J. DIFFERENTIAL GEOMETRY
74 (2006) 425-505

NONABELIAN JACOBIAN OF SMOOTH PROJECTIVE SURFACES

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Abstract

The paper proposes a nonabelian version of the Jacobian for a smooth complex projective surface X. Our version possesses all the classical features: it is the parameter space for a "canonical" family of torsion-free sheaves over X having fixed Chern invariants and rank 2 , it carries a distinguished divisor (a "theta-divisor"), a "package" of nonabelain "theta-functions". But it also has a new feature: our Jacobian carries a distinguished family of Higgs bundles. The parameter space H, called (nonabelian) Albanese, of this family is a projective toric (singular) Fano variety whose hyperplane sections are (singular) Calabi-Yau varieties. In particular, it comes with a distinguished degenerate hyperplane section H_0 equipped with degenerate symplectic structure, i.e., H_0 is the union of projectivized Lagrangian subspaces of a certain symplectic vector space naturally associated with H_0.

Our Jacobian and its Albanese H are related by two correspondences:

(i) a geometric correspondence which sends points of the nonabelian Jacobian to a cycle of Calabi-Yau varieties,

(ii) a cohomological correspondence, which is a Fourier-Mukai functor from the Higgs category on the Jacobian (algebraic/holomorphic side) to the so called F-category on H (algebraic/symplectic side).

Furthermore, there is a "quantum" correspondence which associates an operator-valued series with points of our Jacobian. The operator coefficients of this series are most naturally considered as elements of the universal enveloping algebra of a certain Lie algebra canonically associated to every point of the Jacobian. This gives a sheaf of Lie algebras on our Jacobian which could be viewed as a natural analogue of the Lie algebraic structure of the classical Jacobian.

The basic Lie algebraic properties of this sheaf are established and a dictionary between its representation theory and geometry of the underlying points is given.

Received 12/16/2004.

0. Introduction

It is well-known that the Jacobian of a smooth projective curve occupies the central place in the theory of curves. Via the Jacobian and the Abel-Jacobi map one has the theory of (special) divisors. A curve (of genus ≥ 2) itself is determined, up to an isomorphism, by its Jacobian and its theta-divisor (the classical Theorem of Torelli (see, e.g., [**G-H**])). In this paper we propose a (nonabelian) version of the Jacobian for smooth complex projective surfaces. Our approach can be viewed as a natural generalization of the classical theory. Namely, viewing the Jacobian of a curve as a parameter space for line bundles with a fixed Chern class and the theta-divisor as a section of a distinguished bundle on the Jacobian one is tempted to generalize this set-up for an n-dimensional ($n \geq 2$) smooth projective variety along the following lines:

1) fix the Chern classes for a holomorphic vector bundle of rank n over an n-dimensional smooth projective variety X and construct a "canonical" family of such bundles. The parameter space of such a family might be considered as a nonabelian version of Jacobian for n-dimensional varieties;
2) this nonabelian Jacobian should have an analogue of the theta-divisor or, more generally, a "package" of nonabelian "theta-functions", i.e., a collection of sections of some sheaves naturally associated to the nonabelian Jacobian.

This paper considers the case of smooth complex projective surfaces. A nonabelian Jacobian which we propose possesses all the "classical" attributes: it is the parameter space of a "canonical" family of torsion-free sheaves of rank 2 with fixed Chern invariants, it carries a distinguished "theta-divisor" and a "package" of nonabelian "theta-functions". But it also exhibits new features such as Higgs bundles, relation to Calabi-Yau varieties and quantum-type invariants. The main goal of the paper is to give a unified account of a construction of our nonabelian Jacobian and its various features.

This somewhat long introduction is intended to be an overview of a circle of ideas in their (hopefully) logical sequence as well as a guide for "navigation" through various parts of our constructions. We will try to summarize here what represents to our mind essential moments/ideas of our constructions with a bare minimum of technical details. This way we hope a reader can obtain a global understanding of the properties of our Jacobian and its potential applications without being bogged down by the technical side of the story.

Part I: Definition of a nonabelian Jacobian and its theta-divisor

We fix a line bundle $\mathcal{O}_X(L)$ over a smooth complex projective surface X and a positive integer d. It will be assumed that $H^i(\mathcal{O}_X(-L)) = 0$, for $i = 0, 1$ (e.g. the divisor L is ample). We are aiming at geometric applications so this is a reasonable assumption.

Conceptually our nonabelian Jacobian can be introduced using the language of categories. Consider the category **Pairs** whose objects are pairs $(\mathcal{E}, e)$, where $\mathcal{E}$ is a torsion-free sheaf over X having rank 2 and Chern classes (L, d), and e is a global section of $\mathcal{E}$, whose zero-locus $Z_e = (e = 0)$ is 0-dimensional. The morphisms between two objects of **Pairs** are morphisms of sheaves which preserve the marked sections. For any scheme B over $\mathbf{C}$ let **Pairs** (B) be the category of pairs over B, i.e. the category whose objects are pairs $(\tilde{\mathcal{E}}, \tilde{e})$, where $\tilde{\mathcal{E}}$ is a torsion-free sheaf over $X \times B$ and $\tilde{e}$ is its section whose zero-locus $Z_{\tilde{e}}$ is a subscheme of $X \times B$ finite over B and such that the restriction of $(\tilde{\mathcal{E}}, \tilde{e})$ to each slice $X \times \{b\}$, for any closed point $b \in B$, is an object of **Pairs**. The morphisms in **Pairs** (B) are again the morphisms of sheaves on $X \times B$ preserving the marked sections. With this category in mind our Jacobian is a scheme having a certain universal property with respect to the category **Pairs**.

Proposition-Definition 0.1. There exists a scheme denoted $\mathbf{J}(X; L, d)$ whose closed points are in one-to-one correspondence with objects of the category **Pairs** and having the following universal property:
For every scheme B and every object $(\tilde{\mathcal{E}}, \tilde{e})$ in **Pairs**(B) there is unique morphism $f_B : B \longrightarrow \mathbf{J}(X; L, d)$ which sends every closed point $b \in B$ to the closed point of $\mathbf{J}(X; L, d)$ corresponding to the restriction of $(\tilde{\mathcal{E}}, \tilde{e})$ to the slice $X \times \{b\}$.
The scheme $\mathbf{J}(X; L, d)$ is called nonabelian Jacobian of X (if no ambiguity is likely we denote $\mathbf{J}(X; L, d)$ simply by $\mathbf{J}$).

Alternatively, one can think of $\mathbf{J}(X; L, d)$ as the scheme of pairs $([Z], [\alpha])$, where $[Z]$ is a point in the Hilbert scheme $X^{[d]}$ of 0-dimensional subschemes of X of length d, and $[\alpha] \in \mathbf{P}(\mathrm{Ext}^1_Z)$, where Ext^1_Z is the group of extensions $\mathrm{Ext}^1(\mathcal{J}_Z(L), \mathcal{O}_X)$ ($\mathcal{J}_Z$ is the sheaf of ideals of Z in X). This point of view gives a constructive definition of $\mathbf{J}(X; L, d)$.

Theorem 0.2. *There is a coherent sheaf $\mathcal{S}$ over the Hilbert scheme $X^{[d]}$ such that $\mathbf{J}(X; L, d) = \mathbf{Proj}(\mathcal{S})$. In particular, $\mathbf{J}(X; L, d)$ is equipped with the natural projection $\pi : \mathbf{J}(X; L, d) \longrightarrow X^{[d]}$ and an invertible sheaf $\mathcal{O}_{\mathbf{J}(X;L,d)}(1)$ such that its direct image $\pi_*(\mathcal{O}_{\mathbf{J}(X;L,d)}(1)) = \mathcal{S}$.*

The sheaf $\mathcal{S}$ induces a natural rank stratification of $X^{[d]}$

$$X^{[d]} \supset \Gamma^0_d(L) \supset \Gamma^1_d(L) \supset \cdots \supset \Gamma^r_d(L) \supset \dots$$

where $\Gamma^r_d(L) = \{[Z] \in X^{[d]} \mid rk(\mathcal{S}_{[Z]}) \geq r+1\}$, where $\mathcal{S}_{[Z]}$ is the fibre of the sheaf $\mathcal{S}$ at $[Z] \in X^{[d]}$. The natural projection $\pi : \mathbf{J}(X; L, d) \longrightarrow X^{[d]}$ induces the stratification

$$\mathbf{J}(X; L, d) = \mathbf{J}^0 \supset \mathbf{J}^1 \supset \cdots \supset \mathbf{J}^r \supset \ldots \tag{0.1}$$

where $\mathbf{J}^r = \mathbf{Proj}(\mathcal{S} \otimes \mathcal{O}_{\Gamma^r_d(L)})$. In particular, each stratum is a closed subscheme of $\mathbf{J}(X; L, d)$ and the open stratum $\overset{\circ}{\mathbf{J}^r} = \mathbf{J}^r \setminus \mathbf{J}^{r+1}$ is a $\mathbf{P}^r$-bundle over $\overset{\circ}{\Gamma^r_d}(L) = \Gamma^r_d(L) \setminus \Gamma^{r+1}_d(L)$. Of course, we get something new only for $r \geq 1$. This will be assumed, unless said otherwise, for the rest of the introduction.

An important property of the strata $\mathbf{J}^r$ is that they admit a universal extension, or in category language, $\mathbf{Pairs}(\mathbf{J}^r)$ has a distinguished object. To state our result consider the product $X \times \mathbf{J}^r$ with the natural projections $f_1 : X \times \mathbf{J}^r \longrightarrow X$, $f_2 : X \times \mathbf{J}^r \longrightarrow \mathbf{J}^r$ and $\tilde{\pi} : X \times \mathbf{J}^r \longrightarrow X \times \Gamma^r_d(L)$. Let $\mathcal{Z} \subset X \times \Gamma^r_d(L)$ be the universal cluster and $\tilde{\mathcal{Z}}$ its pullback via the projection $\tilde{\pi}$. Then the following holds.

Theorem 0.3. *Over $X \times \mathbf{J}^r$ there is a torsion free sheaf $\mathbf{E}$ which fits into the following exact sequence*

$$0 \longrightarrow f_2^* \mathcal{O}_{\mathbf{J^r}}(1) \longrightarrow \mathbf{E} \longrightarrow \tilde{\pi}^* \mathcal{J}_{\mathcal{Z}} \otimes f_1^* \mathcal{O}(L) \longrightarrow 0 \tag{0.2}$$

where $\mathcal{J}_{\mathcal{Z}}$ is the ideal sheaf of $\mathcal{Z}$. Furthermore, for every closed point $([Z], [\alpha]) \in \mathbf{J}^r$ the restriction of the sequence (0.2) *to the slice $X \times \{([Z], [\alpha])\}$ corresponds to the extension defined by α.*

At this stage we accomplished the first part of our construction which we summarize in the form of a theorem.

Theorem 0.4.
1) $\mathbf{J}(X; L, d)$ *is a scheme whose closed points are in one-to-one correspondence with objects in the category* **Pairs***.*
2) $\mathbf{J}(X; L, d)$ *has natural stratification* (0.1) *by closed subschemes* $\mathbf{J}^r$*.*
3) *Each stratum* $\mathbf{J}^r$ *comes with a universal extension described in Theorem* 0.3*.*

The properties listed above present an obvious parallel with the classical Jacobian. This analogy persists since it turns out that over an appropriate part of the Hilbert scheme $X^{[d]}$ the scheme $\mathbf{J}(X; L, d)$ carries a distinguished divisor.

Proposition-Definition 0.5. Let $\Gamma = \Gamma^s_{lci}$ be the part of $\Gamma^0_d(L)$ parametrizing L-stable clusters (see Definition 1.1) which are local complete intersections. Then the scheme $\mathbf{J}_\Gamma = \mathbf{Proj}(\mathcal{S} \otimes \mathcal{O}_\Gamma)$ carries a distinguished divisor denoted $\mathbf{\Theta}(X; L, d)$. This is a Cartier divisor corresponding to a distinguished section of an invertible sheaf $\mathcal{O}_{\mathbf{J}_\Gamma}(d) \otimes \pi^*(\mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on Γ. The closed points of $\mathbf{\Theta}(X;L,d)$ correspond to pairs $(\mathcal{E},e)$, where the sheaf $\mathcal{E}$ is not locally free. The divisor $\mathbf{\Theta}(X;L,d)$ will be called the theta-divisor of $\mathbf{J}(X;L,d)$.

At this stage the pair $(\mathbf{J}(X;L,d),\mathbf{\Theta}(X;L,d))$ is a rather precise analogue of the classical Jacobian. However, there is a new feature: $\mathbf{J}$ carries a variation of Hodge-like structure.

Part II: Variation of Hodge-like structure on $\mathbf{J}(X;L,d)$

By a variation of Hodge-like structure we mean a decreasing filtration of sheaves equipped with a "derivative" which shifts the index of the filtration at most by 1 (an analogue of Griffiths' transversality property for the Infinitesimal variation of Hodge structure, [**G**]). This seems to us a qualitatively new feature of our Jacobian (with respect to the classical one). It reflects at the same time the fact that we are in dimension >1 and are dealing with higher rank bundles.

Consider the following sheaves:
$\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ and $\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}}\otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$, where $\mathcal{Z}\subset X\times X^{[d]}$ is the universal cluster. These sheaves are locally free over $\mathbf{J}(X;L,d)$. It is obvious but crucial for all our subsequent constructions to observe that $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ is a sheaf of commutative rings and $\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}}\otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$ is a sheaf of modules over $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$.

Proposition-Definition 0.6. On every sratum $\mathbf{J}^r,(r\geq 1)$, the following holds.

1) The sheaf of rings $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ admits a distinguished filtration

$$0=\tilde{\mathbf{H}}_1\subset\tilde{\mathbf{H}}_0\subset\tilde{\mathbf{H}}_{-1}\subset\cdots\subset\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)\tag{0.3}$$

subject to the following properties.

a) $\tilde{\mathbf{H}}_0=\mathcal{O}_{\mathbf{J}^r}$.

b) Set $\tilde{\mathbf{H}}_{-1}=\tilde{\mathbf{H}}$, then the multiplication in $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ induces the morphisms

$$\mathbf{m_k}:S^k\tilde{\mathbf{H}}\longrightarrow\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$$

for every $k\geq 0$. The sheaf $\tilde{\mathbf{H}}_{-k}$ is defined to be the image of $\mathbf{m_k}$. In particular, one obtains the multiplication morphisms

$$\tilde{\mathbf{H}}\otimes\tilde{\mathbf{H}}_{-k}\longrightarrow\tilde{\mathbf{H}}_{-k-1}\tag{0.4}$$

for every $k\geq 0$.

2) The sheaf $\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}}\otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$ of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$-modules admits a distinguished filtration

$$\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}}\otimes p_1^*(\mathcal{O}_X(K_X+L))\big))=\mathbf{F_0}\supset\mathbf{F_1}\supset\mathbf{F_2}\supset\ldots\tag{0.5}$$

Furthermore the $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$-module structure induces the following morphisms

$$\tilde{\mathbf{H}}\otimes\mathbf{F_k}\longrightarrow\mathbf{F_{k-1}}\tag{0.6}$$

for every $k \geq 1$.

The algebro-geometric meaning of the filtration (0.3) is easy to understand: let $\tilde{\mathbf{H}}_{[Z],[\alpha]}$ be the fibre of $\tilde{\mathbf{H}}$ at the closed point $([Z],[\alpha])$ of $\mathbf{J}^r$. This is a subspace of $H^0(\mathcal{O}_Z)$. So it can be viewed as a linear system on Z. Moreover, the linear system is base point free since it contains the subspace $\tilde{\mathbf{H}}_0([Z],[\alpha])$, the fibre of $\tilde{\mathbf{H}}_0$ at $([Z],[\alpha])$, which is simply the subspace of the constant functions on Z. So $\tilde{\mathbf{H}}_{[Z],[\alpha]}$ defines a morphism

$$\kappa_{[Z],[\alpha]} : Z \longrightarrow \mathbf{P}(\tilde{\mathbf{H}}^*_{[Z],[\alpha]}).$$

Thus the filtration (0.3) captures the geometry of the morphism $\kappa_{[Z],[\alpha]}$. In particular, the ranks of the sheaves $\tilde{\mathbf{H}}_{-k}$ at $([Z],[\alpha])$ determine the Hilbert function of the image of $\kappa_{[Z],[\alpha]}$.

On the reduction of every irreducible component of $\mathbf{J}^r$ the sheaves in both filtrations are torsion free. So the rank p_k of $\tilde{\mathbf{H}}_{-k}$ on such a component is well-defined. Setting $P(k) = p_k$, for every $k \geq 0$, we obtain the Hilbert function P attached to every reduced irreducible component of $\mathbf{J}^r$. This way we arrive to a collection of admissible Hilbert functions associated to $\mathbf{J}^r$. Fix one of them, say P, and denote by $\mathbf{J}^r_P$ the reduced subscheme of $\mathbf{J}^r$, where the ranks of sheaves in the filtration (0.3) are constant and determined by the function P. On such a locus all sheaves of the filtration $\tilde{\mathbf{H}}_{-\bullet}$ are locally free. Furthermore the filtrations $\tilde{\mathbf{H}}_{-\bullet}$ and $\mathbf{F}_\bullet$ are related as follows.

Proposition 0.7. *For every $k \geq 1$ there is an isomorphism*

$$\tilde{\mathbf{H}}_{-k-1}/\tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\mathbf{J}^r_P}(-1) \cong \left(\mathbf{F_k}/\mathbf{F_{k+1}}\right)^*.$$

From Proposition-Definition 0.6 it follows that the sheaf $\tilde{\mathbf{H}}$ plays a special role. In fact it is closely related to the sheaf $\pi^*(\mathcal{S})$, where $\mathcal{S}$ is as in Theorem 0.2.

Proposition 0.8.

1) *There is a natural morphism*

$$\tilde{\mathbf{M}} : \tilde{\mathbf{H}} \otimes \mathcal{O}_{\mathbf{J}^r}(-1) \longrightarrow \mathcal{H}om((\pi^*\mathcal{S}) \otimes \mathcal{O}_{\mathbf{J}^r}, \mathcal{O}_{\mathbf{J}^r}) \tag{0.7}$$

which descends to the morphism

$$\mathbf{M} : \tilde{\mathbf{H}}/\tilde{\mathbf{H}}_0 \longrightarrow \mathcal{T}_{\mathbf{J}/X^{[d]}}$$

where $\mathcal{T}_{\mathbf{J}/X^{[d]}}$ is the relative tangent sheaf of $\pi : \mathbf{J}(X;L,d) \longrightarrow X^{[d]}$.

2) *Let Γ be as in Proposition-Definition 0.5 and let $\dot{\mathbf{J}}^r$ be the complement of the theta-divisor $\mathbf{\Theta}(X;L,d)$ in $\mathbf{J}^r_\Gamma$. Then the morphisms $\tilde{\mathbf{M}}$ and $\mathbf{M}$ are isomorphisms over $\dot{\mathbf{J}}^r$.*

The point of the above identification is as follows. The multiplication morphisms in Proposition-Definition 0.6 shift the index of our filtrations by 1. So these morphisms can be viewed as "formal derivatives" inducing formal Griffiths' transversality condition. However, the isomorphism in Proposition 0.8 allows to show that over $\dot{\mathbf{J}}^r_P = \dot{\mathbf{J}}^r \cap \mathbf{J}^r_P$ the "formal derivatives" coincide, up to a nonzero factor, with actual differentiation along the fibre directions of the fibration $\dot{\mathbf{J}}^r_P$ over $X^{[d]}$.

Part III: The direct sum decomposition of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ and its Higgs structures

Let $(\mathbf{J}^r_P)'$ be the part of $\dot{\mathbf{J}}^r_P$ lying over smooth points of the Hilbert scheme $X^{[d]}$, i.e., we consider the points $([Z],[\alpha]) \in \mathbf{J}^r_P$ such that Z is d distinct points of X and α determines a locally free extension. The main observation is that over such points there is a natural identification of $H^0(\mathcal{O}_Z(K_X+L))$ with $H^0(\mathcal{O}_Z)$. Thus the two filtrations in Proposition-Definition 0.6 can be put together to define two filtrations on $H^0(\mathcal{O}_Z)$. Furthermore there is a nonempty Zariski open subset $\breve{\mathbf{J}}^r_P$ of $(\mathbf{J}^r_P)'$ over which the two filtrations will be opposed. This leads to a direct sum decomposition

$$H^0(\mathcal{O}_Z) = \bigoplus_{p=0}^{l} \mathbf{H}^{\mathbf{p}}([Z],[\alpha]). \tag{0.8}$$

Varying $([Z],[\alpha])$ in $\breve{\mathbf{J}}^r_P$ yields the direct sum decomposition of the sheaf $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$.

Theorem 0.9. *Over $\breve{\mathbf{J}}^r_P$ the sheaf $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ admits a distinguished direct sum decomposition*

$$\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) = \bigoplus_{p=0}^{l} \mathbf{H}^{\mathbf{p}}. \tag{0.9}$$

The number of summands l will be called the weight of the decomposition.

In this decomposition the summand $\mathbf{H}^{\mathbf{0}} = \tilde{\mathbf{H}}$ and we will now consider its multiplicative action on the direct sum (0.9). This is essential for all subsequent constructions.

Proposition 0.10. *The multiplication by $\tilde{\mathbf{H}}$ gives rise to the sheaf morphisms*

$$D_p : \mathbf{H}^{\mathbf{p}} \longrightarrow \tilde{\mathbf{H}}^* \otimes (\mathbf{H}^{\mathbf{p-1}} \oplus \mathbf{H}^{\mathbf{p}} \oplus \mathbf{H}^{\mathbf{p+1}}). \tag{0.10}$$

In particular, the morphism D_p admits the following decomposition

$$D_p = D^-_p + D^0_p + D^+_p$$

where each component is obtained by composing D_p with the projection onto the corresponding summand in (0.10).

Thus the multiplication in $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ can be reinterpreted as the sheaf morphism together with the following decomposition

$$D = \sum_{p=0}^{l} D_p = D^- + D^0 + D^+, \tag{0.11}$$

where $D^{\pm} = \sum_{p=0}^{l} D_p^{\pm}$ and $D^0 = \sum_{p=0}^{l} D_p^0$. Furthermore the commutativity of the multiplication in $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ translates into the fact that D is a Higgs morphism, i.e. $D^2 = 0$. This in turn imposes relations on the components D^-, D^0, D^+.

Proposition 0.11. *The decomposition* (0.11) *is subject to the following identities:*

(i) $D^2 = (D^-)^2 = (D^+)^2 = 0$,
(ii) $D^- \wedge D^0 + D^0 \wedge D^- = D^+ \wedge D^0 + D^0 \wedge D^+ = 0$,
(iii) $(D^0)^2 + D^- \wedge D^+ + D^+ \wedge D^- = 0$.

Thus D, $D^{\pm}$ are natural Higgs structures of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$. But these are just three particular Higgs morphisms in a large family of such structures. This family of Higgs structures constitutes the next feature of our construction.

Part IV: A nonabelian Albanese of $\mathbf{J}(X; L, d)$

We will assume that the weight l in (0.9) is ≥ 2. To construct our family of Higgs morphisms we take a sufficiently general "perturbation" of D in (0.11), i.e., we consider the morphisms of the form

$$\sigma(t,x,y) = \sum_{p=0}^{l-1} t_p D_p^0 + \sum_{p=0}^{l-2} x_p D_p^+ \quad + \sum_{p=0}^{l-2} y_p D_{p+1}^-$$

where $t = (t_p) \in \mathbf{C^l}$, $x = (x_p)$, $y = (y_p) \in \mathbf{C^{l-1}}$ and we seek the conditions on the parameters (t,x,y) for which the morphism $\sigma(t,x,y)$ is Higgs.

Proposition-Definition 0.12. The subset $\hat{H}$ of $\mathbf{C} \times \mathbf{C^{l-1}} \times \mathbf{C^{l-1}}$ defined by the quadratic relations $X_p Y_p = T^2$ $(p = 0, ..., l-2)$ parametrizes a family of Higgs morphisms of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$, i.e., every point $(z, x_0, \ldots, x_{l-2}, y_0, \ldots, y_{l-2}) \in \hat{H}$ corresponds to the Higgs morphism

$$\sigma(z,x,y) = zD^0 + \sum_{p=0}^{l-2} x_p D_p^+ \quad + \sum_{p=0}^{l-2} y_p D_{p+1}^-.$$

The multiplication of a Higgs morphism by a nonzero scalar induces an obvious $\mathbf{C}^*$-action on $\hat{H}$. Set $\hat{H}' = \hat{H} \setminus \{0\}$. The variety $H = \hat{H}'/\mathbf{C}^*$ is a variety of the homothety equivalent non-zero Higgs morphisms of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ and it is called nonabelian Albanese of $\mathbf{J}_P^r$.

The Albanese H is a projective variety with the following properties.

Proposition 0.13. *Let $\mathbf{P^{2(l-1)}}$ be a projective space with the homogeneous coordinates T, X_p, Y_p, $(p = 0, \ldots, l-2)$. Then H is a complete intersection of $l-1$ quadrics $X_pY_p = T^2\,(p = 0, \ldots, l-2)$ in $\mathbf{P^{2(l-1)}}$. In particular, H is a Fano variety of dimension $(l-1)$ and of degree 2^{l-1} with the dualizing sheaf $\omega_H = \mathcal{O}_H(-1)$.*

The Albanese H comes together with a distinguished divisor $H_0 = (T = 0)$, corresponding to the Higgs morphisms composed only of morphisms of type $D^{\pm}_k$. Projectively, the divisor H_0 is a degenerate divisor: it is a union of projective spaces. But it is also degenerate from the symplectic point of view. Namely, from Proposition 0.13 it follows that the hyperplane sections of H are (singular) Calabi-Yau varieties and the point is that the divisor H_0 comes with a *degenerate* symplectic structure: the projective spaces composing it are the projectivized Lagrangian subspaces of a certain symplectic vector space naturally associated to H_0 (Lemma 4.7, Definition 4.8). For this reason we call H_0 a "Lagrangian" cycle of the Albanese H and the irreducible components of H_0 are called "Lagrangian" manifolds.

The Albanese H admits a natural torus action which gives it a structure of a toric variety.

Proposition 0.14. *The nonabelian Albanese H is a projective toric variety with an action of the torus $S = (\mathbf{C}^*)^{\mathbf{l-1}}$. Its fan Δ is the fan in $\mathbf{R^{l-1}}$ generated by the vertices of the cube $[-1,1]^{l-1}$ and the vertices of the cube are in 1-to-1 correspondence with the irreducible components of the divisor H_0. In particular, $Pic(H)$ is generated by the irreducible components of the "Lagrangian" cycle H_0.*

It is important to observe that the Albanese H and its Lagrangian cycle depend only on the weight l of the direct sum decomposition (0.9) and the relations between the morphisms of different degrees composing the Higgs morphisms in Proposition-Definition 0.12. So it "forgets" about our surface X. We remedy this situation in two ways: geometric and cohomological correspondences between $\breve{\mathbf{J}}^r_P$ and its Albanese H.

Part V: Geometric and Cohomological Correspondences between $\breve{\mathbf{J}}^r_P$ and H

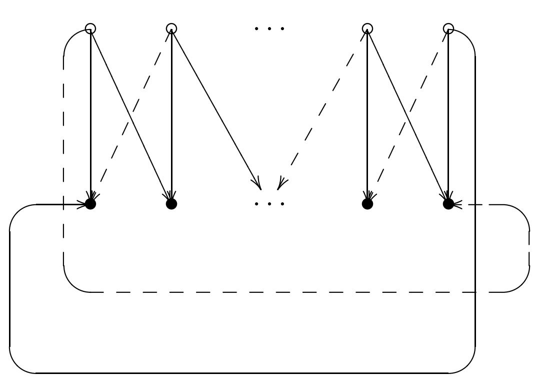
Our geometric correspondence sends a point $([Z],[\alpha]) \in \breve{\mathbf{J}}^r_P$ to a cycle of Calabi-Yau varieties. More precisely, with every point $(z,[\alpha]) \in (Z,[\alpha])$ we associate a particular hyperplane section $H_{(z,[\alpha])}$ of H (distinct from H_0) which is, of course, a Calabi-Yau variety. Then $([Z],[\alpha])$ is sent to the divisor $CY([Z],[\alpha]) = \sum_{z \in Z} H_{(z,[\alpha])}$ which is called a Calabi-Yau cycle of $([Z],[\alpha])$ and CY is a Calabi-Yau cycle map (Proposition

5.1, Proposition 5.3). Thus what comes out of this construction is that behind appropriately chosen points $[Z]$ of the Hilbert scheme $X^{[d]}$ one can "see" Calabi-Yau varieties. These Calabi-Yau varieties come in families: there is a continuous parameter $[\alpha] \in \mathbf{P}(\mathrm{Ext}^1_Z)$ and for a fixed $[\alpha]$ and $z \in Z$ there is also a discrete dynamic of the hyperplane sections $H_{(z,[\alpha])}$ (see Remark 5.2). This gives a possibility of "linking" distinct points of Z in X via the corresponding Calabi-Yau varieties as well as a possibility to "transit" via geometry/topology of Calabi-Yau varieties between points on *different* surfaces. These issues will be discussed in more detail elsewhere.

We turn now to a cohomological correspondence between $\breve{\mathbf{J}}^r_P$ and its Albanese H. This correspondence is a Fourier-Mukai functor between what we call the Higgs category of weight l on $\breve{\mathbf{J}}^r_P$ and the so called F-category on H. The objects of the Higgs category are of algebraic/holomorphic nature. These are graded $\mathcal{O}_{\breve{\mathbf{J}}^r_P}$-modules equipped with Higgs morphisms which shift the grading at most by 1 (see Definition 5.11). The objects of the F-category on H are graded $\mathcal{O}_H$-modules with extra-data subordinate to the structure of the "Lagrangian" cycle H_0 of H. This data is a collection of complexes naturally associated with every Lagrangian manifold of H_0 and some natural relations between these complexes for every pair of the transversally intersecting Lagrangian manifolds (see Definition 5.12). This evokes a certain analogy with Fukaya category, hence the names - F-category and Fukaya type data for the category and for the extra-data, respectively. Thus the cohomological correspondence between $\breve{\mathbf{J}}^r_P$ and its Albanese fits the general philosophy of the homological mirror symmetry conjecture of Kontsevich,[**K**].

Part VI: The trivalent graph of $\breve{\mathbf{J}}^r_P$ and quantum invariants

The direct sum decomposition in Theorem 0.9 and the decomposition of morphisms D_p in Proposition 0.10 can be encapsulated in the following trivalent graph:



The upper (resp. lower) vertices of the graph represent the first l summands of the direct sum decomposition (0.9) ordered by the index set $I = \{0, 1, \dots, l-1\}$ from left to right, the vertical edges are the decomposition preserving morphisms D^0, while other edges of the graph represent the degree-shifting morphisms $D^{\pm}$ (the morphism D^+ (resp. D^-) is depicted by the edges going from left (resp. right) to right (resp. left)). This graph and a technique reminiscent of Reshetikhin-Turaev knot invariants, [**Tu**], allow to go from a "classical observable"= function $t \in \mathbf{H^0}([Z], [\alpha])$ to a "quantum operator" = operator-valued generating series. By choosing functions in $\mathbf{H^0}([Z], [\alpha])$ intrinsically associated to the points of Z we arrive to the "quantum operators" or "quantum" invariants of $([Z], [\alpha])$. The details of this construction with the form of the generating series are given in §6 so we do not reproduce it here. To go on to the next aspect of our Jacobian we just need to know that the coefficients of these generating series are the compositions of the operators of the form $D^0(t), D^{\pm}(t)$, for $t \in \mathbf{H^0}([Z], [\alpha])$, where $D^0(t)$ (resp. $D^{\pm}(t)$) denotes the value of the morphism D^0 (resp. $D^{\pm}$) at the vector $t \in \mathbf{H^0}([Z], [\alpha])$. Hence these coefficients are most naturally regarded as elements of the universal enveloping algebra of the Lie subalgebra $\tilde{\mathbf{g}}([Z], [\alpha])$ of $\mathrm{End}\,(H^0(\mathcal{O}_Z))$ generated by the following set of endomorphisms,

$$\{D^{\pm}(t), D^0(t) \mid t \in \mathbf{H^0}([Z], [\alpha])\}.$$

This Lie algebraic point of view constitutes the next aspect of our constructions.

Part VII: Lie algebras associated to $\breve{\mathbf{J}}^r_P$

As $([Z], [\alpha])$ moves in $\breve{\mathbf{J}}^r_P$ the Lie algebras $\tilde{\mathbf{g}}([Z], [\alpha])$ fit together to form a sheaf of Lie algebras $\tilde{\mathcal{G}}$ over $\breve{\mathbf{J}}^r_P$. This structure could be regarded as an analogue of the (abelian) Lie algebra structure of the classical Jacobian.

We investigate the basic properties of the Lie algebras $\tilde{\mathbf{g}}([Z], [\alpha])$. In particular, it is shown that $\tilde{\mathbf{g}}([Z], [\alpha])$ is reductive and its representation theory is intimately related to the geometry of the underlying subscheme Z (Proposition 7.2, Theorem 7.11, Corollary 7.13)).

It should be mentioned that the Lie algebras $\tilde{\mathbf{g}}([Z], [\alpha])$ could be also regarded as an analogue of the sl_2 representation in the Hard Lefschetz theorem (see, e.g., [**G-H**]): in our sitution the role of a line bundle is played by $[\alpha] \in \mathbf{P}(\mathrm{Ext}^1_Z)$, the usual Hodge decomposition of a projective manifold is replaced by the decomposition (0.8), while the roles of Kähler metric, the associated $(1,1)$-operator L and its adjoint Λ are taken over by a choice of $t \in \mathbf{H^0}([Z], [\alpha])$, the operator $D^+(t)$ and its adjoint $D^-(t)$, respectively. In fact our Lie algebra $\tilde{\mathbf{g}}([Z], [\alpha])$ is very close in spirit to the Lie algebra introduced by Looijenga and Lunts in [**L-L**].

Part VIII: Concluding remarks and speculations

In the last few decades the idea that the Hilbert scheme of points of a given variety should be viewed as a "boundary" of a larger space made its appearance in various guises: it shows up, for example, in the theory of Donaldson's invariants of 4-dimensional manifolds in relation with compactification of the instanton moduli space, [**D-K**], and more recently in the works of Grojnowski and Nakajima relating the representation of Kac-Moody algebras with the cohomology of the Hilbert schemes of complex projective surfaces (see [**N**] and references therein for other related works). The nonabelian Jacobian proposed in this paper follows the same line of thought. Indeed, we have started with the Hilbert scheme $X^{[d]}$ and "extended" it to $\mathbf{J}(X; L, d)$. So one might think of $X^{[d]}$ as the "classical" level. In particular, when we take a smooth point $[Z]$ in the strata $\Gamma^r_d(L)$ all we see is a set of d distinct points (particles) with the space of functions $H^0(\mathcal{O}_Z)$ on Z as the space of "classical observables" of Z . Going over to the Jacobian $\mathbf{J}(X; L, d)$ has an effect of attaching the space of parameters $\mathbf{P}(\mathrm{Ext}^1_Z)$ to Z. One of the consequences of this is that Z becomes a dynamic object. This dynamics manifests itself in the emergence of a variation of Hodge-like structures which in turn leads to various correspondences described in **Part V** and Lie algebras and their representations in **Part VII**. So the points cease to be points: they "open up" to become either Calabi-Yau varieties (geometric correspondence) or linear operators (quantum correspondence). This certainly resembles the quantum gravity picture according to the String theory. The emergence of the trivalent graph in **Part VI** also points in the same direction. It also indicates connections with low-dimensional topology (invariants of knots, 3-manifold invariants) and the moduli spaces of curves.

Another obvious consequence of our considerations is that the representation theory of Lie algebras is naturally attached to subvarieties of higher codimension. It seems to us that the construction presented here could provide a general mechanism for relating subvarieties of codimension ≥ 2 with Lie algebras and their representations. This could be useful in unlocking some of the mysteries about algebraic cycles.

The constructions developed in the paper allow to view vector bundles from a somewhat new perspective. Namely, in our setting a vector bundle $\mathcal{E}$ of rank 2 over X with Chern invariants (L, d) gives rise to a kind of "normal" function defined on a Zariski open subset of $\mathbf{P}(H^0(\mathcal{E}))$ and taking values in $\mathbf{J}(X; L, d)$. This way one can attach to $\mathcal{E}$ new invariants of geometric, Lie algebraic and quantum nature coming respectively from the Calabi-Yau cycle map in **Part V**, Lie algebras in **Part VII** and quantum-type invariants in **Part VI**. This opens up new lines of inquiry on vector bundles such as classification questions according to the properties of these new invariants.

The contents of the paper is as follows:

Parts I, II are treated in §1:
- the nonabelian Jacobian $\mathbf{J}(X; L, d)$ is defined in §1.1 and its "theta-divisor" in §1.2,
- the two variations of Hodge-like structure are introduced in §1.3;
- the cohomological invariant $\mathbf{C}_{\mathbf{J}(X;L,d)}$ is discussed in §1.4 (this is our "package" of nonabelian "theta-functions").

§1 is the most technical part of the paper (for a more detailed account of its contents see [**R2**]).

Part III is the contents of §2 and §3:
- the direct sum decomposition (0.9) is treated in §2 (Lemma 2.1, Corollary 2.4);
- the decompositions of the morphisms D_p in (0.10) and the decomposition of D in (0.11) is the subject of §3 (Lemma 3.6, Lemma 3.9);

Part IV- the nonabelian Albanese is treated in §4 (Proposition 4.3, Proposition 4.9).

Part V- the two correspondences between $\breve{\mathbf{J}}^r_P$ and H are given in §5:
- §5.1 treats the geometric correspondence (the Calabi-Yau cycle map is constructed in Proposition 5.1 (see also Proposition 5.3));
- §5.2 is an example of complete intersection which illustrates our general considerations;
- in §5.3 we discuss a cohomological correspondence between $\breve{\mathbf{J}}^r_P$ and H: the Higgs and F-categories are defined and the Fourier-Mukai functor $\mathcal{F}$ is constructed;

Part VI- the trivalent graph $G(\breve{\mathbf{J}}^r_P)$ and quantum type invariants for points in $\breve{\mathbf{J}}^r_P$ are treated in §6;

Part VII- the Lie algebraic aspect of our construction is the subject of §7.

It is a pleasure to thank Vladimir Roubtsov for his interest in this project, numerous fruitful discussions and valuable comments. We also would like to thank the referees of the paper whose comments and criticism helped to improve the paper.

1. Constructions

This section treats the material summarized in **Parts I, II** of the introduction: we construct our nonabelian Jacobian (§1.1) and its "theta-divisor" (§1.2). Then we define the variation of Hodge-like structures on our Jacobian (§1.3) and its cohomologiacal invariant (§1.4). Most of the present ideas about a nonabelian Jacobian of a smooth projective surface have implicitly appeared in [**R2**].

1.1. A nonabelian Jacobian. Let X be a smooth complex projective surface. Fix a divisor L with $h^0(-L) = h^1(-L) = 0$ and a positive integer d. Our version of the nonabelian Jacobian of X is the parameter space of a "canonical" family of torsion-free sheaves of rank 2 over X with Chern classes (L, d). This family is constructed as follows. We start with $X^{[d]}$, the Hilbert scheme of 0-dimensional subschemes of X, clusters of length d (the term "cluster" was proposed, we believe, by Miles Reid around 1987). Set $\mathcal{Z}^{[d]} \subset X \times X^{[d]}$ to be the universal cluster and p_i $(i = 1, 2)$ the projection of $X \times X^{[d]}$ on the i-th factor and consider the morphism of sheaves on $X^{[d]}$:

$$(1.1)\quad H^0(X, \mathcal{O}_X(L + K_X)) \otimes \mathcal{O}_{X^{[d]}} \xrightarrow{\rho} p_{2*}(\mathcal{O}_{\mathcal{Z}^{[d]}} \otimes p_1{}^*\mathcal{O}_X(L + K_X)).$$

Defining $\Gamma^r_d(L) = \left\{\xi \in X^{[d]} \mid \dim(\operatorname{coker}\rho(\xi)) \geq r + 1\right\}$ we obtain a stratification of $X^{[d]}$:

$$(1.2)\qquad X^{[d]} \supset \Gamma^0_d(L) \supset \Gamma^1_d(L) \supset \cdots \supset \Gamma^r_d(L) \supset \ldots.$$

The strata $\Gamma^r_d(L)$ are naturally closed subschemes of $X^{[d]}$ (they are degeneracy loci of a morphism between locally free sheaves on a projective variety). They will be considered, unless said otherwise, with their *reduced* scheme structure. Denote by $\overset{\circ}{\Gamma}{}^r_d(L)$ the open stratum $\Gamma^r_d(L) \setminus \Gamma^{r+1}_d(L)$. This is a Zariski open subset of $\Gamma^r_d(L)$, for every $r \geq 0$.

If $\xi \in X^{[d]}$ we denote Z_ξ the corresponding cluster of X and for a cluster Z on X we let $[Z]$ to be the corresponding point of the Hilbert scheme of clusters.

Following Tyurin,[**Ty**], we define

Definition 1.1.

(i) A cluster Z is called special with respect to L, or, L-special , iff $[Z] \in \Gamma^0_d(L)$, where $d = degZ$.

(ii) The number $\delta(L, Z) = degZ - rk\,\rho([Z])$ is called the index of L-speciality of Z.

(iii) A cluster Z is called L-stable iff $\delta(L, Z') < \delta(L, Z)$ for any proper subscheme Z' of Z.

For $[Z] \in \Gamma^r_d(L)$ consider the group of extensions

$$\operatorname{Ext}^1_Z := \operatorname{Ext}^1(\mathcal{I}_Z(L), \mathcal{O}_X)$$

where $\mathcal{I}_Z$ is the sheaf of ideals of Z in X. By Serre duality it can be identified with $H^1(\mathcal{I}_Z(L + K_X))^*$. So $\mathbf{P}_Z := \mathbf{P}(H^1(\mathcal{I}_Z(L + K_X))^*)$ can be viewed as a parameter space of torsion-free sheaves of rank 2 over X with Chern classes (L, d). Our nonabelian Jacobian $\mathbf{J}(X; L, d)$ is defined as the union of $\mathbf{P}_Z$ as $[Z]$ varies in $X^{[d]}$. More precisely, define $\mathbf{J}(X; L, d)$ to be the **Proj** of $\operatorname{coker}\rho$, where ρ is the morphism in (1.1), i.e.,

$$\mathbf{J}(X;L,d) = \mathbf{Proj}(S^{\bullet}\mathrm{coker}\,\rho) \tag{1.3}$$

where $S^{\bullet}\mathrm{coker}\,\rho$ is the symmetric algebra of $\mathrm{coker}\,\rho$. By definition $\mathbf{J}(X;L,d)$ comes with the natural projection $\pi : \mathbf{J}(X;L,d) \longrightarrow X^{[d]}$ and an invertible sheaf $\mathcal{O}_{\mathbf{J}}(1)$ such that its direct image $\pi_*(\mathcal{O}_{\mathbf{J}}(1)) = \mathrm{coker}\,\rho$ (when X, L and d are fixed and no ambiguity is likely we will omit these parameters in the definition of the Jacobian and write $\mathbf{J}$ instead of $\mathbf{J}(X;L,d)$).

Observe that the set of closed points of the fibre over a point $[Z]$ in $X^{[d]}$ is naturally homeomorphic to the projective space $\mathbf{P}(H^1(\mathcal{I}_Z(L+K_X))^*) = \mathbf{P}(\mathrm{Ext}^1_Z)$. So the set of closed points of the scheme $\mathbf{J}(X;L,d)$ is in one-to-one correspondence with the set of pairs $([Z],[\alpha])$, where $[Z] \in X^{[d]}$ and $[\alpha] \in \mathbf{P}(\mathrm{Ext}^1_Z)$. Alternatively, a pair $([Z],[\alpha])$ can be thought of as the pair $(\mathcal{E},[e])$, where $\mathcal{E}$ is the torsion-free sheaf seating in the middle of the extension sequence defined by the class α

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z(L) \longrightarrow 0 \tag{1.4}$$

and $[e]$ is the point in the projective space $\mathbf{P}(H^0(\mathcal{E}))$ corresponding to the image of $1 \in H^0(\mathcal{O}_X)$ under the monomorphism in (1.4). Thus the closed points of $\mathbf{J}(X;L,d)$ parametrize the set of pairs $(\mathcal{E},[e])$, where $\mathcal{E}$ is a torsion-free sheaf over X having rank 2 and the Chern classes (L,d). This can be expressed more formally by saying that the scheme $\mathbf{J}(X;L,d)$ has the following universal property:

consider the category **Pairs** whose objects are pairs $(\mathcal{E},e)$, where $\mathcal{E}$ is a torsion-free sheaf over X having rank 2 and Chern classes (L,d), and e is a global section of $\mathcal{E}$, whose zero-locus $Z_e = (e=0)$ is 0-dimensional. The morphisms between two objects of **Pairs** are morphisms of sheaves which preserve the marked sections. Let B be a scheme over $\mathbf{C}$ together with a pair $(\tilde{\mathcal{E}},\tilde{e})$ over B, i.e. $\tilde{\mathcal{E}}$ is a torsion-free sheaf over $X \times B$, whose restriction to each slice $X \times \{b\}$, $b \in B$, is a torsion-free sheaf of rank 2 with fixed Chern classes (L,d), and $\tilde{e}$ is a global section of $\tilde{\mathcal{E}}$ whose zero-locus $Z_{\tilde{e}} \subset X \times B$ is a scheme which is finite and flat over B. We claim that there exists unique morphism

$$f_B : B \longrightarrow \mathbf{J}(X;L,d)$$

which takes every closed point $b \in B$ to the point $[\alpha_b] \in \mathbf{P}(\mathrm{Ext}^1_{Z_{\tilde{e}(b)}})$ corresponding to the extension defined by the Koszul sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\tilde{e}(b)} \tilde{\mathcal{E}}_b \xrightarrow{\wedge\tilde{e}(b)} \mathcal{I}_{Z_{\tilde{e}(b)}}(L) \longrightarrow 0$$

where $\tilde{\mathcal{E}}_b$ and $\tilde{e}(b)$ are the restrictions of $\tilde{\mathcal{E}}$ and $\tilde{e}$ to the slice $X \times \{b\}$.

The morphism f_B is defined as follows. First observe that the section $\tilde{e}$ gives rise to the morphism

$$c_B : B \longrightarrow X^{[d]}.$$

($Z_{\tilde{e}} = (\tilde{e} = 0)$ is a family of clusters of length d on X, parametrized by B; using the universality of the Hilbert scheme $X^{[d]}$ one obtains the morphism c_B.)

Next we lift c_B to $\mathbf{J}(X; L, d)$ using the Koszul sequence on $X \times B$ defined by the pair $(\tilde{\mathcal{E}}, \tilde{e})$:

$$(1.5) \quad 0 \longrightarrow \mathcal{O}_{X\times B} \xrightarrow{\tilde{e}} \tilde{\mathcal{E}} \xrightarrow{\wedge\tilde{e}} \mathcal{J}_{Z_{\tilde{e}}} \otimes p_X^*(\mathcal{O}_X(L)) \otimes p_B^*\mathcal{M} \longrightarrow 0$$

where $\mathcal{J}_{Z_{\tilde{e}}}$ is the ideal sheaf of $Z_{\tilde{e}}$ and $\mathcal{M}$ is an invertible sheaf on B; p_X (resp., p_B) is the projection of $X \times B$ onto X (resp., B). Tensoring (1.5) with $p_X^*(\mathcal{O}_X(K_X)) \otimes p_B^*\mathcal{M}^{-1}$ and taking the direct image of the resulting sequence with respect to p_B we obtain the following surjection

(1.6)
$$\mathcal{R}^1 p_{B*}(\mathcal{J}_{Z_{\tilde{e}}} \otimes p_X^*(\mathcal{O}_X(L + K_X))) \longrightarrow H^2(\mathcal{O}_X(K_X)) \otimes \mathcal{M}^{-1} \longrightarrow 0.$$

By definition of the morphism ρ in (1.1) we have

$$c_B{}^*(\operatorname{coker}\rho) = \mathcal{R}^1 p_{B*}(\mathcal{J}_{Z_{\tilde{e}}} \otimes p_X^*(\mathcal{O}_X(L + K_X))).$$

This together with (1.6) give a surjective morphism

$$c_B{}^*(\operatorname{coker}\rho) \longrightarrow \mathcal{M}^{-1} = H^2(\mathcal{O}_X(K_X)) \otimes \mathcal{M}^{-1},$$

which, by Proposition 7.12,II, [**H**], defines a unique morphism

$$f_B : B \longrightarrow \mathbf{J}(X; L, d)$$

such that $f_B{}^*\mathcal{O}_{\mathbf{J}}(1) = \mathcal{M}^{-1}$ and the diagram

$$\begin{tikzcd} & \mathbf{J}(X;L,d) \arrow[dd, "\pi"] \\ B \arrow[ur, "f_B"] \arrow[dr, "c_B"'] & \\ & X^{[d]} \end{tikzcd}$$

commutes.

The stratification in (1.2) can be lifted by the projection $\pi : \mathbf{J}(X; L, d) \longrightarrow X^{[d]}$ to define the stratification

$$(1.7) \qquad \mathbf{J}(X; L, d) = \mathbf{J}^0 \supset \mathbf{J}^1 \supset \cdots \supset \mathbf{J}^r \supset \dots$$

where $\mathbf{J}^r = \mathbf{Proj}\Big(\operatorname{coker}(\rho) \otimes \mathcal{O}_{\Gamma_d^r(L)}\Big)$. In particular,

$$\overset{\circ}{\mathbf{J}}{}^r = \mathbf{Proj}\left(\operatorname{coker}(\rho) \otimes \mathcal{O}_{\overset{\circ}{\Gamma}{}_d^r(L)}\right) = \mathbf{J}^r \setminus \mathbf{J}^{r+1}$$

is a $\mathbf{P}^r$-bundle over $\overset{\circ}{\Gamma}{}^r_d(L)$ since the sheaf coker(ρ) is locally free of rank $r+1$ over the open strata $\overset{\circ}{\Gamma}{}^r_d(L)$.

Fix $r \geq 0$. We will now construct a universal extension over $X \times \mathbf{J}^r$, i.e., a sheaf over $X \times \mathbf{J}^r$ whose restriction to the slice $X \times \{([Z],[\alpha])\}$, for every closed point $([Z],[\alpha]) \in \mathbf{J}^r$, is isomorphic to the sheaf defined by the extension class $\alpha \in \mathrm{Ext}^1_Z$ (recall that the set of closed points of the fibre of $\mathbf{J}^r$ over $[Z]$ is homeomorphic to the projective space $\mathbf{P}(\mathrm{Ext}^1_Z)$). To do this consider the diagram

$$\begin{tikzcd}
 & X \times \mathbf{J}^r \arrow[r, "f_2"] \arrow[dl, "f_1"'] \arrow[d, "\tilde{\pi}"'] & \mathbf{J}^r \arrow[d, "\pi"] \\
X & X \times \Gamma^r_d \arrow[r, "p_2"] \arrow[l, "p_1"] & \Gamma^r_d
\end{tikzcd}$$

where $\tilde{\pi} = id_X \times \pi$ and all other arrows represent the obvious projections. Let $\mathcal{Z} \subset X \times \Gamma^r_d$ be the universal cluster and $\mathcal{J}_{\mathcal{Z}}$ its sheaf of ideals. Our construction of a universal extension is based on the following.

Lemma 1.2. *Let* $\mathrm{Ext}^1_{\mathbf{J}^r}$ *be the group of extensions*

$$\mathrm{Ext}^1(\tilde{\pi}^*\mathcal{J}_{\mathcal{Z}}, f_1^*\mathcal{O}_X(-L) \otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1)).$$

There is a natural inclusion

$$\mathrm{End}\,(\mathrm{coker}\,(\rho) \otimes \mathcal{O}_{\Gamma^r_d}) \hookrightarrow \mathrm{Ext}^1_{\mathbf{J}^r}.$$

Furthermore, this injection is an isomorphism over the open strata $\overset{\circ}{\Gamma}{}^r_d$.

Proof. This is essentially a relative version of the computation of the group Ext^1_Z in (1.16). Start from the short exact sequence defining $\mathcal{Z}$

$$0 \longrightarrow \mathcal{J}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{X\times\Gamma^r_d} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0.$$

Taking its pullback to $X \times \mathbf{J}^r$ and applying the global Ext $(\cdot, f_1^*\mathcal{O}_X(-L) \otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1))$ functor we obtain the following exact sequence of the global Ext-groups

(1.8)

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbf{J}^r} \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^2(\tilde{\pi}^*\mathcal{O}_{\mathcal{Z}}, f_1^*\mathcal{O}_X(-L) \otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1)) \longrightarrow \mathrm{Ext}^2(\mathcal{O}_{X\times\mathbf{J}^r}, f_1^*\mathcal{O}_X(-L) \otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1)).$$

We have used the fact that the term in this sequence preceding $\mathrm{Ext}^1_{\mathbf{J}^r}$ is the group

$$\mathrm{Ext}^1(\mathcal{O}_{X\times\mathbf{J}^r}, f_1^*\mathcal{O}_X(-L) \otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1))) = H^1(f_1^*\mathcal{O}_X(-L) \otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1)))$$

whose vanishing is insured by our hypothesis

$$h^0(\mathcal{O}_X(-L)) = h^1(\mathcal{O}_X(-L)) = 0.$$

Next we unravel the two Ext^2 terms in the sequence (1.8). The term on the right is the cohomology group $H^2(X\times \mathbf{J}^r, f_1^*\mathcal{O}(-L)\otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1)))$ which is equal to $H^0(\mathbf{J}^r, H^2(\mathcal{O}_X(-L))\otimes \mathcal{O}_{\mathbf{J^r}}(1))$ (this can be seen by using Leray spectral sequence for the morphism f_2 together with the vanishing hypothesis on $\mathcal{O}_X(-L)$).

The middle term of (1.8) can be identified with the group $H^0(X\times \mathbf{J}^r, \mathcal{E}xt^2(\tilde{\pi}^*\mathcal{O}_{\mathcal{Z}}, f_1^*\mathcal{O}(-L)\otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1)))$ (this can be seen by using the local-to-global spectral sequence for Ext^2 together with the fact that $\mathcal{Z}$ is a codimension 2 subvariety of $X\times\Gamma^r_d$). The sheaf

$$\mathcal{E}xt^2(\tilde{\pi}^*\mathcal{O}_{\mathcal{Z}}, f_1^*\mathcal{O}_X(-L)\otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1))$$

can be rewritten as follows:

$$\begin{aligned}(1.9)\qquad &\mathcal{E}xt^2(\tilde{\pi}^*\mathcal{O}_{\mathcal{Z}}, f_1^*\mathcal{O}_X(-L)\otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1))\\ &= \tilde{\pi}^*\left(\mathcal{E}xt^2\left(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L)\right)\right)\otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1).\end{aligned}$$

Taking its direct image with respect to f_2 we obtain

$$\begin{aligned}(1.10)\qquad &\mathrm{Ext}^2(\tilde{\pi}^*\mathcal{O}_{\mathcal{Z}}, f_1^*\mathcal{O}(-L)\otimes f_2^*\mathcal{O}_{\mathbf{J^r}}(1))\\ &= H^0\left(\mathbf{J}^r, f_{2*}\tilde{\pi}^*\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right).\end{aligned}$$

We have the natural morphism
(1.11)

$$\pi^*p_{2*}\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\longrightarrow f_{2*}\tilde{\pi}^*\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right),$$

which is an isomorphism over the open strata $\overset{\circ}{\Gamma}{}^r_d$ (the principle of commutation of cohomology with flat base extension is used here, see Remark 9.3.1, Proposition 9.3,III,[**H**]). This gives the injection on the level of the global sections

$$\begin{aligned}(1.12)\quad &H^0\left(\pi^*p_{2*}\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right)\\ &\qquad\longrightarrow H^0\left(f_{2*}\tilde{\pi}^*\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right),\end{aligned}$$

which yields the following commutative diagram
(1.13)

$$\begin{tikzcd} & & H^0\left(\pi^*p_{2*}\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right) \arrow[r]\arrow[d] & H^0\left(H^2(\mathcal{O}_X(-L))\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right) \arrow[d, equal]\\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathbf{J}^r} \arrow[r] & H^0\left(f_{2*}\tilde{\pi}^*\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right) \arrow[r] & H^0\left(H^2(\mathcal{O}_X(-L))\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right)\end{tikzcd}$$

This implies that the kernel of the top line in (1.13) injects into $\mathrm{Ext}^1_{\mathbf{J}^r}$. We claim that this kernel is $\mathrm{End}\left(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}\right)$. To see this consider the restriction of the morphism ρ in (1.1) to Γ^r_d. Dualizing we obtain

$$0 \longrightarrow \mathcal{H}om(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}, \mathcal{O}_{\Gamma^r_d}) \longrightarrow p_{2*}\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right) \longrightarrow$$

$$\longrightarrow H^2(\mathcal{O}_X(-L))\otimes\mathcal{O}_{\Gamma^r_d}$$

Taking the pullback of this sequence by π^* and then tensoring with $\mathcal{O}_{\mathbf{J}^r}(1)$ yields the following:

$$(1.14) \qquad 0 \longrightarrow \mathcal{H}om(\pi^*\left(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}\right), \mathcal{O}_{\mathbf{J^r}}(1)) \longrightarrow$$
$$\pi^* p_{2*}\left(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}}, p_1^*\mathcal{O}_X(-L))\right)\otimes\mathcal{O}_{\mathbf{J^r}}(1) \longrightarrow H^2(\mathcal{O}_X(-L))\otimes\mathcal{O}_{\mathbf{J^r}}(1).$$

Comparing this with the top line in (1.13) we deduce that its kernel is

$$\begin{aligned}
&\mathrm{Hom}\left(\pi^*\left(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}\right), \mathcal{O}_{\mathbf{J^r}}(1)\right)\\
&= \mathrm{Hom}\left(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}, \mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}\right)\\
&= \mathrm{End}(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d})
\end{aligned}$$

as claimed. q.e.d.

Taking the identity endomorphism $id_{\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}}$ of $\mathrm{coker}\,(\rho)\otimes\mathcal{O}_{\Gamma^r_d}$ and using the inclusion of Lemma 1.2 we obtain our universal extension

$$(1.15) \qquad 0 \longrightarrow f_2^*\mathcal{O}_{\mathbf{J^r}}(1) \longrightarrow \mathbf{E} \longrightarrow \tilde{\pi}^*\mathcal{J}_{\mathcal{Z}}\otimes f_1^*\mathcal{O}(L) \longrightarrow 0.$$

1.2. A Theta-divisor of $\mathbf{J}(X;L,d)$. Let $\Gamma = \Gamma^s_{lci}$ be the subset of $X^{[d]}$ parametrizing L-stable clusters which are local complete intersections. These two conditions are open. So the intersection of Γ with every open stratum $\overset{\circ}{\Gamma^r_d}(L)$ is a Zariski open subset of $\Gamma^r_d(L)$. Thus Γ can be written as a finite disjoint union of locally closed subsets (in the Zariski topology) of $X^{[d]}$. In particular, Γ is a constructable subset of $X^{[d]}$.

Denote $\mathbf{Proj}(\mathrm{coker}\,(\rho)\otimes\mathcal{O}_\Gamma)$ by $\mathbf{J}_\Gamma$. This is the part of $\mathbf{J}(X;L,d)$ lying over Γ. In $\mathbf{J}_\Gamma$ we have the locus parametrizing the extensions which are *not* locally free. We claim that this is a divisor in $\mathbf{J}_\Gamma$ which will be denoted by $\mathbf{\Theta}(X;L,d)$ and called the theta-divisor of $\mathbf{J}(X;L,d)$. Let us first consider the situation for a point $[Z]\in\Gamma$. Then $\pi^{-1}([Z]) = \mathbf{P}(\mathrm{Ext}^1_Z)$. We recall a characterization of $[\alpha]\in\mathbf{P}(\mathrm{Ext}^1_Z)$ corresponding to locally free sheaves.

The group of extensions $\mathrm{Ext}^1_Z = \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_X(-L))$ is computed from the following exact sequence (see [**G-H**])

(1.16)

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_X(-L)) \longrightarrow \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X(-L)) \longrightarrow H^2(-L).$$

By Serre duality this is dual to

$$H^0(\mathcal{O}_X(K_X+L)) \xrightarrow{\rho_Z} H^0(\mathcal{O}_Z(K_X+L)) \longrightarrow H^1(\mathcal{I}_Z(K_X+L)).$$

Given an extension class $\alpha \in \operatorname{Ext}^1(\mathcal{I}_Z, \mathcal{O}_X(-L))$ we have the cup-product

$$(1.17) \qquad H^0(\mathcal{O}_Z) \xrightarrow{\alpha} \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X(-L)) = H^0(\mathcal{O}_Z(K_X+L))^*$$

coming from the Yoneda pairing

$$\operatorname{Ext}^0(\mathcal{O}_Z,\, \mathcal{O}_Z) \otimes \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X(-L)) \longrightarrow \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X(-L))$$

and the inclusions

$$\begin{aligned} &H^0(\mathcal{O}_Z) \otimes \operatorname{Ext}^1(\mathcal{I}_Z, \mathcal{O}_X(-L)) \hookrightarrow H^0(\mathcal{O}_Z) \otimes \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X(-L)) \\ &= \operatorname{Ext}^0(\mathcal{O}_Z,\, \mathcal{O}_Z) \otimes \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X(-L)). \end{aligned}$$

Furthermore, if α corresponds to a locally free sheaf, then the cup-product with α in (1.17) is an isomorphism. Putting the homomorphisms (1.17) together as $[\alpha]$ varies in $\mathbf{P}(\operatorname{Ext}^1_Z)$ we obtain the following morphism of sheaves on $\mathbf{P}(\operatorname{Ext}^1_Z)$

(1.18)

$$\tilde{r}_Z : H^0(\mathcal{O}_Z) \otimes \mathcal{O}_{\mathbf{P}(\operatorname{Ext}^1_Z)}(-1) \longrightarrow \big(H^0(\mathcal{O}_Z(K_X+L))\big)^* \otimes \mathcal{O}_{\mathbf{P}(\operatorname{Ext}^1_Z)}$$

and the locus where $\tilde{r}_Z$ fails to be an isomorphism is a divisor $\mathbf{\Theta}_Z$ in $\mathbf{P}(\operatorname{Ext}^1_Z)$ determined by $\det(\tilde{r}_Z) = 0$. This is the fibre of $\mathbf{\Theta}(X; L, d)$ over $[Z] \in \Gamma$. To define $\mathbf{\Theta}(X; L, d)$ we take the relative version (with respect to the projection π) of the morphism $\tilde{r}_Z$

(1.19)

$$\tilde{\mathbf{R}} : \pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\mathbf{J}_\Gamma}(-1) \longrightarrow \pi^*\Big(p_{2*}\big(\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*\mathcal{O}_X(L), \mathcal{O}_{X\times\Gamma})\big)\Big).$$

Then we define the theta-divisor $\mathbf{\Theta}(X; L, d)$ by the formula

$$\mathbf{\Theta}(X; L, d) = (\det(\tilde{\mathbf{R}}) = 0).$$

From (1.19) it follows that $\mathbf{\Theta}(X; L, d)$ is a Cartier divisor of relative (with respect to the projection π) degree d, i.e., the fibre $\mathbf{\Theta}_Z$ of $\mathbf{\Theta}_{(X;L,d)}$ over $[Z]$ is a hypersurface of degree d in $\mathbf{P}(\operatorname{Ext}^1_Z)$. More precisely, one can show that the support $\mathbf{\Theta}_Z$ is the union of hyperplanes H_z in $\mathbf{P}(\operatorname{Ext}^1_Z)$ as z runs through the set of closed points in Z. Thus the divisor $\mathbf{\Theta}(X; L, d)$ captures the geometry of the underlying 0-dimensional subschemes and one could view the pair $(\mathbf{J}(X; L, d), \mathbf{\Theta}(X; L, d))$ as a rather precise analogue of the classical Jacobian and its theta-divisor.

1.3. Two filtrations on $\mathbf{J}(X; L, d)$. In this section we construct Hodge-like structure as discussed in **Part II** of the introduction.

Consider the stratum $\mathbf{J}^r$. We will construct two filtrations over it: one is a filtration on $\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}})$ and the other on $\pi^*(p_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X + L))))$.

Our starting point is the universal extension (1.15). Tensoring it with $f_1^*\mathcal{O}_X(-L)$ and taking the direct image with respect to f_2 we obtain

$$(1.20) \quad 0 \longrightarrow \mathcal{R}^1 f_{2*}\mathbf{E}' \longrightarrow \mathcal{R}^1 f_{2*}\big(\tilde{\pi}^*\mathcal{J}_{\mathcal{Z}}\big) \longrightarrow H^2(-L) \otimes \mathcal{O}_{\mathbf{J^r}}(1)$$

where $\mathbf{E}' = \mathbf{E} \otimes f_1^* \mathcal{O}_X(-L)$.

On the other hand we have

$$0 \longrightarrow \tilde{\pi}^* \mathcal{J}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{X \times \mathbf{J}^r} \longrightarrow \tilde{\pi}^* \mathcal{O}_{\mathcal{Z}} \longrightarrow 0.$$

Combining its direct image with respect to f_2 with (1.20) we obtain

(1.21)

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & \mathcal{O}_{\mathbf{J^r}} \arrow[d] & \\
 & f_{2*}\tilde{\pi}^* \mathcal{O}_{\mathcal{Z}} \arrow[d] \arrow[dr, "\mathbf{R^r}"] & \\
0 \arrow[r] & \mathcal{R}^1 f_{2*} \mathbf{E}' \arrow[r] \arrow[dr, "\mathbf{c}"'] & \mathcal{R}^1 f_{2*}\big(\tilde{\pi}^* \mathcal{J}_{\mathcal{Z}}\big) \arrow[r] \arrow[d] & H^2(-L) \otimes \mathcal{O}_{\mathbf{J^r}}(1) \\
 & & H^1(\mathcal{O}_X) \otimes \mathcal{O}_{\mathbf{J^r}} \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

Let $\tilde{\mathbf{H}} = \ker \mathbf{R^r}$ and $\mathbf{H} = \ker \mathbf{c}$. Then (1.21) implies the following exact sequence

(1.22) $$0 \longrightarrow \mathcal{O}_{\mathbf{J^r}} \longrightarrow \tilde{\mathbf{H}} \longrightarrow \mathbf{H} \longrightarrow 0.$$

The sheaf $f_{2*}\tilde{\pi}^* \mathcal{O}_{\mathcal{Z}} = \pi^* p_{2*} \mathcal{O}_{\mathcal{Z}}$ comes equipped with the trace morphism

$$\mathbf{Tr} : \pi^* p_{2*} \mathcal{O}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{\mathbf{J^r}}$$

providing a distinguished splitting of (1.22). In particular, we have a direct sum decomposition

(1.23) $$\tilde{\mathbf{H}} = \mathcal{O}_{\mathbf{J^r}} \oplus \mathbf{H}.$$

Using the multiplicative structure of $\pi^* p_{2*} \mathcal{O}_{\mathcal{Z}}$ we consider the multiplication morphisms

(1.24) $$\mathbf{m_k} : S^k \tilde{\mathbf{H}} \longrightarrow \pi^* p_{2*} \mathcal{O}_{\mathcal{Z}}.$$

For every integer $k \geq 0$ we define

$$\tilde{\mathbf{H}}_{-k} = im(\mathbf{m_k}).$$

Let $\tilde{h}_0$ be the image of $1 \in H^0(\mathcal{O}_{\mathbf{J}^r})$ under the monomorphism in (1.22). The multiplication by $\tilde{h}_0$ induces an inclusion

$$\tilde{\mathbf{H}}_{-k} \xrightarrow{\tilde{h}_0} \tilde{\mathbf{H}}_{-k-1}$$

yielding the filtration

$$0 = \tilde{\mathbf{H}}_1 \subset \tilde{\mathbf{H}}_0 \subset \tilde{\mathbf{H}}_{-1} \subset \cdots \subset \pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \tag{1.25}$$

Remark 1.3. By definition we have $\tilde{\mathbf{H}}_0 = \mathcal{O}_{\mathbf{J^r}}$ and $\tilde{\mathbf{H}}_{-1} = \tilde{\mathbf{H}}$

From the description of our filtration it is clear that the sheaf $\tilde{\mathbf{H}}$ plays a special role. In fact it is closely related to the dual of the sheaf $\pi^*(\mathrm{coker}\,(\rho))$, where $\mathrm{coker}\,(\rho)$ is as in (1.3).

Proposition 1.4. *Let* $\mathbf{Ext^1} = \mathcal{H}om\big(\mathrm{coker}\,(\rho) \otimes \mathcal{O}_{\Gamma^r_d}, \mathcal{O}_{\Gamma^r_d}\big)$ *be the dual of* $\mathrm{coker}\,(\rho) \otimes \mathcal{O}_{\Gamma^r_d}$. *Then the following holds.*

1) *There is a natural morphism*

$$\tilde{\mathbf{M}} : \tilde{\mathbf{H}} \otimes \mathcal{O}_{\mathbf{J}^r}(-1) \longrightarrow \pi^*\mathbf{Ext^1} \tag{1.26}$$

which descends to the morphism

$$\mathbf{M} : \mathbf{H} = \tilde{\mathbf{H}}/\tilde{\mathbf{H}}_0 \longrightarrow \mathcal{T}_{\mathbf{J}/X^{[d]}}$$

where $\mathbf{H}$ *is as in* (1.22) *and* $\mathcal{T}_{\mathbf{J}/X^{[d]}}$ *is the relative tangent sheaf of* $\pi : \mathbf{J}(X; L, d) \longrightarrow X^{[d]}$.

2) *Let* Γ *be as in Proposition-Definition* 0.5 *and let* $\dot{\mathbf{J}}^r$ *be the complement of the theta-divisor* $\mathbf{\Theta}(X; L, d)$ *in* $\mathbf{J}^r_\Gamma$. *Then the morphisms* $\tilde{\mathbf{M}}$ *and* $\mathbf{M}$ *are isomorphisms over* $\dot{\mathbf{J}}^r$.

Proof. Consider the following commutative diagram

(1.27)

$$\begin{tikzcd}
\tilde{\mathbf{H}} \otimes \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[d] & \pi^*\mathbf{Ext^1} \arrow[d] \\
\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}}) \otimes \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[r, "\tilde{\mathbf{R}}"] \arrow[d, "\mathbf{R^r(-1)}"'] & \pi^*(p_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))))^* \arrow[d, "\pi^*(\rho^*)"'] \\
H^0(\mathcal{O}_X(K_X+L))^* \otimes \mathcal{O}_{\mathbf{J^r}} \arrow[r, equal] & H^0(\mathcal{O}_X(K_X+L))^* \otimes \mathcal{O}_{\mathbf{J^r}}
\end{tikzcd}$$

where $\tilde{\mathbf{R}}$ is as in (1.19) and $\mathbf{R^r}(-1)$ is the morphism $\mathbf{R^r}$ in (1.21) tensored with $\mathcal{O}_{\mathbf{J^r}}(-1)$. This yields the first morphism in 1)

$$\tilde{\mathbf{M}} : \tilde{\mathbf{H}} \otimes \mathcal{O}_{\mathbf{J^r}}(-1) \longrightarrow \pi^*\mathbf{Ext^1}. \tag{1.28}$$

By definition of $\mathbf{\Theta}(X; L, d)$ the morphism $\tilde{\mathbf{R}}$ is an isomorphism over $\dot{\mathbf{J}}^r$. This implies that $\tilde{\mathbf{M}}$ is an isomorphism over $\dot{\mathbf{J}}^r$ as well.

Turning to the second morphism of 1) we combine (1.28) with (1.22) to obtain

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[d] \\
(1.29) & \tilde{\mathbf{H}} \otimes \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[r, "\tilde{\mathbf{M}}"] \arrow[d] & \pi^*\mathbf{Ext^1} \arrow[d] \\
 & \mathbf{H} \otimes \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[d] & \mathcal{T}_{\mathbf{J}/X^{[d]}} \otimes \mathcal{O}_{\mathbf{J^r}}(-1) \arrow[d] \\
 & 0 & 0
\end{tikzcd}
$$

where the column on the right is the relative Euler sequence tensored with $\mathcal{O}_{\mathbf{J^r}}(-1)$. From this it follows that $\tilde{\mathbf{M}}$ descends to the morphism

$$\mathbf{M} : \mathbf{H} \longrightarrow \mathcal{T}_{\mathbf{J}/X^{[d]}} \otimes \mathcal{O}_{\mathbf{J^r}},$$

which is an isomorphism over $\dot{\mathbf{J}}^r$. q.e.d.

All nonzero sheaves in the filtration (1.25) are torsion-free on every reduced irreducible component of $\mathbf{J}^r$. So the rank p_k of $\tilde{\mathbf{H}}_{-k}$ on such a component is well-defined. Setting $P(k) = p_k$, for every $k \geq 0$, we obtain the Hilbert function P attached to every reduced irreducible component of $\mathbf{J}^r$. This way we arrive to a collection of admissible Hilbert functions associated to $\mathbf{J}^r$. Fix one of such functions, say P, and denote by $\mathbf{J}^r_P$ the reduced subscheme of $\mathbf{J}^r$, where the ranks of sheaves in the filtration (1.25) are constant and determined by the function P. Thus all sheaves in the filtration $\tilde{\mathbf{H}}_{-\bullet}$ are locally free. From now on, unless stated otherwise, we will be working over $\mathbf{J}^r_P$.

Remark 1.5.

1) The ranks $P(k)$ of the sheaves $\tilde{\mathbf{H}}_{-k}$ are related to the geometry of the underlying points in the Hilbert scheme. More precisely, let $([Z],[\alpha])$ be a point in $\mathbf{J}^r_P$. The fibre $\tilde{\mathbf{H}}([Z],[\alpha])$ of $\tilde{\mathbf{H}}$ at $([Z],[\alpha])$ defines the morphism

$$\kappa([Z],[\alpha]) : Z \longrightarrow \mathbf{P}(\tilde{\mathbf{H}}^*([Z],[\alpha])).$$

Let $P_{([Z],[\alpha])}$ be the Hilbert function of the image of $\kappa([Z],[\alpha])$. Then $P(k) = P_{([Z],[\alpha])}(k)$, for every positive integer k. So $\mathbf{J}^r_P$ can be characterized as the locus of points $([Z],[\alpha]) \in \mathbf{J}^r$ whose Hilbert function $P_{([Z],[\alpha])}$ of the image of $\kappa([Z],[\alpha])$ is fixed and equal to the Hilbert function P of the filtration (1.25).

2) Let $\dot{\mathbf{J}}^r_P = \dot{\mathbf{J}}^r \cap \mathbf{J}^r_P$ be the complement of the theta-divisor in $\mathbf{J}^r_P$. The isomorphism $\tilde{\mathbf{M}}$ in Proposition 1.4 implies that $P(1) = r+1$. In particular, for $r = 0$ the filtration $\tilde{\mathbf{H}}_{-\bullet}$ is reduced to $\tilde{\mathbf{H}}_0$. So we will always assume that $r \geq 1$.

Next we turn to the the filtration on $\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$. We begin by returning to the multiplication morphism $\mathbf{m_k}$ in (1.24). Composing it with the morphism $\mathbf{R^r}$ in (1.21) we obtain

$$\tilde{\mathbf{R}}^{\mathbf{r}}_{\mathbf{k}} : S^k\tilde{\mathbf{H}} \longrightarrow H^2(-L) \otimes \mathcal{O}_{\mathbf{J^r}}(1). \tag{1.30}$$

Dualizing this morphism and then tensoring with $\mathcal{O}_{\mathbf{J^r}}(1)$ we obtain the morphism

$$H^0(K_X+L) \otimes \mathcal{O}_{\mathbf{J^r}} \longrightarrow \big(S^k\tilde{\mathbf{H}}\big)^* \otimes \mathcal{O}_{\mathbf{J^r}}(1).$$

Let $\tilde{\mathbf{F}}_{\mathbf{k}}$ be its kernel. Then we obtain the following filtration

$$H^0(K_X+L) \otimes \mathcal{O}_{\mathbf{J^r}} = \tilde{\mathbf{F}}_{\mathbf{1}} \supset \tilde{\mathbf{F}}_{\mathbf{2}} \supset \dots. \tag{1.31}$$

Proposition 1.6. *For every $k \geq 1$ there is an inclusion*

$$\tilde{\mathbf{F}}_{\mathbf{k}} \supset \pi^*(p_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L)))).$$

Proof. The dual of the diagram (1.27) and the definition of $\tilde{\mathbf{F}}_{\mathbf{k}}$ imply the assertion. q.e.d.

Factoring out by $\pi^*(p_{2*}\big(\mathcal{I}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$ in the filtration (1.31) we obtain

$$\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))\big)) = \mathbf{F_0} \supset \mathbf{F_1} \supset \mathbf{F_2} \supset \dots \tag{1.32}$$

where $\mathbf{F_k} = \tilde{\mathbf{F}}_{\mathbf{k}}/\pi^*(p_{2*}\big(\mathcal{I}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$, for every $k \geq 1$.

The filtrations $\tilde{\mathbf{H}}_{-\bullet}$ and $\mathbf{F}_\bullet$ are related as follows:

Proposition 1.7. *For every $k \geq 1$ there is an isomorphism*

$$\tilde{\mathbf{H}}_{-k-1}/\tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\mathbf{J}^r_P}(-1) \cong \big(\mathbf{F_k}/\mathbf{F_{k+1}}\big)^*.$$

Proof. By definition of $\tilde{\mathbf{R}}^{\mathbf{r}}_{\mathbf{k}}$ in (1.30) we have the following commutative diagram

$$
\begin{tikzcd}
 & & & (\tilde{\mathbf{F}}_{\mathbf{k}}/\tilde{\mathbf{F}}_{\mathbf{k+1}})^* \arrow[d] & \\
 & 0 \arrow[d] & & 0 \arrow[u, leftarrow] & \\
0 \arrow[r] & \tilde{\mathbf{H}}_{-k}/\tilde{\mathbf{H}}_{-1} \otimes \mathcal{O}_{\mathbf{J}^r_P}(-1) \arrow[r] \arrow[d] & H^0(\mathcal{O}_X(K_X+L))^* \otimes \mathcal{O}_{\mathbf{J}^r_P} \arrow[r] \arrow[d, equal] & (\tilde{\mathbf{F}}_{\mathbf{k}})^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\mathbf{H}}_{-k-1}/\tilde{\mathbf{H}}_{-1} \otimes \mathcal{O}_{\mathbf{J}^r_P}(-1) \arrow[r] \arrow[d] & H^0(\mathcal{O}_X(K_X+L))^* \otimes \mathcal{O}_{\mathbf{J}^r_P} \arrow[r] & (\tilde{\mathbf{F}}_{\mathbf{k+1}})^* \arrow[r] \arrow[d] & 0 \\
 & \tilde{\mathbf{H}}_{-k-1}/\tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\mathbf{J}^r_P}(-1) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

This yields the isomorphism

$$\tilde{\mathbf{H}}_{-k-1}/\tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\mathbf{J}^r_P}(-1) \cong \left(\tilde{\mathbf{F}}_{\mathbf{k}}/\tilde{\mathbf{F}}_{\mathbf{k+1}}\right)^*.$$

Observing the equality

$$\tilde{\mathbf{F}}_{\mathbf{k}}/\tilde{\mathbf{F}}_{\mathbf{k+1}} \cong \mathbf{F}_{\mathbf{k}}/\mathbf{F}_{\mathbf{k+1}}$$

we obtain the asserted isomorphisms. q.e.d.

A more conceptual way to see the relation between the filtrations $\mathbf{F}_{\bullet}$ and $\tilde{\mathbf{H}}_{-\bullet}$ is to observe that $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ acts on $\mathbf{F_0} = \pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))\big))$ via multiplication. Furthermore, the subsheaf $\tilde{\mathbf{H}}$ acts not only on $\mathbf{F_0}$ but on the whole filtration $\mathbf{F}_{\bullet}$. Namely, we have the following morphism

$$\tilde{\mathbf{H}} \otimes \mathbf{F_k} \longrightarrow \mathbf{F_{k-1}} \tag{1.33}$$

for every $k \geq 1$. Combining with (1.22) yields

$$\mathbf{H} \longrightarrow \mathcal{H}om(\mathbf{F_k}/\mathbf{F_{k+1}}, \mathbf{F_{k-1}}/\mathbf{F_k}). \tag{1.34}$$

Furthermore by Lemma 1.4 the sheaf $\mathbf{H}$ is isomorphic to the relative tangent sheaf $\mathcal{T}_{\mathbf{J}/X^{[d]}}$ of $\mathbf{J}^r$ over the complement of the theta-divisor. So over $\dot{\mathbf{J}}^r_P = \dot{\mathbf{J}}^r \cap \mathbf{J}^r_P$ we obtain the morphism

$$\mathbf{a_k} : \mathcal{T}_{\mathbf{J}/X^{[d]}} \otimes \mathcal{O}_{\dot{\mathbf{J}}^r_P} \longrightarrow \mathcal{H}om(\mathbf{F_k}/\mathbf{F_{k+1}}, \mathbf{F_{k-1}}/\mathbf{F_k}). \tag{1.35}$$

This is reminiscent of Griffiths transversality condition for the Variation of Hodge structure (see [**G**]) and will be referred to as the formal Griffiths transversality. In fact it has been shown in §4.3, [**R2**] that the filtration $\mathbf{F}_\bullet$ can be obtained from $\mathbf{F_2}$ by successive differentiation along the fibres of the $X^{[d]}$-maps starting from the map

$$\mathbf{J}^r_P \overset{\phi_2}{\dashrightarrow} \mathbf{Gr}(f_2, \mathbf{F_1})$$

where $f_2 = rk\mathbf{F_2}$, $\mathbf{Gr}(f_2, \mathbf{F_1})$ is a relative Grassmannian over $\mathbf{J}^r_P$ and ϕ_2 takes a point $([Z],[\alpha]) \in \mathbf{J}^r_P$ to the fibre of $\mathbf{F_2}$ at $([Z],[\alpha])$. More generally, once $\mathbf{F_k}$ is defined we consider a map of $X^{[d]}$-schemes

$$\mathbf{J}^r_P \overset{\phi_k}{\dashrightarrow} \mathbf{Gr}(f_k, \mathbf{F_{k-1}})$$

where $f_k = rk\mathbf{F_k}$. Taking the relative differential of ϕ_k we obtain

$$d_{\mathbf{J}^r_P/X^{[d]}}\phi_k : \mathcal{T}_{\mathbf{J}/X^{[d]}} \otimes \mathcal{O}_{\mathbf{J}^r_P} \longrightarrow \mathcal{H}om(\mathbf{F_k}/\mathbf{F_{k+1}},\, \mathbf{F_{k-1}}/\mathbf{F_k})$$

where the sheaf $\mathbf{F_{k+1}}$ is defined to be the kernel of the morphism

$$\mathbf{F_k} \longrightarrow \left(\mathcal{T}_{\mathbf{J}/X^{[d]}} \otimes \mathcal{O}_{\mathbf{J}^r_P}\right)^* \otimes \mathbf{F_{k-1}}/\mathbf{F_k}.$$

What we are saying is that the formal morphisms $\mathbf{a_k}$ in (1.35) are equal (up to a constant factor) to $d_{\mathbf{J}^r_P/X^{[d]}}\phi_k$ (see Lemma 4.9,[**R2**], where it was shown that $d_{\mathbf{J}^r_P/X^{[d]}}\phi_k = (k-1)\mathbf{a_k}$, for every $k \geq 2$).

1.4. Cohomological invariant of $\mathbf{J}^r$. Here we take a somewhat different point of view on our construction. Rather than concentrating on the filtrations we shift our attention to the morphisms defining those filtrations. Namely, we consider the morphisms

$$\tilde{\mathbf{R}}^{\mathbf{r}}_{\mathbf{k}} : S^k\tilde{\mathbf{H}} \longrightarrow H^2(-L) \otimes \mathcal{O}_{\mathbf{J^r}}(1)$$

defined in (1.30). The decomposition (1.23) implies that $S^k\mathbf{H}$ is a direct summand of $S^k\tilde{\mathbf{H}}$. Restricting $\tilde{\mathbf{R}}^{\mathbf{r}}_{\mathbf{k}}$ to it yields the morphisms

$$\mathbf{R}^{\mathbf{r}}_{\mathbf{k}} : S^k\mathbf{H} \longrightarrow H^2(-L) \otimes \mathcal{O}_{\mathbf{J^r}}(1).$$

By definition the sheaf $\mathbf{H}$ is of cohomological nature (see (1.21)). This is the reason for the following terminology.

Definition 1.8. The sequence of morphisms $\mathbf{C}_{\mathbf{J}^r} := \{\mathbf{R}^{\mathbf{r}}_{\mathbf{k}}\}_{r,k\in\mathbf{N}}$ is called the cohomological invariant of $\mathbf{J}^r$.

The essential property of the cohomological invariant is that it captures the geometry of the clusters underlying $\mathbf{J}^r$. This point of view has been adopted in [**R2**], where the properties of the cohomological invariant of vector bundles are discussed in details. The main point is that a rank 2 bundle $\mathcal{E}$ over X with the Chern invariants (L,d) defines a kind of "normal (meromorphic) function" from $\mathbf{P}(H^0(\mathcal{E}))$ to $\mathbf{J}(X;L,d)$ (see §4, [**R2**]) and the cohomological invariant of $\mathcal{E}$ either determines the universal cluster $\mathcal{Z}_\mathcal{E}$ over $\mathbf{P}(H^0(\mathcal{E}))$ or detects its special geometry (see §3, [**R2**]). Thus what we have at our disposal is the sequence $\mathbf{C}_{\mathbf{J}^r}$ of sections of distinguished sheaves on $\mathbf{J}^r$, namely, $\mathcal{H}om\left(S^k\mathbf{H}, H^2(-L)\otimes\mathcal{O}_{\mathbf{J^r}}(1)\right)$, which either determines the clusters underlying $\mathbf{J}^r$ or detects their special geometry.

2. Orthogonal decomposition of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$

In this section we show that there exists a nonempty Zariski open subset of $\mathbf{J}^r_P$ where $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ admits a canonical direct sum decomposition (see Corollary 2.4).

We begin by observing that the trace morphism

$$\mathbf{Tr} : \pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \longrightarrow \mathcal{O}_{\mathbf{J^r}}$$

induces a bilinear symmetric pairing $\mathbf{q}$ on $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$

$$\mathbf{q}(f,g) = \mathbf{Tr}(fg) \tag{2.1}$$

for any local sections f, g of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$. Furthermore, this pairing is nondegenerate outside of the ramification locus of the projection $p_2 : \mathcal{Z} \longrightarrow X^{[d]}$.

From now on we assume that the projection p_2 is generically smooth over $\Gamma^r_d(P) = \pi(\mathbf{J}^r_P)$, i.e., Z is a set of d distinct points for general $[Z] \in \Gamma^r_d(P)$. Let $\dot{\Gamma}^r_d(P)$ be the complement in $\Gamma^r_d(P)$ of the branch locus of p_2. Put $\ddot{\mathbf{J}}^r_P = \pi^{-1}(\dot{\Gamma}^r_d(P))$. The pairing $\mathbf{q}$ is nondegenerate over $\ddot{\mathbf{J}}^r_P$ so it induces an isomorphism

$$\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \longrightarrow \big(\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)\big)^* \tag{2.2}$$

where $\mathcal{Z}$ here stands for the incidence cluster over $\dot{\Gamma}^r_d(P)$. This isomorphism combined with $\tilde{\mathbf{R}}$ in (1.27) yields a morphism

$$\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X + L))\big)) \otimes \mathcal{O}_{\ddot{\mathbf{J}}^r_P}(-1) \longrightarrow \pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \tag{2.3}$$

which according to Proposition 1.4, 2), is an isomorphism on

$$(\mathbf{J}^r_P)' = \ddot{\mathbf{J}}^r_P \setminus \mathbf{\Theta}(X; L, d).$$

If there is no ambiguity we will often omit the indexes in the above notation and write simply $\mathbf{J}'$ for $(\mathbf{J}^r_P)'$.

Using the isomorphism in (2.3) we transfer the filtration $\mathbf{F}_\bullet \otimes \mathcal{O}_{\mathbf{J}'}(-1)$ in (1.32)) from $\pi^*(p_{2*}\big(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\mathcal{O}_X(K_X+L))\big)) \otimes \mathcal{O}_{\mathbf{J}'}(-1)$ to $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\mathbf{J}'}$. The resulting filtration will be denoted by $\{\mathbf{F^k}\}$.

Lemma 2.1. *There is a nonempty Zariski open subset $\check{\mathbf{J}}^r_P$ of $(\mathbf{J}^r_P)'$ such that for every $k \geq 1$ there is a direct sum decomposition*

$$\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\check{\mathbf{J}}} = \tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\check{\mathbf{J}}} \oplus \mathbf{F^k} \otimes \mathcal{O}_{\check{\mathbf{J}}}$$

which is orthogonal with respect to $\mathbf{q}$ (*as usual, if no ambiguity is likely, we write* $\check{\mathbf{J}}$ *instead of* $\check{\mathbf{J}}^r_P$).

Proof. First we observe that the sheaves $\tilde{\mathbf{H}}_{-k}$ and $\mathbf{F^k}$ are orthogonal with respect to $\mathbf{q}$. This can be seen fibrewise. So we fix $[Z] \in \dot{\Gamma}^r_d(P)$ and an extension class $[\alpha] \in \mathbf{P}(\mathrm{Ext}^1_Z)$ corresponding to a locally free sheaf (recall that Ext^1_Z denotes the group of extensions $\mathrm{Ext}^1(\mathcal{I}_Z(L), \mathcal{O}_X)$). Consider the pairing defined by $\mathbf{q}$ at the point $([Z],[\alpha]) \in \mathbf{J}'$. Let $x \in (\mathbf{F^k})_{([Z],[\alpha])}$, where $(\mathbf{F^k})_{([Z],[\alpha])}$ is the fibre of $\mathbf{F^k}$ at the point $([Z],[\alpha])$. By definition there exists $f \in (\mathbf{F_k})_{([Z],[\alpha])}$ such that $f \otimes \alpha$ is sent to x under the morphism in (2.3). This implies

$$\mathbf{q}(x,t) = (\alpha \cdot t, f)$$

for every $t \in H^0(\mathcal{O}_Z)$ and where $(\cdot,\cdot)$ is the pairing defined by $\tilde{\mathbf{R}}$. By definition $(\mathbf{F_k})_{([Z],[\alpha])}$ annihilates $(\tilde{\mathbf{H}}_{-k})_{([Z],[\alpha])}$ implying $\mathbf{q}(x,t) = 0$, for every $x \in (\mathbf{F^k})_{([Z],[\alpha])}$ and every $t \in (\tilde{\mathbf{H}}_{-k})_{([Z],[\alpha])}$. Hence the orthogonality $(\tilde{\mathbf{H}}_{-k})_{([Z],[\alpha])} \perp (\mathbf{F^k})_{([Z],[\alpha])}$.

Once we have the orthogonality of our subsheaves the statement about direct sum decomposition becomes equivalent to the nondegeneracy of the quadratic form $\mathbf{q}$ on the subsheaves $\tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\mathbf{J}'}$ of the filtration $\tilde{\mathbf{H}}_{-\bullet}$.

Claim 2.2. There exists a nonempty Zariski open subset $\mathbf{J}'(k)$ of $\mathbf{J}'$ such that the restriction of the quadratic form $\mathbf{q}$ to the subsheaf $\tilde{\mathbf{H}}_{-k} \otimes \mathcal{O}_{\mathbf{J}'(k)}$ is again a nondegenerate quadratic form.

A proof of this fact is given in §8 - it is based on an explicit calculation of the quadratic form $\mathbf{q}$ on $\tilde{\mathbf{H}}_{-k}$.

The result of the lemma now follows immediately by taking $\check{\mathbf{J}}$ to be the intersection of the open sets $\mathbf{J}'(k)$ in Claim 2.2 as k runs through the indexes of the filtration $\tilde{\mathbf{H}}_{-\bullet}$. q.e.d.

Let l be the length of the filtration $\{\mathbf{F}^\bullet\}$. Define

$$\mathbf{H^p} = \begin{cases} \mathbf{F^p} \cap \tilde{\mathbf{H}}_{-p-1}, & \text{for} \quad 0 \le p \le l-1, \\ \mathbf{F^l}, & \text{for} \quad p = l. \end{cases} \tag{2.4}$$

Lemma 2.3. *For every $0 \le p \le l-1$ there is a $\mathbf{q}$-orthogonal decomposition*

$$\mathbf{F^p} = \mathbf{F^{p+1}} \oplus \mathbf{H^p}.$$

Proof. From Lemma 2.1 it follows

$$\mathbf{F^p} = \mathbf{F^{p+1}} \oplus (\mathbf{F^{p+1}})^\perp \cap \mathbf{F^p}$$

and $(\mathbf{F^{p+1}})^\perp = \tilde{\mathbf{H}}_{-p-1}$. This and the definition of $\mathbf{H^p}$ in (2.4) imply the assertion. q.e.d.

Corollary 2.4. *For every $0 \leq p \leq l$ there is a* $\mathbf{q}$*-orthogonal decomposition*

$$\mathbf{F^p} = \bigoplus_{s=p}^{l} \mathbf{H^s}.$$

In particular, over $\breve{\mathbf{J}}$ *the vector bundle* $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ *has the following* $\mathbf{q}$*-orthogonal decomposition*

$$\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\breve{\mathbf{J}}} = \bigoplus_{p=0}^{l} \mathbf{H^p}. \tag{2.5}$$

Proof. Follows immediately from Lemma 2.3 and (2.4). q.e.d.

Definition 2.5. The decomposition in (2.5) will be called the orthogonal cohomology decomposition of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$. The rank of $\mathbf{H^p}$ will be denoted by h^p and l in (2.5) will be called the weight of the orthogonal cohomology decomposition of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$.

Remark 2.6.

1) From Lemma 2.3 it follows

$$h^p = rk\mathbf{H^p} = rk(\mathbf{F^p}/\mathbf{F^{p+1}}) = \Delta P(p) \stackrel{def}{=} P(p+1) - P(p)$$

where P is the Hilbert function of the filtration (1.25) and the third equality follows from Proposition 1.7.

2) $h^l = d - P(l)$. In particular, $h^l = 0$ if and only if the morphism $\kappa([Z],[\alpha])$ in Remark 1.5 is an embedding.

3) From (2.4) it follows

$$\mathbf{H^0} = \tilde{\mathbf{H}}_{-1} \otimes \mathcal{O}_{\breve{\mathbf{J}}} = \tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}}.$$

In particular, $h^0 = r+1 = \delta(L,Z)$, the index of L-speciality of Z (see Definition 1.1) for $[Z] \in \dot{\Gamma}^r_d(P)$.

3. Relative Higgs structures on $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$

The subsheaf $\tilde{\mathbf{H}}$ acts on $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) = \mathbf{F^0}$ via multiplication, i.e. we have the morphism

$$\tilde{\mathbf{H}} \otimes \mathbf{F^0} \longrightarrow \mathbf{F^0} \tag{3.1}$$

induced by the multiplicative structure of $\mathbf{F^0}$. We want to see how this action effects the orthogonal cohomology decomposition of $\mathbf{F^0}$ (see Definition 2.5).

We begin by rewriting (3.1) as follows

$$D : \mathbf{F^0} \longrightarrow \tilde{\mathbf{H}}^* \otimes \mathbf{F^0}. \tag{3.2}$$

To fix our notation and terminology we will need some generalities about morphisms written in this form.

Let $\mathcal{F}$ and $\mathcal{G}$ be two vector bundles over a scheme S (as usual we make identification of vector bundles over S and locally free $\mathcal{O}_S$-modules) and let

$$A : \mathcal{F} \longrightarrow \mathcal{G}^* \otimes \mathcal{F} \tag{3.3}$$

be a morphism of $\mathcal{O}_S$-modules. For a section g of $\mathcal{G}$ over an open subset $U \subset S$ we denote

$$A(g) : \mathcal{F} \otimes \mathcal{O}_U \longrightarrow \mathcal{F} \otimes \mathcal{O}_U$$

the endomorphism of $\mathcal{F} \otimes \mathcal{O}_U$ induced by A (we often omit the reference to an open subset in the above notation).
Given two morphisms A, $B : \mathcal{F} \longrightarrow \mathcal{G}^* \otimes \mathcal{F}$ we define

$$A \wedge B : \mathcal{F} \longrightarrow \wedge^2 \mathcal{G}^* \otimes \mathcal{F}$$

as follows:

$$(A \otimes id_{\mathcal{G}^*}) \circ B - (B \otimes id_{\mathcal{G}^*}) \circ A. \tag{3.4}$$

Remark 3.1.

1) We write A^2 for $A \wedge A$.
2) For a local section $g \wedge g'$ of $\wedge^2 \mathcal{G}$ the morphism $A \wedge B$ is given by the following formula

$$(A \wedge B)(g \wedge g') = [A(g'),\ B(g)]$$

where the bracket is the commutator of endomorphisms. In particular, $A^2 = 0$ if and only if $A(g)$ and $A(g')$ are commuting endomorphisms for any local sections g, g' of $\mathcal{G}$.

Definition 3.2. Let A be as in (3.3). It is said to be a Higgs endomorphism of $\mathcal{F}$ with values in $\mathcal{G}^*$ if $A^2 = 0$.

Remark 3.3. In our terminology a Higgs bundle over S (see [**S**]) is a bundle with a Higgs endomorphism having its values in the cotangent bundle of S. More generally, let $f : S \longrightarrow B$ be a smooth morphism of relative dimension ≥ 1. We say that a bundle $\mathcal{F}$ over S is a relative Higgs bundle if it has a Higgs endomorphism with values in the relative cotangent bundle of f. In this case a Higgs endomorphism of $\mathcal{F}$ will be called a relative Higgs field.

We will now return to our considerations of the action of $\tilde{\mathbf{H}}$ on $\mathbf{F^0}$.

Lemma 3.4. *The decomposition*

$$\mathbf{F^0} = \mathbf{F^l} \oplus \tilde{\mathbf{H}}_{-\mathbf{l}}$$

is invariant with respect to the action of $\tilde{\mathbf{H}}$.

Proof. Since $\tilde{\mathbf{H}} \otimes \tilde{\mathbf{H}}_{-l} \longrightarrow \tilde{\mathbf{H}}_{-l-1} = \tilde{\mathbf{H}}_{-l}$ we obtain that $\tilde{\mathbf{H}}_{-l}$ is $\tilde{\mathbf{H}}$-invariant. The multiplication is self-adjoint with respect to $\mathbf{q}$ so $(\tilde{\mathbf{H}}_{-l})^{\perp} = \mathbf{F^l}$ is $\tilde{\mathbf{H}}$-invariant as well. q.e.d.

Remark 3.5. Let $([Z],[\alpha]) \in \breve{\mathbf{J}}$ and let $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ be the fibre of $\tilde{\mathbf{H}}_{-l}$ at $([Z],[\alpha])$. The geometric meaning of $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ is as follows. The fibre of $\tilde{\mathbf{H}}$ at $([Z],[\alpha])$ defines the evaluation morphism

$$\kappa([Z],[\alpha]) : Z \longrightarrow \tilde{\mathbf{H}}^*_{([Z],[\alpha])}$$

which sends $p \in Z$ to the linear functional $\kappa(p) : \tilde{\mathbf{H}}_{([Z],[\alpha])} \longrightarrow \mathbf{C}$, the evaluation at p (this is is an affine version of the morphism $\kappa([Z],[\alpha])$ in Remark 1.5). Let Z' be its image and let $I_{Z'}$ be its ideal in the coordinate ring $S^{\bullet}\tilde{\mathbf{H}}_{([Z],[\alpha])}$. Then

$$\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = S^{\bullet}\tilde{\mathbf{H}}_{([Z],[\alpha])}/I_{Z'} = H^0(\mathcal{O}_{Z'}).$$

The fibre $\mathbf{F}^{\mathbf{l}}([Z],[\alpha])$ of $\mathbf{F}^{\mathbf{l}}$ at $([Z],[\alpha])$ is the subspace of $H^0(\mathcal{O}_Z)$ which is $\mathbf{q}$-orthogonal to $(\kappa([Z],[\alpha]))^*(H^0(\mathcal{O}_{Z'}))$. More explicitly, letting $Z_{p'} = (\kappa([Z],[\alpha]))^{-1}(p')$, for $p' \in Z'$, we have

$$\mathbf{F}^{\mathbf{l}}([Z],[\alpha]) = \left\{ f \in H^0(\mathcal{O}_Z) \;\middle|\; \sum_{q \in Z_{p'}} f(q) = 0,\ \forall p' \in Z' \right\}.$$

Lemma 3.6. *The action of* $\tilde{\mathbf{H}}$ *on* $\mathbf{H}^{\mathbf{p}}$ *is determined by the following morphisms.*

(i) *For* $1 \leq p \leq l-2$:

$$\tilde{\mathbf{H}} \otimes \mathbf{H}^{\mathbf{p}} \longrightarrow \mathbf{H}^{\mathbf{p-1}} \oplus \mathbf{H}^{\mathbf{p}} \oplus \mathbf{H}^{\mathbf{p+1}}.$$

(ii) *For* $p = 0$:

$$\tilde{\mathbf{H}} \otimes \mathbf{H}^{\mathbf{0}} \longrightarrow \mathbf{H}^{\mathbf{0}} \oplus \mathbf{H}^{\mathbf{1}}.$$

(iii) *For* $p = l-1$:

$$\tilde{\mathbf{H}} \otimes \mathbf{H}^{\mathbf{l-1}} \longrightarrow \mathbf{H}^{\mathbf{l-2}} \oplus \mathbf{H}^{\mathbf{l-1}}.$$

Proof. Consider first the case $p = 0$. By definition $\mathbf{H}^{\mathbf{0}} = \tilde{\mathbf{H}}$ and the multiplication by $\tilde{\mathbf{H}}$ sends $\mathbf{H}^{\mathbf{0}}$ to $\tilde{\mathbf{H}}_{-2} = \mathbf{H}^{\mathbf{0}} \oplus \mathbf{H}^{\mathbf{1}}$. For $1 \leq p \leq l-2$ the multiplication by $\tilde{\mathbf{H}}$ sends $\mathbf{H}^{\mathbf{p}} = \mathbf{F}^{\mathbf{p}} \cap \tilde{\mathbf{H}}_{\mathbf{-p-1}}$ to $\mathbf{F}^{\mathbf{p-1}} \cap \tilde{\mathbf{H}}_{\mathbf{-p-2}}$. By Corollary 2.4 we have

$$\mathbf{F}^{\mathbf{p-1}} \cap \tilde{\mathbf{H}}_{\mathbf{-p-2}} = \mathbf{H}^{\mathbf{p-1}} \oplus \mathbf{H}^{\mathbf{p}} \oplus \mathbf{H}^{\mathbf{p+1}}$$

for every $p \leq l-2$.

For $p = l-1$ the multiplication by $\tilde{\mathbf{H}}$ sends $\mathbf{H}^{\mathbf{l-1}}$ to $\mathbf{F}^{\mathbf{l-2}} \cap \tilde{\mathbf{H}}_{\mathbf{-l}} = (\mathbf{F}^{\mathbf{l-1}} \oplus \mathbf{H}^{\mathbf{l-2}}) \cap \tilde{\mathbf{H}}_{\mathbf{-l}} = \mathbf{H}^{\mathbf{l-1}} \oplus \mathbf{H}^{\mathbf{l-2}}$ (see Lemma 2.3 for the first equality). q.e.d.

Let D_p be the restriction of the morphism D in (3.2) to the summand $\mathbf{H}^{\mathbf{p}}$. Then the result of Lemma 3.6 implies

$$D_p : \mathbf{H}^{\mathbf{p}} \longrightarrow \tilde{\mathbf{H}}^* \otimes \mathbf{H}^{\mathbf{p-1}} \oplus \tilde{\mathbf{H}}^* \otimes \mathbf{H}^{\mathbf{p}} \oplus \tilde{\mathbf{H}}^* \otimes \mathbf{H}^{\mathbf{p+1}}.$$

Denote by D_p^-, D_p^0 and D_p^+ the projections of D_p on the first, second and third summands respectively. Set $D^{\pm} = \sum_{p=0}^{l} D_p^{\pm}$ (resp., $D^0 = \sum_{p=0}^{l} D_p^0$). Then we obtain the following decomposition

$$D = D^- + D^0 + D^+. \tag{3.5}$$

The components of this decomposition have the following property with respect to the pairing $\mathbf{q}$.

Lemma 3.7.

(i) *The morphisms D^-, D^+ are adjoint to each other with respect to $\mathbf{q}$, i.e.,*

$$\mathbf{q}(D^+(t)x, y) = \mathbf{q}(x, D^-(t)y)$$

for any local sections x, y of $\mathbf{F^0}$ and any local section t of $\tilde{\mathbf{H}}$.

(ii) *The morphism D^0 is self-adjoint with respect to $\mathbf{q}$, i.e.,*

$$\mathbf{q}(D^0(t)x, y) = \mathbf{q}(x, D^0(t)y)$$

for any local sections x, y of $\mathbf{F^0}$ and any local section t of $\tilde{\mathbf{H}}$.

Proof. Since D is self-adjoint with respect to $\mathbf{q}$ the second assertion of the lemma follows from the first and the decomposition in (3.5). The first assertion can be seen as follows. Let t be a local section of $\tilde{\mathbf{H}}$ and let x be a local section of $\mathbf{H^p}$ and y be a local section of $\mathbf{F^0}$. Then $\mathbf{q}(D^+(t)x, y)$ depends only on the component of y of degree $p+1$. Thus we can assume that y is a local section of $\mathbf{H^{p+1}}$. From Lemma 3.6 and orthogonality it follows

$$\begin{aligned}\mathbf{q}(D^+(t)x, y) &= \mathbf{q}(D(t)x - D^-(t)x - D^0(t)x, y)\\ &= \mathbf{q}(D(t)x, y) = \mathbf{q}(x, D(t)y) = \mathbf{q}(x, D^-(t)y),\end{aligned}$$

yielding the first assertion of the lemma. The second assertion is proved in the same way. q.e.d.

Remark 3.8. By definition D^- (resp., D^+) shifts the index of $\mathbf{H^p}$'s by -1 (resp., $+1$) and D^0 leaves the index unchanged. So D^-, D^0 and D^+ can be viewed as operators of degree (-1), 0 and $(+1)$ respectively.

We will now consider relations between the components of the decomposition in (3.5).

Lemma 3.9. *The decomposition* (3.5) *is subject to the following identities*

(i) $D^2 = (D^-)^2 = (D^+)^2 = 0$,
(ii) $D^- \wedge D^0 + D^0 \wedge D^- = D^+ \wedge D^0 + D^0 \wedge D^+ = 0$,
(iii) $(D^0)^2 + D^- \wedge D^+ + D^+ \wedge D^- = 0$.

Proof. Commutativity of the multiplication in $\mathbf{F^0}$ and Remark 3.1, 2) imply $D^2 = 0$. This together with the decomposition of D in (3.5) yield

$$0 = D^2 = \underbrace{(D^-)^2}_{(-2)} + \underbrace{D^- \wedge D^0 + D^0 \wedge D^-}_{(-1)}$$
$$+ \underbrace{(D^0)^2 + D^- \wedge D^+ + D^+ \wedge D^-}_{(0)} + \underbrace{D^+ \wedge D^0 + D^0 \wedge D^+}_{(+1)} + \underbrace{(D^+)^2}_{(+2)}$$

where the terms are grouped according to their degree. Thus, we obtain the vanishing of every group. This gives the asserted identities. q.e.d.

As a consequence of Lemma 3.9 we see that the components D^0_p, $D^{\pm}_p$ of D^0 and $D^{\pm}$ satisfy the following relations

$$\begin{aligned} D^{\pm}_{p\pm1} \wedge D^{\pm}_p &= 0 \\ D^{\pm}_p \wedge D^0_p + D^0_{p\pm1} \wedge D^{\pm}_p &= 0 \\ (D^0_p)^2 + D^-_{p+1} \wedge D^+_p + D^+_{p-1} \wedge D^-_p &= 0 \end{aligned} \tag{3.6}$$

for every $p = 0, \ldots, l-1$ and where we use the convention that $D^{\pm}_p$ (resp. D^0_p) is zero whenever the index p is not in the above range.

4. A nonabelian Albanese

From Lemma 3.9 it follows that the sheaf $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ comes together with distinguished Higgs morphisms $D, D^{\pm}$. Taking sufficiently general deformation of D we will obtain a large family of Higgs structures on $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$. This, in essence, is our nonabelian Albanese. We will see shortly that the definition of this variety depends only on the weight l of the decomposition in Corollary 2.4 and the relations (3.6). The case of weight $l = 1$ is special (see Remark 7.10) and will be treated elsewhere. So from now on we assume that the weight l of the orthogonal cohomology decomposition of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ is ≥ 2.

We consider a sufficiently general deformation of D of the following form

$$\sigma(t,x,y) = \sigma^0(t) + \sigma^+(x) + \sigma^-(y)$$

where

$$\sigma^0(t) = \sum_{p=0}^{l-1} t_p D^0_p, \quad \sigma^+(x) = \sum_{p=0}^{l-2} x_p D^+_p, \quad \sigma^-(y) = \sum_{p=0}^{l-2} y_p D^-_{p+1} \tag{4.1}$$

and $t = (t_p) \in \mathbf{C^l}$, $x = (x_p)$, $y = (y_p) \in \mathbf{C^{l-1}}$ are deformation parameters. We will now derive sufficient conditions for the morphism $\sigma(t,x,y)$ to be Higgs. For this we expand

$$\begin{aligned} \sigma^2(t,x,y) &= \sigma^0(t) \wedge \sigma^+(x) + \sigma^+(x) \wedge \sigma^0(t) + (\sigma^0(t))^2 + \sigma^+(x) \wedge \sigma^-(y) \\ &\quad + \sigma^-(y) \wedge \sigma^+(x) + \sigma^0(t) \wedge \sigma^-(x) + \sigma^-(x) \wedge \sigma^0(t) \end{aligned}$$

according to the degree as in the proof of Lemma 3.9. Then $\sigma^2(t,x,y) = 0$ yields the vanishing in each degree

$$\left\{\begin{array}{ll} \sigma^0(t)\wedge\sigma^+(x)+\sigma^+(x)\wedge\sigma^0(t) & =0,\\ (\sigma^0(t))^2+\sigma^+(x)\wedge\sigma^-(y)+\sigma^-(y)\wedge\sigma^+(x) & =0,\\ \sigma^0(t)\wedge\sigma^-(x)+\sigma^-(x)\wedge\sigma^0(t) & =0.\end{array}\right. \tag{4.2}$$

Substituting in the expressions of (4.1) and using the relations (3.6) we arrive at the following system of equations

$$\left\{\begin{array}{lcl} x_p(t_{p+1}-t_p) & = & 0,\\ y_p(t_{p+1}-t_p) & = & 0,\\ x_py_p-t_p^2 & = & x_py_p-t_{p+1}^2=0\end{array}\right. \tag{4.3}$$

for $p=0,\dots,l-2$. This yields the set of solutions

$$\hat{H}=\Big\{(z,x,y)\in\mathbf{C}\times\mathbf{C}^{\mathbf{l-1}}\times\mathbf{C}^{\mathbf{l-1}}\mid x=(x_p),\quad y=(y_p),\ x_py_p=z^2,\ p=0,\dots,l-2\Big\}. \tag{4.4}$$

For $(z,x,y)\in\hat{H}$ denote by $\sigma(z,x,y)=zD^0+\sum_{p=0}^{l-2}x_pD_p^+ +\sum_{p=0}^{l-2}y_pD_{p+1}^-$ the corresponding Higgs morphism. It is clear that scaling $\sigma(z,x,y)$ by $\lambda\in\mathbf{C}^*$ gives a $\mathbf{C}^*$- action on $\hat{H}$. Furthermore, conjugating $\sigma(z,x,y)$ by an automorphism

$$g=\sum_{p=0}^{l-1}g_p id_{\mathbf{H^p}},\ (g_p\in\mathbf{C}^*,\ p=0,\dots,l-1)$$

of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ gives a gauge equivalent Higgs morphism

$$g\sigma(z,x,y)g^{-1}=zD^0+\sum_{p=0}^{l-2}\frac{g_{p+1}}{g_p}x_pD_p^+ +\sum_{p=0}^{l-2}\frac{g_p}{g_{p+1}}y_pD_{p+1}^-. \tag{4.5}$$

All together this defines an action of the torus $\hat{S}=(\mathbf{C}^*)^{\mathbf{l}}$ on the variety $\hat{H}$. From (4.5) we deduce that the action has the following form. For $\tau=(\lambda,\lambda_0,\dots,\lambda_{l-2})\in\hat{S}$ and $(z,x,y)\in\hat{H}$ we have

$$\tau\cdot(z,x,y)=\lambda(z,\lambda_0x_0,\dots,\lambda_{l-2}x_{l-2},\lambda_0^{-1}y_0,\dots,\lambda_{l-2}^{-1}y_{l-2}). \tag{4.6}$$

If we factor out $\hat{H}$ by the scaling $\mathbf{C}^*$-action we obtain the projectivization of $\hat{H}$ which will be denoted by H.

Definition 4.1. *H is a variety of the homothety equivalent non-zero Higgs morphisms of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$. This variety will be called nonabelian Albanese of $\mathbf{J}^r_P$.*

Remark 4.2. Let h be in $\hat{H}$ and let $\sigma(h) : \mathbf{F^0} \longrightarrow (\tilde{\mathbf{H}})^* \otimes \mathbf{F^0}$ be the corresponding Higgs morphism. Using the splitting in (1.23) we view $\mathbf{H}$ as a subbundle of $\tilde{\mathbf{H}}$. Restricting $\sigma(h)$ to $\mathbf{H}$ yields the morphisms

$$\mathbf{F^0} \longrightarrow (\mathbf{H})^* \otimes \mathbf{F^0}.$$

Combining this with Lemma 1.4 we obtain the morphisms

$$D_h : \mathbf{F^0} \longrightarrow \Omega_{\breve{\mathbf{J}}/\dot{\Gamma}^r_d(P)} \otimes \mathbf{F^0}$$

where $\Omega_{\breve{\mathbf{J}}/\dot{\Gamma}^r_d(P)}$ is the relative cotangent bundle. Thus in view of terminology of Remark 3.3 we can think of $\hat{H}$ (resp. H) as a variety parametrizing the relative Higgs fields (resp. homothety equivalence classes of the relative Higgs fields) of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$.

We will now consider the projective description of the Albanese H. From (4.4) one easily obtains the following.

Proposition 4.3. *Let $\mathbf{P^{2(l-1)}}$ be a projective space with the homogeneous coordinates T, X_p, Y_p, $(p = 0, \dots, l-2)$. Then H is a complete intersection of $l-1$ quadrics $X_pY_p = T^2\,(p = 0, \dots, l-2)$ in $\mathbf{P^{2(l-1)}}$. In particular, H is a Fano variety of dimension $(l-1)$ and degree 2^{l-1} with the dualizing sheaf $\omega_H = \mathcal{O}_H(-1)$.*

Corollary 4.4. *The hyperplane sections of H are Calabi-Yau varieties of dimension $l-2$.*

Proof. The statement about the dualizing sheaf ω_H of H in Proposition 4.3 and the adjunction formula yield the assertion. q.e.d.

The Albanese H comes together with the distinguished divisor H_0 corresponding to the hyperplane section defined by $T = 0$. This divisor corresponds to Higgs morphisms having components of degree ± 1 only (see Remark 3.8). To give a projective description of this divisor we begin with a more intrinsic definition of the projective space in Proposition 4.3. Introduce the symbols $V_0, V^{\pm}_p$, $p = 0, \dots, l-2$ (one could think of these symbols as labels of vertices of a certain graph (this is our trivalent graph of **Part VI** of the introduction) which will be eventually colored by D^0, D^+_p and D^-_{p+1}, $p = 0, \dots, l-2$, respectively) and let V be $\mathbf{C}$-vector space generated by these symbols. Let V^* to be the space dual to V and define T, X_p, Y_p, $(p = 0, \dots, l-2)$ to be its dual basis. Clearly, the projective space $\mathbf{P^{2(l-1)}}$ is just $\mathbf{P}(V)$ and the divisor H_0 is defined by the equations $T = 0$ and $X_pY_p = 0\,(p = 0, \dots, l-2)$. Then we have the following description of H_0.

Lemma 4.5. *For any subset A of the set of indicies $I = \{0, \dots, l-2\}$ let $\hat{\Pi}_A$ be the subspace of V spanned by the vectors $\{V^+_i, V^-_j \mid i \in A, j \in$*

$I \setminus A\}$ and let Π_A be its projectivization. Then $H_0 = \bigcup_A \Pi_A$, where the union is taken over all subsets A of the set I.

Proof. From the equations defining H_0 it follows that $\Pi_A \subset H_0$ for every A. This gives an inclusion $\bigcup_A \Pi_A \subset H_0$. Since the degree of $\bigcup_A \Pi_A$ is equal to $2^{l-1} = \deg(H_0)$ (this follows from Proposition 4.3) we deduce the equality asserted in the lemma. q.e.d.

From the definition of the irreducible components Π_A of H_0 and the equations defining H (Proposition 4.3) one easily obtains the following.

Lemma 4.6.

(i) *For any two subsets A, B of I the intersection $\Pi_A \cap \Pi_B$ is the projectivization of the vector space spanned by the set $\{V_i^+, V_j^- \mid i \in A \cap B,\ j \in A^- \cap B^-\}$, where A^- denotes the complement of A in I.*

(ii) *The singularity locus of H is $Sing(H) = \bigcup_{A \neq B} \Pi_A \cap \Pi_B$, where the union is taken over all pairs of distinct subsets A, B of the index set I.*

We have seen in Corollary 4.4 that the hyperplane sections of H are Calabi-Yau varieties. We would like to argue that the divisor H_0 is degenerate from this perspective as well. Namely, the divisor H_0 can be thought of as a "Lagrangian" cycle in the following sense.

Lemma 4.7. *Let V' be the subspace of V spanned by the vectors $V_p^{\pm}$, $(p = 0, \ldots, l-2)$. It admits a natural symplectic structure ω with respect to which the subspaces $\hat{\Pi}_A$ in Lemma* 4.5 *are Lagrangian subspaces of the symplectic space (V', ω).*

Proof. Define $\omega(V_p^{\pm}, V_q^{\pm}) = 0, \forall p, q \in \{0, \ldots, l-2\}$ and $\omega(V_p^+, V_q^-) = -\omega(V_q^-, V_p^+) = \delta_{pq}$. Extending it by linearity we obtain the symplectic product $\omega : V' \times V' \longrightarrow \mathbf{C}$, i.e. (V', ω) is a symplectic space. It follows immediately from the definition of ω that $\hat{\Pi}_A$ in Lemma 4.5 are Lagrangian subspaces. q.e.d.

Definition 4.8. The divisor H_0 in Lemma 4.5 is called the Lagrangian cycle of H and its irreducible components Π_A are called Lagrangian manifolds of H.

We turn now to a toric description of $\hat{H}$ and H. From (4.6) it follows that $\hat{H}$ (resp. H) comes naturally with an action of the torus $\hat{S} = (\mathbf{C}^*)^{\mathbf{l}}$ (resp. $S = (\mathbf{C}^*)^{\mathbf{l-1}}$). Furthermore the open set defined by the condition $T \neq 0$ is isomorphic to $(\mathbf{C}^*)^{\mathbf{l}}$ (resp. $(\mathbf{C}^*)^{\mathbf{l-1}}$). This follows immediately from the equations in Proposition 4.3 and the action of the torus $\hat{S}$ (resp.

S) in (4.6) is compatible with the group multiplication of $\hat{S}$ (resp. S). Thus $\hat{H}$ (resp. H) is a toric variety.

We will now seek to determine a fan $\hat{\Delta}$ (resp. Δ) in $\mathbf{R^l}$ (resp. $\mathbf{R^{l-1}}$) defining $\hat{H}$ (resp. H) (see [**F**] for a general reference on toric varieties).

Proposition 4.9.

1) *$\hat{H}$ is an affine toric variety whose fan $\hat{\Delta}$ is generated by the vectors $\hat{E}_A = (1, m_0, \ldots, m_{l-2})$ of the lattice $\mathbf{Z^l}$, labeled by subsets A in $I = \{0, \ldots, l-2\}$ and where $m_i = -1$, if $i \in A$ and $m_i = 1$, if $i \notin A$, i.e., the fan $\hat{\Delta}$ is a cone in $\mathbf{R^l}$ generated by the vertices of the cube $[-1, 1]^l$ whose first coordinate is equal to 1.*
2) *H is a projective toric variety whose fan Δ of H is generated by the vectors $E_A = (m_0, \ldots, m_{l-2})$, where A and the coordinates m_i are as in the definition of $\hat{E}_A$, i.e., Δ is the fan in $\mathbf{R^{l-1}}$ generated by the vertices of the cube $[-1, 1]^{l-1}$ and the vertices of the cube are in 1-to-1 correspondence with the irreducible components of the divisor H_0. In particular, $Pic(H)$ is generated by the irreducible components of the Lagrangian cycle H_0.*

Proof. The fact that $\hat{H}$ is affine follows directly from its definition in (4.4). Let $\mathbf{R^l}$ (resp.$\mathbf{Z^l}$) be the real vector space (resp. the integral lattice) spanned by some vectors $e_0, e^+_p, p = 0, \ldots, l-2$. From now on the vectors in $\mathbf{R^l}$ (resp.$\mathbf{Z^l}$) will be given in this basis.

To determine the vectors in $\mathbf{Z^l}$ which are the edges of the fan $\hat{\Delta}$ we look at the points in $\hat{H}$ whose $\hat{S}$-orbits are of codimension 1. From the defining equations of $\hat{H}$ these points are

$$\hat{p}_A = \sum_{i \in A} V^+_i + \sum_{j \notin A} V^-_j \tag{4.7}$$

where $V^{\pm}_i$ are as in Lemma 4.5. Now we look at the one-dimensional subgroup of $\hat{S}$ corresponding to a vector $v = (m, m^+_0, \ldots, m^+_{l-2}) \in \mathbf{Z^l}$. From (4.6) it follows that the corresponding orbit has the form

$$\lambda_v(z) = (z^m, z^{m+m^+_0}, \ldots, z^{m+m^+_{l-2}}, z^{m-m^+_0}, \ldots, z^{m-m^+_{l-2}}) \tag{4.8}$$

for $z \in \mathbf{C}$. Now we use the fact that the vector v is an edge of $\hat{\Delta}$ if and only if the limit of (4.8) is $\hat{p}_A$ as the variable z approaches 0. This implies that $m > 0$, $m^+_i = -m$ if $i \in A$, and $m^+_i = m$ if $i \notin A$. Hence the first vector of the lattice belonging to $\hat{\Delta}$ is as asserted.

The fan Δ of H can be determined by applying the above argument to the affine patches $H^{\pm}_i$ of H, where H^+_i (resp. H^-_i) is determined by the condition $X_i \neq 0$ (resp. $Y_i \neq 0$), for $i = 0, \ldots, l-2$.

The last assertion in 2) follows from the fact that the irreducible components of H_0 are precisely the closures of the S-orbits of the points p_A, the projectivization of the vectors $\hat{p}_A$ in (4.7). On the other hand these S-orbits of codimension 1 correspond to the edges of the fan Δ and it is well known that these closures are the divisors of H which generate $Pic(H)$ (see Corollary, p. 64, [**F**]). q.e.d.

5. Two correspondences between $\breve{\mathbf{J}}^r_P$ and H

The Albanese H depends only on the weight l of the orthogonal decomposition (2.5) and the relations (3.6) between the morphisms of different degrees composing the Higgs morphisms

$$\sigma(z,x,y) = zD^0 + \sum_{p=0}^{l-2} x_p D^+_p + \sum_{p=0}^{l-2} y_p D^-_{p+1}.$$

So it completely "forgets" the relationship of the Jacobian $\mathbf{J}(X;L,d)$ with our surface. To remedy in part this situation we will construct two correspondences as described in **Part V** of the introduction.

5.1. A geometric correspondence. We construct a natural morphism from $\breve{\mathbf{J}}$ to $\mathbf{P}(H^0(\mathcal{O}_H(d))$. Actually the image will be contained in the subvariety $P^{(d)}$ of $\mathbf{P}(H^0(\mathcal{O}_H(d))$ whose points correspond to the sections of $\mathcal{O}_H(d)$ which can be written as a product of d sections of $\mathcal{O}_H(1)$, i.e.,

$$P^{(d)} = \left\{ [s] \in \mathbf{P}(H^0(\mathcal{O}_H(d)) \mid s = \prod_{i=1}^{d} s_i, \text{ for some } s_i \in H^0(\mathcal{O}_H(1)) \right\}.$$

Proposition 5.1. *There is a distinguished morphism*

$$CY : \breve{\mathbf{J}} \longrightarrow P^{(d)} \subset \mathbf{P}(H^0(\mathcal{O}_H(d))).$$

Proof. Let $([Z],[\alpha]) \in \breve{\mathbf{J}}$. To define $CY([Z],[\alpha])$ we recall that $H^0(\mathcal{O}_H(1))$ comes with a particular basis $T, X_p, Y_p\,(p = 0,\dots,l-2)$ (see Proposition 4.3 and the subsequent discussion). So our strategy is as follows. For every point $z \in Z$ we will produce the constants $t(z,[\alpha]), x_p(z,[\alpha]), y_p(z,[\alpha])$, for $p = 0,\dots,l-2$, depending holomorphically on $([Z],[\alpha])$. Then we use these constants to define the section

(5.1)
$$s(z,[\alpha]) = t(z,[\alpha])T + \sum_{p=0}^{l-2} x_p(z,[\alpha])X_p + \sum_{p=0}^{l-2} y_p(z,[\alpha])Y_p \in H^0(\mathcal{O}_H(1)).$$

Then we define $CY([Z],[\alpha])$ to be the product $\prod_{z\in Z} s(z,[\alpha])$. Thus our argument comes down to defining the constants $t(z,[\alpha]), x_p(z,[\alpha]), y_p(z,[\alpha])$, for $p = 0,\dots,l-2$. This is done by using the orthogonal decomposition (2.5) at the point $([Z],[\alpha])$

$$H^0(\mathcal{O}_Z) = \bigoplus_{p=0}^{l} \mathbf{H^p}([Z],[\alpha]) \tag{5.2}$$

where $\mathbf{H^p}([Z],[\alpha])$ denotes the fibre of $\mathbf{H^p}$ at $([Z],[\alpha])$.

Let δ_z be the delta-function on Z supported at z, i.e., $\delta_z(z)=1$ and vanishes at other points of Z. Denote by δ^0_z the component of δ_z in $\mathbf{H^0}([Z],[\alpha])$. Applying to it the operator $D^+(\delta^0_z)$ we obtain (the right) moving string of functions $\delta^{(p)}_z = (D^+(\delta^0_z))^p(\delta^0_z)$, for $p=0,\dots,l-1$. Once we arrive to $\delta^{(l-1)}_z \in \mathbf{H^{l-1}}([Z],[\alpha])$ we apply the operator $D^-(\delta^0_z)$ to create (the left) moving string of functions $\delta^{(l-1),(l-1-m)}_z = (D^-(\delta^0_z))^m(\delta^{(l-1)}_z)$, for $m=0,\dots,l-1$. The desired constants are obtained essentially by evaluating all these functions at z. More precisely, define

$$x_p(z,[\alpha]) = \exp(\delta^{(p)}_z(z)), \quad y_p(z,[\alpha]) = \exp(\delta^{(l-1),(p+1)}_z(z)),$$
$$t(z,[\alpha]) = \exp(\delta^{(l-1),(0)}_z(z)),$$

where $p=0,\dots,l-2$.

The above construction depends only on the orthogonal decomposition (5.2). Since the latter varies holomorphically with $([Z],[\alpha])$ we obtain that $CY([Z],[\alpha])$ depends holomorphically on $([Z],[\alpha])$ as well. q.e.d.

Remark 5.2.

(i) By Corollary 4.4 the hyperplane sections of H are Calabi-Yau varieties. So the construction in the proof of Proposition 5.1 assigns the Calabi-Yau variety $H(z,[\alpha]) = (s(z,[\alpha])=0)$ to every point $z\in Z$ and associates the cycle $\sum_{z\in Z} H(z,[\alpha])$ of Calabi-Yau varieties to $([Z],[\alpha])$. Hence the notation CY of the morphism in Proposition 5.1. We will often identify $CY([Z],[\alpha])$ with the corresponding divisor and refer to it as a Calabi-Yau cycle of $([Z],[\alpha])$.

(ii) The Calabi-Yau $H(z,[\alpha])$ can be viewed as a result of an "oscillation" of δ^0_z (see the proof of Proposition 5.1 for notation) according to the rules provided by the operators $D^{\pm}, D^0$. Organizing these rules in the trivalent graph discussed in **Part VI** of the introduction, one can think of $H(z,[\alpha])$ as being created as a result of "vibration" of δ^0_z along our trivalent graph.

(iii) The construction in the proof can be iterated. Indeed, when we return back to $\mathbf{H^0}([Z],[\alpha])$ with the function $\delta^{(l-1),(0)}_z$ (the end of the first cycle) we can apply the operator $D^0(\delta^0_z)$ to it and renew our construction to obtain another hyperplane section of H. Continuing in this fashion we obtain an a priori infinite sequence of Calabi-Yau varieties associated to every point of Z.

(iv) Our construction indicates that two distinct points in Z can "interact" via the intersection of their respective Calabi-Yau varieties. Thus one could naturally define "linking" invariants for any pair of distinct points in Z as the invariants of those intersections. These questions will be taken up elsewhere.

It is obvious that in the construction in the proof of Proposition 5.1 one could "average" the constants $t(z,[\alpha]), x_p(z,[\alpha]), y_p(z,[\alpha])$ over Z, thus obtaining a distinguished Calabi-Yau variety associated to $([Z],[\alpha])$. More generally, we have the following.

Proposition 5.3. *Let S_d be the group of permutations on d letters and let $R[A_1,\dots,A_d]^{S_d}$ be the ring of symmetric polynomials in d indeterminates $A_1,\dots,A_d$ with coefficients in any subring R of $\mathbf{C}$. Fix nonzero symmetric polynomials $c, g_p, h_p \in R[A_1,\dots,A_d]^{S_d}$, $p = 0,\dots,l-2$. Then there is a morphism*

$$CY(c, g_0,\dots,g_{l-2}, h_0,\dots,h_{l-2}) : \breve{\mathbf{J}} \longrightarrow \mathbf{P}(H^0(\mathcal{O}_H(1))).$$

Proof. Choose an ordering $j : \{1,\dots,d\} \longrightarrow Z$ of points in Z and set $z_k = j(k)$, for all $k \in \{1,\dots,d\}$. Using the notation introduced in the proof of Proposition 5.1 for every z_k we obtain the functions $\delta^{(p)}_{z_k}$, $\delta^{(l-1),(p)}_{z_k}$, where $p = 0,\dots,l-1$. Evaluating them at z_k we obtain the constants $\xi^{(p)}_k = \delta^{(p)}_{z_k}(z_k)$ and $\xi^{(l-1),(p)}_k = \delta^{(l-1),(p)}_{z_k}(z_k)$. By evaluating the symmetric polynomials we obtain

$$\begin{aligned} t([Z],[\alpha]) &= \exp(c(\xi^{(l-1),(0)}_1,\dots,\xi^{(l-1),(0)}_d)), \\ x([Z],[\alpha])_p &= \exp(g_p(\xi^{(p)}_1,\dots,\xi^{(p)}_d)), \\ y([Z],[\alpha])_p &= \exp(h_p(\xi^{(l-1),(p+1)}_1,\dots,\xi^{(l-1),(p+1)}_d)) \end{aligned}$$

where $p = 0,\dots,l-2$. We use these constants to define the section

$$s([Z],[\alpha]) = t([Z],[\alpha])T + \sum_{p=0}^{l-2} x([Z],[\alpha])_p X_p + \sum_{p=0}^{l-2} y([Z],[\alpha])_p Y_p$$

and take the corresponding point in $\mathbf{P}(H^0(\mathcal{O}_H(1)))$ to be the image of $CY(c, g_0,\dots,g_{l-2}, h_0,\dots,h_{l-2})$ at $([Z],[\alpha])$. q.e.d.

5.2. Complete intersections. We illustrate our general construction by considering complete intersections of sufficiently ample divisors on a surface.

Let X be a smooth projective surface with irregularity $q(X) = h^1(\mathcal{O}_X) = 0$. Fix a very ample line bundle $\mathcal{O}_X(L)$ on X and consider a 0-dimensional complete intersection subscheme Z of two smooth irreducible members C_1, C_2 of the linear system $\mid L \mid$. From the Koszul sequence

$$0 \longrightarrow \mathcal{O}_X(-2L) \longrightarrow \mathcal{O}_X(-L) \oplus \mathcal{O}_X(-L) \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

for the ideal sheaf $\mathcal{I}_Z$ of Z in X we obtain

$$\mathrm{Ext}^1_Z = H^0(\mathcal{O}_X(L))/\mathbf{C}\{s_1, s_2\}$$

where s_1 and s_2 are the sections of $H^0(\mathcal{O}_X(L))$ corresponding to C_1 and C_2 respectively. Thus $[Z] \in \overset{\circ}{\Gamma}{}^r_d(L)$, where $d = L^2$ and $r = h^0(L) - 3$.

Assume $r \geq 1$ and set $\mathbf{P}$ to be the codimension 2 subspace of $\mathbf{P}(H^0(\mathcal{O}_X(L))^*)$ spanned by Z. We will now compute the Hilbert function P of Z in $\mathbf{P}$. By Remark 1.5 this will determine the Hilbert function of the filtration (1.25) over the points of $X^{[d]}$ corresponding to the complete intersections of divisors in the linear system $\mid L \mid$. Let $\mathcal{J}_Z$ be the ideal sheaf of Z in $\mathbf{P}$. Then we have

(5.3)

$$P(k) = \dim(\tilde{\mathbf{H}}_{-k}([Z],[\alpha])) = \deg(Z) - h^1(\mathcal{J}_Z(k)) = L^2 - h^1(\mathcal{J}_Z(k))$$

where $\tilde{\mathbf{H}}_{-k}([Z],[\alpha])$ is the fibre of $\tilde{\mathbf{H}}_{-k}$ in (1.25) at the point $([Z],[\alpha])$. In order to compute $h^1(\mathcal{J}_Z(k))$ we will make several assumptions on L:

1) the line bundle $\mathcal{O}_C(L)$ gives a projectively normal embedding for smooth irreducible curves C in the linear system $\mid L \mid$;
2) $h^2(\mathcal{O}_X(kL)) = 0$, for all $k \geq 1$.

(Observe that these assumptions are satisfied for a sufficiently high multiple of any ample divisor on X.)

Choose a smooth irreducible curve C in $\mid L \mid$ passing through Z. Set $\mathcal{J}_C$ to be the ideal sheaf of C in $\mathbf{P}(H^0(\mathcal{O}_C(L))^*)$. Then the ideal sheaves $\mathcal{J}_Z$ and $\mathcal{J}_C$ are related by the following exact sequence

$$0 \longrightarrow \mathcal{J}_C(-1) \longrightarrow \mathcal{J}_C \longrightarrow \mathcal{J}_Z \longrightarrow 0.$$

This sequence and projective normality of C imply

$$\begin{aligned} h^1(\mathcal{J}_Z(k)) &= h^2(\mathcal{J}_C(k-1)) - h^2(\mathcal{J}_C(k)) \\ &= h^1(\mathcal{O}_C((k-1)L))) - h^1(\mathcal{O}_C(kL)) \end{aligned}$$

for all $k \geq 0$. Substituting in (5.3) we obtain

$$P(k) = rk(\mathbf{H}_{-k}) = L^2 + h^1(\mathcal{O}_C(kL)) - h^1(\mathcal{O}_C((k-1)L))). \tag{5.4}$$

From 1) of the assumptions it follows that $h^1(\mathcal{O}_X(kL)) = 0$, for all $k \geq 0$. This implies that $h^1(\mathcal{O}_C(kL)) = h^2(\mathcal{O}_X((k-1)L)) - h^2(\mathcal{O}_X(kL))$. Substituting in (5.4) yields

(5.5)

$$P(k) = \deg(Z) - h^2(\mathcal{O}_X(kL)) + 2h^2(\mathcal{O}_X((k-1)L)) - h^2(\mathcal{O}_X((k-2)L))$$

for all $k \geq 0$. Finally, using 2) of the assumptions we obtain that the filtration (1.25) at $([Z],[\alpha])$ has the following form

$$\begin{aligned} (5.6)\quad & H^0(\mathcal{O}_X) \\ &= \mathbf{H}_0([Z],[\alpha]) \subset \mathbf{H}_{-1}([Z],[\alpha]) \subset \mathbf{H}_{-2}([Z],[\alpha]) \subset \mathbf{H}_{-3}([Z],[\alpha]) \\ &= H^0(\mathcal{O}_Z). \end{aligned}$$

We now assume that Z is reduced, i.e. it is L^2 distinct points. Then the filtration (5.6) splits to yield the following direct sum decomposition.

$$H^0(\mathcal{O}_Z) = \mathbf{H^0}([Z],[\alpha]) \oplus \mathbf{H^1}([Z],[\alpha]) \oplus \mathbf{H^2}([Z],[\alpha]) \tag{5.7}$$

and we have

$$\begin{aligned} &\dim(\mathbf{H^0}([Z],[\alpha])) = P(1) = h^0(\mathcal{O}_X(L)) - 2 \\ &\dim(\mathbf{H^1}([Z],[\alpha])) = P(2) - P(1) = L^2 - h^2(\mathcal{O}_X) - h^0(\mathcal{O}_X(L)) + 2 \\ &\dim(\mathbf{H^2}([Z],[\alpha])) = P(3) - P(2) = h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X)). \end{aligned} \tag{5.8}$$

Let $\breve{\mathbf{J}}$ be the part of $\mathbf{J}(X;L,L^2)$ where the summands of the decomposition (5.7) have dimensions given by (5.8). If the geometric genus $p_g(X) = h^0(\mathcal{O}_X(K_X)) \neq 0$, then Proposition 4.3 implies that the Albanese H of $\breve{\mathbf{J}}$ is a complete intersection of two quadrics

$$X_0Y_0 = T^2, \;\; X_1Y_1 = T^2$$

in $\mathbf{P}^4$ and its Lagrangian cycle H_0 is the union of 4 lines

$$H_0 = \Pi_\emptyset \cup \Pi_0 \cup \Pi_1 \cup \Pi$$

where $\Pi_\emptyset = \{X_0 = X_1 = 0\}$, $\Pi_0 = \{X_1 = Y_0 = 0\}$, $\Pi_1 = \{X_0 = Y_1 = 0\}$ and $\Pi = \{Y_0 = Y_1 = 0\}$.

From Lemma 4.6 we obtain that H is singular at 4 points $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$, $(1:0:0:0)$. It is easy to see, either from projective or toric description of H, that resolving the singularities of H we obtain $\mathbf{P}^1 \times \mathbf{P}^1$ blown-up at 4 distinct points corresponding to the points of intersection of two reduced and reducible divisors $F_1 + F'_1$, $F_2 + F'_2$, where F and F' are the divisor classes of the distinct rulings of $\mathbf{P}^1 \times \mathbf{P}^1$.

The Calabi-Yau cycle map CY in Proposition 5.1 in this case sends points $([Z],[\alpha]) \in \breve{\mathbf{J}}$ to a cycle of elliptic curves. More precisely, the construction in the proof of Proposition 5.1 associates a smooth elliptic curve with every point $(z,[\alpha]) \in (Z,[\alpha])$.

Thus our construction implies that behind points on a smooth complex projective surface X with $p_g \neq 0$ are "hidden" elliptic curves. To "reveal" them one has to make a point on X to be a part of a (reduced) complete intersection of curves in the linear system of a sufficiently high multiple of any ample divisor L on X (observe that if X is a K3-surface then taking *any* very ample L will be enough). This can be viewed as an answer to the question posed by Nakajima about a possibility that "elliptic curves are hidden in the Hilbert schemes" (see p. 2, [**N**]).

5.3. Fourier-Mukai functor. In this section we consider a cohomological correspondence between $\breve{\mathbf{J}}$ and its Albanese. Namely, we will construct a functor (Fourier-Mukai functor) from a certain category which we call the Higgs category of weight l (see Definition 5.11), where l is as in the decomposition (2.5), to the category of $\mathcal{O}_H$-modules enriched by a Fukaya type data on the Lagrangian manifolds composing the Lagrangian cycle H_0 (see Definition 4.8). To motivate our definitions we begin with the case of the sheaf $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$.

Set $\mathbf{F^0} = \pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$. We know that it comes with the orthogonal decomposition (2.5) of weight l and the Higgs fields $D_h : \mathbf{F^0} \longrightarrow \Omega_{\breve{\mathbf{J}}/\dot{\Gamma}^r_d(P)} \otimes \mathbf{F^0}$, for $[h] \in H$. If no confusion is likely we will denote the relative cotangent bundle simply by $\mathbf{\Omega}$.

The Higgs field D_h can be used to define the complex

(5.9)

$$C_h = (\mathbf{F^0} \otimes \mathbf{\Omega}^\bullet, D_h) : \qquad \mathbf{F^0} \xrightarrow{D_h} \Omega \otimes \mathbf{F^0} \xrightarrow{D_h} \cdots \xrightarrow{D_h} \Omega^r \otimes \mathbf{F^0}$$

where r is the dimension of the fibres of $\breve{\mathbf{J}}$ over $\dot{\Gamma}^r_d(P)$. Thus on the product $\breve{\mathbf{J}} \times H$ one has the universal complex $\tilde{C} = (\pi^*_{\breve{\mathbf{J}}}(\mathbf{F^0} \otimes \Omega), \tilde{D})$ defined by the property that $\tilde{C}$ restricted to the slice $\breve{\mathbf{J}} \times \{[h]\}$, $(\forall [h] \in H)$ coincides with the complex C_h (here $\pi_{\breve{\mathbf{J}}}$ (resp. π_H) denotes the projection of $\breve{\mathbf{J}} \times H$ onto $\breve{\mathbf{J}}$ (resp. H)). Taking the cohomology of the complex $\tilde{C}$ we obtain the graded sheaf $\mathcal{H}^\bullet(\tilde{D}) = \bigoplus_{i=0}^{r} \mathcal{H}^i(\tilde{C})$. Its direct image with respect to the projection π_H yields a graded $\mathcal{O}_H$-module

$$\mathcal{F}(\mathbf{F^0}) = \bigoplus_{i=0}^{r} (\pi_H)_* \mathcal{H}^i(\tilde{C}) \tag{5.10}$$

which we call the Fourier-Mukai transformation of $\mathbf{F^0}$.

Remark 5.4. The fibre $\mathcal{F}(\mathbf{F^0})_{[h]}$ of $\mathcal{F}(\mathbf{F^0})$ at a closed point $[h] \in H$ is the graded vector space

$$\mathcal{F}(\mathbf{F^0})_{[h]} = \Gamma(\mathcal{H}^\bullet(D_h)) = \bigoplus_{i=0}^{r} \Gamma(\mathcal{H}^i(C_h)).$$

We will now explain what we mean by "enrichment" of $\mathcal{F}(\mathbf{F^0})$ by a Fukaya type data. This will consist of a collection of complexes associated to each irreducible component Π_A of H_0 (see Lemma 4.5) and for every pair of "Lagrangians" Π_A and Π_B intersecting transversally, i.e., $\Pi_A \cap \Pi_B$ is a point, a certain natural relation between the complexes corresponding to the point of their intersection.

To define complexes associated to the component Π_A we recall that it comes with a preferred basis $\{V^+_i, V^-_j \mid i \in A, j \in A^-\}$. Thus if $[h]$ is a point of Π_A corresponding to one of these generators it comes colored by "+" or "−". Define $col([h]) = 1$ (resp. -1) if the color of $[h]$ is "+" (resp. "−"). Furthermore, the vectors $V^\pm_i$ are ordered by the index set $I = \{0, \ldots, l-2\}$, so if $[h]$ corresponds to one of the generators of Π_A we define $ord([h])$ to be the order of the corresponding generator. With these definitions in mind we now define a collection of complexes associated to Π_A. There will be two complexes, colored by "$\pm$", for each point $[h] \in \Pi_A$ corresponding to the generators of Π_A. To define them take the Higgs field

$$D_A = \sum_{s\in A} D^+_s + \sum_{q\in A^-} D^-_{q+1} \tag{5.11}$$

corresponding to the point $o = \left[\sum_{s\in A} V^+_s + \sum_{q\in A^-} V^-_q\right]$ in Π_A (this should be viewed as a "general" Higgs field in Π_A). Let $C_A = (\mathbf{F^0} \otimes \mathbf{\Omega^\bullet}, D_A)$ be the corresponding complex. Let $[h]$ be a point of Π_A corresponding to one of its generators and let $p = ord([h])$. At i-th term of the complex C_A we consider the restriction of the differential D_A to the summand $\mathbf{H^{c(p)}} \otimes \mathbf{\Omega^i}$ of $\mathbf{F^0} \otimes \mathbf{\Omega^\bullet}$, where the degree c(p) of the summand is defined according to the color of $[h]$ as follows:

$$c(p) = \begin{cases} p, & if\ col([h]) = 1, \\ p+1, & if\ col([h]) = -1. \end{cases} \tag{5.12}$$

This will be called the colored degree of $[h]$ and denoted by $cdeg([h])$. We define $\mathcal{H}^+_{\Pi_A,[h]}(i)$ to be the resulting cohomology sheaf, i.e.

$$\mathcal{H}^+_{\Pi_A,[h]}(i) = \frac{ker(\mathbf{\Omega^i} \otimes \mathbf{H^{c(p)}} \longrightarrow \mathbf{\Omega^{i+1}} \otimes \mathbf{F^0})}{im(\mathbf{\Omega^{i-1}} \otimes \mathbf{F^0} \longrightarrow \mathbf{\Omega^i} \otimes \mathbf{F^0}) \cap (\mathbf{\Omega^i} \otimes \mathbf{H^{c(p)}})}. \tag{5.13}$$

Define

$$\mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h]) = \bigoplus_{i=0}^{r} \Gamma(\mathcal{H}^+_{\Pi_A,[h]}(i)). \tag{5.14}$$

This is the graded vector space (colored by "+") which we associate to $[h] \in \Pi_A$. The graded vector space colored by "$-$" is obtained by performing the above construction with the Higgs field adjoint to D_A (with respect to $\mathbf{q}$) (see Lemma 3.7), i.e., with the Higgs field

$$D_{A^-} = \sum_{s\in A^-} D^+_s + \sum_{q\in A} D^-_{q+1}.$$

To see that the graded vector spaces in (5.14) are complexes we need to put the cohomology sheaves $\mathcal{H}^\pm_{\Pi_A,[h]}(i)$ in a more explicit form.

Lemma 5.5.

1)

$$\mathcal{H}^+_{\Pi_A,[h]}(i) = \begin{cases} \dfrac{\ker(D^{col([h])}_{c(p)}(i))}{im(D^{col([h])}_{c(p)-col([h])}(i-1))}, & if\ p - col([h]) \in A^{col([h])} \\ \\ \ker(D^+_{c(p)}(i)) \cap \ker(D^-_{c(p)}(i)), & if\ p - col([h]) \notin A^{col([h])}. \end{cases}$$

2)

$$\mathcal{H}^{-}_{\Pi_A,[h]}(i)=\begin{cases}\frac{\ker(D^{-col([h])}_{c(p)}(i))}{im(D^{-col([h])}_{c(p)+col([h])}(i-1))}, & if\ p-col([h])\in A^{col([h])}\\ \\ \frac{\mathbf{\Omega^i}\otimes\mathbf{H^{c(p)}}}{im(D^{+}_{c(p)-1}(i-1))+im(D^{-}_{c(p)+1}(i-1))}, & if\ p-col([h])\notin A^{col([h])}\end{cases}$$

where $D^{\pm}_k(j)$ stands for the morphism $D^{\pm}_k(j):\mathbf{\Omega^j}\otimes\mathbf{H^k}\longrightarrow\mathbf{\Omega^{j+1}}\otimes\mathbf{H^{k\pm1}}$.

(In the above (and subsequent) notation an appearance of $col([h])$ as a superscript should be read as the sign $(\pm)$ of $col([h])$).

3) *The morphism D^0 defines the differentials $d^{\pm}$ on the graded sheaves $\mathcal{H}^{\pm}_{\Pi_A,[h]}=\bigoplus_{i=0}^{r}\mathcal{H}^{\pm}_{\Pi_A,[h]}(i)$. The corresponding homomorphisms induced on $\mathcal{F}^{\pm}(\mathbf{F^0})_{\Pi_A}([h])$ define the differentials which also will be denoted by $d^{\pm}$.*

Proof. Assume the point $[h]$ to be colored by "+". Then $c(p)=p$ and the restriction of D_A to $\mathbf{H^p}$ is D^+_p, if $p-1\in A$, and $(D^+_p+D^-_p)$, if $p-1\notin A$. This implies

$$\ker(\mathbf{\Omega^i}\otimes\mathbf{H^p}\xrightarrow{D_A}\mathbf{\Omega^{i+1}}\otimes\mathbf{F^0})=\begin{cases}\ker(D^+_p(i)), & \text{if } p-1\in A\\ \\ \ker(D^+_p(i))\cap\ker(D^-_p(i)), & \text{if } p-1\notin A\end{cases}$$

and

$$\begin{aligned}&im(\mathbf{\Omega^{i-1}}\otimes\mathbf{F^0}\xrightarrow{D_A}\mathbf{\Omega^i}\otimes\mathbf{F^0})\cap\mathbf{\Omega^i}\otimes\mathbf{H^p}\\ &=\begin{cases}im(D^+_{p-1}(i-1)), & \text{if } p-1\in A\\ \\ 0, & \text{if } p-1\notin A\end{cases}\end{aligned}$$

which yields the first assertion of the lemma.

In the second assertion the differential is D_{A^-}. Its restriction to $\mathbf{H^p}$ is D^-_p, if $p-1\in A$, and 0 otherwise. This implies

$$\ker(\mathbf{\Omega^i}\otimes\mathbf{H^p}\xrightarrow{D_{A^-}}\mathbf{\Omega^{i+1}}\otimes\mathbf{F^0})=\begin{cases}\ker(D^-_p(i)), & \text{if } p-1\in A\\ \\ \mathbf{\Omega^i}\otimes\mathbf{H^p}, & \text{if } p-1\notin A\end{cases}$$

and

$$
\begin{aligned}
& im(\mathbf{\Omega^{i-1}} \otimes \mathbf{F^0} \xrightarrow{D_{A^-}} \mathbf{\Omega^i} \otimes \mathbf{F^0}) \cap \mathbf{\Omega^i} \otimes \mathbf{H^p} \\
& = \begin{cases} im(D^-_{p+1}(i-1)), & \text{if } p-1 \in A \\ \\ im(D^-_{p+1}(i-1)) + im(D^+_{p-1}(i-1)), & \text{if } p-1 \notin A \end{cases}
\end{aligned}
$$

which yields the second assertion of the lemma. The case of $[h]$ colored by "$-$" is completely analogous.

To prove the third assertion we use the relations between D^0 and $D^\pm$ established in Lemma 3.9. The relation (ii) of Lemma 3.9 implies that D^0 defines morphisms

$$d^\pm(i) : \mathcal{H}^\pm_{\Pi_A,[h]}(i) \longrightarrow \mathcal{H}^\pm_{\Pi_A,[h]}(i+1)$$

while the relation (iii) insures that $d^\pm(i+1) \circ d^\pm(i) = 0$. q.e.d.

In view of the above result it will be convenient to introduce the notion of valence for $[h]$ in Π_A:

$$v_{\Pi_A}([h]) = \begin{cases} 2, & ord([h]) - col([h]) \in A^{col([h])} \\ 1, & ord([h]) - col([h]) \notin A^{col([h])}. \end{cases} \tag{5.15}$$

Next we consider the relation of the vector spaces $\mathcal{F}^\pm(\mathbf{F^0})_{\Pi_A}([h])$ and the fibre of $\mathcal{F}(\mathbf{F^0})$ at $[h]$. The differential at $[h]$ is $D_h = D^{col([h])}_{c(p)}$, where $c(p) = cdeg([h])$ is the colored degree of $[h]$. So

$$\mathcal{F}(\mathbf{F^0})_{[h]} = \bigoplus_{i=0}^{r} \Gamma(\mathcal{H}^i(D^{col([h])}_{c(p)})). \tag{5.16}$$

It is easy to see that the cohomology sheaf $\mathcal{H}^i(D^{col([h])}_{c(p)})$ has the following form

$$\mathcal{H}^i(D^{col([h])}_{c(p)}) = \left(\bigoplus_{q \neq c(p)} \mathbf{\Omega^i} \otimes \mathbf{H^q} \right) \oplus \ker(D^{col([h])}_{c(p)}(i)). \tag{5.17}$$

Thus the fibre $\mathcal{F}(\mathbf{F^0})_{[h]}$ is bigraded: it has the (external) grading given by the degree of the cohomology of the complex and the (internal) grading coming from the grading of the sheaf $\mathbf{F^0}$ and the differential D_h which acts on this grading by shifting the index by $col([h])$. Taking the piece of $\mathcal{F}(\mathbf{F^0})_{[h]}$ corresponding to the internal degree $c(p)$, i.e., the colored degree of $[h]$ (recall: $p = ord([h])$), yields

$$\mathcal{F}(\mathbf{F^0})^{\bullet, c(p)}_{[h]} = \bigoplus_{i=0}^{r} \Gamma(\ker(D^{col([h])}_{c(p)}(i))). \tag{5.18}$$

Remark 5.6. From the second equation in (3.6) it follows that the morphism D^0 restricted to $\ker(D^{col([h])}_{c(p)}(i))$ gives rise to a morphism

$$d^0 : \ker(D^{col([h])}_{c(p)}(i)) \longrightarrow \ker(D^{col([h])}_{c(p)}(i+1)).$$

This induces the homomorphism on $\mathcal{F}(\mathbf{F^0})^{\bullet,c(p)}_{[h]}$ which we will also denote by d^0.

Lemma 5.7. *Set* $\mathcal{F}(\mathbf{F^0})^{\bullet,c(p)}_{[h]} = \mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]}$*. Then* $\mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]}$ *is related to* $\mathcal{F}^{\pm}(\mathbf{F^0})_{\Pi_A}([h])$ *as follows.*

1) *If the valence* $v_{\Pi_A}([h]) = 2$*, then there is a natural homomorphism of graded vector spaces*

$$\mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]} \longrightarrow \mathcal{F}^{+}(\mathbf{F^0})_{\Pi_A}([h]).$$

2) *If the valence* $v_{\Pi_A}([h]) = 1$*, then there are natural homomorphisms of graded vector spaces*

$$\mathcal{F}^{+}(\mathbf{F^0})_{\Pi_A}([h]) \longrightarrow \mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]}$$

and

$$\mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]} \longrightarrow \mathcal{F}^{-}(\mathbf{F^0})_{\Pi_A}([h]).$$

3) *The homomorphisms in* $1) - 2)$ *commute with the morphisms* d^0 *in Remark* 5.6 *and the differentials* $d^{\pm}$ *of* $\mathcal{F}^{\pm}(\mathbf{F^0})_{\Pi_A}([h])$.

Proof. The assertions 1) and 2) follow immediately from the definition of $\mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]}$ in (5.18) and the formulas for $\mathcal{H}^{\pm}_{\Pi_A,[h]}(i)$ in 1) and 2) of Lemma 5.5.

The assertion 3) is obvious since both d^0 and the differentials $d^{\pm}$ are induced by the same morphism D^0. q.e.d.

We now turn to the transversal intersection of two "Lagrangians" Π_A and Π_B. From Lemma 4.6 we know that $\Pi_A \cap \Pi_B$ is a point if either $A \cap B = \{p\}$ and $A^- \cap B^- = \emptyset$ or $A^- \cap B^- = \{p\}$ and $A \cap B = \emptyset$. The first case gives $\Pi_A \cap \Pi_B = [V^+_p]$ while the second $\Pi_A \cap \Pi_B = [V^-_p]$.

Let $\Pi_A \cap \Pi_B = \{[h]\}$. For such a pair of "Lagrangians" we define the index of $[h]$ in the pair (Π_A, Π_B)

$$ind_{\Pi_A,\Pi_B}([h]) = v_{\Pi_B}([h]) - v_{\Pi_A}([h]). \tag{5.19}$$

Lemma 5.8. $ind_{\Pi_A,\Pi_B}([h]) = \pm 1$.

Proof. Let $p = ord([h])$. Then $p = A^{col([h])} \cap B^{col([h])}$ and $A^{col([h])} \cup B^{col([h])} = I$. Assume $v_{\Pi_A}([h]) = 1$. From the definition of the valence in (5.15) it follows $p - col([h]) \notin A^{col([h])}$. Hence $p - col([h]) \in B^{col([h])}$ which yields $v_{\Pi_B} = 2$. q.e.d.

Lemma 5.9. *Let $\Pi_A \cap \Pi_B = \{[h]\}$ and let $ind_{\Pi_A,\Pi_B}([h]) = 1$. Then there are natural homomorphisms*

$$\mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h]) \longrightarrow \mathcal{F}^\pm(\mathbf{F^0})_{\Pi_B}([h])$$

and

$$\mathcal{F}^\pm(\mathbf{F^0})_{\Pi_B}([h]) \longrightarrow \mathcal{F}^-(\mathbf{F^0})_{\Pi_A}([h]).$$

Furthermore, these homomorphisms are morphisms of complexes, i.e., they commute with the differentials $d^\pm$.

Proof. By definition of the index $ind_{\Pi_A,\Pi_B}([h])$ in (5.19) we have $v_{\Pi_B}([h]) = 2$ and $v_{\Pi_A}([h]) = 1$. Let $p = ord([h])$ and let $c(p)$ be the colored degree of $[h]$ (see (5.12)). From Lemma 5.5 it follows

$$\begin{array}{rcl} \mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h]) & = & \bigoplus_{i=0}^r \Gamma\left(\ker\left(D^+_{c(p)}(i)\right) \cap \ker\left(D^-_{c(p)}(i)\right)\right) \\ & and & \\ \mathcal{F}^\pm(\mathbf{F^0})_{\Pi_B}([h]) & = & \bigoplus_{i=0}^r \Gamma\left(\frac{\ker\left(D^{\pm col([h])}_{c(p)}(i)\right)}{im(D^{\pm col([h])}_{c(p)\mp col([h])}(i-1))}\right). \end{array}$$

So the first two homomorphisms are induced by the natural inclusion of sheaves

$$\ker\left(D^+_{c(p)}(i)\right) \cap \ker\left(D^-_{c(p)}(i)\right) \longrightarrow \ker\left(D^{\pm col([h])}_{c(p)}(i)\right)$$

while the other two homomorphisms of the lemma come from the obvious morphisms

$$\frac{\ker\left(D^{\pm col([h])}_{c(p)}(i)\right)}{im(D^{\pm col([h])}_{c(p)\mp col([h])}(i-1))} \longrightarrow \frac{\Omega^i \otimes \mathbf{H^{c(p)}}}{im(D^+_{c(p)-1}(i-1)) + im(D^-_{c(p)+1}(i-1))}.$$

The commutation assertion follows from the naturality of the above morphisms and the fact that the differentials in all complexes are induced by the same morphism D^0. q.e.d.

Next we derive a compatibility relation between the vector space $\mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]}$ and the complexes $\mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h])$, $\mathcal{F}^\pm(\mathbf{F^0})_{\Pi_B}([h])$.

Lemma 5.10.
Let $\Pi_A \cap \Pi_B = \{[h]\}$. Then the vector space $\mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]}$ is related to the complexes $\mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h])$ and $\mathcal{F}^+(\mathbf{F^0})_{\Pi_B}([h])$ (resp., $\mathcal{F}^-(\mathbf{F^0})_{\Pi_B}([h])$) by the following natural commutative triangle

$$\begin{tikzcd} & \mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]} \arrow[dr] & \\ \mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h]) \arrow[ur] \arrow[rr] & & \mathcal{F}^+(\mathbf{F^0})_{\Pi_B}([h]) \end{tikzcd}$$

if $ind_{\Pi_A,\Pi_B}([h]) = 1$ (*resp., by*

$$\begin{tikzcd}
& \mathcal{F}(\mathbf{F^0})^{col([h])}_{[h]} \arrow[dl] \arrow[dr] & \\
\mathcal{F}^+(\mathbf{F^0})_{\Pi_A}([h]) \arrow[rr] & & \mathcal{F}^-(\mathbf{F^0})_{\Pi_B}([h])
\end{tikzcd}$$

if $ind_{\Pi_A,\Pi_B}([h]) = -1$), *where the horizontal arrows are the morphisms of complexes in Lemma* 5.9 *and the slanted arrows are as in Lemma* 5.7.

Proof. If $ind_{\Pi_A,\Pi_B}([h]) = 1$, then, arguing as in the proof of Lemma 5.9, we deduce the asserted triangle from the following triangle of morphisms of sheaves

$$\begin{tikzcd}
& ker(D^{col[h]}_{c(p)}(i)) \arrow[dr] & \\
ker(D^+_{c(p)}(i)) \cap ker(D^-_{c(p)}(i)) \arrow[ur] \arrow[rr] & & \frac{ker(D^{col[h]}_{c(p)}(i))}{im(D^{col[h]}_{c(p)}(i-1))}
\end{tikzcd}$$

where p and $c(p)$ are the order and the colored degree of $[h]$ and all the arrows are the obvious morphisms.

If $ind_{\Pi_A,\Pi_B}([h]) = -1$, then the asserted triangle is deduced from the following one

$$\begin{tikzcd}
& ker(D^{col[h]}_{c(p)}(i)) \arrow[dl] \arrow[dr] & \\
\frac{ker(D^{col[h]}_{c(p)}(i))}{im(D^{col[h]}_{c(p)}(i-1))} \arrow[rr] & & \frac{\Omega^i \otimes \mathbf{H^{c(p)}}}{im(D^+_{c(p)-1}(i-1)) + im(D^-_{c(p)+1}(i-1))}
\end{tikzcd}$$

where the arrows again are the obvious morphisms. q.e.d.

The properties of the sheaf $\mathbf{F^0}$ will serve us to define the Higgs category of weight l on the side of $\breve{\mathbf{J}}$ while the properties of its transform $\mathcal{F}(\mathbf{F^0})$ will provide the axiomes for objects of a category on the side of the Albanese H.

Definition 5.11. Let $f : S \longrightarrow B$ be a smooth morphism of quasi-projective complex varieties of relative (complex) dimension $n \geq 1$. The Higgs category $\mathbf{Higgs}_w(S)$ of weight $w \geq 2$ on S is a category whose objects are torsion-free $\mathcal{O}_S$-modules $\mathcal{M}$ equipped with a direct sum decomposition

$$\mathcal{M} = \bigoplus_{p=0}^{w-1} \mathcal{M}^p$$

and a relative Higgs field

$$d_{\mathcal{M}} : \mathcal{M} \longrightarrow \Omega_{S/B} \otimes \mathcal{M}$$

with the property that the restriction of $d_{\mathcal{M}}$ to the p-th summand of the direct sum decomposition takes values in $\Omega_{S/B} \otimes (\mathcal{M}^{p-1} \oplus \mathcal{M}^p \oplus \mathcal{M}^{p+1})$, i.e.,

$$d^p_{\mathcal{M}} = d_{\mathcal{M}} \mid_{\mathcal{M}^p} : \mathcal{M}^p \longrightarrow \Omega_{S/B} \otimes (\mathcal{M}^{p-1} \oplus \mathcal{M}^p \oplus \mathcal{M}^{p+1}).$$

A morphism of two objects $(\mathcal{M}_1, d_{\mathcal{M}_1})$, $(\mathcal{M}_2, d_{\mathcal{M}_2})$ in $\mathbf{Higgs}_w(S)$ is a morphism $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ which respects the decomposition and commutes with the Higgs fields, i.e., $\phi(\mathcal{M}^p_1) \subset \mathcal{M}^p_2$, $\forall p$, and $\phi d_{\mathcal{M}_1} = d_{\mathcal{M}_2}\phi$.

It is clear that we can define the Albanese $H = H_w(S)$ as in Definition 4.1 for the Higgs category $\mathbf{Higgs}_w(S)$. As before it comes together with the "Lagrangian" cycle $H_0 = \bigcup\limits_{A \subset I} \Pi_A$, where $I = \{0, \ldots, w-2\}$ and the union is taken over all subsets A of I. The irreducible components Π_A are called "Lagrangian" submanifolds of H. They are spanned by "canonical" generators $\{V^+_i, V^-_j \mid i \in A, j \in A^-\}$ (see Lemma 4.5 and Definition 4.8). We will now define a category of $\mathcal{O}_H$-modules enriched by a Fukaya type data. This will be called F-category of H.

Definition 5.12. An F-category on H (denoted $F(H)$) is a category whose objects are $\mathcal{O}_H$-modules $\mathcal{F}$ together with a finite collection $\mathcal{F}(\Pi_A)$ of complexes, for every "Lagrangian" manifold Π_A of H. The complexes in the collections $\mathcal{F}(\Pi_A)$ are called Lagrangian complexes. The module $\mathcal{F}$ and the Lagrangian complexes are subject to the following axioms.

Module axiom. An $\mathcal{O}_H$-module $\mathcal{F}$ is equipped with a grading $\mathcal{F} = \bigoplus\limits_{i=0}^{m} \mathcal{F}^i$ and an $\mathcal{O}_H$-morphism $d_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}$ of degree 1, i.e., $d_{\mathcal{F}}(\mathcal{F}^i) \subset \mathcal{F}^{i+1}$.

Fibre axiom. The fibre $\mathcal{F}^i_{[h]}$ of $\mathcal{F}^i$ at points $[h]$ corresponding to the canonical generators $\{V^{\pm}_p\}$ admits a direct sum decomposition $\mathcal{F}^i_{[h]} = \bigoplus\limits_{p=0}^{w-1} \mathcal{F}^{i,p}_{[h]}$ such that $d_{\mathcal{F}}$ at $[h]$ preserves this decomposition, i.e., $d_{\mathcal{F}}(\mathcal{F}^{i,p}_{[h]}) \subset \mathcal{F}^{i+1,p}_{[h]}$, $\forall p$.

The vector space $\mathcal{F}^{col([h])}_{[h]} = \bigoplus\limits_{i=0}^{m} \mathcal{F}^{i,cdeg([h])}_{[h]}$ composed of the summands of the colored degree $cdeg([h])$ of $[h]$ is called the colored fibre of $\mathcal{F}$ at $[h]$.

The axiom of Lagrangian complexes. For a Lagrangian manifold Π_A the collection $\mathcal{F}(\Pi_A)$ consists of graded vector spaces

$$\mathcal{F}^{\pm}_{\Pi_A}([h]) = \bigoplus_{i=0}^{m} (\mathcal{F}^{\pm}_{\Pi_A}([h]))^i$$

labeled by the points $[h]$ corresponding to the canonical generators of Π_A and colored by "+" or "−". The vector spaces $\mathcal{F}^{\pm}_{\Pi_A}([h])$ are equipped with the differentials

$$d_{\mathcal{F}^{\pm}_{\Pi_A}([h])} : \mathcal{F}^{\pm}_{\Pi_A}([h]) \longrightarrow \mathcal{F}^{\pm}_{\Pi_A}([h])$$

i.e., $d^i_{\mathcal{F}^{\pm}_{\Pi_A}([h])} : (\mathcal{F}^{\pm}_{\Pi_A}([h]))^i \longrightarrow (\mathcal{F}^{\pm}_{\Pi_A}([h]))^{i+1}$ and $d^{i+1}_{\mathcal{F}^{\pm}_{\Pi_A}([h])} \circ d^i_{\mathcal{F}^{\pm}_{\Pi_A}([h])} = 0,\ \forall i$. Furthermore, the complexes $(\mathcal{F}^{\pm}_{\Pi_A}([h]), d_{\mathcal{F}^{\pm}_{\Pi_A}([h])})$ are subject to a compatibility relation with the colored fibres $\mathcal{F}^{col([h])}_{[h]}$: there are natural homomorphism(s)

$$\mathcal{F}^{col([h])}_{[h]} \longrightarrow \mathcal{F}^{+}_{\Pi_A}([h]) \tag{5.20}$$

if $v_{\Pi_A}([h]) = 2$, and

$$\mathcal{F}^{+}_{\Pi_A}([h]) \longrightarrow \mathcal{F}^{col([h])}_{[h]} \longrightarrow \mathcal{F}^{-}_{\Pi_A}([h]) \tag{5.21}$$

if $v_{\Pi_A}([h]) = 1$.

These morphisms commute with $d_{\mathcal{F}}$ and $d_{\mathcal{F}^{\pm}_{\Pi_A}([h])}$, i.e., the diagrams

$$\begin{tikzcd}
\mathcal{F}^{col([h])}_{[h]} \arrow[r] \arrow[d, "d_{\mathcal{F}}"'] & \mathcal{F}^{\pm}_{\Pi_A}([h]) \arrow[d, "d_{\mathcal{F}^{\pm}_{\Pi_A}([h])}"] \\
\mathcal{F}^{col([h])}_{[h]} \arrow[r] & \mathcal{F}^{\pm}_{\Pi_A}([h])
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}^{+}_{\Pi_A}([h]) \arrow[r] \arrow[d, "d_{\mathcal{F}^{+}_{\Pi_A}([h])}"'] & \mathcal{F}^{col([h])}_{[h]} \arrow[d, "d_{\mathcal{F}}"] \\
\mathcal{F}^{+}_{\Pi_A}([h]) \arrow[r] & \mathcal{F}^{col([h])}_{[h]}
\end{tikzcd}$$

commute.

Axiom of transversal Lagrangian manifolds. For every pair of Lagrangian manifolds Π_A and Π_B intersecting transversally at a point $[h]$, whose index (see (5.19) for definition) $ind_{\Pi_A,\Pi_B}([h]) = 1$, there are natural morphisms of complexes

$$(\mathcal{F}^{+}_{\Pi_A}([h]), d_{\mathcal{F}^{+}_{\Pi_A}([h])}) \longrightarrow (\mathcal{F}^{\pm}_{\Pi_B}([h]), d_{\mathcal{F}^{\pm}_{\Pi_B}([h])})$$

$$(\mathcal{F}^{\pm}_{\Pi_B}([h]), d_{\mathcal{F}^{\pm}_{\Pi_B}([h])}) \longrightarrow (\mathcal{F}^{-}_{\Pi_A}([h]), d_{\mathcal{F}^{-}_{\Pi_A}([h])})$$

These morphisms are subject to a compatibility relation with the colored fibre $\mathcal{F}^{col([h])}_{[h]}$ at $[h]$: the diagram

$$\begin{tikzcd} & (\mathcal{F}^{col([h])}_{[h]}, d_{\mathcal{F}}) \arrow[dr] & \\ (\mathcal{F}^{+}_{\Pi_A}([h]), d_{\mathcal{F}^{+}_{\Pi_A}([h])}) \arrow[ur] \arrow[rr] & & (\mathcal{F}^{\pm}_{\Pi_B}([h]), d_{\mathcal{F}^{\pm}_{\Pi_B}([h])}) \end{tikzcd}$$

commutes (here the slanted arrows are the compatibility homomorphisms in (5.20) and (5.21) respectively).

The morphisms in $F(H)$ are defined as morphisms of graded $\mathcal{O}_H$-modules together with morphisms of Lagrangian complexes. The morphisms of graded $\mathcal{O}_H$-modules must be compatible with the degree 1 morphisms of the **Module axiom** and respect the **Fibre axiom** in order to induce the homomorphism of the colored fibres. These induced morphisms together with the morphisms of Lagrangian complexes must respect the compatibility relations in the remaining axioms.

An easy generalization of the construction of $\mathcal{F}(\mathbf{F^0})$ gives us the Fourier-Mukai functor

$$\mathcal{F} : \mathbf{Higgs}_w(S) \longrightarrow F(H_w(S)).$$

Applying these general considerations to $S = \breve{\mathbf{J}}$, $B = \dot{\Gamma}^r_d(P)$, $f = \pi : \breve{\mathbf{J}} \longrightarrow \dot{\Gamma}^r_d(P)$ and $w = l$, where l is as in (2.5), we obtain the Fourier-Mukai functor

$$\mathcal{F} : \mathbf{Higgs}_l(\breve{\mathbf{J}}) \longrightarrow F(H) \tag{5.22}$$

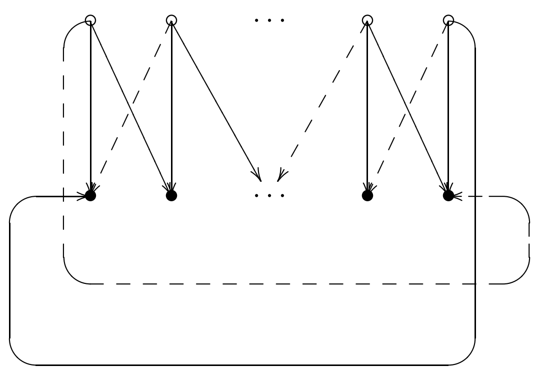
where H is the Albanese of $\breve{\mathbf{J}}$. This functor could be viewed as a go-between algebraic/holomorphic side (torsion-free $\mathcal{O}_{\breve{\mathbf{J}}}$-modules together with relative Higgs fields) and "symplectic" side ($\mathcal{O}_H$-sheaves together with Lagrangian complexes).

6. A trivalent graph of $\breve{\mathbf{J}}$ and quantum invariants

The previous sections show that the nonabelian Jacobian and its Albanese can be related geometrically by a Calabi-Yau cycle map (Proposition 5.1, Proposition 5.3) and cohomologically by the Fourier-Mukai functor (5.22). The latter relates an algebraic/holomorphic data on $\breve{\mathbf{J}}$ (torsion-free sheaves with Higgs fields) to a symplectic type data on H. This fits the general philosophy of Homological Mirror Symmetry conjecture of Kontsevich,[**K**]. These two aspects of our construction make it plausible that our nonabelian Jacobian has something to do with quantum gravity (see e.g., [**B**], for an introduction to the subject). In this section we give concrete evidence of this by observing that one can naturally associate a trivalent graph $G(\breve{\mathbf{J}})$ to $\breve{\mathbf{J}}$. Then using this graph one can associate to every point $([Z],[\alpha])$ in $\breve{\mathbf{J}}$ a generating series with operator-valued coefficients. This generating series could be viewed as a quantum invariant of a pair $([Z],[\alpha])$ or, equivalently, of a pair $(\mathcal{E},e)$, where $\mathcal{E}$ is a locally free sheaf of rank 2 over X with Chern invariants (L,d) and e is a global section of $\mathcal{E}$ whose zero-locus is d distinct points of X.

The graph $G = G(\breve{\mathbf{J}})$ which we associate to $\breve{\mathbf{J}}$ is basically a pictorial representation of the orthogonal decomposition (2.5) and the action of the morphisms $D^0, D^{\pm}$ in (3.5) on the summands of this decomposition. More precisely, we take l parallel vertical edges with upper (resp. lower) vertices aligned on a horizontal line (see (6.1)). These vertical segments should be thought of as a pictorial representation of the morphism D^0 preserving the summands of the decomposition (2.5). The segments are naturally ordered, from left to right, by the index set $I = \{0,1,\ldots,l-1\}$. The vertices of the i-th edge will be labeled by i_u and i_d, for the upper and lower one respectively. We now connect the neighboring vertices as follows. For every $i \in I$ connect i_u to $(i-1)_d$ and $(i+1)_d$ using the convention that $(-1)_d = (l-1)_d$ and $l_d = 0_d$. This gives the following trivalent graph

(6.1)



which will be denoted $G(\breve{\mathbf{J}})$ or simply by G if no ambiguity is likely. We agree that all edges of G are oriented from "up" to "down". The graph G with this orientation will be denoted by $\vec{G}$. In this orientation the edges incident to an upper (resp. lower) vertex of G are always out (resp. in)-going. At every upper vertex of G we fix a counterclockwise order of incident edges and color them, starting with the vertical one, by "0", "+", and "−", respectively. This will be called the *natural coloring of* G. It is chosen to correspond to the morphisms D^0, D^+, D^- and the orientation of edges of $\vec{G}$ matches the sense of action of these operators on a vector of pure degree p (in the decomposition (2.5)) placed at the vertex p_u of G.

Our strategy of extracting from G an operator-valued generating function is reminiscent of the Reshetikhin-Turaev construction of the knot invariants, [**Tu**]. We color every *oriented* edge e of G by an edge-operator $D_e^{c(e)}$, where $c(e)$ is the color of the operator ("0", or "+", or "−") determined by the orientation of e. Given a connected path γ in G stretching between two chosen vertical levels of G (see (6.1)) we obtain the path-operator D_γ defined as the composition of the edge-operators $D_e^{c(e)}$, where e runs through the edges composing γ. A "quantum" operator $Q^{k,m}$ from the vertical level k to the vertical level m of G is given by the generating series

$$Q^{k,m} = \sum_{n=1}^{\infty} \frac{1}{n!} Q_n^{k,m} q^{n-1} \tag{6.2}$$

where the coefficients $Q_n^{k,m}$ are the sums of the path-operators D_γ taken over the set $L_n(k,m)$ of all connected paths γ of length n in G stretching from the level k to the level m, i.e., $Q_n^{k,m} = \sum\limits_{\gamma \in L_n(k,m)} D_\gamma$. Of course, as it stands this is merely an heuristic recipe. To make it work we need some precisions about path-operators D_γ.

First recall that a path γ (always assumed to be connected unless said otherwise) in G is a finite set of oriented edges $\{e_1, \ldots, e_n\}$ such that the 'target' of the edge e_i coincides with the 'source' of the edge e_{i+1}, for every $i = 1, \ldots, n-1$. The number of edges composing a path γ is called its length and denoted $l(\gamma)$. A path γ is said to stretch from the vertical level k to the vertical level m of G if the 'source' of γ is a vertex of the vertical edge of order k and the 'target' of γ is a vertex of the vertical edge of order m of the graph G (recall that the vertical edges of G are ordered by the index set $I = \{0, 1, \ldots, l-1\}$).

Let $\gamma = \{e_1, \ldots, e_n\}$ be a path in G. Then the path-operator D_γ is given by the identity

$$D_\gamma = D_{e_n}^{c(e_n)} \circ \cdots \circ D_{e_1}^{c(e_1)}$$

where the color $c(e)$ of an *oriented* edge e in G is determined as follows. If the orientation of e coincides with the one in $\vec{G}$ (i.e. e is oriented from "up" to "down") then $c(e)$ is the color of e given by the natural coloring of G; otherwise $c(e)$ is taken to be the opposite of the natural color of e (of course, for the "neutral" color "0" we agree that $\pm 0 = 0$).

Next we explain the meaning of the edge-operators $D_e^{c(e)}$. This is simply the morphism $D^{c(e)}$ in (3.5) viewed as a morphism $\tilde{\mathbf{H}} \longrightarrow \mathcal{E}nd(\mathbf{F^0})$ (compare with (3.2)). In particular, given a point $([Z], [\alpha])$ of $\breve{\mathbf{J}}$, the edge-operator $D_e^{c(e)}$ becomes the operator

$$D^{c(e)}(t_e) : H^0(\mathcal{O}_Z) \longrightarrow H^0(\mathcal{O}_Z)$$

once we assign to the (nonoriented) edge e an element t_e in the fibre $\tilde{\mathbf{H}}([Z],[\alpha])$ of $\tilde{\mathbf{H}}$ at $([Z],[\alpha])$. Thus to evaluate D_γ one should color the set of nonoriented edges $E(G)$ by elements of $\tilde{\mathbf{H}}([Z],[\alpha])$, i.e. given a map

$$f : E(G) \longrightarrow \tilde{\mathbf{H}}([Z],[\alpha])$$

we define an endomorphism $D_\gamma([Z],[\alpha])(f)$ of $H^0(\mathcal{O}_Z)$ by the following formula

$$D_\gamma([Z],[\alpha])(f) = D^{c(e_n)}(f(e_n)) \circ \cdots \circ D^{c(e_1)}(f(e_1)). \tag{6.3}$$

Thus the path-operator D_γ at $([Z],[\alpha])$ is a map

$$D_\gamma([Z],[\alpha]) : Map(E(G), \tilde{\mathbf{H}}([Z],[\alpha])) \longrightarrow End(H^0(\mathcal{O}_Z)) \tag{6.4}$$

which sends an element $f \in Map(E(G), \tilde{\mathbf{H}}([Z],[\alpha]))$ to $D_\gamma([Z],[\alpha])(f)$ determined by (6.3). Now the "quantum" operator at $([Z],[\alpha])$ proposed in (6.2) acquires precise meaning. It is a map

$$Q^{k,m}([Z],[\alpha]) : Map(E(G), \tilde{\mathbf{H}}([Z],[\alpha])) \longrightarrow End(H^0(\mathcal{O}_Z)) \otimes \mathbf{C}[[q]]$$

defined by the formula

$$Q^{k,m}(([Z],[\alpha])(f) = \sum_{n=1}^{\infty} \frac{1}{n!} Q^{k,m}_n([Z],[\alpha])(f) q^{n-1}, \qquad \forall f \in Map(E(G), \tilde{\mathbf{H}}([Z],[\alpha])) \tag{6.5}$$

where $Q^{k,m}_n([Z],[\alpha])(f) = \sum\limits_{\gamma \in L_n(k,m)} D_\gamma([Z],[\alpha])(f)$ and $D_\gamma([Z],[\alpha])(f)$ is defined by (6.3).

We will now specialize the above considerations to the constant maps $f \in Map(E(G), \tilde{\mathbf{H}}([Z],[\alpha]))$. In this case we identify such a function with its value, say t in $\tilde{\mathbf{H}}([Z],[\alpha])$, and write $D_\gamma([Z],[\alpha])(t)$ instead of $D_\gamma([Z],[\alpha])(f)$, i.e. we have the following formula

$$D_\gamma([Z],[\alpha])(t) = D^{c(e_n)}(t) \circ \cdots \circ D^{c(e_1)}(t). \tag{6.6}$$

This gives us a *polarized "quantum" operator* at $([Z],[\alpha])$

$$Q^{k-m}([Z],[\alpha]) : \tilde{\mathbf{H}}([Z],[\alpha]) \longrightarrow End(H^0(\mathcal{O}_Z)) \otimes \mathbf{C}[[q]] \tag{6.7}$$

defined by the following formula

$$Q^{k-m}([Z],[\alpha])(t) = \sum_{n=1}^{\infty} \frac{1}{n!} Q^{k-m}_n([Z],[\alpha])(t) q^{n-1} \quad \forall t \in \tilde{\mathbf{H}}([Z],[\alpha]) \tag{6.8}$$

where $Q^{k-m}_n([Z],[\alpha])(t) = \dfrac{1}{2} \sum\limits_{\gamma \in L_n(k,m)} D_\gamma([Z],[\alpha])(t)$. Observe that due to the circular symmetry of the graph G the polarized operators depend only on the *relative* position of the vertical levels k and m, i.e. they depend only on $(k-m)$ (the coefficient $\frac{1}{2}$ takes into account the symmetry of paths in G with respect to exchanging upper and lower vertices).

To arrive to our quantum invariants of a point $([Z],[\alpha])$ we evaluate $Q^{k-m}([Z],[\alpha])$ at elements of $\tilde{\mathbf{H}}([Z],[\alpha])$ intrinsically determined by Z. These are the elements $\{\delta^0_z\}_{z\in Z}$ which were introduced in the proof of Proposition 5.1. Thus we define a "quantum" operator $QC_s(z,[\alpha])$ of degree s of a point $(z,[\alpha]) \in (Z,[\alpha])$ by the following formula

$$QC_s(z,[\alpha]) = Q^s([Z],[\alpha])(\delta^0_z) \tag{6.9}$$

where $s \in \{0, \pm 1, \ldots, \pm(l-1)\}$ and $Q^s([Z],[\alpha])(\delta^0_z)$ is determined by (6.8). Summing over the points z in Z we obtain a "quantum operator"

$$QC_s([Z],[\alpha]) = \sum_{z\in Z} QC_s(z,[\alpha]), \tag{6.10}$$

which will be called *"quantum" Chern operator of degree* s of $([Z],[\alpha])$ in $\breve{\mathbf{J}}$.

The polarized quantum operators $Q^0([Z],[\alpha])(t)$ of degree 0 are of a particular interest. This is because on the one hand they are related to the topology of the graph G and on the other hand the coefficients $Q^0_n([Z],[\alpha])(t)$ of $Q^0([Z],[\alpha])(t)$ are self-adjoint operators of $H^0(\mathcal{O}_Z)$ with respect to $\mathbf{q}$ in (2.1). The former is because we are summing over the relative loops in (G, e^0_m), i.e., the paths in G whose beginning and end are on the m-th vertical edge e^0_m of G. The latter follows from the fact that every path $\gamma \in L_n(m,m)$ gives rise to the "opposite" path $\bar{\gamma} \in L_n(m,m)$ obtained from γ by retracing it in the opposite direction. From the definition of path-operator and Lemma 3.7 it follows that $D_{\bar{\gamma}}(t)$ is $\mathbf{q}$-adjoint to $D_\gamma(t)$.

We will now give an algebraic description of the polarized quantum operator $Q^0([Z],[\alpha])(t)$. Set

$$D_u(t) = u^{-1}D^-(t) + D^0(t) + uD^+(t)$$

where u is a formal variable. The n-th power $D^n_u(t)$ of $D_u(t)$ is a Laurent polynomial in u with coefficients in $End(H^0(\mathcal{O}_Z))$. More precisely, the coefficient of u^k is an endomorphism of $H^0(\mathcal{O}_Z)$ of degree k with respect to the orthogonal cohomology decomposition

$$H^0(\mathcal{O}_Z) = \bigoplus_{p=0}^{l} \mathbf{H^p}([Z],[\alpha]) \tag{6.11}$$

i.e., the coefficient of u^k is an endomorphism of $H^0(\mathcal{O}_Z)$ which shifts the grading in the decomposition (6.11) by k. In particular, the constant term of $D^n_u(t)$ is an operator of degree 0, which is given as a sum of monomials of the form $\prod_{i=1}^{n} D^{\varepsilon_i}(t)$, where $\varepsilon_i \in \{+,0,-\}$ and the number of plus signs is equal to the number of minus signs.

Lemma 6.1. *Denote the constant term of* $D_u^n(t)$ *by* $(t^n)_0$. *Then* $(t^n)_0 = Q_n^0([Z],[\alpha])(t)$, $\forall n \in \mathbf{N}$.

Proof. An exchange of upper and lower vertices of the graph G induces an involution i on $L_n(m,m)$. Furthermore, for a path $\gamma \in L_n(m, m)$ and its involution $i(\gamma)$ we have

$$D_\gamma([Z],[\alpha])(t) = D_{i(\gamma)}([Z],[\alpha])(t).$$

This implies

$$Q_n^0([Z],[\alpha])(t) = \sum_{[\gamma]\in L_n(m,m)/\langle i\rangle} D_{[\gamma]}([Z],[\alpha])(t)$$

where $[\gamma]$ is the equivalence class of the path γ in $L_n(m,m)/\langle i\rangle$ and $D_{[\gamma]}([Z],[\alpha])(t) = D_\gamma([Z],[\alpha])(t)$. The path-operators $D_{[\gamma]}([Z],[\alpha])(t)$ obviously have the form of the monomials composing $(t^n)_0$. Conversely, a monomial $\prod_{i=1}^{n} D^{\varepsilon_i}(t)$ in $(t^n)_0$ defines a unique, up to the involution i, path $\gamma \in L_n(m,m)$ whose path-operator $D_\gamma([Z],[\alpha])(t) = \prod_{i=1}^{n} D^{\varepsilon_i}(t)$.

q.e.d.

From Lemma 6.1 it follows

$$Q^0([Z],[\alpha])(t) = \sum_{n=1}^{\infty} \frac{(t^n)_0}{n!} q^{n-1}. \tag{6.12}$$

Remark 6.2.

(i) Viewing the multiplication by t^n as an endomorphism of $H^0(\mathcal{O}_Z)$ one obtains that $(t^n)_0$ is the component of t^n of degree 0.

(ii) For $\xi \in \tilde{\mathbf{H}}([Z],[\alpha])$ define

$$[\exp(\xi)]_0 = \sum_{n=0}^{\infty} \frac{(\xi^n)_0}{n!},$$

then the quantum operator $Q^0([Z],[\alpha])(t)$ can be written as follows

$$Q^0([Z],[\alpha])(t) = \frac{[\exp(qt)]_0 - 1}{q}. \tag{6.13}$$

Using the trace operation we can pass from "quantum" operators to formal q-series.

Definition 6.3. Let V be a finite dimensional vector space over a field k and let $End(V)[[q]]$ be the ring of formal power series in the indeterminant q with coefficients in $End(V)$. For a formal power series $F(q) = \sum_{n=0}^{\infty} f_n q^n$ in $End(V)[[q]]$ define the q-trace $Tr_q(F)$ as follows:

$$Tr_q(F) = \sum_{k=0}^{\infty} Tr(f_k) q^k$$

where $Tr : End(V) \longrightarrow k$ is the usual trace homomorphism.

Taking the q-trace of $Q^0([Z],[\alpha])(t)$ yields the following q-series

$$\begin{aligned} Tr_q(Q^0([Z],[\alpha])(t)) &= \sum_{n=1}^{\infty} \frac{Tr((t^n)_0)}{n!} q^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{Tr(t^n)}{n!} q^{n-1} = Tr_q\left(\frac{\exp(qt)-1}{q}\right) \end{aligned} \tag{6.14}$$

where the second equality follows from the fact that the trace of endomorphisms of nonzero degree vanishes.

Proposition-Definition 6.4. The q-trace of the quantum Chern operator $QC_0([Z],[\alpha])$ (see (6.10)) is the q-series

$$\begin{aligned} \tilde{C}_q([Z],[\alpha]) &= Tr_q(QC_0([Z],[\alpha]) \\ = \sum_{z\in Z} Tr_q\left(\frac{\exp(q\delta_z^0)-1}{q}\right) &= d + \sum_{n=2}^{\infty} \tilde{c}_n q^{n-1} \end{aligned}$$

where $\tilde{c}_n = \sum_{z\in Z} \frac{Tr((\delta_z^0)^n)}{n!}$. The q-series $\tilde{C}_q([Z],[\alpha])$ is a natural q-deformation of $d = \deg(Z)$ and it will be called the q-Chern number of $([Z],[\alpha])$.

Proof. The second equality of the proposition follows from the definition of $QC_0([Z],[\alpha])$ in (6.10) and the last equality in (6.14). To establish the last equality of the proposition we expand

$$\sum_{z\in Z} Tr_q\left(\frac{\exp(q\delta_z^0)-1}{q}\right)$$

in powers of q.

$$\sum_{z\in Z} Tr_q\left(\frac{\exp(q\delta_z^0)-1}{q}\right) = \sum_{z\in Z}\left(\sum_{n=1}^{\infty} \frac{Tr((\delta_z^0)^n)}{n!} q^{n-1}\right) = \sum_{n=1}^{\infty} \tilde{c}_n q^{n-1}$$

where $\tilde{c}_n = \sum_{z\in Z} \frac{Tr((\delta^0_z)^n)}{n!}$. In particular,

$$\tilde{c}_1 = \sum_{z\in Z} Tr(\delta^0_z) = Tr(\sum_{z\in Z} \delta^0_z) = Tr(1) = d$$

where the third identity follows from the fact that

$$\sum_{z\in Z} \delta^0_z = \sum_{z\in Z} \delta_z = 1.$$

q.e.d.

Varying $t \in \tilde{\mathbf{H}}([Z],[\alpha])$ and $([Z],[\alpha]) \in \breve{\mathbf{J}}$ in (6.12) we obtain a quantum operator Q^0 which is a function on the total space of the bundle $\tilde{\mathbf{H}}$ taking values in $\mathbf{End}(\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}}))[[q]]$. In fact, we have seen that the coefficients of the power series of Q^0 at every point $(t,([Z],[\alpha]))$ of $\tilde{\mathbf{H}}$ are the compositions of the endomorphisms $D^+(t), D^0(t), D^-(t)$. This naturally leads us to consider the Lie algebras generated by the set

$$\{D^+(t), D^0(t), D^-(t) \mid t \in \tilde{\mathbf{H}}([Z],[\alpha])\}.$$

As $([Z],[\alpha])$ varies in $\breve{\mathbf{J}}$ we obtain a sheaf of Lie algebras over $\breve{\mathbf{J}}$ which can be viewed as a (nonabelian) analogue of the (abelian) Lie algebraic structure of the classical Jacobian.

7. Lie algebras associated to $\mathbf{J}(X;L,d)$

In this section we construct and study the sheaf of Lie algebras over $\breve{\mathbf{J}}$ mentioned in the end of the previous section.

Definition 7.1. Let f be a section of $\mathbf{End}(\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}}))$ over a subscheme $U \subset \breve{\mathbf{J}}$. We will say that f has degree m if it shifts the degree of the cohomology decomposition (2.5) by m, i.e., $f(\mathbf{H^p} \otimes \mathcal{O}_U) \subset \mathbf{H^{p+m}} \otimes \mathcal{O}_U$ for every $0 \le p \le l$.

The multiplication in $\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}})$ defines an inclusion $\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}}) \subset \mathbf{End}(\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}}))$. In particular, D in (3.2) induces an inclusion

$$D : \tilde{\mathbf{H}} \longrightarrow \mathbf{End}(\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}})). \tag{7.1}$$

Using the decomposition (3.5) we consider $D^{\pm}(\tilde{\mathbf{H}})$ and $D^0(\tilde{\mathbf{H}})$ and define $\tilde{\mathcal{G}}$ to be the subsheaf of Lie subalgebras of $\mathbf{End}(\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}}))$ generated by $D^+(\tilde{\mathbf{H}})$, $D^0(\tilde{\mathbf{H}})$, $D^-(\tilde{\mathbf{H}})$.

Proposition 7.2. *$\tilde{\mathcal{G}}$ is a sheaf of reductive Lie algebras generated by germs of sections of degree ± 1 and 0.*

Proof. The second assertion is obvious since $\tilde{\boldsymbol{\mathcal{G}}}$ is generated by germs of sections in $D^+(\tilde{\mathbf{H}})$, $D^0(\tilde{\mathbf{H}})$, $D^-(\tilde{\mathbf{H}})$ which have degree $+1$, 0, -1, respectively.

Turning to the first assertion we recall that $\tilde{\mathbf{H}}_{-l}$ is a $\tilde{\boldsymbol{\mathcal{G}}}$-module (Lemma 3.4). So it is enough to show that every $\tilde{\boldsymbol{\mathcal{G}}}$-submodule of $\tilde{\mathbf{H}}_{-l}$ has a $\tilde{\boldsymbol{\mathcal{G}}}$-invariant complement. Let $\mathcal{M}$ be a $\tilde{\boldsymbol{\mathcal{G}}}$-submodule of $\tilde{\mathbf{H}}_{-l}$ and let $\mathcal{M}^{\perp}$ be the subsheaf of $\tilde{\mathbf{H}}_{-l}$ which is $\mathbf{q}$-orthogonal to $\mathcal{M}$. From Lemma 3.7 it follows that $\mathcal{M}^{\perp}$ is also a $\tilde{\boldsymbol{\mathcal{G}}}$-submodule of $\tilde{\mathbf{H}}_{-l}$. It remains to show that $\mathcal{M}^{\perp}$ is complementary to $\mathcal{M}$. Let v be a local section of $\mathcal{M}\cap\mathcal{M}^{\perp}$ in a neighborhood U of some point in $\breve{\mathbf{J}}$. Then $\mathbf{q}(t\cdot v, v) = 0$ for any section t of $\tilde{\mathbf{H}}\otimes\mathcal{O}_U$. Hence $\mathbf{q}(h\cdot v, v) = \mathbf{q}(h, v^2) = 0$, for any section h of $\tilde{\mathbf{H}}_{-l}\otimes\mathcal{O}_U$. Lemma 3.4 implies that v^2 is a section of $\mathbf{F^l}\otimes\mathcal{O}_\mathbf{U}$. But v^2 is also a section of $\tilde{\mathbf{H}}_{-l}\otimes\mathcal{O}_U$. Since $\mathbf{F^l}\cap\tilde{\mathbf{H}}_{\mathbf{-l}} = 0$ we deduce that $v^2 = 0$. Hence $v = 0$ yielding $\mathcal{M}\cap\mathcal{M}^{\perp} = 0$. q.e.d.

The structure theorem of reductive Lie algebras (see, e.g.,[**Bour**]) implies the following decomposition

$$\tilde{\boldsymbol{\mathcal{G}}} = \boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{G}} \tag{7.2}$$

where $\boldsymbol{\mathcal{C}}$ is the center of $\tilde{\boldsymbol{\mathcal{G}}}$ and $\boldsymbol{\mathcal{G}}$ is a sheaf of semisimple Lie algebras. Furthermore, $\boldsymbol{\mathcal{C}}$ is composed of germs of semisimple endomorphisms of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$.

We will regard $\tilde{\mathbf{H}}$ as a subsheaf of $\tilde{\boldsymbol{\mathcal{G}}}$, i.e., using the inclusion in (7.1) we identify $\tilde{\mathbf{H}}$ with its image $D(\tilde{\mathbf{H}})$ in $\tilde{\boldsymbol{\mathcal{G}}}$. From this point of view $\tilde{\mathbf{H}}$ is a subsheaf of abelian Lie subalgebras of $\tilde{\boldsymbol{\mathcal{G}}}$. Furthermore, the action of $\tilde{\mathbf{H}}$ on $\tilde{\mathbf{H}}_{-l}$ is diagonalizable. Let $\boldsymbol{\Lambda}$ be the scheme of weights of this action. Then $\boldsymbol{\Lambda}$ has the following geometric interpretation. Consider the diagram

$$\begin{tikzcd} & \tilde{\mathbf{H}}^* \arrow[d, "pr"] \\ \breve{\mathcal{Z}} \arrow[r, "\tilde{p}_2"] \arrow[d] & \breve{\mathbf{J}} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r, "p_2"] & \dot{\Gamma}^r_d(P) \end{tikzcd}$$

where $\breve{\mathcal{Z}} = \mathcal{Z}\times_{\dot{\Gamma}^r_d(P)}\breve{\mathbf{J}}$ is the fibre product and $\tilde{\mathbf{H}}^*$ is viewed as the total space of a vector bundle over $\breve{\mathbf{J}}$ and pr is its natural projection. Observe that $\tilde{p}_2$ admits a lifting

$$\begin{tikzcd} & \tilde{\mathbf{H}}^* \arrow[d, "pr"] \\ \breve{\mathcal{Z}} \arrow[ur, "\kappa"] \arrow[r, "\tilde{p}_2"] & \breve{\mathbf{J}} \end{tikzcd} \tag{7.3}$$

defined by the evaluation (see details in the proof below).

Lemma 7.3. *The scheme of weights* $\mathbf{\Lambda} = \kappa(\breve{\mathcal{Z}})$.

Proof. We will work fiberwise. Let $([Z],[\alpha]) \in \breve{\mathbf{J}}$ and let $\tilde{\mathbf{H}}_{([Z],[\alpha])}$ be the fibre of $\tilde{\mathbf{H}}$ at $([Z],[\alpha])$. Then $\kappa|_{p_2^{-1}([Z],[\alpha])} : Z \longrightarrow \tilde{\mathbf{H}}^*_{([Z],[\alpha])}$ is the evaluation morphism as in Remark 3.5. Setting $Z' = \kappa(Z)$ and letting $I_{Z'}$ be its ideal in $S^{\bullet}\tilde{\mathbf{H}}_{([Z],[\alpha])}$ we obtain the following identification

$$\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = S^{\bullet}\tilde{\mathbf{H}}_{([Z],[\alpha])}/I_{Z'} = H^0(\mathcal{O}_{Z'}) = \kappa^*(H^0(\mathcal{O}_{Z'}) \subset H^0(\mathcal{O}_Z)$$

where $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ is the fibre of $\tilde{\mathbf{H}}_{-l}$ at $([Z],[\alpha])$. Thus $\tilde{\mathbf{H}}_{([Z],[\alpha])}$ is a subspace of $H^0(\mathcal{O}_{Z'})$ and it acts on $H^0(\mathcal{O}_{Z'})$ by multiplication. The set of weights of this action is $\Lambda_{([Z],[\alpha])}$, the fibre of $\mathbf{\Lambda}$ over $([Z],[\alpha])$. We want to identify it with Z'.

Let $\lambda \in \Lambda_{([Z],[\alpha])}$ and let V_λ be the corresponding weight subspace of $H^0(\mathcal{O}_{Z'})$. The action of $\tilde{\mathbf{H}}_{([Z],[\alpha])}$ on V_λ is given by the following equation

$$t(p')v(p') - t(\lambda)v(p') = 0,\ \forall p' \in Z' \tag{7.4}$$

for every $t \in \tilde{\mathbf{H}}_{([Z],[\alpha])}$ and $v \in V_\lambda$. This extends to the action of $S^{\bullet}\tilde{\mathbf{H}}_{([Z],[\alpha])}$ by the formula

$$A(p')v(p') - A(\lambda)v(p') = 0,\ \forall p' \in Z'$$

for every $A \in S^{\bullet}\tilde{\mathbf{H}}_{([Z],[\alpha])}$ and $v \in V_\lambda$. This implies that $A(\lambda) = 0$, $\forall A \in I_{Z'}$. Hence $\lambda \in Z'$ implying an inclusion $\Lambda_{([Z],[\alpha])} \subset Z'$.

Conversely, for $p' \in Z'$ take $\delta_{p'}$ to be the characteristic function with support at p', i.e., $\delta_{p'}(p') = 1$ and $\delta_{p'}(q') = 0$, $\forall q' \in Z' \setminus \{p'\}$. Then $t \cdot \delta_{p'} = t(p')\delta_{p'}$, for every $t \in \tilde{\mathbf{H}}_{([Z],[\alpha])}$. Thus $p' \in \Lambda_{([Z],[\alpha])}$ yielding $Z' \subset \Lambda_{([Z],[\alpha])}$. q.e.d.

Remark 7.4. The proof of Lemma 7.3 implies that the weight space decomposition

$$\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = \oplus_{p' \in Z'} \mathbf{C}\delta_{p'}.$$

On the sheaf level the proof implies $pr_*\big(\mathcal{O}_{\mathbf{\Lambda}}\big) = \tilde{\mathbf{H}}_{-l}$.

Lemma 7.5. *Let* $\boldsymbol{\mathcal{N}}(\tilde{\mathbf{H}})$ *be the normalizer of* $\tilde{\mathbf{H}}$ *in* $\tilde{\boldsymbol{\mathcal{G}}}$. *Then* $\boldsymbol{\mathcal{N}}(\tilde{\mathbf{H}}) = \boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{H}}$, *where* $\boldsymbol{\mathcal{H}}$ *is a subsheaf of Cartan subalgebras of* $\boldsymbol{\mathcal{G}}$.

Proof. The inclusion $\boldsymbol{\mathcal{C}} \subset \boldsymbol{\mathcal{N}}(\tilde{\mathbf{H}})$ is obvious. Setting $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{N}}(\tilde{\mathbf{H}}) \cap \boldsymbol{\mathcal{G}}$ and using (7.2) we obtain the direct sum

$$\boldsymbol{\mathcal{N}}(\tilde{\mathbf{H}}) = \boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{H}}.$$

The last assertion follows from the fact that $\boldsymbol{\mathcal{H}}_{([Z],[\alpha])}$, the fibre of $\boldsymbol{\mathcal{H}}$ at $([Z],[\alpha])$, must preserve the weight decomposition in Remark 7.4. q.e.d.

Next we turn to the subsheaves $D^{\pm}(\tilde{\mathbf{H}}) \subset \tilde{\boldsymbol{\mathcal{G}}}$.

Lemma 7.6. *$D^{\pm}(\tilde{\mathbf{H}})$ are the subsheaves of abelian nilpotent Lie subalgebras of $\boldsymbol{\mathcal{G}}$.*

Proof. Let t be a section of $\tilde{\mathbf{H}}$ over an open set $U \subset \breve{\mathbf{J}}$. Then $D^{\pm}(t)$ is a section of $\mathbf{End}(\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_U)$ of degree ± 1 (see Definition 7.1). Hence $D^{\pm}(t)$ is nilpotent. The commutativity of $D^{\pm}(\tilde{\mathbf{H}})$ follows from Lemma 3.9, (i) and Remark 3.1.

To see that $D^{\pm}(t)$ is a section of $\boldsymbol{\mathcal{G}}\otimes\mathcal{O}_U$ we may assume that $D^{\pm}(t) \neq 0$. It is not a section of $\boldsymbol{\mathcal{C}}\otimes\mathcal{O}_U$, since $D^{\pm}(t)$ is nilpotent. So (7.2) implies

$$D^{\pm}(t) = C^{\pm}(t) + X^{\pm}(t)$$

where $C^{\pm}(t)$ (resp., $X^{\pm}(t)$) is a section of $\boldsymbol{\mathcal{C}}\otimes\mathcal{O}_U$ (resp., $\boldsymbol{\mathcal{G}}\otimes\mathcal{O}_U$) and $X^{\pm}(t) \neq 0$. Taking the Jordan decomposition of $X^{\pm}(t)$ we obtain

$$D^{\pm}(t) = C^{\pm}(t) + X^{\pm}_{s}(t) + X^{\pm}_{n}(t) \tag{7.5}$$

where $X^{\pm}_{s}(t)$ (resp., $X^{\pm}_{n}(t)$) is the semisimple (resp., nilpotent) component of $X^{\pm}(t)$ and both are sections of $\boldsymbol{\mathcal{G}}\otimes\mathcal{O}_U$. Observe that (7.5) is the Jordan decomposition of $D^{\pm}(t)$ with the semisimple part $(C^{\pm}(t) + X^{\pm}_{s}(t))$. The nilpotency of $D^{\pm}(t)$ and uniqueness of Jordan decomposition imply that $D^{\pm}(t) = X^{\pm}_{n}(t)$ is a section of $\boldsymbol{\mathcal{G}}\otimes\mathcal{O}_U$. q.e.d.

Lemma 7.7. *The center $\boldsymbol{\mathcal{C}}$ of $\tilde{\boldsymbol{\mathcal{G}}}$ is the image of the subspace $D^0(\tilde{\mathbf{H}}) \subset \tilde{\mathcal{G}}$ under the natural projection*

$$\tilde{\boldsymbol{\mathcal{G}}} = \boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{G}} \longrightarrow \boldsymbol{\mathcal{C}}.$$

Proof. From Lemma 7.6 and the definition of $\tilde{\boldsymbol{\mathcal{G}}}$ it follows $\tilde{\mathcal{G}} = D^0(\tilde{\mathbf{H}}) + [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}] = D^0(\tilde{\mathbf{H}}) + \mathcal{G}$. This yields the assertion. q.e.d.

Remark 7.8. The sheaf of nilpotent abelian Lie algebras $D^{-}(\tilde{\mathbf{H}})$ defines an increasing filtration on $\tilde{\mathbf{H}}_{-l}$ whose p-th step is the subsheaf of germs of sections of $\tilde{\mathbf{H}}_{-l}$ annihilated by $S^p(D^{-}(\tilde{\mathbf{H}}))$. We claim that this is $\tilde{\mathbf{H}}_{-p}$ of the filtration (1.25).

Proof. From Corollary 2.4 it follows that $\tilde{\mathbf{H}}_{-p}$ is annihilated by $S^p(D^{-}(\tilde{\mathbf{H}}))$. To show the opposite inclusion we argue by induction on p.

Let ϕ be a section of $\tilde{\mathbf{H}}_{-l}$ over an open set $U \subset \breve{\mathbf{J}}$ which is annihilated by $\Gamma(U, D^{-}(\tilde{\mathbf{H}}))$ and let $\phi = \sum_{i=0}^{l-1} \phi^i$, $\phi^i \in \Gamma(U, \mathbf{H^i})$, be its orthogonal cohomology decomposition. The condition $D^{-}(t)(\phi) = 0$, $\forall t \in \Gamma(U, \tilde{\mathbf{H}})$, implies $D^{-}(t)(\phi^i) = 0$, for every $i \geq 1$ and all $t \in \Gamma(U, \tilde{\mathbf{H}})$. This yields

$$\mathbf{q}(t \cdot h, \phi^i) = \mathbf{q}(h, t \cdot \phi^i) = 0$$

for all $h \in \Gamma(U, \tilde{\mathbf{H}}_{-i})$ and all $t \in \Gamma(U, \tilde{\mathbf{H}})$. Hence ϕ^i is orthogonal to $\Gamma(U, \tilde{\mathbf{H}}_{-i-1})$. From Lemma 2.1 it follows that $\phi^i \in \Gamma(U, \mathbf{F^{i+1}})$. This and Corollary 2.4 imply $\phi^i = 0$, for every $i \geq 1$. So $\phi = \phi^0$ is a section of $\mathbf{H^0} \otimes \mathcal{O}_{\mathbf{U}} = \tilde{\mathbf{H}}_{\mathbf{-1}} \otimes \mathcal{O}_{\mathbf{U}}$. This proves the claim for $p = 1$. Assuming it to be true for $p \geq 1$ we will prove it for $p + 1$.

Let ϕ be a section of $\tilde{\mathbf{H}}_{-l}$ over an open set $U \subset \breve{\mathbf{J}}$ which is annihilated by $\Gamma(U, S^{p+1}D^-(\tilde{\mathbf{H}}))$. Observe that for every $t \in \Gamma(U, \tilde{\mathbf{H}})$ the section $D^-(t)(\phi)$ is annihilated by $\Gamma(U, S^p D^-(\tilde{\mathbf{H}}))$. By induction hypothesis it is a section of $\tilde{\mathbf{H}}_{-p} \otimes \mathcal{O}_U$. Taking as above the orthogonal cohomology decomposition of ϕ we obtain $D^-(t)(\phi^i) = 0$, for every $i \geq p + 1$ and every $t \in \Gamma(U, \tilde{\mathbf{H}})$. By the first part of the proof $\phi^i \in \Gamma(U, \tilde{\mathbf{H}}_{-1})$, for every $i \geq p + 1$. This implies $\phi^i = 0$, for every $i \geq p + 1$, yielding $\phi \in \Gamma(U, \tilde{\mathbf{H}}_{-p-1})$. This completes the proof of the claim.

Corollary 7.9. *Let $\breve{\mathbf{J}}_0$ be a connected component of $\breve{\mathbf{J}}$ on which $\boldsymbol{\mathcal{G}}$ is zero. Then $\tilde{\boldsymbol{\mathcal{G}}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0} = \tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$. Furthermore, $\tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$, viewed as a subsheaf of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$, is a sheaf of subrings of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$.*

Proof. From Lemma 7.6 it follows $D^{\pm}(\tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}) = 0$ implying that D in (7.1) induces an isomorphism between $\tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$ and $\tilde{\boldsymbol{\mathcal{G}}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$. Furthermore, for any two sections t, t' of $\tilde{\mathbf{H}}$ over an open set $U \subset \breve{\mathbf{J}}_0$, the product $t \cdot t' = D^0(t)(t')$ is again a section of $\tilde{\mathbf{H}} \otimes \mathcal{O}_U$. Thus $\tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$ is a sheaf of rings. q.e.d.

Remark 7.10.

(i) Let $\breve{\mathbf{J}}_0$ be as in Corollary 7.9. Then the orthogonal cohomology decomposition over $\breve{\mathbf{J}}_0$ has the form
$$\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big) \otimes \mathcal{O}_{\breve{\mathbf{J}}_0} = \mathbf{H^0} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0} \oplus \mathbf{H^1} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}.$$
Conversely, the orthogonal cohomology decomposition of weight 1 (see Definition 2.5) over a connected component $\breve{\mathbf{J}}_0$ implies vanishing of $\boldsymbol{\mathcal{G}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$. Indeed, since a weight 1 decomposition means that $\tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$ is closed under multiplication we obtain $D^+(t) = 0$, for any local section t of $\tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$. Since $D^-(t)$ is adjoint to $D^+(t)$ with respect to $\mathbf{q}$ (Lemma 3.7), we obtain $D^-(t) = 0$ as well. This implies that $D : \tilde{\mathbf{H}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0} \longrightarrow \tilde{\boldsymbol{\mathcal{G}}} \otimes \mathcal{O}_{\breve{\mathbf{J}}_0}$ in (7.1) is an isomorphism.

(ii) The fact that $\tilde{\mathbf{H}}$ is a subring of $\pi^*\big(p_{2*}\mathcal{O}_{\mathcal{Z}}\big)$ over a connected component of $\breve{\mathbf{J}}$ also means that the cohomological invariant $\mathbf{C_J}$ (see Definition 1.8) vanishes over such a component. In view of Corollary 7.9 we may say that the cohomological invariant detects noncomutativity of $\tilde{\boldsymbol{\mathcal{G}}}$. This situation can be viewed as an analogue of hyperellipticity in the theory of curves.

Next we consider the components of $\breve{\mathbf{J}}$ where the sheaf of semisimple Lie algebras $\boldsymbol{\mathcal{G}}$ does not vanish. These components will be denoted by $\breve{\mathbf{J}}_s$. On such components the sheaf $\tilde{\mathbf{H}}_{-l}$ is a $\mathcal{G}$-module and one can relate the theory of representations of semisimple Lie algebras with geometry of underlying clusters. Such a relationship is given by the following dictionary.

Let $([Z],[\alpha])$ *be a point in* $\breve{\mathbf{J}}_{\mathbf{s}}$ *and let* $\mathbf{g}([Z],[\alpha])$ *be the fibre of* $\boldsymbol{\mathcal{G}}$ *at* $([Z],[\alpha])$. *Then the decomposition of the fibre* $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ *of* $\tilde{\mathbf{H}}_{-l}$ *at* $([Z],[\alpha])$, *into the sum of irreducible* $\mathbf{g}([Z],[\alpha])$*-modules gives rise to a decomposition of* Z *into a disjoint union of* L*-special clusters.*

To state our result recall the morphism $\kappa : \breve{\mathcal{Z}} \longrightarrow \mathbf{\Lambda} \subset \tilde{\mathbf{H}}^*$ in (7.3). Let $\Lambda_{([Z],[\alpha])}$ (resp., $\tilde{\mathbf{H}}_{([Z],[\alpha])}$) be the fibre of $\mathbf{\Lambda}$ (resp., $\tilde{\mathbf{H}}$) over $([Z],[\alpha])$. Then the precise version of the dictionary between the decomposition of $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ into the sum of irreducible $\mathbf{g}([Z],[\alpha])$-modules and geometry of Z is the following.

Theorem 7.11.

(i) *Let* $\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = \bigoplus_i V_i([Z],[\alpha])$ *be a decomposition of* $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ *into the direct sum of irreducible* $\mathbf{g}([Z],[\alpha])$*-modules.*

Then every $V_i([Z],[\alpha])$ *is an ideal of* $H^0(\mathcal{O}_{\Lambda_{([Z],[\alpha])}})$. *Furthermore,* $\tilde{\mathbf{H}}_{([Z],[\alpha])}$ *admits the following direct sum decomposition*

$$\tilde{\mathbf{H}}_{([Z],[\alpha])} = \bigoplus_i V_i^0([Z],[\alpha])$$

where $V_i^0([Z],[\alpha]) = \tilde{\mathbf{H}}_{([Z],[\alpha])} \cap V_i([Z],[\alpha])$.

(ii) *Let* $\Lambda^i_{([Z],[\alpha])}$ *be the subscheme of* $\Lambda_{([Z],[\alpha])}$ *defined by* $V_i([Z],[\alpha])$. *Then* $\Lambda_{([Z],[\alpha])} = \bigcup_i \Lambda^i_{([Z],[\alpha])}$ *is a decomposition of* $\Lambda_{([Z],[\alpha])}$ *into the union of pairwise disjoint subclusters. Furthermore, let* $\Pi_i([Z],[\alpha])$ *be the subspace of* $\tilde{\mathbf{H}}^*_{([Z],[\alpha])}$ *annihilating* $V_i^0([Z],[\alpha])$*; then* $\Lambda^i_{([Z],[\alpha])} = \Pi_i([Z],[\alpha]) \bigcap \Lambda_{([Z],[\alpha])}$. *In particular, the index of* L*-speciality (see Definition* 1.1*) of* $Z_i = \kappa^*(\Lambda^i_{([Z],[\alpha])})$ *is* $\delta(L,Z_i) = \dim\big(\Pi_i([Z],[\alpha])\big)$.

Proof.

(i) To simplify the notation we will omit the reference to $([Z],[\alpha])$ whenever no confusion is likely. From the proof of Lemma 7.3 it follows $\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = H^0(\mathcal{O}_\Lambda)$. Let $\tilde{\mathbf{g}}$ and $\mathbf{c}$ be, respectively, the fibre of $\tilde{\boldsymbol{\mathcal{G}}}$ and $\boldsymbol{\mathcal{C}}$ at $([Z],[\alpha])$. Let $H^0(\mathcal{O}_\Lambda) = \bigoplus_{\lambda\in\mathbf{c}^*} W_\lambda$ be the decomposition of $H^0(\mathcal{O}_\Lambda)$ into the weight spaces of $\mathbf{c}$. Let V be a $\mathbf{g}$-submodule of the wieght space W_λ. Then it is also a $\tilde{\mathbf{g}}$-submodule of $H^0(\mathcal{O}_\Lambda)$. This implies that V is an $S^\bullet\tilde{\mathbf{H}}_{([Z],[\alpha])}$-module and hence an $H^0(\mathcal{O}_\Lambda)$-module (see the proof of Lemma 7.3). Thus V is an ideal of $H^0(\mathcal{O}_\Lambda)$.

Let $V^\perp$ be the subspace of $H^0(\mathcal{O}_\Lambda)$ $\mathbf{q}$-orthogonal to V. Then it is also a $\tilde{\mathbf{g}}$-submodule of $H^0(\mathcal{O}_\Lambda)$ and Proposition 7.2 yields

$$H^0(\mathcal{O}_\Lambda) = V \oplus V^\perp. \tag{7.6}$$

The first part of the argument implies that $V^{\perp}$ is an ideal of $H^0(\mathcal{O}_{\Lambda})$ as well.

Let $V^0 = V \cap \tilde{\mathbf{H}}_{([Z],[\alpha])}$ and $(V^{\perp})^0 = V^{\perp} \cap \tilde{\mathbf{H}}_{([Z],[\alpha])}$. For $h \in \tilde{\mathbf{H}}_{([Z],[\alpha])}$ let h_V (resp., $h_{V^{\perp}}$) be its component in V (resp., $V^{\perp}$). Since $D^-(t)(h) = 0$, for all $t \in \tilde{\mathbf{H}}_{([Z],[\alpha])}$, we obtain

$$D^-(t)(h_V) + D^-(t)(h_{V^{\perp}}) = 0, \ \forall t \in \tilde{\mathbf{H}}_{([Z],[\alpha])}.$$

Since the decomposition (7.6) is preserved by $D^-(t)$ it follows $D^-(t)(h_V) = D^-(t)(h_{V^{\perp}}) = 0$, for all $t \in \tilde{\mathbf{H}}_{([Z],[\alpha])}$. By Remark 7.8 h_V and $h_{V^{\perp}}$ are in $\tilde{\mathbf{H}}_{([Z],[\alpha])}$. This yields the decomposition

$$\tilde{\mathbf{H}}_{([Z],[\alpha])} = V^0 \oplus (V^{\perp})^0.$$

(ii) Let Λ_V (resp., $\Lambda_{V^{\perp}}$) be the subscheme of Λ defined by the ideal V (resp., $V^{\perp}$). The fact that $V \cdot V^{\perp} = 0$ implies $\Lambda_V \cup \Lambda_{V^{\perp}} = \Lambda$, while (7.6) gives $\Lambda_V \cap \Lambda_{V^{\perp}} = \emptyset$.

Set Π_V (resp., $\Pi_{V^{\perp}}$) to be the subspace of $\tilde{\mathbf{H}}^*_{([Z],[\alpha])}$ annihilating V^0 (resp., $(V^{\perp})^0$). Then the definition of V^0 (resp., $(V^{\perp})^0$) implies that Π_V (resp., $\Pi_{V^{\perp}}$) is the linear span of Λ_V (resp., $\Lambda_{V^{\perp}}$).

Let $Z_V = \kappa^*(\Lambda_V)$ (resp., $Z_{V^{\perp}} = \kappa^*(\Lambda_{V^{\perp}})$) and consider the morphism $\mathbf{R^r}$ in (1.21) at the point $([Z],[\alpha])$

$$\mathbf{R^r}([Z],[\alpha]) : H^0(\mathcal{O}_Z) \longrightarrow H^0(K_X + L)^*.$$

Recall that $\tilde{\mathbf{H}}_{([Z],[\alpha])} = \ker(\mathbf{R^r}([Z],[\alpha]))$ and $\delta(L, Z) = \mathrm{dim}(\tilde{\mathbf{H}}_{([Z],[\alpha])})$. From the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \kappa^* V^{\perp} \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \kappa^* V \arrow[r] \arrow[dr] & H^0(\mathcal{O}_Z) \arrow[r] \arrow[d] & H^0(\mathcal{O}_{Z_V}) \arrow[r] & 0 \\
 & & H^0(\mathcal{O}_{Z_{V^{\perp}}}) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

we can identify $\kappa^* V^{\perp} = H^0(\mathcal{O}_{\Lambda_V})$ (resp., $\kappa^* V = H^0(\mathcal{O}_{\Lambda_{V^{\perp}}})$). This induces the morphism

$$\mathbf{R}(Z_V) : H^0(\mathcal{O}_{Z_V}) \longrightarrow H^0(K_X + L)^*$$

$$(\text{resp., } \mathbf{R}(\mathrm{Z_{V^{\perp}}}) : H^0(\mathcal{O}_{Z_{V^{\perp}}}) \longrightarrow H^0(K_X + L)^*).$$

From this it follows

$$\begin{aligned}\delta(L, Z_V) &= \dim\big(\ker(\mathbf{R}(Z_V))\big) = \dim\big(V^{\perp}\big)^0 \\ &= \delta(L,Z) - \dim\big(V^0\big) = \dim\big(\Pi_V\big) \\ (\text{resp.}, \delta(\mathrm{L}, \mathrm{Z}_{\mathrm{V}^{\perp}}) &= \dim\big(\ker(\mathbf{R}(Z_{V^{\perp}}))\big) = \dim\big(V^0\big)).\end{aligned}$$

q.e.d.

Remark 7.12. The decomposition $\tilde{\mathbf{H}}_{([Z],[\alpha])} = \bigoplus_i V_i^0$ in Theorem 7.11 has the property $V_i^0 \cdot V_j^0 = 0$, for all $i \neq j$. Letting $\tilde{\mathbf{g}}_{\mathbf{i}}$ to be the Lie subalgebra of $\tilde{\mathbf{g}}$ generated by $D^{\pm}(V_i^0)$, $D^0(V_i^0)$ we obtain

$$\tilde{\mathbf{g}} = \bigoplus_{\mathbf{i}} \tilde{\mathbf{g}}_{\mathbf{i}}.$$

Furthermore, the proof of Theorem 7.11 implies that the center of $\tilde{\mathbf{g}}_{\mathbf{i}}$ is $\mathbf{C} \cdot \mathbf{Id}_{V_i}$ and $\tilde{\mathbf{g}}_{\mathbf{i}}$ acts irreducibly on V_i and by zero on every V_j with $j \neq i$.

We have seen in the proof of Theorem 7.11 how reducibility of $\mathbf{g}$-module $\tilde{\mathbf{H}}_{-l}([Z],[\alpha])$ implies an L-special decomposition of Z. Reversing the argument we can relate the classical algebro-geometric notion of points in general position (see e.g., [**G-H**]) and irreducible representations of $\mathbf{g}$.

Corollary 7.13. *Let $[Z] \in \dot{\Gamma}_d^r(P)$, where $r \geq 1$ and $d \geq r+2$. If Z is in general position with respect to the linear system $\mid K_X + L \mid$, then $\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = H^0(\mathcal{O}_Z)$, for all $([Z],[\alpha]) \in \check{\mathbf{J}}$, and $H^0(\mathcal{O}_Z)$ is an irreducible $\mathbf{g}([Z],[\alpha])$-module.*

Proof. Recall the morphism $\kappa([Z],[\alpha]) : Z \longrightarrow \mathbf{P}(\tilde{\mathbf{H}}^*_{([Z],[\alpha])}) = \mathbf{P}^r$ (see Remark 1.5). We claim that Z being in general position with respect to $\mid K_X + L \mid$ implies that $\kappa([Z],[\alpha])$ is an embedding and its image is a set of d points in general position in $\mathbf{P}^r$.

Assume this is not the case. Then there exists a subset of $(r+1)$ points $Z_0 \subset Z$ whose image under $\kappa([Z],[\alpha])$ is contained in a hyperplane, i.e., $\tilde{\mathbf{H}}_{([Z],[\alpha])} \cap H^0(Z, \mathcal{I}_{Z_0}) \neq 0$, where $\mathcal{I}_{Z_0} \subset \mathcal{O}_Z$ is the subsheaf of ideals of Z_0. Set $Z_0^c = Z \setminus Z_0$. Repeating the considerations in the proof of Theorem 7.11, (ii) we deduce

$$\delta(L, Z_0^c) = \dim\big(\tilde{\mathbf{H}}_{([Z],[\alpha])} \cap H^0(Z, \mathcal{I}_{Z_0})\big) \geq 1.$$

This implies that the linear span of Z_0^c with respect to the linear system $\mid K_X + L \mid$ is of dimension $\deg(Z_0^c) - \delta(L, Z_0^c) - 1 \leq d - r - 3$. This contradicts the general position of Z with respect to $\mid K_X + L \mid$.

The fact that $\kappa([Z],[\alpha])$ is an embedding and Remark 2.6, 3) imply $\tilde{\mathbf{H}}_{-l}([Z],[\alpha]) = H^0(\mathcal{O}_Z)$. The irreducibility of $H^0(\mathcal{O}_Z)$ can be seen as follows. Let V be a nontrivial $\mathbf{g}$-submodule of $H^0(\mathcal{O}_Z)$. Using the notation of the proof of Theorem 7.11, (ii) we obtain the decomposition

$Z = Z_V \cup Z_{V^\perp}$. Furthermore, the linear span of the image of Z_V (resp., $Z_{V^\perp}$) under $\kappa([Z],[\alpha])$ is of dimension $r - dim(V^0)$ (resp., $r - dim(V^\perp)^0$). The general position of the image of Z in $\mathbf{P}^r$ implies

$$\deg(Z_V) = r - \dim(V^0) + 1, \quad \deg(Z_{V^\perp}) = r - \dim(V^\perp)^0 + 1.$$

This yields $d = r + 1$, contradicting the assumption on d. q.e.d.

The sheaf of Lie algebras $\boldsymbol{\mathcal{G}}$ is naturally graded by the notion of the degree of sections (see Definition 7.1). Let $\boldsymbol{\mathcal{G}^p}$ be the subsheaf of $\boldsymbol{\mathcal{G}}$ generated by germs of its sections of degree $\boldsymbol{p}$. Then we have

$$\text{(7.7)} \qquad 1)\ \boldsymbol{\mathcal{G}} = \bigoplus_{\boldsymbol{p=-(l-1)}}^{\boldsymbol{l-1}} \boldsymbol{\mathcal{G}^p}, \qquad 2)\ \left[\boldsymbol{\mathcal{G}^p}, \boldsymbol{\mathcal{G}^q}\right] \subset \boldsymbol{\mathcal{G}^{p+q}}.$$

Setting $\boldsymbol{\mathcal{G}^-} = \bigoplus_{\boldsymbol{p<0}} \boldsymbol{\mathcal{G}^p}$ (resp., $\boldsymbol{\mathcal{G}^+} = \bigoplus_{\boldsymbol{p>0}} \boldsymbol{\mathcal{G}^p}$) we obtain

$$\text{(7.8)} \qquad \boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}^-} \oplus \boldsymbol{\mathcal{G}^0} \oplus \boldsymbol{\mathcal{G}^+}.$$

Proposition 7.14.

(i) $\boldsymbol{\mathcal{G}^\pm}$ *are subsheaves of nilpotent Lie subalgebras of* $\boldsymbol{\mathcal{G}}$*.*

(ii) *The fiberwise Killing form on* $\boldsymbol{\mathcal{G}}$ *induces an isomorphism*

$$\boldsymbol{\mathcal{G}^-} \longrightarrow \left(\boldsymbol{\mathcal{G}^+}\right)^*.$$

Proof. The first assertion is obvious since $\boldsymbol{\mathcal{G}^\pm}$ consist of germs of nilpotent endomorphisms. The second assertion can be seen as follows. From 2) in (7.7) it follows that $\boldsymbol{\mathcal{G}^0}$ is a subsheaf of Lie subalgebras of $\boldsymbol{\mathcal{G}}$ and $[\boldsymbol{\mathcal{G}^0}, \boldsymbol{\mathcal{G}^\pm}] \subset \boldsymbol{\mathcal{G}^\pm}$. This and (i) imply that for any open set $U \subset \breve{\mathbf{J}}_s$ and any sections $x \in \Gamma(U, \boldsymbol{\mathcal{G}^0})$ and $y \in \Gamma(U, \boldsymbol{\mathcal{G}^\pm})$ the endomorphism $ad(x)ad(y)$ is nilpotent. This implies that $\boldsymbol{\mathcal{G}^0}$ is orthogonal to $\boldsymbol{\mathcal{G}^\pm}$ with respect to the Killing form on $\boldsymbol{\mathcal{G}}$. Thus $\boldsymbol{\mathcal{G}^0}$ is orthogonal to $\left(\boldsymbol{\mathcal{G}^-} \oplus \boldsymbol{\mathcal{G}^+}\right)$. Since the Killing form is nondegenerate on $\boldsymbol{\mathcal{G}}$, it follows that its restriction to $\left(\boldsymbol{\mathcal{G}^-} \oplus \boldsymbol{\mathcal{G}^+}\right)$ is nondegenerate. Furthermore, it is zero on the nilpotent subalgebras $\boldsymbol{\mathcal{G}^\pm}$. Thus the Killing form induces a nondegenerate pairing

$$\boldsymbol{\mathcal{G}^-} \otimes \boldsymbol{\mathcal{G}^+} \longrightarrow \boldsymbol{\mathcal{O}_{\breve{\mathbf{J}}_s}}$$

yielding the asserted isomorphism. q.e.d.

Remark 7.15. $\boldsymbol{\mathcal{G}^p}$ is orthogonal to $\boldsymbol{\mathcal{G}^q}$ with respect to the Killing form on $\boldsymbol{\mathcal{G}}$, for all $\boldsymbol{p, q}$ with $\boldsymbol{p + q \neq 0}$. Thus the Killing form induces a nondegenerate pairing

$$\text{(7.9)} \qquad \boldsymbol{\mathcal{G}^p} \otimes \boldsymbol{\mathcal{G}^{-p}} \longrightarrow \boldsymbol{\mathcal{O}_{\breve{\mathbf{J}}_s}}$$

for all $\boldsymbol{\mathcal{G}^p \neq 0}$. This implies that the morphism

$$\text{(7.10)} \qquad \boldsymbol{\mathcal{G}^p} \otimes \boldsymbol{\mathcal{G}^{-p}} \longrightarrow \boldsymbol{\mathcal{G}^0}$$

given by the Lie bracket is nonzero, for every nonzero $\boldsymbol{p}$ with $\boldsymbol{\mathcal{G}^p} \neq \mathbf{0}$. In fact, we show that the Lie bracket in (7.10) is nonzero fibrewise.

Let $([Z],[\alpha]) \in \breve{\mathbf{J}}_s$ and let $\mathbf{g^p} = \mathbf{g^p}([Z],[\alpha])$ be the fibre of $\boldsymbol{\mathcal{G}^p}$ at $([Z],[\alpha])$. Then the following holds.

$$ad_p(x) := ad(x)|_{\mathbf{g^{-p}}} : \mathbf{g^{-p}} \longrightarrow \mathbf{g^0} \text{ is nonzero}, \forall x \in \mathbf{g^p} \setminus \{0\}. \tag{7.11}$$

This is a well-known fact in the theory of Lie algebras. Indeed, assume $ad_p(x) = 0$. Then

$$[ad(x),[ad(x),ad(y)]] = [ad(x),ad([x,y])] = 0, \forall y \in \mathbf{g^{-p}}.$$

Furthermore, $ad(x)$ is nilpotent ($x \in \mathbf{g^p}$, for $p \neq 0$). By a result of Kostant (Lemma 3.2, [**Kos**]) $ad(x)ad(y)$ is nilpotent, for all $y \in \mathbf{g^{-p}}$. This implies $(x,y) = 0$, $\forall y \in \mathbf{g^{-p}}$, where $(\cdot,\cdot)$ stands for the Killing form on $\mathbf{g}$, the fibre of $\boldsymbol{\mathcal{G}}$ at $([Z],[\alpha])$. The nondegeneracy of the Killing form on $\mathbf{g^p} \otimes \mathbf{g^{-p}}$ (known from (7.9)) implies $x = 0$.

The fact that the morphism in (7.10) is nonzero and Lemma 7.6 yield $\boldsymbol{\mathcal{G}^0} \neq \mathbf{0}$.

Next we turn to a consideration of $\boldsymbol{\mathcal{G}^0}$.

Definition 7.16. Let $\boldsymbol{x}$ be a section of $\mathbf{End}(\tilde{\mathbf{H}}_{-l})$ over an open set $U \subset \breve{\mathbf{J}}$. Its adjoint (with respect to $\mathbf{q}$) is the section of $\mathbf{End}(\tilde{\mathbf{H}}_{-l})$ over U denoted by $\boldsymbol{x}^\dagger$ and uniquely determined by the following identity

$$\mathbf{q}(\boldsymbol{x}(\boldsymbol{t}),\boldsymbol{t}') = \mathbf{q}(\boldsymbol{t},\boldsymbol{x}^\dagger(\boldsymbol{t}'))$$

for all $t,t' \in \Gamma(U,\tilde{\mathbf{H}}_{-l})$.

The operation of taking adjoint defines an anti-involution on $\tilde{\boldsymbol{\mathcal{G}}}$. This anti-involution preserves the decomposition (7.2). From Lemma 7.7 we also see that the center $\boldsymbol{\mathcal{C}}$ is the sheaf of germs of self-adjoint endomorphisms. Furthermore, from Lemma 3.7 it follows that the operation of taking adjoint on $\boldsymbol{\mathcal{G}}$ is subject to the following:

$$(\boldsymbol{\mathcal{G}}^{\pm})^\dagger = \boldsymbol{\mathcal{G}}^{\mp}, \ \ (\boldsymbol{\mathcal{G}^0})^\dagger = \boldsymbol{\mathcal{G}^0}. \tag{7.12}$$

Proposition 7.17. $\boldsymbol{\mathcal{G}^0}$ *is a subsheaf of reductive Lie algebras.*

Proof. By Remark 7.15 the Killing form on $\boldsymbol{\mathcal{G}}$ induces on $\boldsymbol{\mathcal{G}^0}$ a nondegenerate bilinear form and this is equivalent to being a reductive Lie algebra (Proposition 5,I.6.4,[**Bour**]). q.e.d.

By the structure theorem of reductive Lie algebras we have

$$\boldsymbol{\mathcal{G}^0} = \boldsymbol{\mathcal{C}^0} \oplus \boldsymbol{\mathcal{G}^0_s} \tag{7.13}$$

where $\boldsymbol{\mathcal{C}^0}$ is the center of $\boldsymbol{\mathcal{G}^0}$ and $\boldsymbol{\mathcal{G}^0_s} = [\boldsymbol{\mathcal{G}^0},\boldsymbol{\mathcal{G}^0}]$ is a sheaf of semisimple Lie algebras.

The action of $\boldsymbol{\mathcal{G}^0}$ (resp,. $\boldsymbol{\mathcal{C}^0}$ and $\boldsymbol{\mathcal{G}^0_s}$) on $\tilde{\mathbf{H}}_{-\boldsymbol{l}}$ preserves the orthogonal cohomology decomposition in (2.5). Thus the sheaves $\mathbf{H^p}$'s are $\boldsymbol{\mathcal{G}^0}$-modules. Furthermore, the morphisms

$$\boldsymbol{\mathcal{G}^q} \otimes \mathbf{H^p} \longrightarrow \mathbf{H^{p+q}} \qquad \boldsymbol{\mathcal{G}^q} \otimes \boldsymbol{\mathcal{G}^p} \longrightarrow \boldsymbol{\mathcal{G}^{p+q}} \tag{7.14}$$

are $\boldsymbol{\mathcal{G}^0}$-morphisms.

Lemma 7.18. *The morphism $\boldsymbol{\mathcal{G}^1} \otimes \mathbf{H^p} \longrightarrow \mathbf{H^{p+1}}$ is surjective for every $p = 0, \dots, l-2$.*

Proof. By definition $\tilde{\mathbf{H}} \otimes \tilde{\mathbf{H}}_{-\boldsymbol{p-1}} \longrightarrow \tilde{\mathbf{H}}_{-\boldsymbol{p-2}}$ is onto for all $p \leq l-2$. This implies that for a local section $\boldsymbol{x}$ of the sheaf $\mathbf{H^{p+1}}$ there exist local sections $\boldsymbol{t_i, y_i} (i = 1, \dots, n)$ of $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}_{-\boldsymbol{p-1}}$ respectively, such that

$$\boldsymbol{x} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{t_i y_i} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{D^+(t_i)(y_i)} + \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{D^0(t_i)(y_i)} + \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{D^-(t_i)(y_i)}.$$

Decomposing $\boldsymbol{y_i} = \sum_{\boldsymbol{s=0}}^{\boldsymbol{p}} \boldsymbol{y_i^s}$ according to the components of the direct sum $\mathbf{H_{-p-1}} = \bigoplus_{\mathbf{s=0}}^{\mathbf{p}} \mathbf{H^s}$ (see Corollary 2.4) we obtain

$$\boldsymbol{x} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{D^+(t_i)(y_i^p)}.$$

This together with Lemma 7.6 yields the assertion. q.e.d.

Let $\boldsymbol{\mathcal{U}(\mathcal{G})}$ be the sheaf of the universal enveloping algebras of $\boldsymbol{\mathcal{G}}$. It inherits a grading from the grading of $\boldsymbol{\mathcal{G}}$ in 1) of (7.7). We denote $\boldsymbol{(\mathcal{U}(\mathcal{G}))^n}$ the subsheaf of germs of sections of $\boldsymbol{\mathcal{U}(\mathcal{G})}$ of degree $\boldsymbol{n}$ with respect to this induced grading. In particular, the multiplication in $\boldsymbol{\mathcal{U}(\mathcal{G})}$ has the property

$$\boldsymbol{(\mathcal{U}(\mathcal{G}))^p \otimes (\mathcal{U}(\mathcal{G}))^q \longrightarrow (\mathcal{U}(\mathcal{G}))^{p+q}}.$$

This implies that $\boldsymbol{(\mathcal{U}(\mathcal{G}))^0}$ is a subalgebra of $\boldsymbol{\mathcal{U}(\mathcal{G})}$ and the sheaves $\mathbf{H^p (p = 0, \dots, l-1)}$ are $\boldsymbol{(\mathcal{U}(\mathcal{G}))^0}$-modules.

Proposition 7.19. *Let $\mathbf{H^p} = \bigoplus_{\mathbf{i}} \mathbf{H^p_i}$ be the decomposition of $\mathbf{H^p}$ into the direct sum of irreducible $\boldsymbol{(\mathcal{U}(\mathcal{G}))^0}$-modules. Then*

$$\mathbf{H^p_i \cdot H^p_j = 0,} \ \ \forall \mathbf{i \neq j}$$

where $(\cdot)$ is the multiplication in $\boldsymbol{\pi^*(p_{2*}\mathcal{O}_{\mathcal{Z}})}$.

Proof. Under the action of the center $\boldsymbol{\mathcal{C}}$ of $\tilde{\boldsymbol{\mathcal{G}}}$ the sheaf $\mathbf{H^p}$ decomposes into the orthogonal direct sum of weight subsheaves, since $\boldsymbol{\mathcal{C}}$ acts on $\mathbf{H^p}$ by commuting self-adjoint endomorphisms (see Lemma 7.7). Each of the weight subsheaves is an $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-module. Let $\boldsymbol{\mathcal{V}}$ be a $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-submodule contained in one of the weight subsheaves. Denote by $\boldsymbol{\mathcal{V}}^{\perp}$ the subsheaf of $\mathbf{H^p}$ orthogonal to $\boldsymbol{\mathcal{V}}$. Then from (7.12) it follows that $\boldsymbol{\mathcal{V}}^{\perp}$ is also a $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-submodule of $\mathbf{H^p}$. For all local sections $\boldsymbol{v}$ of $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{w}$ of $\boldsymbol{\mathcal{V}}^{\perp}$ we have

$$\mathbf{q}(\boldsymbol{Pv}, \boldsymbol{w}) = \mathbf{0} \tag{7.15}$$

for all local sections $\boldsymbol{P}$ of $\boldsymbol{S^{\bullet}}\tilde{\mathbf{H}}$. This follows from the fact that the component of degree 0 of $\boldsymbol{P}$ is a local section of $\boldsymbol{S^{\bullet}\mathcal{C}} \otimes (\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$.

We will now show that the relation in (7.15) implies $\boldsymbol{v} \cdot \boldsymbol{w} = \mathbf{0}$ in $\boldsymbol{\pi}^*\big(\boldsymbol{p_{2*}\mathcal{O}_{\mathcal{Z}}}\big)$. It is enough to check it for every point $([Z], [\alpha]) \in \breve{\mathbf{J}}$. For this we use the proof of Lemma 7.3. There we have seen that the fibre $\mathbf{H_{-1}}([Z], [\alpha])$ of $\mathbf{H_{-1}}$ at $([Z], [\alpha])$ can be identified with $H^0(\mathcal{O}_{\Lambda_{([Z],[\alpha])}})$ (see the proof of Lemma 7.3 for notation). This implies that there exists $\boldsymbol{P} \in \boldsymbol{S^{\bullet}}\tilde{\mathbf{H}}_{([\boldsymbol{Z}],[\alpha])}$ such that $\boldsymbol{P(\lambda)} = \mathbf{1}$ and $\boldsymbol{P(\mu)} = \mathbf{0}$, $\forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}_{([\boldsymbol{Z}],[\alpha])} \setminus \{\boldsymbol{\lambda}\}$. Substituting such $\boldsymbol{P}$ in (7.15) we deduce

$$\boldsymbol{v(\lambda)w(\lambda)} = \mathbf{0},\ \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{([\boldsymbol{Z}],[\alpha])}.$$

Thus $\boldsymbol{v} \cdot \boldsymbol{w} = \mathbf{0}$ in $\boldsymbol{\pi}^*\big(\boldsymbol{p_{2*}\mathcal{O}_{\mathcal{Z}}}\big)$, for all local sections $\boldsymbol{v}$ of $\boldsymbol{\mathcal{V}}$ and all local sections $\boldsymbol{w}$ of $\boldsymbol{\mathcal{V}}^{\perp}$. In particular, we deduce $\boldsymbol{\mathcal{V}} \cap \boldsymbol{\mathcal{V}}^{\perp} = \mathbf{0}$. This yields the orthogonal direct sum decomposition

$$\mathbf{H^p} = \boldsymbol{\mathcal{V}} \oplus \boldsymbol{\mathcal{V}}^{\perp}$$

where $\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{V}}^{\perp} = \mathbf{0}$ in $\boldsymbol{\pi}^*\big(\boldsymbol{p_{2*}\mathcal{O}_{\mathcal{Z}}}\big)$. If $\boldsymbol{\mathcal{V}}$ (resp. $\boldsymbol{\mathcal{V}}^{\perp}$) is not irreducible $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-submodule we renew the procedure until we arrive at a decomposition of $\mathbf{H^p}$ into irreducible $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-submodules subject to the property asserted in the proposition. q.e.d.

Corollary 7.20. *The decompositions of* $\mathbf{H^0}$ *and* $\mathbf{H^{l-1}}$ *given in Proposition* 7.19 *are decompositions into irreducible* $\boldsymbol{\mathcal{G}}^{\mathbf{0}}$*-modules.*

Proof. This follows from the fact that on $\mathbf{H^0}$ (resp. $\mathbf{H^{l-1}}$) the product

$$\boldsymbol{D^-(t)D^+(t')} = [\boldsymbol{D^-(t)}, \boldsymbol{D^+(t')}]$$

(resp. $\boldsymbol{D^+(t)D^-(t')} = [\boldsymbol{D^+(t)}, \boldsymbol{D^-(t')}]$, for all local sections $\boldsymbol{t}$ and $\boldsymbol{t'}$ of $\tilde{\mathbf{H}}$. q.e.d.

Lemma 7.21. *Let* $p \leq l-2$ *and let* $\mathbf{H^p}$ *be an irreducible* $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$*-module. Then* $\mathbf{H^{p+1}}$ *is also an irreducible* $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$*-module.*

Proof. Let $\mathbf{H^{p+1}_1}$ and $\mathbf{H^{p+1}_2}$ be two distinct irreducible components in the decomposition of $\mathbf{H^{p+1}}$ given by Proposition 7.19. Consider the morphisms

$$(\boldsymbol{\mathcal{U}(\mathcal{G})})^{-\mathbf{1}} \otimes \mathbf{H^{p+1}_i} \longrightarrow \mathbf{H^p} \tag{7.16}$$

for $\boldsymbol{i} = \mathbf{1}, \mathbf{2}$. Since $\mathbf{H^p}$ is an irreducible $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-module it follows that these morphisms are either surjective or zeros. We claim that they can not be both surjective. Indeed, if it were the case we could write any two local sections $\boldsymbol{w}, \boldsymbol{w'}$ of $\mathbf{H^p}$ as follows

$$\boldsymbol{w} = \sum_{s} \boldsymbol{A_s x_s} \quad \boldsymbol{w'} = \sum_{s'} \boldsymbol{A'_{s'} x_{s'}}$$

for some local sections $\boldsymbol{A_s}, \boldsymbol{A'_{s'}}$ of $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{-1}$ and some local sections $\boldsymbol{x_s}$ of $\mathbf{H_1^{p+1}}$ and $\boldsymbol{x_{s'}}$ of $\mathbf{H_2^{p+1}}$. Taking $\mathbf{q}$-pairing of $\boldsymbol{w}$ and $\boldsymbol{w'}$ we obtain

$$\mathbf{q}(\boldsymbol{w}, \boldsymbol{w'}) = \sum_{s} \mathbf{q}(\boldsymbol{A_s x_s}, \boldsymbol{w'}) \tag{7.17}$$

$$= \sum_{s} \mathbf{q}(\boldsymbol{x_s}, \boldsymbol{A_s^{\dagger} w'}) = \sum_{s,s'} \mathbf{q}(\boldsymbol{x_s}, \boldsymbol{A_s^{\dagger} A'_{s'} x_{s'}}) = \mathbf{0}$$

where the last equality follows from the fact that $\boldsymbol{A_s^{\dagger} A'_{s'}}$ are local sections of $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$ and $\mathbf{H_1^{p+1}}$ and $\mathbf{H_2^{p+1}}$ are orthogonal $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{0}}$-submodules of $\mathbf{H^{p+1}}$. From (7.17) it follows that $\mathbf{q}$ is degenerate on $\mathbf{H^p}$, which is impossible. Thus at least one of the morphisms in (7.16) must be zero. Assume the morphism for $\boldsymbol{i} = \mathbf{2}$ to be zero. But then the image of the morphism $(\boldsymbol{\mathcal{U}(\mathcal{G})})^{\mathbf{1}} \otimes \mathbf{H^p} \longrightarrow \mathbf{H^{p+1}}$ lies in the the submodule of $\mathbf{H^{p+1}}$ orthogonal to $\mathbf{H_2^{p+1}}$. This contradicts the surjectivity of this morphism proved in Lemma 7.18 q.e.d.

We will now relate the representation theory of $\boldsymbol{\mathcal{G}^0}$ with geometry of underlying clusters.

Let $([Z],[\alpha]) \in \breve{\mathbf{J}}_s$ and let $\mathbf{g^0}([Z],[\alpha])$ (resp., $\mathbf{c^0}([Z],[\alpha])$ and $\mathbf{g_s^0}([Z],[\alpha])$) be the fibre of $\boldsymbol{\mathcal{G}^0}$ (resp,. $\boldsymbol{\mathcal{C}^0}$ and $\boldsymbol{\mathcal{G}_s^0}$) at $([Z],[\alpha])$. We consider the decomposition of the fibre $\mathbf{H^0}([Z],[\alpha])$ of $\mathbf{H^0}$ at $([Z],[\alpha])$ given by Proposition 7.19

$$\mathbf{H^0}([Z],[\alpha]) = \bigoplus_{i} \mathbf{H_i^0}([Z],[\alpha]). \tag{7.18}$$

(If there is no ambiguity we will omit the reference to $([Z],[\alpha])$ in the notation above.)

A geometric interpretation of the decomposition (7.18) can again be given in terms of geometry of the morphism $\kappa : \breve{\tilde{\mathcal{Z}}} \longrightarrow \mathbf{\Lambda} \subset \tilde{\mathbf{H}}^*$ in (7.3) and it is exactly the same as the one in Theorem 7.11.

Theorem 7.22.

(i) *The decomposition* (7.18) *has the following property*

$$\mathbf{H}_i^0([Z],[\alpha]) \cdot \mathbf{H}_j^0([Z],[\alpha]) = 0, \, \forall i \neq j$$

i.e., $x \cdot y = 0$ in $H^0(\mathcal{O}_Z)$, for all $x \in \mathbf{H}_i^0([Z],[\alpha])$ and $y \in \mathbf{H}_j^0$ with $i \neq j$.

(ii) *Let* $\Pi_i([Z],[\alpha])$ *be the subspace of* $\tilde{\mathbf{H}}^*_{([Z],[\alpha])}$ *annihilating* $\mathbf{H}^0_i([Z],[\alpha])$ *and let* $\Lambda^i_{([Z],[\alpha])} = \Pi_i([Z],[\alpha]) \bigcap \Lambda_{([Z],[\alpha])}$, *where* $\Lambda_{([Z],[\alpha])}$ *is the fibre of* $\mathbf{\Lambda}$ *at* $([Z],[\alpha])$. *Then*

$$\Lambda_{([Z],[\alpha])} = \bigcup_i \Lambda^i_{([Z],[\alpha])}$$

is a decomposition of $\Lambda_{([Z],[\alpha])}$ *into the union of pairwise disjoint subclusters, where for every* $\mathbf{H}^0_i([Z],[\alpha]) \neq 0$, *or* $\mathbf{H^0}([Z],[\alpha])$, *the subcluster* $\Lambda^i_{([Z],[\alpha])} \neq \emptyset, \Lambda_{([Z],[\alpha])}$.

(iii) *Let* $Z_i = \kappa^*(\Lambda^i_{([Z],[\alpha])})$. *Then*

$$Z = \bigcup_i Z_i$$

is a decomposition of Z *into the union of pairwise disjoint* L*-special subclusters* Z_i *whose index of* L*-speciality*

$$\delta(L, Z_i) = \dim\big(\Pi_i([Z],[\alpha])\big).$$

Proof. The property (i) comes from Proposition 7.19. One obtains (ii) and (iii) from (i) by arguing as in the proof of Theorem 7.11,(ii). q.e.d.

Using this result we can strengthen Corollary 7.13 in the following way.

Corollary 7.23. *Let* $[Z] \in \dot{\Gamma}^r_d(P)$, *where* $r \geq 1$ *and* $d \geq r+2$. *If* Z *is in general position with respect to the linear system* $\mid K_X + L \mid$, *then* $\mathbf{H^p}([Z],[\alpha])$ *are irreducible* $(\mathbf{U}(\mathbf{g}))^{\mathbf{0}}$*-modules* ($\mathbf{U}(\mathbf{g})$ *is the universal enveloping algebra of* $\mathbf{g}$), *for all* $p = 0,\ldots,l-1$. *In particular,* $\mathbf{H^0}[Z],[\alpha])$ *and* $\mathbf{H^{l-1}}([Z],[\alpha])$ *are irreducible* $\mathbf{g^0}([Z],[\alpha])$*-modules, for all* $([Z],[\alpha]) \in \breve{\mathbf{J}}$.

Proof. Arguing as in the proof of Corollary 7.13 we obtain that $\mathbf{H^0}([Z],[\alpha])$ is an irreducible $\mathbf{g^0}([Z],[\alpha])$-module. This together with the proof of Corollary 7.20 implies that $\mathbf{H^0}([Z],[\alpha])$ is an irreducible $(\mathbf{U}(\mathbf{g}))^{\mathbf{0}}$-module. From this and Lemma 7.21 we deduce that $\mathbf{H^p}([Z],[\alpha])$ are irreducible $(\mathbf{U}(\mathbf{g}))^{\mathbf{0}}$-modules, for all $p = 0,\ldots,l-1$. Applying once again Corollary 7.20 we obtain that $\mathbf{H^{l-1}}([Z],[\alpha])$ is an irreducible $\mathbf{g^0}([Z],[\alpha])$-module. q.e.d.

To illustrate our results we calculate the Lie algebras $\mathbf{g^0}([Z],[\alpha])$ and $\mathbf{g}([Z],[\alpha])$ over the first nontrivial (with respect to our constructions) stratum $\overset{\circ}{\Gamma}{}^1_d(L)$.

Example 7.24. Let $[Z] \in \overset{\circ}{\Gamma}{}^1_d(L)$ be as in Corollary 7.23. Then the filtration (1.25) at $([Z],[\alpha])$ is a maximal ladder

$$0 \subset \tilde{\mathbf{H}}_0([Z],[\alpha]) \subset \tilde{\mathbf{H}}_{-1}([Z],[\alpha]) \subset \cdots \subset \tilde{\mathbf{H}}_{-(d-1)}([Z],[\alpha]) = H^0(\mathcal{O}_Z)$$

where $\tilde{\mathbf{H}}_0([Z],[\alpha]) = \mathbf{C}\cdot 1$ (Remark 1.3). This implies that the orthogonal cohomology decomposition in (2.5) has the following form

$$H^0(\mathcal{O}_Z) = \bigoplus_{p=0}^{d-2} \mathbf{H}^{\mathbf{p}}([Z],[\alpha]) \tag{7.19}$$

where $\dim(\mathbf{H}^{\mathbf{0}}([Z],[\alpha])) = 2$ and $\dim(\mathbf{H}^{\mathbf{p}}([Z],[\alpha])) = 1$, for $1 \leq p \leq d-2$. We fix an orthonormal basis $\{v^1, \ldots, v^{d-2}\}$ of $\mathbf{F}^{\mathbf{1}}([Z],[\alpha]) = \bigoplus_{p=1}^{d-2} \mathbf{H}^{\mathbf{p}}([Z],[\alpha])$ compatible with the orthogonal cohomology decomposition. On $\mathbf{H}^{\mathbf{0}}([Z],[\alpha])$ we choose an orthonormal basis $\{t_0,\, t_1\}$ so that $t_0 \in \tilde{\mathbf{H}}_0([Z],[\alpha])$ (i.e., $t_0 = \frac{1}{\sqrt{d}}\cdot 1$ and $Tr(t_1) = \mathbf{q}(1,t_1) = 0$, $Tr((t_1)^2) = \mathbf{q}(t_1,t_1) = 1$). Then

$$B = \{t_0,\, t_1; v^1, \ldots, v^{d-2}\} \tag{7.20}$$

is an orthonormal basis of $H^0(\mathcal{O}_Z)$.

We will compute explicitly the graded direct sum 1) in (7.7) at $([Z],[\alpha])$

$$\mathbf{g} = \bigoplus_{\mathbf{p}=-(\mathbf{d}-\mathbf{2})}^{\mathbf{d}-\mathbf{2}} \mathbf{g}^{\mathbf{p}}. \tag{7.21}$$

All the matrices in our computations will be written with respect to the basis B in (7.20).

Claim 7.25. Put $\mathbf{g}^{\mathbf{0}} = \mathbf{g}^{\mathbf{0}}([Z],[\alpha])$. Then $[\mathbf{g}^{\mathbf{0}}, \mathbf{g}^{\mathbf{0}}] = \mathbf{sl}(\mathbf{H}^{\mathbf{0}}([Z],[\alpha])) = \mathbf{sl_2 C}$. Furthermore, the Chevalley generators

$$e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathbf{sl_2 C}$ are given with respect to the orthonormal basis $\{t_0,\, t_1\}$.

Proof. The inclusion $[\mathbf{g}^{\mathbf{0}}, \mathbf{g}^{\mathbf{0}}] \subset \mathbf{sl}(\mathbf{H}^{\mathbf{0}}([Z],[\alpha]))$ is obvious from (7.19). To see the equality consider elements $A = [D^-(t_1), D^+(t_1)] \in \mathbf{g}^{\mathbf{0}}$ and $D^0(t_1) \in \mathbf{c} \oplus \mathbf{g}^{\mathbf{0}}$, where $\mathbf{c}$ is the fibre of the center of $\tilde{\boldsymbol{\mathcal{G}}}$ at $([Z],[\alpha])$). They act on the basis $\{t_0,\, t_1\}$ as follows:

$$A(t_0) = 0,\ A(t_1) = at_1,\ D^0(t_1)(t_0) = \frac{1}{\sqrt{d}}t_1,\ D^0(t_1)(t_1) = \frac{1}{\sqrt{d}}t_0 + bt_1$$

where $a = \mathbf{q}(D^+(t_1)(t_1), D^+(t_1)(t_1))$, $b = \mathbf{q}(t_1^2, t_1)$. Observe that the claim in Remark 7.8 implies that $a \neq 0$. A straightforward computation yields

$$\frac{\sqrt{d}}{a}[A, D^0(t_1)] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \frac{\sqrt{d}}{a^2}[A,[A, D^0(t_1)]] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with respect to the basis $\{t_0,\, t_1\}$. This implies the assertion of the claim. q.e.d.

From the claim and the decomposition (7.13) it follows

$$\mathbf{g^0} = \mathbf{c^0} \oplus \mathbf{sl_2C}$$

where $\mathbf{c^0}$ is the center of $\mathbf{g^0}$. To describe the graded pieces $\mathbf{g^p}$, for $p \neq 0$, in the decomposition (7.21) we consider the adjoint action of $\mathbf{sl}(\mathbf{H^0}([Z],[\alpha]))$. Denote by $\mathbf{g^p_n}$ the eigenspace of h_0 in $\mathbf{g^p}$ corresponding to the eigenvalue n.

Claim 7.26.

$$\mathbf{g^p} = \mathbf{g^p_{-1}} \oplus \mathbf{g^p_0} \oplus \mathbf{g^p_1}$$

for every $p \neq 0$, where $dim\mathbf{g^p_{\pm 1}} = 1$.

Proof. Let T be an element of $\mathbf{g^p_n}$, for $n \neq 0$. Consider two cases according to the sign of $\mathbf{p}$.

(i) **Case** $p < 0$**:** observe T and $[h_0, T]$ are zero on $\mathbf{H^0}([Z],[\alpha])$ and $[h_0, T](v^k) = 0$, for all $k \neq (-p)$. The eigenvalue relation implies $T(v^k) = 0$, for all $k \neq (-p)$. For $k = -p$, let $T(v^{-p}) = \tau_0 t_0 + \tau_1 t_1$. Then

$$n(\tau_0 t_0 + \tau_1 t_1) = [h_0, T](v^{-p}) = \tau_0 t_0 - \tau_1 t_1.$$

This implies that $n = \pm 1$ and $\mathbf{g^p_1}$ (resp., $\mathbf{g^p_{-1}}$) admits for a basis an element T^p_1 (resp., T^p_{-1}) represented by the endomorphism of $H^0(\mathcal{O}_Z)$ which takes v^{-p} to t_0 (resp., t_1) and sends to zero all other vectors in B.

(ii) **Case** $p > 0$**:** observe $[h_0, T](v) = nT(v) = 0$, unless $v \in \mathbf{H^0}([Z],[\alpha])$. For t_0 and t_1 we have

$$nT(t_0) = -T(t_0) \textit{ and } nT(t_1) = T(t_1).$$

This implies that $n = \pm 1$ and $\mathbf{g^p_1}$ (resp., $\mathbf{g^p_{-1}}$) admits for a basis an element T^p_1 (resp., T^p_{-1}) represented by the endomorphism of $H^0(\mathcal{O}_Z)$ which takes t_1 to v^p (resp., t_0 to v^p) and sends to zero all other vectors in B.

Observe that $(ad(D^-(t_1)))^p(h_0)$ and $(ad(D^+(t_1)))^p(h_0)$ are nonzero elements, respectively, of $\mathbf{g^{-p}_{-1}}$ and $\mathbf{g^p_1}$, for $p = 1, \ldots, d-2$. This completes the argument. q.e.d.

Let $T^p_{\pm 1}$ be as in the proof of Claim 7.26. The Lie brackets of these elements are as follows

1) $[T^p_\varepsilon, T^q_\varepsilon] = 0$, $\forall p + q \neq 0$ and $\forall \varepsilon \in \{-1, +1\}$.
2) $[T^p_\varepsilon, T^q_{\varepsilon'}] = 0$, for all p, q having the same sign and all $\varepsilon, \varepsilon' \in \{-1, +1\}$.
3) For $p, q > 0$, put $E_{p,q}$ to be the endomorphism of $H^0(\mathcal{O}_Z)$ which takes v^q to v^p and sends to zero all other vectors of the basis B.

Denote by H_i, for $i = 0, 1$, the endomorphism of $H^0(\mathcal{O}_Z)$ which takes t_i to itself and kills all other elements of B. Then we have

$$[T^p_1, T^{-q}_{-1}] = E_{p,q} - \delta_{pq}H_1 \textit{ and } [T^p_{-1}, T^{-q}_1] = E_{p,q} - \delta_{pq}H_0 \tag{7.22}$$

for all $p, q > 0$, where δ_{pq}'s are the Kronecker symbols.

4) $[T^p_1, T^{-p}_1] = -e_0$ *and* $[T^p_{-1}, T^{-p}_{-1}] = -f_0$, for every $p > 0$.

From (7.22) it follows that $\mathbf{g} = \mathbf{sl}(H^0(\mathcal{O}_Z))$. This implies that

$$\{\, T^k_{-1},\, T^k_1,\, E_{p,p-k}\,(k+1 \le p \le d-2)\}$$

is a basis of $\mathbf{g^k}$ and $\{\, T^{-k}_{-1},\, T^{-k}_1,\, E_{p-k,p}\,(k+1 \le p \le d-2)\}$ is a basis of $\mathbf{g^{-k}}$, for $1 \le k \le d-2$, while the center

$$\mathbf{c^0} = \left\{ \lambda(H_0 + H_1) + \sum_{p=1}^{d-2} c_p E_{p,p} \;\middle|\; 2\lambda + \sum_{p=1}^{d-2} c_p = 0, \right.$$
$$\left. c_p\ (1 \le p \le d-2),\ \lambda \in \mathbf{C} \right\}.$$

It should be mentioned that the sheaf of Lie algebras $\tilde{\boldsymbol{\mathcal{G}}}$ (resp. $\boldsymbol{\mathcal{G}}$) is an object of the Higgs category on $\check{\mathbf{J}}$ (see Definition 5.11). Its grading is defined as in 1) of (7.7) (in particular, the weight of $\tilde{\boldsymbol{\mathcal{G}}}$ (and $\boldsymbol{\mathcal{G}}$) is $(2l-1)$. The Higgs field $\boldsymbol{d}_{\tilde{\boldsymbol{\mathcal{G}}}}$ (resp. $\boldsymbol{d}_{\boldsymbol{\mathcal{G}}}$) is determined by some distinguished sections of $\boldsymbol{\Omega} \otimes \tilde{\boldsymbol{\mathcal{G}}}$ (resp. $\boldsymbol{\Omega} \otimes \boldsymbol{\mathcal{G}}$) (as in §4 we denote by $\boldsymbol{\Omega}$ the relative cotangent bundle of $\check{\mathbf{J}}$ over $\dot{\Gamma}^r_d(P)$). Namely, the morphism (7.1) which served us to define $\tilde{\boldsymbol{\mathcal{G}}}$ can now be viewed, using Remark 4.2, as a section $\tilde{\boldsymbol{s}}$ of $\boldsymbol{\Omega} \otimes \tilde{\boldsymbol{\mathcal{G}}}$. Decomposing it according to type determined by the grading of $\tilde{\boldsymbol{\mathcal{G}}}$ we obtain

$$\tilde{\boldsymbol{s}} = \boldsymbol{s}^- + \tilde{\boldsymbol{s}}^{\mathbf{0}} + \boldsymbol{s}^+ \in \boldsymbol{H}^{\mathbf{0}}(\boldsymbol{\Omega} \otimes (\tilde{\boldsymbol{\mathcal{G}}})^{-\mathbf{1}}) \oplus \boldsymbol{H}^{\mathbf{0}}(\boldsymbol{\Omega} \otimes (\tilde{\boldsymbol{\mathcal{G}}})^{\mathbf{0}}) \oplus \boldsymbol{H}^{\mathbf{0}}(\boldsymbol{\Omega} \otimes (\tilde{\boldsymbol{\mathcal{G}}})^{\mathbf{1}}) \tag{7.23}$$

where $\boldsymbol{s}^{\pm}(\boldsymbol{t}) = \boldsymbol{D}^{\pm}(\boldsymbol{t})$ and $\tilde{\boldsymbol{s}}^{\mathbf{0}}(\boldsymbol{t}) = \boldsymbol{D}^{\mathbf{0}}(\boldsymbol{t})$, for every local section $\boldsymbol{t}$ of the relative tangent bundle $\boldsymbol{\mathcal{T}}$ of $\check{\mathbf{J}}$ over $\dot{\Gamma}^r_d(P)$.

Remark 7.27. Taking the image $\boldsymbol{s}^{\mathbf{0}}$ of $\tilde{\boldsymbol{s}}^{\mathbf{0}}$ under the morphism $\boldsymbol{\Omega} \otimes \tilde{\boldsymbol{\mathcal{G}}} \longrightarrow \boldsymbol{\Omega} \otimes \boldsymbol{\mathcal{G}}$ induced by the projection $\tilde{\boldsymbol{\mathcal{G}}} \longrightarrow \boldsymbol{\mathcal{G}}$ defined by the decomposition $\tilde{\boldsymbol{\mathcal{G}}} = \boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{G}}$ in (7.2), we obtain a distinguished section $\boldsymbol{s}$ of $\boldsymbol{\Omega} \otimes \boldsymbol{\mathcal{G}}$ together with the type decomposition $\boldsymbol{s} = \boldsymbol{s}^- + \boldsymbol{s}^{\mathbf{0}} + \boldsymbol{s}^+$.

We will now show that the section $\tilde{\boldsymbol{s}}$ (resp. $\boldsymbol{s}$) defines a relative Higgs field on $\tilde{\boldsymbol{\mathcal{G}}}$ (resp. $\boldsymbol{\mathcal{G}}$). First we recall the bracket operation

$$(\boldsymbol{\Omega}^{\boldsymbol{p}} \otimes \tilde{\boldsymbol{\mathcal{G}}}) \otimes (\boldsymbol{\Omega}^{\boldsymbol{q}} \otimes \tilde{\boldsymbol{\mathcal{G}}}) \longrightarrow \boldsymbol{\Omega}^{\boldsymbol{p}+\boldsymbol{q}} \otimes \tilde{\boldsymbol{\mathcal{G}}}$$

defined on the local sections of the form $\boldsymbol{\omega} \otimes \boldsymbol{x}$ as follows

$$[\boldsymbol{\omega} \otimes \boldsymbol{x}, \boldsymbol{\omega}' \otimes \boldsymbol{x}'] = \boldsymbol{\omega} \wedge \boldsymbol{\omega}' \otimes [\boldsymbol{x}, \boldsymbol{x}'].$$

Using the identities in Lemma 3.9 one easily obtains the following relations between the components of the decomposition in (7.23)

$$[\boldsymbol{s}^{\pm},\boldsymbol{s}^{\pm}]=\mathbf{0},\qquad [\boldsymbol{s}^{\pm},\tilde{\boldsymbol{s}}^{\mathbf{0}}]=[\boldsymbol{s}^{\pm},\boldsymbol{s}^{\mathbf{0}}]=\mathbf{0},$$
$$[\boldsymbol{s}^{-},\boldsymbol{s}^{+}]=-\tfrac{\mathbf{1}}{\mathbf{2}}[\tilde{\boldsymbol{s}}^{\mathbf{0}},\tilde{\boldsymbol{s}}^{\mathbf{0}}]=-\tfrac{\mathbf{1}}{\mathbf{2}}[\boldsymbol{s}^{\mathbf{0}},\boldsymbol{s}^{\mathbf{0}}]. \tag{7.24}$$

In particular we deduce

$$[\tilde{\boldsymbol{s}},\tilde{\boldsymbol{s}}]=[\boldsymbol{s},\boldsymbol{s}]=\mathbf{0}. \tag{7.25}$$

This condition is equivalent to having a relative Higgs field on $\tilde{\boldsymbol{\mathcal{G}}}$ (resp. $\boldsymbol{\mathcal{G}}$). Indeed, given a global section $\boldsymbol{e}$ of $\boldsymbol{\Omega}\otimes\tilde{\boldsymbol{\mathcal{G}}}$ we obtain the morphism

$$[\boldsymbol{e},\cdot]:\tilde{\boldsymbol{\mathcal{G}}}\longrightarrow\boldsymbol{\Omega}\otimes\tilde{\boldsymbol{\mathcal{G}}}$$

defined by taking the bracket with $\boldsymbol{e}$.

Lemma 7.28. *The morphism* $[\boldsymbol{e},\cdot]$ *is Higgs if and only if* $[\boldsymbol{e},\boldsymbol{e}]=\mathbf{0}$.

Proof. It is enough to check it locally. Let $\{\boldsymbol{\omega_i}\}$ be a local frame of $\boldsymbol{\Omega}$ and let $\{\boldsymbol{t_i}\}$ be its dual frame. Then $\boldsymbol{e}$ has the following local expression

$$\boldsymbol{e}=\sum_{\boldsymbol{i}}\boldsymbol{\omega_i}\otimes\boldsymbol{e}(\boldsymbol{t_i}).$$

This implies $[\boldsymbol{e},\boldsymbol{e}]=\mathbf{2}\sum_{\boldsymbol{i<j}}\boldsymbol{\omega_i}\wedge\boldsymbol{\omega_j}\otimes[\boldsymbol{e}(\boldsymbol{t_i}),\boldsymbol{e}(\boldsymbol{t_j})]$. From this it follows that the vanishing of the bracket $[\boldsymbol{e},\boldsymbol{e}]$ is equivalent to

$$[\boldsymbol{e}(\boldsymbol{t_i}),\boldsymbol{e}(\boldsymbol{t_j})]=\mathbf{0},\forall\boldsymbol{i},\boldsymbol{j}. \tag{7.26}$$

Let $\boldsymbol{x}$ be a local section of $\tilde{\boldsymbol{\mathcal{G}}}$. Computing $[\boldsymbol{e},[\boldsymbol{e},\boldsymbol{x}]]$ in our local frame we obtain

$$[\boldsymbol{e},[\boldsymbol{e},\boldsymbol{x}]]=\sum_{\boldsymbol{i<j}}\boldsymbol{\omega_i}\wedge\boldsymbol{\omega_j}\otimes[[\boldsymbol{e}(\boldsymbol{t_i}),\boldsymbol{e}(\boldsymbol{t_j})],\boldsymbol{x}].$$

This together with (7.26) yields the assertion. q.e.d.

Lemma 7.28 together with (7.25) implies that the morphism $\boldsymbol{d_{\tilde{\mathcal{G}}}}:\tilde{\boldsymbol{\mathcal{G}}}\longrightarrow\boldsymbol{\Omega}\otimes\tilde{\boldsymbol{\mathcal{G}}}$ (resp. $\boldsymbol{d_{\mathcal{G}}}:\boldsymbol{\mathcal{G}}\longrightarrow\boldsymbol{\Omega}\otimes\boldsymbol{\mathcal{G}}$) defined by the bracket with $\tilde{\boldsymbol{s}}$ (resp. $\boldsymbol{s}$) is a relative Higgs field of $\tilde{\boldsymbol{\mathcal{G}}}$ (resp. $\boldsymbol{\mathcal{G}}$). Thus $(\tilde{\boldsymbol{\mathcal{G}}},\boldsymbol{d_{\tilde{\mathcal{G}}}})$ and $(\boldsymbol{\mathcal{G}},\boldsymbol{d_{\mathcal{G}}})$ are objects of the Higgs category on $\breve{\mathbf{J}}$.

8. Proof of Claim 2.2

Let $[Z]\in\dot{\Gamma}^r_d(P)$ and let $[\alpha]\in\mathbf{P}(\mathrm{Ext}^1_Z)$ be an extension class corresponding to a locally free sheaf (see Lemma 2.1 for notation). Let $\tilde{\mathbf{H}}_{-k}([Z],[\alpha])$ be the fibre of $\tilde{\mathbf{H}}_{-k}$ at the point $([Z],[\alpha])\in\mathbf{J}'$. We begin by recalling the isomorphism (1.26) in Proposition 1.4 which identifies the fibre $\tilde{\mathbf{H}}([Z],[\alpha])$ of $\tilde{\mathbf{H}}$ at $([Z],[\alpha])$ with the group of extensions Ext^1_Z

$$\tilde{\mathbf{H}}([Z],[\alpha])\xrightarrow{\alpha}\mathrm{Ext}^1_Z\subset H^0(\mathcal{E}xt^2(\mathcal{O}_Z(L),\mathcal{O}_X)). \tag{8.1}$$

Observe that $H^0(\mathcal{E}xt^2(\mathcal{O}_Z(L),\mathcal{O}_X)) = H^0(\omega_Z \otimes \mathcal{O}_X(-L-K_X))$, where ω_Z is the dualizing sheaf of Z. Since the latter is invertible we think of Ext^1_Z as a linear subspace of sections of an invertible sheaf on Z. This and (8.1) imply that the subspace $\tilde{\mathbf{H}}([Z],[\alpha])$ consists of functions of the form $\frac{\gamma}{\alpha}$, where γ runs through Ext^1_Z, i.e., there is the following identification

$$\tilde{\mathbf{H}}([Z],[\alpha]) = \left\{ \frac{\gamma}{\alpha} \mid \gamma \in \mathrm{Ext}^1_Z \right\} \tag{8.2}$$

(the assumption that α corresponds to a locally free sheaf insures that $\alpha(z) \neq 0$, for all $z \in Z$; hence the quotient $\frac{\gamma}{\alpha}$ is a well-defined function on Z).

We will now consider a one-parameter deformation of $\tilde{\mathbf{H}}_{-k}([Z],[\alpha])$ in the direction given by a nonzero vector $\beta \in \mathrm{Ext}^1_Z$. In particular, we will calculate the quadratic form $\mathbf{q}$ as a function of the deformation parameter and vector $\tau = \frac{\beta}{\alpha} \in \tilde{\mathbf{H}}([Z],[\alpha])$.

Let $\alpha(\epsilon) = \alpha + \epsilon\beta$ be an arc in Ext^1_Z passing through α in the direction of a nonzero vector $\beta \in \mathrm{Ext}^1_Z$ and where ϵ is a deformation parameter. From (8.2) it follows that

$$\begin{aligned}\tilde{\mathbf{H}}(\epsilon) &= \tilde{\mathbf{H}}([Z],[\alpha(\epsilon)])\\ &= \left\{ \frac{\gamma}{\alpha(\epsilon)} = \frac{\gamma}{\alpha+\epsilon\beta} \mid \gamma \in \mathrm{Ext}^1_Z \right\} = \frac{1}{1+\epsilon\tau}\tilde{\mathbf{H}}([Z],[\alpha])\end{aligned}$$

where $\tau = \frac{\beta}{\alpha}$ and the last equality is obtained by factoring out α in the denominator of $\frac{\gamma}{\alpha+\epsilon\beta}$ and using the identification in (8.2). This implies that the fibre $\tilde{\mathbf{H}}_{-k}(\epsilon)$ of $\tilde{\mathbf{H}}_{-k}$ at $([Z],[\alpha(\epsilon)])$ has the following form

$$\tilde{\mathbf{H}}_{-k}(\epsilon) = \frac{1}{(1+\epsilon\tau)^k}\tilde{\mathbf{H}}_{-k}([Z],[\alpha]). \tag{8.3}$$

We are now in the position to calculate the quadratic form $\mathbf{q}$ on $\tilde{\mathbf{H}}_{-k}(\epsilon)$. Let $f_1,\ldots,f_{p_k}$ be a basis of $\tilde{\mathbf{H}}_{-k}([Z],[\alpha])$, where p_k is the dimension of $\tilde{\mathbf{H}}_{-k}([Z],[\alpha])$. From (8.3) it follows that the functions $\frac{1}{(1+\epsilon\tau)^k}f_i$, $i = 1,\ldots,p_k$, form a basis of $\tilde{\mathbf{H}}_{-k}(\epsilon)$ and the quadratic form $\mathbf{q}$ is determined by the matrix

$$Q(\epsilon,\tau) = (\mathbf{q}_{ij}(\epsilon,\tau))_{i,j=1,\ldots,p_k} \tag{8.4}$$

where

(8.5)

$$\mathbf{q}_{ij}(\epsilon,\tau) = \mathbf{q}\left(\frac{1}{(1+\epsilon\tau)^k}f_i, \frac{1}{(1+\epsilon\tau)^k}f_j\right) = \sum_{z\in Z}(1+\epsilon\tau(z))^{-2k}f_i(z)f_j(z)$$

for $i, j \in \{1, \ldots, p_k\}$.

Next we calculate the determinant of the matrix $Q(\epsilon, \tau)$. For this choose an ordering of the points in Z by the set $\{1, \ldots, d\}$, i.e. $Z = \{z_1, \ldots, z_d\}$. Then from (8.5) it follows that

$$F(\epsilon, \tau) = \det Q(\epsilon, \tau) = \sum_S \Delta_S(\epsilon, \tau) \tag{8.6}$$

where the sum is taken over all ordered subsets S of $\{1, \ldots, d\}$ with the cardinality $\mid S \mid$ of S equals to p_k, and where for $S = \{s_1, \ldots, s_{p_k}\}$, the function $\Delta_S(\epsilon, \tau)$ is as follows

$$\Delta_S(\epsilon, \tau) = \left(\prod_{s \in S} (1 + \epsilon\tau(z_s))\right)^{-2k} A_S,$$

where A_S is the determinant of the following matrix

$$Q(S) = \begin{pmatrix} f_1(z_{s_1})f_1(z_{s_1}) & f_1(z_{s_2})f_2(z_{s_2}) & \cdots & f_1(z_{s_{p_k}})f_{p_k}(z_{s_{p_k}}) \\ f_2(z_{s_1})f_1(z_{s_1}) & f_2(z_{s_2})f_2(z_{s_2}) & \cdots & f_2(z_{s_{p_k}})f_{p_k}(z_{s_{p_k}}) \\ \cdots & \cdots & \cdots & \cdots \\ f_{p_k}(z_{s_1})f_1(z_{s_1}) & f_{p_k}(z_{s_2})f_2(z_{s_2}) & \cdots & f_{p_k}(z_{s_{p_k}})f_{p_k}(z_{s_{p_k}}) \end{pmatrix}.$$

In particular, $A_S = \det Q(S) = \left(\prod_{i=1}^{p_k} f_i(z_{s_i})\right) d_S$, where d_S is the determinant of the matrix

$$M_S = \begin{pmatrix} f_1(z_{s_1}) & f_1(z_{s_2}) & \cdots & f_1(z_{s_{p_k}}) \\ f_2(z_{s_1}) & f_2(z_{s_2}) & \cdots & f_2(z_{s_{p_k}}) \\ \cdots & \cdots & \cdots & \cdots \\ f_{p_k}(z_{s_1}) & f_{p_k}(z_{s_2}) & \cdots & f_{p_k}(z_{s_{p_k}}) \end{pmatrix} \tag{8.7}$$

i.e., M_S is the matrix whose column vectors correspond to the images of the points $z_{s_1}, \ldots, z_{s_{p_k}}$ under the morphism

$$\kappa_k : Z \longrightarrow \left(\tilde{\mathbf{H}}_{-k}([Z], [\alpha])\right)^*$$

determined by the subspace $\tilde{\mathbf{H}}_{-k}([Z], [\alpha])$ of $H^0(\mathcal{O}_Z)$. This implies that $\Delta_S(\epsilon, \tau) \neq 0$ for the subsets $S = \{s_1, \ldots, s_{p_k}\}$ such that the images of $z_{s_1}, \ldots, z_{s_{p_k}}$ under the morphism κ_k span $\left(\tilde{\mathbf{H}}_{-k}([Z], [\alpha])\right)^*$ and all diagonal entries of the matrix M_S in (8.7) are nonzero.

Let $S_1, \ldots, S_N$ be distinct ordered subsets of $\{1, \ldots, d\}$ whose corresponding subsets $Z_1, \ldots, Z_N$ of Z are subject to the above conditions. Then the function $F(\epsilon, \tau)$ in (8.6) has the following form

$$F(\epsilon, \tau) = \sum_{n=1}^{N} \Delta_{S_n}(\epsilon, \tau) = \sum_{n=1}^{N} F_n(\epsilon, \tau) A_{S_n} \tag{8.8}$$

where

$$(8.9)\qquad F_n(\epsilon,\tau)=\left(\prod_{s\in S_n}(1+\epsilon\tau(z_s))\right)^{-2k},\quad A_{S_n}=\left(\prod_{i=1}^{p_k}f_i(z_{s_i})\right)d_{S_n}$$

where $d_{S_n}=\det M_{S_n}$ with the matrix M_{S_n} as in (8.7). Observing that any two matrices M_{S_n},M_{S_t} are related by the transition matrix $G_{nt}\in\mathbf{GL}(\mathbf{C},\mathbf{p_k})$, we obtain

$$d_{S_n}=g_{nt}d_{S_t}$$

where g_{nt} is the determinant of G_{nt}. Fixing t and substituting these expressions into (8.9) and then into (8.8) we obtain

$$(8.10)\qquad F(\epsilon,\tau)=\left(\sum_{n=1}^{N}c_{nt}F_n(\epsilon,\tau)\right)d_t$$

where $c_{nt}=\left(\prod_{i=1}^{p_k}f_i(z_{s_i})\right)g_{nt}$. Our claim now is reduced to showing that the function $F(\epsilon,\tau)$ is nonzero on $\mathbf{C}\times\tilde{\mathbf{H}}([Z],[\alpha])$.

We argue by contradiction. Assume that $F(\epsilon,\tau)$ is identically zero. From (8.10) it follows that the functions $F_n(\epsilon,\tau)$ are subject to the following linear relation over $\mathbf{C}$,

$$(8.11)\qquad \sum_{n=1}^{N}c_{nt}F_n=0.$$

From (8.9) it follows that for a general choice of $\tau\in\tilde{\mathbf{H}}([Z],[\alpha])$ the expression $F_n(\epsilon,\tau)$, viewed as a rational function of ϵ, has poles at $-(\tau(p))^{-1}$, where $p\in Z_n$. This implies that the function F_n is uniquely determined by the image $\kappa_k(Z_n)$. So the relation (8.11) should be interpreted geometrically as a relation between the images of the sets $Z_1,\dots,Z_N$. This is a key to obtaining a contradiction. To this end we choose $f_1,\dots,f_{p_k}$ to be the basis of $\tilde{\mathbf{H}}_{-k}([Z],[\alpha])$ dual to the basis $\kappa_k(Z_1)=\{\kappa_k(z_{s_1}),\dots,\kappa_k(z_{s_{p_k}})\}$ of $\left(\tilde{\mathbf{H}}_{-k}([Z],[\alpha])\right)^*$. Then the matrix M_{S_1} is the identity matrix and the relation (8.11) for $t=1$ yields

$$(8.12)\qquad F_1+\sum_{n=2}^{N}c_{n1}F_n=0.$$

Let I be a smallest subset (with respect to inclusion of sets) of $\{2,\dots,d\}$ such that the relation

$$(8.13)\qquad F_1+\sum_{i\in I}c_{i1}F_i=0$$

holds. From what has been said before this implies that for general $\tau\in\tilde{\mathbf{H}}([Z],[\alpha])$ every pole $-(\tau(p))^{-1}$ $(p\in Z_1)$ of $F_1(\epsilon,\tau)$ is contained in the union of poles of the functions $F_i(\epsilon,\tau),\ i\in I$. From this it follows that the image $\kappa_k(Z_1)\subset \bigcup_{i\in I}\kappa_k(Z_i)$. So the image of every point $p\in Z_1$ is contained in $\kappa_k(Z_i)$, for some $i\in I$. We claim that $\kappa_k(p)$ must be contained in $\kappa_k(Z_i)$, for all $i\in I$. Indeed, let I_p be the subset of I such that the subsets $\kappa_k(Z_j), \forall j\in I_p$, contain $\kappa_k(p)$. Then we have

$$F_1+\sum_{j\in I_p} c_{j1}F_j = -\sum_{i\notin I_p} c_{i1}F_i.$$

For general $\tau\in\tilde{\mathbf{H}}([Z],[\alpha])$ the right hand side is a function of ϵ which has no pole at $-(\tau(p))^{-1}$, while the left hand side, if nonzero, does have pole at $-(\tau(p))^{-1}$. Hence the expression on the left must be identically zero. The minimality condition on I implies that $I_p=I$. Since this argument holds for every $p\in Z_1$ we conclude that $\kappa_k(Z_1)=\kappa_k(Z_i)$, for all $i\in I$. But this implies that the matrices M_{S_i} in (8.7) are obtained from the identity matrix M_{S_1} by permutations of its columns. Furthermore the coefficients c_{i1} involve the products of the diagonal entries of M_{S_i} which must be all nonzero. But this is possible if the permutations in question are all equal to the identity. Hence $M_{S_i}=M_{S_1}$, for all $i\in I$, and the relation (8.13) becomes

$$(\mid I\mid +1)F_1=0,$$

which is impossible. This completes the proof of Claim 2.2.

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