)$ constructed above defines a map

$$\mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger) \longrightarrow \mathcal{G}^\alpha_{r,\chi'}(X^\dagger). \tag{2.5}$$

On the other hand, by [**1**] the moduli space of α-stable GPBs (V, V^0) on X^+ of rank r and $\chi(V) = \chi' = \chi + r$ form a fine moduli space $\mathbf{G}^\alpha_{r,\chi'}(X^+)$. We now show that the above correspondence induces a morphism

$$\mathbf{M}^\alpha_{r,\chi}(\mathfrak{X}^\dagger) \longrightarrow \mathbf{G}^\alpha_{r,\chi'}(X^+). \tag{2.6}$$

Later we will show that in case $r = 3$ this is the composition of two blow-ups along smooth subvarieties.

Let $\mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ and $\mathfrak{F}^\alpha_{r,\chi'}(X^+)$ be the moduli functors of the sets $\mathcal{V}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ and $\mathcal{G}^\alpha_{r,\chi'}(X^+)$. To prove the statement, it suffices to show that the map (2.5) defines a transformation of functors $\mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger) \Rightarrow \mathfrak{F}^\alpha_{r,\chi'}(X^+)$. Namely, to any scheme S and a family $\mathcal{E} \in \mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)(S)$ it associates a unique family $\mathcal{V}^G \in \mathfrak{F}^\alpha_{r,\chi'}(X^+)(S)$, compatible to (2.5), and satisfies the base change property. Let $\mathcal{E} \in \mathfrak{F}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)(S)$ be any family over $(W_S, q^\dagger)$. Let $\rho : \tilde{W}_S \to W_S$ be the normalization along $q^\dagger(S) \subset W_S$ and let $\pi : \tilde{W}_S \to X \times S$ the contraction of all rational curves on the fibers. Let $\mathbf{q}_-$ and $\mathbf{q}_+ \subset \tilde{W}_S$ be the two sections of $\tilde{W}_S/S$ that are the pre-images of $q^\dagger(S)$. We then denote $\mathbf{p}_i = p_i \times S \subset X \times S$. As usual, we index $\mathbf{q}_\pm$ so that $\pi(\mathbf{q}_-) = \mathbf{p}_1$. We define $\mathcal{V} = \big(\pi_*\rho^*\mathcal{E}^\vee\big)^\vee$. Because $\mathcal{E}$ is a family of α-stable sheaves, $R^i\pi_*\rho^*\mathcal{E}^\vee = 0$ for $i > 0$. Hence $\pi_*\rho^*\mathcal{E}^\vee$ and $\mathcal{V}$ are locally free sheaves on $X \times S$. Next, there are canonical homomorphisms

$$\pi_*\rho^*\mathcal{E}^\vee|_{\mathbf{p}_1} \longrightarrow \rho^*\mathcal{E}^\vee|_{\mathbf{q}_-} \quad \text{and} \quad \pi_*\rho^*\mathcal{E}^\vee|_{\mathbf{p}_2} \longrightarrow \rho^*\mathcal{E}^\vee|_{\mathbf{q}_+}.$$

Coupled with the identity $\rho^*\mathcal{E}|_{\mathbf{q}_-} \equiv \rho^*\mathcal{E}|_{\mathbf{q}_+}$, we obtain homomorphisms

$$\mathcal{E}|_{q^\dagger(S)} \stackrel{\text{diag}}{\longrightarrow} \rho^*\mathcal{E}|_{\mathbf{q}_- + \mathbf{q}_+} \longrightarrow \big(\pi_*\rho^*\mathcal{E}^\vee\big)^\vee|_{\mathbf{p}_1+\mathbf{p}_2} \equiv \mathcal{V}|_{\mathbf{p}_1+\mathbf{p}_2}.$$

Here $\rho^*\mathcal{E}|_{\mathbf{q}_-+\mathbf{q}_+}$ is $\rho^*\mathcal{E}|_{\mathbf{q}_-} \oplus \rho^*\mathcal{E}|_{\mathbf{q}_+}$, considered as a sheaf of $\mathcal{O}_S$-modules. We define $\mathcal{V}^0$ to be the image sheaf of the composition of the above arrows. It is direct to check that this construction $\mathcal{E} \Rightarrow (\mathcal{V}, \mathcal{V}^0)$ satisfies the base change property. Hence as we argued before (in constructing $V^G = (V, V^0)$) for each closed $\xi \in S$ the induced

$$\big(\mathcal{E}|_{q^\dagger(S)}\big) \otimes \mathbf{k}(\xi) \longrightarrow \mathcal{V}|_{\mathbf{p}_1+\mathbf{p}_2} \otimes \mathbf{k}(\xi)$$

is injective. Hence $\mathcal{V}^0$ is a subvector bundle of $\mathcal{V}|_{\mathbf{p}_1+\mathbf{p}_2}$. Consequently, by Lemma 2.4 and the base change property the pair $\mathcal{V}^G = (\mathcal{V}, \mathcal{V}^0)$ is a family of α-stable GPB in $\mathfrak{F}^\alpha_{r,\chi'}(X^+)$. This defines the desired transformation of the functors.

2.4. $\mathbf{M}^\alpha_{r,\chi}(\mathfrak{X}^\dagger)$ as a blow-up of $\mathbf{G}^\alpha_{r,\chi'}(X^+)$. In this subsection, we will restrict ourselves to the case $r = 3$ and $\chi = 4$. We will show that $\mathbf{M}^\alpha_{3,4}(\mathfrak{X}^\dagger)$ is a blow up of $\mathbf{G}^\alpha_{3,7}(X^+)$. We will prove this by looking at the inverse of (2.6):

$$\Phi : \mathbf{G}^\alpha_{3,7}(X^+) \dashrightarrow \mathbf{M}^\alpha_{3,4}(\mathfrak{X}^\dagger), \tag{2.7}$$

and prove that after two blow-ups of the domain we can resolve the indeterminacy and the resulting morphism is an isomorphism.

For simplicity, in the remainder of this paper we will denote $\mathbf{G}^\alpha_{3,7}(X^+)$ by $\mathbf{G}^\alpha$ and denote $\mathbf{M}^\alpha_{3,4}(\mathfrak{X}^\dagger)$ by $\mathbf{M}^\alpha$. By [**1**, Theorem 2], we know that there is a universal family of GPBs $(\mathcal{V}, \mathcal{V}^0)$ on $\mathbf{G}^\alpha \times X$, where $\mathcal{V}$ is a rank 3 vector bundle over $\mathbf{G}^\alpha \times X$ and $\mathcal{V}^0$ is a rank 3 subbundle of $\mathcal{V}|_{p_1+p_2} \triangleq \mathcal{V}|_{\mathbf{G}^\alpha \times p_1} \oplus \mathcal{V}|_{\mathbf{G}^\alpha \times p_2}$ over $\mathbf{G}^\alpha$. On the open dense subset $U \subset \mathbf{G}^\alpha$ where the induced $\mathcal{V}^0 \to \mathcal{V}|_{p_1}$ and $\mathcal{V}^0 \to \mathcal{V}|_{p_2}$ are isomorphisms, we get a family of vector bundles over $U \times X_0$ by taking the kernel of

$$(1 \times \rho)_* \mathcal{V} \longrightarrow (1 \times \rho)_* \mathcal{V}|_q / \mathcal{V}^0 \equiv \big(\mathcal{V}|_{p_1+p_2}\big)/\mathcal{V}^0,$$

where $\rho : X \to X_0$ is the normalization map and $q \in X_0$ is its node. By Lemma 2.4, the resulting family is a family of α-stable sheaves on X_0.

The resulting U-family of α-stable sheaves defines a morphism $U \to \mathbf{M}^\alpha$, which defines a rational map $\mathbf{G}^\alpha - \to \mathbf{M}^\alpha$, inverse to the given $\mathbf{M}^\alpha \to \mathbf{G}^\alpha$. We now show how to eliminate indeterminacy by blowing up the domain $\mathbf{G}^\alpha$ twice.

Let $\mathbf{Y}_i$ (resp. $\mathbf{Z}_i$) be the subvariety of $\mathbf{G}^\alpha$ consisting of GPB (V, V^0) such that $V^0 \to V|_{p_1}$ (resp. $V^0 \to V|_{p_2}$) have ranks at most i. Clearly $\mathbf{Y}_0 \subset \mathbf{Y}_1 \subset \mathbf{Y}_2$ and $\mathbf{Z}_0 \subset \mathbf{Z}_1 \subset \mathbf{Z}_2$ are chains of subvarieties with $\mathbf{Y}_0$ and $\mathbf{Z}_0$ smooth. Further, because $\dim V^0 = 3$, we know that $\mathbf{Z}_0 \cap \mathbf{Y}_2 = \mathbf{Y}_0 \cap \mathbf{Z}_2 = \mathbf{Y}_1 \cap \mathbf{Z}_1 = \emptyset$ and $\mathbf{Y}_1$ intersects $\mathbf{Z}_2$ and $\mathbf{Z}_1$ intersects $\mathbf{Y}_2$ transversally. We now blow up $\mathbf{G}^\alpha$ along $\mathbf{Y}_0 \cup \mathbf{Z}_0$. We denote the blow-up of $\mathbf{G}^\alpha$ by $\mathbf{G}^\alpha_1$ and denote the proper transforms of $\mathbf{Y}_i$ and $\mathbf{Z}_i$ by $\mathbf{Y}_{i,1}$ and $\mathbf{Z}_{i,1}$. Because of the intersection property mentioned, the proper transforms $\mathbf{Y}_{1,1}$ and $\mathbf{Z}_{1,1}$ are smooth, satisfying similar intersection properties. We next blow up $\mathbf{G}^\alpha_1$ along $\mathbf{Y}_{1,1} \cup \mathbf{Z}_{1,1}$. We denote the blow-up by $\tilde{\mathbf{G}}^\alpha$ and denote the corresponding total transforms by $\tilde{\mathbf{Y}}_i$ and $\tilde{\mathbf{Z}}_i$. This time, all $\tilde{\mathbf{Y}}_i$ and $\tilde{\mathbf{Z}}_i$ are smooth normal crossing divisors.

We now show that $\mathbf{G}^\alpha - \to \mathbf{M}^\alpha$ lifts to a morphism $\tilde{\mathbf{G}}^\alpha \to \mathbf{M}^\alpha$. Such a morphism will be induced by a family of α-stable sheaves parameterized by $\tilde{\mathbf{G}}^\alpha$. We now construct such a family. First, we blow up the codimension 2 subvarieties $\tilde{\mathbf{Y}}_0 \times p_1$ and $\tilde{\mathbf{Z}}_0 \times p_2 \subset \tilde{\mathbf{G}}^\alpha \times X$. We denote the resulting family (the blown-up) by W_1. Let

$$\pi_1 : W_1 \to \tilde{\mathbf{G}}^\alpha, \qquad \Phi_1 : W_1 \longrightarrow \tilde{\mathbf{G}}^\alpha \times X \quad \text{and} \quad \phi : \tilde{\mathbf{G}}^\alpha \longrightarrow \mathbf{G}^\alpha$$

be the obvious projections. Note that the fibers of π_1 are one of X, $X_{0,1}$ and $X_{1,0}$. Let $\mathbf{q}_1 \subset W_1$ (resp. $\mathbf{q}_1'$) be the proper transform of $\tilde{\mathbf{G}}^\alpha \times p_1$ (resp. $\tilde{\mathbf{G}}^\alpha \times p_2$) and let $\mathbf{D}_1$ (resp. $\mathbf{D}_1'$) be the exceptional divisor over $\tilde{\mathbf{Y}}_0 \times p_1$ (resp. $\tilde{\mathbf{Z}}_0 \times p_2$).

Next let $(\mathcal{V}, \mathcal{V}^0)$ be the universal bundle of $\mathbf{G}^\alpha$ given by a vector bundle $\mathcal{V}$ over $\mathbf{G}^\alpha \times X$ and a subbundle $\mathcal{V}^0$ of $\mathcal{V}|_{\mathbf{p}_1+\mathbf{p}_2}$, where $\mathbf{p}_i = \mathbf{G}^\alpha \times p_i \subset \mathbf{G}^\alpha \times X$. We introduce a new locally free sheaf on W_1:

$$\mathcal{V}_1 \triangleq \ker\{\Phi_1^*\mathcal{V} \longrightarrow \Phi_1^*\mathcal{V}|_{\mathbf{D}_1} \oplus \Phi_1^*\mathcal{V}|_{\mathbf{D}_1'}\}.$$

Because $\tilde{\mathbf{Y}}_0$ is the locus where $V^0 \to V|_{p_1}$ are zeros, and likewise for $\tilde{\mathbf{Z}}_0$, the pull back $\phi_1^*\mathcal{V}^0 \to \Phi_1^*\mathcal{V}|_{\mathbf{q}_1+\mathbf{q}_1'}$ factors through

$$\mathcal{V}_1^0 \triangleq \phi^*\mathcal{V}^0 \longrightarrow \mathcal{V}_1|_{\mathbf{q}_1+\mathbf{q}_1'}. \tag{2.8}$$

The pair $(\mathcal{V}_1, \mathcal{V}_1^0)$ is a family of GPBs on $(W_1, \mathbf{q}_1, \mathbf{q}_1')$, parameterized by $\tilde{\mathbf{G}}^\alpha$. As argued in [**4**], for each $\xi \in \tilde{\mathbf{G}}^\alpha$ the homomorphisms

$$\mathcal{V}_1^0|_{W_{1,\xi}} \longrightarrow \mathcal{V}_1|_{W_{1,\xi}\cap\mathbf{q}_1} \quad \text{and} \quad \mathcal{V}_1^0|_{W_{1,\xi}} \longrightarrow \mathcal{V}_1|_{W_{1,\xi}\cap\mathbf{q}_1'}$$

have ranks at least 1.

We now modify this family along the rank 1 degeneracy loci $\tilde{\mathbf{Y}}_1$ and $\tilde{\mathbf{Z}}_1$. The construction is similar. We first blow up W_1 along the disjoint union of $\pi_1^{-1}(\mathbf{Y}_1) \cap \mathbf{q}_1$ and $\pi_1^{-1}(\mathbf{Z}_1) \cap \mathbf{q}_1'$. Let W_2 be the blow-up, let $\mathbf{q}_2$ and $\mathbf{q}_2'$ be the proper transforms of $\mathbf{q}_1$ and $\mathbf{q}_1'$ and let $\mathbf{D}_2$ and $\mathbf{D}_2'$ be the exceptional divisors of $\Phi_2 : W_2 \to W_1$. As argued in [**4**], the cokernel

$$\mathcal{L}_1 \triangleq \mathrm{Coker}\{\mathcal{V}_1^0|_{\pi_1^{-1}(\tilde{\mathbf{Y}}_1)} \longrightarrow \mathcal{V}_1|_{\pi_1^{-1}(\tilde{\mathbf{Y}}_1)\cap\mathbf{q}_1}\}$$

is a rank two locally free sheaf on $\pi_1^{-1}(\tilde{\mathbf{Y}}_1) \cap \mathbf{q}_1$. Similarly, let $\mathcal{L}_1'$ be defined with $\tilde{\mathbf{Y}}_1$ replaced by $\tilde{\mathbf{Z}}_1$ and with $\mathbf{q}_1$ replaced by $\mathbf{q}_1'$. It is also a rank two locally free sheaf on $\pi_1^{-1}(\tilde{\mathbf{Z}}_1) \cap \mathbf{q}_1'$. Similarly to the rank 0 case, we define $\mathcal{V}_2$ to be the kernel of $\Phi_2^*\mathcal{V}_1 \to \Phi_2^*\mathcal{L}_1 \oplus \Phi_2^*\mathcal{L}_1'$. It is a locally free sheaf over W_2. Further the homomorphism (2.8) induces a homomorphism

$$\mathcal{V}_2^0 \triangleq \phi^*\mathcal{V}^0 \longrightarrow \mathcal{V}_2|_{\mathbf{q}_2+\mathbf{q}_2'}. \tag{2.9}$$

The pair $(\mathcal{V}_2, \mathcal{V}_2^0)$ is a family of GPBs on W_2 over $\tilde{\mathbf{G}}^\alpha$.

Lastly, we resolve the rank 2 degeneracy. Let $\pi_2 : W_2 \to \tilde{\mathbf{G}}^\alpha$ be the projection and let W_3 be the blow up of W_2 along the disjoint union of $\pi_2^{-1}(\tilde{\mathbf{Y}}_2) \cap \mathbf{q}_2$ and $\pi_2^{-1}(\tilde{\mathbf{Z}}_2) \cap \mathbf{q}_2'$. Let $\mathbf{q}_3$, $\mathbf{q}_3'$ be the proper transforms of $\mathbf{q}_2$ and $\mathbf{q}_2'$ and let $\mathbf{D}_3$ and $\mathbf{D}_3'$ be the exceptional divisors of $\Phi_3 : W_3 \to W_2$. Again the cokernel $\mathcal{L}_2$ of $\mathcal{V}_2^0|_{\pi_2^{-1}(\tilde{\mathbf{Y}}_2)} \to \mathcal{V}_2|_{\pi_2^{-1}(\tilde{\mathbf{Y}}_2)\cap\mathbf{q}_2}$ and the similarly defined $\mathcal{L}_2'$ are rank one locally free sheaves on $\pi_2^{-1}(\tilde{\mathbf{Y}}_2)\cap\mathbf{q}_2$ and $\pi_2^{-1}(\tilde{\mathbf{Z}}_2)\cap\mathbf{q}_2'$. We define $\mathcal{V}_3$ to be the kernel of $\Phi_3^*\mathcal{V}_2 \to \Phi_3^*\mathcal{L}_2 \oplus \Phi_3^*\mathcal{L}_2'$. Then we have the canonical

$$\mathcal{V}_3^0 \triangleq \phi^*\mathcal{V}^0 \longrightarrow \mathcal{V}_3|_{\mathbf{q}_3+\mathbf{q}_3'}. \tag{2.10}$$

Lemma 2.8. *The family of GPBs $(\mathcal{V}_3, \mathcal{V}_3^0)$ on W_3 has the property that the induced homomorphisms $\mathcal{V}^0 \to \mathcal{V}_3|_{\mathbf{q}_3}$ and $\mathcal{V}_3^0 \to \mathcal{V}_3|_{\mathbf{q}_3'}$ are both isomorphisms.*

Proof. We omit the proof since it is similar to that in [**4**] and follows directly from the construction. q.e.d.

For simplicity, we denote $\mathbf{q}_3$ and $\mathbf{q}_3' \subset W_3$ by $\mathbf{q}$ and $\mathbf{q}' \subset \tilde{W}$, and denote $(\mathcal{V}_3, \mathcal{V}_3^0)$ by $(\tilde{\mathcal{V}}, \tilde{\mathcal{V}}^0)$, respectively. We then glue the two sections $\mathbf{q}$ and $\mathbf{q}'$ of $\tilde{W}$ to obtain a family $W^\dagger$ over $\tilde{\mathbf{G}}^\alpha$ with the marked section $\mathbf{q}^\dagger$, the gluing locus. Clearly, $W^\dagger$ is a family of based semistable model of X_0 and the tautological projection $\rho : \tilde{W} \to W^\dagger$ is the normalization of $W^\dagger$ along $\mathbf{q}^\dagger$. Over $W^\dagger$, we define $\mathcal{E}$ via the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \rho_*\tilde{\mathcal{V}} \longrightarrow \big(\tilde{\mathcal{V}}|_{\mathbf{q}} \oplus \tilde{\mathcal{V}}|_{\mathbf{q}'}\big)/\tilde{\mathcal{V}}^0 \longrightarrow 0.$$

We now show that the family $\mathcal{E}$ is a family of α-stable vector bundles over $W^\dagger/\tilde{\mathbf{G}}^\alpha$. We begin with a closed $\tilde{\xi} \in \tilde{\mathbf{G}}^\alpha$, with $E \to X_{n+m}^\dagger$ the restriction of $\mathcal{E} \to W^\dagger$ to the fiber over $\tilde{\xi}$. We will follow the notation introduced before. In particular, let D be the chain of rational curves in $X_{n+m}^\dagger$ with $q^\dagger = q_n$ its based node, let $X_{n,m}$ be its normalization along $q^\dagger$ and let R and R' be its two rational tails. By our construction, E is the gluing of a GPB $(\tilde{V}, \tilde{V}^0)$ on $X_{n,m}$. Let $\xi \in \mathbf{G}^\alpha$ be the image of $\tilde{\xi}$ with (V, V^0) the corresponding GPB.

Lemma 2.9. *The vector bundle $\tilde{V}|_R$ is a regular vector bundle, as defined in Definition* 2.1. *Further, if we let $A = \ker\{V^0 \to V|_{p_1}\}$, then $\operatorname{Im}\{A \to V^0 \cong \tilde{V}^0 \to \tilde{V}|_{q_n}\}$ is exactly the transfer $T_\to \subset \tilde{V}|_{q_n}$ defined in* (2.1).

Proof. We define the type of the left tail $\tilde{V}|_R$ to be the triple (i_0, i_1, i_2) defined by $i_j = 1$ if $\tilde{\xi} \in \tilde{\mathbf{Y}}_j$ and $i_j = 0$ otherwise. Clearly, $n = i_0 + i_1 + i_3$. We first study the case $(i_0, i_1, i_2) = (0, 1, 1)$. Since we only want to understand $\tilde{V}|_R$, we can assume without loss of generality that $m = 0$. We begin with an explicit description of the construction of $\tilde{V} \to X_{2,0}$. First, we let $B_1 \cong \mathbb{C}^2$ be the cokernel of $V^0 \to V|_{p_1}$ and let $V' = \ker\{V \to B_1 \otimes \mathbf{k}(p_1)\}$. The definition of V' induces a canonical filtration $B_1 \subset V'|_{p_1}$. Next, let U_2 be $\mathcal{O}_{D_1}(1)^{\oplus 2} \oplus \mathcal{O}_{D_1}$. Again the canonical inclusion $\mathcal{O}_{D_1}(1)^{\oplus 2} \subset U_2$ defines a filtration $\mathbf{k}(q_0)^{\oplus 2} \subset U_2|_{q_0}$. We fix an isomorphism $U_2|_{q_0} \cong V'|_{p_1}$ so that it preserves the two subspaces $\mathbb{C}^2$ just mentioned. We then define V_2 by the induced exact sequence on $X_{1,0}$:

$$0 \longrightarrow V_2 \longrightarrow j_* V' \oplus j'_* U_2 \longrightarrow V'|_{p_1} \otimes \mathbf{k}(q_0) \longrightarrow 0.$$

By our construction of $\mathcal{V}_2$, V_2 is the restriction of $\mathcal{V}_2$ to $W_{2,\tilde{\xi}}$. The restriction of $\mathcal{V}_2^0 \to \mathcal{V}_2|_{\mathbf{q}_2}$ induces a homomorphism $V^0 \to V_2|_{q_1}$. Since $i_2 = 1$, its cokernel B_2 has dimension 1.

We define V_3 similarly. Let V_2' be the kernel of $V_2 \to B_2$. Note that $V_2'|_{q_1}$ has a filtration $\mathbb{C} \subset \mathbb{C}^3$. Let $U_3 = \mathcal{O}_{D_2}(1) \oplus \mathcal{O}_{D_2}^{\oplus 2}$. Then $U_3|_{q_1}$ also has a filtration $\mathbb{C} \subset \mathbb{C}^3$. We fix an isomorphism $V_2'|_{q_1} \cong U_3|_{q_1}$, preserving the two filtrations. We then define V_3 on $X_{2,0}$ by the exact sequence

$$0 \longrightarrow V_3 \longrightarrow \bar{j}_* V_2 \oplus \bar{j}'_* U_3 \longrightarrow V_2'|_{q_1} \otimes \mathbf{k}(q_1) \longrightarrow 0.$$

Here $\bar{j} : X_{1,0} \to X_{2,0}$ and $\bar{j}' : D_2 \to X_{2,0}$ are the obvious inclusions. Again V_3 is the restriction of $\mathcal{V}_3$ to the fiber of $W_3 = \tilde{W}$ over $\tilde{\xi}$. Also, the restriction of $\mathcal{V}_3^0 \to \mathcal{V}_3|_{\mathbf{q}_3}$ gives us $V^0 \to V_3|_{q_2}$, which must be an isomorphism.

We now check that $V_3|_{R_2}$ is regular. First, $V_3|_{D_2}$ has one degree 1 factor and two trivial factors. We claim that $V_3|_{D_1}$ also has one degree 1 factor and two trivial factors. By our construction, this will be true if $\operatorname{Im}\{V^0 \to V_2|_{q_1}\} \subset V_2|_{q_1}$ is different from the $\mathbb{C}^2 \subset V_2|_{q_2}$ induced by the canonical $\mathcal{O}_{D_1}(1)^{\oplus 2} \subset V_2|_{D_1}$. Indeed, if they are identical, then $V^0 \to V|_{p_1}$ has rank 0, a contradiction to $i_0 = 0$. Hence $V_3|_{D_1}$ has only one degree 1 factor.

It remains to show that $\dim T_{\rightarrow} \subset \dim V_3|_{q_2} = 2$. We claim that $T_{\rightarrow} = \operatorname{Im}\{A \to V^0 \to V_3|_{q_2}\}$, where $A = \ker\{V^0 \to V|_{p_1}\} \cong \mathbb{C}^2$. But this can be checked directly based on our explicit construction, and will be left to the readers.

The other cases are trivial except when the type of $\tilde{V}|_R$ is of type $(1,1,1)$. The study of this case is parallel to the case studied, and will be omitted. This proves the lemma. q.e.d.

Lemma 2.10. *Let the notation be as before. Then E on $X^\dagger_{n+m}$ is α-stable.*

Proof. We first check that $E|_D$ is regular. By the previous lemma, the restriction of E to each rational curve has only degree 1 and 0 factors. Hence to show $E|_D$ is regular we only need to show that there is no non-trivial section $s \in H^0(E)$ that vanishes on $X^0 \subset X^\dagger_{n+m}$. Let s be any such section and let $\tilde{s}$ be its lift in $H^0(X_{n,m}, \tilde{V})$. Then $\tilde{s}(q_n) \in T_{\rightarrow}$ and $\tilde{s}(q'_m) \in T'_{\rightarrow}$. On the other hand, if we let $v \in V^0$ be the lift of $s(q^\dagger)$ via the canonical $V^0 \cong E|_{q^\dagger}$, $\tilde{s}(q_n) = \tilde{s}(q'_m) = v$, under the canonical $\tilde{V}|_{q_n} \cong \tilde{V}|_{q'_m} \cong V^0$. By the previous lemma, $\tilde{s}(q_n)$ lies in the kernel of

$V^0 \to V|_{p_1}$ and $\tilde{s}(q'_m)$ lies in the kernel of $V^0 \to V|_{p_2}$. This is impossible unless $v = 0$ since $V^0 \to V|_{p_1+p_2}$ is injective. This shows that $s(q^\dagger) = 0$. Then $s = 0$ since $\tilde{V}|_{R_n}$ and $\tilde{V}|_{R'_m}$ are both regular. This proves that $E|_R$ is regular.

Once we proved that $E|_D$ is regular, we can apply Lemma 2.4 to conclude that E is α-stable. This completes the proof. q.e.d.

Corollary 2.11. *The family of locally free sheaves $\mathcal{E}$ on $W^\dagger$ over $\tilde{\mathbf{G}}^\alpha$ is a family of α-stable vector bundles.*

As a consequence, we get a morphism $\lambda : \tilde{\mathbf{G}}^\alpha \to \mathbf{M}^\alpha$ over $\mathbf{G}^\alpha$. Now let $\mathbf{U} \subset \tilde{\mathbf{G}}^\alpha$ be the largest open subset so that $\lambda|_{\mathbf{U}}$ is one-one. Since both $\tilde{\mathbf{G}}^\alpha$ and $\mathbf{M}^\alpha$ are smooth, λ will be an isomorphism if $\mathrm{Codim}(\tilde{\mathbf{G}}^\alpha - \mathbf{U}) \geq 2$. The complement of the 6 divisors $\mathbf{Y}_i, \mathbf{Z}_i$ ($i = 0, 1, 2$) represents GPBs (V, V^0) such that $V^0 \to V|_{p_1}$ and $V^0 \to V|_{p_2}$ are isomorphisms. Obviously this is mapped isomorphically by λ onto the open subset in $\mathbf{M}^\alpha$ whose points represent α-stable bundles over X_0. By construction, the complement of this open set in $\mathbf{M}^\alpha$ consists of 6 divisors whose generic points are bundles E over $X_1^\dagger$ (2 choices for $q^\dagger$) such that the restriction of E to the rational component is $\mathcal{O}^a \oplus \mathcal{O}(1)^{3-a}$ (3 choices for a). From our construction of the family of α-stable bundles over $\tilde{\mathbf{G}}^\alpha$, it is easy to see that λ maps the 6 divisors of $\tilde{\mathbf{G}}^\alpha$ to the 6 divisors of $\mathbf{M}^\alpha$. Hence the restriction of λ to any of the 6 divisors $\mathbf{Y}_i, \mathbf{Z}_i$ is generically finite. Since λ is injective on the complement of the 6 divisors, λ is a local homeomorphism at generic points of the divisors and hence λ is injective on an open set U whose complement has codimension ≥ 2. Therefore we have proved

Corollary 2.12. $\tilde{\mathbf{G}}^\alpha \cong \mathbf{M}^\alpha$.

3. Variations of $\mathbf{M}^\alpha_{r,d}(\mathfrak{X}^\dagger)$ in α

The goal of this section is to investigate how $\mathbf{M}^\alpha_{r,d}(\mathfrak{X}^\dagger)$ varies when α varies in $[0, 1)$. Following the work of [**2**, **15**], it is expected that there is a finite set $A \subset (0, 1)$ so that $\mathbf{M}^\alpha_{r,d}(\mathfrak{X}^\dagger)$ is a constant family when α varies in a connected component of $(0, 1) - A$. Further, for $\alpha \in A$ the two moduli spaces

$$\mathbf{M}^{\alpha^-}_{r,d}(\mathfrak{X}^\dagger) \longleftarrow -- \longrightarrow \mathbf{M}^{\alpha^+}_{r,d}(\mathfrak{X}^\dagger) \tag{3.1}$$

are birational and differ by a series of flips. In this section we will give detailed description of the flips of (3.1).

3.1. The jumping loci. We begin with determining the set A. We continue to assume $(r, \chi) = 1$ and $\chi > r$ throughout this section. Let $\alpha \in [0, 1) - \Lambda_r$ be any real number, let $\delta > 0$ be a sufficiently small number and let $\alpha^\pm = \alpha \pm \delta$. We suppose $\mathbf{M}^{\alpha^-}_{r,\chi}(\mathfrak{X}^\dagger)$ is different from

$\mathbf{M}^{\alpha^+}_{r,\chi}(\mathfrak{X}^\dagger)$. Then there is a locally free sheaf E on $X_n^\dagger$ that is in $\mathbf{M}^{\alpha^-}_{r,\chi}(\mathfrak{X}^\dagger)$ but not in $\mathbf{M}^{\alpha^+}_{r,\chi}(\mathfrak{X}^\dagger)$. Namely, there is a proper subsheaf $F \subset E$ so that

$$\mu_{\mathbf{d}}(F,\alpha^-) < \mu_{\mathbf{d}}(E,\alpha^-) \quad \text{while} \quad \mu_{\mathbf{d}}(F,\alpha^+) \geq \mu_{\mathbf{d}}(E,\alpha^+). \tag{3.2}$$

Since $E|_D$ is regular, $r_0(F) > 0$. Then (3.2) implies that

$$\frac{\chi(F) - \alpha r^\dagger(F)}{r_0(F)} = \frac{\chi(E) - \alpha r}{r}.$$

This means that $\alpha \in \Lambda_r$. Hence the set A can be chosen to be Λ_r. We summarize it as a lemma.

Lemma 3.1. *For any two α_1, α_2 in a connected component of $[0,1) - \Lambda_r$, the birational map (3.1) is an isomorphism.*

It is direct to check that $\Lambda_3 = \{1/3, 2/3\}$.

In the remainder of this section, we will restrict ourselves to the case where $r = 3$. Since for $\alpha \notin \Lambda_3$ the moduli space $\mathbf{M}^\alpha_{3,\chi}$ is a blow-up of $\mathbf{G}^\alpha_{3,\chi'}$, the previous lemma suggests the following lemma.

Lemma 3.2. *When α varies in a connected component of $[0,1) - \Lambda_3$ the moduli spaces $\mathbf{G}^\alpha_{3,\chi}$ are all isomorphic.*

Proof. The proof is straightforward and will be omitted. q.e.d.

As in [**4**], it is relatively easy to prove a vanishing result of the top Chern classes of a certain vector bundle on $\mathbf{M}^{1^-}_{3,\chi}$. What we need is the vanishing result on $\mathbf{M}^0_{3,\chi}$. One strategy to achieve this is to give an explicit description of the flips involved in the birational maps

$$\mathbf{M}^0_{3,\chi} \longleftarrow - \rightarrow \mathbf{M}^{1/2}_{3,\chi} \longleftarrow - \rightarrow \mathbf{M}^{1^-}_{3,\chi}.$$

It turns out the two arrows are similar. So we only need to study the first arrow in detail.

3.2. Variation of $\mathbf{G}^\alpha_{r,\chi'}$.

Since $\mathbf{M}^\alpha_{r,\chi}$ is a blow-up of $\mathbf{G}^\alpha_{r,\chi'}$, it is natural to study the variation of $\mathbf{G}^\alpha$ in detail, which we will do now.

As we will see, we need to study GPB (V, V^0) with $\dim V^0 \neq \operatorname{rank} V$. Here is our convention. We denote by $\mathbf{G}^\alpha_{r,\chi,a}$ the moduli space of α-stable GPBs (V, V^0) of rank r vector bundles V with $\chi(V) = \chi$ and a-dimensional subspaces $V^0 \subset V|_{p_1+p_2}$. We will still use $\mathbf{G}^\alpha_{r,\chi}$ to denote $\mathbf{G}^\alpha_{r,\chi,r}$, i.e., when $\dim V^0 = \operatorname{rank} V$. Also, in the remainder of this paper we will mostly be interested in the case $r = 3$ and $\chi = 4$; for convenience we will abbreviate $\mathbf{G}^\alpha_{3,7}$ to $\mathbf{G}^\alpha$ and abbreviate $\mathbf{M}^\alpha_{3,4}$ to $\mathbf{M}^\alpha$.

We first investigate how $\mathbf{G}^\alpha$ varies when α varies. Recall that a GPB $(V, V^0) \in \mathbf{G}^\alpha_{r,\chi,a}$ on X is α-stable (α-semistable) if for any nontrivial proper subsheaf $F \subset V$, we have

$$\frac{\chi(F) + (1-\alpha)\dim V^0 \cap F|_{p_1+p_2}}{r(F)} < \frac{\chi(V) + (1-\alpha)a}{r} \qquad (\leq).$$

Since both sides of the above inequality are linear, if a GPB (V, V^0) on X is α_1-stable but α_2-unstable for some $\alpha_1 < \alpha_2$ in the interval $[0, 1)$, then we get the equality

$$\text{(3.3)} \qquad \frac{\chi(F) + (1-\alpha)\dim V^0 \cap F|_{p_1+p_2}}{\mathrm{rank}(F)} = \frac{7 + (1-\alpha)3}{3}$$

for some α between α_1 and α_2. It is elementary to check that the equality can hold only when $\alpha = 1/3$ or $2/3$. Hence $\mathbf{G}^\alpha$ varies only at $\alpha = 1/3$ and $2/3$ and thus it suffices to consider only the moduli spaces $\mathbf{G}^{0^+}$, $\mathbf{G}^{1/2}$ and $\mathbf{G}^{1^-}$.

The variation of $\mathbf{G}^\alpha$ near $\alpha = 1/3$ can be described as follows: the equation (3.3) holds at $\alpha = 1/3$ only if we have a subbundle F such that

$$\text{(3.4)} \qquad \mathrm{rank}(F) = 2, \quad \chi(F) = 4, \quad \dim V^0 \cap F|_{p_1+p_2} = 3$$

or a subbundle L such that

$$\text{(3.5)} \qquad \mathrm{rank}(L) = 1, \quad \chi(L) = 3, \quad \dim V^0 \cap L_{p_1+p_2} = 0.$$

Suppose a GPB (V, V^0) is 0^+-stable but $1/2$-unstable. Then V has a subbundle L satisfying (3.5). The quotient bundle $F = V/L$ is equipped with a 3-dimensional subspace F^0 of $F|_{p_1+p_2}$ that is the image of V^0. Let $L^0 = 0$. Then both GPBs (L, L^0) and (F, F^0) are $1/3$-stable with the same parabolic slopes. Notice that the $1/3$-semistability is equivalent to the $1/3$-stability for $\mathbf{G}^{1/3}_{2,4,3}$ and $\mathbf{G}^{1/3}_{1,3,0}$.

We now let $\mathbf{A} = \mathbf{G}^{1/3}_{2,4,3} \times \mathbf{G}^{1/3}_{1,3,0}$. The previous argument shows that there are maps

$$\mathbf{G}^0 - \mathbf{G}^{1/2} \xrightarrow{\eta^-} \mathbf{A} \xleftarrow{\eta^+} \mathbf{G}^{1/2} - \mathbf{G}^0$$

that send (V, V^0) to pairs $((F, F^0), (L, L^0))$. We now show that there are two vector bundles W^- and W^+ over $\mathbf{A}$ so that

$$\text{(3.6)} \qquad \mathbf{G}^0 - \mathbf{G}^{1/2} = \mathbb{P}W^- \quad \text{and} \quad \mathbf{G}^{1/2} - \mathbf{G}^0 = \mathbb{P}W^+.$$

Let $(F^G, L^G) = ((F, F^0), (L, L^0)) \in \mathbf{A}$ be any pair. Let $\mathrm{Ext}^1(F^G, L^G)$ be the space of all extensions of GPBs

$$0 \longrightarrow L^G \longrightarrow V^G \longrightarrow F^G \longrightarrow 0.$$

It is a $\mathbb{C}$-vector space which fits into the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(F^G, L^G) \longrightarrow \mathrm{Hom}(F, L) \longrightarrow \mathrm{Hom}(F^0, L|_{p_1+p_2}/L^0) \longrightarrow \mathrm{Ext}^1(F^G, L^G) \longrightarrow \mathrm{Ext}^1(F, L) \longrightarrow 0.$$

Thus we have

$$(\eta^-)^{-1}((F^G, L^G)) = \mathbf{P}\,\mathrm{Ext}^1(F^G, L^G).$$

Since L^G and F^G are both 1/3-stable with the same slope, $\mathrm{Hom}(F^G, L^G) = 0$ by a standard argument. Hence by the Riemann-Roch theorem, we have

$$\dim \mathrm{Ext}^1(F^G, L^G) = -\chi(\mathrm{Ext}^\cdot(F, L)) + 6 = 2g.$$

(Recall $g(X) = g - 1$.) As to the base $\mathbf{A}$, we have

$$\dim \mathbf{G}^{1/3}_{2,4,3} = \dim \mathrm{Ext}^1(F^G, F^G) = -\chi(\mathrm{Ext}^\cdot(F, F)) + 1 + 3 = 4g - 4$$

and

$$\dim \mathbf{G}^{1/3}_{1,3,0} = \dim \mathrm{Ext}^1(L^G, L^G) = -\chi(\mathrm{Ext}^\cdot(L, L)) + 1 = g - 1.$$

Thus $\dim \mathbf{A} = 5g - 5$ and

$$\dim\big(\mathbf{G}^0 - \mathbf{G}^{1/2}\big) = (2g - 1) + (5g - 5) = 7g - 6.$$

Following the standard procedure, we pick a universal family $\mathcal{F}^G = (\mathcal{F}, \mathcal{F}^0)$ over $\mathbf{G}^{1/3}_{2,4,3} \times X^+$ and a universal family $\mathcal{L}^G = (\mathcal{L}, \mathcal{L}^0)$ over $\mathbf{G}^{1/3}_{1,3,0} \times X^+$. We let π_{ij} be the projection from $\mathbf{A} \times X = \mathbf{G}^{1/3}_{2,4,3} \times \mathbf{G}^{1/3}_{1,3,0} \times X$ to the product of the i-th and the j-th factor. We then form the locally free sheaves

$$W^- = \mathrm{Ext}^1_{\pi_{12}}(\pi_{13}^*\mathcal{F}^G, \pi_{23}^*\mathcal{L}^G) \quad \text{and} \quad W^+ = \mathrm{Ext}^1_{\pi_{12}}(\pi_{23}^*\mathcal{L}^G, \pi_{13}^*\mathcal{F}^G).$$

Note that fibers of W^- and W^+ over (F^G, L^G) are exactly $\mathrm{Ext}^1(F^G, L^G)$ and $\mathrm{Ext}^1(L^G, F^G)$, respectively. Again, as in [**14**] one shows that

$$\mathbf{G}^0 - \mathbf{G}^{1/2} = \mathbb{P}W^- \quad \text{and} \quad N_{\mathbb{P}W^-/\mathbf{G}^0} \cong \pi_-^* W^+ \otimes \mathcal{O}_{\mathbb{P}W^-}(-1).$$

Here we use $N_{A/B}$ to denote the normal bundle of $A \subset B$ and $\pi_- : \mathbb{P}W^- \to \mathbf{A}$ is the projection. Notice that

$$\dim \mathbf{M}^0 = \dim \mathbf{G}^0 = (7g - 6) + \dim \mathrm{Ext}^1(L^G, F^G) = 9g - 8$$

(the dimension of Ext^1 is calculated below), which is exactly the dimension of the moduli of rank three vector bundles over a genus g curve.

Similarly, the vector space $\mathrm{Ext}^1(L^G, F^G)$ that parameterize all extensions

$$0 \longrightarrow F^G \longrightarrow V^G \longrightarrow L^G \longrightarrow 0$$

satisfies a similar long exact sequence and by the stability of F^G we have

$$\dim \mathrm{Ext}^1(L^G, F^G) = -\chi(\mathrm{Ext}^\cdot(L, F)) = 2g - 2.$$

Hence for the same reason,

$$\mathbf{G}^{1/2} - \mathbf{G}^0 = \mathbb{P}W^+ \quad \text{and} \quad N_{\mathbb{P}W^+/\mathbf{G}^{1/2}} \cong \pi_+^* W^- \otimes \mathcal{O}_{\mathbb{P}W^+}(-1).$$

Again, following the work of Thaddeus [**14**], one checks that the blow-up of $\mathbf{G}^0$ along $\mathbb{P}W^-$ is isomorphic to the blow-up of $\mathbf{G}^{1/2}$ along $\mathbb{P}W^+$, extending the birational map $\mathbf{G}^0 - \to \mathbf{G}^{1/2}$. Since the details are routine, we omit them here. This is the explicit description of the flip between $\mathbf{G}^0$ and $\mathbf{G}^{1/2}$.

3.3. Flip loci in $\mathbf{M}^\alpha$–First approach. In this subsection, we will study the flip loci of $\mathbf{M}^0 \sim_{\rm bir} \mathbf{M}^{1/2}$ utilizing the fact that both are moduli of stable vector bundles over $X^\dagger$.

Let $\Sigma^- \subset \mathbf{M}^0$ and $\Sigma^+ \subset \mathbf{M}^{1/2}$ be the indeterminacy loci of the above birational map, namely the smallest closed subsets so that

$$\mathbf{M}^0 - \Sigma^- \xrightarrow{\cong} \mathbf{M}^{1/2} - \Sigma^+ \tag{3.7}$$

is an isomorphism. Our first approach to determine the set $\Sigma^\pm$ is to characterize all members in $\Sigma^\pm$.

Lemma 3.3. *Let $E \in \Sigma^\pm$ be any member and $F \subset E$ its $1/3^\mp$-destabilizing subsheaf. Then*

$$(r_0(F), r^\dagger(F), \chi(F)) = \begin{cases} (1,0,1) & \text{if } E \in \Sigma^-, \\ (2,3,3) & \text{if } E \in \Sigma^+. \end{cases}$$

Proof. By Lemma 2.5, we must have $r_0(F) = 1$ or 2. In the case $r_0(F) = 1$, $r^\dagger(F)$ and $\chi(F)$ must satisfy the equation $\chi(F) - r^\dagger(F)/3 = (4-1)/3$, which is possible only if $r^\dagger(F) = 0$ and $\chi(F) = 1$. To determine if E is in Σ^- or Σ^+, we only need to compute

$$\mu(F,0) \sim 1 < 4/3 \quad \text{and} \quad \mu(F,1/2) \sim 1 > 4/3 - 1/2,$$

which implies that $E \notin \mathbf{M}^{1/2}$. Thus $E \in \Sigma^-$.

Similarly, when $r_0(F) = 2$, the restraint is $\chi(F)/2 - r^\dagger(F)/6 = (4-1)/3$, which has solution $r^\dagger(F) = 3$ and $\chi(F) = 3$. A simple computation shows that $E \in \Sigma^+$. q.e.d.

We now give a more detailed description of pairs $F \subset E \in \Sigma^+$ which must have $(r_0(F), r^\dagger(F), \chi(F)) = (2,3,3)$. We assume E is over $X^\dagger_{n+m}$. As before, let $q_0, q_1, \ldots$ be the nodes of $X^\dagger_{n+m}$ with $q_0 = p_-$ in X. We define $r(F, q_i) = \dim\{F|_{q_i} \to E|_{q_i}\}$. Then since $r_0(F) = 2$, we must have $r(F,q_0)$, $r(F,q_n) \le 2$. On the other hand, let $F|_D^0 = \oplus_1^k \mathcal{L}_i$ [5] be the decomposition given by Lemma 2.6. Then because $r^\dagger(F) = 3$, $k \ge 3$. We claim that $k = 3$. First, because

$$4 \ge r(F,q_0) + r(F,q_n) = \sum r(\mathcal{L}_i, q_0) + r(\mathcal{L}_i, q_n),$$

and $r(\mathcal{L}_i, q_0) + r(\mathcal{L}_i, q_n) \ge 1$, $k > 4$ is impossible. When $k = 4$, by Lemma 3.3

$$(F|_D)^{\rm t.f.} \cong \mathcal{O}_{[0,i_1)} \oplus \mathcal{O}_{[0,i_2)} \oplus \mathcal{O}_{(j_1,n]} \oplus \mathcal{O}_{(j_2,n]}.$$

[5] $F|_D^0$ is the torsion free part of $F|_D$.

Clearly, this is possible only if $\chi(E|_D) \geq 4 + 4\chi(\mathcal{O}_D)$, which contradicts the regularity of $E|_D$. Hence $k = 3$. In this case we must have

$$(F|_D)^{\text{t.f.}} \cong \mathcal{O}_{[0,i)} \oplus \mathcal{O}_{(j,n]} \oplus \mathcal{O}_{[0,n]} \quad \text{or} \quad \mathcal{O}_{[0,i)} \oplus \mathcal{O}_{(j,n]} \oplus \mathcal{O}_{[0,n]}^{[k]}.$$

Now let the marked node of $X_n^\dagger$ be q_l. Since $r(F, q_l) > r(F, q_0)$ and $r(F, q_n)$, we must have $0 < l < n$; thus $n = 2$ or 3. When $n = 2$, $F|_D^0$ must be one of the list

$$\mathcal{O}_{[0,2)} \oplus \mathcal{O}_{(0,2]} \oplus \mathcal{O}_{[0,2]},\ \mathcal{O}_{[0,2)} \oplus \mathcal{O}_{(0,2]} \oplus \mathcal{O}_{[0,2]}^{[1]},\ \mathcal{O}_{[0,2)} \oplus \mathcal{O}_{(0,2]} \oplus \mathcal{O}_{[0,2]}^{[2]}. \tag{3.8}$$

When $n = 3$, the $\alpha^{\pm}$-stable condition on E forces $E|_{D_i}$ to have at least one degree 1 factor and combined there are at most three degree 1 factors. Hence each $E|_{D_i}$ has exactly one degree 1 factor. Following this, it is easy to see that when $q^\dagger = q_1$, $F|_D^0$ must be one of

$$\mathcal{O}_{[0,2)} \oplus \mathcal{O}_{(0,3]} \oplus \mathcal{O}_{[0,3]}^{[3]},\ \mathcal{O}_{[0,3)} \oplus \mathcal{O}_{(0,3]} \oplus \mathcal{O}_{[0,3]}^{[2]} \tag{3.9}$$

and when $q^\dagger = q_2$, $F|_D$ must be one of

$$\mathcal{O}_{[0,3)} \oplus \mathcal{O}_{(1,3]} \oplus \mathcal{O}_{[0,3]}^{[1]},\ \mathcal{O}_{[0,3)} \oplus \mathcal{O}_{(0,3]} \oplus \mathcal{O}_{[0,3]}^{[2]}.$$

To make our presentation easier to follow, we represent such subsheaves by graphs. Here is the rule we will follow: for each $D_k \subset D$, we will encounter invertible subsheaves $\sigma : \mathcal{L} \to E|_{D_k}$, where $\mathcal{L}$ is either $\mathcal{O}_{D_k}$ or $\mathcal{O}_{D_k}(1)$. There are two cases, depending on whether the image sheaf $\sigma(\mathcal{L})$ lies in a factor $\mathcal{O}_{D_k}$ or a factor $\mathcal{O}_{D_k}(1)$ of $E|_{D_k}$. In case $\sigma(q_{k-1}) = 0$ (resp. $\neq 0$) we will attach a circle (resp. dot) to the left end point of this line segment. We attach a circle or a dot to the right end point of the line segment according to whether $\sigma(q_k) = 0$ or $\neq 0$. Following this rule, we will represent a sheaf of $\mathcal{O}_D$-modules whose restriction to each D_k is as mentioned by a chain of line segments, with dots or circles attached. The following is the list of such subsheaves on $D = D_{[0,2]}$:

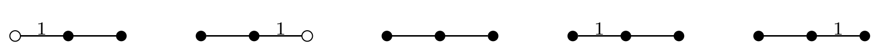


Figure 1. These represent subsheaves $\mathcal{O}_{(0,2]}$, $\mathcal{O}_{[0,2)}$, $\mathcal{O}_{[0,2]}$, $\mathcal{O}_{[0,2]}^{[1]}$ and $\mathcal{O}_{[0,2]}^{[2]}$.

We next indicate how to represent a pair of sheaves $F \subset E$ near $D \subset X_2^\dagger$. Let U_1 and U_2 be small (analytic) disks containing p_1 and $p_2 \in X$ and let $\hat{D} = U_1 \cup D \cup U_2 \subset X_2$ be an analytic neighborhood of $D \subset X_2$. The following are three examples of pairs $F \subset E$ of subsheaves in invertible sheaves E:

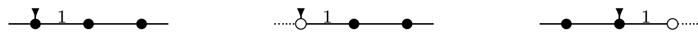


Figure 2. The sheaves E in all three cases are invertible sheaves of $\mathcal{O}_{\hat{D}}$-modules. Their degrees along D_1 and D_2 are 1 and 0 (abbreviated $\mathcal{O}_{\hat{D}}^{[1]}$) in the first two cases and 0 and 1 (abbreviated $\mathcal{O}_{\hat{D}}^{[2]}$) in the last case. In the first case, $F \cong E$ and $E/F = 0$; in the second case, $F = \mathcal{O}_{D\cup U_2}$ and $E/F \cong \mathcal{O}_{U_1}$; in the last case, $F = \mathcal{O}_{U_1\cup D}$ and $E/F \cong \mathcal{O}_{U_2}$. The three arrows indicate that the marked node of the first two examples is q_0 and of the last example is q_1.

Accordingly, a pair $F \subset E$ with $\operatorname{rank} E = 3$ along $\hat{D}$ will be represented by three horizontal lines, each representing a direct summand of $E|_{\hat{D}}$. The following is such an example:

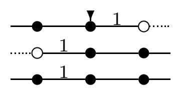


Figure 3. In this example, $E|_{\hat{D}}$ is a direct sum of (from top to bottom) $\mathcal{O}_{\hat{D}}^{[2]} \oplus \mathcal{O}_{\hat{D}}^{[1]} \oplus \mathcal{O}_{\hat{D}}^{[1]}$; the solid lines represent the subsheaf $F|_{\hat{D}} \subset E|_{\hat{D}}$, which is a direct sum of $\mathcal{O}_{U_1\cup D} \subset \mathcal{O}_{\hat{D}}^{[2]}$, $\mathcal{O}_{D\cup U_2} \subset \mathcal{O}_{\hat{D}}^{[1]}$ and $\mathcal{O}_{\hat{D}}^{[1]} \subset \mathcal{O}_{\hat{D}}^{[1]}$.

By analyzing the possible structures of $\{F \subset E\} \in \Sigma^+$ over $X^\dagger_{n+m}$, we arrive at the following complete lists of such sheaves:

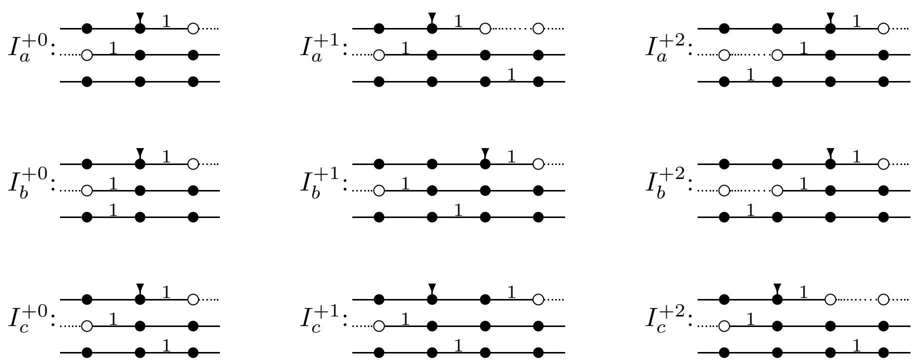


Figure 4. This is the complete list of sheaves in Σ^+. Note that I_c^{+i} are the reflections of I_b^{+i}.

Lemma 3.4. *The above is a complete list of sheaves in Σ^+. The sheaves of types I_i^{+j} can be (small) deformed to sheaves of type I_i^{+0}. Let $\mathbb{I}_a^+$, $\mathbb{I}_b^+$ and $\mathbb{I}_c^+$ be the set of sheaves in Σ^+ of types $I_a^{+\cdot}$, $I_b^{+\cdot}$ and $I_c^{+\cdot}$, respectively. Then $\mathbb{I}_a^+$, $\mathbb{I}_b^+$ and $\mathbb{I}_c^+$ are three irreducible components of Σ^+. Finally, $\mathbb{I}_b^+ \cap \mathbb{I}_c^+ = \emptyset$ while $\mathbb{I}_a^+ \cap \mathbb{I}_b^+$ (resp. $\mathbb{I}_a^+ \cap \mathbb{I}_c^+$) is the set consisting of sheaves of type $I_a^{+1} = I_c^{+2}$ (resp. $I_a^{+2} = I_b^{+2}$).*

Proof. We have already shown that sheaves of types in the above list can not be 0-stable. On the other hand, it is easy to construct examples of sheaves of each of the type in the list that are 1/2-stable. Hence, all types in the list classifies some sheaves in Σ^+. It remains to show that this is a complete list. If $\{F \subset E\} \in \Sigma^+$ is over $X_2^\dagger$, then we have already shown that it must be from the list $\{I_a^{+0}, I_b^{+0}, I_c^{+0}\}$. The case where E is over $X_3^\dagger$ is similar, and will be omitted.

The statement that sheaves in I_a^{+1} can be deformed to sheaves in I_a^{+0} is straightforward. Let E be such a sheaf, over $X_3^\dagger$ with $q^\dagger = q_1$. Clearly, by smoothing the node $q_3 \in X_3^\dagger$ we can deform $X_3^\dagger$ to $X_2^\dagger$. Then it is direct to see that we can deform E to E' on $X_2^\dagger$ so that E' has type I_a^{+0}. The proof of the remaining statements are similar. We omit the details here since a direct construction will be given when we study the flips of $\mathbf{M}^\alpha$ later. q.e.d.

We next give the graphs of all types of sheaves in Σ^-. Since the proof is parallel, we will omit it here.

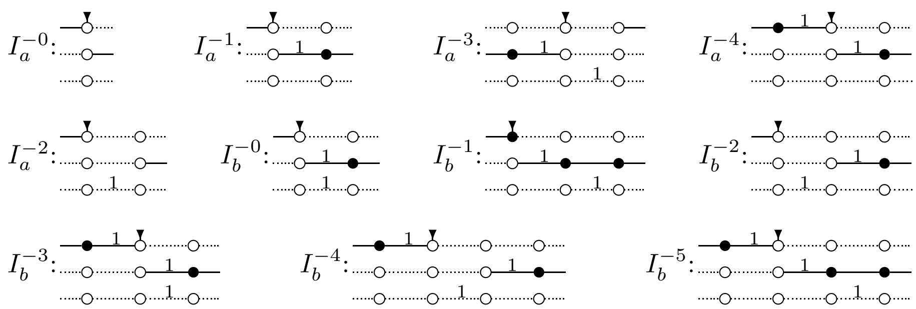


Figure 5. These are graphs of some of the sheaves in Σ^-.

Proposition 3.5. *The sheaves in Σ^- can be divided into subsets:*

1. I_a^{-i} *and* $I_a^{-i'}$ *where* $i = 0, \dots, 4$. *The graph* $I_a^{-i'}$ *is the reflection of* I_a^{-i} *along the vertical axis passing through the arrow;*
2. I_b^{-i} *where* $i = 0, \dots, 5$*;*
3. I_c^{-i}*, where* $i = 0, \dots, 5$. *Here the graph of* I_c^{-i} *is the reflection of* I_b^{-i} *along the axis passing the arrow.*

The set Σ^- is an irreducible subvariety of $\mathbf{M}^0$.

3.4. Flip loci in $\mathbf{M}^\alpha$–Second approach. In this subsection we will give an alternative description of the flip loci $\Sigma^\pm$ of $\mathbf{M}^0 \sim_{\text{bir}} \mathbf{M}^{1/2}$, based on the flip loci of the moduli of GPBs described before.

First let us describe Σ^+. We know that $\mathbf{M}^{1/2}$ is the result of blowing-up $\mathbf{G}^{1/2}$ along $\mathbf{Y}_0 \cup \mathbf{Z}_0$ and then blowing-up the proper transform of $\mathbf{Y}_1 \cup \mathbf{Z}_1$. It is obvious that Σ^+ lies in the inverse image of $\mathbf{G}^{1/2} - \mathbf{G}^0 = \mathbb{P}W^+$. It is easy to see that the varieties $\mathbf{Y}_0$ and $\mathbf{Z}_0$ are disjoint from $\mathbb{P}W^+$. Thus they do not contribute to the flip loci Σ^+. Let (V, V^0) be a non-split extension of $(L,0)\in \mathbf{G}^{1/3}_{1,3,0}=M_{1,3}(X)$ by $(F,F^0)\in \mathbf{G}^{1/3}_{2,4,3}$. Then (V,V^0) lies in $\mathbf{Y}_1$ (resp. $\mathbf{Z}_1$) if and only if $F|_{p_1}\subset F^0$ (resp. $F|_{p_2}\subset F^0$). Let

$$\Xi_1=\{\big((F,F^0),(L,0)\big)\mid F|_{p_2}\subset F^0\}\quad\text{and}$$
$$\Xi_2=\{\big((F,F^0),(L,0)\big)\mid F|_{p_1}\subset F^0\},$$

both subsets of $\mathbf{A}$. Since $\mathbf{Y}_1$ is defined to be the loci where $V^0\to V|_{p_1}$ have dimensions at most one, $\mathbf{Y}_1\cap\mathbb{P}W^+$ is the preimage of $\Xi_1\subset\mathbf{A}$, which has codimension 2 in $\mathbb{P}W^+$ and the normal bundle $N_{\mathbf{Y}_1\cap\mathbb{P}W^+/\mathbb{P}W^+}$ is the pull-back of the normal bundle $N_{\Xi_1/\mathbf{A}}$. A similar statement holds for $\mathbf{Z}_1\cap\mathbb{P}W^+$ as well. To determine the preimage of $\mathbb{P}W^+$ in $\mathbf{M}^{1/2}$, we need to know the normal bundle $N_{\mathbf{Y}_1/\mathbf{G}^{1/2}}$, especially its restriction to $\mathbf{Y}_1\cap\mathbb{P}W^+$. Now let $\xi\in\mathbf{Y}_1\cap\mathbb{P}W^+$ be any point lying over $(F^G,L^G)\in\mathbf{A}$. It is direct to see that

$$T_\xi\mathbf{G}^{1/2}/\big(T_\xi\mathbf{Y}_1+T_\xi\mathbb{P}W^+\big)\cong\operatorname{Hom}(F|_{p_1},L|_{p_2}) \tag{3.10}$$

canonically. Since $\mathbf{Y}_1\cap\mathbb{P}W^+$ has codimension two in $\mathbb{P}W^+$,

$$\dim T_\xi\mathbf{G}^{1/2}/T_\xi\mathbf{Y}_1=4.$$

A similar picture holds for the intersection $\mathbf{Z}_1\cap\mathbb{P}W^+$.

Now let $\mathbf{B}$ be the blow-up of $\mathbf{A}$ along $\Xi_1\cup\Xi_2$ with Υ_1 and Υ_2 the two exceptional divisors in $\mathbf{A}$ over Ξ_1 and Ξ_2 respectively. Then the preimage of $\mathbb{P}W^+$ in $\mathbf{M}^{1/2}$ is the union of three smooth irreducible varieties: the first is the blow-up of $\mathbb{P}W^+$ along $\mathbb{P}W^+\cap(\mathbf{Y}_1\cup\mathbf{Z}_1)$, which is $\mathbb{P}W^+\times_{\mathbf{A}}\mathbf{B}$, a projective bundle over $\mathbf{B}$; the second is $\mathbf{P}^3$-bundles over $\mathbb{P}W^+\times_{\mathbf{A}}\Xi_1$ and the third is $\mathbf{P}^3$-bundles over $\mathbb{P}W^+\times_{\mathbf{A}}\Xi_2$. We denote these three components by $\mathbb{I}^+_a$, $\mathbb{I}^+_b$ and $\mathbb{I}^+_c$, respectively. Note that

$$\dim\mathbf{B}=5g-5,\quad \dim\mathbb{I}^+_a=7g-8\quad\text{and}\quad\dim\mathbb{I}^+_b=\dim\mathbb{I}^+_c=7g-7. \tag{3.11}$$

The intersections are

$$\mathbb{I}^+_a\cap\mathbb{I}^+_b=\mathbb{I}^+_a\times_{\mathbf{B}}\Upsilon_1\quad\text{and}\quad\mathbb{I}^+_a\cap\mathbb{I}^+_c=\mathbb{I}^+_a\times_{\mathbf{B}}\Upsilon_1.$$

We close this subsection by showing that the subsets $\mathbb{I}^\pm_\cdot$ just defined are exactly the corresponding subsets described in the previous subsection. Indeed, a general sheaf $[\mathcal{E}]\in\mathbb{I}_a$ defined in this section has associated GPB E^G fitting into the exact sequence

$$0\longrightarrow(F,F^0)\longrightarrow(E,E^0)\longrightarrow(L,0)\longrightarrow 0$$

so that $F^0\cap F|_{p_1}$ and $F^0\cap F|_{p_2}$ are 1-dimensional. Hence the associated subsheaf $\mathcal{F}\subset\mathcal{E}$ has no $\mathcal{O}_{\mathbf{P}^1}(1)$ when restricted to any rational component in the base curve of $\mathcal{E}$. This shows that the two $\mathbb{I}_a$ defined in the above two subsections are identical. Now consider a general sheaf $[\mathcal{E}]$ in the component $\mathbb{I}_b$ defined in this section. Since its associated GPB E^G

is in $\mathbf{Y}_1$, it still fits into the above exact sequence with $F|_{p_2} \subset F^0$. In particular, there is a rational $\mathbf{P}^1$ in the base of $\mathcal{E}$ which is to the left of the marked node so that $\mathcal{F}|_{\mathbf{P}^1}$ has a factor $\mathcal{O}_{\mathbf{P}^1}(1)$. This shows that the two definitions of $\mathbb{I}_b$ are identical.

3.5. Flipping $\mathbf{M}^{1/2}$. The goal of this subsection is to show that we can flip $\mathbf{M}^{1/2}$ along $\mathbb{I}_a^+$ and then flip the resulting variety along the proper transform of $\mathbb{I}_a^+ \cup \mathbb{I}_c^+$. We will show in the next section that the resulting variety is isomorphic to $\mathbf{M}^0$.

We begin with determining the normal bundles of $\mathbb{I}_\bullet^\pm$. In the following we adopt the convention that for $S \subset P$ we denote by $T_S P$ the restriction of the tangent bundle TP to S.

Lemma 3.6. *Let P, Q be smooth subvarieties of a nonsingular variety R such that $S \triangleq P \cap Q$ is smooth. Let $\pi : \tilde{R} \to R$ be the blowing-up along Q and $\tilde{P}$ be the proper transform of P. Then we have an exact sequence of vector bundles*

$$0 \longrightarrow N_{\tilde{P}/\tilde{R}} \longrightarrow \pi^* N_{P/R} \longrightarrow \pi^*[T_S R/(T_S P + T_S Q)] \longrightarrow 0.$$

Proof. It follows from Lemma 15.4 (i), (iv) in [**3**]. q.e.d.

Our first application of the lemma is a description of the normal bundle to $\mathbb{I}_a^+$. We put $R = \mathbf{M}^{1/2}$, $P = \mathbb{P}W^+$ and $Q = \mathbf{Y}_1 \cup \mathbf{Z}_1$. Since $S \triangleq P \cap Q$ is $\mathbb{P}W^+ \times_{\mathbf{A}} (\Xi_1 \cup \Xi_2)$, we have $\tilde{P} = \mathbb{P}\tilde{W}^+$, where $\tilde{W}^+$ is the pull back of W^+ to $\mathbf{B}$, and where the latter is the blow-up of $\mathbf{A}$ along $\Xi_1 \cup \Xi_2$. To determine the normal bundle $N_{\tilde{P}/\tilde{R}}$, we need to find the other two terms in the above exact sequence. The normal bundle $N_{P/R} \cong \rho^* W^-(-1)$. Also a globalized version of the isomorphism (3.10) shows that the quotient bundle $T_S R/(T_S P + T_S Q)$ is the pull-back of a vector bundle on $\Xi_1 \cup \Xi_2$, tensored with $\mathcal{O}_{\mathbb{P}W^+}(-1)$. Thus by Lemma 3.6, the normal bundle $N_{\mathbb{I}_a^+/\mathbf{M}^{1/2}}$, which is $N_{\tilde{P}/\tilde{R}}$ in the statement of lemma, becomes the pull-back of a vector bundle over $\mathbf{B}$ tensored with $\mathcal{O}_{\mathbb{P}\tilde{W}^+}(-1)$. Hence by the standard theory in birational geometry we can flip $\mathbf{M}^{1/2}$ along $\mathbb{I}_a^+$. Let $\mathbf{M}_1$ be the result of this flip, let $\tilde{\mathbb{I}}_a$ be the flipped loci and let $\tilde{\mathbb{I}}_b$ and $\tilde{\mathbb{I}}_c$ be the proper transforms of $\mathbb{I}_b$ and $\mathbb{I}_c$.

Our next step is to show that we can flip $\mathbf{M}_1$ along $\tilde{\mathbb{I}}_b \cup \tilde{\mathbb{I}}_c$. We begin with a detailed description of $\tilde{\mathbb{I}}_b$. As mentioned before, $\mathbb{I}_a^+ \cap \mathbb{I}_b^+$ is the projective bundle $\mathbb{P}_{\Upsilon_1} W^+$ while $\mathbb{I}_b^+$ is a $\mathbf{P}^3$-bundle over $\mathbb{P}_{\Xi_1} W^+$. To proceed, we need a more detailed description of $\mathbb{I}_b^+$. Let $\mathcal{F}^G = (\mathcal{F}, \mathcal{F}^0)$ and $\mathcal{L}^G = (\mathcal{L}, 0)$ be the restrictions to Ξ_1 of the pull-backs of the universal families of $\mathbf{G}^{1/3}_{2,4,3}$ and $\mathbf{G}^{1/3}_{1,3,0}$, respectively. Since $\mathcal{F}_\xi|_{p_2} \subset \mathcal{F}^0_\xi$ for each $\xi \in \Xi_1$, there is a tautological line subbundle $\ell \subset \mathcal{F}|_{p_1}$ so that for each ξ

$$\mathcal{F}^0_\xi = \mathcal{F}_\xi|_{p_2} \oplus \ell_\xi.$$

Then the normal bundle $N_{\mathbf{Y}_1/\mathbf{G}}$ to $\mathbf{Y}_1$ in $\mathbf{G} \triangleq \mathbf{G}^{1/2}$, restricting to $\mathbb{P}_{\Xi_1}W^+ \subset \mathbf{G}$, is

$$N_{\mathbf{Y}_1/\mathbf{G}}|_{\mathbb{P}_{\Xi_1}W^+} = \mathcal{H}om\big(\psi_1^*\mathcal{F}|_{p_2}, \psi_1^*\mathcal{F}|_{p_1}/\ell \oplus \psi_1^*\mathcal{L}|_{p_1}(-1)\big)$$

where $\psi_1 : \mathbb{P}_{\Xi_1}W^+ \to \Xi_1$ is the projection and the sheaf $\psi_1^*\mathcal{L}|_{p_1}$ is twisted by $\mathcal{O}_{\mathbb{P}W^+}(-1)$ because the universal GPB over $\mathbb{P}_{\Xi_1}W^+$ is given by

$$0 \longrightarrow \psi_1^*\mathcal{F}^G \longrightarrow \mathcal{E}^G \longrightarrow \psi_1^*\mathcal{L}^G \otimes \mathcal{O}_{\mathbb{P}W^+}(-1) \longrightarrow 0.$$

Therefore,

$$\mathbb{I}_b^+ = \mathbb{P}\big(\psi_1^*\mathcal{F}|_{p_2}^\vee \otimes (\psi_1^*\mathcal{F}|_{p_1}/\ell \oplus \psi_1^*\mathcal{L}|_{p_1}(-1))\big), \tag{3.12}$$

as a $\mathbf{P}^3$-bundle over $\mathbb{P}_{\Xi_1}W^+$, which itself is a smooth subvariety of $\mathbf{M}^{1/2}$.

Based on this description, it is easy to see that $\mathbb{I}_a^+ \cap \mathbb{I}_b^+$ is the subbundle

$$\mathbb{I}_a^+ \cap \mathbb{I}_b^+ = \mathbb{P}\big(\psi_1^*(\mathcal{F}|_{p_2}^\vee \otimes (\mathcal{F}|_{p_1}/\ell))\big) = \mathbb{P}\big(\mathcal{F}|_{p_2}^\vee \otimes \mathcal{F}|_{p_1}/\ell\big) \times_{\Xi_1} \mathbb{P}_{\Xi_1}W^+.$$

Let

$$\begin{array}{ccc} \mathbb{P}_{\Xi_1}W^+ \times_{\Xi_1} \mathbb{P}(\mathcal{F}|_{p_2}^\vee) & \xrightarrow{\pi_2} & \mathbb{P}\mathcal{F}|_{p_2}^\vee \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \psi_2} \\ \mathbb{P}_{\Xi_1}W^+ & \xrightarrow{\psi_1} & \Xi_1 \end{array}$$

be the projections and let $\mathcal{K}$ be the tautological line subbundle of $\mathcal{F}|_{p_2}^\vee$ on $\mathbb{P}\mathcal{F}|_{p_2}^\vee$. Then after we blow up $\mathbf{M}^{1/2}$ along $\mathbb{I}_a^+$, the proper transform $\mathbf{E}_b$ is the blowing up of $\mathbb{I}_b^+$ along $\mathbb{I}_a^+ \cap \mathbb{I}_b^+$:

$$\mathbf{E}_b = \mathbb{P}\big(\pi_1^*\psi_1^*\mathcal{F}|_{p_2}^\vee \otimes (\mathcal{F}|_{p_1}/\ell) \oplus \pi_2^*\mathcal{K} \otimes \pi_2^*\psi_2^*\mathcal{L}|_{p_1} \otimes \pi_1^*\mathcal{O}(-1)\big), \tag{3.13}$$

as a $\mathbf{P}^2$-bundle over $\mathbb{P}_{\Xi_1}W^+ \times_{\Xi_1} \mathbb{P}\mathcal{F}|_{p_2}^\vee$. Inside this $\mathbf{P}^2$-bundle there is a subbundle

$$\mathbb{P}\big(\pi_1^*\psi_1^*\mathcal{F}|_{p_2}^\vee \otimes (\mathcal{F}|_{p_1}/\ell)\big) \quad \text{over} \quad \mathbb{P}W^+ \times_{\Xi_1} \mathbb{P}\mathcal{F}|_{p_2}^\vee,$$

which is the intersection $\mathbf{E}_a \cap \mathbf{E}_b$. Viewed as a bundle over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$, we have

$$\mathbf{E}_a \cap \mathbf{E}_b = \big(\mathbb{P}W^+ \times_{\Xi_1} \mathbb{P}\mathcal{F}|_{p_2}^\vee\big) \times_{\Xi_1} \mathbb{P}\mathcal{F}|_{p_2}^\vee.$$

The proper transform $\tilde{\mathbb{I}}_b \subset \mathbf{M}_1$ is then the contraction of $\mathbf{E}_a \cap \mathbf{E}_b$ along all $\mathbb{P}W^+$ factors, which is a bundle over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$. We claim that $\tilde{\mathbb{I}}_b$ is a projective bundle over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$. Indeed, $\mathbf{E}_b$, considered as a bundle over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$, has a subbundle $\mathbb{P}_{\Xi_1}W^+ \times_{\Xi_1} \mathbb{P}\mathcal{F}|_{p_2}^\vee$. Further, using the explicit expression (3.13) its normal bundle in $\mathbf{E}_b$ along each slice

$$\mathbb{P}W_\xi^+ \times \eta \subset \mathbb{P}_{\Xi_1}W^+ \times_{\Xi_1} \mathbb{P}\mathcal{F}|_{p_2}^\vee,$$

where $\eta \in \mathbb{P}\mathcal{F}|_{p_2}^\vee$ is over $\xi \in \Xi_1$, is isomorphic to $\mathcal{O}_{\mathbb{P}W_\xi^+}(1)^{\oplus 2}$. Hence $\tilde{\mathbb{I}}_b$ is a projective bundle over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$. By a moment of thought, one sees that

$$\tilde{\mathbb{I}}_b = \mathbb{P}W_+ \quad \text{where} \quad W_+ = \mathcal{K}^\vee \otimes \psi_2^*(\mathcal{L}|_{p_1}^\vee \otimes \mathcal{F}|_{p_2}^\vee \otimes \mathcal{F}|_{p_1}/\ell) \oplus \psi_2^* W^+.$$

Let $\pi\colon\tilde{\mathbb{I}}_b = \mathbb{P}W_+ \to \mathbb{P}\mathcal{F}|_{p_2}^\vee$ be the projection.

Lemma 3.7. *There is a vector bundle W_- over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$ so that the normal bundle $N_{\tilde{\mathbb{I}}_b/\mathbf{M}_1}$ is isomorphic to $\pi^* W_- \otimes \mathcal{O}_{\mathbb{P}W_+}(-1)$. The same conclusion holds for $\tilde{\mathbb{I}}_c$ as well.*

As a corollary, we can flip $\mathbf{M}_1$ along $\tilde{\mathbb{I}}_b \cup \tilde{\mathbb{I}}_c$ to obtain a new smooth variety $\mathbf{M}_2$.

Proof. We continue to use the notation developed earlier. First, the exceptional divisor $\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}$ of the blowing up of $\mathbf{G} = \mathbf{G}^{1/2}$ is a fiber bundle over $\mathbf{Y}_1$. Because $\mathbb{I}_b^+ = \mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}} \times_{\mathbf{Y}_1} (\mathbf{Y}_1 \cap \mathbb{P}W^+)$, we have the exact sequence of vector bundles

$$0 \longrightarrow N_{\mathbb{I}_b^+/\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}} \longrightarrow N_{\mathbb{I}_b^+/\mathbf{M}^{1/2}} \longrightarrow N_{\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}/\mathbf{M}^{1/2}} \longrightarrow 0$$

and the identity

$$N_{\mathbb{I}_b^+/\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}} = \pi^* N_{\mathbf{Y}\cap\mathbb{P}W^+/\mathbf{Y}_1},$$

where

$$\pi : \mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}} \times_{\mathbf{Y}_1} (\mathbf{Y}_1 \cap \mathbb{P}W^+) \to \mathbf{Y}_1 \cap \mathbb{P}W^+$$

is the projection. Let $\eta \in \mathbb{P}\mathcal{F}|_{p_2}^\vee$ be any point over $\xi \in \Xi_1$. Then based on the description (3.12) η defines naturally a subvariety

$$\mathbf{P}_\eta \triangleq \mathbb{P}\big(0 \oplus \eta \otimes \mathcal{L}_\xi|_{p_1}(-1)\big) \subset \mathbb{I}_b^+$$

which is a section of

$$\mathbb{I}_b^+ \times_{\Xi_1} \xi \longrightarrow \mathbb{P}_{\Xi_1} W^+ \times_{\Xi_1} \xi \triangleq \mathbb{P}W_\xi^+.$$

Hence $\mathbf{P}_\eta$ is isomorphic to $\mathbb{P}W_\xi^+$.

We claim that the normal bundle $N_{\mathbb{I}_b^+/\mathbf{M}^{1/2}}$ restricting to $\mathbf{P}_\eta$ is isomorphic to a vector space V tensored by $\mathcal{O}_{\mathbf{P}_\eta}(-1)$. Indeed, since the normal bundle $N_{\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}/\mathbf{M}^{1/2}}$ is the tautological sub-line bundle of the pull back of $N_{\mathbf{Y}_1/\mathbf{G}}$ over $\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}$, its restriction to $\mathbf{P}_\eta = \mathbb{P}\big(0 \oplus \eta \otimes \mathcal{L}_\xi|_{p_1}(-1)\big)$ is isomorphic to $\mathcal{O}_{\mathbf{P}_\eta}(-1)$. As to the term $N_{\mathbf{Y}_1\cap\mathbb{P}W^+/\mathbf{Y}_1}$, it is clear that its restriction to $\mathbb{P}W_\xi^+$ is isomorphic to $V \otimes \mathcal{O}(-1)$ for a linear subspace $V \subset \operatorname{Ext}^1(\mathcal{F}_\xi^G, \mathcal{L}_\xi^G)$. Hence $N_{\mathbb{I}_b^+/\mathbb{P}N_{\mathbf{Y}_1/\mathbf{G}}}|_{\mathbf{P}_\eta}$, and therefore $N_{\mathbb{I}_b^+/\mathbf{M}^{1/2}}|_{\mathbf{P}_\eta}$, are of the forms $V' \otimes \mathcal{O}_{\mathbf{P}_\eta}(-1)$ for some vector spaces V'.

Finally, since the flip loci of $\mathbf{M}^{1/2} \sim \mathbf{M}_1$ are away from $\iota(\mathbb{P}W_\xi^+)$,

$$N_{\tilde{\mathbb{I}}_b/\mathbf{M}_1}|_{\mathbf{P}_\eta} \cong N_{\mathbb{I}_b^+/\mathbf{M}^{1/2}}|_{\mathbf{P}_\eta}.$$

By a theorem in [**9**], the restriction of $N_{\tilde{\mathbb{I}}_b/\mathbf{M}_1}$ to the fiber of $\mathbb{P}W_+$ over $\eta \in \mathbb{P}\mathcal{F}|_{p_2}^\vee$ is of the form $W' \otimes \mathcal{O}(-1)$. Because this is true for all

$\eta \in \mathbb{P}\mathcal{F}|^{\vee}_{p_2}$, there must be a vector bundle W_- satisfying the requirement of the lemma. The case for $\tilde{\mathbb{I}}_c$ is exactly the same and will be omitted. q.e.d.

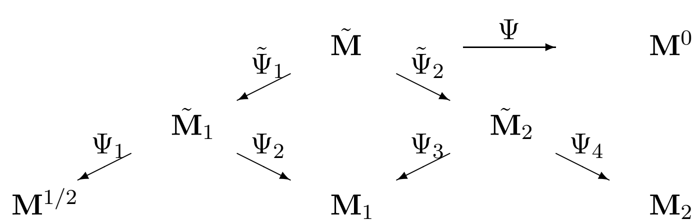
4. The isomorphism of the two flips

The goal of this section is to prove the following.

Proposition 4.1. *The birational* $\mathbf{M}^0 \sim \mathbf{M}^{1/2}$ *induces an isomorphism* $\mathbf{M}^0 \cong \mathbf{M}_2$.

We first briefly outline the strategy. As argued in Section 3, we can flip $\mathbf{M}^{1/2}$ along $\mathbb{I}_a^+$ to obtain a new variety $\mathbf{M}_1$. Let $\tilde{\mathbb{I}}_a$, etc., be the flipped loci of $\mathbb{I}_a^+$, etc. Then Lemma 3.7 tells us that we can flip $\mathbf{M}_1$ again along $\tilde{\mathbb{I}}_b$ and $\tilde{\mathbb{I}}_c$ to obtain a new variety $\mathbf{M}_2$. The key is to show that the birational map $\mathbf{M}_2 \sim \mathbf{M}^0$ extends to a morphism $\mathbf{M}_2 \to \mathbf{M}^0$, because it is an isomorphism away from a subset of codimension at least two, it is an isomorphism.

We now list the main steps in constructing the morphism $\mathbf{M}_2 \to \mathbf{M}^0$. We first blow up $\mathbf{M}^{1/2}$ along $\mathbb{I}_a^+$ to obtain the variety $\tilde{\mathbf{M}}_1$, and then contract its exceptional divisor to get the flip $\mathbf{M}_1$. Let $\tilde{\mathbb{I}}_b$ and $\tilde{\mathbb{I}}_c$ be the flipped loci of $\mathbb{I}_b^+$ and $\mathbb{I}_c^+$. We then blow up $\mathbf{M}_1$ along $\tilde{\mathbb{I}}_a \cup \tilde{\mathbb{I}}_c$ and contract the exceptional divisor to get the second flip $\mathbf{M}_2$, as shown below. It is easy to see that if we blow up $\tilde{\mathbf{M}}_1$ along $\pi_2^{-1}(\tilde{\mathbb{I}}_b \cup \tilde{\mathbb{I}}_c)$, the resulting variety $\tilde{\mathbf{M}}$ fits into the diagram below. The first main technical part of the proof is to show that the birational maps extend to a morphism $\phi\colon \tilde{\mathbf{M}} \to \mathbf{M}^0$. This is achieved by first picking a (local) universal family of $\mathbf{M}^{1/2}$ and performing an elementary modification to it to obtain a family on $\tilde{\mathbf{M}}_1$, and then performing another elementary modification to the new family to get a family over $\tilde{\mathbf{M}}$. We will show that the latter is a family of 0^+-stable vector bundles, and thus induces a morphism Ψ, extending the birational map.



In the end, we will show that Ψ descends to a morphism $\mathbf{M}_2 \to \mathbf{M}^0$, as desired.

4.1. The family over $\tilde{\mathbf{M}}_1$. Our first step is to construct the (local) tautological family over $\tilde{\mathbf{M}}_1$. Let $\xi \in \mathbb{I}_a^+ \cap \mathbb{I}_b^+$ be any point and let $\mathbf{U} \subset \mathbf{M}^{1/2} - \mathbb{I}_c$ be an open subset containing ξ. Without loss of generality, we can assume that the moduli space $\mathbf{M}^{1/2}$ admits a universal family

$\mathfrak{E}$ over $\mathbf{U}$ that is a sheaf over a family of nodal curves $\mathcal{W}$ over $\mathbf{U}$. The desired family over $\tilde{\mathbf{M}}$ will be the result of an elementary modification to the pull back of $\mathfrak{E}$ to $\tilde{\mathbf{M}}$.

To this end, we first need to construct the associated family $\tilde{\mathcal{W}}$ and $\mathcal{W}^{\mathrm{st}}$ over $\tilde{\mathbf{M}}_1$. Let $\tilde{\mathbf{M}}_1$ be the blow up of $\mathbf{M}^{1/2}$ along $\mathbb{I}_a^+$, let $\tilde{\mathbf{U}} \subset \tilde{\mathbf{M}}_1$ be the pre-image of $\mathbf{U} \subset \mathbf{M}^{1/2}$, and let $\bar{\mathcal{W}} = \mathcal{W} \times_{\mathbf{U}} \tilde{\mathbf{U}}$ be the pull back family. The family $\bar{\mathcal{W}}$ is singular, and hence needs to be smoothed first. We now set up the notation for the singular loci of the fibers of $\mathcal{W}/\mathbf{U}$. Let $\xi \in \mathbf{U}$ and $\mathcal{W}/\mathbf{U}$ be as before, and let $\mathcal{N} \subset \mathcal{W}$ be the singular loci of the fibers of $\mathcal{W}/\mathbf{U}$. Then $\mathcal{N} \cap \mathcal{W}_\xi$ consists of four nodes: q_0, q_1, $q^\dagger = q_2$ and q_3 of $\mathcal{W}_\xi$. By shrinking $\mathbf{U}$ if necessary, we can assume that $\mathcal{N}$ is a disjoint union of four varieties $\mathcal{N}_0, \ldots, \mathcal{N}_3$ indexed so that $\mathcal{N}_i \cap \mathcal{W}_\xi = q_i$. Clearly, $\mathcal{N}_2$ is a section over $\mathbf{U}$ while all others are codimension two smooth subvarieties of $\mathcal{W}$. Let $\mathbf{D}_i \subset \mathbf{U}$ be the image of $\mathcal{N}_i$ under the projection $\mathcal{W} \to \mathbf{U}$. Then $\mathbb{I}_a^+ \cap \mathbf{U} \subset \mathbf{D}_1 \cap \mathbf{D}_3$ while $(\mathbb{I}_a^+ \cap \mathbb{I}_b^+) \cap \mathbf{U} = \mathbb{I}_a^+ \cap \mathbf{D}_0$.

Next, let $\mathbf{E}_a \subset \tilde{\mathbf{M}}_1$ be the exceptional divisor of Ψ_1, let $\tilde{\mathbf{D}}_i \subset \tilde{\mathbf{U}}$ be the proper transform of $\mathbf{D}_i$ and let $\bar{\mathcal{N}}_i = \mathcal{N}_i \times_{\mathbf{U}} \tilde{\mathbf{U}} \subset \bar{\mathcal{W}}$ be the associated subscheme. Since $\mathbf{D}_0$ intersects transversally with $\mathbb{I}_a^+$, the total family $\bar{\mathcal{W}}$ is smooth along $\bar{\mathcal{N}}_0$. Obviously, $\bar{\mathcal{N}}_2$ remains a section over $\tilde{\mathbf{U}}$. As to $\bar{\mathcal{N}}_1$ and $\bar{\mathcal{N}}_3$, first of all, $\bar{\mathcal{W}}$ remains smooth along $\bar{\mathcal{N}}_1$ (resp. $\bar{\mathcal{N}}_3$) away from $\bar{\mathcal{P}}_1 \triangleq \bar{\mathcal{N}}_1|_{\mathbf{E}_a \cap \tilde{\mathbf{D}}_1}$ (resp. $\bar{\mathcal{P}}_3 \triangleq \bar{\mathcal{N}}_3|_{\mathbf{E}_a \cap \tilde{\mathbf{D}}_3}$). Secondly, the normal slice to $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_3$ in $\bar{\mathcal{W}}$ is isomorphic to the singularity of $z_1 z_2 = z_3 z_4$. Hence we can find a small resolution of the singularities of $\bar{\mathcal{W}}$ to obtain a new family $\tilde{\mathcal{W}}$. It is known that the small resolution is obtained by first blowing up the singular loci of $\bar{\mathcal{W}}$ and then contracting one $\mathbf{P}^1$ factor of the exceptional divisors[6] . Since there are two $\mathbf{P}^1$ factors, to proceed we need to specify our choice of contraction. Let $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_3$ be the exceptional loci of $\tilde{\mathcal{W}} \to \bar{\mathcal{W}}$, which are $\mathbf{P}^1$-bundles over $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_3$ respectively. We next pick a lift $\eta \in \mathbf{E}_a \cap \tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{D}}_3$ of $\xi \in \mathbb{I}_a^+ \cap \mathbb{I}_b^+$ and consider the fiber $\tilde{\mathcal{W}}_\eta$ of $\tilde{\mathcal{W}}$ over η. As it stands, it contains five rational curves, indexed by R_1, R_{1+}, R_2, R_3 and R_{3+} so that the first intersects with X at p_1 while any two consecutive $R_\bullet$'s insect at one point. The small resolution is the one so that $R_{1+} = \tilde{\mathcal{W}}_\eta \cap \tilde{\mathcal{P}}_1$ and $R_{3+} = \tilde{\mathcal{W}}_\eta \cap \tilde{\mathcal{P}}_3$, and that if we let $S \subset \tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{D}}_3$ be a smooth curve which contains η and is transversal to $\mathbf{E}_a$, then the family $\tilde{\mathcal{W}}_S = \tilde{\mathcal{W}} \times_{\tilde{\mathbf{U}}} S$ smooth the nodes $q_{1+} = R_{1+} \cap R_2$ and $q_{3-} = R_3 \cap R_{3+}$ of the central fiber $\tilde{\mathcal{W}}_\eta$. (See the figure below.)

[6]See [**8**, §1] for details.

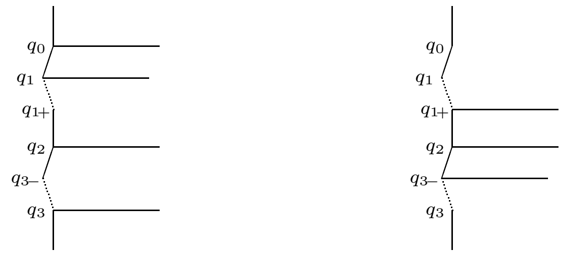


Figure 6. The left one represents the total family over S: the vertical chain of lines is the central fiber $\tilde{\mathcal{W}}_\eta$ with each corner representing a node, as labelled, the top and the bottom lines representing the main component X while others are rational curves. The two dotted lines represent the two $\mathbf{P}^1$ that are contracted under $\tilde{\mathcal{W}}_\eta \to \mathcal{W}_\eta$, and the horizontal lines show that the associated nodes are not smoothed in the family S. The right figure represents the total space over a curve $\eta \in S' \subset \tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{D}}_3$, $S' \subset \mathbf{E}_a$ and is transversal to $\tilde{\mathbf{D}}_0$.

The family $\mathcal{W}^{\mathrm{st}}$ is a contraction of $\tilde{\mathcal{W}}$. Let $\tilde{\mathcal{R}}_2$ and $\tilde{\mathcal{R}}_3$ be the two irreducible components of $\tilde{\mathcal{W}} \times_{\tilde{\mathcal{U}}} \mathbf{E}_a$ that contain R_2 and $R_3 \subset \tilde{\mathcal{W}}_\eta$ respectively. It is easy to see that each is isomorphic to $(\mathbf{E}_a \cap \tilde{\mathbf{U}}) \times \mathbf{P}^1$ and its normal bundle in $\tilde{\mathcal{W}}$ has degree -1 along its fibers. Therefore, we can contract $\tilde{\mathcal{W}}$ along $\tilde{\mathcal{R}}_2$ and $\tilde{\mathcal{R}}_3$ to obtain a new family of nodal curves. We denote this new family by $\mathcal{W}^{\mathrm{st}}$ with projection

$$\mathrm{pr} : \tilde{\mathcal{W}} \longrightarrow \mathcal{W}.$$

Next we investigate the associated families of sheaves over $\tilde{\mathcal{W}}$ and $\mathcal{W}^{\mathrm{st}}$. Let $\mathfrak{E}$ be the universal sheaf over $\mathcal{W}$. By our description of the sheaves in $\mathbb{I}_a^+$, for each $\zeta \in \mathbb{I}_a^+ \cap \mathbf{U}$ the sheaf $\mathfrak{E}_\zeta = \mathfrak{E} \otimes_{\mathcal{O}_{\mathcal{W}}} \mathcal{O}_{\mathcal{W}_\zeta}$ has a canonical subsheaf $\mathfrak{F}_\zeta \subset \mathfrak{E}_\zeta$ and the associated quotient sheaf $\mathfrak{L}_\zeta = \mathfrak{E}_\zeta / \mathfrak{F}_\zeta$. Let $\mathcal{Z}_\zeta \subset \mathcal{W}_\zeta$ be the support of $\mathfrak{L}_\zeta$. Then $\mathfrak{L}_\zeta$ is a rank one locally free sheaf of $\mathcal{O}_{\mathcal{Z}_\zeta}$-modules. Further, it is direct to check that the union $\cup_{\zeta \in \mathbf{U} \cap \mathbb{I}_a^+} \mathcal{Z}_\zeta$ forms a smooth subvariety of $\mathcal{W}$ and is the irreducible component of $\mathcal{W} \times_{\mathbf{U}} (\mathbb{I}_a^+ \cap \mathbf{U})$ that contains $X \times (\mathbb{I}_a^+ \cap \mathbf{U})$. We denote this by $\mathcal{Z}$ with inclusion $\iota : \mathcal{Z} \subset \mathcal{W}$. Further, there is a locally free sheaf $\mathfrak{L}$ of $\mathcal{O}_{\mathcal{Z}}$-modules and a quotient sheaf homomorphism $\mathfrak{E} \to \iota_* \mathfrak{L}$ so that its restriction to each fiber $\mathcal{W}_\zeta$ is exactly the pair $\mathfrak{E}_\zeta \to \mathfrak{L}_\zeta$ mentioned before.

Now we are ready to perform an elementary modification on the pull back sheaf over $\tilde{\mathcal{W}}$. Let $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{W}}$ be the pre-image of $\mathcal{Z} \subset \mathcal{W}$ under the projection $\tilde{\mathcal{W}} \to \mathcal{W}$. By our choice of the small resolution, $\tilde{\mathcal{Z}}$ is a smooth divisor of $\tilde{\mathcal{W}}$ and the total space $\tilde{\mathcal{W}} \times_{\tilde{\mathbf{U}}} (\mathbf{E}_a \cap \tilde{\mathbf{U}})$ is a union of three irreducible components: $\tilde{\mathcal{R}}_2$, $\tilde{\mathcal{R}}_3$ and $\tilde{\mathcal{Z}}$. We consider the pull-back family $\tilde{\mathfrak{E}} \triangleq \tilde{\mathrm{pr}}^*\mathfrak{E}$ and the associated surjective homomorphism $\tilde{\mathfrak{E}} \to \tilde{\mathfrak{L}} \triangleq \tilde{\mathrm{pr}}^*\iota_*\mathfrak{L}$. Let $\tilde{\mathfrak{E}}'$ be the kernel of this homomorphism.

Lemma 4.2. *The sheaf $\tilde{\mathfrak{E}}'$ is a locally free sheaf of $\mathcal{O}_{\tilde{\mathcal{W}}}$-modules. Further, for any $\eta \in \mathbf{E}_a \cap \tilde{\mathbf{U}}$, the restriction of $\tilde{\mathfrak{E}}'$ to $\tilde{\mathcal{R}}_2 \times_{\tilde{\mathbf{U}}} \eta$ and $\tilde{\mathcal{R}}_3 \times_{\tilde{\mathbf{U}}} \eta \cong \mathbf{P}^1$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}^{\oplus 3}$.*

Proof. Since $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{W}}$ is a smooth divisor and $\tilde{\mathfrak{L}}$ is an invertible sheaf of $\mathcal{O}_{\tilde{\mathcal{Z}}}$-modules, the kernel of $\tilde{\mathrm{pr}}^*\mathfrak{E} \to \tilde{\mathfrak{L}}$ is locally free.

We now prove the second part. We first consider the case $\eta \in \mathbf{E}_a \cap \tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{D}}_3$ in detail. Let $S \subset \tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{D}}_3$ be a smooth curve that contains η and is transversal to $\mathbf{E}_a$. Since $\tilde{\mathbf{D}}_0$ is transversal to $\mathbf{E}_a$, we can assume $S \subset \tilde{\mathbf{D}}_0$ (see Figure 6). Then the irreducible component V_1 of $\tilde{\mathcal{W}}_S$ contains R_{1+} and R_2 as (-1)-curves. Similarly, R_3 and R_{3+} are (-1)-curves in the irreducible component V_2 as shown in Figure 6. Now let $\tilde{\mathfrak{E}}_S \triangleq \tilde{\mathfrak{E}} \otimes_{\mathcal{O}_{\tilde{\mathcal{W}}}} \mathcal{O}_{\tilde{\mathcal{W}}_S}$ be the pull-back family. As before, we denote by $\tilde{\mathfrak{E}}_\eta \to \tilde{\mathfrak{L}}_\eta$ the restriction of $\tilde{\mathfrak{E}} \to \tilde{\mathfrak{L}}$ to $\tilde{\mathcal{W}}_\eta$. Since $\mathbf{E}_a$ is a smooth divisor, the sheaf $\tilde{\mathfrak{E}}'_\eta \triangleq \tilde{\mathfrak{E}}'|_{\tilde{\mathcal{W}}_\eta}$ is canonically isomorphic to $\ker\{\tilde{\mathfrak{E}}_S \to \tilde{\mathfrak{L}}_\eta\}|_{\tilde{\mathcal{W}}_\eta}$. Following our convention, the pair $\tilde{\mathfrak{E}}_\eta \to \tilde{\mathfrak{L}}_\eta$ can be represented by the left graph in Figure 7 below.

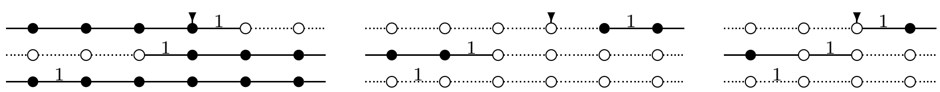


Figure 7. The last graph represents the type I_b^{-5} in Figure 5. The quotient sheaf $\tilde{\mathfrak{L}}_\eta$ is represented by the dotted lines.

We now show that the middle one represents the sheaf $\tilde{\mathfrak{E}}'_\eta$. First, since $\tilde{\mathfrak{E}}_S$ fits into the exact sequence

$$0 \longrightarrow \tilde{\mathfrak{E}}'_S \longrightarrow \tilde{\mathfrak{E}}_S \longrightarrow \tilde{\mathfrak{L}}_\eta \longrightarrow 0,$$

after tensoring with $\mathcal{O}_{\tilde{\mathcal{W}}_\eta}$, we obtain

$$0 \longrightarrow \tilde{\mathfrak{L}}_\eta \longrightarrow \tilde{\mathfrak{E}}'_\eta \longrightarrow \tilde{\mathfrak{E}}_\eta \longrightarrow \tilde{\mathfrak{L}}_\eta \longrightarrow 0.$$

Because $\tilde{\mathfrak{E}}'_\eta$ is locally free, $\tilde{\mathfrak{E}}'_\eta$ must be of the type shown in the middle figure above. Here we used the fact that the total space $\tilde{\mathcal{W}}_S$ is smooth at the non-locally free loci of $\tilde{\mathfrak{L}}_\eta$ (as sheaves of $\mathcal{O}_{\tilde{\mathcal{W}}_\eta}$-modules) and the curves R_{2+} and R_{3+} are (-1)-curves. Consequently, the restriction of $\tilde{\mathfrak{E}}'_\eta$ to the two rational curves R_2 and R_3 are of the firm $\mathcal{O}_{\mathbf{P}^1}^{\oplus 3}$.

The study of the sheaves $\mathfrak{E}'_\eta$ for $\eta \in \mathbf{E}_a$ belonging to $\tilde{\mathbf{D}}_1 - \tilde{\mathbf{D}}_3$, $\tilde{\mathbf{D}}_3 - \tilde{\mathbf{D}}_1$ and in the complement of $\tilde{\mathbf{D}}_1 \cup \tilde{\mathbf{D}}_3$ are similar and will be omitted.

q.e.d.

For completeness, we list their stable modifications as follows.

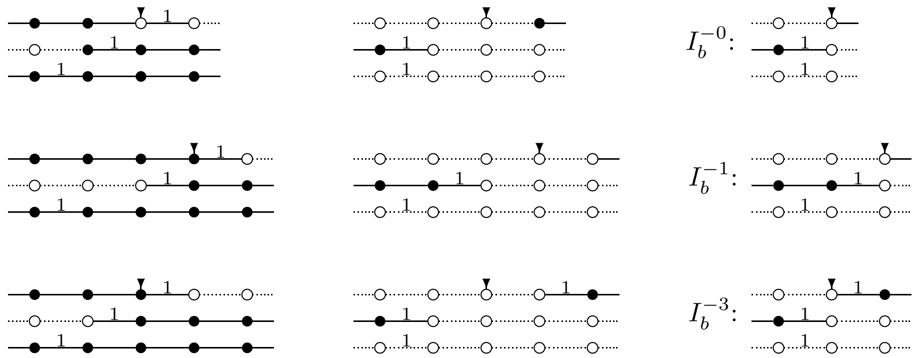


Figure 8. The top, middle and the bottom figures represent the process of stable modifications of $\mathfrak{E}_\eta$ for η in the complement of $\tilde{\mathbf{D}}_1 \cup \tilde{\mathbf{D}}_3$, in $\tilde{\mathbf{D}}_3 - \tilde{\mathbf{D}}_1$ and in $\tilde{\mathbf{D}}_1 - \tilde{\mathbf{D}}_3$ respectively.

We now construct the stable modification of $\tilde{\mathfrak{E}}$. We first contract $\tilde{\mathcal{W}}$ along $\tilde{\mathcal{R}}_2$ and $\tilde{\mathcal{R}}_3$. Since the restriction of $\tilde{\mathfrak{E}}'$ to fibers of $\tilde{\mathcal{R}}_2$ and $\tilde{\mathcal{R}}_3$ are isomorphic to $\mathcal{O}_{\mathbf{P}^1}^{\oplus 3}$, there is a unique sheaf $\mathfrak{E}^{\mathrm{st}}$ on $\mathcal{W}^{\mathrm{st}}$ whose pull back to $\tilde{\mathcal{W}}$ is $\tilde{\mathfrak{E}}'$. The sheaf $\mathfrak{E}^{\mathrm{st}}$ is called the stable modification of $\tilde{\mathfrak{E}}$. The restriction of $\tilde{\mathfrak{E}}^{\mathrm{st}}$ to fibers over $\mathbf{E}_a$ are represented by the right figures in Figure 7.

In the following, for any $\eta \in \mathbf{E}_a$ we denote by $\mathfrak{E}^{\mathrm{st}}_\eta$ the restriction of $\mathfrak{E}^{\mathrm{st}}$ to $\mathcal{W}^{\mathrm{st}}_\eta$. Note that by applying the same construction to different open subsets $\mathbf{U}$, we can construct sheaves $\mathfrak{E}^{\mathrm{st}}_\eta$ for all $\eta \in \mathbf{E}_a$, and it is independent of the choices of $\mathbf{U}$.

Let $\mathbf{E}_b, \mathbf{E}_c \subset \tilde{\mathbf{M}}_1$ be the proper transforms of $\mathbb{I}_b^+$ and $\mathbb{I}_c^+$.

Lemma 4.3. *The sheaves $\mathfrak{E}^{\mathrm{st}}_\eta$ are 0^+-stable for all $\eta \in \mathbf{E}_a - \mathbf{E}_b \cup \mathbf{E}_c$.*

Proof. Let $\eta \in \mathbf{E}_a$ be any closed point with $\mathfrak{E}^{\mathrm{st}}_\eta$ the associated sheaf on $X^\dagger_{n,m}$ for some appropriate integers n and m. Let $\pi^\dagger : X_{n,m} \to X^\dagger_{n,m}$ be the desingularization of the marked node and let $\pi : X_{n,m} \to X$ be the contraction of all rational curves. Then $E_\eta = \pi_* \pi^{\dagger *} \mathfrak{E}^{\mathrm{st}}_\eta$ is a rank three locally free sheaf of $\mathcal{O}_X$-modules with a GPB structure $E^0_\eta \subset E_\eta|_{p_1+p_2}$ as described in Section 2. Furthermore, the subsheaf $\mathfrak{L}_\eta \subset \mathfrak{E}^{\mathrm{st}}_\eta$ and the quotient sheaf $\mathfrak{E}^{\mathrm{st}}_\eta \to \mathfrak{F}_\eta$ defines a sub and a quotient GPB bundle, $L^G_\eta \subset E^G_\eta$ and $E^G_\eta \to F^G_\eta$. According to Proposition 2.4, $\mathfrak{E}^{\mathrm{st}}_\eta$ is 0-stable if and only if E^G_η is 0-stable, which is the case when the extension

$$0 \longrightarrow L^G \longrightarrow E^G_\eta \longrightarrow F^G \longrightarrow 0 \tag{4.1}$$

is non-trivial. Since E^G_η is never $F^G_\eta \oplus L^G_\eta$ when $\mathfrak{E}^{\rm st}_\eta$ is of types I_b^{-0} and I_b^{-1}, $\mathfrak{E}^{\rm st}_\eta$ could be 0^+-unstable only when it was of type I_b^{-3} or of I_b^{-5}. (Here recall that there are no strictly 0-semistable vector bundles.)

We now demonstrate that η must belong to $\mathbf{E}_a \cap \mathbf{E}_b$ when $\mathfrak{E}^{\rm st}_\eta$ is of type I_b^{-3} and its associated E^G_η is a split extension. Let $S \subset \tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_3 - \tilde{\mathbf{D}}_1$ be a smooth curve containing η and transversal to $\mathbf{E}_a$, with $\tilde{\mathcal{W}}_S$ the restriction of $\tilde{\mathcal{W}}$ over S. Let $X \times S \subset \tilde{\mathcal{W}}_S$ be the main irreducible component with $\iota : X \times \eta \subset X \times S$ the central fiber. Let $\tilde{\mathfrak{E}}_S$ and $\tilde{\mathfrak{E}}'_S$ be the associated sheaf on $\tilde{\mathcal{W}}_S$ constructed before (Figure 7). Then the fact that $\mathfrak{E}^{\rm st}_\eta$ is 0-unstable, which is the case when (4.1) splits, implies that there must be a surjective sheaf homomorphism $\tilde{\mathfrak{E}}'_S \to \iota_* L(-p_1 - p_2)$ so that the composite

$$\tilde{\mathfrak{L}}_\eta \longrightarrow \tilde{\mathfrak{E}}'_S|_{\tilde{\mathcal{W}}_\eta} \longrightarrow \iota_* L(-p_1 - p_2) \tag{4.2}$$

is surjective. Let $\tilde{\mathcal{L}}_2$ be the sheaf of $\mathcal{O}_{\tilde{\mathcal{W}}_S}$-modules that fits into the commutative diagram with the lower sequence exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathfrak{E}}'_S & \longrightarrow & \tilde{\mathfrak{E}}_S & \longrightarrow & \tilde{\mathcal{L}}_\eta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \iota_* L(-p_1 - p_2) & \longrightarrow & \tilde{\mathcal{L}}_2 & \longrightarrow & \tilde{\mathcal{L}}_\eta & \longrightarrow & 0 \end{array} \tag{4.3}$$

Now let $X_2 = X \times \operatorname{Spec} \mathbf{k}[t]/(t^2)$ and let $\iota_2 : X_2 \to \tilde{\mathcal{W}}_S$ be the immersion extending $\iota : X \to \tilde{\mathcal{W}}_S$. Since (4.3) is exact while the composition of (4.2) is surjective, the pull-back sheaf $\iota_2^* \tilde{\mathcal{L}}_2$ is an invertible sheaf of $\mathcal{O}_{X_2}$-modules and is an extension of $\iota_* L(-p_1 - p_2)$ by $\iota_* L(-p_1 - p_2)$. Now let $S' \subset \mathbf{M}^{1/2}$ be the image of $S \subset \tilde{\mathbf{M}}_1$ under the projection $\tilde{\mathbf{M}}_1 \to \mathbf{M}^{1/2}$ with $\eta' \in S'$ the image of $\eta \in S$. Since $\eta \in \mathbf{E}_a \cap \mathbf{E}_b$, η' belongs to $\mathbb{I}_a^+ \cap \mathbb{I}_b^+$. A direct check shows that the existence of $\tilde{\mathfrak{E}}_S \to \tilde{\mathcal{L}}_2$ implies that the tangent $T_{\eta'} S'$ must be contained in the span of the tangent spaces

$$T_{\eta'} \mathbb{I}_a^+ + T_{\eta'} \mathbb{I}_b^+ \subset T_{\eta'} \mathbf{M}^{1/2}. \tag{4.4}$$

Since $\mathbf{E}_a$ is the blown-up locus of $\mathbb{I}_a^+$ while $\mathbf{E}_b$ is the proper transform of $\mathbb{I}_b^+$, the curve S must specialize to a point in the intersection $\mathbf{E}_a \cap \mathbf{E}_b$, and hence $\eta \in \mathbf{E}_a \cap \mathbf{E}_b$. This proves the claim.

In case $\mathfrak{E}^{\rm st}_\eta$ is of type I_b^{-5}, a similar argument shows that (4.4) also holds. But then since those $\mathfrak{E}^{\rm st}_\eta$ of type I_b^{-5} are elementary modification along direction inside $\tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{D}}_3$, it must be inside $T_{\eta'} \mathbb{I}_a^+$. But this is impossible, which proves that if $\mathfrak{E}^{\rm st}_\eta$ is of type I_b^{-5} then it must be 0-stable. q.e.d.

4.2. The family over $\tilde{\mathbf{M}}$. In this subsection, we will construct a family of sheaves over $\tilde{\mathbf{M}}$ after performing an elementary modification to the family constructed in the previous subsection, and we will then show that all members of this family are 0-stable. This way we can prove the proposition by applying the universal property of the moduli space $\mathbf{M}^0$:

Proposition 4.4. *The birational map* $\tilde{\mathbf{M}} - \to \mathbf{M}^0$ *extends to a morphism* $\tilde{\mathbf{M}} \to \mathbf{M}^0$.

We now prove this proposition. We will sketch the steps that are parallel to the proof in the previous subsection and provide details when called for. As mentioned before, the main strategy is to pull back the (local) tautological families over $\tilde{\mathbf{M}}_1$ to $\tilde{\mathbf{M}}$, find a small resolution of the base variety of these families over $\tilde{\mathbf{M}}$, and then perform an elementary modification to these new families. The members of the resulting families are 0^+-stable, and hence induce a morphism $\tilde{\mathbf{M}} \to \mathbf{M}^0$.

We begin with any analytic neighborhood $\tilde{\mathbf{U}} \subset \tilde{\mathbf{M}}_1 - \mathbf{E}_c$ and the tautological family $\mathfrak{E}^{\mathrm{st}}$ on $\tilde{\mathcal{W}}$ over $\tilde{\mathbf{U}}$ that was constructed in the previous subsection. When $\xi \in \mathbf{E}_b$ is away from $\mathbf{E}_a$, $\mathfrak{E}^{\mathrm{st}}_\xi$ must be of the type I_b^{+0} or I_b^{+1}, as shown in Figure 4, and accordingly it has an associated quotient sheaf $\mathfrak{L}_\xi$ and subsheaf $\mathfrak{F}_\xi$ that fits into the exact sequence

$$0 \longrightarrow \mathfrak{F}_\xi \longrightarrow \mathfrak{E}^{\mathrm{st}}_\xi \longrightarrow \mathfrak{L}_\xi \longrightarrow 0. \tag{4.5}$$

Note that sheaves of types I_b^{+2} are in $\mathbb{I}_a \cap \mathbb{I}_b$, and thus won't appear in $\mathbf{E}_b - \mathbf{E}_b$. If $\xi \in \mathbf{E}_b \cap \mathbf{E}_a$, then it is of type $\mathbb{I}_b^{-3}$ with the additional property that the associated GPB short exact sequence of $\mathfrak{E}^{\mathrm{st}}_\xi$ splits, as proved in Lemma 4.3. We now show that we can pick a new associated quotient sheaf of $\mathfrak{E}^{\mathrm{st}}_\xi$ so as to make it of type I_b^{+0} as well: since $\mathfrak{E}^{\mathrm{st}}_\xi$ is of type $\mathbb{I}_b^{-3}$, it is a sheaf on $X^\dagger_{1,1}$ and fits into the exact sequence

$$0 \longrightarrow \mathcal{L}_\xi \longrightarrow \mathfrak{E}^{\mathrm{st}}_\xi \longrightarrow \mathcal{F}_\xi \longrightarrow 0,$$

according to the proof of Lemma 4.2. Because the associated GPB splits, if we let $\mathfrak{L}_\xi$ be the cokernel of $\mathcal{O}_{D_1}(-1) \oplus \mathcal{O}_{D_2}(-1) \to \mathcal{L}_\xi$ we obtain a unique surjective

$$\mathfrak{E}^{\mathrm{st}}_\xi \longrightarrow \mathfrak{L}_\xi \tag{4.6}$$

so that the composite $\mathcal{L}_\xi \to \mathfrak{E}^{\mathrm{st}}_\xi \to \mathfrak{L}_\xi$ is exactly the defining quotient homomorphism $\mathcal{L}_\xi \to \mathfrak{L}_\xi$. Let $\mathfrak{F}_\xi$ be the kernel of $\mathfrak{E}^{\mathrm{st}}_\xi \to \mathfrak{L}_\xi$. Then $\mathfrak{E}^{\mathrm{st}}_\xi$ fits into the exact sequence (4.5) as well, and the latter will be called the associated exact sequence of $\mathfrak{E}^{\mathrm{st}}_\xi$ while the sheaves $\mathfrak{F}_\xi$ and $\mathfrak{L}_\xi$ will be called the associated sub and quotient sheaves of $\mathfrak{E}^{\mathrm{st}}_\xi$. Clearly, (4.5) makes such $\mathfrak{E}^{\mathrm{st}}_\xi$ a sheaf of type I_b^{+0}.

As before, we can make the quotients $\mathfrak{E}^{\mathrm{st}}_\xi \to \mathfrak{L}_\xi$ into a family of quotients. Let $\tilde{\mathcal{Z}}_\xi \subset \tilde{\mathcal{W}}_\xi$ be the support of $\mathfrak{L}_\xi$; $\tilde{\mathcal{Z}}_\xi$ is $X \subset X^\dagger_{1,1}$ when $\mathfrak{E}^{\mathrm{st}}_\xi$ is of type I_b^{+0}, and is $X_{1,0} \subset X^\dagger_{2,1}$ when $\mathfrak{E}^{\mathrm{st}}_\xi$ is of type I_b^{+1}. The union

$\tilde{\mathcal{Z}} = \mathrm{Supp}_{\xi\in\mathbf{E}_b}\tilde{\mathcal{Z}}_\xi$ is an irreducible variety and there is an invertible sheaf $\mathfrak{L}$ of $\mathcal{O}_{\tilde{\mathcal{Z}}}$-modules and a surjective homomorphism

$$\mathfrak{E}^{\mathrm{st}} \longrightarrow \iota_*\mathfrak{L}, \tag{4.7}$$

where $\iota : \tilde{\mathcal{Z}} \hookrightarrow \tilde{\mathcal{W}}$ is the inclusion, so that its restriction to each $\tilde{\mathcal{W}}_\xi$ is exactly the associated homomorphism in (4.5).

Our next step is to pull back the family $\mathfrak{E}^{\mathrm{st}}$ to a family over $\tilde{\mathbf{M}}$ and perform elementary modification to it. Let $\tilde{\mathbf{V}}$ be $\tilde{\Psi}_1^{-1}(\tilde{\mathbf{U}})$, which is the blowing up of $\tilde{\mathbf{U}}$ along $\mathbf{E}_b$; let $\pi : \tilde{\mathbf{V}} \to \tilde{\mathbf{U}}$ be the projection; let $\tilde{\mathbf{E}}_b$ be the exceptional divisor and let $\tilde{\mathbf{E}}_a$ be the proper transform of $\mathbf{E}_a$. Let $\mathcal{X} = \tilde{\mathcal{W}} \times_{\tilde{\mathbf{U}}} \tilde{\mathbf{V}}$ be the pull-back family over $\tilde{\mathbf{V}}$ and let $\mathcal{Y} = \tilde{\mathcal{Z}} \times_{\tilde{\mathbf{U}}} \tilde{\mathbf{V}}$ be the associated subvariety of $\mathcal{X}$. As before, the total space of $\mathcal{X}$ is not smooth, and we need to small resolve its singularity. For the moment, we consider the case where all sheaves over $\tilde{\mathbf{U}}$ are of types I_b^{+0}. Hence, by shrinking $\tilde{\mathcal{W}}$ if necessary we can assume that the singular locus of the fibers of $\tilde{\mathcal{W}}/\tilde{\mathbf{U}}$ consists of three smooth connected codimension two subvarieties: $\mathcal{N}_0$, $\mathcal{N}_1$ and $\mathcal{N}_2$ that are ordered so that for $\xi \in \mathbf{E}_b$ the intersection $\tilde{\mathcal{W}}_\xi \cap \mathcal{N}_i$ is the i-th nodal point of $\tilde{\mathcal{W}}_\xi$. Now let $\tilde{\mathcal{N}}_i = \mathcal{N}_i \times_{\tilde{\mathbf{U}}} \tilde{\mathbf{V}}$, let $\mathbf{D}_i \subset \tilde{\mathbf{U}}$ be the image divisor of $\mathcal{N}_i$ under $\mathcal{X} \to \tilde{\mathbf{U}}$ and let $\tilde{\mathbf{D}}_i \subset \tilde{\mathbf{V}}$ be the proper transform of $\tilde{\mathbf{D}}_i$. Then $\tilde{\mathcal{N}}_1$ are the marked nodes of the whole family $\mathcal{X}$, that $\mathcal{X}$ smooth along $\tilde{\mathcal{N}}_0$ except over those $\xi \in \tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{E}}_b$ and smooth along $\tilde{\mathcal{N}}_2$ except over those $\xi \in \tilde{\mathbf{D}}_2 \cap \tilde{\mathbf{E}}_b$. Again, we blow up $\mathcal{X}$ along $\tilde{\mathcal{N}}_0 \times_{\tilde{\mathbf{V}}} (\tilde{\mathbf{D}}_1 \cap \tilde{\mathbf{E}}_b)$ and $\tilde{\mathcal{N}}_2 \times_{\tilde{\mathbf{V}}} (\tilde{\mathbf{D}}_3 \cap \tilde{\mathbf{E}}_b)$; we then contract one $\mathbf{P}^1$-factor from each of the two exceptional divisors to obtain a family of nodal curves $\tilde{\mathcal{X}}$. As before, we choose the contraction so that if we let $\tilde{\mathcal{Y}} \subset \tilde{\mathcal{X}}$ be the proper transform of $\mathcal{Y} \subset \mathcal{X}$, and let $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{D}}_2$ be the two exceptional loci of $\tilde{\mathcal{X}} \to \mathcal{X}$, then the intersections $\tilde{\mathcal{Y}} \cap \tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{Y}} \cap \tilde{\mathcal{D}}_2$ are finite over $\tilde{\mathbf{V}}$; namely they contain no curves that lie inside a single fiber $\tilde{\mathcal{X}}_\xi$ over some $\xi \in \tilde{\mathbf{V}}$.

We now pull back the family $\mathfrak{E}^{\mathrm{st}}$ to $\tilde{\mathcal{X}}$ and perform an elementary modification. Let $p : \tilde{\mathcal{X}} \to \tilde{\mathcal{W}}$ be the projection, and let

$$p^*\mathfrak{E}^{\mathrm{st}} \to p^*\iota_*\mathfrak{L} \tag{4.8}$$

be the pull-back of the pair (4.7). Then the kernel $\tilde{\mathfrak{E}}$ of the homomorphism (4.8) is the modification we seek for.

It remains to prove that all members in $\tilde{\mathfrak{E}}$ are 0^+-stable. Let $\xi \in \mathbf{E}_b$ be any point and let $\eta \in \tilde{\mathbf{E}}_b$ be any of its lifts. Since $\xi \in \mathbf{D}_0 \cap \mathbf{D}_2$, η can possibly be in $\tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_2$, in $\tilde{\mathbf{D}}_0 - \tilde{\mathbf{D}}_2$, in $\tilde{\mathbf{D}}_2 - \tilde{\mathbf{D}}_0$ or is away from $\tilde{\mathbf{D}}_0 \cup \tilde{\mathbf{D}}_2$. Now we investigate in detail the sheaf $\tilde{\mathfrak{E}}_\eta$ when $\eta \in \tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_2$. First, the curve $\tilde{\mathcal{X}}_\eta$ is $X_{2,2}^\dagger$ with $\tilde{\mathcal{X}}_\eta \to \mathcal{X}_\eta$ the contraction of the first and the last rational curves; when η_t is curve in $\tilde{\mathbf{V}}$ with $\eta_0 = \eta$ and is normal to $\tilde{\mathbf{E}}_b$, the family $\tilde{\mathcal{X}}_{\eta_t}$ smooths the first[7] and the third node of $\tilde{\mathcal{X}}_\eta$. The pull back

$$(p^*\tilde{\mathfrak{E}})_\eta \longrightarrow (p^*\iota_*\mathfrak{L})_\eta$$

is represented by the left graph below with $(p^*\iota_*\mathfrak{L})_\eta$ represented by the dotted lines. The modified sheaf $\tilde{\mathfrak{E}}_\eta$ is represented by the middle graph. After contracting the third rational curve in $\tilde{\mathcal{X}}_\eta$ we obtain a sheaf shown in the right graph that is of type I_b^{-4}:

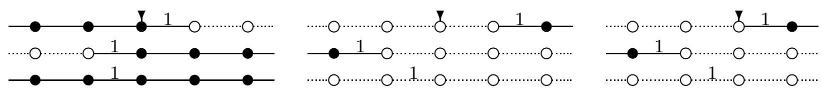


Figure 9. The graphs represent the sheaf over $\eta \in \tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_2$ before the modification, after the modification and after the stabilization. The resulting sheaf is of type I_b^{-4}.

We will call the sheaf obtained after contracting the third rational curve the stable modification of $(p^*\mathfrak{E}^{\mathrm{st}})_\eta$, and will denote it by $\tilde{\mathfrak{E}}^{\mathrm{st}}_\eta$.

We can derive the other stable modifications $\tilde{\mathfrak{E}}^{\mathrm{st}}_\eta$ similarly and will give the graphs sketching their respective process as follows:

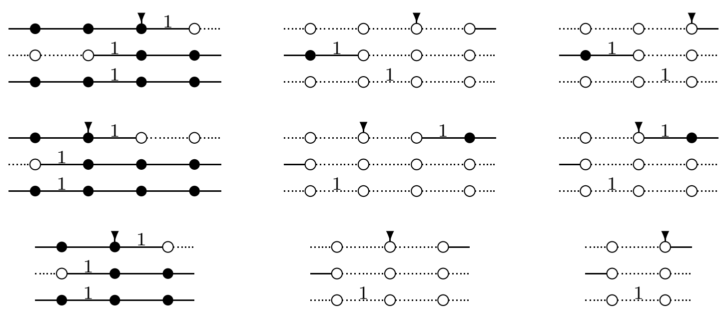


Figure 10. From top to bottom, they represent the derivation of $\tilde{\mathfrak{E}}^{\mathrm{st}}_\eta$ in case η is in $\eta \in \tilde{\mathbf{D}}_2 - \tilde{\mathbf{D}}_0$, in $\tilde{\mathbf{D}}_0 - \tilde{\mathbf{D}}_2$ and is away from $\tilde{\mathbf{D}}_0 \cup \tilde{\mathbf{D}}_2$.

Exactly as in the case before, we can contract all those rational curves that are immediately to the right of the marked nodes in $\tilde{\mathcal{X}}_\eta$ for all $\eta \in \tilde{\mathbf{E}}_b$ simultaneously and obtain a new family $\tilde{\mathcal{X}}^{\mathrm{st}}$ that has smooth total space. Let $\varphi : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}^{\mathrm{st}}$ be the stabilization; then $\tilde{\mathfrak{E}}^{\mathrm{st}} = \varphi_*\tilde{\mathfrak{E}}$ is locally free and the sheaves $\tilde{\mathfrak{E}}^{\mathrm{st}}_\eta$ for $\eta \in \mathbf{E}_b$ are exactly the sheaves shown in the right column in Figures 9 and 10.

Lemma 4.5. *All stable modifications $\tilde{\mathfrak{E}}^{\mathrm{st}}_\eta$ derived so far are* 0*-stable.*

[7] Recall that our convention is to label the nodes from the 0-th to the 3-rd if there are four nodes.

Proof. We will only sketch the proof here, since the details are exactly the same as in the proof of Lemma 4.3. First, $\tilde{\mathfrak{E}}_\eta$ could be 0-unstable only if its associated GPB were split. This is possible only when it is of type I_b^{-4}. Because $\eta \in \tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_2$, the split of its associated GPB implies that the tangent in $T_\xi \tilde{\mathbf{M}}_1$ associated to η lies in $T_\xi \mathbf{E}_b$, which is impossible. Thus the associated GPB is irreducible and hence all stable modification derived are 0-stable. q.e.d.

We now consider the case where $\mathfrak{E}^{\mathrm{st}}_\xi$ is of type I_b^{+1}. Let $\eta \in \tilde{\mathbf{E}}_b$ be a lift of ξ, let $\mathcal{X}/\tilde{\mathbf{V}}$ be an analytic neighborhood of η as before and let $\tilde{\mathcal{X}} \to \mathcal{X}$ be the small resolution constructed according to a similar rule. We will have similar quotient family $p^*\mathfrak{E}^{\mathrm{st}} \to p^*\iota_*\mathfrak{L}$, and we will take the $\tilde{\mathfrak{E}}$ be the kernel of this homomorphism. We still need to determine the types of members in $\tilde{\mathfrak{E}}$. As before, we let $\mathcal{N}_0, \ldots, \mathcal{N}_3$ be the loci of singular points of the fibers of $\mathcal{W}/\tilde{\mathbf{U}}$, let $\mathcal{D}_0, \ldots, \mathcal{D}_3$ be their respective images in $\tilde{\mathbf{U}}$ and let $\tilde{\mathcal{D}}_0, \ldots, \tilde{\mathcal{D}}_3$ be their proper transforms in $\tilde{\mathbf{V}}$. The resulting type of the stable modification will depend on whether η is in $\tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_2$, in $\tilde{\mathbf{D}}_2 - \tilde{\mathbf{D}}_0$, in $\tilde{\mathbf{D}}_0 - \tilde{\mathbf{D}}_2$ or is away from $\tilde{\mathbf{D}}_0 \cup \tilde{\mathbf{D}}_2$. We will show their respective stable modification by providing their associated graphes as before:

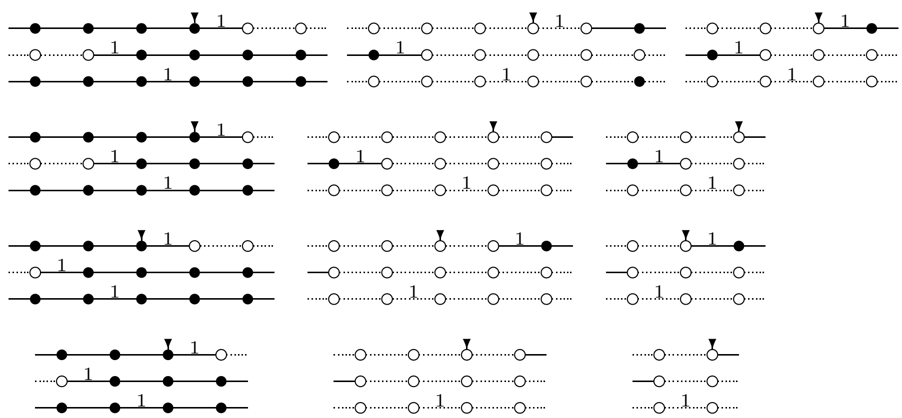


Figure 11. The graphs represent the stable modifications of type I_b^{+1} sheaves over $\tilde{\mathbf{D}}_0 \cap \tilde{\mathbf{D}}_2$, over $\tilde{\mathbf{D}}_2 - \tilde{\mathbf{D}}_0$, over $\tilde{\mathbf{D}}_0 - \tilde{\mathbf{D}}_2$ or away from $\tilde{\mathbf{D}}_0 \cup \tilde{\mathbf{D}}_2$.

Lemma 4.6. *Let $\tilde{\mathfrak{E}}$ over $\tilde{\mathcal{X}}$ be the result of the stable elementary modification of $p^*\mathfrak{E}^{\mathrm{st}}$; then all its members are weakly 0-stable and hence their stabilization are 0-stable.*

Proof. The proof is similar to the argument before and will be omitted. q.e.d.

The family $\tilde{\mathfrak{E}}^{\mathrm{st}}$ over each $\tilde{\mathbf{V}}$ induces a morphism $\tilde{\mathbf{V}} \to \mathbf{M}^0$ that is the local extension of the birational $\tilde{\mathbf{M}} - \to \mathbf{M}^0$. Since both $\tilde{\mathbf{M}}$ and

$\mathbf{M}^0$ are smooth, the local extension patch together to form a morphism $\tilde{\mathbf{M}} \to \mathbf{M}^0$, as desired.

4.3. The existence of the descent $\tilde{\mathbf{M}} \to \mathbf{M}^0$. In this subsection, we will show that the morphism $\Psi : \tilde{\mathbf{M}} \to \mathbf{M}^0$ descends to a morphism $\mathbf{M}_2 \to \mathbf{M}^0$.

We begin with a brief outline of our strategy. First, we know that the flipped loci $\tilde{\mathbb{I}}_b$ is $\mathbb{P}W_+$ and is the result of $\mathbf{E}_b$ after contracting $\mathbf{E}_a \cap \mathbf{E}_b$, and that the exceptional divisor of $\tilde{\mathbf{M}}_2 \to \mathbf{M}_1$ over $\tilde{\mathbb{I}}_b$ is $\mathbb{P}W_+ \times_{\mathbb{P}\mathcal{F}|_{p_2}^\vee} \mathbb{P}W_-$. Since $\mathbf{M}_2$ is a flip of $\mathbf{M}_1$, the projection Ψ_4 is the result of contracting all fibers of $\mathbb{P}W_+$. On the other hand, by our description of the contraction $\tilde{\mathbf{M}}_1 \to \mathbf{M}_1$, the exceptional divisor of $\tilde{\mathbf{M}} \to \tilde{\mathbf{M}}_1$ over $\Psi_3^{-1}(\tilde{\mathbb{I}}_b)$ is $\mathbf{E}_b \times_{\mathbf{M}_1} \mathbb{P}W_-$. Since both $\mathbf{M}_2$ and $\mathbf{M}^0$ are smooth and since $\mathbf{E}_b$ is proper, to show that Ψ descends to a morphism $\mathbf{M}_2 \to \mathbf{M}^0$ it suffices to show that there is an open subset $\mathbf{U} \subset \mathbf{E}_b$ and an open $\mathbf{V} \subset \mathbb{P}W^-$ so that the restriction of Ψ to $\mathbf{U} \times_{\mathbf{M}_1} \mathbf{V}$ is a composition of the second projection $\mathbf{U} \times_{\mathbf{M}_1} \mathbf{V} \to \mathbf{V}$ with a morphism $\mathbf{V} \to \mathbf{M}^0$.

To prove the last statement, we need a description of the normal bundle $N_{\mathbb{P}W_+/\mathbf{M}_1}$ that relates directly to the elementary modification we shall perform. It is expressed in terms of relative extension sheaves; hence a tautological family on $\mathbf{M}_1$ is required. There is one more technical difficulty: the space $\mathbf{M}_1$ is not a moduli space per se, thus we can not use deformation theory to derive its tangent bundle. Nevertheless, $\mathbf{M}_1$ is birational to $\mathbf{M}^{1/2}$, and thus over a dense open subset its tangent bundle is given by the deformation theory of sheaves.

Our first step is to construct a tautological family over an open subset of $\mathbb{P}W_+ \subset \mathbf{M}_1$. Since $\mathbb{P}W_+$ is a projective bundle over $\mathbb{P}\mathcal{F}|_{p_2}^\vee$, we shall content ourselves with constructing such a family over an open subset of the fiber $\mathbb{P}W_{+\eta}$ of $\mathbb{P}W_+$ over a general $\eta \in \mathbb{P}\mathcal{F}|_{p_2}^\vee$. Such a family will be the universal extension of a sheaf $\mathcal{F}_\eta$ by another sheaf $\mathcal{L}_\eta$ over $X_{1,1}^\dagger$.

We begin with constructing $\mathcal{F}_\eta$ and $\mathcal{L}_\eta$. Let $\eta_0 \in \Xi_1$ be any point associated to a pair of GPBs (F^G, L^G) with $F^0 = \ell \oplus F|_{p_2}$ for a line $\ell \subset F|_{p_1}$, and let $\beta_1 : F \to F|_{p_1}/\ell = \mathbf{k}(p_1)$ be the induced homomorphism. Let $\tilde{X}_{\mathbf{A}^1}$ be the blowing up of $(p_1, 0) \in X \times \mathbf{A}^1$, let $\varphi : \tilde{X}_{\mathbf{A}^1} \to X$ be the projection, and define

$$\tilde{F} = \ker\{\varphi^* F \xrightarrow{\varphi^*\beta_1} \varphi^*\mathbf{k}(p_1)\}|_{\tilde{X}_0}, \quad \text{where } \tilde{X}_0 = \tilde{X}_{\mathbf{A}^1} \times_{\mathbf{A}^1} 0.$$

The sheaf $\tilde{F}$ is a locally free sheaf on $X_{1,0} \cong \tilde{X}_0$ whose restriction to the unique rational curve $D_1 \subset X_{1,0}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$. To obtain a sheaf on $X_{1,1}$ we consider the map $\varphi_2 : X_{1,1} \to X_{1,0}$ contracting the rational curve $D_2 \subset X_{1,1}$ attached to p_2. By abuse of notation, we still denote by D_1 the other rational curve in $X_{1,1}$. The sheaf we intend to construct is the direct sum

$$F' = \varphi_2^* \tilde{F} \oplus \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}(1), \tag{4.9}$$

where $\mathcal{O}_{D_i} = \iota_{i*}\mathcal{O}_{D_i}$ with $\iota_i : D_i \to X_{1,1}$ the inclusion.

Our next step is to glue F' along the two marked points of $X_{1,1}$ using a lift $\eta \in \mathbb{P}\mathcal{F}|_{p_2}^{\vee}$ of $\eta_0 \in \Xi_1$ that is defined by a homomorphism $\beta_2 : F \to \mathbf{k}(p_2)$. Let $\psi : X_{1,1} \to X_{1,1}^{\dagger}$ be the obvious morphism, let $q_- = D_1 \cap \psi^{-1}(q^{\dagger})$ and $q_+ = D_2 \cap \psi^{-1}(q^{\dagger})$ be the two marked points of $X_{1,1}$, and let $q_0 = D_1 \cap X$ and $q_2 = D_2 \cap X$. We consider the space $K_- = \mathrm{Hom}_{X_{1,1}}(\mathcal{O}_{D_1}, F')$ and the subspaces of $F'|_{q_-}$:

$$V_{-,1} = \{f(q_-) \mid f \in K_-, f(q_0) = 0\} \quad \text{and} \quad V_{-,2} = \{f(q_-) \mid f \in K_-\}.$$

Obviously,

$$0 \neq V_{-,1} \subsetneq V_{-,2} \subsetneq F'|_{q_-}$$

form a filtration that depends only on F^G. Similarly, we define a filtration

$$0 \neq V_{+,1} \subsetneq V_{+,2} \subsetneq F'|_{q_+}$$

via

$$V_{+,1} = \ker\{\varphi_2^* \tilde{F}|_{q_+} \xrightarrow{\beta_2} \varphi_2^* \mathbf{k}(p_2)|_{q_+}\} \quad \text{and} \quad V_{+,2} = V_{+,1} \oplus \mathcal{O}_{D_2}(1)|_{q_+}.$$

This filtration depends on η. After that, we pick an isomorphism $h : F'|_{q_-} \to F'|_{q_+}$ that preserves these two filtrations, and identify F' along the two marked points q_- and q_+ via this isomorphism to form a vector bundle on $X_{1,1}^{\dagger}$. We denote the resulting vector bundle by $\mathcal{F}_h$.

Given two such homomorphisms h_1 and h_2, we say $\mathcal{F}_{h_1} \sim \mathcal{F}_{h_2}$ if there is an automorphism σ of $X_{1,1}^{\dagger}$ and an isomorphism $\sigma^*\mathcal{F}_{h_2} \cong \mathcal{F}_{h_1}$.

Lemma 4.7. *Let $\eta \in \mathbb{P}\mathcal{F}|_{p_2}^{\vee}$ be any element and let $\mathcal{F}_h$ be the sheaf on $X_{1,1}^{\dagger}$ so constructed. Then $\mathcal{F}_h$, modulo the equivalence relation so defined, is independent of the choice of h. Let $\mathcal{F}_\eta$ be a representative of this equivalence class. Then for any $\eta' \in \mathbb{P}\mathcal{F}|_{p_2}^{\vee}$, $\mathcal{F}_\eta \sim \mathcal{F}_{\eta'}$ if and only if $\eta = \eta'$.*

Proof. Let G be the group of pairs (v, σ) where σ is an automorphism of the pointed curve $(X_{1,1}, q_\pm)$ and v is an isomorphism $\sigma^*F' \xrightarrow{v} F'$. It is direct to check that the tautological homomorphism $G \to \mathrm{Aut}(F'|_{p_2})$ preserves the filtration $V_{+,\bullet}$ while the image of $G \to \mathrm{Aut}(F'|_{p_1})$ is exactly the subgroup of automorphisms that preserve the filtration $V_{-,\bullet}$. It follows that the equivalence class of the sheaf $\mathcal{F}_h$ is independent of the choice of h.

The proof of the second part is straightforward and will be omitted. q.e.d.

Finally, let $\iota_0 : X \to X_{1,1}^{\dagger}$ be the tautological inclusion and let $\mathcal{L}_\eta = \iota_{0*}L(-p_1 - p_2)$.

Next, we will construct a vector space $\mathcal{W}_\eta$ and a family of sheaves over $\mathbb{P}\mathcal{W}_\eta$. Later we will show that $\mathcal{W}_\eta$ is canonically isomorphic to

$W_{+\eta}$ and the family over $\mathbb{P}\mathcal{W}_\eta$ and the tautological family over $\mathbb{P}W_{+\eta}$ coincide over a dense open subset of $\mathbb{P}\mathcal{W}_\eta \cong \mathbb{P}W_{+\eta}$.

The vector space $\mathcal{W}_\eta$ is the kernel of the canonical homomorphism

$$\operatorname{Ext}^1_{X^\dagger_{1,1}}(\mathcal{F}_\eta, \mathcal{L}_\eta) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{F}_\eta, \mathcal{L}_\eta)) \equiv H^0(\mathbf{k}(q_0) \oplus \mathbf{k}(q_2)) \xrightarrow{\text{proj}} \mathbf{k}(q_2).$$

Next we construct a family of curves over $\mathbb{P}\mathcal{W}_\eta$. Let $\mathcal{D}_1 = \mathbf{P}^1 \times \mathbb{P}\mathcal{W}_\eta$, let $\mathcal{O}(1)$ be the degree one line bundle over $\mathbb{P}\mathcal{W}_\eta$ and let $\mathcal{D}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ be the associated projective bundle over $\mathbb{P}\mathcal{W}_\eta$. We fix two sections

$$\mathcal{Q}_- = 0 \times \mathbb{P}\mathcal{W}_\eta \quad \text{and} \quad \mathcal{Q}_0 = \infty \times \mathbb{P}\mathcal{W}_\eta$$

of $\mathcal{D}_1$ and pick two sections

$$\mathcal{Q}_2 = \mathbb{P}(0 \oplus \mathcal{O}(1)) \quad \text{and} \quad \mathcal{Q}_+ = \mathbb{P}(\mathcal{O} \oplus 0)$$

of $\mathcal{D}_2$. We then glue $\mathcal{D}_1$ to $X \times \mathbb{P}\mathcal{W}_\eta$ by identifying $\mathcal{Q}_0$ with $p_1 \times \mathbb{P}\mathcal{W}_\eta$, and glue $\mathcal{D}_2$ to $\mathcal{X}_{1,0}$ by identifying $\mathcal{Q}_2$ with $p_2 \times \mathbb{P}\mathcal{W}_\eta$. We denote the family from the first gluing by $\mathcal{X}_{1,0}$ and denote the family resulting from both gluings by $\mathcal{X}_{1,1}$. The first is a constant family of $X_{1,0}$ and the second is a non-constant family of $X_{1,1}$ over $\mathbb{P}\mathcal{W}_\eta$. Let $\mathcal{X}^\dagger_{1,1}$ be the result of gluing the two sections $\mathcal{Q}_-$ and $\mathcal{Q}_+$ of $\mathcal{X}_{1,1}$. It is a family of $X^\dagger_{1,1}$ over $\mathbb{P}\mathcal{W}_\eta$.

We now construct a sheaf $\mathfrak{F}$ over $\mathcal{X}^\dagger_{1,1}$. Let $\mathcal{X}_{1,1} \to \mathcal{X}_{1,0}$ be the contraction of the component $\mathcal{D}_2$, and let $\Phi : \mathcal{X}_{1,1} \to X_{1,1}$ be the composition of the contraction $\mathcal{X}_{1,1} \to \mathcal{X}_{1,0}$ with the projection $\mathcal{X}_{1,0} \to X_{1,0} \subset X_{1,1}$. We consider the sheaf of $\mathcal{O}_{\mathcal{X}_{1,1}}$-modules

$$\mathfrak{F}' = \Phi^* \varphi_2^* \tilde{F} \oplus \mathcal{O}_{\mathcal{D}_1} \oplus \mathcal{O}_{\mathcal{D}_2}(\mathcal{Q}_2), \tag{4.10}$$

according to the convention of (4.9). Clearly, there are canonical isomorphisms

$$\mathfrak{F}'|_{\mathcal{Q}_\pm} \cong F'|_{q_\pm} \otimes_{\mathbf{k}} \mathcal{O}_{\mathcal{Q}_\pm}.$$

Hence any isomorphism $h : F'|_{q_-} \cong F'|_{q_+}$ induces a canonical isomorphism $\tilde{h} : \mathfrak{F}'|_{\mathcal{Q}_-} \cong \mathfrak{F}'|_{\mathcal{Q}_+}$. Using an isomorphism h that preserves the two filtrations as defined in Lemma 4.7, we can glue $\mathfrak{F}'$ along the two marked sections $\mathcal{Q}_-$ and $\mathcal{Q}_+$ to obtain a new sheaf $\mathfrak{F}$ over $\mathcal{X}^\dagger_{1,1}$. Clearly, restricting to each fiber of $\mathcal{X}_{1,1}$ over $\mathbb{P}\mathcal{W}_\eta$, the sheaf $\mathfrak{F}'$ is merely the sheaf F' constructed before, and the sheaf $\mathfrak{F}$ is isomorphic to the $\mathcal{F}$ constructed in Lemma 4.7.

We are ready to construct the desired tautological family $\mathfrak{E}$ over $\mathcal{X}^\dagger_{1,1}$. First, recall that $\mathfrak{F}'$ has a direct summand $\mathcal{O}_{\mathcal{D}_2}(\mathcal{Q}_2)$. Since $\mathcal{O}_{\mathcal{D}_2}(\mathcal{Q}_2)|_{\mathcal{Q}_2} \cong \mathcal{O}_{\mathcal{Q}_2}(-1)$[8] , the inclusion $\mathcal{O}_{\mathcal{D}_2}(\mathcal{Q}_2) \subset \mathfrak{F}'$ defines a subsheaf

$$g_1 : \mathcal{O}_{\mathcal{Q}_2}(-1) \to \mathfrak{F}'|_{\mathcal{Q}_2} \equiv \mathfrak{F}|_{\mathcal{Q}_2}.$$

[8]In case $\pi : Z \to \mathbb{P}\mathcal{W}_\eta$ is a family and $\mathcal{E}$ is a sheaf of $\mathcal{O}_Z$-modules, we will use $\mathcal{E}(1)$ to denote the sheaf $\mathcal{E} \otimes \pi^* \mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(1)$.

Let $\mathfrak{L}$ be the sheaf of $\mathcal{O}_{\mathcal{X}_{1,1}^\dagger}$-modules $\mathfrak{L}=\bar\iota_* L(-p_1-p_2)$, where $\bar\iota: X\times\mathbb{P}\mathcal{W}_\eta\to\mathcal{X}_{1,1}$ is the tautological inclusion. Next let $\pi:\mathcal{X}_{1,1}^\dagger\to\mathbb{P}\mathcal{W}_\eta$ be the projection and consider the following two relative extension sheaves and the natural homomorphism between them:

$$\mathcal{E}xt^1_{\mathcal{X}_{1,1}^\dagger/\mathbb{P}\mathcal{W}_\eta}(\mathfrak{L}(-1),\mathfrak{F})\longrightarrow\mathcal{E}xt^1_{\hat{\mathcal{Q}}_2}(\hat{\mathfrak{L}}(-1),\hat{\mathfrak{F}}),\tag{4.11}$$

where the latter is canonically isomorphic to

$$\operatorname{Ext}^1_{W[2]_0^\dagger}(\mathcal{L}_\eta,\mathcal{F}_\eta)\otimes\mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(-1).\tag{4.12}$$

Here $\hat{\mathcal{Q}}_2$ is the formal completion of $\mathcal{X}_{1,1}^\dagger$ along $\mathcal{Q}_2$ while $\hat{\mathfrak{L}}=\mathfrak{L}\otimes_{\mathcal{O}_{\mathcal{X}_{1,1}^\dagger}}\mathcal{O}_{\hat{\mathcal{Q}}_2}$, etc. Because the support of $\mathfrak{L}$ only intersects $\mathcal{D}_2$ along $\mathcal{Q}_2$ and the restriction of $\mathfrak{F}$ to $X_{1,0}\times\mathbb{P}\mathcal{W}_\eta\subset\mathcal{X}_{1,1}^\dagger$ is the pull-back of $F'|_{X_{1,0}}$, the kernel of the above homomorphism is canonically isomorphic to $\mathcal{W}_\eta\otimes\mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(1)$. Let ϵ be a tautological section of $\mathcal{W}_\eta\otimes\mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(1)$. It can be viewed as a section of the relative extension sheaf, and thus defines an extension sheaf $\mathfrak{E}'$ fitting into the exact sequence

$$0\longrightarrow\mathfrak{F}\longrightarrow\mathfrak{E}'\longrightarrow\mathfrak{L}(-1)\longrightarrow 0.$$

Because ϵ is in the kernel of (4.11), $\mathfrak{E}'$ is not locally free along $\mathcal{D}_2$. Not only that, but there is a homomorphism $\mu_1:\mathfrak{E}'\longrightarrow\mathcal{O}_{\mathcal{Q}_2}(-1)$ so that the composite $\mathfrak{F}\to\mathfrak{E}'\to\mathcal{O}_{\mathcal{Q}_2}(-1)$ is identical to the composite $\mathfrak{F}\to\mathcal{O}_{\mathcal{D}_2}(\mathcal{D}_2)|_{\mathcal{Q}_2}\equiv\mathcal{O}_{\mathcal{Q}_2}(-1)$ induced by (4.10). Let $\mu_2:\mathfrak{E}'\to\mathcal{O}_{\mathcal{Q}_2}(-1)$ be induced by $\mathfrak{E}'\to\mathfrak{L}(-1)|_{\mathcal{Q}_2}$ and define $\mathfrak{E}''$ by the exact sequence

$$0\longrightarrow\mathfrak{E}''\longrightarrow\mathfrak{E}'\xrightarrow{(\mu_1,\mu_2)}\mathcal{O}_{\mathcal{Q}_2}(-1)\longrightarrow 0.$$

The resulting sheaf $\mathfrak{E}''$ is locally free along $\mathcal{Q}_2$.

The sheaf $\mathfrak{E}'$ is still non-locally free along some points of $\mathcal{Q}_0$. Indeed, the section ϵ composed with the homomorphism

$$\mathcal{E}xt^1_{\mathcal{X}_{1,1}^\dagger/\mathbb{P}\mathcal{W}_\eta}(\mathfrak{L}(-1),\mathfrak{F})\longrightarrow\mathcal{E}xt^1_{\hat{\mathcal{Q}}_0}(\hat{\mathfrak{L}}(-1),\hat{\mathfrak{F}})\cong\mathcal{O}_{\mathcal{Q}_0}(1)$$

defines a section of $\mathcal{O}_{\mathcal{Q}_0}(1)$ whose vanishing locus is exactly where $\mathfrak{E}''$ is not locally free. Let $s\in H^0(\mathcal{O}_{\mathcal{Q}_0}(1))$ be this section. Before we proceed, we need to resolve the non-local freeness of $\mathfrak{E}''$. We first glue $\mathcal{D}_2$ to $X\times\mathbb{P}\mathcal{W}_\eta$ by identifying $\mathcal{Q}_2$ with $p_2\times\mathbb{P}\mathcal{W}_\eta$, and then glue $\mathcal{D}_1$ to the resulting family by identifying $\mathcal{Q}_-$ with $\mathcal{Q}_+$ in the obvious way. Let $\varphi_1:\mathcal{X}'\to\mathcal{X}_{1,1}^\dagger$ be the projection, which is the smoothing of $\mathcal{X}_{1,1}^\dagger$ along $\mathcal{Q}_0\subset\mathcal{X}_{1,1}^\dagger$. Clearly, $\mathcal{X}'$ is a locally constant family of $X_{0,2}$'s. We next blow up $\mathcal{X}'$ along $p_1\times s^{-1}(0)$, where s is the section of $\mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(1)$ mentioned before. We denote the blowing up by $\tilde{\mathcal{X}}$, and let $\mathcal{Q}_0'\subset\tilde{\mathcal{X}}$ be the proper transform of $p_1\times\mathbb{P}\mathcal{W}_\eta\subset\mathcal{X}'$. Lastly, we construct a new family $\mathcal{X}_{1,1}^\dagger$ by identifying (gluing) the two sections $\mathcal{Q}_0'$ and $\mathcal{Q}_0$ of $\mathcal{X}'$ in the obvious way, and we keep the section $\mathcal{Q}_-=\mathcal{Q}_+\subset\mathcal{X}^\dagger$ as its marked section. This way $\mathcal{X}^\dagger$ is a family whose members are either $X^\dagger_{1,1}$ or $X^\dagger_{2,1}$. Let $\varphi\colon\mathcal{X}^\dagger\to\mathcal{X}^\dagger_{1,1}$ be the tautological projection.

Lemma 4.8. *There is a unique family of locally free sheaves* $\tilde{\mathfrak{E}}$ *over* $\mathcal{X}^\dagger$ *so that* $\varphi_*\tilde{\mathfrak{E}}=\mathfrak{E}''$ *and* $R^1\varphi_*\tilde{\mathfrak{E}}=0$.

Proof. The proof is straightforward and will be omitted. q.e.d.

For $\xi\in\mathbb{P}\mathcal{W}_\eta$ we denote by $\tilde{\mathfrak{E}}_\xi$ the restriction of $\tilde{\mathfrak{E}}$ to the fiber $\mathcal{X}^\dagger_\xi$ of $\mathcal{X}^\dagger$ over ξ.

Lemma 4.9. *There is a line* $\Sigma_\eta\subset\mathbb{P}\mathcal{W}_\eta$ *so that for each* $\xi\in\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta$ *the sheaf* $\tilde{\mathfrak{E}}_\eta$ *is* $\frac{1}{2}$*-stable.*

Proof. We will postpone the proof until the next subsection. q.e.d.

Since for $\xi\in\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta$ the sheaf $\tilde{\mathfrak{E}}_\xi$ is $\frac{1}{2}$-stable, by the universal property of $\mathbf{M}^{1/2}$ the family $\tilde{\mathfrak{E}}$ induces a canonical morphism $u\colon\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta\to\mathbf{M}^{1/2}$. Further, by the construction of the family, it is clear that u factor through $\mathbb{I}_b-\mathbb{I}_a$, and hence to $\tilde{\mathbb{I}}_b-\tilde{\mathbb{I}}_a\subset\mathbf{M}_1$. Since $\tilde{\mathbb{I}}_b=\mathbb{P}W_+$, the morphism u induces a morphism

$$\tilde{u}:\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta\longrightarrow\mathbb{P}W_+.$$

On the other hand, the construction of the family $\tilde{\mathfrak{E}}$ ensures that the composition of $\tilde{u}$ with the projection $\mathbb{P}W_+\to\mathbb{P}\mathcal{F}|^\vee_{p_2}$ maps $\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta$ to the point $\eta\in\mathbb{P}\mathcal{F}|^\vee_{p_2}$. Thus $\tilde{u}$ factor through

$$u_\eta:\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta\longrightarrow\mathbb{P}W_{+\eta}.$$

Lemma 4.10. *The morphism* u_η *extends to an isomorphism* $\mathbb{P}\mathcal{W}_\eta\to\mathbb{P}W_{+\eta}$.

Proof. By construction u_η is one-one. Since $\dim\mathbb{P}\mathcal{W}_\eta=\dim\mathbb{P}W_{+\eta}\geq 3$, u_η is an isomorphism away from a line. It is direct to check that u_η maps lines in $\mathbb{P}\mathcal{W}_\eta-\Sigma_\eta$ to lines in $\mathbb{P}W_{+\eta}$. Hence u_η automatically extends to an isomorphism. q.e.d.

By the argument in Section 3, there is a vector bundle W_- over $\mathbb{P}\mathcal{F}|^\vee_{p_2}$ so that the normal bundle

$$N_{\tilde{\mathbb{I}}_b/\mathbf{M}_1}\cong\varphi^*W_-\otimes\mathcal{O}_{\mathbb{P}W_+}(-1),$$

where $\varphi\colon\mathbb{P}W_+\to\mathbb{P}\mathcal{F}|^\vee_{p_2}$ is the projection.

Lemma 4.11. *The normal bundle* $N_{\tilde{\mathbb{I}}_b/\mathbf{M}_1}|_{\mathbb{P}\mathcal{W}_\eta}$ *is canonically isomorphic to*

$$\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta,\mathcal{F}_\eta)\otimes\mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(-1).$$

Lemma 4.12. *The restriction of the morphism* Ψ *to the preimage of* $\mathbb{P}W_{+\eta}$*, say* $\phi_\eta : \mathbb{P}\mathcal{W}_\eta \times \mathbb{P}\,\mathrm{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta) \to \mathbf{M}^0$ *is the composite of the second projection with a morphism* $h : \mathbb{P}\,\mathrm{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta) \to \mathbf{M}^0$.

Proof. We will postpone the proofs of these two lemmas until the next subsection. q.e.d.

Since $\mathbf{M}_2$ is a flip of $\mathbf{M}_1$ along $\tilde{\mathbb{I}}_b \cup \tilde{\mathbb{I}}_c$, the restriction of Ψ_3 to the exceptional divisor over $\tilde{\mathbb{I}}_b$, which is $\mathbb{P}W_+ \times \mathbb{P}W_-$, is the composite of the second projection with the morphism $\mathbb{P}W_- \to \mathbf{M}_2$. In particular, this proves that

Lemma 4.13. *For any* $z \in \mathrm{Im}\,\phi_\eta$*, the image set* $(\Psi_4 \circ \tilde{\Psi}_2)(\phi_\eta^{-1}(z))$ *is a single point set in* $\mathbf{M}_2$.

Since $\eta \in \mathbb{P}\mathcal{F}_\eta|^\vee_{p_2}$ is an arbitrary point, this proves

Lemma 4.14. *For any closed* $z \in \mathbf{M}_2$ *there is a unique point* $z' \in \mathbf{M}^0$ *so that* $(\Psi_4 \circ \tilde{\Psi}_2)^{-1}(z) = \Psi^{-1}(z')$.

As a corollary, this proves the equivalence result we set out to prove:

Proposition 4.15. *The induced birational map* $\mathbf{M}^0 \sim \mathbf{M}_2$ *is an isomorphism of varieties.*

In this subsection, we will give the proof of Lemmas 4.9, 4.11 and 4.12.

Proof of Lemma 4.9. We need to investigate when the sheaf $\tilde{\mathfrak{E}}_\xi$ is $\frac{1}{2}$-stable. For the moment, we assume ξ is away from the vanishing locus $s^{-1}(0)$. Then $\tilde{\mathfrak{E}}_\xi$ is of type $\mathbb{I}_b^{+0}$ that fits into the exact sequence

$$0 \longrightarrow \mathcal{F}_\eta \longrightarrow \tilde{\mathfrak{E}}_\xi \longrightarrow \mathcal{L}_\eta \longrightarrow 0. \tag{4.13}$$

Following the discussion in Section 3, $\tilde{\mathfrak{E}}_\xi$ is not $\frac{1}{2}$-stable if and only if there is a sheaf $\bar{\mathcal{L}}_\eta$ that is locally free away from the marked node $q_1 = q^\dagger$ so that it fits into the diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_\eta & \longrightarrow & \tilde{\mathfrak{E}}_\xi & \longrightarrow & \mathcal{L}_\eta & \longrightarrow & 0 \\ & & \uparrow\subset & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{D_1}(-1) \oplus \mathcal{O}_{D_2}(-1) & \longrightarrow & \bar{\mathcal{L}}_\eta & \longrightarrow & \mathcal{L}_\eta & \longrightarrow & 0. \end{array}$$

Because $\bar{\mathcal{L}}_\eta$ is unique and the left square is a push-out, $\bar{\mathfrak{E}}_\xi$ is uniquely determined by the left vertical inclusion. On the other hand, the subsheaf $\mathcal{O}_{D_2}(-1) \hookrightarrow \mathcal{F}_\eta$ is unique and there is a $\mathbf{P}^1$ family of subsheaves $\mathcal{O}_{D_1}(-1) \hookrightarrow \mathcal{F}_\eta$. Hence there is a $\mathbf{P}^1$ family of extensions (4.13) that are derived from the diagram above. Further, it is easy to see that there is one choice of $\mathcal{O}_{D_1}(-1) \hookrightarrow \mathcal{F}_\eta$ so that the associated sheaf $\tilde{\mathfrak{E}}_\xi$ is not locally free at q_0. A quick reasoning shows that this corresponds exactly to the case where $\xi \in s^{-1}(0)$. Combined, this shows that there is a line $\Sigma_\eta \subset \mathbb{P}\mathcal{W}_\eta$ so that for all $\xi \in \mathbb{P}\mathcal{W}_\eta - \Sigma_\eta \cup s^{-1}(0)$ the associated sheaves $\tilde{\mathfrak{E}}_\xi$ are $\frac{1}{2}$-stable.

In case $\xi \in s^{-1}(0)$, a similar argument shows that $\tilde{\mathfrak{E}}_\xi$ is $\frac{1}{2}$-stable unless $\xi \in s^{-1}(0) \cap \Sigma_\eta$. This proves the lemma. q.e.d.

Proof of Lemma 4.11. We now prove that

$$N_{\tilde{\mathbb{I}}_b/\mathbf{M}_1}|_{\mathbb{P}\mathcal{W}_\eta} \cong \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta) \otimes \mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(-1). \tag{4.14}$$

Let $W[2]^\dagger/\mathbf{A}^2$ be the family of marked curves containing $W[2]^\dagger_0 \cong X^\dagger_{1,1}$ as its central fiber. Then $\tilde{\mathfrak{E}}_\xi$, where $\xi \in \mathbb{P}\mathcal{W}_\eta - s^{-1}(0) \cup \Sigma_\eta$, is a sheaf over $W[2]^\dagger_0$. It is known that the first order deformations of $\tilde{\mathfrak{E}}_\xi$ as sheaves of $\mathcal{O}_{w[2]^\dagger}$-modules are $\operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \tilde{\mathfrak{E}}_\xi)$, which fits into the diagram

$$\begin{array}{ccccc}
\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta) & & & & \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{L}_\eta) \\
\downarrow & & & & \downarrow \\
\operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \mathcal{F}_\eta) & \longrightarrow & \operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \tilde{\mathfrak{E}}_\xi) & \xrightarrow{\phi_1} & \operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \mathcal{L}_\eta) \\
\downarrow & & & & \downarrow \phi_2 \\
\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{F}_\eta) & & & & \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta)
\end{array}$$

Because the standard $(\mathbb{C}^*)^{\times 2}$ action on $\mathbf{A}^2$ lifts to an action on $W[2]^\dagger \to \mathbf{A}^2$, it induces a homomorphism $\mathbb{C}^{\oplus 2} \equiv T_0\mathbf{A}^2 \to \operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \tilde{\mathfrak{E}}_\xi)$. Since $\tilde{\mathfrak{E}}_\xi$ is $\frac{1}{2}$-stable, $[\tilde{\mathfrak{E}}_\xi] \in \mathbf{M}^{1/2}$, and hence lies in $\mathbb{I}_b^+$. Then the tangent space $T_{[\tilde{\mathfrak{E}}_\xi]}\mathbf{M}^{1/2}$ at $[\tilde{\mathfrak{E}}_\xi]$ is canonically isomorphic to $\operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \tilde{\mathfrak{E}}_\xi)/T_0\mathbf{A}^2$.

We now claim that the kernel of $\phi = \phi_2 \circ \phi_1$ contains the image of $T_0\mathbf{A}^2 \to \operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \tilde{\mathfrak{E}}_\xi)$ and the tangent space of $\mathbb{I}_b^+$ at $[\tilde{\mathfrak{E}}_\xi]$ is the quotient $\ker(\phi)/T_0\mathbf{A}^2$. Indeed, the groups $\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta,\mathcal{L}_\eta), \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{F}_\eta)$ and $\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta)$ parameterize the first order deformations of $\mathcal{L}_\eta$, of $\mathcal{F}_\eta$ and the space of extensions of $\mathcal{L}_\eta$ by $\mathcal{F}_\eta$. It is direct to check that the kernel of ϕ is the tangent space at $[\tilde{\mathfrak{E}}_\xi]$ of the space of all sheaves of type $\mathbb{I}_b^+$. Because the $(\mathbb{C}^*)^{\times 2}$ action preserves this space, we have $\operatorname{Im}(T_0\mathbf{A}^2) \subset \ker(\phi)$ and hence $\ker(\phi)/T_0\mathbf{A}^2 \equiv T_{[\tilde{\mathfrak{E}}_\xi]}\mathbb{I}_b^+$.

We now show that ϕ is surjective. Once this is established, then the normal vector space to $\mathbb{I}_b^+$ at $\tilde{\mathfrak{E}}_\eta$ is canonically isomorphic to

$$N_{\mathbb{I}_b^+/\mathbf{M}^{1/2}}|_{\tilde{\mathfrak{E}}_\xi} \equiv \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta). \tag{4.15}$$

First, since $T_{[\tilde{\mathfrak{E}}_\phi]}\mathbb{I}_a^+ = \ker(\xi)/T_0\mathbf{A}^2$, the image of ϕ is the normal vector space to $\mathbb{I}_a^+$ in $\mathbf{M}^{1/2}$ at $[\tilde{\mathfrak{E}}_\xi]$. By (3.11), we know that the normal vector space has dimension $2g-1$. Thus to prove the lemma it suffices to show that

$$\dim \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta) = 2g-1.$$

Recall that $q^\dagger = q_1$. By a direct computation we have the exact sequence

$$0 \longrightarrow \mathbf{k}(q_0) \oplus \mathbf{k}(q_2) \longrightarrow \mathcal{E}xt^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta) \longrightarrow \mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta) \otimes_{\mathbf{k}} T_0\mathbf{A}^2 \longrightarrow 0.$$

We claim that $H^0(\mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta)) = 0$. First, let $F^G = (F, F^0)$ and $L^G = (L, 0)$ be the associated GPB vector bundles of $[\tilde{\mathfrak{E}}_\xi]$ in $\mathbf{G}^{1/3}_{2,4,3}$ and $\mathbf{G}^{1/3}_{1,3,0}$, respectively; that is the image of the morphism $\mathbb{I}^+_a \to \mathbf{G}^{1/3}_{2,4,3} \times \mathbf{G}^{1/3}_{1,3,0}$ introduced in Section 3.2. By abuse of notation, we let $j : X \to W[2]^\dagger_0$ be the main irreducible component. Then we have exact sequences of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{F}_\eta \otimes_{\mathcal{O}_{W[2]^\dagger}} \mathcal{O}_X \longrightarrow F \longrightarrow \mathbf{k}(p_1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{L}_\eta \otimes_{\mathcal{O}_{W[2]^\dagger}} \mathcal{O}_X \longrightarrow L \longrightarrow \mathbf{k}(p_1) \oplus \mathbf{k}(p_2) \longrightarrow 0.$$

Further, a direct check shows that

$$\mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta) = j_* \ker\{F^\vee \otimes L \longrightarrow \mathbf{k}(p_1) \otimes L|_{p_1} \oplus F^\vee \otimes L|_{p_2}\}.$$

Hence any nontrivial homomorphism $F \to L$ in $H^0(\mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta))$ is a homomorphism $F^G \to L^G$ of GPBs. But both F^G and L^G are $\frac{1}{3}$-stable GPBs and thus there are no nontrivial homomorphisms between them. This proves that $H^0(\mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta)) = 0$. Combined with the exact sequence

$$0 \longrightarrow H^1(\mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta)) \longrightarrow \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta) \longrightarrow H^0(\mathcal{E}xt^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta)) \longrightarrow 0,$$

we obtain

$$\dim \operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta) = 2 + h^1(\mathcal{H}om(\mathcal{F}_\eta, \mathcal{L}_\eta) = 2g-1.$$

This proves that the arrow ϕ is surjective.

Because the isomorphism (4.15) is canonical, restricting to $\mathbb{P}\mathcal{W}_\eta - s^{-1}(0) \cup \Sigma_\eta$ we have canonical isomorphism

$$N_{\mathbb{I}^+_b/\mathbf{M}^{1/2}}|_{\mathbb{P}\mathcal{W}_\eta - s^{-1}(0)\cup\Sigma_\eta} \cong \mathcal{E}xt^1_{\mathcal{X}^\dagger_{1,1}/\mathbb{P}\mathcal{W}_\eta}(\mathfrak{L}(-1), \mathfrak{F})|_{\mathbb{P}\mathcal{W}_\eta - s^{-1}(0)\cup\Sigma_\eta}.$$

Because $\operatorname{Codim}\Sigma_\eta \geq 2$, $s^{-1}(0)$ is a hypersurface and both $N_{\mathbb{I}^+_b/\mathbf{M}^{1/2}}$ and $\mathcal{E}xt^1_{\mathcal{X}^\dagger_{1,1}/\mathbb{P}\mathcal{W}_\eta}(\mathfrak{L}(-1), \mathfrak{F})$ are of the forms $\mathbb{C}^{2g-1} \otimes \mathcal{O}_{\mathbb{P}\mathcal{W}_\eta}(-1)$, the above isomorphism must extend to an isomorphism (4.14), as desired. This proves the lemma. q.e.d.

Proof of Lemma 4.12. We pick an element

$$(\xi, v) \in \mathbb{P}\mathcal{W}_\eta \times \mathbb{P}\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta).$$

We assume that the sheaf $\tilde{\mathfrak{E}}_\xi$ is a sheaf over $W[2]^\dagger_0$, which is $\frac{1}{2}$-stable, and the image of v in

$$H^0\big(\mathcal{E}xt^1_{W[2]^\dagger}(\mathcal{F}_\eta, \mathcal{L}_\eta)\big) = \mathbf{k}(q_0) \oplus \mathbf{k}(q_2)$$

is not contained in either $\mathbf{k}(q_0)$ or in $\mathbf{k}(q_2)$. Now let $B = \operatorname{Spec}\mathbf{k}[u]/(u^2)$ and let $B_0 \subset B$ be the closed point. Then a lift $\tilde{v} \in \mathbb{P}\operatorname{Ext}^1_{W[2]^\dagger}(\tilde{\mathfrak{E}}_\xi, \tilde{\mathfrak{E}}_\xi)$ of v, namely $\phi(\tilde{v}) = v$, defines a sheaf of $\mathcal{O}_{W[2]^\dagger \times B}$-modules $\tilde{\mathfrak{E}}_\xi(\tilde{v})$ that is the extension of $\tilde{\mathfrak{E}}_\xi$ by $\tilde{\mathfrak{E}}_\xi \otimes \mathcal{I}$ defined by the class $\tilde{v}$. Here $\mathcal{I}$ is the ideal sheaf of $W[2]^\dagger \times B_0 \subset W[2]^\dagger \times B$. Then a direct local calculation of extension sheaves shows that there is an embedding $\omega : B \to \mathbf{A}^2$ that does not lie in the two coordinate lines of $\mathbf{A}^2$, and so $\tilde{\mathfrak{E}}_\xi(\tilde{v})$ is a rank three locally free sheaf of $\mathcal{O}_{W[2]^\dagger \times_{\mathbf{A}^2} B}$-modules.

We now suppose $\Psi_2^{-1}(\xi)$ is a single point. Then $(\xi, v) \in \tilde{\mathbf{M}}_1$ lifts to a unique element in $\tilde{\mathbf{M}}$, which we denote by (ξ, v) as well. Then following the discussion in subsection 4.1, the image $\Psi((\xi, v)) \in \mathbf{M}^0$ is the point associated to the sheaf $\tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v})$ that was constructed by first taking the kernel of the composite

$$\mathfrak{F}_\xi(\tilde{v}) = \ker\big\{\tilde{\mathfrak{E}}_\xi(\tilde{v}) \longrightarrow \tilde{\mathfrak{E}}_\xi(\tilde{v}) \otimes_{\mathcal{O}_{W[2]^\dagger \times_{\mathbf{A}^2} B}} \mathcal{O}_{W[2]^\dagger_0} \cong \tilde{\mathfrak{E}}_\xi \longrightarrow \mathcal{L}_\eta\big\}$$

and then restricting to the closed fiber $W[2]^\dagger_0$

$$\tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v}) \equiv \mathfrak{F}_\xi(\tilde{v}) \otimes_{\mathcal{O}_{W[2]^\dagger \times_{\mathbf{A}^2} B}} \mathcal{O}_{W[2]^\dagger_0}.$$

First of all, since $B \to \mathbf{A}^2$ does not lie in any of the two coordinate lines, to perform the elementary modification we do not need to modify any of the nodes in $W[2]^\dagger_0$ and the sheaf $\tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v})$ is locally free. On the other hand, $\tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v})$ is the cokernel of the composite $\mathcal{F}_\eta \equiv \mathcal{F}_\eta \otimes \mathcal{I} \longrightarrow \mathfrak{F}_\xi(\tilde{v})$ that is the unique lifting of

$$\mathcal{F}_\eta \otimes_{\mathcal{O}_{W[2]^\dagger_0}} \mathcal{I} \longrightarrow \tilde{\mathfrak{E}}_\xi \otimes_{\mathcal{O}_{W[2]^\dagger_0}} \mathcal{I} \longrightarrow \tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v}).$$

Hence $\tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v})$ fits into the exact sequence

$$0 \longrightarrow \mathcal{L}_\eta \longrightarrow \tilde{\tilde{\mathfrak{F}}}_\xi(\tilde{v}) \longrightarrow \mathcal{F}_\eta \longrightarrow 0$$

and the extension class of this exact sequence is a multiple of

$$v \in \mathbb{P}\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta)$$

we started with. In particular the image $\Psi((\xi, v))$ depends only on v. Now we pick an (analytic) open subset U_η of $\xi \in \mathbb{P}\mathcal{W}_\eta$ and $V_\eta \subset \mathbb{P}\operatorname{Ext}^1_{W[2]^\dagger}(\mathcal{L}_\eta, \mathcal{F}_\eta)$ so that

$$\tilde{\Psi}_2|_{\tilde{\Psi}_2^{-1}(U_\eta \times V_\eta)} : \tilde{\Psi}_2^{-1}(U_\eta \times V_\eta) \to U_\eta \times V_\eta$$

is an isomorphism. Then the fact that $\Psi((\xi, v))$ depends only on v implies that the restriction of Ψ to $\tilde{\Psi}_2^{-1}(U_\eta \times V_\eta)$ is the composite of the second projection $U_\eta \times V_\eta \to V_\eta$ with a morphism $V_\eta \to \mathbf{M}^0$. Since Ψ is a morphism and

$$\tilde{\Psi}_2^{-1}(\mathbb{P}W_+ \times_{\mathbb{P}\mathcal{F}|_{p_2}^\vee} \mathbb{P}W_-) \cong \mathbf{E}_b \times_{\mathbb{P}\mathcal{F}|_{p_2}^\vee} \mathbb{P}W_-,$$

its restriction to $\tilde{\Psi}_2^{-1}(\mathbb{P}W_+ \times_{\mathbb{P}\mathcal{F}|_{p_2}^\vee} \mathbb{P}W_-)$ must be the composite of the second projection with a morphism $\mathbb{P}W_- \to \mathbf{M}^0$. This proves the lemma. q.e.d.

5. The vanishing results

The purpose of this section is to prove the main theorem of this paper.

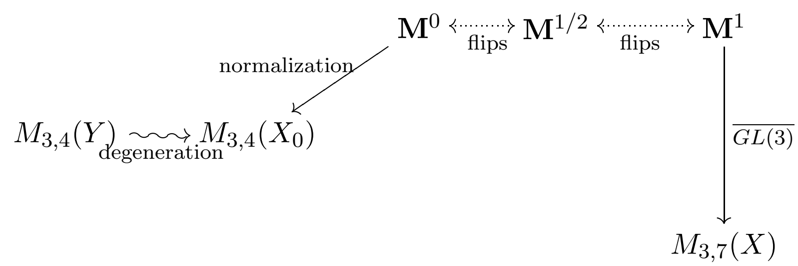
Theorem 5.1. *Let $M_{3,\chi}(Y)$ be the moduli space of stable vector bundles over a smooth irreducible curve Y of genus g for $\chi \equiv 1, 2 \bmod 3$. Then we have*

$$c_i(M_{3,\chi}(Y)) = 0 \quad \textit{for } i > 6g - 5.$$

Since $M_{3,1}(Y) \cong M_{3,2}(Y)$ by $E \to E^\vee \otimes L$ for a fixed line bundle L of degree 1, we may assume $\chi \equiv 1 \bmod 3$, say $\chi = 4$.

Our proof is induction on the genus g. When $g = 1$, $M_{3,4}(Y) \cong Y$ by Atiyah's theorem and hence we have the vanishing result. We assume from now on that $g \geq 2$.

In the previous sections, we established the following diagram:



Suppose $c_i(M_{3,4}(X)) = 0$ for $i > 6g - 11$. We want to show that $c_i(M_{3,4}(Y)) = 0$ for $i > 6g - 5$.

5.1. Chern classes of $\mathbf{M}^0$.

Let $S_0 = M_{3,7}(X)$ and $E \to S_0 \times X$ be a universal bundle. Recall that a vector bundle on X_n is α-stable if and only if its associated GPB (V, V^0) is α-stable. When $\alpha = 1^-$, this is equivalent to V being stable. Hence $\mathbf{M}^{1^-}$ is a fiber bundle over S_0 obtained by blowing up $\mathbf{G}^{1^-} = Gr(3, E|_{p_1+p_2})$, the Grassmannian bundle over S_0. Let $\pi_1 : S_1 = \mathbb{P}\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \to S_0$ be the projectivization of the bundle $\mathrm{Hom}(E|_{p_1}, E|_{p_2})$.

We blow up S_1 along the locus of rank 1 homomorphisms

$$B := \mathbb{P}E|_{p_1}^\vee \times_{S_0} \mathbb{P}E|_{p_2}$$

and let $\pi_2 : S_2 \to S_1$ be the blow-up map.

The exceptional divisor $\Delta_1 := \tilde{B}$ of $\pi_2 : S_2 \to S_1$ and the proper transform Δ_2 of the locus of rank 2 homomorphisms in S_1 are normal crossing divisors. Let $\Delta = \Delta_1 + \Delta_2$.

Lemma 5.2. *$c_i(\Omega_{S_2/S_0}(\log\Delta)) = 0$ for $i > 6$. Consequently, if $c_i(\Omega_{S_0}) = 0$ for $i > 6g-11$, then $c_i(\Omega_{S_2}(\log\Delta)) = 0$ for $i > 6g-5$.*

Since $\Omega_{S_2/S_0}(\log\Delta)$ is locally free of rank 8, it suffices to check that the 7th and 8th Chern classes vanish. The proof is a lengthy computation. See the Appendix.

Let S_3 be the result of two blow-ups of $\mathbb{P}\big(\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \oplus \mathcal{O}_{S_0}\big)$ first along the section $\mathbb{P}\mathcal{O}_{S_0}$ and then along the closure of the locus of rank 1 homomorphisms in $\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \subset \mathbb{P}\big(\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \oplus \mathcal{O}_{S_0}\big)$. The obvious rational map

$$\mathbb{P}\big(\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \oplus \mathcal{O}_{S_0}\big) \dashrightarrow \mathbb{P}\mathrm{Hom}(E|_{p_1}, E|_{p_2}) = S_1$$

becomes a $\mathbb{P}^1$-bundle after the above first blow-up and the preimage of B is the center of the second blow-up. Hence we get a $\mathbb{P}^1$-bundle projection

$$\pi_3 : S_3 \to S_2$$

and S_2 naturally embeds into S_3.

Next, we blow up S_3 along $\Delta_1 = \tilde{B} \subset S_2$, which lies in S_3 as a codimension 2 subvariety. Let $\pi_4 : S_4 \to S_3$ be the blow-up. Then by local computation, S_4 is the same as the result of the blow-ups of $\mathbb{P}\big(\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \oplus \mathcal{O}_{S_0}\big)$, first along $\mathbb{P}\mathcal{O}_{S_0}$, second along B, which lies in $\mathbb{P}\mathrm{Hom}(E|_{p_1}, E|_{p_2})$, and finally along the proper transform of the closure of the locus rank 1 homomorphisms in

$$\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \subset \mathbb{P}\big(\mathrm{Hom}(E|_{p_1}, E|_{p_2}) \oplus \mathcal{O}_{S_0}\big).$$

So if we finally blow up S_4 along the proper transform of $\Delta_2 \subset S_2$ which lies in S_4 as a codimension 2 subvariety, then we obtain the moduli space $\mathbf{M}^{1^-}$ of 1^--stable bundles which we also denote by S_5 and the last blow-up is denoted by $\pi_5 : S_5 \to S_4$. Recall that $\mathbf{M}^{1^-}$ has six divisors $\tilde{\mathbf{Y}}_0, \tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2, \tilde{\mathbf{Z}}_0, \tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2$. Let D be the sum of these.

Lemma 5.3. *$c_i(\Omega_{\mathbf{M}^1}(\log D)) = 0$ for $i > 6(g-1)$ if and only if $c_i(\Omega_{S_2}(\log\Delta)) = 0$ for $i > 6(g-1)$.*

Proof. The proof is due to Gieseker [**4**]. Notice that we have four divisors $\widetilde{\Delta}_0, \widetilde{\Delta}_1, \widetilde{\Delta_2}, \widetilde{\Delta}_3$ in S_3 which are the images of $\tilde{\mathbf{Z}}_0, \tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2, \tilde{\mathbf{Y}}_0$ respectively. Let $\widetilde{\Delta}$ be the sum of $\widetilde{\Delta}_i$'s. Notice that $\widetilde{\Delta}_1 = \pi_3^{-1}(\Delta_1)$ and $\widetilde{\Delta}_2 = \pi_3^{-1}(\Delta_2)$. Hence we have an exact sequence

$$0 \to \pi_3^*\Omega_{S_2}(\log\Delta) \to \Omega_{S_3}(\log\widetilde{\Delta}) \to \Omega_{S_3/S_2}(\log(\widetilde{\Delta}_0 + \widetilde{\Delta}_3)) \to 0.$$

But the line bundle $\Omega_{S_3/S_2}(\log(\widetilde{\Delta}_0+\widetilde{\Delta}_3))$ is trivial since we can find a nowhere vanishing section as follows: as π_3 is a $\mathbb{P}^1$-bundle there is an open covering $\{U_i\}$ of S_2 and rational functions z_i on $\pi_3^{-1}(U_i)$ with a simple pole at $\widetilde{\Delta}_0$ and a simple zero at $\widetilde{\Delta}_3$. Because $z_i = z_j f_{ij}$ for a nowhere vanishing function f_{ij} on $U_i \cap U_j$, dz_i/z_i gives a well-defined section of $\Omega_{S_3/S_2}(\log(\widetilde{\Delta}_0+\widetilde{\Delta}_3))$.

Next, S_4 was obtained by blowing up along the intersection of two divisors $\widetilde{\Delta}_0$ and $\widetilde{\Delta}_1$ in S_3. Let Z_1' be the exceptional divisor of π_4. By a local computation, we get

$$\pi_4^*\Omega_{S_3}(\log\widetilde{\Delta}) \cong \Omega_{S_4}(\log(\widetilde{\Delta}+Z_1')).$$

By the same argument, we see that

$$\pi_5^*\Omega_{S_4}(\log(\widetilde{\Delta}+Z_1')) \cong \Omega_{\mathbf{M}^1}(\log D).$$

The lemma now follows immediately. q.e.d.

By Lemmas 5.2 and 5.3, we deduce the vanishing of Chern classes for $\mathbf{M}^{1^-}$.

Corollary 5.4. *$c_i(\Omega_{\mathbf{M}^{1^-}}(\log D)) = 0$ for $i > 6g-5$*

5.2. From $\mathbf{M}^{1^-}$ to $\mathbf{M}^0$. The goal of this subsection is to show the following.

Proposition 5.5.
$c_i(\Omega_{\mathbf{M}^{1^-}}(\log D)) = 0$ for $i > 6g-5$ iff $c_i(\Omega_{\mathbf{M}^0}(\log D)) = 0$ for $i > 6g-5$.

Recall that $\mathbf{M}^0$ is obtained from $\mathbf{M}^{1^-}$ by a sequence of flips along subvarieties, each of which lies in the intersection of two of the six divisors, and the blow-up center is not contained in any other divisor.

We use a lemma from [**4**]. Let J be the base of a flip. In other words, there are two vector bundles E and F over J and a variety S into which $\mathbb{P}E$ is embedded. And the normal bundle to $Z = \mathbb{P}E$ is the pull-back of F tensored with $\mathcal{O}_{\mathbb{P}E}(-1)$. Let $\tilde{S}$ be the blow-up of S along Z and S' be the blow-down of $\tilde{S}$ along the $\mathbb{P}E$-direction. Then $\tilde{S}$ is the blow-up of S' along $Z' = \mathbb{P}F$. Suppose that there are normally crossing smooth divisors D_i in S such that Z is contained in $D_1 \cap D_2$ as a smooth subvariety but no other divisor contains Z. Let $D = \sum D_i$ and D' be its proper transform in S'. The following lemma is from [**4**, §12].[9]

Lemma 5.6. *Suppose the top k Chern classes of J vanish. Then $c_i(\Omega_S(\log D)) = 0$ for $i > \dim S - k - 1$ iff $c_i(\Omega_{S'}(\log D')) = 0$ for $i > \dim S - k - 1$.*

[9] Gieseker assumed that $Z \subset D_1 \cap D_2$ and $Z \cap D_k = \emptyset$ for $k \neq 1, 2$. But the same proof works as long as $Z \cap D$ is a smooth divisor in Z.

The flip bases for $\alpha = 2/3$ are as follows: The moduli spaces $\mathbf{P}^{2/3}_{2,5,p_i}$ of stable parabolic bundles with parabolic weight 1/3 and quasi-parabolic structure at p_i for $i = 1, 2$ lie in the moduli of GPBs $\mathbf{G}^{2/3}_{2,5,1}$. Let $\widetilde{\mathbf{G}}^{2/3}_{2,5,1}$ be the blow-up of $\mathbf{G}^{2/3}_{2,5,1}$ along $\mathbf{P}^{2/3}_{2,5,p_1} \cup \mathbf{P}^{2/3}_{2,5,p_2}$. Then the flip bases are

- $\widetilde{\mathbf{G}}^{2/3}_{2,5,1} \times Jac(X)$
- a Jacobian of X times a $\mathbb{P}^1$ bundle over $\mathbf{P}^{2/3}_{2,5,p_1}$
- a Jacobian of X times a $\mathbb{P}^1$ bundle over $\mathbf{P}^{2/3}_{2,5,p_2}$.

Because the underlying vector bundle of a parabolic bundle or a GPB above is stable, all these three moduli spaces are fiber bundles over $M_{2,5}(X)$. By Gieseker's theorem [**4**], we know that the top $2g-3$ Chern classes of $M_{2,5}(X)$ vanish. Hence the top $3g-4$ Chern classes of the flip bases for $\alpha = 2/3$ vanish. From the above lemma, we have

$$c_i(\Omega_{\mathbf{M}^{1/2}}(\log D)) = 0 \quad \text{for } i > 6g-5.$$

We can similarly deal with the flip bases for $\alpha = 1/3$: the moduli spaces $\mathbf{P}^{1/3}_{2,4,p_i}$ of stable parabolic bundles with parabolic weight 2/3 and quasi-parabolic structure at p_i for $i = 1, 2$ lie in the moduli of GPBs[10] $\mathbf{G}^{1/3}_{2,4,3}$. Let $\widetilde{\mathbf{G}}^{1/3}_{2,4,3}$ be the blow-up of $\mathbf{G}^{1/3}_{2,4,3}$ along $\mathbf{P}^{1/3}_{2,4,p_1} \cup \mathbf{P}^{1/3}_{2,4,p_2}$. Then the flip bases are

- $\widetilde{\mathbf{G}}^{1/3}_{2,4,3} \times Jac(X)$
- a Jacobian of X times a $\mathbb{P}^1$ bundle over $\mathbf{P}^{1/3}_{2,4,p_1}$
- a Jacobian of X times a $\mathbb{P}^1$ bundle over $\mathbf{P}^{1/3}_{2,4,p_2}$.

Because the underlying vector bundle of a parabolic bundle above is stable, the moduli spaces $\mathbf{P}^{1/3}_{2,4,p_1}$ and $\mathbf{P}^{1/3}_{2,4,p_2}$ are fiber bundles over $M_{2,3}(X)$. By Gieseker's theorem [**4**], we know that the top $2g-3$ Chern classes of $M_{2,3}(X)$ vanish.

The moduli space $\widetilde{\mathbf{G}}^{1/3}_{2,4,3}$ is not a fiber bundle over $M_{2,3}(X)$ but this is isomorphic to a divisor in Gieseker's moduli space: consider the universal family $\mathcal{F}$ over $\widetilde{\mathbf{G}}^{1/3}_{2,4,3} \times X$. Blow up this space along $\mathbf{P}^{1/3}_{2,4,p_1} \times p_1$ and $\mathbf{P}^{1/3}_{2,4,p_2} \times p_2$. Perform elementary modifications as in §2.4 so that we get a family of curves over $\widetilde{\mathbf{G}}^{1/3}_{2,4,3}$ and a vector bundle on the family of curves. The restriction of this vector bundle to the proper transforms of $\widetilde{\mathbf{G}}^{1/3}_{2,4,3} \times p_1$ and $\widetilde{\mathbf{G}}^{1/3}_{2,4,3} \times p_2$ is equipped with a choice of basis and we can glue the rank 2 bundle $\mathcal{O} \oplus \mathcal{O}(1)$ over a rational curve $\mathbb{P}^1$ to get a vector bundle over the family of nodal genus g curves. It is elementary to check that this is a family of bundles in the Gieseker's moduli space

[10] The choice of 1 dimensional subspace V_1 of $E|_{p_1}$ gives rise to the 3 dimensional subspace $V = V_1 + E|_{p_2}$. This is a GPB in $\mathbf{G}^{1/3}_{2,4,3}$.

$\mathbf{M}_{2,3}^{1/3}$ for the rank 2 case and so we get a morphism

$$\widetilde{\mathbf{G}}_{2,4,3}^{1/3} \to \mathbf{M}_{2,3}^{1/3}.$$

It is now an easy matter to check that this morphism is bijective onto a divisor of rank 1 locus in the Gieseker's moduli space. Hence, $\widetilde{\mathbf{G}}_{2,4,3}^{1/3}$ becomes a fiber bundle over $M_{2,1}(X)$ after a flip whose base is the product of two Jacobians over X. By Gieseker's lemma again, we deduce that the top $2g-3$ Chern classes of $\widetilde{\mathbf{G}}_{2,4,3}^{1/3}$ vanish and hence the top $3g-4$ Chern classes of all the flip bases for $\alpha = 1/3$ vanish. From Gieseker's lemma, we have

$$c_i(\Omega_{\mathbf{M}^0}(\log D)) = 0 \quad \text{ for } i > 6g-5.$$

The argument at the end of §13 in [**4**] enables us to deduce the vanishing Chern classes of the general member of the family $\mathbf{M}_{3,4}(\mathfrak{W})$ from the vanishing of the Chern classes of $\Omega_{\mathbf{M}^0}(\log D)$. So we conclude that

$$c_i(M_{3,4}(Y)) = 0 \quad \text{ for } i > 6g-5.$$

6. Appendix

The purpose of this appendix is to prove Lemma 5.2.

Notice that $E|_{p_1}$ is isomorphic to $E|_{p_2}$. Let a_1, a_2, a_3 be the Chern roots of $E|_{p_1} \cong E|_{p_2}$ and let $\xi = c_1(\mathcal{O}_{S_1}(-1))$. Then the Chern roots of $\mathrm{Hom}(E|_{p_1}, E|_{p_2})$ are $0, 0, 0, \pm(a_1-a_2), \pm(a_2-a_3), \pm(a_3-a_1)$ and thus the cohomology ring $H^*(S_1)$ is the polynomial algebra $H^*(S_0)[\xi]$ over $H^*(S_0)$ modulo the relation

$$\xi^3(\xi^2-(a_1-a_2)^2)(\xi^2-(a_2-a_3)^2)(\xi^2-(a_3-a_1)^2). \tag{6.1}$$

From the exact sequence

$$0 \to \Omega_{S_1/S_0} \to \pi_1^* \mathrm{Hom}(E|_{p_1}, E|_{p_2}) \otimes \mathcal{O}_{S_1}(-1) \to \mathcal{O} \to 0$$

we deduce that the total Chern class of the relative cotangent bundle of S_1 over S_0 is

$$c(\Omega_{S_1/S_0}) = (1+\xi)^3((1+\xi)^2-(a_1-a_2)^2)((1+\xi)^2-(a_2-a_3)^2)((1+\xi)^2-(a_3-a_1)^2)).$$

Similarly, we can describe the cohomology rings and the total Chern classes of the relative tangent bundles of $\mathbb{P}E|_{p_1}^\vee$ and $\mathbb{P}E|_{p_2}$ over S_0. Let

$$u = c_1(\mathcal{O}_{\mathbb{P}E|_{p_1}^\vee}(-1)) + \frac{1}{3}c_1(E|_{p_1})$$

$$v = c_1(\mathcal{O}_{\mathbb{P}E|_{p_2}}(-1)) - \frac{1}{3}c_1(E|_{p_1}).$$

We intentionally shifted the generators to make our computation simpler. Then, we have

$$H^*(\mathbb{P}E|_{p_1}^\vee) = H^*(S_0)[u]/\langle u^3 + \alpha u + \beta\rangle$$
$$H^*(\mathbb{P}E|_{p_2}) = H^*(S_0)[v]/\langle v^3 + \alpha v - \beta\rangle$$

where

$$\alpha = c_2(E|_{p_1}) - \frac{1}{3}c_1^2(E|_{p_1})$$
$$\beta = c_3(E|_{p_1}) - \frac{1}{3}c_1(E|_{p_1})c_2(E|_{p_1}) + \frac{2}{27}c_1^3(E|_{p_1}).$$

Also we have

$$c(T_{\mathbb{P}E|_{p_1}^\vee/S_0}) = (1-u)^3 + \alpha(1-u) - \beta = 1 - 3u + 3u^2 + \alpha$$
$$c(T_{\mathbb{P}E|_{p_2}/S_0}) = (1-v)^3 + \alpha(1-v) + \beta = 1 - 3v + 3v^2 + \alpha.$$

Using α and β, we can rewrite

$$c(\Omega_{S_1/S_0}) = (1+\xi)^3((1+\xi)^6 + 6\alpha(1+\xi)^4 + 9\alpha^2(1+\xi)^2 + 4\alpha^3 + 27\beta^2)$$

as one can check by direct computation.

From [**3**], we get the exact sequences

$$0 \to \mathcal{O}_{\tilde{B}}(-1) \to g^* N_{B/S_1} \to F \to 0$$
$$0 \to T_{S_2} \to \pi_2^* T_{S_1} \to \jmath_* F \to 0$$

where $g : \tilde{B} \to B$ is the restriction of π_2 to the exceptional divisor $\tilde{B}$ and $\jmath$ is the inclusion of $\tilde{B}$. Therefore, we have

$$c(T_{S_2}) = \pi_2^* c(T_{S_1})/c(\jmath_* F) \tag{6.2}$$

$$c(F) = g^* c(N_{B/S_1})/c(\mathcal{O}_{\tilde{B}}(-1)). \tag{6.3}$$

Since $\mathcal{O}_{S_1}(-1)$ restricts to $\mathcal{O}_{\mathbb{P}E|_{p_1}^*}(-1) \boxtimes \mathcal{O}_{\mathbb{P}E|_{p_2}}(-1)$, ξ restricts to

$$c_1(\mathcal{O}_{\mathbb{P}E|_{p_1}^*}(-1)) + c_1(\mathcal{O}_{\mathbb{P}E|_{p_2}}(-1)) = u + v.$$

The restriction of the relative tangent bundle T_{S_1/S_0} to B has total Chern class

$$(1-u-v)^3((1-u-v)^6 + 6\alpha(1-u-v)^4 + 9\alpha^2(1-u-v)^2 + 4\alpha^3 + 27\beta^2)$$

while the total Chern class of the relative tangent bundle T_{B/S_0} is

$$(1 - 3u + 3u^2 + \alpha)(1 - 3v + 3v^2 + \alpha).$$

Hence the total Chern class of the normal bundle N_{B/S_1} is

$$\frac{(1-u-v)^3((1-u-v)^6 + 6\alpha(1-u-v)^4 + 9\alpha^2(1-u-v)^2 + 4\alpha^3 + 27\beta^2)}{(1 - 3u + 3u^2 + \alpha)(1 - 3v + 3v^2 + \alpha)}$$

which is by direct computation equal to

$$1 - 6\xi + (15\xi^2 + 4\alpha - 3uv) - (15\xi^3 + 9\alpha\xi) + (6\xi^4 + 6\alpha\xi^2)$$

with $\xi|_B = u + v$ understood by abuse of notations.

Let $\eta = c_1(\mathcal{O}_{S_2}(\tilde{B})) \in H^2(S_2)$. Then $\eta|_{\tilde{B}} = c_1(\mathcal{O}_{\tilde{B}}(-1))$ since $\mathcal{O}_{S_2}(\tilde{B})|_{\tilde{B}} \cong \mathcal{O}_{\tilde{B}}(-1)$. So we have

$$H^*(\tilde{B}) \cong H^*(B)[\eta]/\Big\langle \eta^4 + 6\xi\eta^3 + (15\xi^2 + 4\alpha - 3uv)\eta^2 + (15\xi^3 + 9\alpha\xi)\eta + (6\xi^4 + 6\alpha\xi^2)\Big\rangle.$$

In fact, it is easy to see that the above relation lifts to a relation

$$\eta^4 + 6\xi\eta^3 + (15\xi^2 + 4\alpha)\eta^2 - 3\jmath_*(uv)\eta + (15\xi^3 + 9\alpha\xi)\eta + (6\xi^4 + 6\alpha\xi^2) = 0 \tag{6.4}$$

in $H^*(S_2)$.

From (6.3), we have

$$\begin{aligned} c(F) &= c(N_{B/S_1})/(1+\eta) \\ &= 1 - (6\xi + \eta) + (15\xi^2 + 4\alpha - 3uv + 6\xi\eta + \eta^2) \\ &\quad - (15\xi^3 + 9\alpha\xi + 15\xi^2\eta + 4\alpha\eta + 6\xi\eta^2 + \eta^3 - 3uv\eta). \end{aligned}$$

By local computation we have

$$c(\mathcal{O}_{S_2}(\Delta_1)) = 1 + \eta, \quad c(\mathcal{O}_{S_2}(\Delta_2)) = 1 - 3\xi - 2\eta.$$

Hence we have

(6.5)

$$\begin{aligned} &c(\Omega_{S_2/S_0}(\log\Delta)) \\ &= \frac{(1+\xi)^3((1+\xi)^6 + 6\alpha(1+\xi)^4 + 9\alpha^2(1+\xi)^2 + 4\alpha^3 + 27\beta^2)}{(1-\eta)(1+3\xi+2\eta)} c(\Omega_{S_2/S_1}). \end{aligned}$$

For $c(\Omega_{S_2/S_1})$, we compute

$$c(T_{S_2/S_1}) = c(T_{S_2})/\pi_2^* c(T_{S_1}) = \frac{1}{c(\jmath_*F)} \tag{6.6}$$

and change the signs of the terms of degree $\equiv 2 \pmod 4$.

It is a consequence of the Grothendieck-Riemann-Roch theorem that

$$c(\jmath_*F) = 1 - \jmath_*\left(\frac{1}{\eta}\left(1 - \prod \frac{1+b_i}{1+b_i-\eta}\right)\right) \tag{6.7}$$

where b_i are the Chern roots of F, i.e.,

$$\begin{aligned} &\prod(1+b_i) = 1 - (6\xi+\eta) + (15\xi^2 + 4\alpha - 3uv + 6\xi\eta + \eta^2) \\ &-(15\xi^3 + 9\alpha\xi + 15\xi^2\eta + 4\alpha\eta + 6\xi\eta^2 + \eta^3 - 3uv\eta). \end{aligned}$$

By expanding, we see that

$$\begin{aligned} &\prod(1+b_i-\eta) = 1 - (6\xi + 4\eta) + (15\xi^2 + 4\alpha - 3uv + 18\xi\eta + 6\eta^2) \\ &-(15\xi^3 + 9\alpha\xi + 30\xi^2\eta + 8\alpha\eta + 18\xi\eta^2 + 4\eta^3 - 6uv\eta). \end{aligned}$$

Hence, we have

$$\frac{1}{\eta}\left(1-\prod\frac{1+b_i}{1+b_i-\eta}\right)=\frac{F}{1-D}. \tag{6.8}$$

Here

$$\begin{aligned}A&=(6\xi+4\eta)-(15\xi^2+4\alpha+18\xi\eta+6\eta^2)\\&\quad+(15\xi^3+9\alpha\xi+30\xi^2\eta+8\alpha\eta+18\xi\eta^2+4\eta^3-6uv\eta),\\B&=-3+(12\xi+5\eta)-(15\xi^2+12\xi\eta+3\eta^2+4\alpha),\end{aligned}$$

$D=A+3uv$ and $F=B+3uv$. Then by expanding[11] (6.8) and collecting all terms of degrees up to 14, we obtain

$$\begin{aligned}&27u^3v^3+9u^2v^2+3uv+18BA^5uv+12BA^3uv+3Auv\\&+3A^2uv+18Au^2v^2+3A^3uv+27A^2u^2v^2\\&+81Au^3v^3+3A^4uv+36A^3u^2v^2\\&+162A^2u^3v^3+BA^3+BA^4+B+3A^5uv+45A^4u^2v^2\\&+270A^3u^3v^3+54A^5u^2v^2+405A^4u^3v^3+3uvA^7+567u^3v^3A^5\\&+63u^2v^2A^6+27Bu^3v^3+9Bu^2v^2+3Buv+3A^6uv+BA^2\\&+BA+BA^5+BA^6+BA^7+6BAuv+9BA^2uv\\&+27BAu^2v^2+54BA^2u^2v^2+108BAu^3v^3+15BA^4uv+90BA^3u^2v^2\\&+270BA^2u^3v^3+135BA^4u^2v^2+540BA^3u^3v^3+189BA^5u^2v^2\\&+945BA^4u^3v^3+21BA^6uv.\end{aligned}$$

By the projection formula and $\jmath_*1=\eta$,

$$\jmath_*\left(\frac{1}{\eta}\left(1-\prod\frac{1+b_i}{1+b_i-\eta}\right)\right)=\jmath_*\left(F/(1-D)\right)$$

is, up to degree 16, equal to

$$\begin{aligned}&27\jmath_*(u^3v^3)+9\jmath_*(u^2v^2)+3\jmath_*(uv)+18BA^5\jmath_*(uv)\\&+12BA^3\jmath_*(uv)+3A\jmath_*(uv)\\&+3A^2\jmath_*(uv)+18A\jmath_*(u^2v^2)+3A^3\jmath_*(uv)+27A^2\jmath_*(u^2v^2)\\&+81A\jmath_*(u^3v^3)+3A^4\jmath_*(uv)+36A^3\jmath_*(u^2v^2)\\&+162A^2\jmath_*(u^3v^3)+BA^3\eta+BA^4\eta+B\eta+3A^5\jmath_*(uv)+45A^4\jmath_*(u^2v^2)\\&+270A^3\jmath_*(u^3v^3)+54A^5\jmath_*(u^2v^2)+405A^4\jmath_*(u^3v^3)\\&+3\jmath_*(uv)A^7+567\jmath_*(u^3v^3)A^5\\&+63\jmath_*(u^2v^2)A^6+27B\jmath_*(u^3v^3)+9B\jmath_*(u^2v^2)\\&+3B\jmath_*(uv)+3A^6\jmath_*(uv)+BA^2\eta\\&+BA\eta+BA^5\eta+BA^6\eta+BA^7\eta+6BA\jmath_*(uv)+9BA^2\jmath_*(uv)\\&+27BA\jmath_*(u^2v^2)+54BA^2\jmath_*(u^2v^2)+108BA\jmath_*(u^3v^3)\\&+15BA^4\jmath_*(uv)+90BA^3\jmath_*(u^2v^2)\\&+270BA^2\jmath_*(u^3v^3)+135BA^4\jmath_*(u^2v^2)\\&+540BA^3\jmath_*(u^3v^3)+189BA^5\jmath_*(u^2v^2)\\&+945BA^4\jmath_*(u^3v^3)+21BA^6\jmath_*(uv).\end{aligned}$$

[11] We used Maple 7 for the computations in this section.

Substitute the above expression into (6.7) and expand (6.6) up to degree 16. Change the signs of the terms of degree $\equiv 2 \bmod 4$ and plug it into (6.5).

Now we can compute the Chern classes by direct computation from (6.5). The 7th Chern class is, up to sign, equal to

$$\begin{aligned}
&-378\xi^5\eta^2 + 36\alpha\xi\jmath_*(uv)\eta - 72\alpha\xi^2\eta^3 - 138\alpha^2\xi\eta^2\\
&-492\alpha^2\xi^2\eta - 516\alpha\xi^3\eta^2\\
&-1056\alpha\xi^4\eta - 540\xi^2\jmath_*(uv)\eta^2 + 558\alpha\xi^2\jmath_*(uv)\\
&+108\alpha\jmath_*(u^2v^2) - 810\xi^2\jmath_*(u^2v^2)\\
&-630\xi^3\jmath(uv)\eta - 564\xi^6\eta - 252\xi^7 - 504\alpha\xi^5 - 252\alpha^2\xi^3 - 72\xi^4\eta^3\\
&-54\alpha^2\jmath_*(uv) + 72\xi^4\jmath_*(uv) - 126\xi\eta^3\jmath_*(uv) - 54\eta\beta^2 - 54\jmath_*(u^3v^3)
\end{aligned}$$

and the 8th Chern class is

$$\begin{aligned}
&1332\jmath_*(uv)\eta^2\alpha\xi + 2178\jmath_*(uv)\xi^2\alpha\eta - 2835\jmath_*(u^2v^2)\xi^3\\
&-1143\jmath_*(u^2v^2)\eta^3 + 2214\jmath_*(uv)\xi^5\\
&-6939\jmath_*(u^2v^2)\xi^2\eta - 4968 * \jmath_*(u^2v^2)\xi\eta^2\\
&+1485\jmath_*(u^2v^2)\alpha\xi + 774\jmath_*(u^2v^2)\alpha\eta\\
&-72\xi^5\eta^3 + 132\xi^6\eta^2 - 336\alpha^2\xi\eta^3 - 588\alpha^2\xi^2\eta^2 - 408\alpha\xi^3\eta^3\\
&-456\alpha\xi^4\eta^2 + 1152\jmath_*(u^2v^2)\eta^3\\
&+6372\xi^2\jmath_*(u^2v^2)\eta - 828\alpha\jmath_*(u^2v^2)\eta + 3942\eta^2\jmath_*(u^2v^2)\xi\\
&+846\jmath_*(uv)\xi^4\eta + 30\jmath_*(uv)\alpha^2\eta\\
&+1008\jmath_*(uv)\eta^2\xi^3 + 72\jmath_*(uv)\xi^2\eta^3 - 6\jmath_*(uv)\alpha\eta^3 + 3150\jmath_*(uv)\alpha\xi^3\\
&-828\jmath_*(uv)\alpha^2\xi - 567\eta\jmath_*(u^3v^3) - 648\alpha\xi^5\eta - 498\alpha^2\xi^3\eta\\
&+132\alpha^3\xi\eta - 18\xi^7\eta + 153\xi^8\\
&-810\xi\jmath_*(u^3v^3) + 54\beta^2\eta^2 + 165\alpha^2\xi^4 + 6\xi^2\alpha^3 + 81\xi^2\beta^2\\
&+312\alpha\xi^6 + 594\jmath_*(u^2v^2)\jmath_*(uv).
\end{aligned}$$

Notice that the 7th Chern class is the image by $\jmath_*$ of

$$\begin{aligned}
&-378\xi^5\eta + 36\alpha uv\xi\eta - 72\alpha\xi^2\eta^2 - 138\alpha^2\xi\eta - 492\alpha^2\xi^2\\
&-516\alpha\xi^3\eta - 1056\alpha\xi^4\\
&-540\xi^2\eta^2uv + 558\alpha uv\xi^2 + 108\alpha u^2v^2 - 810\xi^2u^2v^2\\
&-630\xi^3uv\eta - 564\xi^6\\
&-72\xi^4\eta^2 - 54\alpha^2uv + 72\xi^4uv - 126\xi\eta^3uv - 54\beta^2 - 54u^3v^3\\
&+42\alpha\xi(\eta^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha)\eta - 3uv\eta + 15\xi^3 + 9\alpha\xi)\\
&+42\xi^3(\eta^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha)\eta - 3uv\eta + 15\xi^3 + 9\alpha\xi)
\end{aligned} \tag{6.9}$$

with $\xi^4 + \alpha\xi^2$ replaced by

$$\jmath_*[-\left(\eta^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha)\eta - 3uv\eta + 15\xi^3 + 9\alpha\xi\right)/6]$$

from the relation (6.4).

This is a class in $H^*(\widetilde{B})$ which is a polynomial algebra over $H^*(S_0)$ generated by u, v, and η with the relations $\xi = u + v$,

$$u^3 + \alpha u + \beta = 0$$

$$v^3 + \alpha v - \beta = 0$$

$$\eta^4 + 6\xi\eta^3 + (15\xi^2 + 4\alpha - 3uv)\eta^2 + (15\xi^3 + 9\alpha\xi)\eta + (6\xi^4 + 6\alpha\xi^2) = 0.$$

Using Gröbner package, one can check that the class (6.9) is zero. Therefore we have proved that the 7th Chern class vanishes.

We apply the same strategy for the 8th Chern class. The only term we cannot express as the image of $\jmath_*$ in the above fashion using (6.4) is the term $81\xi^2\beta^2$. Let

$$\mu = (\xi^4 + \alpha\xi^2)(\xi^2 + \alpha)(\xi^2 + 4\alpha) + 27\xi^2\beta^2.$$

This is exactly the relation (6.1) divided by ξ and thus we have

$$\xi\mu = 0.$$

Then by the relation (6.4) as above $c_8(\Omega_{S_2/S_0}(\log\Delta)) - 3\mu$ is the image by $\jmath_*$ of

$$\begin{aligned}
&-72\xi^5\eta^2 - 336\alpha^2\xi\eta^2 - 588\alpha^2\xi^2\eta - 408\alpha\xi^3\eta^2\\
&-456\alpha\xi^4\eta + 132\xi^6\eta - 18\xi^7\\
&-648\alpha\xi^5 + 1008\xi^3uv\eta^2 - 498\alpha^2\xi^3 + 54\eta\beta^2 + 132\xi\alpha^3\\
&-\tfrac{51}{2}\xi^4(y^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha - 3uv)\eta + 15\xi^3 + 9\alpha\xi)\\
&-\alpha^2(y^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha - 3uv)\eta + 15\xi^3 + 9\alpha\xi)\\
&+2214\xi^5uv - 2835\xi^3u^2v^2\\
&-\tfrac{53}{2}\alpha\xi^2(y^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha - 3uv)\eta + 15\xi^3 + 9\alpha\xi)\\
&-810u^3v^3\xi + 27u^3v^3\eta\\
&+3150\alpha uv\xi^3 + 1332\alpha uv\xi\eta^2 + 2178\alpha uv\xi^2\eta - 54\alpha u^2v^2\eta\\
&+9\eta^3u^2v^2 + 30\alpha^2uv\eta + 1485\alpha u^2v^2\xi - 828\alpha^2uv\xi + 846\xi^4uv\eta\\
&-1026\eta^2u^2v^2\xi - 567\xi^2u^2v^2\eta + 72\xi^2\eta^3uv - 6\eta^3\alpha uv\\
&+\tfrac{1}{2}(\xi^2 + \alpha)(\xi^2 + 4\alpha)(\eta^3 + 6\xi\eta^2 + (15\xi^2 + 4\alpha - 3uv)\eta + 15\xi^3 + 9\alpha\xi).
\end{aligned}$$

If we simplify this expression using the Gröbner package for the ring $H^*(\widetilde{B})$, we get

(6.10)
$$\begin{aligned}
&3\eta^3u^2v^2 + \alpha u^2\eta^3 + 2\eta^3\alpha uv - 3\eta^3u\beta + \alpha v^2\eta^3 + 3\eta^3v\beta + a^4\eta^3\\
&+6\alpha u^2v^2\eta + 9\eta u^2v\beta + 4\alpha^2\eta u^2 - 9\eta uv^2\beta + 2\alpha^2uv\eta - 6\eta\alpha u\beta\\
&+4\alpha^2\eta v^2 + 6\eta\alpha v\beta + 4\alpha^3\eta + 9\beta^2\eta.
\end{aligned}$$

If we multiply η to this expression (6.10), we get zero! Hence $c_8(\Omega_{S_2/S_0}(\log\Delta)) - 3\mu$ lies in $H^*(S_1)$ because its restriction to $\widetilde{B}$ is exactly the above expression multiplied by η and is equal to zero.

If we multiply $\xi = u + v$ to (6.10) and simplify using the Gröbner package, we get zero! By the projection formula, this implies that $c_8(\Omega_{S_2/S_0}(\log\Delta)) - 3\mu$ lies in the kernel of multiplication by ξ

$$\xi : H^*(S_1) \to H^{*+2}(S_1)$$

which is exactly $H^*(S_0)\mu$. Therefore, we deduce that

$$c_8(\Omega_{S_2/S_0}(\log\Delta)) - 3\mu = c\mu$$

for some rational number c. To compute c, we restrict the image by $\jmath_*$ of (6.10) to a fiber of $\pi_1 \circ \pi_2 : S_2 \to S_0$ so that $\alpha = 0$ and $\beta = 0$.

Using the explicit relations it is now an elementary exercise to check that $c = -3$. Hence, we conclude that $c_8(\Omega_{S_2/S_0}(\log\Delta)) = 0$.

References

[1] U.N. Bhosle, *Generalized parabolic bundles and applications*, II, Proc. Indian Acad. Sci. Math. Sci. **106**(**4**) (1996) 403–420, MR 1425615, Zbl 0879.14012.

[2] I.V. Dolgachev & Y. Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes tudes Sci. Publ. Math. **87** (1998) 5–56, MR 1659282, Zbl 1001.14018.

[3] W. Fulton, *Intersection theory*, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998, MR 1644323, Zbl 0885.14002.

[4] D. Gieseker, *A degeneration of the moduli space of stable bundles*, J. Differential Geom. **19**(**1**) (1984) 173–206, MR 0739786, Zbl 0557.14008.

[5] D. Gieseker & J. Li, *Irreducibility of moduli of rank two vector bundles*, J. Differential. Geom. **40** (1994) 23-104, MR 1285529, Zbl 0827.14008.

[6] D. Gieseker & I. Morrison, *Hilbert stability of rank-two bundles on curves*, J. Differential Geom. **19**(**1**) (1984) 1–29, MR 0739780, Zbl 0573.14005.

[7] I. Kausz, *A modular compactification of the general linear group*, Doc. Math. **5** (2000) 553–594 (electronic), MR 1796449, Zbl 0971.14029.

[8] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57**(**3**) (2001) 509–578, MR 1882667, Zbl 1015.14026.

[9] C. Okonek, M. Schneider, & H. Spindler, *Vector and Heinz Vector bundles on complex projective spaces*, Progress in Mathematics, **3**, Birkhäuser, Boston, Mass., 1980, MR 0778380, Zbl 0598.32022.

[10] D. Mumford, J. Fogarty, & J. Kirwan, *Geometric invariant theory*, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), **34**, Springer-Verlag, Berlin, 1994, MR 1304906, Zbl 0797.14004.

[11] D.S. Nagaraj & C.S. Seshadri, *Degenerations of the moduli spaces of vector bundles on curves.* II. *Generalized Gieseker moduli spaces*, Proc. Indian Acad. Sci. Math. Sci. **109**(**2**) (1999) 165–201, MR 1687729, Zbl 0957.14021.

[12] C.S. Seshadri, *Degenerations of the moduli spaces of vector bundles on curves*, School on Algebraic Geometry (Trieste, 1999), 205-265, ICTP Lect. Notes, **1**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000, MR 1795864, Zbl 0986.14018.

[13] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety*, II, Inst. Hautes Études Sci. Publ. Math. **80** (1994) 5–79, MR 1320603, Zbl 0891.14006.

[14] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117**(**2**) (1994) 317–353, MR 1273268, Zbl 0882.14003.

[15] ______, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9**(**3**) (1996) 691–723, MR 1333296, Zbl 0874.14042.

[16] D. Zagier, *On the cohomology of moduli spaces of rank two vector bundles over curves*, The moduli space of curves (Texel Island, 1994), 533–563, Progr. Math., **129**, Birkhauser, 1995, MR 1363070, Zbl 0845.14016.

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