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GINZBURG-WEINSTEIN VIA GELFAND-ZEITLIN

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Abstract

Let $\mathrm{U}(n)$ be the unitary group, and $\mathfrak{u}(n)^*$ the dual of its Lie algebra, equipped with the Kirillov Poisson structure. In their 1983 paper, Guillemin-Sternberg introduced a densely defined Hamiltonian action of a torus of dimension $(n-1)n/2$ on $\mathfrak{u}(n)^*$, with moment map given by the Gelfand-Zeitlin coordinates. A few years later, Flaschka-Ratiu described a similar, 'multiplicative' Gelfand-Zeitlin system for the Poisson Lie group $\mathrm{U}(n)^*$.

By the Ginzburg-Weinstein theorem, $\mathrm{U}(n)^*$ is isomorphic to $\mathfrak{u}(n)^*$ as a Poisson manifold. Flaschka-Ratiu conjectured that one can choose the Ginzburg-Weinstein diffeomorphism in such a way that it intertwines the linear and nonlinear Gelfand-Zeitlin systems. Our main result gives a proof of this conjecture, and produces a *canonical* Ginzburg-Weinstein diffeomorphism.

1. Introduction and statement of results

A theorem of Ginzburg-Weinstein [**14**] states that for any compact Lie group K with its standard Poisson structure, the dual Poisson Lie group K^* is Poisson diffeomorphic to the dual of the Lie algebra $\mathfrak{k}^*$, with the Kirillov Poisson structure. The result of [**14**] does not, however, give a constructive way for obtaining such a diffeomorphism. For the case of the unitary group $K = \mathrm{U}(n)$, Flaschka-Ratiu [**13**] (see also their preprint [**12**]) suggested the existence of a *distinguished* Ginzburg-Weinstein diffeomorphism, intertwining Gelfand-Zeitlin systems on $\mathfrak{u}(n)^*$ and $\mathrm{U}(n)^*$, respectively. In this paper, we will give a proof of the Flaschka-Ratiu conjecture. The main result has the following 'linear algebra' implications, which may be stated with no reference to Poisson geometry.

Let $\mathrm{Sym}(n)$ denote the space of real symmetric $n \times n$ matrices. For $k \leq n$ let $A^{(k)} \in \mathrm{Sym}(k)$ denote the kth principal submatrix (upper left $k \times k$ corner) of $A \in \mathrm{Sym}(n)$, and $\lambda_i^{(k)}(A)$ its ordered set of eigenvalues, $\lambda_1^{(k)}(A) \leq \cdots \leq \lambda_k^{(k)}(A)$. The map

$$\lambda\colon \mathrm{Sym}(n) \to \mathbb{R}^{\frac{n(n+1)}{2}}, \tag{1}$$

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taking A to the collection of numbers $\lambda_i^{(k)}(A)$ for $1 \le i \le k \le n$, is a continuous map called the *Gelfand-Zeitlin map.* Its image is the *Gelfand-Zeitlin cone* $\mathfrak{C}(n)$, cut out by the 'interlacing' inequalities,

$$\lambda_i^{(k+1)} \le \lambda_i^{(k)} \le \lambda_{i+1}^{(k+1)}, \quad 1 \le i \le k \le n-1. \tag{2}$$

Now let $\mathrm{Sym}^+(n) \subset \mathrm{Sym}(n)$ denote the subset of positive definite symmetric matrices, and define a logarithmic Gelfand-Zeitlin map

$$\mu\colon \mathrm{Sym}^+(n) \to \mathbb{R}^{\frac{n(n+1)}{2}}, \tag{3}$$

taking A to the collection of numbers $\mu_i^{(k)}(A) = \log(\lambda_i^{(k)}(A))$. Then μ is a continuous map from $\mathrm{Sym}^+(n)$ onto $\mathfrak{C}(n)$.

Theorem 1.1. *There is a unique continuous map* $\psi\colon \mathrm{Sym}(n) \to \mathrm{SO}(n)$*, with* $\psi(0) = I$*, such that the map*

$$\gamma = \exp \circ \mathrm{Ad}_\psi\colon \mathrm{Sym}(n) \to \mathrm{Sym}^+(n), \quad \mathrm{Ad}_\psi(A) \equiv \mathrm{Ad}_{\psi(A)} A \tag{4}$$

intertwines the Gelfand-Zeitlin maps λ *and* μ*. In fact,* ψ *is smooth and* γ *is a diffeomorphism.*

Remark. For a general real semi-simple Lie group G with Cartan decomposition $G = KP$, Duistermaat [**10**] proved the existence of a smooth map $\psi\colon \mathfrak{p} \to K$ such that the map $\gamma = \exp \circ \mathrm{Ad}_\psi\colon \mathfrak{p} \to P$ intertwines the 'diagonal projection' with the 'Iwasawa projection'. Theorem 1.1 gives canonical maps with this property for the case $G = \mathrm{SL}(n, \mathbb{R})$.

Example. The case $n = 2$ can be worked out by hand (see also [**13**, Example 3.27]). Even in this case, smoothness of the map γ is not entirely obvious. Since $\gamma(A + tI) = e^t\gamma(A)$, it is enough to consider trace-free matrices,

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

The matrix A has Gelfand-Zeitlin variables

$$\lambda_1^{(2)}(A) = -r, \quad \lambda_2^{(2)}(A) = r, \quad \lambda_1^{(1)}(A) = a$$

with $r := \sqrt{a^2 + b^2}$. Hence, the matrix $\gamma(A)$ should have eigenvalues e^{-r}, e^r and upper left entry e^a. This gives

$$\gamma(A) = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} \end{pmatrix}$$

with

$$\tilde{a} = e^a, \quad \tilde{b} = \pm\sqrt{2e^a\cosh(r) - e^{2a} - 1}, \quad \tilde{c} = 2\cosh(r) - e^a.$$

To obtain a continuous map, one has to take the sign of $\tilde{b}$ equal to the sign of b. The matrix $\psi(A) \in \mathrm{SO}(2)$ is a rotation matrix by some angle

$\theta(A)$. A calculation gives

$$\cos(2\theta(A)) = \frac{a}{r} \pm \sqrt{1 - \left(\frac{e^a - \cosh(r)}{\sinh(r)}\right)^2}.$$

One can consider similar questions for the space $\mathrm{Herm}(n)$ of complex Hermitian $n \times n$-matrices, and its subset $\mathrm{Herm}^+(n)$ of positive definite matrices. Define surjective maps

$$\lambda\colon \mathrm{Herm}(n) \to \mathfrak{C}(n), \quad \mu\colon \mathrm{Herm}^+(n) \to \mathfrak{C}(n)$$

in terms of eigenvalues of principal submatrices, as before. Let

$$\mathrm{Herm}_0(n) = \lambda^{-1}(\mathfrak{C}_0(n))$$

denote the subset where all of the eigenvalue inequalities (2) are strict. The k-torus $T(k) \subset U(k)$ of diagonal matrices acts on $\mathrm{Herm}_0(n)$ as follows

$$t \bullet A = \mathrm{Ad}_{U^{-1}tU} A, \qquad t \in T(k),\ A \in \mathrm{Herm}_0(n). \tag{5}$$

Here $U \in \mathrm{U}(k) \subset U(n)$ is a unitary matrix such that $\mathrm{Ad}_U A^{(k)}$ is diagonal, with entries $\lambda_1^{(k)}, \ldots, \lambda_k^{(k)}$. The action is well-defined since $U^{-1}tU$ does not depend on the choice of U, and preserves the Gelfand-Zeitlin map (1). The actions of the various $T(k)$'s commute, hence they define an action of the *Gelfand-Zeitlin torus*

$$T(n-1) \times \cdots \times T(1) \cong \mathrm{U}(1)^{(n-1)n/2}.$$

Here the torus $T(n)$ is excluded, since the action (5) is trivial for $k = n$.

Let $\mathrm{Herm}_0^+(n)$, $\mathrm{Sym}_0(n)$ and $\mathrm{Sym}_0^+(n)$ denote the intersections of $\mathrm{Herm}_0(n)$ with $\mathrm{Herm}^+(n)$, $\mathrm{Sym}(n)$ and $\mathrm{Sym}^+(n)$. Thus $\mathrm{Herm}_0^+(n) = \mu^{-1}(\mathfrak{C}_0(n))$.

Theorem 1.2. *There is a unique continuous map*

$$\gamma\colon \mathrm{Herm}(n) \to \mathrm{Herm}^+(n)$$

with the following three properties:

a) γ *intertwines the Gelfand-Zeitlin maps:* $\mu \circ \gamma = \lambda$.
b) γ *intertwines the Gelfand-Zeitlin torus actions on* $\mathrm{Herm}_0(n)$ *and* $\mathrm{Herm}_0^+(n)$.
c) *For any connected component* S *of* $\mathrm{Sym}_0(n) \subset \mathrm{Herm}(n)$, $\gamma(S) \subset S$.

In fact, γ *is a diffeomorphism, and has the following additional properties:*

d) γ *is equivariant for the conjugation action of* $T(n) \subset \mathrm{U}(n)$,
e) $\gamma(A + uI) = e^u \gamma(A)$,
f) $\gamma(\overline{A}) = \overline{\gamma(A)}$ (*where the bar denotes complex conjugation*).

g) *For $k \le n$, the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Herm}(n) & \longrightarrow & \mathrm{Herm}(k) & \longrightarrow & \mathrm{Herm}(n) \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\gamma} \\ \mathrm{Herm}^+(n) & \longrightarrow & \mathrm{Herm}^+(k) & \longrightarrow & \mathrm{Herm}^+(n). \end{array}$$

Here the left horizontal maps take a matrix to its kth principal submatrix, while the right horizontal maps are the obvious inclusions as the upper left corner, extended by 0's, respectively 1's, along the diagonal.

Similar to the statement for real symmetric matrices, Theorem 1.1, the map γ can be written in the form $\gamma = \exp \circ \mathrm{Ad}_\psi$ for a suitable map $\psi\colon \mathrm{Herm}(n) \to \mathrm{SU}(n)$. To fix the choice of ψ, we have to impose an equivariance condition under the Gelfand-Zeitlin torus action. Given $A \in \mathrm{Herm}_0(n)$, let $U_k \in U(k)$ be matrices diagonalizing $A^{(k)}$, and $V_k \in U(k)$ matrices diagonalizing $\gamma(A)^{(k)} = \gamma(A^{(k)})$. Then the Gelfand-Zeitlin action of $t = (t_{n-1}, \ldots, t_1) \in T(n-1) \times \cdots \times T(1)$ is given by

$$t \bullet A = \mathrm{Ad}_{\chi(t,A)} A, \quad t \bullet \gamma(A) = \mathrm{Ad}_{\tilde{\chi}(t,A)} \gamma(A)$$

where

$$\begin{aligned} \chi(t,A) &= U_1^{-1} t_1 U_1 \cdots U_{n-1}^{-1} t_{n-1} U_{n-1}, \\ \tilde{\chi}(t,A) &= V_1^{-1} t_1 V_1 \cdots V_{n-1}^{-1} t_{n-1} V_{n-1}. \end{aligned}$$

Note that $\chi(t,A)$, $\tilde{\chi}(t,A)$ are independent of the choice of U_i, V_i.

Theorem 1.3. *The map $\psi\colon \mathrm{Sym}(n) \to \mathrm{SO}(n)$, $\psi(0) = I$ from Theorem 1.1 extends uniquely to a continuous (in fact, smooth) map $\psi\colon \mathrm{Herm}(n) \to \mathrm{SU}(n)$ with the equivariance property*

$$\psi(t \bullet A) = \tilde{\chi}(t,A)\psi(A)\chi(t,A)^{-1} \tag{6}$$

for all $A \in \mathrm{Herm}_0(n)$, $t \in T(n-1) \times \cdots \times T(1)$. The map γ from Theorem 1.2 is expressed in terms of ψ as $\gamma = \exp \circ \mathrm{Ad}_\psi$. Furthermore,

a) *ψ is equivariant for the conjugation action of $T(n) \subset \mathrm{U}(n)$.*
b) *$\overline{\psi(A)} = \psi(\overline{A})$.*
c) *For all $k \le n$, the following diagram commutes,*

$$\begin{array}{ccc} \mathrm{Herm}(k) & \longrightarrow & \mathrm{Herm}(n) \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi} \\ \mathrm{SU}(k) & \longrightarrow & \mathrm{SU}(n). \end{array}$$

Note that the equivariance property (6) of ψ implies the equivariance of γ:

$$\gamma(t \bullet A) = \exp(\mathrm{Ad}_{\psi(t\bullet A)\chi(t,A)} A) = \exp(\mathrm{Ad}_{\tilde{\chi}(t,A)\psi(A)} A) = t \bullet \gamma(A).$$

Let us now place these results into the context of Poisson geometry. Let $\mathfrak{u}(n)$ be the Lie algebra of $\mathrm{U}(n)$, consisting of skew-Hermitian matrices, and identify

$$\mathrm{Herm}(n) \cong \mathfrak{u}(n)^*,$$

using the pairing $\langle A, \xi\rangle = 2\,\mathrm{Im}(\mathrm{tr}\, A\xi)$. Then $\mathrm{Herm}(n)$ inherits a Poisson structure from the Kirillov-Poisson structure on $\mathfrak{u}(n)^*$. It was proved by Guillemin-Sternberg in [**15**] that the action of each $T(k)$ on $\mathrm{Herm}_0(n)$ is Hamiltonian, with moment map the corresponding Gelfand-Zeitlin variables, $(\lambda_1^{(k)}, \ldots, \lambda_k^{(k)})$. On the other hand, the unitary group $\mathrm{U}(n)$ carries a standard structure as a Poisson Lie group, with dual Poisson Lie group $\mathrm{U}(n)^*$ the group of complex upper triangular matrices with strictly positive diagonal entries. $\mathrm{U}(n)^*$ may be identified with $\mathrm{Herm}^+(n)$, by the map taking the upper triangular matrix $X \in U(n)^*$ to the positive Hermitian matrix $(X^*X)^{1/2} \in \mathrm{Herm}^+(n)$. Flaschka-Ratiu [**12**] proved that the $T(k)$ action on $\mathrm{Herm}_0^+(n)$ is Hamiltonian for the Poisson structure induced from $\mathrm{U}(n)^*$, with moment map the logarithmic Gelfand-Zeitlin variables $(\mu_1^{(k)}, \ldots, \mu_k^{(k)})$.

Theorem 1.4. *The map $\gamma\colon \mathfrak{u}(n)^* \to \mathrm{U}(n)^*$ described in Theorem* 1.2 *is a Poisson diffeomorphism.*

That is, for the group $K = \mathrm{U}(n)$ we have found a fairly explicit description of a Ginzburg-Weinstein diffeomorphism, in Gelfand-Zeitlin coordinates. By contrast, no coordinate expressions are known for the Ginzburg-Weinstein maps constructed in [**14**, **1**, **4**, **11**].

Remark. A recent paper of Kostant-Wallach [**17**] studies in detail the holomorphic (i.e., complexified) Gelfand-Zeitlin system, for the full space $\mathfrak{gl}(n, \mathbb{C})$. It may be interesting to consider a nonlinear version of the holomorphic system, and to generalize our results to that setting.

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2. Uniqueness of the map γ

In this section, we construct a map γ over the open dense subset $\mathrm{Herm}_0(n) = \lambda^{-1}(\mathfrak{C}_0(n))$, satisfying all the properties listed in Theorem 1.2. The existence of a smooth extension to $\mathrm{Herm}(n)$ will be proved in the subsequent sections. We denote by $T_{\mathbb{R}}(k) = T(k) \cap \mathrm{O}(k) \cong (\mathbb{Z}_2)^k$ the 'real part' of the torus. The action of the Gelfand-Zeitlin torus on $\mathrm{Herm}_0(n)$ restricts to an action of $T_{\mathbb{R}}(n-1) \times \cdots T_{\mathbb{R}}(1) \cong (\mathbb{Z}_2)^{(n-1)n/2}$ on $\mathrm{Sym}^+(n)$. The following facts concerning the Gelfand-Zeitlin map are standard; we include the proof since we are not aware of a convenient reference.

Proposition 2.1. *The restriction of the Gelfand–Zeitlin map to* $\mathrm{Herm}_0(n)$ *defines a principal bundle*

$$\lambda\colon \mathrm{Herm}_0(n) \to \mathfrak{C}_0(n) \tag{7}$$

with structure group the Gelfand-Zeitlin torus $T(n-1)\times\cdots\times T(1)$*. It further restricts to a principal bundle*

$$\lambda\colon \mathrm{Sym}_0(n) \to \mathfrak{C}_0(n) \tag{8}$$

with structure group $T_{\mathbb{R}}(n-1)\times\cdots\times T_{\mathbb{R}}(1)$*. Similarly for the restriction of the logarithmic Gelfand-Zeitlin map* $\mu\colon \mathrm{Herm}^+(n)\to\mathfrak{C}(n)$ *to* $\mathrm{Herm}_0^+(n)$ *and* $\mathrm{Sym}_0^+(n)$*.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Herm}_0(n) & \longrightarrow & \mathfrak{C}_0(n) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Herm}_0(n-1) & \longrightarrow & \mathfrak{C}_0(n-1) \end{array} \tag{9}$$

where the horizontal maps are the Gelfand-Zeitlin maps, the left vertical map is $A\mapsto A^{(n-1)}$, and the right vertical map $\mathfrak{C}_0(n)\to\mathfrak{C}_0(n-1)$ is the obvious projection, forgetting the variables $\lambda_i^{(n)}$. The Gelfand-Zeitlin map $\mathrm{Herm}_0(n)\to\mathfrak{C}_0(n)$ factorizes as

$$\mathrm{Herm}_0(n)\to \mathrm{Herm}_0(n-1)\times_{\mathfrak{C}_0(n-1)}\mathfrak{C}_0(n)\to\mathfrak{C}_0(n), \tag{10}$$

where the middle term is the fiber product. By induction, we may assume that the map $\mathrm{Herm}_0(n-1)\to\mathfrak{C}_0(n-1)$, and hence the last map in (10), is a principal bundle for the action of the Gelfand-Zeitlin torus $T(n-2)\times\cdots\times T(1)$. Hence, it suffices to show that the first map in (10) is a principal $T(n-1)$ bundle for the Gelfand-Zeitlin action. Thus, let $\lambda_i^{(k)}$, $1\le i\le k\le n$ be the components of a given point $\lambda\in\mathfrak{C}_0(n)$, and let $A^{(n-1)}\in\mathrm{Herm}_0(n-1)$ be a matrix with Gelfand-Zeitlin parameters $\lambda_i^{(k)}$ for $1\le i\le k\le n-1$. Let us try to find $b_1,\ldots,b_{n-1}\in\mathbb{C}$ and $c\in\mathbb{R}$ such that the matrix

$$A=\begin{pmatrix} A^{(n-1)} & b\\ b^* & c\end{pmatrix} \tag{11}$$

has eigenvalues $\lambda_i^{(n)}$. (Here b denotes the $(n-1)\times 1$-matrix with entries b_i.) Choose $U\in U(n-1)$ such that the matrix $\Lambda^{(n-1)}=UA^{(n-1)}U^{-1}$ is diagonal, with entries $\lambda_i^{(n-1)}$ down the diagonal. Then

$$UAU^{-1}=\begin{pmatrix} \Lambda^{(n-1)} & \tilde b\\ \tilde b^* & c\end{pmatrix}$$

where $\tilde b = U\,b$. (As before, we think of $\mathrm{U}(k)$ for $k\le n$ as a subgroup of $\mathrm{U}(n)$, using the inclusion as the upper left corner.) The characteristic polynomial $\det(A-uI)$ is therefore given by

$$\det(A-uI)=(c-u)\prod_j(\lambda_j^{(n-1)}-u)-\sum_i|\tilde{b}_i|^2\prod_{j\neq i}(\lambda_j^{(n-1)}-u).$$

Setting this equal to $\det(A-uI)=\prod_r(\lambda_r^{(n)}-u)$, and evaluating at $u=\lambda_i^{(n-1)}$ and at $u=\lambda_n^{(n)}$, one finds

$$|\tilde{b}_i|^2=-\frac{\prod_r(\lambda_r^{(n)}-\lambda_i^{(n-1)})}{\prod_{j\neq i}(\lambda_j^{(n-1)}-\lambda_i^{(n-1)})},\quad c=\lambda_n^{(n)}-\sum_i\frac{\prod_{r\neq n}(\lambda_r^{(n)}-\lambda_i^{(n-1)})}{\prod_{j\neq i}(\lambda_j^{(n-1)}-\lambda_i^{(n-1)})}.$$

The eigenvalue inequalities ensure that the right hand side of the expression for $|\tilde{b}_i|^2$ is >0. This shows that the first map in (10) is onto. Furthermore, since c is uniquely determined while $\tilde{b}_i$ are determined up to a phase, this map defines a principal $T(n-1)$ bundle. Since left matrix multiplication of $\tilde{b}$ by an element of $T(n-1)$ is exactly the Gelfand-Zeitlin action, the proof for $\mathrm{Herm}_0(n)$ is complete. The proof for $\mathrm{Sym}_0(n)$ is similar, considering only matrices with entries in $\mathbb{R}$. The parallel statements for the map μ are a direct consequence of the statements for λ. q.e.d.

Lemma 2.2. *There exists a unique continuous map* $\gamma\colon \mathrm{Herm}_0(n)\to\mathrm{Herm}_0^+(n)$*, satisfying* (a)–(c) *from Theorem* 1.2. *Furthermore, this map also has the properties* (d)–(g) *from Theorem* 1.2.

Proof. The choice of a connected component $S\subset\mathrm{Sym}_0(n)$ defines a cross-section, hence a trivialization, of the principal bundle $\lambda\colon \mathrm{Herm}_0(n)\to\mathfrak{C}_0(n)$. The intersection

$$S^+=S\cap\mathrm{Sym}_0^+(n)$$

is a connected component of $\mathrm{Sym}_0^+(n)$, which likewise trivializes the bundle $\mu\colon \mathrm{Herm}_0^+(n)\to\mathfrak{C}_0(n)$. Thus, we obtain a unique map γ satisfying (a)–(b), with $\gamma(S)=S_+$ for the given S. By equivariance, the property $\gamma(S)=S_+$ holds true for *all* components $S\subset\mathrm{Sym}_0(n)$, which gives (c). We claim that property (d) follows from (b). Indeed, the Gelfand-Zeitlin action of any element in $t_k\in Z(U(k))\subset T(k)\subset T(n)$ coincides with the conjugation action, since the functions $\chi,\tilde{\chi}$ in (6) are simply $\chi(t_k,A)=\tilde{\chi}(t_k,A)=t_k$. The collection of these subgroups $Z(U(k))\cong\mathrm{U}(1)$ of $T(n)$, together with the center $Z(\mathrm{U}(n))$ (which acts trivially) generate $T(n)$, proving the claim. Properties (e) and (f) follow from the uniqueness, since the maps

$$A\mapsto e^{-u}\gamma(A+uI),\qquad A\to\overline{\gamma(\overline{A})}$$

satisfy (a)–(c). Finally (g) holds by the commutativity of the diagram

$$\begin{array}{ccccc}
\mathrm{Herm}(n) & \longrightarrow & \mathrm{Herm}(k) & \longrightarrow & \mathrm{Herm}(n) \\
\Big\downarrow \lambda & & \Big\downarrow \lambda & & \Big\downarrow \lambda \\
\mathfrak{C}(n) & \longrightarrow & \mathfrak{C}(k) & \longrightarrow & \mathfrak{C}(n)
\end{array}$$

and of the similar diagram for map μ. q.e.d.

Lemma 2.3. *There is a continuous function $\psi\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$, with the property that the map $\gamma = \exp \circ \mathrm{Ad}_\psi\colon \mathrm{Sym}_0(n) \to \mathrm{Sym}_0^+(n)$ intertwines the Gelfand-Zeitlin maps. The map ψ is uniquely determined by the additional condition $\psi(uA) \to I$ for $u \to 0$.*

Proof. We have seen above that there is a unique continuous map $\gamma\colon \mathrm{Sym}_0(n) \to \mathrm{Sym}_0^+(n)$ which intertwines the Gelfand-Zeitlin maps and satisfies $\gamma(S) = S^+$ for any component $S \subset \mathrm{Sym}_0(n)$. Since $\gamma(A)$ and $\exp(A)$ have the same eigenvalues, and since $S \cong \mathfrak{C}_0$ is contractible, one can always choose a continuous map $\psi\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$ with $\gamma = \exp \circ \mathrm{Ad}_\psi$.

Conversely, suppose $\psi\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$ is a continuous map, such that $\gamma = \exp \circ \mathrm{Ad}_\psi$ intertwines the Gelfand-Zeitlin maps. Suppose $\psi(uA) \to I$ for $u \to 0$. We will show that (i) the map γ has the property $\gamma(S) = S^+$ for any connected component S, and (ii) the map ψ with these properties is unique. Proof of (i): It suffices to show that the restriction of γ to $\mathrm{Sym}_0(n)$ is homotopic to the identity map of $\mathrm{Sym}_0(n)$. Define

$$[0,1] \times \mathrm{Sym}_0(n) \to \mathrm{Sym}_0(n),$$
$$(u, A) \mapsto A_u = \begin{cases} A & \text{for } u = 0 \\ \frac{1}{u}(\gamma(uA) - I) + uI & \text{for } 0 < u \le 1. \end{cases}$$

This is a well-defined continuous map since

$$\lim_{u \to 0} \Big(\frac{1}{u}(\gamma(uA) - I) + uI \Big) = \lim_{u \to 0} \Big(\frac{1}{u}\big(\exp(\mathrm{Ad}_{\psi(uA)} A) - I \big) \Big) = A.$$

Furthermore $A_u \in \mathrm{Sym}_0(n)$, since $\mathrm{Sym}_0(n)$ is invariant under scalar multiplication by nonzero numbers, as well as under addition of a scalar multiple of the identity matrix. Clearly $A_1 = \gamma(A)$. Proof of (ii): Suppose $\psi'\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$ is another map with $\gamma(A) = \exp(\mathrm{Ad}_{\psi'(A)} A)$ and $\lim_{u\to 0} \psi'(uA) = I$. Then $\psi'(A) = \psi(A)\chi(A)$ where $\chi(A)$ centralizes A and $\lim_{u \to 0} \chi(uA) = I$. Since the centralizer subgroup $\mathrm{O}(n)_A$ of any $A \in \mathrm{Sym}_0(n)$ is discrete, and $\mathrm{O}(n)_A = \mathrm{O}(n)_{uA}$ for $u > 0$, we have $\chi(A) = \chi(uA) \xrightarrow[u \to 0]{} I$. Thus $\chi(A) = I$, proving uniqueness of $\psi\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$. q.e.d.

Note that we have not yet shown that it is actually possible to satisfy the normalization condition $\lim_{u\to 0} \psi(uA) = I$. This can be proved 'by hand', but will in any case be automatic for the map constructed below (cf. Section 5.3).

Lemma 2.4. *The map* $\psi\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$, $\lim_{u\to 0}\psi(uA) = I$ *described in Lemma* 2.3 *admits a unique extension* $\psi\colon \mathrm{Herm}_0(n) \to \mathrm{SU}(n)$ *with the equivariance property* (6). *The composition* $\gamma = \exp\circ\mathrm{Ad}_\psi\colon \mathrm{Herm}_0(n) \to \mathrm{Herm}_0^+(n)$ *coincides with the map described in Lemma* 2.2. *Furthermore, this map also has the properties* (a)–(c) *described in Theorem* 1.3.

Proof. By construction, the map $\gamma\colon \mathrm{Sym}_0(n) \to \mathrm{Sym}_0^+(n)$ has the equivariance property $\gamma(t\bullet A) = t\bullet\gamma(A)$ for all $t \in T_\mathbb{R}(n-1)\times\cdots\times T_\mathbb{R}(1)$. This implies the equivariance property (6) for the map $\psi\colon \mathrm{Sym}_0(n) \to \mathrm{SO}(n)$, using the uniqueness part of Lemma 2.3. Hence, ψ admits a unique $T(n-1)\times\cdots\times T(1)$-equivariant extension to a map $\mathrm{Herm}_0(n) \to \mathrm{SU}(n)$, and the property $\gamma = \exp\circ\mathrm{Ad}_\psi$ follows by equivariance. Let us now check the additional properties from Theorem 1.3.

(a) As mentioned above, the Gelfand-Zeitlin action of $Z(\mathrm{U}(k)) \subset T(k) \subset T(n)$ for $k < n$ coincides with the action by conjugation. Hence, (6) gives $\psi(\mathrm{Ad}_{t_k} A) = \mathrm{Ad}_{t_k}\psi(A)$ for $t_k \in Z(U(k))$. Since the collection of these subgroups, together with $Z(U(n))$, generates $T(n)$, it follows that ψ is $T(n)$-equivariant.

(b) $\overline{\psi(A)} = \psi(\overline{A})$ follows from the uniqueness properties, since both ψ and $A \to \overline{\psi(\overline{A})}$ are $T(n-1)\times\cdots\times T(1)$-equivariant extensions of the given map over $\mathrm{Sym}_0(n)$.

(c) Let $\psi^{(k)}\colon \mathrm{Herm}(k) \to \mathrm{SU}(k)$ denote the analogue of the map ψ, for given $k < n$. Since ψ is equivariant for the conjugation by $T(n)$, it is in particular equivariant for the subgroup $T(n-k)$ embedded as the *lower right* corner. Since $\mathrm{Herm}_0(k)$ (as the upper left corner) is fixed under this action, it follows that the restriction of ψ takes values in $\mathrm{S}(\mathrm{U}(k)\times T(n-k))$. Similarly, the restriction to $\mathrm{Sym}_0(k)$ takes values in $\mathrm{S}(\mathrm{O}(k)\times T_\mathbb{R}(n-k))$. Since $T_\mathbb{R}(n-k)$ is discrete, the property $\lim_{u\to 0}\psi(uA) = I$ implies that $\psi|\,\mathrm{Sym}_0(k)$ must take values in $\mathrm{SO}(k)$. From the uniqueness properties, it therefore follows that it coincides with $\psi^{(k)}|\,\mathrm{Sym}_0(k)$. The more general statement $\psi|\,\mathrm{Herm}_0(k) = \psi^{(k)}|\,\mathrm{Herm}_0(k)$ now follows by equivariance under the Gelfand-Zeitlin action of $T(k-1)\times\cdots\times T(1)$. q.e.d.

To complete the proof of Theorems 1.1, 1.2, and 1.3, it suffices to find a smooth map $\psi\colon \mathrm{Herm}(n) \to \mathrm{SU}(n)$, $\psi(0) = I$ with the following properties:

(i) $\psi(\overline{A}) = \overline{\psi(A)}$,
(ii) $\gamma = \exp\circ\mathrm{Ad}_\psi$ is a diffeomorphism intertwining the Gelfand-Zeitlin maps and the Gelfand-Zeitlin torus actions,
(iii) ψ has the equivariance property (6).

The construction of a map ψ with these properties, using Poisson-geometric techniques, will be finished in Section 5.3.

3. Poisson-geometric techniques

In this section we discuss various tools that are needed for our construction of Ginzburg-Weinstein diffeomorphisms.

3.1. Bisections. Suppose

$$\mathcal{A}\colon K \to \mathrm{Diff}(M) \tag{12}$$

is an action of a Lie group K on a manifold M. We will often write $k.x := \mathcal{A}(k)(x)$ for $k \in K,\ x \in M$. Consider the action groupoid

$$K \times M \rightrightarrows M$$

with face maps $\partial_0(k,x) = x$ and $\partial_1(k,x) = k.x$. A *bisection* [**7**, Chapter 15] of $K \times M \rightrightarrows M$ is a submanifold $N \subset K \times M$ such that both maps ∂_i restrict to diffeomorphisms $N \to M$. Any bisection has the form $N = \{(x, \psi(x)) | \, x \in M\}$ where $\psi \in C^\infty(M,K)$ is a map such that

$$\mathcal{A}(\psi)(x) := \mathcal{A}(\psi(x))(x)$$

defines a diffeomorphism $\mathcal{A}(\psi) \in \mathrm{Diff}(M)$. Henceforth, we will refer to the map ψ itself as a bisection. [1] Let $\Gamma(M,K) \subset C^\infty(M,K)$ denote the set of bisections. The map

$$\Gamma(M,K) \to \mathrm{Diff}(M),\ \ \psi \mapsto \mathcal{A}(\psi) \tag{13}$$

is a group homomorphism for the following product on $\Gamma(M,K)$,

$$(\psi_1 \odot \psi_2)(x) = \psi_1(\mathcal{A}(\psi_2)(x))\psi_2(x).$$

The inverse of a bisection ψ for this product is given by

$$\psi^{-1}(x) = \psi(\mathcal{A}(\psi)^{-1}(x))^{-1}.$$

The group homomorphism (13) extends the action (12) of K, and has kernel the bisections satisfying $\psi(x) \in K_x$ for all $x \in M$. For later reference we note the following easy fact:

Lemma 3.1. *Suppose $\psi \in \Gamma(M,K)^K$ is an equivariant bisection (that is, $k \odot \psi = \psi \odot k$ for all $k \in K$). Then $\psi \odot \phi = \phi \odot \psi$ for all bisections ϕ satisfying $\mathcal{A}(\psi)^*\phi = \phi$. Furthermore, $(\phi \odot \psi)(x) = \phi(x)\psi(x)$.*

---

[1] It can be shown that a smooth map $\psi\colon M \to K$ is a bisection, if and only if for all $x \in M$, the map $k \mapsto \psi(k.x)k$ is a diffeomorphism of K. In this case, $\psi^{-1}(x) =: h$ is obtained as the unique solution of $\psi(h.x)h = 1$.

Proof. Since $\mathcal{A}(\psi)^*\phi = \phi$, the product $(\phi \odot \psi)(x)$ coincides with the pointwise product $\phi(x)\psi(x)$. On the other hand, since ψ is K-equivariant,

$$(\psi \odot \phi)(x) = \psi(\mathcal{A}(\phi)(x))\phi(x) = \mathrm{Ad}_{\phi(x)}(\psi(x))\phi(x) = \phi(x)\psi(x).$$

q.e.d.

The Lie algebra $\Gamma(M, \mathfrak{k})$ corresponding to $\Gamma(M, K)$ may be described as follows. Let

$$\mathfrak{k} \to \mathfrak{X}(M), \ \xi \mapsto \mathcal{A}(\xi) \tag{14}$$

denote the infinitesimal generators of the K-action, i.e., $\mathcal{A}(\xi)$ is the vector field with flow [2] $F_t = \mathcal{A}(\exp(t\xi))$. Then (14) is a homomorphism of Lie algebras. For $\beta \in C^\infty(M, \mathfrak{k})$ let $\mathcal{A}(\beta) \in \mathfrak{X}(M)$ be the vector field $\mathcal{A}(\beta)(x) = \mathcal{A}(\beta(x))(x)$. The map $\beta \mapsto \mathcal{A}(\beta)$ is a Lie algebra homomorphism for the 'action algebroid' [**7**] Lie bracket

$$[\beta_1, \beta_2](x) = [\beta_1(x), \beta_2(x)] + (L_{\mathcal{A}(\beta_1)}\beta_2)(x) - (L_{\mathcal{A}(\beta_2)}\beta_1)(x) \tag{15}$$

on $C^\infty(M, \mathfrak{k})$. Let $\Gamma(M, \mathfrak{k})$ denote the space $C^\infty(M, \mathfrak{k})$ with this Lie bracket.

To see more clearly how $\Gamma(M, \mathfrak{k})$ is the infinitesimal counterpart of $\Gamma(M, K)$, it is useful to realize $\Gamma(M, K)$ as a group of diffeomorphisms of $K \times M$. Define two commuting actions on $K \times M$, by setting

$$\tilde{\mathcal{A}}(k)(h, x) = (hk^{-1}, k.x), \quad \tilde{\mathcal{A}}'(k)(h, x) = (kh, x).$$

Then the map

$$\Gamma(M, K) \hookrightarrow \mathrm{Diff}(K \times M), \quad \psi \mapsto \tilde{\mathcal{A}}(\partial_0^*\psi)$$

identifies $\Gamma(M, K)$ with the group of diffeomorphism of $K \times M$ which commute with the action $\tilde{\mathcal{A}}'$ and preserve the $\tilde{\mathcal{A}}$-orbits. Similarly,

$$\Gamma(M, \mathfrak{k}) \hookrightarrow \mathfrak{X}(K \times M), \quad \beta \mapsto \tilde{\mathcal{A}}(\partial_0^*\beta)$$

identifies $\Gamma(M, \mathfrak{k})$ with the Lie algebra of vector fields on $K \times M$ which are invariant under the action $\tilde{\mathcal{A}}'$ and are tangent to the $\tilde{\mathcal{A}}$-orbits.

Let us now assume for simplicity that K is compact. For any $\beta \in \Gamma(M, \mathfrak{k})$, the vector field $\tilde{\mathcal{A}}(\partial_0^*\beta)$ is complete, since it is tangent to orbits. Hence, its time one flow exists, and defines an element of $\Gamma(M, K)$. We have thus extended the exponential map $\exp\colon \mathfrak{k} \to K$ to a map

$$\exp\colon \Gamma(M, \mathfrak{k}) \to \Gamma(M, K),$$

where $\psi = \exp(\beta)$ is the unique element such that $\tilde{\mathcal{A}}(\partial_0^*\psi)$ is the time one flow of $\tilde{\mathcal{A}}(\partial_0^*\beta)$.

[2] In this paper, we follow the convention that the flow F_t of a (possibly time dependent) vector field X_t is defined in terms of its action on functions by $(X_t f)(F_t^{-1}(x)) = \frac{\partial}{\partial t} f(F_t^{-1}(x))$. The Lie derivative L_{X_t} on differential forms is then characterized by $F_t^* \circ L_{X_t} = -\frac{\partial}{\partial t} F_t^*$.

More generally, one can 'integrate' families of maps $\beta_t \in \Gamma(M, \mathfrak{k})$ to families of bisections ψ_t, by viewing β_t as a time dependent vector field $\tilde{\mathcal{A}}(\partial_0^* \beta_t)$ on $K \times M$ and identifying ψ_t with the corresponding flow $\tilde{F}_t$ on $K \times M$. Equivalently, let F_t be the flow of the vector field $\mathcal{A}(\beta_t)$ on M. Then $F_t = \mathcal{A}(\psi_t)$, where the bisection $\psi_t \in \Gamma(M, K)$ is the solution of the ordinary differential equation on K,

$$\beta_t(F_t(x)) = \frac{\partial \psi_t(x)}{\partial t} \psi_t(x)^{-1}, \qquad \psi_0(x) = 1. \tag{16}$$

3.2. Gauge transformations of Poisson structures. Let M be a Poisson manifold, with Poisson bivector field π. The group of Poisson diffeomorphisms of (M, π) will be denoted $\mathrm{Diff}_\pi(M)$, and the group of Poisson vector fields by $\mathfrak{X}_\pi(M)$. Let $\sigma \in \Omega^2(M)$ be a closed 2-form, with the property that the bundle map

$$1 + \sigma^\flat \circ \pi^\sharp : \ T^*M \to T^*M \tag{17}$$

is invertible everywhere on M. (Here $\sigma^\flat \colon TM \to T^*M$ and $\pi^\sharp \colon T^*M \to TM$ are the bundle maps defined by a 2-form σ and bivector field π, respectively.) Then the formula

$$\pi_\sigma^\sharp := \pi^\sharp \circ (1 + \sigma^\flat \circ \pi^\sharp)^{-1} \tag{18}$$

defines a new Poisson structure π_σ on M, called the *gauge transformation of* π *by* σ [**20**, **5**]. The symplectic leaves of π_σ coincide with those of π, while the leafwise symplectic forms change by the pull-back of σ.

Gauge transformations of Poisson structures arise in the context of Hamiltonian group actions. A Poisson action $\mathcal{A} \colon K \to \mathrm{Diff}_\pi(M)$ is called *Hamiltonian*, if there exists a *moment map* $\Phi \colon M \to \mathfrak{k}^*$, equivariant relative to the coadjoint action on $\mathfrak{k}^*$, such that the generating vector fields for the action are

$$\mathcal{A}(\xi) = -\pi^\sharp \left\langle \mathrm{d}\Phi, \xi \right\rangle. \tag{19}$$

The moment map condition (19) shows in particular that Hamiltonian actions preserve the symplectic leaves. From the equivariance condition, it follows that Φ is a Poisson map. Conversely, any Poisson map $\Phi \colon M \to \mathfrak{k}^*$ defines a Lie algebra action by Equation (19). If K is connected, and if the infinitesimal $\mathfrak{k}$-action integrates to a K-action, then the latter is Hamiltonian with Φ as its moment map.

Proposition 3.2. *Let* (M, π) *be a Hamiltonian* K*-manifold with moment map* Φ*. For any bisection* $\psi \in \Gamma(M, K)$ *let*

$$\sigma_\psi = -d \left\langle \Phi, (\psi^{-1})^* \theta^L \right\rangle, \tag{20}$$

where $\theta^L \in \Omega^1(K) \otimes \mathfrak{k}$ *denotes the left-invariant Maurer-Cartan form. Then* σ_ψ *defines a gauge transformation of* π*, and*

$$\mathcal{A}(\psi)_* \pi = \pi_{\sigma_\psi}.$$

Proof. Since it suffices to prove this identity leafwise, we may assume that π is the inverse of a symplectic form ω. The moment map condition (19) translates into $\iota(\mathcal{A}(\xi))\omega + \langle \mathrm{d}\Phi, \xi\rangle = 0$. We will show

$$\mathcal{A}(\psi^{-1})^*\omega = \omega + \sigma_\psi, \tag{21}$$

thus in particular $\omega + \sigma_\psi$ is symplectic. One easily checks that the pull-back of ω under the map $\partial_1\colon K \times M \to M,\ (k, x) \mapsto k.x$ is

$$\partial_1^*\omega = \omega - \mathrm{d}\left\langle \Phi, \theta^L \right\rangle \in \Omega^2(K \times M).$$

Equation (21) follows, since $\mathcal{A}(\psi^{-1})$ is a composition of ∂_1 with the inclusion $M \to K \times M,\ x \mapsto (\psi^{-1}(x), x)$. q.e.d.

We collect some other formulas for the 2-form σ_ψ which will become useful later.

Proposition 3.3. *Let (M, π, Φ) be as in Proposition* 3.2*.*

a) *For any bisection $\psi \in \Gamma(M, K)$,*

$$\mathcal{A}(\psi)^*\sigma_\psi = d\left\langle \Phi, \psi^*\theta^L \right\rangle.$$

b) *If $\psi_1, \psi_2 \in \Gamma(M, K)$ are bisections,*

$$\sigma_{\psi_1 \odot \psi_2} = \sigma_{\psi_1} + \mathcal{A}(\psi_1^{-1})^*\sigma_{\psi_2}.$$

Proof. (a) Using the equivariance of Φ we have

$$\begin{aligned}\mathcal{A}(\psi^{-1})^* \left\langle \Phi, \psi^*\theta^L \right\rangle &= \left\langle \mathcal{A}(\psi^{-1})^*\Phi, \mathcal{A}(\psi^{-1})^*\psi^*\theta^L) \right\rangle \\ &= \left\langle \Phi, \mathrm{Ad}_{\psi^{-1}}^{-1}\left(\mathcal{A}(\psi^{-1})^*\psi^*\theta^L\right) \right\rangle.\end{aligned}$$

But $\mathcal{A}(\psi^{-1})^*\psi^*\theta^L = -(\psi^{-1})^*\theta^R = -\mathrm{Ad}_{\psi^{-1}}\left((\psi^{-1})^*\theta^L\right)$.

(b) From the definition $\psi_1 \odot \psi_2 = (\mathcal{A}(\psi_2)^*\psi_1)\psi_2$ we obtain

$$(\psi_1 \odot \psi_2)^*\theta^L = \psi_2^*\theta^L + \mathrm{Ad}_{\psi_2(\cdot)^{-1}}(\mathcal{A}(\psi_2)^*\psi_1^*\theta^L)$$

where $\psi_2(\cdot)^{-1}$ denotes the function $x \mapsto \psi_2(x)^{-1}$ (not to be confused with $\psi_2^{-1}(x)$). Therefore,

$$\mathrm{d}\left\langle \Phi, (\psi_1 \odot \psi_2)^*\theta^L \right\rangle = \mathrm{d}\left\langle \Phi, \psi_2^*\theta^L \right\rangle + \mathcal{A}(\psi_2)^*\mathrm{d}\left\langle \Phi, \psi_1^*\theta^L \right\rangle.$$

Now apply $\mathcal{A}((\psi_1 \odot \psi_2)^{-1})^* = \mathcal{A}(\psi_1^{-1})^*\mathcal{A}(\psi_2^{-1})^*$ to this result, and use (a). q.e.d.

Proposition 3.3(b) shows in particular that

$$\Gamma_0(M, K) = \{\psi \in \Gamma(M, K) |\ \sigma_\psi = 0\}$$

is a subgroup of the group of bisections. By Proposition 3.2, the homomorphism $\Gamma(M, K) \to \mathrm{Diff}(M),\ \psi \mapsto \mathcal{A}(\psi)$ restricts to a group homomorphism,

$$\Gamma_0(M, K) \to \mathrm{Diff}_\pi(M).$$

3.3. Moser's method for Poisson manifolds. Let (M, π) be a Poisson manifold, and σ_t a smooth family of closed 2-forms on M, with $\sigma_0 = 0$, such that $1 + \sigma_t^\flat \circ \pi^\sharp$ is invertible for all t. Consider the family of gauge-transformed Poisson structures, $\pi_t = \pi_{\sigma_t}$. Suppose

$$\frac{\partial}{\partial t}\sigma_t = -\mathrm{d}a_t \tag{22}$$

for a smooth family of 1-forms $a_t \in \Omega^1(M)$, and define a time dependent *Moser vector field* $v_t \in \mathfrak{X}(M)$ by $v_t = -\pi_t^\sharp(a_t)$. Assume that the time dependent vector field v_t is complete (this is automatic if the symplectic leaves of M are compact), and let F_t be the flow with initial condition $F_0 = \mathrm{id}$. One has [**2**],

$$\pi_t = (F_t)_*\pi.$$

The following alternative expression for the Moser vector field is useful:

Lemma 3.4. *The Moser vector field is given by $v_t = -\pi^\sharp(b_t)$ where $b_t = a_t + \iota(v_t)\sigma_t$. The 1-form b_t satisfies*

$$\frac{\partial}{\partial t}\big(F_t^*\sigma_t\big) = -d(F_t^* b_t), \tag{23}$$

where F_t is the flow of v_t.

Proof. By definition $v_t = -\pi^\sharp(\tilde{b}_t)$ where $\tilde{b}_t = (1 + \sigma_t^\flat \circ \pi^\sharp)^{-1} a_t$. The calculation

$$a_t = (1 + \sigma_t^\flat \circ \pi^\sharp)\tilde{b}_t = \tilde{b}_t - \sigma_t^\flat v_t = \tilde{b}_t - \iota(v_t)\sigma_t$$

shows $\tilde{b}_t = b_t$. From the definition of b_t and the formula for $\mathrm{d}a_t$, we find

$$\mathrm{d}b_t = \mathrm{d}a_t + L(v_t)\sigma_t = -\Big(\frac{\partial}{\partial t} - L(v_t)\Big)\sigma_t = -(F_t^{-1})^*\frac{\partial}{\partial t}(F_t^*\sigma_t).$$

q.e.d.

We will refer to b_t as the *Moser 1-form*. Note that for any given Poisson manifold (M, π) the list of data $v_t, F_t, a_t, b_t, \sigma_t, \pi_t$ is determined by b_t (and also by a_t).

The following proposition describes a situation where the twist flows $\mathcal{A}(\psi_t)$ from Section 3.1 can be viewed as Moser flows.

Proposition 3.5. *Suppose K is a compact Lie group, and (M, π) is a Hamiltonian Poisson K-manifold with moment map $\Phi\colon M \to \mathfrak{k}^*$. Let $\beta_t \in \Gamma(M, \mathfrak{k})$ define (cf. (16)) the family of bisections $\psi_t \in \Gamma(M, K)$ with $\psi_0 = 1$. Then the 2-form σ_t determined by the Moser 1-form*

$$b_t = \langle d\Phi, \beta_t\rangle$$

coincides with the form σ_{ψ_t}. Hence, the Moser flow F_t coincides with $\mathcal{A}(\psi_t)$, and the gauge transformed Poisson structure $\pi_t = \pi_{\sigma_t}$ equals $\mathcal{A}(\psi_t)_*\pi$.

Proof. By the moment map property, $v_t = -\pi^\sharp(b_t) = \mathcal{A}(\beta_t)$, with flow $F_t = \mathcal{A}(\psi_t)$. We have to verify Equation (23). Observe (cf. (16))

$$\mathrm{d}F_t^*\beta_t = \mathrm{d}\left(\frac{\partial \psi_t}{\partial t}\psi_t(\cdot)^{-1}\right) = \mathrm{Ad}_{\psi_t}\frac{\partial}{\partial t}\left(\psi_t^*\theta^L\right).$$

Since $F_t^*\Phi = \mathrm{Ad}^*_{\psi_t(\cdot)^{-1}}\Phi$ by equivariance of the moment map, this gives

$$F_t^*\left\langle \Phi, \mathrm{d}\beta_t\right\rangle = \left\langle \mathrm{Ad}^*_{\psi_t(\cdot)^{-1}}\Phi, \mathrm{d}F_t^*\beta\right\rangle = \left\langle \Phi, \frac{\partial}{\partial t}(\psi_t^*\theta^L)\right\rangle = \frac{\partial}{\partial t}\left\langle \Phi, \psi_t^*\theta^L\right\rangle.$$

Therefore, using Proposition 3.3(a),

$$F_t^*\mathrm{d}b_t = -\mathrm{d}F_t^*\left\langle \Phi, \mathrm{d}\beta_t\right\rangle = -\frac{\partial}{\partial t}\mathrm{d}\left\langle \Phi, \psi_t^*\theta^L\right\rangle = -\frac{\partial}{\partial t}F_t^*\sigma_t.$$

q.e.d.

3.4. Stability of Poisson actions. A well-known argument due to Palais shows that actions of compact Lie groups K on compact manifolds M are stable. That is, any deformation of such an action is obtained via conjugation by a family of diffeomorphisms of M. This result extends to the Poisson category:

Proposition 3.6 (Stability of Poisson actions of compact Lie groups)**.** *Let (M,π) be a Poisson manifold, K a compact Lie group, and $\mathcal{A}_t\colon K \to \mathrm{Diff}_\pi(M)$ a family of K-actions by Poisson diffeomorphism of M. Let $w_t \in \mathfrak{X}(M)$ be the time dependent vector field, given in terms of its action on functions by*

$$w_t = -\int_K dk\, \mathcal{A}_t(k^{-1})^*\frac{\partial}{\partial t}\mathcal{A}_t(k)^* \tag{24}$$

where dk denote the normalized Haar measure on K. Then w_t is a Poisson vector field. If the flow $F_t \in \mathrm{Diff}_\pi(M)$ of w_t exists (e.g., if M is compact, or if the K-orbits are independent of t), then

$$\mathcal{A}_t(k)\circ F_t = F_t \circ \mathcal{A}_0(k),\ \ k \in K. \tag{25}$$

Proof. The vector field w_t given by (24) has the property,

$$\frac{\partial}{\partial t}\mathcal{A}_t(k)^* + w_t \circ \mathcal{A}_t(k)^* - \mathcal{A}_t(k)^* \circ w_t = 0.$$

Assuming that the flow F_t of w_t is defined, this integrates to

$$F_t^* \circ \mathcal{A}_t(k)^* \circ (F_t^{-1})^* = \mathcal{A}_0(k)^*,$$

which gives (25). Since $\mathcal{A}_t(k)$ are Poisson diffeomorphisms, each vector field $w_t(k) = -\mathcal{A}_t(k^{-1})^*\frac{\partial}{\partial t}\mathcal{A}_t(k)^*$ is a Poisson vector field, and hence so is the K-average (24). q.e.d.

Remark. Note that if the actions $\mathcal{A}_t\colon K \to \mathrm{Diff}_\pi(M)$ commute with another (fixed) action of a compact Lie group H, then the vector field w_t and hence the diffeomorphisms F_t are H-equivariant.

3.5. Poisson diffeomorphisms of $\mathfrak{k}^*$. Of particular importance is the case $M = \mathfrak{k}^*$, with $\mathcal{A}\colon K \to \mathrm{Diff}_\pi(\mathfrak{k}^*)$ the coadjoint action. We begin with the following simple observation:

Lemma 3.7. *For any compact Lie group K, the center of the group $\Gamma(\mathfrak{k}^*, K)$ of bisections is the subgroup $\Gamma(\mathfrak{k}^*, K)^K$ of equivariant bisections, and is contained in the kernel of the map $\Gamma(\mathfrak{k}^*, K) \to \mathrm{Diff}(\mathfrak{k}^*)$, $\psi \mapsto \mathcal{A}(\psi)$.*

Proof. Suppose ψ is a K-equivariant bisection, i.e. $k \odot \psi = \psi \odot k$ for all $k \in K$. Equivalently, $\psi(k.\mu) = \mathrm{Ad}_k\, \psi(\mu)$ for all $\mu \in \mathfrak{k}^*, k \in K$. Specializing to $k \in K_\mu$, this shows that $\psi(\mu)$ is in the centralizer of K_μ. Since K is compact, this implies $\psi(\mu) \in K_\mu$. We have thus shown $\mathcal{A}(\psi) = \mathrm{id}$ for all $\psi \in \Gamma(\mathfrak{k}^*, K)^K$. Now suppose $\psi, \phi \in \Gamma(\mathfrak{k}^*, K)$, where ψ is K-equivariant. Then

$$\begin{aligned}(\psi \odot \phi)(\mu) &= \psi(\mathcal{A}(\phi)(\mu))\phi(\mu) = \mathrm{Ad}_{\phi(\mu)}(\psi(\mu))\phi(\mu)\\ &= \phi(\mu)\psi(\mu) = (\phi \odot \psi)(\mu)\end{aligned}$$

(for the last equality, we used that $\mathcal{A}(\psi) = \mathrm{id}$). This shows that $\Gamma(\mathfrak{k}^*, K)^K$ is contained in the center of $\Gamma(\mathfrak{k}^*, K)$. The converse is obvious, since central elements commute in particular with elements of K. q.e.d.

Remark. A similar statement holds for invariant open subsets of $\mathfrak{k}^*$, with the same proof.

Consider k^* as a Hamiltonian K-space, with Φ the identity map. The subgroup $\Gamma_0(\mathfrak{k}^*, K)$ of bisections ψ with $\sigma_\psi = 0$ is the group of *Lagrangian bisections*. (One can show that a bisection is Lagrangian if and only if its graph is a Lagrangian submanifold of the symplectic groupoid $K \times \mathfrak{k}^* = T^*K$.) The diffeomorphism $\mathcal{A}(\psi)$ defined by a Lagrangian bisection is a Poisson diffeomorphism preserving symplectic leaves.

Proposition 3.8. *The kernel of the homomorphism*

$$\Gamma_0(\mathfrak{k}^*, K) \longrightarrow \mathrm{Diff}_\pi(\mathfrak{k}^*), \quad \psi \to \mathcal{A}(\psi) \tag{26}$$

is the group of invariant Lagrangian bisections $\Gamma_0(\mathfrak{k}^*, K)^K$, while its image is the normal subgroup of Poisson diffeomorphisms preserving symplectic leaves.

Proof. By Lemma 3.7 above, $\mathcal{A}(\psi) = \mathrm{id}$ for all $\psi \in \Gamma_0(\mathfrak{k}^*, K)^K$. Suppose conversely that $\psi \in \Gamma_0(\mathfrak{k}^*, K)$ is a Lagrangian bisection with $\mathcal{A}(\psi) = \mathrm{id}$. Then each $\psi_t = r_t^*\psi$ generates the trivial action. In particular, this is true for the constant map $\psi_0 \equiv \psi(0)$. Hence $\psi(0)$ is in the center of K. Replacing ψ with $\psi' = \psi(0)^{-1}\psi$, we may assume $\psi(0) = 1$. Let $\beta_t = t^{-1} r_t^* \beta \in \Gamma(\mathfrak{k}^*, \mathfrak{k})^K$ be the $\mathfrak{k}$-valued functions generating ψ_t (cf. (16)), and $b_t = t^{-2} r_t^* b$ the associated family of closed 1-forms. Since

$\mathcal{A}(\psi) = \mathrm{id}$, the vector field $v = -\pi^\sharp(b)$ is zero. Hence b is K-invariant, and therefore ψ is K-equivariant.

Let $F \in \mathrm{Diff}_\pi(\mathfrak{k}^*)$ be any Poisson diffeomorphism preserving symplectic leaves. (In particular, $F(0) = 0$.) We have to show $F = \mathcal{A}(\psi)$ for some Lagrangian bisection ψ.

Suppose first that F is a *linear* Poisson diffeomorphism of $\mathfrak{k}^*$. Then F is dual to a Lie algebra automorphism $f \in \mathrm{Aut}(\mathfrak{k})$. Since F preserves orbits, the same is true for the map f. This implies that f is an *inner* automorphism, $f = \mathrm{Ad}_k$ for some $k \in K$. Hence $F = \mathrm{Ad}_k^* = \mathcal{A}(k^{-1})$ is given by a Lagrangian bisection $\psi \equiv k^{-1}$.

Consider now the general case. For $t \in \mathbb{R}$ let $r_t \colon \mathfrak{k}^* \to \mathfrak{k}^*$ denote the map $r_t(\mu) = t\mu$. Let $F_t = r_{t^{-1}} \circ F \circ r_t$ for $t \neq 0$, so that the limit for $t \to 0$ is the linearization $F_0 = \mathrm{d}_0 F$ of F at the origin. Since $(r_t)_*\pi = t\pi$, each F_t is a Poisson diffeomorphism preserving leaves. By the linear case considered above, we may assume $F_0 = \mathrm{id}$.

Let $v_t \in \mathfrak{X}_\pi(M)$ be the time-dependent vector field, given in terms of its action on functions by $v_t = -(F_t^{-1})^* \circ \frac{\partial}{\partial t} F_t^*$. Write $v = v_1$. Then $v_t = t^{-1}(r_{t^{-1}})_* v$ for $t \neq 0$. The vector fields v_t vanish to second order at 0, since $F_t(0) = 0$ and $\mathrm{d}_0 F_t \equiv \mathrm{id}$ for all t. In particular, $v_0 = 0$. We now use the well-known fact that a Poisson vector field on $\mathfrak{k}^*$ is Hamiltonian if and only if it is tangent to the symplectic leaves (which is automatic if $\mathfrak{k}$ is semi-simple). This follows from the description of the first Poisson cohomology of $\mathfrak{k}^*$ (see e.g., [**14**])

$$H^1_\pi(\mathfrak{k}^*) \cong (\mathfrak{k}^*)^K \otimes C^\infty(\mathfrak{k}^*)^K.$$

Hence, we may write $v = -\pi^\sharp(b)$ for some exact 1-form $b \in \Omega^1(\mathfrak{k}^*)$. The 1-form b can be normalized by requiring that its K-average be zero. (Note that exact 1-forms on $\mathfrak{k}^*$ generate the zero vector field if and only if they are K-invariant.) Letting $b_t = t^{-2} r_t^* b$, and denoting by $\beta_t = t^{-1} r_t^* \beta \in \Gamma(\mathfrak{k}^*, \mathfrak{k})$ the corresponding $\mathfrak{k}$-valued functions, we get

$$v_t = -\pi^\sharp(b_t) = \mathcal{A}(\beta_t).$$

Let $\psi_t \in \Gamma(\mathfrak{k}^*, K)$, $\psi_0 = 1$ be the family of bisections obtained by integrating β_t (see (16)). We have $\psi_t = r_t^* \psi$ with $\psi = \psi_1$. Since the 1-forms b_t are closed, the corresponding 2-forms $\sigma_t = \sigma_{\psi_t}$ (cf. (23) and Proposition 3.5) vanish. That is, the bisections ψ_t are Lagrangian. We have $F_t = \mathcal{A}(\psi_t)$ by construction, and in particular $F = \mathcal{A}(\psi)$. q.e.d.

Remark 3.9. Let (M, π) be a Poisson manifold admitting a symplectic realization $S \rightrightarrows M$. In Bursztyn-Weinstein [**6**, Section 5], the Poisson diffeomorphisms of M which are generated by Lagrangian bisections of S are referred to as *inner automorphisms* of M. Proposition 3.8 characterizes the inner automorphisms for the case $T^*K \rightrightarrows \mathfrak{k}^*$.

Proposition 3.10. *Suppose $\sigma \in \Omega^2(\mathfrak{k}^*)$ is a closed 2-form, defining a gauge transformation of the Kirillov-Poisson structure π on $\mathfrak{k}^*$. Then* *there exists a bisection $\psi \in \Gamma(\mathfrak{k}^*, K)$, $\psi(0) = 1$, such that $\sigma_\psi = \sigma$. In particular, $\mathcal{A}(\psi)_*\pi = \pi_\sigma$. ψ is unique up to multiplication from the right by a Lagrangian bisection. If σ is invariant under the action of $H \subset K$, the bisection ψ can be taken H-invariant.*

Proof. The assumption on σ means that the bundle map $A = 1 + \sigma^\flat \circ \pi^\sharp$ is invertible everywhere. Define a smooth family of closed 2-forms σ_t, by letting $\sigma_0 = 0$ and $\sigma_t = t^{-1} r_t^* \sigma$ for $t \neq 0$. Introduce the corresponding operators

$$A_t = 1 + \sigma_t^\flat \circ \pi^\sharp$$

on $T^*\mathfrak{k}^*$, connecting $A_1 = A$ with $A_0 = 1$. Using $(r_t)_*\pi = t\pi$, one finds $A_t = r_t^* \circ A \circ r_{t^{-1}}^*$ for $t \neq 0$. Since A is invertible, it follows that the operator A_t is invertible for all t. Hence, each σ_t defines a gauge transformation. Now let a_t be the family of 1-forms, obtained by applying the homotopy operator to $-\frac{\partial}{\partial t}\sigma_t$, and b_t the corresponding family of Moser 1-forms. By Proposition 3.5, the bisections ψ_t corresponding to b_t satisfy $\sigma_{\psi_t} = \sigma_t$. Thus $\psi = \psi_1$ has the desired property $\sigma_\psi = \sigma$. Uniqueness of ψ up to Lagrangian bisections follows directly from Proposition 3.3(b). If σ is H-invariant, then the bisection ψ just constructed is H-invariant. q.e.d.

For any compact, connected Lie group K, we denote by $Z(K) \subset K$ the identity component of the center, and by K^{ss} its semi-simple part (commutator subgroup). Thus $\hat{K} = K^{ss} \times Z(K) \to K$ is a finite covering of K, and $\mathfrak{k} = \mathfrak{k}^{ss} \oplus \mathfrak{z}(\mathfrak{k})$ on the level of Lie algebras.

Proposition 3.11. *Let K_1, K_2 be compact Lie groups, and suppose $\Phi\colon \mathfrak{k}_2^* \to \mathfrak{k}_1^*$ is the moment map for a Hamiltonian action $\mathcal{A}\colon K_1 \to \mathrm{Diff}_\pi(\mathfrak{k}_2^*)$. Suppose that the composition of Φ with the projection $\mathfrak{k}_1^* \to \mathfrak{z}(\mathfrak{k}_1)^*$ is a linear map, $\mathfrak{k}_2^* \to \mathfrak{z}(\mathfrak{k}_1)^*$. Then there exists a Lie algebra homomorphism $\tau\colon \mathfrak{k}_1 \to \mathfrak{k}_2$ and a Lagrangian bisection $\psi \in \Gamma_0(\mathfrak{k}_2^*, K_2)$ such that $\Phi = \tau^* \circ \mathcal{A}(\psi)$.*

Proof. Let us first of all observe that $\Phi(0) = 0$. Indeed, for the $\mathfrak{z}(\mathfrak{k}_1)^*$-component of Φ this follows by linearity, while for the $(\mathfrak{k}_1^{ss})^*$-component it follows since moment maps are equivariant by definition.

For all $t \neq 0$, the scaled Poisson homomorphism $\Phi_t = r_t^{-1} \circ \Phi \circ r_t$ is a moment map for the scaled action $k \mapsto \mathcal{A}_t(k) = r_t^{-1} \circ \mathcal{A}(k) \circ r_t$. Note that the $\mathfrak{z}(\mathfrak{k}_1)^*$-component of Φ_t, and hence the $Z(K_1) \subset K_1$-action, do not depend on t. The limit Φ_0 for $t \to 0$ equals the linearization of Φ at 0, and is a moment map for the linearized action $\mathcal{A}_0$. By Proposition 3.6 and the subsequent remark, there exists a $Z(K_1)$-equivariant Poisson diffeomorphism $F \in \mathrm{Diff}_\pi(\mathfrak{k}_2^*)$ with $\mathcal{A}(k) = F \circ \mathcal{A}_0(k) \circ F^{-1}$, $k \in K_1$. Since moment maps for semisimple Lie groups (in this case K_1^{ss}) are unique, and since the $\mathfrak{z}(\mathfrak{k}_1)^*$-component of Φ does not depend on t, this implies $\Phi = \Phi_0 \circ F^{-1}$. By Proposition 3.8, $F^{-1} = \mathcal{A}(\psi)$ for some Lagrangian bisection ψ. Since Φ_0 is a linear Poisson map, it is of the form $\Phi_0 = \tau^*$ for a Lie algebra homomorphism τ^*. q.e.d.

4. Ginzburg-Weinstein diffeomorphisms

The main result of this section is Theorem 4.7, showing that Ginzburg-Weinstein diffeomorphisms can be arranged to be compatible with given Poisson Lie group homomorphisms.

4.1. Poisson Lie groups. We briefly review Poisson Lie groups, referring to [**9**, **8**, **19**] for more detailed information. A Poisson Lie group is a Lie group K, equipped with a Poisson structure π^K for which the product map is Poisson. The linearization of π^K at the group unit is a Lie algebra 1-cocycle $\delta^K\colon \mathfrak{k} \to \wedge^2\mathfrak{k}$, with the property that the dual map $(\delta^K)^*$ defines a Lie bracket on $\mathfrak{k}^*$. Conversely, if K is connected, the cobracket δ^K determines π^K. One refers to the Lie algebra $\mathfrak{k}$ together with δ^K as the *tangent Lie bialgebra* of the Poisson Lie group K. The *dual Poisson Lie group* K^* is the connected, simply connected Poisson Lie group with tangent Lie bialgebra $\mathfrak{k}^*$. If $\pi^K = 0$, the dual Poisson Lie group is simply $\mathfrak{k}^*$ with the Kirillov Poisson structure.

A *Poisson Lie group action* of (K, π^K) on a Poisson manifold (M, π^M) is a K-action such that the action map $\partial_1\colon K \times M \to M,\ (k, x) \mapsto k.x$ is a Poisson map. A K^**-valued moment map* [**18**] for such an action is a Poisson map $\Psi\colon M \to K^*$ such that the generating vector fields are given by

$$\mathcal{A}(\xi) = -(\pi^M)^\sharp \Psi^* \left\langle \theta^R, \xi \right\rangle. \tag{27}$$

Here $\theta^R \in \Omega^1(K^*) \otimes \mathfrak{k}^*$ is the right-invariant Maurer-Cartan form on K^*. Equation (27) reduces to the usual moment map condition (19) if K carries the zero Poisson structure. According to Lu [**18**], *any* Poisson map $\Psi\colon M \to K^*$ defines an infinitesimal Poisson Lie group action via (27). In particular, the identity map of K^* defines an infinitesimal *dressing action* of K on K^*. In nice cases, it integrates to a global action of K.

Compact Lie group K carries a standard Poisson structure π^K structure, constructed by Lu and Weinstein in [**19**]. Let $G = K^{\mathbb{C}}$ be the complexification of K, viewed as a real Lie group, and $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ its Lie algebra. Consider the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad G = KAN$$

relative to a choice of maximal torus $T \subset K$ and fundamental Weyl chamber. That is, $\mathfrak{a} = \sqrt{-1}\mathfrak{t}$ while $\mathfrak{n}$ is the direct sum of root spaces for the positive roots. Let $B(\cdot, \cdot)$ be an invariant scalar product on $\mathfrak{k}$, and $B^{\mathbb{C}}(\cdot, \cdot)$ its complexification. Then $2 \operatorname{Im} B^{\mathbb{C}}(\cdot, \cdot)$ is an invariant scalar product on $\mathfrak{g}$, and restricts to a non-degenerate pairing between

$\mathfrak{k}$ and the Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$. In this way $\mathfrak{k}^* \cong \mathfrak{a} \oplus \mathfrak{n}$ acquires a Lie algebra structure, making $\mathfrak{k}$ into a Lie bialgebra. Thus K is a Poisson Lie group, with $K^* = AN$ the dual Poisson Lie group. The action of K on G by left multiplication descends to the dressing action $\mathcal{A}_{K^*}$ on K^*, viewed as a homogeneous space G/K. To analyze the dressing action, it is convenient to work with the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad G = K\,P \tag{28}$$

where $\mathfrak{p} = \sqrt{-1}\mathfrak{k}$ and $P = \exp \mathfrak{p}$. Recall that the exponential map $\exp\colon \mathfrak{g} \to G$ restricts to a diffeomorphism $\mathfrak{p} \to P$. Let $e\colon \mathfrak{k}^* \to K^*$ be the diffeomorphism defined by the compositions,

$$\mathfrak{k}^* \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p} \xrightarrow{\exp} P \cong G/K \cong K^*.$$

Then e intertwines the coadjoint action $\mathcal{A}_{\mathfrak{k}^*}$ with the dressing action:

$$e \circ \mathcal{A}_{\mathfrak{k}^*}(k) = \mathcal{A}_{K^*}(k) \circ e.$$

Example. Let $K = \mathrm{U}(n)$, with maximal torus $T = T(n)$ and the usual choice of positive roots. Then $G = \mathrm{GL}(n, \mathbb{C})$ (viewed as a real Lie group), N are the upper triangular matrices with 1's down the diagonal, and A are the diagonal matrices with positive entries. Hence $K^* = AN$ are the upper triangular matrices with positive diagonal entries. Furthermore, $\mathfrak{p} = \mathrm{Herm}(n)$ and $P = \mathrm{Herm}^+(n)$. The isomorphism $K^* \cong P$ is explicitly given by $X \mapsto (X^*X)^{1/2}$, and identifies the dressing action with the conjugation action on $\mathrm{Herm}^+(n)$.

4.2. Ginzburg-Weinstein diffeomorphisms. Let K be a compact Lie group with the standard Poisson structure, and consider the map $e\colon \mathfrak{k}^* \to K^*$ constructed above. In [**1**], it was observed that the Poisson structure $\pi_1^{\mathfrak{k}^*} = (e^{-1})_* \pi^{K^*}$ is gauge equivalent to the Kirillov-Poisson structure $\pi_0^{\mathfrak{k}^*} = \pi^{\mathfrak{k}^*}$.

Theorem 4.1 ([**1**]). *There is a canonical T-invariant closed 2-form $\sigma \in \Omega^2(\mathfrak{k}^*)$, with the property*

$$(e^{-1})_* \pi^{K^*} = \pi_\sigma^{\mathfrak{k}^*}.$$

We can now state a refined version of the Ginzburg-Weinstein theorem [**14**]. A similar result was obtained by Enriquez-Etingof-Marshall in [**11**], for *formal* Poisson Lie groups.

Theorem 4.2 (Ginzburg-Weinstein diffeomorphisms). *Let K be a compact Lie group with the standard Poisson structure. Then there exists a bisection $\psi \in \Gamma(\mathfrak{k}^*, K)$, with $\psi(0) = 1$, such that the map*

$$\gamma = e \circ \mathcal{A}(\psi)\colon \mathfrak{k}^* \to K^*$$

is a Poisson diffeomorphism. The bisection ψ can be chosen to be T-equivariant and to take values in the semi-simple part K^{ss}.

Proof. By Proposition 3.10, there exists a bisection $\psi \in \Gamma(\mathfrak{k}^*, K)$, $\psi(0) = 1$ with $\sigma_\psi = \sigma$. For any such bisection $\mathcal{A}(\psi)_* \pi^{\mathfrak{k}^*} = \pi_\sigma^{\mathfrak{k}^*} = (e^{-1})_* \pi^{K^*}$. Since σ is T-invariant, one can take ψ to be T-equivariant.

The map ψ lifts to a unique map $\hat{\psi} \in \Gamma(M, \hat{K}), \hat{\psi}(0) = 1$ with values in the finite cover $\hat{K} = K^{ss} \times Z(K)$ of K. Replacing ψ with the K^{ss}-component of $\hat{\psi}$, we arrange that ψ takes values in K^{ss}. q.e.d.

Definition 4.3. A bisection $\psi \in \Gamma(\mathfrak{k}^*, K)$ will be called a *Ginzburg-Weinstein twist* if it has the properties $\psi(0) = 1$ and $\sigma_\psi = \sigma$.

By Proposition 3.3(b), Ginzburg-Weinstein twists are unique up to a Lagrangian bisection.

Ginzburg-Weinstein twists can be used to turn ordinary $\mathfrak{k}^*$-valued moment maps into K^*-valued moment maps, and vice versa. However, the change of the moment map produces a twisted action.

Definition 4.4. Given an $K \to \mathrm{Diff}(M)$ on a manifold M, and a bisection $\psi \in \Gamma(M, K)$, the *ψ-twisted action* of K on M is defined as follows,

$$\mathcal{A}^\psi \colon K \to \mathrm{Diff}(M), \ \mathcal{A}^\psi(k) = \mathcal{A}(\psi) \circ \mathcal{A}(k) \circ \mathcal{A}(\psi)^{-1}. \tag{29}$$

Proposition 4.5. *Suppose $\psi \in \Gamma(\mathfrak{k}^*, K)$ is a Ginzburg-Weinstein twist, and let $\gamma = e \circ \mathcal{A}(\psi)$. Let (M, π) be a Poisson manifold, and $\Phi \colon M \to \mathfrak{k}^*$ and $\Psi \colon M \to K^*$ two Poisson maps related by $\Psi = \gamma \circ \Phi$. Then Φ is the moment map for a Hamiltonian K-action $\mathcal{A}$ if and only if Ψ is the moment map for a Poisson Lie group K-action $\mathcal{A}'$. The two actions are related as follows,*

$$\mathcal{A}' = \mathcal{A}^{\Phi^* \psi^{-1}}, \quad \mathcal{A} = (\mathcal{A}')^{(e^{-1} \circ \Psi)^* \psi}. \tag{30}$$

Proof. Suppose Φ generates a K-action $\mathcal{A}$. We will show that Ψ generates the action $\mathcal{A}' = \mathcal{A}^{\psi^{-1} \circ \Phi}$. By [**1**, **3**], the map

$$e \circ \Phi \colon M \to K^*$$

is the moment map for a Poisson-Lie group action of (K, π^K) on M, where M is equipped with the gauge transformed Poisson structure $\pi_{\Phi^* \sigma}$. Since $\sigma = \sigma_\psi$, the diffeomorphism $\mathcal{A}(\Phi^* \psi^{-1})$ takes the gauge transformed Poisson structure $\pi_{\Phi^* \sigma}$ structure back to π. Furthermore, $\mathcal{A}(\Phi^* \psi^{-1})$ intertwines $\mathcal{A}$ with the twisted action $\mathcal{A}' = \mathcal{A}^{\psi^{-1} \circ \Phi}$, and takes $e \circ \Phi$ to

$$(e \circ \Phi) \circ \mathcal{A}(\Phi^* \psi) = e \circ \mathcal{A}(\psi) \circ \Phi = \Psi.$$

It follows that Ψ is a moment map for the twisted action $\mathcal{A}' = \mathcal{A}^{\Phi^* \psi^{-1}}$ on (M, π).

Conversely, assume that Ψ generates an action $\mathcal{A}'$. Then $e^{-1} \circ \Psi : M \to \mathfrak{k}^*$ is a moment map for a Hamiltonian action on $(M, \pi_{-(e^{-1} \circ \Psi)^* \sigma})$.

Applying $\mathcal{A}((e^{-1}\circ\Psi)^*\psi$ to restore the Poisson structure π, and arguing as above, we see that Φ is a moment map for the action $\mathcal{A} = (\mathcal{A}')^{(e^{-1}\circ\Psi)^*\psi}$ on (M,π). (Alternatively, one can also use Lemma 4.9 below to argue that the two formulas (12) are equivalent.) q.e.d.

4.3. Functorial properties of Ginzburg-Weinstein maps. A homomorphism of Poisson Lie groups K_1, K_2 is a Lie group homomorphism

$$\mathcal{T}\colon K_1 \to K_2$$

which is also a Poisson map. On the infinitesimal level, a homomorphism of Poisson Lie groups defines a homomorphism of Lie bialgebras, $\tau\colon \mathfrak{k}_1 \to \mathfrak{k}_2$. That is, the dual map $\tau^*\colon \mathfrak{k}_2^* \to \mathfrak{k}_1^*$ is a Lie algebra homomorphism, and in particular exponentiates to a dual Poisson Lie group homomorphism $\mathcal{T}^*\colon K_2^* \to K_1^*$. Given a Poisson Lie group action $\mathcal{A}$ of K_2 on a Poisson manifold M, with moment map $\Psi\colon M \to K_2^*$, the composition $\mathcal{T}^*\circ\Psi$ is a moment map for the K_1-action $\mathcal{A}\circ\mathcal{T}$.

For any compact Lie group K with the standard Poisson structure, the maximal torus T with the zero Poisson structure is a Poisson-Lie subgroup. That is, the inclusion $\mathcal{T}\colon T \to K$ is a Poisson-Lie group homomorphism.

Lemma 4.6. *Suppose $\psi \in \Gamma(\mathfrak{k}^*, K)$ is a T-equivariant Ginzburg-Weinstein twist, and let $\gamma = e\circ\mathcal{A}(\psi)$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{k}^* & \xrightarrow[\tau^*]{} & \mathfrak{t}^* \\ \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \cong} \\ K^* & \xrightarrow[\mathcal{T}^*]{} & T^* \end{array}$$

Proof. Let $\mathcal{T}\colon T \to K$ denote the inclusion, and consider the Poisson map $\Upsilon\colon \mathfrak{k}^* \to \mathfrak{t}^*$ given as the composition of the Poisson maps $\gamma\colon \mathfrak{k}^* \to K^*$ and $\mathcal{T}^*\colon K^* \to T^* \cong \mathfrak{t}^*$. Proposition 4.5 shows that γ is a moment map for the twisted K-action $\mathcal{A}^{\psi^{-1}}$, and hence Υ is a moment map for the twisted T-action, $\mathcal{A}^{\psi^{-1}}\circ\mathcal{T}$. Since ψ is T-equivariant, the twisted and untwisted T-actions coincide. Hence, Υ and τ^* are moment maps for the same T-action. It follows that their difference is a K-invariant function $\mathfrak{k}^* \to \mathfrak{t}^*$. It hence suffices to show that Υ and τ^* coincide on $\mathfrak{t}^* = (\mathfrak{k}^*)^T \subset \mathfrak{k}^*$ (fixed point set for the coadjoint action of T on $\mathfrak{k}^*$). That is, we have to show that Υ restricts to the identity map of $\mathfrak{t}^*$.

Since ψ is T-equivariant, it takes $\mathfrak{t}^* = (\mathfrak{k}^*)^T$ to $T = K^T$ (fixed point set for the conjugation action of T on K). In particular, $\mathcal{A}(\psi)$ acts trivially on $\mathfrak{t}^*$, and hence γ coincides with e on $\mathfrak{t}^* \subset \mathfrak{k}^*$. Since $e\colon \mathfrak{k}^* \to K^*$ restricts to the natural identification $\mathfrak{t}^* \cong T^*$, we conclude that Υ restricts to the identity map of $\mathfrak{t}^*$. q.e.d.

Theorem 4.7 (Compatible Ginzburg-Weinstein maps)**.** *Let K_1, K_2 be compact Poisson Lie groups with the standard Poisson structure, and $\mathcal{T}\colon K_1 \to K_2$ a Poisson Lie group homomorphism. Given any Ginzburg-Weinstein twist $\psi_1 \in \Gamma(\mathfrak{k}_1^*, K_1)$, there exists a Ginzburg-Weinstein twist $\psi_2 \in \Gamma(\mathfrak{k}_2^*, K_2)$, for which the diagram*

$$\begin{tikzcd} \mathfrak{k}_2^* \arrow[r, "\tau^*"] \arrow[d, "\gamma_2"] & \mathfrak{k}_1^* \arrow[d, "\gamma_1"] \\ K_2^* \arrow[r, "\mathcal{T}^*"'] & K_1^* \end{tikzcd} \tag{31}$$

with $\gamma_i = e_i \circ \mathcal{A}_i(\psi_i)$ commutes. Here $\mathcal{A}_i$ denotes the coadjoint action of K_i on $\mathfrak{k}_i^*$. One can arrange that ψ_2 takes values in the semi-simple part K_2^{ss}.

Proof. We may assume, passing to a finite cover of K_1 if necessary, that the semi-simple part K_1^{ss} is simply connected. We begin by choosing an arbitrary T_2-equivariant Ginzburg-Weinstein twist ψ_2. We will show how to modify ψ_2 (possibly destroying the T_2-equivariance), in such a way that the above diagram commutes. The idea is to apply Proposition 3.11 to the Poisson map

$$\Upsilon = \gamma_1^{-1} \circ \mathcal{T}^* \circ \gamma_2 \colon \mathfrak{k}_2^* \to \mathfrak{k}_1^*.$$

To apply this proposition, we have to verify that the $\mathfrak{z}(\mathfrak{k}_1)^*$-component of Υ is given by a linear map. In fact, we will show that the $\mathfrak{z}(\mathfrak{k}_1)^*$-components of Υ and τ^* are equal. Since ψ_2 is T_2-equivariant, Lemma 4.6 shows that γ_2 restricts to the natural identification $\mathfrak{t}_2^* \to T_2^*$. Similarly, γ_1 restricts to the natural identification $\mathfrak{z}(\mathfrak{k}_1)^* \to Z(K_1)^* = \mathfrak{z}(\mathfrak{k}_1)^*$, since this is true for e_2, and since the action of K_1 (hence of $\mathcal{A}(\psi_1)$) on $\mathfrak{z}(\mathfrak{k}_1)^*$ is trivial. Since the Poisson bivector of K_2 vanishes exactly along T_2, the map $\mathcal{T}$ must take $Z(K_1) \subset T_1$ into T_2. Hence, the diagram

$$\begin{tikzcd} \mathfrak{k}_2^* \arrow[r] \arrow[d, "\gamma_2"] & \mathfrak{t}_2^* \arrow[r] \arrow[d, "\gamma_2"] & \mathfrak{z}(\mathfrak{k}_1)^* \arrow[d, "\gamma_1"] \\ K_2^* \arrow[r] & T_2^* \arrow[r] & Z(K_1)^* \end{tikzcd}$$

commutes, proving the claim.

It follows in particular that the $\mathfrak{z}(\mathfrak{k}_1)^*$-component of Υ is a moment map for the action of $Z(K_1) \subset K_1$ via $\mathcal{T}$. On the other hand, the $(\mathfrak{k}_1^{ss})^*$-component is a moment map for some action of K_1^{ss}, since K_1^{ss} is simply connected. Hence, Υ is the moment map for a K_1-action. By Proposition 3.11, there exists a Lagrangian bisection $\phi \in \Gamma_0(\mathfrak{k}_2^*, K_2)$, $\phi(0) = 1$, with the property $\Upsilon \circ \mathcal{A}(\phi) = \Upsilon_0 = \tau^*$. That is, replacing ψ with $\psi' = \psi \circ \phi$ the diagram (31) commutes. As in the proof of Theorem 4.2, one can arrange that the new ψ takes values in K_2^{ss}, without changing $\mathcal{A}(\psi)$. q.e.d.

Let us call two Ginzburg-Weinstein twists $\psi_1 \in \Gamma(\mathfrak{k}_1^*, K_1)$ and $\psi_2 \in \Gamma(\mathfrak{k}_2^*, K_2)$ *compatible* (relative to $\mathcal{T}\colon K_1 \to K_2$) if the corresponding Ginzburg-Weinstein diffeomorphism $\gamma_i = e_i \circ \mathcal{A}(\phi_i)$ defines a commutative diagram (31). The compatibility condition is equivalent to a certain equivariance condition, as the following result shows.

Theorem 4.8. *Suppose $\psi_1 \in \Gamma(\mathfrak{k}_1^*, K_1)$ and $\psi_2 \in \Gamma(\mathfrak{k}_2^*, K_2)$ are compatible Ginzburg-Weinstein twists, and put*

$$\hat{\psi}_1 = \mathcal{T} \circ \psi_1 \circ (e_1^{-1} \circ \mathcal{T}^* \circ e_2) \in \Gamma(\mathfrak{k}_2^*, K_2).$$

Then the 'ratio' $\hat{\psi}_1^{-1} \odot \psi_2 \in \Gamma(\mathfrak{k}_2^*, K_2)$ is K_1-equivariant in the sense that it $\odot$-commutes with all $\mathcal{T}(k)$ for all $k \in K_1$. One has the formula

$$(\hat{\psi}_1^{-1} \odot \psi_2)(\mu) = \mathcal{T}\big(\psi_1(\tau^*\mu)\big)^{-1}\, \psi_2(\mu). \tag{32}$$

Proof. Given *arbitrary* Ginzburg-Weinstein twists ψ_1, ψ_2, consider again the moment map $\Upsilon = \gamma_1^{-1} \circ \mathcal{T}^* \circ \gamma_2 \colon \mathfrak{k}_2^* \to \mathfrak{k}_1^*$ as in the proof of Theorem 4.7. Suppose (M, π) is a Poisson manifold, and $\Phi\colon M \to \mathfrak{k}_2^*$ is the moment map for a Hamiltonian action $\mathcal{A}\colon K_2 \to \mathrm{Diff}_\pi(M)$. Let us compute the K_1-action generated by $\Upsilon \circ \Phi$. Using Proposition 4.5, we have

$$\begin{array}{rrll}
 & \Phi\colon M \to \mathfrak{k}_2^* & \text{is a moment map for} & \mathcal{A} \\
\Rightarrow & \gamma_2 \circ \Phi\colon M \to K_2^* & \text{is a moment map for} & \mathcal{A}^{\Phi^*\psi_2^{-1}} \\
\Rightarrow & \mathcal{T}^* \circ \gamma_2 \circ \Phi\colon M \to K_1^* & \text{is a moment map for} & \mathcal{A}^{\Phi^*\psi_2^{-1}} \circ \mathcal{T} \\
\Rightarrow & \gamma_1^{-1} \circ \mathcal{T}^* \circ \gamma_2 \circ \Phi\colon M \to \mathfrak{k}_1^* & \text{is a moment map for} & \\
 & & & (\mathcal{A}^{\Phi^*\psi_2^{-1}} \circ \mathcal{T})^{\psi_1 \circ e_1^{-1} \circ (\mathcal{T}^* \circ \gamma_2 \circ \Phi)}.
\end{array}$$

We may re-write the result as

$$\begin{aligned}
(\mathcal{A}^{\Phi^*\psi_2^{-1}} \circ \mathcal{T})^{\psi_1 \circ e_1^{-1} \circ \mathcal{T}^* \circ \gamma_2 \circ \Phi} &= (\mathcal{A}^{\Phi^*\psi_2^{-1}})^{\Phi^*(\mathcal{T} \circ \psi_1 \circ e_1^{-1} \circ \mathcal{T}^* \circ \gamma_2)} \circ \mathcal{T} \\
&= (\mathcal{A}^{\Phi^*\psi_2^{-1}})^{\Phi^*(\hat{\psi}_1 \circ \mathcal{A}(\psi_2))} \circ \mathcal{T} \\
&= \mathcal{A}^{\Phi^*(\psi_2^{-1} \odot \hat{\psi}_1)} \circ \mathcal{T}.
\end{aligned}$$

In the last line, we have used Lemma 4.9 below to write an iterated twist as a single twist.

Assume now that ψ_1, ψ_2 are compatible. The commutativity of the diagram (31) means that $\Upsilon = \tau^*$. In particular, for any Hamiltonian K_2-space (M, π, Φ), the twisted K_1-action $\mathcal{A}^{\Phi^*(\psi_2^{-1} \odot \hat{\psi}_1)} \circ \mathcal{T}$ coincides with the untwisted action $\mathcal{A} \circ \mathcal{T}$. By definition of the twisted action, this is equivalent to

$$\mathcal{A}(\Phi^*(\psi_2^{-1} \odot \hat{\psi}_1)) \circ \mathcal{A}(\mathcal{T}(k)) = \mathcal{A}(\mathcal{T}(k)) \circ \mathcal{A}(\Phi^*(\psi_2^{-1} \odot \hat{\psi}_1)), \ \ k \in K_1. \tag{33}$$

Apply this result to $M = K_2 \times \mathfrak{k}_2^*$, with symplectic structure coming from the identification with T^*K_2, and with $\Phi\colon (k, \mu) \mapsto \mu$ the moment map for the K_2-action $\mathcal{A}(k)(h, \mu) = (hk^{-1}, k.\mu)$. Since the map $\Gamma(\mathfrak{k}_2^*, K_2) \to$

$\mathrm{Diff}(K_2 \times \mathfrak{k}_2^*)$, $\psi \mapsto \mathcal{A}(\Phi^*\psi)$ is 1-1, the above equation implies $(\psi_2^{-1} \odot \hat{\psi}_1) \odot \mathcal{T}(k) = \mathcal{T}(k) \odot (\psi_2^{-1} \odot \hat{\psi}_1)$ as desired.

The bisection $\hat{\psi}_1^{-1} \odot \psi_2$ may be re-written, using

$$\hat{\psi}_1^{-1} = \mathcal{T} \circ \psi_1^{-1} \circ (e_1^{-1} \circ \mathcal{T}^* \circ e_2) = \mathcal{T} \circ \psi_1^{-1} \circ \mathcal{A}(\psi_1) \circ \tau^* \circ \mathcal{A}(\psi_2)^{-1}$$

because of the commutativity of the diagram (31). Thus,

$$\hat{\psi}_1^{-1}(\mathcal{A}(\psi_2)(\mu)) = \mathcal{T}\big(\psi_1(\tau^*\mu)\big)^{-1},$$

and (32) follows. q.e.d.

In the proof we used the following lemma:

Lemma 4.9. *Suppose M is a K-manifold with action $\mathcal{A}$. Let $\psi \in \Gamma_{\mathcal{A}}(M, K)$ be a bisection relative to the action $\mathcal{A}$, and $\phi \in \Gamma_{\mathcal{A}^\psi}(M, K)$ a bisection relative to the twisted action $\mathcal{A}^\psi$. Then the iterated twist $(\mathcal{A}^\psi)^\phi$ can be written as a single twist,*

$$(\mathcal{A}^\psi)^\phi = \mathcal{A}^{\psi \odot \mathcal{A}(\psi)^*\phi}.$$

Proof. By definition, $\mathcal{A}^\psi(\phi)(x) = \mathcal{A}(\psi)\mathcal{A}(\phi(x))\mathcal{A}(\psi^{-1})(x)$. Hence

$$\mathcal{A}^\psi(\phi) = \mathcal{A}(\psi \odot \mathcal{A}(\psi)^*\phi \odot \psi^{-1}).$$

Using this formula we calculate, for all $k \in K$,

$$\begin{aligned}(\mathcal{A}^\psi)^\phi(k) &= \mathcal{A}^\psi(\phi) \circ \mathcal{A}^\psi(k) \circ \mathcal{A}^\psi(\phi)^{-1} \\ &= \mathcal{A}(\psi \odot \mathcal{A}(\psi)^*\phi) \circ \mathcal{A}(k) \circ \mathcal{A}(\psi \odot \mathcal{A}(\psi)^*\phi)^{-1} \\ &= \mathcal{A}^{\psi \odot \mathcal{A}(\psi)^*\phi}(k).\end{aligned}$$

q.e.d.

4.4. Anti-Poisson involutions. An *anti-Poisson involution* of a Poisson manifold (M, π) is an involutive diffeomorphism $s \in \mathrm{Diff}(M)$ reversing the Poisson structure, $s_*\pi = -\pi$. An anti-Poisson involution of a Poisson Lie group (K, π^K) is an anti-Poisson involution s_K of the underlying Poisson manifold which is also an automorphism of the group K. In this case, s_K canonically induces an anti-Poisson involution of the dual Poisson Lie group K^*.

Suppose K is a compact Lie group with standard Poisson structure. Then any anti-linear involution of the Lie algebra $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ preserving the Iwasawa decomposition and the bilinear form $2\,\mathrm{Im}\, B^{\mathbb{C}}$ defines an anti-Poisson involution s_K of K. Let $s_{\mathfrak{k}^*}$ be the induced involution of $\mathfrak{k}^*$.

Lemma 4.10. *There exists a Ginzburg-Weinstein twist $\psi \in \Gamma(\mathfrak{k}^*, K)$ which, in addition to the properties from Theorem* 4.2*, satisfies the equivariance property*

$$\psi \circ s_{\mathfrak{k}^*} = s_K \circ \psi.$$

Proof. The Ginzburg-Weinstein twist constructed in the proof of Theorem 4.2 has the required equivariance property under involutions. Indeed, the forms σ_t, a_t on $\mathfrak{k}^*$, hence also the Moser 1-form b_t, change sign under $s_{\mathfrak{k}^*}$ (see [**3**]). It follows that the Moser vector field v_t is $s_{\mathfrak{k}^*}$-invariant, while the function β_t is equivariant, $\beta_t \circ s_{\mathfrak{k}^*} = s_{\mathfrak{k}} \circ \beta_t$.

q.e.d.

Example. If $K = \mathrm{U}(n)$, the complex conjugation operation $s_K(A) = \overline{A}$ is an anti-Poisson involution. The involution $s_{\mathfrak{k}^*}$ is complex conjugation on $\mathfrak{k}^* \cong \mathrm{Herm}(n)$, and s_{K^*} is complex conjugation on upper triangular matrices or, equivalently, on the space $P = \mathrm{Herm}^+(n)$ of positive definite matrices. Compatibility of a Ginzburg-Weinstein twist ψ with these involutions just means $\psi(\overline{A}) = \overline{\psi(A)}$. In particular, ψ restricts to a bisection $\mathrm{Sym}(n) \to \mathrm{SO}(n)$.

The functoriality properties of Ginzburg-Weinstein maps generalize in the obvious way to the presence of such involutions. Thus, suppose $K_i,\ i = 1, \ldots, n$ are compact Poisson Lie groups with standard Poisson structure, and s_{K_i} are anti-Poisson involutions of K_i of the type discussed above. Assume $\mathcal{T}_i \colon K_i \to K_{i+1},\ i = 1 \ldots, n-1$ are Poisson Lie group homomorphisms with

$$\mathcal{T}_i \circ s_{K_i} = s_{K_{i+1}} \circ \mathcal{T}.$$

Then the Ginzburg-Weinstein twists $\psi_{i,t} \in \Gamma(\mathfrak{k}_i^*, K_i)$ constructed in Theorem 4.7 can be arranged to satisfy

$$\psi_{i,t} \circ s_{\mathfrak{k}_i^*} = s_{K_i} \circ \psi_{i,t}.$$

Indeed, the maps obtained in the proof of Theorem 4.7 automatically have this property, since all constructions are compatible with the involutions. It follows that all maps in the commutative diagram (31) intertwine the various involutions. In particular, one obtains a commutative diagram for the fixed point sets of the involutions.

5. Gelfand-Zeitlin systems

5.1. Thimm actions. The following construction of torus actions from non-Abelian group actions appeared in Thimm's work [**21**] on completely integrable systems, and was later clarified by Guillemin-Sternberg in [**15**]. We will present the Thimm actions using the terminology of bisections. Let K be a compact Lie group, with maximal torus T, and let $\mathfrak{k}^*_{\mathrm{reg}} \subset \mathfrak{k}^*$ be the subset of regular elements; that is, elements whose stabilizer is conjugate to T. Pick a fundamental Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$. Then $\mathfrak{k}^*_{\mathrm{reg}} = K/T \times \mathrm{int}(\mathfrak{t}^*_+)$ as K-manifolds. Restriction of equivariant bisections over $\mathfrak{k}^*_{\mathrm{reg}}$ to $\mathrm{int}(\mathfrak{t}^*_+)$ defines a group isomorphism,

$$\Gamma(\mathfrak{k}^*_{\mathrm{reg}}, K)^K \xrightarrow{\cong} \Gamma(\mathrm{int}(\mathfrak{t}^*_+), T). \tag{34}$$

Lemma 5.1. *The isomorphism* (34) *identifies the subgroups of Lagrangian bisections:* $\Gamma_0(\mathfrak{k}^*_{\mathrm{reg}}, K)^K \cong \Gamma_0(\mathrm{int}(\mathfrak{t}^*_+), T)$.

Proof. If $\psi \in \Gamma(\mathfrak{k}^*_{\mathrm{reg}}, K)^K$ is Lagrangian, then clearly so is its restriction to $\mathrm{int}(\mathfrak{t}^*_+)$. For the converse, suppose ψ restricts to a Lagrangian bisection over $\mathrm{int}(\mathfrak{t}^*_+)$. For any $\xi \in \mathfrak{k}$ we have $\iota(\xi)\left\langle \mu, (\psi^{-1})^*\theta^L \right\rangle = \left\langle \mu, \xi - \mathrm{Ad}_{\psi(\mu)}(\xi) \right\rangle = 0$, since $\psi(\mu) \in K_\mu$. Hence also

$$\iota(\xi)\mathrm{d}\left\langle \mu, (\psi^{-1})^*\theta^L \right\rangle = L(\xi)\left\langle \mu, (\psi^{-1})^*\theta^L \right\rangle - \mathrm{d}\iota(\xi)\left\langle \mu, (\psi^{-1})^*\theta^L \right\rangle = 0.$$

Since on the other hand the pull-back of $\mathrm{d}\left\langle \mu, (\psi^{-1})^*\theta^L \right\rangle$ to $\mathrm{int}(\mathfrak{t}^*_+) \subset \mathfrak{k}^*_{\mathrm{reg}}$ is zero, this shows $\mathrm{d}\left\langle \mu, (\psi^{-1})^*\theta^L \right\rangle = 0$. Thus ψ is Lagrangian. q.e.d.

Define a group homomorphism

$$\chi\colon T \to \Gamma_0(\mathfrak{k}^*_{\mathrm{reg}}, K)^K \tag{35}$$

by composing the map inverse to (34) with the inclusion $T \to \Gamma_0(\mathrm{int}(\mathfrak{t}^*_+), T)$ as constant bisections. That is, $\chi(t)\colon \mathfrak{k}^*_{\mathrm{reg}} \to K$ is the unique K-equivariant map with $\chi(t)(\mu) = t$ for $\mu \in \mathrm{int}(\mathfrak{t}^*_+)$.

Recall that by Lemma 3.7, $\Gamma(\mathfrak{k}^*_{\mathrm{reg}}, K)^K$ is the center of $\Gamma(\mathfrak{k}^*_{\mathrm{reg}}, K)$, and that its action on $\mathfrak{k}^*_{\mathrm{reg}}$ is trivial. In particular, $\chi(t)$ acts trivially on $\mathfrak{k}^*_{\mathrm{reg}}$. Non-trivial actions are obtained by pulling $\chi(t)$ back under an equivariant map, $\Phi\colon M \to \mathfrak{k}^*$. Thus let $M_0 = \Phi^{-1}(\mathfrak{k}^*_{\mathrm{reg}}) \subset M$, and $\chi_M(t) = \Phi^*\chi(t) \in \Gamma(M_0, K)^K$. We define the *Thimm action* of $t \in T$ by

$$t \bullet x = \mathcal{A}(\chi_M(t))(x), \quad x \in M_0.$$

By construction, the Thimm action commutes with the K-action, and the map Φ is Thimm-invariant:

$$\Phi(t \bullet x) = t \bullet \Phi(x) = \Phi(x).$$

From now on, we will write $\chi(t)(\mu) \equiv \chi(t; \mu)$ and similarly for χ_M.

Lemma 5.2. *If $\psi \in \Gamma(M_0, K)$ is constant along the fibers of Φ, then ψ commutes (under $\odot$) with all $\chi_M(t)$, and $(\psi \odot \chi_M(t))(x) = \psi(x)\chi_M(t; x)$.*

Proof. Since $\mathcal{A}(\chi_M(t))$ preserves the fibers of Φ, the bisection ψ satisfies $\mathcal{A}(\chi_M(t))^*\psi = \psi$. Hence, Lemma 3.1 applies. q.e.d.

Thimm actions are naturally associated with Hamiltonian group actions.

Lemma 5.3 (Guillemin-Sternberg [**15**])**.** *Suppose (M, π) is a Hamiltonian K-manifold, with moment map $\Phi\colon M \to \mathfrak{k}^*$. Then the Thimm T-action on M_0 is Hamiltonian, with moment map $q \circ \Phi\colon M \to \mathfrak{t}^*$. Here $q\colon \mathfrak{k}^* \to \mathfrak{t}^*_+ \subset \mathfrak{t}^*$ is the unique K-invariant map with $q(\mu) = \mu$ for $\mu \in \mathfrak{t}^*_+$.*

Suppose now that

$$K_1 \xrightarrow{\mathcal{T}_1} K_2 \xrightarrow{\mathcal{T}_2} \cdots \to K_n \tag{36}$$

is a sequence of compact Lie groups and homomorphisms, with differentials $\tau_i \colon \mathfrak{k}_i \to \mathfrak{k}_{i+1}$. For $i < j$ we will write $\mathcal{T}_i^j = \mathcal{T}_{j-1} \circ \cdots \circ \mathcal{T}_i \colon K_i \to K_j$, with differential $\tau_i^j \colon \mathfrak{k}_i \to \mathfrak{k}_j$. Take the maximal tori $T_i \subset K_i$ and positive Weyl chambers $\mathfrak{t}_{i,+}^*$ to be compatible, in the sense that for all $i < n$,

$$\mathcal{T}_i(T_i) \subset T_{i+1}, \quad \tau_i^*(\mathfrak{t}_{i+1,+}^*) \subset \mathfrak{t}_{i,+}^*.$$

Let M be a K_n-manifold, and $\Phi_n \colon M \to \mathfrak{k}_n^*$ an equivariant map. Then each K_i acts on M via $\mathcal{T}_i^n$, and we obtain a K_i-invariant map $\Phi_i = (\tau_i^n)^* \Phi_n \colon M \to \mathfrak{k}_i^*$. Let

$$M_0 = \bigcap_{i=1}^n \Phi_i^{-1}(\mathfrak{k}_{i,reg}^*),$$

and define $\chi_{i,M} \colon T_i \to \Gamma(M_0, K_n)$ by

$$\chi_{i,M}(t_i) = \mathcal{T}_i^n \circ \chi_i(t_i) \circ \Phi_i, \quad t_i \in T_i$$

where $\chi_i(t_i) \in \Gamma(\mathfrak{k}_{i,\mathrm{reg}}^*, K_i)$.

Lemma 5.4. *The images of the homomorphisms $\chi_{i,M} : T_i \to \Gamma(M_0, K_n)$ all commute. Hence, they combine to define a group homomorphism*

$$\chi_M \colon T_n \times \cdots \times T_1 \to \Gamma(M_0, K_n).$$

One has the formula

$$\chi_M(t_n, \ldots, t_1; x) = \chi_{1,M}(t_1; x) \cdots \chi_{n,M}(t_n; x).$$

Proof. Let $t_i \in T_i$, $t_j \in T_j$ where $i < j$. The bisection $\chi_j(t_j) \in \Gamma(\mathfrak{k}_{j,\mathrm{reg}}^*, K_j)$ is K_j-equivariant, while $\mathcal{T}_i^j \circ \chi_i(t_i) \circ (\tau_i^j)^*$ is constant along the fibers of $(\tau_i^j)^*$. Hence, Lemma 5.2 shows that the two bisections commute under $\odot$, and that the product $(\mathcal{T}_i^j \circ \chi_i(t_i) \circ (\tau_i^j)^*) \odot \chi_j(t_j)$ equals the pointwise product. It follows that $\chi_{i,M}(t_i)$ and $\chi_{j,M}(t_j)$ commute and that the product $\chi_{i,M}(t_i) \odot \chi_{j,M}(t_j)$ equals the pointwise product. q.e.d.

We define the Thimm action of $t = (t_n, \ldots, t_1) \in T_n \times \cdots \times T_1$ on M_0 by

$$t \bullet x = \mathcal{A}(\chi_M(t_n, \ldots, t_1))(x).$$

If (M, π) is a Hamiltonian K_n-space, with moment map Φ_n, then the Thimm action of $T_n \times \cdots \times T_1$ on M_0 is Hamiltonian, with moment map

$$(q_n \circ \Phi_n, \ldots, q_1 \circ \Phi_1) \colon M_0 \to \mathfrak{t}_n^* \times \cdots \times \mathfrak{t}_1^*.$$

Here $q_i\colon \mathfrak{k}_i^* \to \mathfrak{t}_{i,+}^* \subset \mathfrak{t}_i^*$ are the unique K_i-invariant maps with $q_i(\mu) = \mu$ for $\mu \in \mathfrak{t}_+^*$. As a special case, the identity map $\Phi\colon \mathfrak{k}_n^* \to \mathfrak{k}_n^*$ gives rise to a Hamiltonian action of $T_{n-1} \times \cdots \times T_1$ on

$$(\mathfrak{k}_n^*)_0 = \bigcap_{i=1}^{n-1} ((\tau_i^n)^*)^{-1}(\mathfrak{k}_{i,reg}^*).$$

(The torus T_n is excluded, since its Thimm action is trivial.)

5.2. Thimm actions for Poisson Lie groups. Let K be a compact Lie group with standard Poisson structure, and $K^*_{\mathrm{reg}} \subset K^*$ the subset of points whose stabilizer under the dressing action of K has maximal rank. Since $e\colon \mathfrak{k}^*_{\mathrm{reg}} \to K^*_{\mathrm{reg}}$ is a K-equivariant diffeomorphism, any K-equivariant map $\Psi\colon M \to K^*$ defines a Thimm T-action, via the composition $e^{-1} \circ \Psi$. Let $\psi \in \Gamma(\mathfrak{k}^*, K)$ be a Ginzburg-Weinstein twist, and $\gamma = e \circ \mathcal{A}(\psi)$. Parallel to Lemma 5.3 we have:

Lemma 5.5. *Suppose M is a Poisson manifold, and $\Psi\colon M \to K^*$ is a moment map for a Poisson Lie group action $\mathcal{A}\colon K \to \mathrm{Diff}(M)$. Then the Thimm T-action on M_0 is Hamiltonian, with moment map*

$$p \circ \Psi\colon M_0 \to \mathfrak{t}^*.$$

Here $p = q \circ e^{-1}\colon K^* \to \mathfrak{t}^*$. If $\Psi = \gamma \circ \Phi$, where $\Phi\colon M \to \mathfrak{k}^*$ is a moment map for a Hamiltonian K-action, then the Thimm actions defined by Φ and Ψ coincide.

Proof. As shown in Proposition 4.5, Φ is the moment map for the twisted action $\mathcal{A}^{\Phi^*\psi}$ on M. Since $\mathcal{A}_{\mathfrak{k}^*}(\psi)$ preserves orbits, $p = q \circ e^{-1} = q \circ \gamma^{-1}$. Thus, $p \circ \Psi = q \circ \Phi$ where $\Phi = \gamma^{-1} \circ \Psi$. Thus, Lemma 5.3 identifies $p \circ \Psi$ as the moment map for the Thimm T-action corresponding to Φ (relative to the *twisted* action $\mathcal{A}^{\Phi^*\psi}$). Since the two K-actions are conjugate under $\mathcal{A}(\Phi^*\psi)$, the same is true for the two Thimm T-actions. But since $\chi_M(t)$ is K-equivariant, Lemma 5.2 shows $\Phi^*\psi \odot \chi_M(t) \circ \Phi^*\psi^{-1} = \chi_M(t)$. Hence, the two Thimm actions coincide. q.e.d.

Suppose (36) is a sequence of homomorphisms of Poisson Lie groups $K_1, \ldots, K_n$, equipped with the standard Poisson structure. Let M be a K_n-manifold, let $\Psi_n\colon M \to K_n^*$ be an equivariant map, and let $\Psi_i\colon M \to K_i^*$ be the composition of Ψ_n with the map $(\mathcal{T}_i^n)^*\colon K_n^* \to K_i^*$. We then obtain commuting Thimm T_i-actions on

$$M_0 = \bigcap_{i=1}^{n} \Psi_i^{-1}(K_{i,\mathrm{reg}}^*).$$

If (M, π) is a Poisson manifold, and Ψ_n is the moment map for a Poisson-Lie group action of K_n, then the Thimm $T_n \times \cdots \times T_1$-action on M_0 is

Hamiltonian, with moment map

$$(p_n \circ \Psi_n, \ldots, p_1 \circ \Psi_1) \colon M_0 \to \mathfrak{t}_n^* \times \cdots \times \mathfrak{t}_1^*.$$

Here $p_i = q_i \circ e_i^{-1}$. In particular, we obtain a Hamiltonian $T_{n-1} \times \cdots \times T_1$-action on

$$(K_n^*)_0 = \bigcap_{i=1}^{n-1} ((\mathcal{T}_i^n)^*)^{-1}(K_{i,\mathrm{reg}}^*).$$

By an inductive application of Theorem 4.7, it is possible to choose Ginzburg-Weinstein twists $\psi_i \in \Gamma(\mathfrak{k}_i^*, K_i)$, with $\psi_i(0) = 1$, which are *compatible* in the sense that the resulting diagram

$$\begin{tikzcd}
\mathfrak{k}_n^* \arrow[r, "\tau_{n-1}^*"] \arrow[d, "\gamma_n"] & \cdots \arrow[r, "\tau_2^*"] & \mathfrak{k}_2^* \arrow[r, "\tau_1^*"] \arrow[d, "\gamma_2"] & \mathfrak{k}_1^* \arrow[d, "\gamma_1"] \\
K_n^* \arrow[r, "\mathcal{T}_{n-1}^*"'] & \cdots \arrow[r, "\mathcal{T}_2^*"'] & K_2^* \arrow[r, "\mathcal{T}_1^*"'] & K_1^*
\end{tikzcd}$$

with $\gamma_i = e_i \circ \mathcal{A}_i(\psi_i)$ commutes.

Proposition 5.6. *For any choice of compatible Ginzburg-Weinstein twists $\psi_i \in \Gamma(\mathfrak{k}_i^*, K_i)$, the map $\gamma_n \colon \mathfrak{k}_n^* \to K_n^*$ intertwines the Thimm $T_{n-1} \times \cdots \times T_1$-actions on $(\mathfrak{k}_n^*)_0$ and $(K_n^*)_0$, as well as their moment maps. The map ψ_n has the following equivariance property under the Thimm action of $t = (t_{n-1}, \ldots, t_1) \in T_{n-1} \times \cdots \times T_1$,*

$$\psi_n(t \bullet \mu) = \tilde{\chi}(t; \mu)\, \psi_n(\mu)\, \chi(t; \mu)^{-1}. \tag{37}$$

Here

$$\chi(t; \mu) = \prod_{i=1}^{n-1} \mathcal{T}_i^n\big(\chi_i(t_i; \mu)\big), \quad \tilde{\chi}(t; \mu) = \prod_{i=1}^{n-1} \mathcal{T}_i^n\Big(\operatorname{Ad}_{\psi_i((\tau_i^n)^*\mu)}\, \chi_i(t_i; \mu)\Big).$$

Proof. For each $i < n$ we obtain commutative diagrams

$$\begin{tikzcd}
\mathfrak{k}_n^* \arrow[r, "(\tau_i^n)^*"] \arrow[d, "\gamma_n"] & \mathfrak{k}_i^* \arrow[r, "q_i"] \arrow[d, "\gamma_i"] & \mathfrak{t}_{i,+}^* \arrow[d, "="] \\
K_n^* \arrow[r, "(\mathcal{T}_i^n)^*"'] & K_i^* \arrow[r, "p_i"'] & \mathfrak{t}_{i,+}^*
\end{tikzcd} \tag{38}$$

It follows that the map γ_n intertwines the moment maps for the actions of $T_{n-1} \times \cdots \times T_1$, as well as the actions themselves. By Theorem 4.8, the commutativity of the diagram (38) implies that the bisection $\hat{\psi}_i^{-1} \odot \psi_n \in \Gamma(\mathfrak{k}_n^*, K_n)$ is K_i-equivariant, and that

$$(\hat{\psi}_i^{-1} \odot \psi_n)(\mu) = \mathcal{T}_i^n\big(\psi_i((\tau_i^n)^*\mu)\big)^{-1}\, \psi_n(\mu).$$

The K_i-equivariance of the bisection $\psi = \hat{\psi}_i^{-1} \odot \psi_n$ implies the Thimm T_i-equivariance,

$$\psi(t_i \bullet \mu) = \mathrm{Ad}_{\mathcal{T}_i^n \circ \chi_i(t_i;\mu)}\, \psi(\mu). \tag{39}$$

Using $(\tau_i^n)^*(t_i \bullet \mu) = (\tau_i^n)^*\mu$, this yields

$$\psi_n(t_i \bullet \mu) = \mathcal{T}_i^n \Big(\mathrm{Ad}_{\psi_i((\tau_i^n)^*\mu)}\, \chi_i(t_i;\mu) \Big)\, \psi_n(\mu)\, \mathcal{T}_i^n \big(\chi_i(t_i;\mu)\big)^{-1},$$

proving (37). q.e.d.

Remark.

a) Throughout this discussion, we can assume that the functions ψ_i take values in the semi-simple part K_i^{ss}.

b) In the presence of anti-Poisson involutions s_{K_i} (of the type discussed in Section 4.4) with $s_{K_{i+1}} \circ \mathcal{T}_i = \mathcal{T}_i \circ s_{K_i}$, one can assume that the maps ψ_i satisfy $s_{K_i} \circ \psi_i = \psi_i \circ s_{\mathfrak{k}_i^*}$. Thus γ_n restricts to a diffeomorphism between the fixed point sets of $s_{\mathfrak{k}_n^*}$ and $s_{K_n^*}$, equivariant for the action of $T'_{n-1} \times \cdots \times T'_1$, where T'_i is the fixed point set of the restriction of s_{K_i} to T_i.

5.3. The $\mathrm{U}(n)$ **Gelfand-Zeitlin system.** Consider the sequence (36) for the special case $K_i = \mathrm{U}(i)$, with the standard choice of maximal tori $T_i = T(i)$, and with $\mathcal{T}_i^j \colon U(i) \to \mathrm{U}(j)$ the inclusions as the upper left corner (extended by 1's along the diagonal). Identifying $\mathfrak{u}(i)^* \cong \mathrm{Herm}(i)$ as above, the standard choice of fundamental Weyl chamber consists of diagonal matrices with decreasing diagonal entries. The maps $(\tau_i^j)^* \colon \mathfrak{u}(j)^* \to \mathfrak{u}(i)^*$ translate into the projection of a Hermitian $j \times j$-matrix onto the ith principal submatrix, and are clearly compatible with these choices of $\mathfrak{t}_{i,+}^*$. As shown by Guillemin-Sternberg [**15**], the Thimm $T_{n-1} \times \cdots \times T_1$-action for the sequence of projections

$$\mathfrak{u}(n)^* \to \cdots \to \mathfrak{u}(2)^* \to \mathfrak{u}(1)^*$$

defines a completely integrable system on $\mathfrak{u}(n)^*$, and coincides with the Gelfand-Zeitlin system described in Section 1.

Let $U(i)$ carry the standard Poisson-Lie group structure corresponding to these choices of T_i, $\mathfrak{t}_{i,+}^*$ and the scalar product $B_i(A', A) = -\operatorname{tr}(A'A)$. The bracket on $\mathfrak{u}(i)^*$ corresponds to its identification with upper triangular matrices, with real diagonal entries. The map

$$(\tau_i^j)^* \colon \mathfrak{u}(j)^* \to \mathfrak{u}(i)^*$$

projects an upper triangular matrix onto the upper left $i \times i$ block, and is easily checked to preserve Lie brackets. Hence, $\mathcal{T}_i^j$ are Poisson-Lie group homomorphisms. The identification

$$\mathrm{U}(i)^* \cong \mathrm{Herm}^+(i)$$

takes the dressing action of $\mathrm{U}(i)$ to the action by conjugation. The maps

$$(\mathcal{T}_i^j)^* \colon \mathrm{U}(j)^* \to \mathrm{U}(i)^*$$

are again identified with projection to the upper left corner, both under the identification with positive definite matrices, and under the identification with the group upper triangular matrices with positive diagonal. The Thimm $T(n-1) \times \cdots \times T(1)$-action for the sequence of maps

$$\mathrm{U}(n)^* \to \cdots \to \mathrm{U}(2)^* \to \mathrm{U}(1)^*$$

is Flaschka-Ratiu's nonlinear Gelfand-Zeitlin system. Let $\psi_i \colon \mathfrak{u}(i)^* \to \mathrm{SU}(i)$ be compatible Ginzburg-Weinstein twists, with $\psi_i(0) = 1$ and $\psi_i(\overline{A}) = \overline{\psi_i(A)}$, and let $\gamma_i \colon \mathfrak{u}(i)^* \to \mathrm{U}(i)^*$ be the corresponding Ginzburg-Weinstein diffeomorphisms. Then $\psi_n \colon \mathfrak{u}(n)^* \to \mathrm{SU}(n)$ has the properties (i)-(iii) listed at the end of Section 2. This finally completes the proof of Theorems 1.1, 1.2, and 1.3. Furthermore, from the uniqueness properties of ψ_n (Theorem 1.3), and since γ_n is a Poisson map by construction, Theorem 1.4 now comes for free.

Remark. While all the arguments in this paper were carried out in the C^∞-category, we could equally well have worked in the C^ω-category of real-analytic maps. In particular, the distinguished 2-form $\sigma \in \Omega^2(\mathfrak{k}^*)$ from Section 4.2 is real-analytic, by the explicit formula given in [**3**]. It follows that the distinguished Ginzburg-Weinstein twist ψ for $\mathrm{U}(n)$ is not only smooth, but is in fact real-analytic.

5.4. Other classical groups. We conclude with some remarks on Gelfand-Zeitlin systems for the other classical groups. Consider first the special orthogonal groups $\mathrm{SO}(n)$, with the standard choice of maximal tori. Guillemin-Sternberg's construction for the series of inclusions

$$\mathrm{SO}(2) \to \mathrm{SO}(3) \to \cdots$$

produces a Gelfand-Zeitlin torus action over an open dense subset of each Poisson manifold $\mathfrak{so}(n)^*$ (not to be confused with the real locus of $\mathfrak{u}(n)^*$, which does not carry a Poisson structure). A dimension count confirms that this defines a completely integrable system. On the other hand, for the symplectic groups the series of inclusions

$$\mathrm{Sp}(1) \to \mathrm{Sp}(2) \to \cdots$$

does not yield a completely integrable system, since the Gelfand-Zeitlin torus does not have sufficiently large dimension. (By a more sophisticated construction, Harada [**16**] was able to obtain additional integrals of motion in this case.) Consider now the standard Poisson structures on the groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$. Unfortunately, the inclusions $\mathrm{SO}(i) \to \mathrm{SO}(i+1)$ are *not* Poisson Lie group homomorphisms, essentially due to the fact that the Dynkin diagram of $\mathrm{SO}(i)$ is not a subdiagram of that of $\mathrm{SO}(i+1)$. However, the inclusions $\mathrm{SO}(i) \to \mathrm{SO}(i+2)$ are Poisson Lie group homomorphisms, and so are the inclusions $\mathrm{Sp}(i) \to \mathrm{Sp}(i+1)$. By the same discussion as for the unitary groups, one obtains Ginzburg-Weinstein diffeomorphisms $\mathfrak{so}(n)^* \to \mathrm{SO}(n)^*$ (resp. $\mathfrak{sp}(n)^* \to \mathrm{Sp}(n)^*$) intertwining the resulting (partial) Gelfand-Zeitlin systems. However, in contrast to the unitary groups, there is no simple uniqueness statement in these cases.

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