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STABLE BRANCHED MINIMAL IMMERSIONS WITH PRESCRIBED BOUNDARY

Leon Simon & Neshan Wickramasekera

Abstract

We describe a method for producing smooth 2-valued minimal graphs over the cylindrical region $(D \setminus \{0\}) \times \mathbb{R}^{n-2}$, where D is the disk in $\mathbb{R}^2$, subject to given continuous 2-valued boundary data on $\partial D \times \mathbb{R}^{n-2}$. Subject to appropriate symmetry assumptions, the construction produces branched minimal immersions in $D \times \mathbb{R}^{n-2} \times \mathbb{R}$ with prescribed boundary and branching at every point of $\{0\} \times \mathbb{R}^{n-2}$, and we also discuss the nature of the possible singularities along $\{0\} \times \mathbb{R}^{n-2}$ in case of general boundary data.

Introduction

Recently the second author ([**Wic04**], [**Wic05**]) has established a regularity and compactness theory for stable branched minimal immersions near points of density less than 3. The work in [**Wic05**] in fact considers a class of immersed minimal hypersurfaces in an open ball $B \subset \mathbb{R}^{n+1}$ which are assumed to have no boundary in B and be immersed away from a set of $K \subset B$ which is relatively closed in B and which has finite $(n-2)$-dimensional Hausdorff measure; K is to be thought of as the singular set, including the branch points if any exist, and one of the main theorems of [**Wic05**] asserts that, near singular points having density not much larger than two, K breaks up into a set of dimension $\leq n-7$ (empty for $n \leq 6$ and discrete for $n = 7$) of genuine singularities and a "branching set" of dimension $\leq n-2$, at each point of which there is a tangent plane of multiplicity 2.

The question therefore naturally arises as to the size of the class of such branched stable immersions. We here present a method which shows that in fact there is a very rich class of such hypersurfaces, each having a branching set equal to an $(n-2)$-dimensional $C^{1,\alpha}$ submanifold for some $\alpha \in (0, 1)$. Indeed one of the main results here (Theorem 2

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of §2) establishes the existence of stable $C^{1,\alpha}$ branched minimal immersions Φ from the cylinder $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : |x| < 1\}$ into $\mathbb{R}^{n+1}$ having prescribed boundary data which is required to have a $\mathbb{Z}_k$ symmetry for some odd $k \geq 3$ but which is otherwise arbitrary bounded continuous; Φ inherits the $\mathbb{Z}_k$ symmetry and has branch points at $(0, y)$ for each $y \in \mathbb{R}^{n-2}$ (so that the actual geometric branch set in the image is the embedded $C^{1,\alpha}$ submanifold $\{\Phi(0, y) : y \in \mathbb{R}^{n-2}\}$).

The case $n = 2$ (when there are no y variables and the examples under consideration have isolated branch points) is also of interest, and appears to be new, although in the case $n = 2$ other techniques for generating branched minimal immersions with isolated branch points are available—for example modifications of the method [**CHS84**] can be used to prove quite general existence theorems which complement the result for symmetric data proved here. The precise conclusion in the case $n = 2$ for symmetric boundary data is given in Corollary 1 of §2.

The proof of Theorem 2 involves construction of a $C^{1,\alpha}(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$ function u_0 as the solution, with prescribed bounded continuous boundary data φ (not necessarily with any symmetry properties in the first instance), of the Euler-Lagrange equation of the (degenerate) functional $\mathcal{F}_0$ (introduced in §1) which maps to the non-parametric area functional under the transformation $T : (re^{i\theta}, y) \mapsto (r^2 e^{2i\theta}, y)$; composition with the inverse transformation takes the single-valued function u_0 to the 2-valued function $u(re^{i\theta}, y) = u_0(r^{1/2}e^{i\theta/2}, y)$, $0 \leq \theta < 4\pi$ (i.e., $u(re^{i\theta}, y) = u_0(\pm r^{1/2}e^{i\theta/2}, y)$, $0 \leq \theta < 2\pi$), and the map Φ is just the map that takes the cylinder $\overline{\mathcal{C}}$ to the graph of the 2-valued function u (explicitly: $\Phi(re^{i\theta}, y) = (re^{i\theta}, y, u_0(r^{1/2}e^{i\theta/2}, y)) = (re^{i\theta}, y, u_0(\pm r^{1/2}e^{i\theta/2}, y)))$.

There is some subtlety involved in checking that the graph of u so obtained is stationary and C^1, and for this some varifold theory is needed—this is where the symmetry condition on φ is used. The discussion in §§2, 3 (in particular Theorem 1, Theorem 3 and Corollary 2) also more or less fully illuminates what happens in general when no symmetry condition on φ is assumed. As discussed in Theorem 3 and Corollary 2, in this case u_0 may have discontinuities and it may not be true that the graph of u is stationary with respect to first variation of area, because the closure of the graph in this case has "vertical pieces" (open regions in the $(n-1)$-dimensional plane $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$) which introduce a varifold boundary and negate the stationarity of the graph.

In §4 we discuss extension of the main results to the case of q-valued (rather than 2-valued) solutions, i.e., branch points of order q rather than of order 2. The main results are given in Theorems 4, 5, which include Theorems 2, 3 as the special case when $q = 2$.

1. The Initial Functional $\mathcal{F}_0$

For $n \ge 2$ we first study the functional which transforms to the non-parametric area functional under the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which takes $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ to $(x_1^2 - x_2^2, 2x_1x_2, y)$. Identifying $x = (x_1, x_2)$ with $x_1 + ix_2$, we can write $x = re^{i\theta}$, $r = \sqrt{x_1^2 + x_2^2}$, and $T(re^{i\theta}, y) = (r^2e^{2i\theta}, y)$. Thus we study the functional

$$\mathcal{F}_0(v) = \int_\Omega 4r^2\sqrt{1 + (4r^2)^{-1}|D_xv|^2 + |D_yv|^2}\,dxdy,$$

Here and subsequently we use the notation that Ω is a bounded open subset of the cylinder

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : |x| < 1\},$$

$D_xv = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2})$, and $D_yv = (\frac{\partial v}{\partial y_1}, \dots, \frac{\partial v}{\partial y_{n-2}})$. (Note that in case $n = 2$ we have $\Omega \subset \mathcal{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ and D_yv is absent from the functional.)

$\mathcal{F}_0$ is of course a degenerate functional, but we can approximate by non-degenerate functionals of the form

$$\mathcal{F}_\delta(v) = \int_\Omega 4r_\delta^2\sqrt{1 + (4r_\delta^2)^{-1}|D_xv|^2 + |D_yv|^2}\,dxdy, \tag{1.1}$$

where, for $\delta \in (0, \frac{1}{2})$, r_δ is a smooth function of the variables $x = (x_1, x_2)$ with $r_\delta \equiv r$ for $r \ge \delta$ and $\delta \ge r_\delta \ge \delta/2$ for $r \in [0, \delta)$. (We'll first prove existence properties for $\mathcal{F}_\delta$ and then let $\delta \downarrow 0$.)

The Euler-Lagrange equation for the functional $\mathcal{F}_\delta$ is

$$\begin{aligned}\sum_{i=1}^{2} D_{x_i}&\left(\frac{D_{x_i}v}{\sqrt{1 + (4r_\delta^2)^{-1}|D_xv|^2 + |D_yv|^2}}\right)\\ &+ 4r_\delta^2\sum_{i=1}^{n-2} D_{y_i}\left(\frac{D_{y_i}v}{\sqrt{1 + (4r_\delta^2)^{-1}|D_xv|^2 + |D_yv|^2}}\right) = 0,\end{aligned} \tag{1.2}$$

which is a quasilinear elliptic equation, and which can be written in weak form

$$\begin{aligned}\int_{\mathcal{C}}&\left(\sum_{i=1}^{2}\frac{D_{x_i}vD_{x_i}\zeta}{\sqrt{1 + (4r_\delta^2)^{-1}|D_xv|^2 + |D_yv|^2}}\right.\\ &\left.+ 4r_\delta^2\sum_{i=1}^{n-2}\frac{D_{y_i}vD_{y_i}\zeta}{\sqrt{1 + (4r_\delta^2)^{-1}|D_xv|^2 + |D_yv|^2}}\right)dxdy = 0, \quad \zeta \in C_c^1(\mathcal{C}).\end{aligned} \tag{1.3}$$

Let $\varphi = \varphi(x, y) : \partial\mathcal{C} \to \mathbb{R}$ be an arbitrary Lipschitz function which is ρ_j-periodic in the variable y_j for some $\rho_j > 0$ and each $j = 1, \dots, n-2$ (the periodicity is imposed for technical convenience and will be removed at the end of this section by letting the length of the period approach ∞), and suppose that u_δ is a $C^2(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$ solution of the Euler-Lagrange equation for $\mathcal{F}_\delta$ which is also ρ_j-periodic in the variable y_j for each $j = 1, \ldots, n-2$, and which attains the boundary values φ on $\partial\mathcal{C}$. (Thus in the case $n = 2$, when there are no variables y_j, u_δ is just a $C^2(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ solution of the Euler-Lagrange equation on the disk $\mathcal{D} = \{x = (x_1, x_2) : |x| < 1\}$ with $u_\delta = \varphi$ on $\partial\mathcal{D}$.)

We claim that for each $n \geq 2$ such u_δ exists by virtue of the gradient estimates [**Sim76**] and standard elliptic theory, and in addition that u_δ is smooth, is continuous up to the boundary $\partial\mathcal{C}$, and has globally bounded derivatives on $\mathcal{C}$ with respect to the variables $y_1, \ldots, y_{n-2}$, as follows:

In fact there is a well established theory for solutions u of quasilinear elliptic equations which arise as the Euler-Lagrange equation of functionals of the form $\int_\Omega F(x, u, Du)\, dx$, where x denotes the independent variables in the given domain $\Omega \subset \mathbb{R}^n$, and where $F(x, t, p)$ is a given smooth function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ which is locally uniformly convex with respect to the variable p. In the present instance we use notation $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ (rather than $x \in \mathbb{R}^n$) for the independent variables $\in \Omega \subset \mathcal{C}$, and $F(x, y, t, p) = 4r_\delta^2 \sqrt{1 + (4r_\delta^2)^{-1}|p_x|^2 + |p_y|^2}$, independent of the variable t, where $p = (p_1, \ldots, p_n)$, $p_x = (p_1, p_2)$, $p_y = (p_3, \ldots, p_n)$. In this case (as in all cases when the integrand $F(x, t, p)$ does not depend on t), we have that any given $C^2(\mathcal{C})$ solution $v = u_\delta$ of (1.2) satisfies a strong maximum principle:

(1.4)
$$v \text{ cannot attain a maximum/minimum in } \mathcal{C} \text{ unless it is constant,}$$

and also the *difference* $v_1 - v_2$ of any two $C^2(\mathcal{C})$ solutions also satisfies a strong maximum principle:

(1.5)
$$v_1 - v_2 \text{ cannot attain a maximum/minimum in } \mathcal{C} \text{ unless it is constant.}$$

We now focus attention on $C^2(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$ solutions $v(x, y) = u_\delta(x, y)$ of (1.2) such that

$$v(x, y) \text{ is periodic with some period } \rho_j > 0 \text{ in each variable } y_j,\ j = 1, \ldots, n-2, \tag{1.6}$$

and we observe that by applying (1.5) to $v_1(x, y) = v(x, y)$ and $v_2(x, y) = v(x, y + h)$ ($h \in \mathbb{R}^{n-2}$ an arbitrary fixed vector), such a solution v satisfies

(1.7)
$$\sup_{(x,y),(x,z)\in\mathcal{C}, y\neq z} \frac{|v(x, y) - v(x, z)|}{|y - z|} \leq \sup_{|x|=1, y,z\in\mathbb{R}^{n-2}, y\neq z} \frac{|v(x, y) - v(x, z)|}{|y - z|}.$$

For the moment we assume that the boundary data $\varphi \equiv v|\partial\mathcal{C}$ is Lipschitz in the y variables, uniformly with respect to $x \in S^1$; that is we assume that there is $L > 0$ such that

(1.8)
$$\begin{cases} \sup_{x\in S^1} |\varphi(x,y) - \varphi(x,z)| \le L|y-z|, \quad y,z \in \mathbb{R}^{n-2}, \\ \varphi(x,y) \text{ is periodic in the variable } y_j \text{ with period } \rho_j,\ j = 1,\ldots,n-2, \end{cases}$$

which means that (1.7) implies

$$\sup_{\mathcal{C}} |D_y v| \le L \tag{1.9}$$

for any $C^2(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$ solution $v = u_\delta$ of (1.2) which satisfies the periodicity conditions (1.6).

Next we observe that for solutions v of (1.2) which satisfy (1.9) we can check the structural conditions 1.1, 1.2, 1.3, 1.4 of [**Sim76**] with structural constants $\beta_j = \beta_j(\delta, n, L)$ and structural functions $\underline{\mu} = \beta\,(1+|Du|^2)^{-1}$, $\beta = \beta(\delta, n, L)$, and $\overline{\mu}, \lambda, \Lambda$ constants depending on δ, n, L, and the dependence on δ, σ can be dropped in favor of a dependence on σ alone if we restrict points (x,y) with $|x| > \sigma \ge \delta$; hence by [**Sim76**, Theorem 1] we have the interior gradient estimates

$$\begin{cases} \sup_\Omega |Dv| \le C(\delta,\sigma,n,L) \\ \sup_{\Omega\setminus\{(x,y):|x|<\sigma\}} |Dv| \le C(\sigma,n,L) \end{cases} \tag{1.10}$$

on any domain $\Omega \subset \mathcal{C}$, and for any $\sigma \in (\delta, 1/2)$, provided $\operatorname{dist}(\Omega, \partial\mathcal{C}) \ge \sigma > 0$. Then standard regularity theory for uniformly elliptic quasilinear equations gives us for each $\ell = 1, 2, \ldots$ that

$$\begin{cases} \sup_\Omega |D^\ell v| \le C(\ell,\delta,\sigma,n,L) \\ \sup_{\Omega\setminus\{(x,y):|x|<\sigma\}} |D^\ell v| \le C(\ell,\sigma,n,L). \end{cases} \tag{1.11}$$

Also, assuming that $\sup_{\partial\mathcal{C}}(|D\varphi| + |D^2\varphi|) \le R$, and keeping in mind that $v(r^{1/2}e^{i\theta/2}, y)$ is a solution of the minimal surface equation (MSE) for $r \in (\delta, 1)$ and $\alpha < \theta < \alpha + \pi$ ($\alpha \in [0, 2\pi)$ arbitrary) we can use standard local barrier constructions for solutions of the MSE to prove that if v is a $C^2(\overline{\mathcal{C}})$ solution of (1.2) which satisfies (1.6), then, there is a boundary gradient estimate $\sup_{\partial\mathcal{C}} |Dv| \le C(R)$, and in this case [**Sim76**, Theorem 1] gives gradient bounds up to the boundary of $\mathcal{C}$:

$$\begin{cases} \sup_{\overline{\mathcal{C}}} |Dv| \le C(\delta,n,L,R) \\ \sup_{\overline{\mathcal{C}}\cap\{(x,y):|x|>\sigma\}} |Dv| \le C(\sigma,n,L,R) \end{cases} \tag{1.10$'$}$$

and there are then also versions of the bounds as in (1.11) up to the boundary: For any $\ell \ge 1$

$$\begin{cases} \sup_{\overline{\mathcal{C}}} |D^\ell v| \le C(\ell,\delta,n,\rho_1,\ldots,\rho_{n-2},L,R) \\ \sup_{\{(x,y)\in\overline{\mathcal{C}}:|x|>\sigma\}} |D^\ell v| \le C(\ell,\sigma,n,\rho_1,\ldots,\rho_{n-2},L,R), \end{cases} \tag{1.11$'$}$$

assuming that $\sigma \in (\delta, 1/2)$ and $\sup_{\partial\mathcal{C}} \sum_{j=1}^{\ell+1} |D^j\varphi| \le R$.

We can therefore apply the Leray-Schauder existence theory as in [**GT83**] (working in the Banach space of $C^{1,\alpha}(\overline{\mathcal{C}})$ functions which are periodic in the y variables with the given periods ρ_j as in (1.6)) in order to conclude that we have a $C^{2,\alpha}(\overline{\mathcal{C}})$ solution v of (1.2) which is periodic in the y variables as in (1.6) and which has boundary data φ. If φ is merely Lipschitz with Lipschitz constant L with respect to the y-variables (and still periodic with respect to the y variables) then we can approximate φ uniformly on $\partial\mathcal{C}$ by a sequence φ_k of smooth functions each periodic in the y variables and with Lipschitz constant L with respect to the y variables, and then use (1.5), (1.10), (1.11) to assert that the corresponding sequence v_k of solutions converges in the C^2 sense locally in $\mathcal{C}$ and uniformly with respect to the sup norm on $\overline{\mathcal{C}}$ to a $C^2(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$ solution u_δ of (1.2) with $u_\delta|\partial\mathcal{C} = \varphi$, and with u_δ satisfying also (1.9).

Finally, using the interior estimates (1.10), (1.11) and the local boundary continuity estimates for the MSE (which follows from the existence of local boundary barriers for solutions of the MSE, as already used above in establishing (1.10′)), we deduce that as $\delta \downarrow 0$ a subsequence of the solutions u_δ converges in the C^2 norm on $\{(x,y) : \sigma < |x| < 1-\sigma\}$ and uniformly on $\{(x,y) : \sigma < |x| \le 1\}$, for each $\sigma \in (0,1/2)$, to a function u_0, where

(1.12)
$$\begin{cases} u_0 \in C^\infty(\mathcal{C} \setminus (\{0\} \times \mathbb{R}^{n-2})) \cap C^0(\overline{\mathcal{C}} \setminus (\{0\} \times \mathbb{R}^{n-2})). \\ u_0|\partial\mathcal{C} = \varphi \\ u_0(x,y) \text{ is periodic in variable } y_j \text{ with period } \rho_j,\ j = 1,\dots,n-2. \\ \sup_{0<|x|\le 1}|u_0(x,y) - u_0(x,z)| \le L|y-z|,\ y,z \in \mathbb{R}^{n-2},\ L \text{ as in (1.8).} \\ u_0 \text{ satisfies the Euler-Lagrange equation for } \mathcal{F}_0 \text{ on } \mathcal{C} \setminus (\{0\} \times \mathbb{R}^{n-2}). \end{cases}$$

In case the boundary data φ is merely bounded ($|\varphi| < M$ for some constant M) and continuous (rather than Lipschitz and periodic as in (1.8)) we can still approximate φ by smooth functions φ_k which are periodic in the y variables with periods $\rho_1 = \rho_2 = \cdots = \rho_{n-2} \to \infty$ and which converge uniformly to φ on each compact subset of $\partial\mathcal{C}$. Then by (1.12) we have a corresponding sequence of C^∞ solutions $u_0^{(k)}$. Using the fact that these transform (via the transformation $T(re^{1\theta}, y) = (r^2 e^{i\theta}, y)$) to 2-valued smooth solutions of the MSE which can be written as the union of two single-valued smooth solutions on each slit domain $\Omega_{\theta_0} = (\mathcal{D} \setminus \{\lambda e^{i\theta_0} : \lambda \ge 0\}) \times \mathbb{R}^{n-2}$, we can use the standard interior estimates for the gradient of solutions of the miminal surface equation to argue that we have uniform estimates $\sup_{\sigma<|x|<\rho} |Du_0^{(k)}| \le C(\sigma,\rho,M)$ for any $\sigma,\rho \in (0,1)$ with $\sigma < \rho$, where $M = \sup_{\partial D \times \mathbb{R}^{n-2}} |\varphi|$. This means in particular that we still have a Lipschitz estimate $|D_y u_0^{(k)}| \le L_\rho$, independent of k for the solutions $u_0^{(k)}$ on the domain $\mathcal{C}_\rho = \{(x,y) : 0 < |x| < \rho, y \in \mathbb{R}^{n-2}\}$, for any $\rho \in (0,1)$, and so we can repeat all the arguments leading to (1.12) on the domain $\mathcal{C}_\rho$. (Technically we are thus applying the previous discussion to the functions $\rho^{-1}u_0^{(k)}(\rho x, \rho y)$.) We can also use local barriers for solutions of the MSE near boundary points (cf. the argument leading to (1.10′)) to establish continuity estimates for $u_0^{(k)}$ at boundary points which are uniform with respect to k. Thus by passing to the limit after selecting a suitable subsequence of $u_0^{(k)}$, we get a limit function which is continuous on $\overline{\mathcal{C}} \setminus \{0\} \times \mathbb{R}^{n-2}$ and which satisfies analogous estimates to those of (1.12) on $\mathcal{C}_\rho$ for each $\rho < 1$, except for the periodicity in the y variables.

Specifically, if φ is merely bounded and continuous on ∂C, then there is a solution u_0 on $\mathcal{C} \setminus (\{0\} \times \mathbb{R}^{n-2})$ with

(1.12′)
$$\begin{cases} u_0 \in C^\infty(\mathcal{C}_\rho \setminus (\{0\} \times \mathbb{R}^{n-2})) \cap C^0(\overline{\mathcal{C}} \setminus (\{0\} \times \mathbb{R}^{n-2})), \quad \rho \in (0,1) \\ \sup_{\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^{n-2}:\sigma<|x|<\rho\}} |D^\ell u_0| \le C(n,\sigma,\rho,\ell),\ 0<\sigma<\rho<1,\ \forall \ell \\ u_0 | \partial\mathcal{C} = \varphi \\ \sup_{0<|x|\le\rho} |u_0(x,y) - u_0(x,z)| \le L_\rho |y-z|, \quad y,z \in \mathbb{R}^{n-2},\ \forall \rho \in (0,1) \\ u_0 \text{ satisfies the Euler-Lagrange equation for } \mathcal{F}_0 \text{ on } \mathcal{C} \setminus (\{0\} \times \mathbb{R}^{n-2}). \\ \text{If } \ell \in \{2,3,\ldots\} \text{ and } \varphi \circ S_\ell = \varphi, \text{ then } u_0 \circ S_\ell = u_0 \text{ also.} \end{cases}$$

In the last property $S_\ell(re^{i\theta}, y) = (re^{i(\theta + 2\pi/\ell)}, y)$, and this last property follows from the fact that if $\varphi \circ S_\ell = \varphi$ then the smooth periodic approximations of φ can be chosen to have the same invariance, and hence the $u_0^{(k)}$ have this invariance also, because, by virtue of the maximum principle (1.5), the Euler-Lagrange equation for each functional $\mathcal{F}_\delta$ has a unique solution subject to smooth data on $\partial\mathcal{C}$ which is periodic in the y-variables.

By construction, the functional $\mathcal{F}_0$ transforms to the area functional $\mathcal{A}$ in any region $\Omega \subset \mathcal{C} \setminus \{(0,0)\} \times \mathbb{R}^{n-2}$ where the transformation

(1.13) $$T : (x,y) \mapsto (x_1^2 - x_2^2, 2x_1x_2, y) \quad \text{i.e., } T : (re^{i\theta}, y) \mapsto (r^2 e^{2i\theta}, y)$$

is 1:1. Thus the relation

(1.14) $$u = u_0 \circ T^{-1}$$

defines a 2-valued function on $\overline{\mathcal{C}} \setminus \{(0,0)\} \times \mathbb{R}^{n-2}$ such that if Ω_{θ_0} is any one of the "slit domains" $\mathcal{C} \setminus (\{\lambda e^{i\theta_0} : \lambda \ge 0\} \times \mathbb{R}^{n-2})$, where $\theta_0 \in [0, 2\pi)$, and if

$$\begin{cases} T_1 = T | \{(re^{i\theta}, y) : 0 < r < 1,\ \theta \in (\theta_0/2, \theta_0/2 + \pi)\} \\ T_2 = T | \{(re^{i\theta}, y) : 0 < r < 1,\ \theta \in (\theta_0/2 - \pi, \theta_0/2)\}, \end{cases}$$

then
(1.15)

$$\begin{cases} u_j = u_0 \circ T_j^{-1} \text{ is a } C^2(\Omega_{\theta_0}) \text{ solution of the MSE, } j = 1, 2, \\ \sup_{\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^{n-2}:\sigma<|x|<\rho\}}|D^\ell u_j| \le C(n,\sigma,\rho,\ell),\ 0<\sigma<\rho<1,\ \forall\ell \\ \operatorname{graph} u|\Omega_{\theta_0} = \operatorname{graph} u_1 \cup \operatorname{graph} u_2, \\ |u_j(x,y) - u_j(x,z)| \le L_\rho|y-z|,\ 0<|x|\le\rho<1, \\ y,z\in\mathbb{R}^{n-2},\ j=1,2. \end{cases}$$

Notice that so far we say nothing of what happens at $r = 0$, and that is the essential issue which we analyze in the next section.

2. Main Results

Here u_0 is the $C^\infty(\mathcal{C} \setminus (\{(0,0)\} \times \mathbb{R}^{n-2})) \cap C^0(\overline{\mathcal{C}} \setminus (\{(0,0)\} \times \mathbb{R}^{n-2}))$ solution of the Euler-Lagrange equation for $\mathcal{F}_0$, constructed as in §1 above. Thus u_0 has prescribed bounded continuous boundary values φ and u_0 satisfies the conditions (1.12′), and $u(re^{i\theta}, y) = u_0(r^{1/2}e^{i\theta/2}, y)$ is the corresponding 2-valued solution of the MSE as in (1.13)–(1.15).

Here and subsequently we use the following notation: G is the graph of u; thus G is covered by the map
(2.1)

$$\Phi(re^{i\theta}, y) = (re^{i\theta}, y, u_0(r^{1/2}e^{i\theta/2}, y)), \quad \theta\in\mathbb{R},\ r\in(0,1],\ y\in\mathbb{R}^{n-2},$$

which is a minimal immersion into $(\mathcal{C}\times\mathbb{R})\setminus(\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})$ with period 4π in θ, and G decomposes, over any slit domain $\Omega_{\theta_0} = \mathcal{C} \setminus (\{\lambda e^{i\theta_0} : \lambda \ge 0\} \times \mathbb{R}^{n-2})$ (where $\theta_0 \in [0, 2\pi)$ is given) into the union of a unique pair of smooth minimal graphs, as in (1.15). Of course then geometric quantities like the second fundamental form of G (which we denote by A_G) and the upward pointing unit normal of G (which we denote by $\nu = (\nu_1, \ldots, \nu_{n+1})$) are well defined smooth quantities on G when G is viewed as an immersion into $(\mathcal{C} \times \mathbb{R}) \setminus (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$. We also have the Jacobi field equation

$$\Delta_G\nu_{n+1} + |A_G|^2\nu_{n+1} = 0 \text{ on } G \tag{2.2}$$

for the $(n+1)$'st component ν_{n+1} of the upward pointing unit normal ν.

The first main theorem we prove here is as follows:

Theorem 1. *With u_0 as in (1.12′), the following 3 properties are all equivalent:*

(i) *u_0 extends across $\{0\} \times \mathbb{R}^{n-2}$ to give a continuous function $\overline{u}_0 \in C^0(\overline{\mathcal{C}})$.*
(ii) *$\mathcal{H}^{n-2}(\overline{G} \cap (\{0\} \times K \times \mathbb{R})) < \infty$ for any compact $K \subset \mathbb{R}^{n-2}$, and G is stable in the sense that the stability inequality $\int_G |A_G|^2\zeta^2\,d\mathcal{H}^n \le$*

$\int_G |\nabla_G \zeta|^2 \, d\mathcal{H}^n$ *holds for all functions* $\zeta \in C^1((\overline{\mathcal{C}} \times \mathbb{R}) \setminus (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}))$ *of bounded support* $\subset \{(x, y, t) : |x| < \sigma\}$ *for some* $\sigma < 1$.

(iii) $\sup_{|x|<\sigma, y \in \mathbb{R}^{n-2}} |Du| \leq C = C(n, M, \sigma) \, (< \infty)$, *and* Du *is uniformly Hölder continuous as a* 2*-valued function on* $\{(x, y) : 0 < |x| < \sigma\} \; \forall \, \sigma < 1$, *in the sense that if* Ω_{θ_0} *is any one of the slit domains as in* (1.15) *then, for each* $\sigma \in (0, 1)$, *each* Du_j *is uniformly Hölder continuous on* $\{(re^{i\theta}, y) \in \Omega_{\theta_0} : |e^{i\theta} - e^{i\theta_0}| > 1 - \sigma, 0 < r < \sigma, y \in \mathbb{R}^{n-2}\}$, *with exponent* $\alpha = \alpha(n, M, \sigma) \in (0, 1)$ *and Hölder coefficient* $\leq C = C(n, M, \sigma)$. *Here* u_j *are as in* (1.15) *and* M *is any upper bound for* $\sup_{\partial \mathcal{C}} \varphi$.

Remarks.

(1) Notice that the above theorem guarantees that if u_0 extends across $\{0\} \times \mathbb{R}^{n-2}$ to give a continuous function $\overline{u}_0$, then the closure of G in $\mathcal{C} \times \mathbb{R}$ is a $C^{1,\alpha}$ stable branched minimal immersion, with the branched immersion being given explicitly by the covering map $\Phi(re^{i\theta}, y) = (re^{i\theta}, y, \overline{u}_0(r^{1/2} e^{i\theta/2}, y))$ for $0 \leq r < 1$ and $\theta \in \mathbb{R}$, which is 4π-periodic in the θ variable.

(2) Of course (iii) trivially implies (i), so to prove the theorem it will be enough to show (i) $\iff$ (ii) and (i)$\Rightarrow$(iii), and this is what we shall do below.

(3) We should remark that in fact (i)$\Rightarrow$(iii) is a direct consequence of the general regularity theory established in [**Wic05**], but the proof in the present context is much simpler and we include it as part of the proof of Theorem 1.

For the second main theorem we need to assume the $\mathbb{Z}_k$ symmetry mentioned in the introduction. The main result is then as follows:

Theorem 2. *If* u_0 *is as in* (1.12′) *with bounded continuous boundary data* φ *satisfying the* $\mathbb{Z}_k$ *symmetry condition* $\varphi \circ S_k = \varphi$ *for some odd* $k \geq 3$, *where* $S_k(e^{i\theta}, y) = (e^{i(\theta + 2\pi/k)}, y)$, *then* (i), (ii), (iii) *of Theorem* 1 *hold, with the additional conclusion* (*in addition to* (iii)) *that*

$$\sup_{0<|x|<\sigma, y \in \mathbb{R}^{n-2}} |x|^{-\alpha} |D_x u(x, y)| \leq C,$$

where $\alpha = \alpha(k, n, \sigma, M) \in (0, 1/2)$ *and* $C = C(k, n, \sigma, M) > 0$, *with* M *any upper bound for* $\sup_{\partial \mathcal{C}} |\varphi|$. *In particular, the closure of* G *in* $\mathcal{C} \times \mathbb{R}$ *is a* $C^{1,\alpha}$ *branched immersion, with the branched immersion being given explicitly by the covering map* (2.1) *which is* 4π*-periodic in the* θ *variable and which has boundary values at* $r = 1$ *equal to* $(e^{i\theta}, y, \varphi(e^{i\theta/2}, y))$.

Remark. Notice that in terms of the single-valued function u_0, the gradient estimate of the above theorem is equivalently written

$$\sup_{0<|x|<\sigma^2, \, y \in \mathbb{R}^{n-2}} |x|^{-1-2\alpha} |D_x u_0(x, y)| \leq C,$$

with the same constants $\alpha = \alpha(k, n, \sigma, M), C = C(k, n, \sigma, M)$.

In the particular case $n = 2$, we have the following:

Corollary 1. *If $n = 2$ and if $\varphi : S^1 \to \mathbb{R}$ is continuous and has the symmetry $\varphi(e^{i\theta}) \equiv \varphi(e^{i(\theta+2\pi/k)})$ for some odd integer $k \geq 3$, then u_0 in (1.12$'$) extends to a continuous map $\overline{\mathcal{D}} \to \mathbb{R}$ such that $\Phi : re^{i\theta} \mapsto (re^{i\theta}, u_0(r^{1/2}e^{i\theta/2})), 0 \leq r < 1, \theta \in \mathbb{R}$, is a $C^{1,\alpha}$ covering map (with period 4π) for a stable branched minimal immersion of the unit disk into $\mathbb{R}^3$ with prescribed boundary values $(e^{i\theta}, \varphi(e^{i\theta/2}))$ and a branch point at 0 (and no other branch points), and $\sup_{0<|x|<\sigma} |x|^{-1-2\alpha}|D_x u_0(x)| \leq C$. Here $\alpha = \alpha(k, n, M, \sigma) \in (0, 1/2)$ and $C = C(k, n, M, \sigma)$, with M any upper bound for $\sup_{S^1} |\varphi|$.*

The following result, needed in the proof of Theorem 2 and of independent interest, further analyzes the local structure of the graph G over points which are close to a discontinuity of u_0.

Theorem 3. *Suppose u_0, as in (1.12$'$), is discontinuous at some point $(0, y_0) \in \{0\} \times \mathbb{R}^{n-2}$, and $\rho_0 \in (0, \frac{1}{4}]$. Then there is a $\rho_1 \in (0, \rho_0]$ and a point $(0, y_1, t_1) \in B_{\rho_0}(0, y_0) \times \mathbb{R}$ such that $B_{\rho_1}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}) \subset \overline{G}$, G (as an n-dimensional integer multiplicity varifold in $\mathbb{R}^{n+1}$) has a unique tangent cone $\mathbb{C}$ at $(0, y_1, t_1)$ of the form*

$$\mathbb{C} = |H_1| + |H_2|,$$

where H_1, H_2 are distinct n-dimensional half-spaces meeting at angle $\neq \pi$ along the common boundary $\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R}$, $|H_j|$ is the multiplicity 1 varifold corresponding to H_j, and

$$\overline{G} \cap B_{\rho_1}(0, y_1, t_1) = L_1 \cup L_2,$$

where each L_j is an embedded C^∞ manifold-with-boundary, with boundary (in the open ball $B_{\rho_1}(0, y_1, t_1)$) $\partial L_j = B_{\rho_1}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})$, L_j has the tangent half-space H_j at the point $(0, y_1, t_1)$, and $(L_1 \setminus \partial L_1) \cap (L_2 \setminus \partial L_2) = \emptyset$.

Remarks.

(1) If the boundary data φ is S_k invariant (i.e., $\varphi \circ S_k = \varphi$) then by the last identity in (1.12$'$) the graph G is also S_k invariant, and hence so is the tangent cone $\mathbb{C}$ of the above theorem. But then $H_1, H_2, S_k(H_1), S_k(H_2)$ consists of at least 3 distinct half-spaces, and so $\mathbb{C}$ is not S_k invariant, a contradiction. That is if φ is S_k invariant then u_0 extends across $\{0\} \times \mathbb{R}^{n-2}$ as a continuous function $\overline{u}_0$, and hence (i)–(iii) of Theorem 1 all hold, and $\overline{G}$ has a multiplicity 2 tangent plane of dimension n at $(0, y, \overline{u}_0(0, y))$ which is S_k invariant, hence contains the subspace $\mathbb{R}^2\times\{0\}\times\{0\} \subset \mathbb{R}^2\times\mathbb{R}^{n-2}\times\mathbb{R}$. Hence using the Hölder continuity of Du guaranteed by (iii) of Theorem 1, we must have $\lim_{|x|\to 0} |D_x u(x, y)| = 0$ and $\sup_{0<|x|<\sigma} |x|^{-\alpha}|D_x u| \leq C(k, n, M, \sigma)$. This explains the special

conclusions of Theorem 2 in case of S_k symmetric boundary data, so we need now only prove Theorem 1 and Theorem 3, which we shall do in the next section.

(2) Notice that the above theorem in particular shows that the graph G is not stationary as an integer multiplicity varifold in $\mathcal{C} \times \mathbb{R}$ if u_0 has discontinuities because H_1, H_2 meet at angle $\neq \pi$. Thus we have the following extension of Theorem 1:

Corollary 2. *If $\varphi : \partial\mathcal{C} \to \mathbb{R}$ is bounded continuous, then u_0, as in $(1.12')$, extends continuously across $\{0\}\times\mathbb{R}^{n-2}$ (so (i), (ii), (iii) of Theorem 1 all hold) if and only if the graph $G = \{(re^{i\theta}, y, u_0(r^{1/2}e^{i\theta/2}, y)) : 0 < r < 1, \theta \in \mathbb{R}, y \in \mathbb{R}^{n-2}\}$, viewed as a multiplicity 1 varifold in $\mathcal{C}\times\mathbb{R}$, is stationary in $\mathcal{C} \times \mathbb{R}$.*

3. Proofs

As we pointed out in Remark 1 following Theorem 3, Theorem 2 follows directly from Theorems 1, 3, so we need only prove Theorems 1, 3.

Proof of Theorem 1. Let u_0 be as in $(1.12')$. We first show that (i) $\Longleftrightarrow$ (ii). So suppose (i) holds and, as in §2, let G be the graph of the 2-valued function u over $\mathcal{C}$, so that

$$G = \{(x, y, u(x, y)) : 0 < |x| < 1,\ y \in \mathbb{R}^{n-2}\}. \tag{1}$$

According to $(1.12')$

$$|\overline{u}_0(x, y) - \overline{u}_0(x, z)| \le L_\rho |y - z|, \quad y, z \in \mathbb{R}^{n-2},\ |x| < \rho, 0 < \rho < 1, \tag{2}$$

and in particular this holds for $x = 0$. Also

$$\overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}) = \{(0, y, \overline{u}_0(0, y)) : y \in \mathbb{R}^{n-2}\}$$

which (by (2) with $x = 0$ and $\rho = \frac{1}{2}$) is the graph of the Lipschitz function $\overline{u}_0(0, y)$ over $\mathbb{R}^{n-2}$ and so

$$\mathcal{H}^{n-2}(\overline{G} \cap (\{0\} \times \Omega \times \mathbb{R})) < \infty \tag{3}$$

for any bounded open subset $\Omega \subset \mathbb{R}^{n-2}$, which is the first claim in (2).

We also need the first variation formula

$$\int_G \operatorname{div}_G \zeta \, d\mathcal{H}^n = 0, \tag{4}$$

valid for any Lipschitz function $\zeta = (\zeta_1, \ldots, \zeta_{n+1})$ on G with compact support in G. Here $\operatorname{div}_G \zeta$ denotes the divergence of ζ on G, which is defined by $\operatorname{div}_G \zeta = \sum_{j=1}^{n+1} e_j \cdot \nabla_G \zeta_j$, where ∇_G is computed via local decomposition into the smooth minimal graphs as in (1.15). It is important to note here that this makes sense, and formula (4) is correct, if either ζ is the restriction to the set G of a Lipschitz function in $\mathbb{R}^{n+1}$ or if we assume ζ is actually 2-valued of the form

$$\zeta(re^{i\theta}, y) = \zeta_0(r^{1/2}e^{i\theta/2}, y), \quad 0 < r < 1,\, \theta \in \mathbb{R},\, y \in \mathbb{R}^{n-2}, \tag{5}$$

with the understanding that near any point

$$p_0 = (re^{i\theta_0}, y, u_0(r^{1/2}e^{i\theta_0/2}, y))$$

of G we use the (single) value $\zeta_0(r^{1/2}e^{i\theta/2}, y)$ for the values of ζ near p_0 on the part of G given by the map $(re^{i\theta}, y) \mapsto (re^{i\theta}, y, u_0(r^{1/2}e^{i\theta/2}, y))$ with θ close to θ_0.

Of course in this case the validity of (4) is easily checked via taking a partition of unity $\beta_1, \beta_2, \ldots$ of $\mathcal{D} \setminus \{0\}$ with each β_j having support in a disk $\subset \mathcal{D}\setminus\{0\}$ and with any given point of $\mathcal{D}\setminus\{0\}$ having a neighborhood which intersects at most finitely many of the supports of the β_j. We can then interpret β_j as a function of the variables $(x, y, t) \in \mathcal{D} \times \mathbb{R}^{n-2} \times \mathbb{R}$ which happens to be independent of the y and t variables, and we note that (4) is valid with $\beta_j\zeta$ in place of ζ. By summing over j we then justify (4) for the given ζ as in (5), provided ζ vanishes on $\partial\mathcal{C}$ and has compact support in $\overline{\mathcal{C}} \setminus (\{0\} \times \mathbb{R})$. (We will eliminate the latter restriction shortly—see (10) below.)

In particular for each $\delta, \rho \in (0, 1)$ and each $(x_0, y_0, t_0) \in \mathcal{C} \times \mathbb{R}$ with $|(x_0, y_0)| < 1 - \rho$ we can insert the choice

$$\zeta(x, y, t) = \beta_\delta(x)\lambda_\rho(x, y)\gamma_\rho(t)e_{n+1}$$

in (5), where (i) β_δ is is a $C^\infty(\mathbb{R}^2)$ function which vanishes identically for $|x| < \delta/2$, which is identically equal to 1 for $|x| \geq \delta$, and $|D\beta_\delta| \leq 3/\delta$, (ii) $\lambda_\rho(x, y) \equiv 1$ for $|(x - x_0, y - y_0)| < \rho/2$, $\lambda_\rho(x, y) \equiv 0$ for $|(x - x_0, y - y_0)| > \rho$, $0 \leq \lambda_\rho \leq 1$ everywhere, and $|D\lambda_\rho| \leq 3\rho^{-1}$, and (iii) $\gamma_\rho(t) \equiv 0$ for $t < t_0 - \rho$, $\gamma_\rho(t) = t - t_0 + \rho$ for $t \in [t_0 - \rho, t_0 + \rho]$, and $\gamma_\rho(t) \equiv 2\rho$ for $t > t_0 + \rho$. Then the identity (4) with this choice of ζ gives

(6)
$$\int_{G\cap(B_{\rho/2}(x_0,y_0)\times(t_0-\rho,t_0+\rho))} \beta_\delta\, e_{n+1} \cdot \nabla_G\, t \leq 2\rho \int_{G\cap(B_\rho(x_0,y_0)\times\mathbb{R})} \Big(|e_{n+1} \cdot \nabla_G\, \lambda_\rho| + |e_{n+1} \cdot \nabla_G\, \beta_\delta|\Big).$$

Now for any C^1 function h on $\mathbb{R}^{n+1}$, $\nabla_G\, h$ is just the orthogonal projection $Dh - (\nu \cdot Dh)\nu$ of the $\mathbb{R}^{n+1}$ gradient of h onto the tangent space of G, so $e_{n+1} \cdot \nabla_G\, t \equiv 1 - \nu_{n+1}^2$ and $e_{n+1} \cdot \nabla_G\, h = -\nu_{n+1}\nu \cdot Dh$ in case h is independent of the last variable t, and hence (6) gives

$$\int_{(G\cap(B_{\rho/2}(x_0,y_0)\times(t_0-\rho,t_0+\rho)))}\beta_\delta\,d\mathcal{H}^n \leq \int_{G\cap(B_\rho(x_0,y_0)\times\mathbb{R})}(1+2\rho(|D\lambda_\rho|+|D\beta_\delta|))\nu_{n+1}\,d\mathcal{H}^n. \tag{7}$$

Now the volume form on G is $\nu_{n+1}^{-1}dxdy$, so (keeping in mind that G is the graph of a two-valued function) the right side is $\leq 2\int_{B_\rho(x_0,y_0)}(1+2\rho(|D\lambda_\rho|+|D\beta_\delta|))\,dxdy$ and the contribution from $D\beta_\delta\to 0$ as $\delta\downarrow 0$, so (7) gives, after letting $\delta\downarrow 0$,

$$\mathcal{H}^n(G\cap(B_{\rho/2}(x_0,y_0)\times(t_0-\rho,t_0+\rho)))\leq C\rho^n, \tag{8}$$

which since $B_{\rho/2}(x_0,y_0)\times(t_0-\rho,t_0+\rho)\supset B_{\rho/2}(x_0,y_0,t_0)$ also gives

$$\mathcal{H}^n(G\cap B_{\rho/2}(x_0,y_0,t_0))\leq C\rho^n,\quad C=C(n),$$

provided only that $B_\rho(x_0,y_0)\subset\mathcal{C}$. (Note that the point (x_0,y_0,t_0) here need not be in $\overline{G}$.)

Now observe we derived (4) subject to the restriction that ζ should have compact support in G and so in particular ζ must vanish in a neighborhood of the closed set $\overline{G}\cap(\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})$. However (3) guarantees that, for any given compact $K\subset\mathbb{R}^{n-2}$,

$$\mathcal{H}^{n-1}(\overline{G}\cap(\{0\}\times K\times\mathbb{R}))=0 \tag{9}$$

and we claim that in fact then

$$\begin{cases}\text{the first variation formula (4) holds for any 2-valued } \zeta \text{ as}\\ \text{in (5) if } \zeta \text{ is locally Lipschitz for } 0<r\leq 1,\ \zeta(re^{i\theta},y)\equiv 0\\ \text{for } r=1 \text{ or } y\notin K, \text{ and } |D\zeta|\in L^1(G).\end{cases} \tag{10}$$

This is easily checked by using (4) with $\beta_\delta\zeta$ in place of ζ, where β_δ is Lipschitz on $\mathbb{R}^{n+1}$ with $\beta_\delta\equiv 0$ in a neighborhood of $\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R}$, $\beta_\delta\equiv 1$ at all points at distance $\geq\delta$ from $\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R}$, and $\int_G|D\beta_\delta|<\delta$, and then letting $\delta\downarrow 0$. (Notice that it is standard that such a β_δ exists because, by (9), $\overline{G}\cap(\{0\}\times K\times\mathbb{R})$ is a compact set of $\mathcal{H}^{n-1}$-measure zero, and hence we can select a finite family of balls $B_{\sigma_j}(0,y_j,t_j)$, $j=1,\ldots,N$, with centers $(0,y_j,t_j)\in\overline{G}\cap(\{0\}\times K\times\mathbb{R})$ and radii $\sigma_j<\delta$ with $G\cap(\{0\}\times K\times\mathbb{R})\subset\cup_j B_{\sigma_j}(0,y_j,t_j)$ and $\sum_j\sigma_j^{n-1}<\delta$. Then we can select non-negative functions $\psi_j\in C^\infty(\mathbb{R}^{n+1})$ with $\psi_j\equiv 0$ in $B_{\sigma_j}(0,y_j,t_j)$, $\psi_j\equiv 1$ on $\mathbb{R}^{n+1}\setminus B_{2\sigma_j}(0,y_j,t_j)$ and $|D\psi_j|\leq 3/\sigma_j$, whereas, by (8), $\mathcal{H}^n(G\cap B_{\sigma_j}(0,y_j,t_j))\leq C\sigma_j^n$ for every j. We can then take $\beta_\delta=\min\{\psi_1,\ldots,\psi_N\}$ and check that β_δ has the desired properties with $C\delta$ in place of δ, $C=C(n)$.

Once we have (3) and (4) for functions as in (5), (10) it is standard to prove the stability inequality: by taking $w=-\log\nu_{n+1}$ on G (which is interpreted as a smooth function when we view G as a smooth immersion as in (1.15)), we first see from (2.2) that

$$-\Delta_G w + (|\nabla_G w|^2 + |A_G|^2) = 0 \text{ on } G, \tag{11}$$

the weak form of which is

$$\int_G \Big((|\nabla_G w|^2 + |A_G|^2)\zeta + \nabla_G w \cdot \nabla_G \zeta \Big)\, d\mathcal{H}^n = 0, \tag{12}$$

for any locally Lipschitz function ζ with compact support in $\{(x, y, t) : 0 < |x| < 1,\, y \in \mathbb{R}^{n-2}, t \in \mathbb{R}\}$ and which can be 2-valued as in (5). This is evidently justified using (4) and (10), together with the fact that $\Delta_G w = \operatorname{div}_G(\nabla_G w)$.

Now because of (3), (8) and the fact that $\overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$ is closed, we can for each $\delta > 0$ select a Lipschitz function β_δ on $\mathbb{R}^{n+1}$ such that $\beta_\delta \equiv 0$ in a neighborhood of $\overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$, with $\beta_\delta \equiv 1$ on the set of points with distance at least δ from $\overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$ and with

$$\int_G |D\beta_\delta|^2 < \delta + C\mathcal{H}^{n-2}(\operatorname{support} \zeta \cap \overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})) < \infty,\ C = C(n). \tag{13}$$

Indeed the same construction for β_δ that we used in the discussion following (10) can be used here, except that now we choose the balls $B_{\sigma_j}(x_j, y_j, t_j)$ with $\sigma_j < \delta$ and $\omega_{n-2} \sum_j \sigma_j^{n-2} \le \delta + 2^{n-2}\mathcal{H}^{n-2}(\operatorname{support} \zeta \cap \overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}))$, which we can do by definition of $\mathcal{H}^{n-2}$. Then for any locally Lipschitz function ζ on $\{(x, y, t) : 0 < |x| \le 1,\, y \in \mathbb{R}^{n-2},\, t \in \mathbb{R}\}$ with bounded support and $\zeta \equiv 0$ on $\partial\mathcal{C} \times \mathbb{R}$, we have that $\beta_\delta \zeta^2$ is Lipschitz with compact support in $\{(x, y, t) : 0 < |x| \le 1,\, y \in \mathbb{R}^{n-2},\, t \in \mathbb{R}\}$, and so we can use (12) with $\beta_\delta \zeta^2$ in place of ζ. This first shows

$$\begin{aligned} &\int_G \Big(|\nabla_G w|^2 + |A_G|^2 \Big) \zeta^2 \beta_\delta \\ &= -\int_G \Big(\zeta^2 \nabla_G w \cdot \nabla_G \beta_\delta + 2\zeta \nabla_G w \cdot \nabla_G \zeta \beta_\delta \Big) \\ &\le \varepsilon \int_G |\nabla_G w|^2 \zeta^2 + C(\varepsilon) \int_G \Big(\zeta^2 |\nabla_G \beta_\delta|^2 + |\nabla \zeta|^2 \Big), \end{aligned} \tag{14}$$

so that by letting $\delta \downarrow 0$ we conclude that

$$\begin{aligned} \int_G \Big(|A_G|^2 + |\nabla_G w|^2 \Big) \zeta^2 &\le C\mathcal{H}^{n-2}(\operatorname{support} \zeta \cap G \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})) \\ &\quad + C \int_G |\nabla_G \zeta|^2 < \infty, \quad C = C(n). \end{aligned}$$

This enables us to let $\delta \downarrow 0$ in the first identity of (14) so that

$$\int_G \Big(|\nabla_G w|^2 + |A_G|^2 \Big) \zeta^2\, d\mathcal{H}^n = -\int_G 2\zeta \nabla_G w \cdot \nabla_G \zeta\, d\mathcal{H}^n$$

and, using Cauchy-Schwarz in the form $2ab \le a^2 + b^2$, we deduce the stability inequality $\int_G |A_G|^2\zeta^2 \le \int_G |\nabla_G \zeta|^2$, as claimed in (ii). By using the Cauchy-Schwarz inequality in the alternative form $2ab \le \frac{1}{2}a^2 + 2b^2$, we also obtain

$$\int_G \Big(|\nabla_G w|^2 + |A_G|^2\Big)\zeta^2 \, d\mathcal{H}^n \le 4 \int_G |\nabla_G \zeta|^2 \, d\mathcal{H}^n. \tag{15}$$

We next prove (ii)⇒(i). For this we do not need the stability condition in (ii); indeed we will show that the hypothesis $\mathcal{H}^{n-1}(\overline{G} \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})) = 0$ suffices to give (i), as follows: Suppose that there is $y_0 \in \mathbb{R}^{n-2}$ such that $\liminf_{|x|\to 0} u_0(x, y_0) < \limsup_{|x|\to 0} u_0(x, y_0)$. Let $m = \limsup_{|x|\to 0} u_0(x, y_0) - \liminf_{|x|\to 0} u_0(x, y_0) > 0$ and using the Lipschitz condition with respect to the y-variables given by (1.12′), with Lipschitz constant $L = L_\rho$ corresponding to $\rho = \frac{1}{2}$, we have $\limsup_{|x|\to 0} u_0(x, y) - t_0 > \frac{m}{2}$ and $t_0 - \liminf_{|x|\to 0} u_0(x, y) > \frac{m}{2}$ whenever $|y - y_0| < \frac{m}{4(L+1)}$, where $t_0 = \frac{1}{2}(\liminf_{|x|\to 0} u_0(x, y_0) + \limsup_{|x|\to 0} u_0(x, y_0))$. This evidently implies that $\overline{G} \cap ((0, y_0, t_0) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$ contains the relatively open subset $B^{n-2}_{\rho_0}(y_0) \times (t_0 - \rho_0, t_0 + \rho_0) \cap ((0, y_0, t_0) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$, $\rho_0 = \min\{\frac{m}{4(L+1)}, \frac{1}{2}\}$, $t_0 = (\liminf_{x\to 0} u_0(x, y_0) + \limsup_{x\to 0} u_0(x, y_0))/2$, and therefore has positive $\mathcal{H}^{n-1}$-measure, in particular contradicting (3), so (ii) fails.

We have thus proved (i) $\iff$ (ii), and in accordance with the Remark (2) following the statement of Theorem 1, we have only now to check that (i)⇒(iii). For this we need to modify some standard PDE arguments from the usual $\mathbb{R}^n$ setting, so that we can instead work on G. We showed already that (i)⇒ implies (3), (4) for any ζ as in (10), so we can use these facts in the remainder of the argument.

The identity (11) guarantees that $\Delta_G w \ge 0$ (and of course $w \ge 0$ because $w = -\log \nu_{n+1}$ and $\nu_{n+1} \le 1$). We also have the Sobolev inequality

$$\left(\int_G \zeta^\kappa\right)^{1/\kappa} \le C \int_G |\nabla \zeta|, \quad \kappa = \frac{n}{n-1}, \tag{16}$$

for any locally Lipschitz function as in (5), (10) assuming we integrate the appropriate values of the 2-valued function ζ as explained in the discussion following (5). This is not quite a direct consequence of the normal Sobolev inequality for minimal submanifolds (e.g., [**MS73**]), because of the requirement that functions ζ as in (5) are included, rather than just the restriction to G of functions which are locally Lipschitz on $\mathcal{C} \times \mathbb{R}$. However since we have already established that the first variation formula (4) is valid for such functions, we can use one of the usual proofs of the Sobolev inequality (e.g., as in [**MS73**]) without change, so (16) is valid as claimed.

The proof of the gradient estimate claimed in (iii) of Theorem 1 will now be proved by modifying one of the standard proofs of the gradient estimate for (single-valued) solutions of the MSE. The gradient estimate for single-valued solutions of the MSE was first established in [**BDM69**], and here we follow essentially the same procedure, with some simplifications suggested in [**Sim76**], [**Tru72**], as follows:

For each $\tau \geq 1$, let $w_\tau = \min\{w, \tau\}$, so that w_τ is a bounded locally Lipschitz function which is 2-valued in the sense of (5), and so we can apply the identity (12) with $w_\tau^{2q}\zeta^2$ in place of ζ and we can also use the Sobolev inequality (16) with $w_\tau^{2q}\zeta^2$ in place of ζ. In view of the volume bounds (8) and the fact that $\Delta_G w \geq 0$ on G by (11), we can then use Moser iteration exactly as in the usual $\mathbb{R}^n$ setting (see [**GT83**]), using the Sobolev inequality (16) in place of the usual Sobolev inequality, in order to conclude that

$$\sup_{G\cap B_{1/8}(0,y_0,t_0)} w_\tau \leq C \int_{G\cap B_{1/6}(0,y_0,t_0)} w_\tau \, d\mathcal{H}^n, \tag{17}$$

where $t_0 = u(0, y_0)$. On the other hand using the identity (4) again with $w_\tau \cdot \gamma \cdot \lambda$ in place of ζ, where $\gamma = \gamma(t), \lambda = \lambda(x, y)$ are the same as the functions $\gamma_\rho(t), \lambda_\rho$ in (6) but with $\rho = 1/3$ and $x_0 = 0$, we conclude (cf. (6))

$$\begin{aligned} &\int_{G\cap(B_{1/6}(0,y_0)\times(t_0-1/6,t_0+1/6))} w_\tau \, e_{n+1}\cdot\nabla_G\, t \, d\mathcal{H}^n \\ &\qquad \leq \int_{G\cap(B_{1/3}(0,y_0)\times\mathbb{R})} \Big(w_\tau\,|e_{n+1}\cdot\nabla_G\,\lambda| + |\nabla_G\, w_\tau|\Big)\, d\mathcal{H}^n. \end{aligned} \tag{18}$$

Also by (15) we have

$$\begin{aligned} &\int_{G\cap(B_{1/3}(0,y_0)\times\mathbb{R})} |\nabla_G\, w| \\ &\leq \big(\mathcal{H}^n(G\cap(B_{1/3}(0,y_0)\times\mathbb{R}))\big)^{1/2}\left(\int_{G\cap(B_{1/3}(0,y_0)\times\mathbb{R})}|\nabla_G\, w|^2\right)^{1/2} \\ &\leq C\mathcal{H}^n(G\cap(B_{1/2}(0,y_0)\times\mathbb{R})) \end{aligned}$$

and (since $e_{n+1}\cdot\nabla_G\, t = 1-\nu_{n+1}^2$ and $|e_{n+1}\cdot\nabla_G\,\lambda| \leq 3\nu_{n+1}$ as in the discussion following (6)) we also have $\int_{G\cap(B_{1/2}(0,y_0)\times\mathbb{R})} w_\tau |e_{n+1}\cdot\nabla_G\,\lambda|\, d\mathcal{H}^n \leq C\int_{G\cap(B_{1/2}(0,y_0)\times\mathbb{R})} w_\tau \nu_{n+1}\, d\mathcal{H}^n$ so in fact (18) gives

$$\int_{G\cap(B_{1/6}(0,y_0)\times(t_0-1/6,t_0+1/6))} w_\tau\, d\mathcal{H}^n \leq C(n)\mathcal{H}^n(G\cap(B_{1/2}(0,y_0)\times\mathbb{R})).$$

Thus, after letting $\tau \uparrow \infty$, (17) in fact yields the bound

$$\sup_{G\cap B_{1/8}(0,y_0,t_0)} w \leq C\mathcal{H}^n(G\cap(B_{1/2}(0,y_0)\times\mathbb{R})),\ C = C(n),$$

and since the set $G\cap(B_{1/2}(0,y_0)\times\mathbb{R}\subset\cup_{j=-N}^{N}G\cap(B_{1/2}(0,y_0)\times[j,j+1)$ with a suitable value of $N\le C(1+\sup_{B_{1/2}(0,y_0)}|u|)$, we deduce from (8) that

$$\mathcal{H}^n(G\cap(B_{1/2}(0,y_0)\times\mathbb{R}))\le C(1+\sup_{B_{1/2}(0,y_0)}|u|),$$

and hence finally the gradient bound

$$\sup_{G\cap B_{1/8}(0,y_0,t_0)} w\le C(1+M),$$

where M is any upper bound for $\sup_{B_{1/2}(0,y_0)}|u|$. By exponentiating each side this gives

$$\sup_{G\cap B_{1/8}(0,y_0,t_0)}|Du|\le C_1\exp(C_2M),\quad C_1=C_1(n),C_2=C_2(n), \tag{19}$$

which has the same form as the gradient bound for single-valued solutions of the MSE.

To complete the proof of (iii) we have to establish a Hölder estimate for the 2-valued functions $D_{x_j}u$, $j=1,2$ and $D_{y_j}u$, $j=1,\ldots,n-2$. Notice these derivatives are 2-valued functions of the form (5), and are smooth on $\mathcal{C}\setminus\{0\}\times\mathbb{R}^{n-2}$ assuming as usual we make the natural selection of value on G as in the discussion following (5). By differentiating the MSE with respect to any one of the variables $x_1,x_2,y_1,\ldots,y_{n-2}$ we get a divergence-form equation

$$\sum_{i,j=1}^{n}D_i(a_{ij}D_jv)=0 \text{ on } \mathcal{C}\setminus(\{0\}\times\mathbb{R}^{n-2}), \tag{20}$$

where v is the derivative of u with respect to the chosen variable, and

$$a_{ij}=\nu_{n+1}(\delta_{ij}-\nu_i\nu_j),\quad i,j=1,\ldots,n. \tag{21}$$

Notice that since the volume element for G is $\nu_{n+1}^{-1}dx$ and since $Dv\in L^2$ locally in $\mathcal{C}$ (by (15)), we can write (20) in the weak form on G (with ζ as in (5)) as

$$\int_G\sum_{i,j=1}^{n}\widetilde{a}_{ij}D_ivD_j\zeta\,d\mathcal{H}^n=0 \tag{22}$$

for any ζ as in (5), (10) with $\nabla_G\zeta\in L^2(G)$, where

$$\widetilde{a}_{ij}=\nu_{n+1}a_{ij}=\nu_{n+1}^2(\delta_{ij}-\nu_i\nu_j).$$

Notice that in fact then

$$\sum_{i,j=1}^{n}\widetilde{a}_{ij}D_ivD_j\zeta=\nu_{n+1}^2\nabla_G\zeta\cdot\nabla_G v$$

and $\lambda\le\nu_{n+1}^2\le1$ for suitable $\lambda=\lambda(n,M)>0$ by (19), and so (22) is in exactly the uniformly elliptic form used (in the case of single valued solutions of the MSE) to establish a Harnack theory in [**BG72**]; we therefore simply need to modify the proof of [**BG72**] to the present 2-valued setting. In fact the discussion in [**BG72**, §§4, 5] carries over without change to the present setting, so the only thing we need to check is that a Poincaré inequality as in [**BG72**, §3] applies here. But, since G is the graph of a 2-valued function u of the form (5) and with bounded gradient, it is an easy exercise to check that such a Poincaré inequality follows directly from the usual Poincaré inequality for functions on $\mathbb{R}^n$.

Hence we do have the required Harnack inequality for non-negative solutions v of the equation (20), and hence solutions v of arbitrary sign (in particular $v =$ any one of the derivatives $D_{x_1}u, D_{x_2}u, D_{y_1}u, \ldots, D_{y_{n-2}}u$) are then Hölder continuous by the usual procedure:

We let $M_\rho = \sup_{G\cap B_\rho(x_0,y_0,t_0)} v$, $m_\rho = \inf_{G\cap B_\rho(x_0,y_0,t_0)} v$ and note that then $M_\rho - v$ and $v - m_\rho$ are non-negative solutions of (20) and hence by the Harnack inequality we have some constant $C = C(n, M) > 1$ such that

$$\sup_{G\cap B_{\rho/2}(x_0,y_0,t_0)} (M_\rho - v) \le C \inf_{G\cap B_{\rho/2}(x_0,y_0,t_0)} (M_\rho - v)$$

and

$$\sup_{G\cap B_{\rho/2}(x_0,y_0,t_0)} (v - m_\rho) \le C \inf_{G\cap B_{\rho/2}(x_0,y_0,t_0)} (v - m_\rho).$$

But this says exactly that

$$M_\rho - m_{\rho/2} \le C(M_\rho - M_{\rho/2}) \text{ and } M_{\rho/2} - m_\rho \le C(m_{\rho/2} - m_\rho),$$

and adding these inequalities gives

$$M_{\rho/2} - m_{\rho/2} \le \frac{C-1}{C+1}\,(M_\rho - m_\rho),$$

and by the usual iteration procedure this shows that v is uniformly Hölder continuous on the set $G \cap B_{1/2}(0, y_0, t_0)$. Since u is Lipschitz, this then of course gives Hölder continuity of v as a 2-valued function on $B^n_\sigma(0, y_0)$ for some fixed $\sigma = \sigma(n, M) \in (0, 1/2)$. This completes the proof of (iii) and hence the proof of Theorem 1.

Proof of Theorem 3. To begin, let $(0, y_0)$ be a point of discontinuity of u_0. As pointed out in the proof of Theorem 1 (in the proof that (ii)⇒(i)), using the Lipschitzness of $u_0(x, y)$ with respect to the y-variable, we have

$$\overline{G} \supset \{0\} \times B^{n-2}_{\rho_0}(y_0) \times (t_0 - \rho_0, t_0 + \rho_0), \tag{1}$$

where $t_0 = (\liminf_{|x|\to 0} u(x, y_0) + \limsup_{|x|\to 0} u(x, y_0))/2$ and $\rho_0 = \min\{\frac{m}{4(L+1)}, \frac{1}{2}\}$ with

$$m = \limsup_{|x|\to 0} u_0(x, y_0) - \liminf_{|x|\to 0} u_0(x, y_0) (> 0).$$

Now the graph G of u, as an integer multiplicity varifold, is not necessarily stationary in $\mathcal{C} \times \mathbb{R}$ (indeed Corollary 2 asserts that it is definitely not stationary in $\mathcal{C} \times \mathbb{R}$ under the present hypothesis that $(0, y_0)$

is a point of discontinuity of u), but it is (by (1.15)) stationary in $(\mathcal{C}\times\mathbb{R})\setminus(\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})$, and hence by the reflection principle ([**All75**, §3.2]—Cf. the discussion in the proof of Lemma A of Appendix A) we see that for each $(y,t)\in B_{\rho_0}(y_0,t_0)$

(2)
$\sigma^{-n}\mathcal{H}^n(G\cap B_\sigma(0,y,t))$ is increasing with σ, $\sigma\in(0,\rho_0-|(y-y_0,t-t_0)|)$,

there exists a tangent cone $\mathbb{C}$ of G at $(0,y,t)$, and the density of G, $\Theta_G(0,y,t)$, defined by

(3) $$\Theta_G(0,y,t)=\lim_{\rho\downarrow 0}(\omega_n\rho^n)^{-1}\mathcal{H}^n(G\cap B_\rho(0,y,t)))$$

exists and satisfies

(4) $$\Theta_G(0,y,t)\geq\frac{1}{2},\quad (0,y,t)\in(\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})\cap B_{\rho_0}(0,y_0,t_0).$$

Now let $\varepsilon>0$, define $\kappa=\inf\{\Theta_G(0,y,t):(0,y,t)\in(\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})\cap B_{\rho_0}(0,y_0,t_0)\}(\geq 1/2$ by (4)), and select a point $(0,y_1,t_1)\in(\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R})\cap B_{\rho_0}(0,y_0,t_0)$ with

(5) $$\Theta_G(0,y_1,t_1)\leq\kappa+\varepsilon/2$$

and take $\rho_1\in(0,\rho_0-|(y_1-y_0,t_1-t_0)|)$ such that

(6) $$(\omega_n\rho_1^n)^{-1}\mathcal{H}^n(G\cap B_{\rho_1}(0,y_1,t_1))<\kappa+3\varepsilon/4,$$

which we can do because $(\omega_n\rho^n)^{-1}\mathcal{H}^n(G\cap B_\rho(0,y_1,t_1))\downarrow\Theta_G(0,y_1,t_1)$ as $\rho\downarrow 0$ by (2). Notice that if $\sigma\leq\sigma_0\in(0,\rho_1/2)$ and if $(0,y,t)\in B_{\sigma_0}(0,y_1,t_1)$ then we have

$$\begin{aligned}\Theta_G(0,y,t)&\leq(\omega_n\sigma^n)^{-1}\mathcal{H}^n(G\cap B_\sigma(0,y,t))\\&\leq(\omega_n(\rho_1-\sigma_0)^n)^{-1}\mathcal{H}^n(G\cap B_{\rho_1-\sigma_0}(0,y,t))\\&\leq(\omega_n(\rho_1-\sigma_0)^n)^{-1}\mathcal{H}^n(G\cap B_{\rho_1}(0,y_1,t_1))\\&\leq(1-\sigma_0/\rho_1)^{-n}(\kappa+\varepsilon/2)\\&\leq\kappa+\varepsilon\text{ if }\sigma\leq\sigma_0=\sigma_0(\kappa,n,\rho_1,\varepsilon).\end{aligned}$$

So assume $\delta>0$ is given and choose $\varepsilon>0$, θ as in Lemma A, and then $\sigma_0=\sigma_0(\delta,\kappa,n,\rho_1)$ so that the above holds with this choice of ε. Then the above shows that the hypotheses of Lemma A of the Appendix A hold if we take $\sigma\in(0,\sigma_0]$, $(0,y_2,t_2)\in B_{\sigma_0}(0,y_1,t_1)$ and if $V=h_\# G$, where $h:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ is defined by $h(x,y,t)=\sigma^{-1}(x,y-y_2,t-t_2)$, uniformly for $(0,y_2,t_2)\in B_{\sigma_0}(0,y_1,t_1)$. Then according to Lemma A we have (possibly with a new $\sigma_0=\sigma_0(\delta,\kappa,n,\rho_1)$) that $\forall\,(0,y,t)\in B_{\sigma_0}(0,y_1,t_1)$ and all $\sigma\in(0,\sigma_0]$ $\exists$ half-spaces $H_1,\ldots,H_p$ (depending on σ,y,t) with $p=p(\sigma,y,t)\in\{1,\ldots,P_0\}$ and

(7) $$\text{Hausdorff distance}(G\cap B_\sigma(0,y,t),\cup_{j=1}^p H_j\cap B_\sigma(0,y,t))<\delta\sigma,$$

where P_0 is a fixed integer (determined by $\mathcal{H}^n(G\cap B_{1/2}(0,y_1,t_1))$).

Now for $\tau > 0$ let N_τ denote the tubular neighborhood, cross-section radius τ, of the subspace $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$; thus

$$N_\tau = \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} \times \mathbb{R} : |x| < \tau\}. \tag{8}$$

In view of the fact that G decomposes into a union of graphs as in (1.15), to each of which we can separately apply the regularity theory for stable embedded minimal surfaces as in [**SS83**], we see that (7) yields (possibly with a new σ_0 still depending only on $\delta, \kappa, n, \rho_1$)

$$\begin{cases} \forall\, \sigma \le \sigma_0 \text{ and } \forall\, (0, y, t) \in B_{\sigma_0}(0, y_1, t_1)\ \exists\, q = q(\sigma, y, t) \in \\ \{1, \ldots, Q_0\} \text{ with } B_\sigma(0, y, t) \cap G \setminus N_{\tilde\delta\sigma} = \cup_{j=1}^{q} L_j(\sigma, y, t), \end{cases} \tag{9}$$

where Q_0 is a fixed integer (determined by $\mathcal{H}^n(G \cap B_{1/2}(0, y_1, t_1))$), where $\tilde\delta = C\delta \in (0, \frac{1}{4}]$, with $C = C(n)$ sufficiently large, and we henceforth adopt the convention that we only consider δ small enough so that $C(n)\delta \le \frac{1}{8}$, and where $L_1(\sigma, y, t), \ldots, L_q(\sigma, y, t)$ are embedded minimal hypersurfaces with each $L_j(\sigma, y, t) (\subset G)$ being representable as a minimal graph. More specifically, for each $j = 1, \ldots, q$ there is an n-dimensional half-space $H_j(\sigma, y, t)$ with boundary $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$ and having unit normal $\eta_j \in S^n$, and a $w_j \in C^\infty(\Omega_j)$, with $\Omega_j \supset H_j(\sigma, y, t) \cap B_\sigma(0, y, t) \setminus N_{\tilde\delta\sigma/2}$ with

$$\begin{aligned} &L_j(\sigma, y, t) \cap (B_\sigma(0, y, t) \setminus N_{\tilde\delta\sigma} \\ &\quad = \{\xi + w_j(\xi)\eta_j : \xi \in \Omega_j\} \cap (B_\sigma(0, y, t) \setminus N_{\tilde\delta\sigma} \subset G, \end{aligned} \tag{10}$$

and

$$\sup_{\xi \in \Omega_j} (\sigma^{-1}|w_j(\xi)| + |Dw_j(\xi)|) \le C\delta, \quad C = C(n). \tag{11}$$

We now fix

$$q_0 = q(\sigma_0, y_1, t_1),\ L_j^1 = L_j(\sigma_0, y_1, t_1),\ j = 1, \ldots, q_0. \tag{12}$$

By an inductive procedure (induction on k), based on application of (9), (10) and (11) to suitable finite collections of points $(0, y, t) \in B_{\sigma_0}(0, y_1, t_1)$ we prove that, for each $k = 2, 3, \ldots$ there are embedded minimal hypersurfaces $L_1^{k-1} \subset L_1^k \subset G \cap B_{\sigma_0}(0, y_1, t_1) \setminus N_{\tilde\delta^k \sigma_0}$ such that

$$\partial L_1^k \cap B_{\sigma_0}(0, y_1, t_1) \subset G \cap B_{\sigma_0}(0, y_1, t_1) \cap \partial N_{\tilde\delta^k \sigma_0}$$

and such that for each $(0, y, t) \in B_{\sigma_0}(0, y_1, t_1)$ and each $\sigma \in [\tilde\delta^{k-1}\sigma_0, \sigma_0]$ there is $j = j(\sigma, y, t) \in \{1, \ldots, q_0\}$ with

$$\begin{aligned} &L_1^k \cap B_\sigma(0, y, t) \cap B_{\sigma_0}(0, y_1, t_1) \setminus N_{\tilde\delta\sigma} \\ &\quad = L_j(\sigma, y, t) \cap B_\sigma(0, y, t) \cap B_{\sigma_0}(0, y_1, t_1) \setminus N_{\tilde\delta\sigma}, \end{aligned}$$

where $L_j(\sigma, y, t)$ as in (10)).

Notice that L_1^k is clearly unique (depending on the choice of L_1^1) for each k, by unique continuation of solutions of the MSE, so then $L_1 = \cup_{k=1}^{\infty} L_1^k$ is an embedded minimal hypersurface with $(\overline{L}_1 \setminus L_1) \cap$

$B_{\sigma_0}(0, y_1, t_1) \subset (0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$. Also, by (10) and (11), the hypotheses of Allard's boundary regularity theorem ([**All75**, §4]) are then satisfied and we have (possibly with a smaller σ_0) that L_1 is a smooth embedded hypersurface-with-boundary, with boundary $\partial L_1 \cap B_{\sigma_0}(0, y_1, t_1) = B_{\sigma_0}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$ and

$$L_1 = \{\xi + w_1(\xi)\eta_1 : \xi \in H_1 \cap B_{\sigma_0}(0, y_1, t_1)\} \cap B_{\sigma_0}(0, y_1, t_1), \tag{13}$$

where $w_1 \in C^\infty(H_1 \cap B_{\sigma_0}(0, y_1, t_1))$ and $\sigma_0^{-1} \sup |w_1| + \sup |Dw_1| \le C\delta$, $C = C(n)$.

Repeating this process with L_j^1 in place of L_1^1 for each $j = 2, \ldots, q_0$, where q_0 and L_j^1 are as in (12), and using (9) again, we then have

$$\overline{G} \cap B_{\sigma_0}(0, y_1, t_1) = \cup_{j=1}^{q_0} L_j, \tag{14}$$

where each L_j is a C^∞ manifold-with-boundary, with boundary ∂L_j (taken in the open ball $B_{\sigma_0}(0, y_1, t_1)$) given by $\partial L_j = \Gamma$, where, here and subsequently,

$$\Gamma = B_{\sigma_0}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}). \tag{15}$$

Let $H_1, \ldots, H_{q_0}$ be the tangent half-spaces of the $L_1, \ldots, L_{q_0}$ respectively at the point $(0, y_1, t_1)$, and note that it is possible that two or more of the H_j are equal (because two or more of the L_j might share a common tangent half-plane at the point $(0, y_1, t_1)$). However it is not possible for a distinct pair L_i, L_j to meet with angle zero everywhere along an open subset $\widetilde{\Gamma}$ of $\Gamma = B_{\sigma_0}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$ because then uniqueness of the Cauchy problem would imply that L_i, L_j agree identically on some open region, and then the whole graph would be a multiplicity 2 version of a single-valued graph. But then u would be a smooth single-valued solution of the minimal surface equation on $\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}$, which would imply that u extends smoothly to all of $\mathcal{C}$, because (single-valued) solutions of the minimal surface equation cannot have singularities on a set of zero $(n-1)$-dimensional Hausdorff measure by [**Sim77**]. However this contradicts the fact that in the present case we have a discontinuity at $(0, y_1)$. Hence we can select a new $\widetilde{y}_1, \widetilde{t}_1$, as close as we please to the y_1, t_1, such that no pair of L_j meet at angle 0 at the point $(0, \widetilde{y}_1, \widetilde{t}_1)$. In particular this means that the tangent half-spaces $\widetilde{H}_1, \ldots, \widetilde{H}_{q_0}$ of $L_1, \ldots, L_{q_0}$ at the point $(0, \widetilde{y}_1, \widetilde{t}_1)$ are distinct, and we can take $\widetilde{L}_j = L_j \cap B_{\widetilde{\sigma}_0}(0, \widetilde{y}_1, \widetilde{t}_1)$ for $j = 1, \ldots, q_0$, with $\widetilde{\sigma}_0 \in (0, \sigma_0 - |(y_1 - \widetilde{y}_1, t_1 - \widetilde{t}_1)|)$ chosen small enough to ensure that $\widetilde{L}_1 \setminus \partial \widetilde{L}_1, \ldots, \widetilde{L}_{q_0} \setminus \partial \widetilde{L}_{q_0}$ are pairwise disjoint and that the $\widetilde{L}_j$ all meet with non-zero angle along the common boundary Γ. Also, by the reflection principle for minimal surfaces, we see that if a pair $\widetilde{L}_i, \widetilde{L}_j$ meet at angle π at each point of an non-empty open subset $\widetilde{\Gamma} = B_{\widehat{\sigma}_0}(0, \widehat{y}_1, \widehat{t}_1) \cap \Gamma$, where $B_{\widehat{\sigma}_0}(0, \widehat{y}_1, \widehat{t}_1) \subset B_{\widetilde{\sigma}_0}(0, \widetilde{y}_1, \widetilde{t}_1)$, then with $\widehat{L}_k = B_{\widehat{\sigma}_0}(0, \widehat{y}_1, \widehat{t}_1) \cap \widetilde{L}_k$,

$k=1,\ldots,q_0$, we would have that $\widehat{L}_i\cup\widehat{L}_j$ is a smooth embedded minimal hypersurface containing $\widetilde{\Gamma}$.

Thus by replacing $\sigma_0, y_1, t_1, H, L_j$ by $\widetilde{\sigma}_0, \widetilde{y}_1, \widetilde{t}_1, \widetilde{H}_j, \widetilde{L}_j$ or $\widehat{\sigma}_0, \widehat{y}_1, \widehat{t}_1, \widehat{H}_j, \widehat{L}_j$ as in the above discussion, we can assume that we have made a selection of base-point $(0, y_1, t_1)$ (as close as we please to the original $(0, y_0, t_0)$) and new scale σ_0 with the properties

$$\text{(16)}\quad \begin{cases} L_1\setminus\partial L_1,\ldots,L_{q_0}\setminus\partial L_{q_0} \text{ are pairwise disjoint} \\ \ \text{and } L_1,\ldots,L_{q_0} \text{ meet with angle} \neq 0 \\ \ \text{along } \Gamma = B_{\sigma_0}(0,y_1,t_1)\cap((0,y_1,t_1)+\{0\}\times\mathbb{R}^{n-2}\times\mathbb{R}), \\ H_j \text{ is the tangent half-space to } L_j \text{ at } (0,y_1,t_1), \\ \forall\ i\neq j, L_i, L_j \text{ either meet with angle} \neq \pi \text{ along } \Gamma \\ \ \text{or meet with angle} \equiv \pi, \\ \ \text{in which case } L_i\cup L_j \text{ is a smooth embedded hypersurface.} \end{cases}$$

The half-spaces H_j can be written in the form $\{(\lambda\omega_j, y, t) : \lambda\geq 0, (y,t)\in \mathbb{R}^{n-2}\times\mathbb{R}\}$ for some unique $\omega_j\in S^1$, so

$$\text{(17)}\qquad \omega_j = e^{i\eta_j}, \quad \eta_j\in[0,2\pi).$$

By applying (1.15) with $\theta_0 = \eta_j+\pi$ we associate two solutions $u_j^{\pm}$ of the MSE, with graphs $G_j^{\pm}$, with each $u_j^{\pm}$ defined over $(\mathcal{D}\setminus\{-\lambda\omega_j : \lambda\geq 0\})\times\mathbb{R}^{n-2}$, and with

$$\text{(18)}\quad G\cap((\mathcal{D}\setminus\{-\lambda\omega_j:\lambda\geq 0\})\times\mathbb{R}^{n-2}\times\mathbb{R}) = G_j^+\cup G_j^-, \quad j=1,\ldots,q_0;$$

of course then for any $k\in\{1,\ldots,q_0\}$ such that $\omega_k\neq\eta_j+\pi$ we have $L_k\setminus\partial L_k$ is either entirely contained in G_j^+ or entirely contained in G_j^-.

We claim that q_0 is even. To see this observe that if $0<\sigma<\sigma_0/\sqrt{2}$ (so that $\{(x,y_1,t) : |x|=\sigma, |t-t_1|\leq\sigma\}\subset B_{\sigma_0}(0,y_1,t_1)$) and if we let $\gamma(\theta) = (\sigma e^{i\theta}, y_1, u_0(\sigma^{1/2}e^{i\theta/2}, y_1))$, then (assuming σ is sufficiently small and appropriately reordering and relabeling the $L_1,\ldots,L_{q_0}$) we can select pairwise disjoint intervals (α_j,β_j) with

$$\text{(19)}\quad \alpha_j<\beta_j<\alpha_{j+1}<\beta_{j+1}, \quad j=1,\ldots,q_0-1, \quad \beta_{q_0}-\alpha_1<4\pi,$$

and such that $\gamma|[\alpha_j,\beta_j]$ is 1:1 and $\gamma([\alpha_j,\beta_j]) = L_j\cap\{(x,y_1,t) : |x|=\sigma, t\in[t_1-\sigma,t_1+\sigma]\}$ for each $j=1,\ldots,q_0$. (We can in fact arrange $\max\{\beta_j-\alpha_j : j=1,\ldots,q_0\}$ is as small as we please by taking σ sufficiently small.) Then $\gamma|[\alpha_1,\alpha_1+4\pi]$ is a closed curve which traverses each of the arcs $L_j\cap\{(x,y_1,t) : |x|=\sigma, t\in[t_1-\sigma,t_1+\sigma]\}$ exactly once, and, by (14), $e_{n+1}\cdot\gamma(\theta)$ is never between $[t_1-\sigma,t_1+\sigma]$ for $\theta\in[\alpha_1,\alpha_1+4\pi]\setminus(\cup_{j=1}^{q_0}[\alpha_j,\beta_j])$, and hence $e_{n+1}\cdot\gamma|[\alpha_j,\beta_j]$ must be alternately increasing and decreasing as a function of θ for $j=1,\ldots,q_0$, and the number of j such that it is increasing must therefore match the number of j such that it is decreasing. So q_0 is even as claimed.

Now again relabel the L_j, and the corresponding H_j, this time to ensure that the angles η_j in (17) satisfy

$$0 \le \eta_1 < \eta_2 < \cdots < \eta_{q_0} < 2\pi,\ \eta_{q_0+1} = \eta_1 + 2\pi, \tag{20}$$

let $L_{q_0+1} = L_1,\ L_0 = L_{q_0},\ H_{q_0+1} = H_1,\ H_0 = H_{q_0}$, and let θ_j be the angle between H_j and it's nearest neighbor in the counter-clockwise direction. Thus

$$\theta_j = \eta_{j+1} - \eta_j, \quad j = 1, \ldots, q_0, \quad \theta_{q_0+1} = \theta_1, \text{ and } \sum_{j=1}^{q_0} \theta_j = 2\pi. \tag{21}$$

In particular $\sum_{i=1}^{q_0}(\theta_i + \theta_{i+1}) = 2\sum_{i=1}^{q_0}\theta_i = 4\pi$, and so we see that if $q_0 \ge 4$ then there must be 3 successive half-spaces $H_{i_0-1}, H_{i_0}, H_{i_0+1}$ such that

(22)
$$\begin{cases} \text{either } \theta_{i_0} + \theta_{i_0+1} < \pi \\ \text{or } \theta_{i_0} + \theta_{i_0+1} = \pi \text{ and } L_{i_0-1}, L_{i_0+1} \text{ meet at angle } \pi \text{ along } \Gamma \\ \quad \text{and } L_{i_0-1} \cup L_{i_0+1} \text{ is a smooth embedded minimal hypersurface.} \end{cases}$$

With such i_0, consider the three corresponding hypersurfaces L_{i_0-1}, L_{i_0}, L_{i_0+1}. Notice that, in the notation of (18), either at least two of these three hypersurfaces lie in $G^+_{i_0}$ or else at least two lie in $G^-_{i_0}$. Let us suppose for convenience of notation that the former possibility holds, let $\widetilde{u} = u^+_{i_0}$ (so graph of $\widetilde{u}$ is $G^+_{i_0}$), and let $L, \widetilde{L}$ be chosen from $L_{i_0-1}, L_{i_0}, L_{i_0+1}$ as follows: If L_{i_0-1} is contained in $G^+_{i_0}$ then take $L = L_{i_0-1}$ and $\widetilde{L} = L_{i_0}$ if $L_{i_0} \subset G^+_{i_0}$ and $\widetilde{L} = L_{i_0+1}$ if L_{i_0} is not contained in $G^+_{i_0}$. If L_{i_0-1} is not contained in $G^+_{i_0}$ then take $L = L_{i_0}$ and $\widetilde{L} = L_{i_0+1}$. Having thus chosen $L, \widetilde{L}$, let $H, \widetilde{H}$ denote the tangent half-spaces of $L, \widetilde{L}$ respectively at the point $(0, y_1, t_1)$.

We first dispense with the possibility that $L = L_{i_0-1}, \widetilde{L} = L_{i_0+1}$ with the second alternative in (22), so that $L \cup \widetilde{L}$ is a smooth embedded hypersurface containing $\Gamma = B_{\sigma_0}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$, and $L \setminus \Gamma, \widetilde{L} \setminus \Gamma$ are both contained in the graph $G^+_{i_0}$. We claim this is impossible because then $L \cup \widetilde{L}$ would be a minimal hypersurface with $(n+1)$'st component ν_{n+1} of the unit normal strictly positive away from Γ and vanishing on Γ, but this would contradict the Hopf maximum principle for ν_{n+1}; the Hopf maximum principle holds because ν_{n+1} satisfies the Jacobi field equation 2.2 which means $\Delta\nu_{n+1} \le 0$. Thus we conclude that $L, \widetilde{L}$ meet with angle $< \pi$ along Γ in all cases. In particular this shows that half-spaces $H, \widetilde{H}$ must meet at angle $< \pi$.

Now $G \cap B_{\sigma_0}(0, y_1, t_1) \setminus (\cup_{i=1}^{q_0} L_i) = \emptyset$ (by (14)), hence there are no points (x, y) with $0 < |x| < \sigma_0$ and $\widetilde{u}(x, y) = t_1$ other than points $(x, y) \in P(\cup_{i=1}^{q_0} L_i)$, where P denotes the projection of (x, y, t) onto (x, y). Since by construction $L, \widetilde{L} \in \{L_j : j = 1, \ldots, q_0\}$ and both $L, \widetilde{L}$ are contained in G, it then follows that, for sufficiently small σ, there is a wedge-shaped domain Ω with $\widetilde{u} \equiv t_1$ on $\partial\Omega \cap B_\rho(0,y_1) \setminus ((0,y_1) + \{0\} \times \mathbb{R}^{n-2})$ and with Ω asymptotic at $(0,y_1)$ to the convex wedge $W(H,\widetilde{H})$ between $(0,y_1) + PH$ and $(0,y_1) + P\tilde{H}$ ($P(x,y,t) = (x,y)$ as above).

Now we can apply Lemma B of Appendix B with $u = \widetilde{u}$, with Ω as above, with $x_0 = (0,y_1)$, $\rho_0 = \rho$ (sufficiently small), $\psi \equiv t_1$ on $\partial\Omega \cap B_\rho(0,y_1)$, and with U being any open half-space in $\mathbb{R}^n$ with $(0,y_1) \in \partial U$ and $(B_\rho(0,y_1) \setminus \{(0,y_1)\}) \cap \overline{W}(H,\widetilde{H}) \subset U$, where $\overline{W}(H,\widetilde{H})$ is the closure of the convex wedge $W(H,\widetilde{H})$ introduced above. (For example, a suitable choice for U would be the half-space $(0,y_1) + \{(x,y) : x \cdot (\omega_{i_0-1} + \omega_{i_0+1}) > 0\}$.)

But then Lemma B asserts that $\widetilde{u}|\Omega$ extends continuously to $\Omega \cup \{(0,y_1)\}$ with value t_1 at $(0,y_1)$. On the other hand $L \cap \{(x,y,t) \in B_\rho(0,y_1,t_1) : x \neq 0, t > t_1\}$ is contained in the graph of $\widetilde{u}|\Omega$, and $\overline{L} \cap B_\rho(0,y_1,t_1)$ contains the vertical segment $(0,y_1) \times (t_1, t_1 + \sigma)$ so the closure of graph $\widetilde{u}|\Omega$ contains this vertical segment, which means that $\widetilde{u}|\Omega$ does not extend continuously to $\Omega \cup \{(0,y_1)\}$. Thus we have a contradiction, so we must have $q_0 = 2$ and there are just two half-spaces H_1, H_2 and two submanifolds L_1, L_2. This completes the proof of Theorem 3, except for the claim that H_1 and H_2 do not meet at angle π.

To check this last point we observe that otherwise, by (16), $L_1 \cup L_2$ is a smooth embedded hypersurface containing Γ. Let u_1, u_2 be the smooth functions such that $\operatorname{graph} u_j = L_j \setminus \Gamma$ for $j = 1,2$, so that $(1 + |Du_j|^2)^{-1/2}(-Du_j, 1)$ is the upward pointing unit normal of L_j. Evidently the domain of u_j is Ω_j such that for small enough σ we have $\{(x,y) \in \Omega_j : |x| < \sigma\} \subset W_{\ell_j} \times \mathbb{R}^{n-2}$, where W_{ℓ_j} is a thin conical neighborhood of ℓ_j, with ℓ_j the ray from the origin in $\mathbb{R}^2$ given by the orthogonal projection of H_j onto $\mathbb{R}^2$. Then ℓ_1, ℓ_2 meet at the origin with angle π. Let $\eta \in \mathbb{R}^2$ be a unit normal to $\ell_1 \cup \ell_2$. Since L_1, L_2 are smooth submanifolds with boundary Γ, the unit normals $(1+|Du_j|^2)^{-1/2}(-Du_j, 1)$, $j = 1,2$, each have asymptotic limit $\pm(\eta,0,0)$ on approach to Γ. In particular this means that the limit of the unit normal $(1 + |Du_1|^2)^{-1/2}(-Du_1, 1)$ of L_1 agrees with $\pm$ the limit of the unit normal $(1 + |Du_2|^2)^{-1/2}(-Du_2, 1)$ of L_2 on approach to Γ, and in fact the plus sign must hold because for sufficiently small σ we know (Cf. the argument following (19)) that $\frac{\partial}{\partial\theta} u_1(re^{i\theta}, y)$ has a constant sign (either large positive or negative with large absolute value) in $\Omega_1 \cap B_\sigma(0,y_1)$ and its sign must be opposite to the sign of $\frac{\partial}{\partial\theta} u_1(re^{i\theta}, y)$ on $\Omega_2 \cap B_\sigma(0,y_1)$, and it follows that $(\eta,0) \cdot Du_1\, (= (\eta,0,0) \cdot (-Du_1, 1))$ and $(\eta,0) \cdot Du_2\, (= (\eta,0,0) \cdot (-Du_2, 1))$ must have the same sign for $|x| < \sigma$ (i.e., either $(\eta,0) \cdot Du_j > 0$ in $\{(x,y) \in \Omega_j : |x| < \sigma\}$ for both $j = 1,2$ or $(\eta,0) \cdot Du_j < 0$ in $\{(x,y) \in \Omega_j : |x| < \sigma\}$ for both $j = 1,2$). That is, there is a continuous unit normal ν of the smooth hypersurface

$L_1 \cup L_2$ which points upward (i.e., $e_{n+1} \cdot \nu \geq 0$) on both L_1 and L_2, hence the maximum principle can again be applied to $e_{n+1} \cdot \nu$ as in the discussion following (22) above and gives to a contradiction. Thus H_1, H_2 do not meet at angle π. This completes the proof of Theorem 3.

4. Extension to the q-valued case

Let $q \in \{2, 3, 4, \ldots\}$. All of the above has a straightforward generalization to the consideration of examples involving q-valued (instead of 2-valued) graphs of the form $u_0(r^{1/q}e^{i\theta/q}, y)$, $0 \leq \theta < 2q\pi$, with prescribed boundary data given by $\varphi(e^{i\theta/q})$, where φ is a given bounded continuous function on $\partial\mathcal{C}$.

Indeed by straightforward modifications of the discussion of §1, using $T(re^{i\theta}, y) = (r^q e^{iq\theta}, y)$ and

$$(*) \qquad \mathcal{F}_0(v) = \int_\Omega (qr^{q-1})^2 \sqrt{1 + (qr^{q-1})^{-2}|D_x v|^2 + |D_y v|^2}\, dxdy$$

in place of the $T, \mathcal{F}_0$ of §1, we prove there is a u_0 as in (1.12′) and a corresponding q-valued solution $u(re^{i\theta}, y) = u_0(r^{1/q}e^{i\theta/q}, y)$ of the MSE on $\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}$ with the prescribed boundary values $\varphi(e^{i\theta/q})$.

In this case the analogue of Theorem 3 is the following:

Theorem 4. *Let u (with $u(re^{i\theta}, y) = u_0(r^{1/q}e^{i\theta/q}, y)$ on $\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}$) be the q-valued solution of the MSE as above and let $\rho_0 \in (0, \frac{1}{4}]$. If u is discontinuous at some point $(0, y_0) \in \{0\} \times \mathbb{R}^{n-2}$ then there is a point $(0, y_1, t_1) \in \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$ with $|y_0 - y_1| < \rho_0$ and $\rho_1 \in (0, \rho_0]$ such that $B_{\rho_1}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}) \subset \overline{G}$, and such that G (as an n-dimensional integer multiplicity varifold in $\mathbb{R}^{n+1}$) has a unique tangent cone $\mathbb{C}$ at $(0, y_1, t_1)$ of the form*

$$\mathbb{C} = |H_1| + \cdots + |H_{q_0}|,$$

where q_0 is even with $q_0 \in \{2, \ldots, 2q-2\}$, and where $H_1, \ldots, H_{q_0}$ are distinct n-dimensional half-spaces with common boundary $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$ and $|H_j|$ is the multiplicity 1 varifold corresponding to H_j, and

$$\overline{G} \cap B_{\rho_1}(0, y_1, t_0) = \cup_{j=1}^{q_0} L_j,$$

where each L_j is an embedded C^∞ manifold-with-boundary, with boundary (taken in the open ball $B_{\rho_1}(0, y_1, t_1)$) $\partial L_j = B_{\rho_1}(0, y_1, t_1) \cap ((0, y_1, t_1) + \{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$, L_j has the tangent half-space H_j at the point $(0, y_1, t_1) \in \partial L_j$, and $L_j \setminus \partial L_j$, $j = 1, \ldots, q_0$, are pairwise disjoint.

The proof is a straightforward modification of the proof of Theorem 3, the fact that $q_0 \leq 2q - 2$ coming from an application of Lemma B exactly analogous to the corresponding part of the proof of Theorem 3, as follows: If $\eta_1, \ldots, \eta_{q_0}, \theta_1, \ldots, \theta_{q_0}$ as in the proof of Theorem 3 and if we adopt the convention that $i + q$ is counted mod-q_0 if $i + q > q_0$, then

$\sum_{i=1}^{q_0}(\theta_i+\cdots+\theta_{i+q-1}) = q(\theta_1+\cdots+\theta_{q_0}) = 2q\pi$, so if $\theta_i+\cdots+\theta_{i+q-1} > \pi$ for each $i = 1, \ldots, q_0$ we would have $q_0\pi < 2q\pi$, i.e., $q_0 < 2q$ and hence $q_0 \leq 2q-2$ because q_0 is even. Thus if $q_0 > 2q-2$ then we would have $\theta_{i_0} + \cdots + \theta_{i_0+q-1} \leq \pi$ for some $i_0 \in \{1, \ldots, q_0\}$, and (cf. the proof of Theorem 3) at least two of the $q+1$ sheets $L_i, i \in \{i_0-1, \ldots, i_0+q-1\}$, are in the same (single-valued) graph $G_{i_0}^{j_0}$ for some $j_0 \in \{1, \ldots, q\}$, where $\{G_{i_0}^1, \ldots, G_{i_0}^q\}$ are the q single-valued graphs whose union is the graph of the q-valued function u over the slit domain $\mathcal{C} \setminus (\{-\lambda e^{i\eta_{i_0}} : \lambda \geq 0\} \times \mathbb{R}^{n-2})$. After eliminating the possibility $\theta_{i_0} + \cdots + \theta_{i_0+q-1} = \pi$ as in the case $q = 2$ (by applying the strong maximum principle to $e_{n+1} \cdot \nu$, where ν is the upward-pointing unit normal of $G_{i_0}^{j_0}$), we can then apply Lemma B of Appendix B as in the proof of Theorem 3 to give a contradiction. Thus $q_0 \leq 2q-2$.

Finally we observe that if k, q are relatively prime (so $\ell k + mq = 1$ for some integers ℓ, m), then the graph of the q-valued function u is S_k invariant if $u_0 \circ S_k = u_0$, and hence the collection of hypersurfaces $L_1, \ldots, L_{q_0}$ and the corresponding half-spaces $H_1, \ldots, H_{q_0}$ (tangent to $L_1, \ldots, L_{q_0}$ respectively at $(0, y_1, t_1)$) are invariant under S_k, and so we must have $q_0 \geq 2k$, because at least k of the L_j have boundary data which is strictly increasing in θ for θ near η_j, and likewise at least k of the L_j have boundary data which is strictly decreasing in θ for θ near η_j. Thus $2q-2 \geq q_0 \geq 2k$ and we have a contradiction if $k > q$.

Thus we obtain the following q-valued generalization of Theorem 2:

Theorem 5. *Suppose $q \geq 2$, $k > q$, k, q are relatively prime, $\varphi \circ S_k = \varphi$ and $\varphi \circ S_q \neq \varphi$, with φ bounded and continuous, and let u_0 be as in* (1.12′)*, where $\mathcal{F}_0$ is now the modified functional as in* $(*)$ *above. Then the q-valued function u defined by $u(re^{i\theta}, y) = u_0(r^{1/q}e^{i\theta/q}, y)$ has a q-valued $C^{1,\alpha}$ graph for some $\alpha \in (0,1)$, with*

$$\sup_{0<|x|<\sigma} |x|^{-\alpha}|D_x u| \leq C,$$

and such that the q-valued analogue of (iii) *of Theorem* 1 *holds.*

Remark. If both $\varphi \circ S_k = \varphi$ and $\varphi \circ S_q = \varphi$ then since k, q are relatively prime u would be a multiplicity q version of a single valued function, so this case is of no interest in the present context; this explains the condition $\varphi \circ S_q \neq \varphi$ in the above theorem.

5. Appendix A

Here we establish the following general varifold lemma, needed in the proof of Theorem 3,4 above:

Lemma A. *For each given $\delta \in (0,1)$ and $M > 0$ there are $\theta = \theta(n, M, \delta) \in (0, 1/2]$ and $\varepsilon = \varepsilon(n, M, \delta) \in (0, 1/4]$ such that if V is an n-dimensional integer multiplicity varifold in the open unit ball $B_1(0) \subset$*

$\mathbb{R}^{n+1}$ *with* $\|V\|(B_1(0)) < M$, *if* $\sigma \in (0, \theta]$ *and if* V *satisfies the conditions*

(a) V *is stationary in* $B_1(0) \setminus (\{0\} \times \mathbb{R}^{n-1})$ *and* $0 \in \operatorname{support} \|V\|$,
(b) $\omega_n^{-1}\|V\|(B_1(0)) \leq \Theta_{\|V\|}(0) + \varepsilon$,
(c) $\Theta_{\|V\|}(\xi) \geq \Theta_{\|V\|}(0) - \varepsilon$ *for each* $\xi \in B_1(0) \cap (\{0\} \times \mathbb{R}^{n-1})$,

then there are n*-dimensional half-spaces* $H_1, \dots, H_q$ *(depending on* σ*) with common boundary* $\{0\} \times \mathbb{R}^{n-1}$ *such that*

$$\text{Hausdorff distance}\big(\operatorname{support}\|V\| \cap B_\sigma(0), \cup_{j=1}^q H_j \cap B_\sigma(0)\big) < \delta\sigma.$$

Proof. Otherwise we have a $\delta_0 > 0$, $M_0 > 0$ and sequences $\sigma_k, \varepsilon_k \downarrow 0$, V_k stationary in $B_1(0) \setminus (\{0\} \times \mathbb{R}^{n-1})$ with mass $< M_0$ and

(1) $$\omega_n^{-1}\|V_k\|(B_1(0)) \leq \Theta_{\|V_k\|}(0) + \varepsilon_k$$

and

(2) $$\Theta_{\|V_k\|}(\xi) \geq \Theta_{\|V_k\|}(0) - \varepsilon_k \text{ for each } \xi \in B_1(0) \cap (\{0\} \times \mathbb{R}^{n-1}),$$

yet such that

(3)
$$\text{Hausdorff distance}\big(\operatorname{support}\|V_k \llcorner B_{\sigma_k}(0)\|, \cup_{j=1}^q H_j \cap B_{\sigma_k}(0)\big) \geq \delta_0\sigma_k$$

for every finite collection $H_1, \dots, H_q$ of n-dimensional half-spaces with with common boundary $\{0\} \times \mathbb{R}^{n-1}$.

Let R be the odd reflection of $\mathbb{R}^{n+1}$ across $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$ (so $R : (x, y, t) \mapsto (-x, y, t)$ for $x \in \mathbb{R}^2$), let η_k be the homothety $(x, y, t) \mapsto \sigma_k^{-1}(x, y, t)$ and define

$$\widetilde{V}_k = \eta_{k\,\#}(V_k + R_\# V_k), \quad \widehat{V}_k = \eta_{k\,\#} V_k,$$

so that $\widetilde{V}_k$ is stationary on $B_{1/\sigma_k}(0) \setminus (\{0\} \times \mathbb{R}^{n-1})$ by the reflection principle of [**All75**, §3.2].

It therefore follows that the monotonicity formula holds for $\widetilde{V}_k$ and hence $\Theta_{\widehat{V}_k}(0)$ exists (and equals $\frac{1}{2}\Theta_{\widetilde{V}_k}(0)$), and the monotonicity identity holds for $\widehat{V}_k = \eta_{k\,\#}V_k$:

(4) $$\int_{B_\rho(0)\setminus B_\sigma(0)} |X|^{-n-2}(\nu_{\widehat{V}_k} \cdot X)^2 \, d\|\widehat{V}_k\| + \sigma^{-n}\|\widehat{V}_k\|(B_\sigma(0)) = \rho^{-n}\|\widehat{V}_k\|(B_\rho(0))$$

for $0 < \sigma < \rho < \sigma_k^{-1}$, where $X = (x, y, t)$ is the general point in $\mathbb{R}^{n+1}$ and $\nu_{\widehat{V}_k}$ is the unit normal for the tangent space of $\widehat{V}_k$. By applying the compactness theorem for integral varifolds we then have

(5) $$\widehat{V}_{k'} \to W$$

where the convergence is in the varifold sense to an integer multiplicity varifold W which is stationary in $\mathbb{R}^{n+1} \setminus (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$, and W is a cone:

$$\eta_{\rho\,\#}W = W \quad \rho > 0 \tag{6}$$

or equivalently, using first variation, $\nu_W(X) \cdot X = 0$ for a.e., $X \in$ support $\|W\|$. (The latter comes from the fact that, by monotonicity of $\rho^{-n}\|\widehat{V}_k\|(B_\rho(0))$, we have that $\rho^{-n}\|W\|(B_\rho(0))$ is constant in the variable ρ, and therefore by applying the monotonicity (4) to W we conclude (6)). By applying the same reasoning to $\tau_{\xi\,\#}W$, where $\tau_\xi(X) = X - \xi$ and ξ is an arbitrary point in $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$, and noting that

$$\begin{aligned}\Theta_W(0) &= \lim_{\rho\to\infty}(\omega_n\rho^n)^{-1}\|W\|(B_\rho(0)) \\ &= \lim_{\rho\to\infty}(\omega_n\rho^n)^{-1}\|\tau_{\xi\,\#}W\|(B_\rho(0)),\end{aligned}$$

we also deduce that

$$\Theta_W(0) \geq \Theta_W(\xi) \text{ with equality } \iff \tau_{\xi\,\#}W = W \tag{7}$$

(because by the monotonicity identity $\Theta_W(0) = \Theta_W(\xi)$ implies $\nu_W \cdot X = 0 = \nu_W \cdot (X - \xi) = 0$ a.e., on support of W, and so $\xi \cdot \nu_W = 0$ a.e., whence it follows from the first variation formula that $\tau_{\xi\,\#}W = W$).

Also using (2) in combination with (1) together with the upper semi-continuity of $\Theta_{\widetilde{V}_{k'}}(\xi)$ we also have

$$\Theta_W(\xi) \geq \Theta_W(0), \quad \xi \in \{0\} \times \mathbb{R}^{n-1}. \tag{8}$$

which in combination with (7) implies that W is invariant under translations in such directions ξ. Thus W is invariant under translations by all elements of $\{0\} \times \mathbb{R}^{n-1}$ and so is a cylinder with 1-dimensional cross section W_0, where W_0 is a 1-dimensional stationary integer multiplicity cone on $\mathbb{R}^2$; that is W_0 is a sum of rays, each with positive integer multiplicity, emanating from 0. Thus

$$W = \sum_{j=1}^{q} |H_j|, \tag{9}$$

where each H_j is an n-dimensional half-space and $H_1, \ldots, H_q$ have common boundary $\{0\} \times \mathbb{R}^{n-1}$, and where we must allow the possibility that some of the $H_1, \ldots, H_q$ are equal. Since varifold convergence (5) of the stationary integral varifolds $\widehat{V}_k$ to W implies Hausdorff distance convergence of support of $\|\widehat{V}_k\|$ to the support of $\|W\|$ on each set $B_\rho(0) \setminus \{(x,y,t) : |x| < \sigma\}$, (9) contradict (3), so the proof of Lemma A is complete.

6. Appendix B

Here we establish a lemma concerning removability of boundary discontinuities for solutions of the MSE. The result here is a modification of the argument in [**Sim77**], which in turn depends on an argument introduced in [**Fin53**] to remove isolated interior singularities of solutions of the MSE.

Lemma B. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega)$ a bounded solution of the MSE in Ω, let $x_0 \in \partial\Omega$, and suppose there is $\rho_0 > 0$ such that $\Omega \cap B_{\rho_0}(x_0) \subset U$ for some open half-space U with $x_0 \in \partial U$. Suppose also that $\psi : \partial\Omega \cap B_{\rho_0}(x_0) \to \mathbb{R}$ is continuous and that there is a compact set $\mathcal{K} \subset \partial\Omega$ with $x_0 \in \mathcal{K}$, $\mathcal{H}^{n-1}(\mathcal{K}) = 0$ and with $\lim_{x\in\Omega, x\to y} u(x) = \psi(y)$ for each $y \in \partial\Omega \cap B_{\rho_0}(x_0) \setminus \mathcal{K}$. Then $\lim_{x\to x_0, x\in\Omega} u(x) = \psi(x_0)$ (i.e., u extends continuously to $\Omega \cup \{x_0\}$).*

Proof. For any function v we let $\nu(v) = \frac{Dv}{\sqrt{1+|Dv|^2}}$, so that the MSE for u can be written

$$\sum_{i=1}^{n} D_i \nu_i(u) = 0,$$

which in weak form is

$$\int_\Omega \nu(u) \cdot D\zeta = 0 \tag{1}$$

for all ζ which are Lipschitz with compact support in Ω. For any given $\varepsilon > 0$ we select $\sigma = \sigma(\varepsilon) \in (0, \rho_0)$ such that $\psi(x) < \psi(x_0) + \varepsilon$ on $\partial\Omega \cap B_\sigma(x_0)$. Then a standard barrier construction (see e.g., [**GT83**, §14.1]) shows that there is $C^2(\overline{U} \cap \overline{B}_\sigma(x_0))$ supersolution v of the MSE with $v(x_0) = \psi(x_0) + \varepsilon$, $v(x) \geq \psi(x_0) + \varepsilon$ for each $x \in \overline{U} \cap \overline{B}_\sigma(x_0)$ and with $v > M$ at each point of $\overline{U} \cap \partial B_\sigma(x_0)$, where $M = \sup_\Omega u$. The requirement that v is a supersolution can be written in weak form as

$$\int_{\Omega \cap B_\sigma(x_0)} \nu(v) \cdot D\zeta \geq 0 \tag{2}$$

for all ζ which are non-negative Lipschitz with compact support in $\Omega \cap B_\sigma(x_0)$.

We now define $w = \max\{\arctan(u - v) - \varepsilon, 0\}$ on $\Omega \cap B_\sigma(x_0)$, and observe (using the local uniform convexity of the function $\sqrt{1 + |p|^2}$ for $p \in \mathbb{R}^n$) that

$$(\nu(u) - \nu(v)) \cdot (Du - Dv) \geq c(x)|D(u - v)|^2 \tag{3}$$

for some function $c(x)$ which is positive and continuous on $\Omega \cap B_\sigma(x_0)$.

Since $\mathcal{H}^{n-1}(\mathcal{K}) = 0$ we can choose (as in the proof of Theorem 1) a Lipschitz function β_δ with $\beta_\delta \equiv 0$ in a neighborhood of $\mathcal{K}$,

$$\int_{\mathbb{R}^n} |D\beta_\delta| < \delta \tag{4}$$

and $\beta_\delta(x) \equiv 1$ whenever $\operatorname{dist}(x,\mathcal{K}) \geq \delta$. Then the function $w\beta_\delta$ is non-negative Lipschitz with compact support in $\Omega \cap B_\sigma(x_0)$.

Taking the difference of (1) and (2) we have

$$\int_{\Omega\cap B_\sigma(x_0)} (\nu(u)-\nu(v))\cdot D\zeta \leq 0$$

for each non-negative Lipschitz function ζ with compact support in $\Omega \cap B_\sigma(x_0)$, and by selecting $\zeta = \beta_\delta w$ we see that

$$\begin{aligned}\int_{\{x\in\Omega\cap B_\sigma(x_0):w(x)>0\}} &\beta_\delta\,(1+(u-v)^2)^{-1}(\nu(u)-\nu(v))\cdot(Du-Dv)\\ &\leq -\int_{\Omega\cap B_\sigma(x_0)} w\,(\nu(u)-\nu(v))\cdot D\beta_\delta,\end{aligned}$$

and since $0 \leq w \leq \pi/2$ and $|\nu(u)-\nu(v)| \leq 2$ we see $|w\,(\nu(u)-\nu(v))\cdot D\beta_\delta| \leq 4|D\beta_\delta|$, and so from (4) the right side here is $\leq 4\delta$, whereas by (3) the integrand in the integral on the left is $\geq c(x)|D(u-v)|^2\beta_\delta$ on $\Omega_\varepsilon \equiv \{x \in \Omega \cap B_\sigma(x_0) : w(x) > 0\}$, so we deduce, after letting $\delta \downarrow 0$, that $u - v = \text{const.}$ on Ω_ε, hence $\arctan(u-v) \leq \varepsilon$ everywhere in $\Omega \cap B_\sigma(x_0)$. Thus we have $\limsup_{x\to x_0} u(x) \leq \psi(x_0) + C\varepsilon$ for each $\varepsilon > 0$. Thus $\limsup_{x\to x_0} u(x) \leq \psi(x_0)$.

By the same argument applied to $-u$ with $-\psi$ in place of ψ, we then conclude that also $\liminf_{x\to x_0} u(x) \geq \psi(x_0)$, and hence $\lim_{x\to x_0, x\in\Omega} u(x) = \psi(x_0)$, as claimed.

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Stanford University
Mathematics, Bldg. 380
450 Serra Mall
Stanford, CA 94305-2125
E-mail address: lms@math.stanford.edu

Department of Mathematics
University of California
San Diego 9500 Gilman Drive
La Jolla, CA 92093-0112
E-mail address: nwickram@ucsd.edu