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ENLARGEABILITY AND INDEX THEORY

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Abstract

Let M be a closed enlargeable spin manifold. We show nontriviality of the universal index obstruction in the K-theory of the maximal C^*-algebra of the fundamental group of M. Our proof is independent of the injectivity of the Baum-Connes assembly map for $\pi_1(M)$ and relies on the construction of a certain infinite dimensional flat vector bundle out of a sequence of finite dimensional vector bundles on M whose curvatures tend to zero.

Besides the well known fact that M does not carry a metric with positive scalar curvature, our results imply that the classifying map $M \to B\pi_1(M)$ sends the fundamental class of M to a nontrivial homology class in $H_*(B\pi_1(M);\mathbb{Q})$. This answers a question of Burghelea (1983).

1. Introduction

1.1. Enlargeability and the universal index obstruction. For a closed spin manifold M^n, Rosenberg in [**16**] constructs an index

$$\alpha^{\mathbb{R}}_{\max}(M) \in KO_n(C^*_{\max,\mathbb{R}}\pi_1(M))$$

in the K-theory of the (maximal) real C^*-algebra of the fundamental group of M. By the Lichnerowicz-Schrödinger-Weitzenböck formula this index is zero if M admits a metric of positive scalar curvature. The Gromov-Lawson-Rosenberg conjecture states that, conversely, the vanishing of $\alpha(M)$ implies that M admits such a metric, if $n \geq 5$. By a result of the second named author, this conjecture is known to be false in general [**17**] or [**4**]. But a stable version of this conjecture is true, if the Baum-Connes assembly map

$$\mu\colon KO_*^{\pi_1(M)}(\underline{E}\pi_1(M)) \to KO_*(C^*_{\max,\mathbb{R}}\pi_1(M))$$

is injective [**20**]. The proof of this (and related results) is based on the existence of a natural map $D\colon KO_*(M) \to KO_*^{\pi_1(M)}(\underline{E}\pi_1(M))$ into the

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equivariant K-homology of the classifying space for $\pi_1(M)$-actions with finite isotropy and of a factorization

$$KO_n(M) \xrightarrow{D} KO_n^{\pi_1(M)}(\underline{E}\pi_1(M)) \xrightarrow{\mu} KO_n(C^*_{\max,\mathbb{R}}\pi_1(M))$$

which sends the KO-fundamental class $[M] \in KO_n(M)$ to $\alpha^{\mathbb{R}}_{\max}(M)$. Therefore, if $\alpha^{\mathbb{R}}_{\max}(M) = 0$ and μ is injective, one knows that $D([M]) = 0$ and this situation can be analyzed by algebraic topological means. (Actually, Stephan Stolz is using the reduced group C^*-algebra; compare the discussion in Section 1.4.)

In this paper, we describe a new method to detect non-vanishing of this universal index obstruction in a nontrivial case. This is independent of the injectivity of the Baum-Connes map. For convenience, we study the complex K-theory index element $\alpha_{\max}(M)$ in the K-theory of the maximal complex C^*-algebra of $\pi_1(M)$. The usual Lichnerowicz argument shows that $\alpha_{\max}(M) = 0$ if M admits a metric of positive scalar curvature.

In the first part of our paper, we prove a weak converse to this statement. Recall:

Definition 1.1. A closed oriented manifold M^n is called *enlargeable* if the following holds: Fix some Riemannian metric g on M. Then, for all $\epsilon > 0$, there is a finite connected cover $\overline{M}$ of M and an ϵ-contracting map $(\overline{M}, \overline{g}) \to (S^n, g_0)$ of non-zero degree, where $\overline{g}$ is induced by g and g_0 is the standard metric on S^n.

M is called *area-enlargeable* if in the above definition ϵ-contracting is replaced by ϵ-area contracting. Here, a map $f\colon M \to N$ between two n-dimensional Riemannian manifolds is called *ϵ-area contracting* if $\left\|\Lambda^2 T_x f\right\| \le \epsilon$ for each $x \in M$, where $\Lambda^2 T_x f\colon \Lambda^2 T_x M \to \Lambda^2 T_{f(x)} N$ is the induced map between the second exterior powers of tangent spaces viewed as normed spaces via the given Riemannian metrics.

Note that every enlargeable manifold is area-enlargeable, but that the converse might not be true.

We remark that in contrast to the definition in [**6**], we do not require that the covers $\overline{M}$ necessarily admit spin structures.

Theorem 1.2. *Let M^n be an enlargeable or area-enlargeable spin manifold. Then*

$$\alpha_{\max}(M) \neq 0 \in K_n(C^*_{\max}\pi_1(M)).$$

By a result of Gromov and Lawson [**6**], enlargeable spin manifolds do not admit metrics of positive scalar curvature. Recall the question posed in the second paragraph of the introduction to the article [**15**]: "Nevertheless, it is not clear, if their results always imply ours or vice versa."

Our paper gives a complete answer in one direction: if M is spin, the index obstruction $\alpha_{\max}(M)$ completely subsumes the enlargeability (and area-enlargeability) obstruction to positive scalar curvature of Gromov and Lawson.

For the applications to positive scalar curvature, we restrict our discussion to spin manifolds M in order to keep the exposition transparent. We conjecture that it is possible to extend the results to the case where only the universal cover of M admits a spin structure.

1.2. Flat bundles of C^*-modules. The idea of our proof can be summarized as follows. We construct a C^*-algebra morphism

$$\phi\colon C^*_{\max}\pi_1(M) \to Q$$

where Q is a (complex) C^*-algebra whose K-theory can be explicitely calculated, and then we study the image of $\alpha_{\max}(M)$ under the induced map in K-homology.

The map ϕ results from the holonomy representation of $\pi_1(M)$ associated to an infinite dimensional flat bundle on M which is obtained in the following way: Because M is enlargeable or area-enlargeable, there is a sequence $E_i \to M$ of (finite dimensional) unitary vector bundles with connections whose curvatures tend to zero, but whose Chern characters are nontrivial. We construct an infinite dimensional smooth bundle $V \to M$ with connection and with the following property: The fiber over $p \in M$ consists of bounded sequences $(v_1, v_2, \ldots)$ with $v_i \in (E_i)_p$ and the connection restricts to the given connection of E_i on each "block". We denote by $W \subset V$ the subbundle consisting of sequences tending to zero. The $\mathrm{End}(V)$-valued curvature form on V sends V to W by the asymptotic curvature property of the sequence (E_i). Hence the quotient bundle $V/W \to M$ with the induced connection is flat.

However, this bundle still encodes the asymptotic non-triviality of the Chern characters of the original bundles in such a way that the index of the Dirac operator on M twisted with this bundle is nontrivial. This index can be expressed in terms of the collection of indices of the Dirac operator twisted with E_i, $i \in \mathbb{N}$.

It should be noted that the precise argument in Section 2 needed to construct the bundle V requires a considerable amount of care.

In this respect, we realize the idea formulated at the end of the introduction to [**7**]: "Passing to the limit, one might expect to find an interesting infinite dimensional, *flat* bundle E_0 over the original manifold, so that one could apply the Bochner method directly to the Dirac operator with coefficients in E_0". In our case, the role of E_0 is played by the bundle $V/W \to M$.

1.3. Almost flat bundles and almost representations. Our construction can be seen in relation to the notions of almost flat bundles as studied by Connes-Gromov-Moscovici [**3**] and of almost representations as studied by Mishchenko and his coauthors (compare e.g. [**14**]). Heath Emerson informed us that he and Jerry Kaminker plan to carry out a systematic study of these notions in the context of the Baum-Connes conjecture. Contrary to the definitions used in the mentioned sources, we do not require the different bundles in the almost flat sequence to define the same K-theory class or the "not quite representations" induced by such a sequence to be related in any way. Keeping this flexibility throughout the argument enables us to prove the general statement of Theorem 1.2.

1.4. The different C^*-indices. In our paper, we use complex C^*-algebras, because this avoids some technicalities and is sufficient for the applications that we have in mind. It should be possible to show real versions of our theorems in a similar way.

Here we want to compare the reduced to the maximal index, and the real to the complex version. In recent literature on the positive scalar curvature question, in most cases the real reduced index $\alpha^{\mathbb{R}}_{\mathrm{red}}(M) \in KO_n(C^*_{\mathrm{red},\mathbb{R}}\pi_1(M))$ is used, whereas Rosenberg [**15**, **16**] uses $\alpha_{\max}(M)$ and $\alpha^{\mathbb{R}}_{\max}(M)$.

Note that for any discrete group π, we have canonical maps

$$\begin{array}{ccc} C^*_{\max,\mathbb{R}}\pi & \xrightarrow{\omega^{\mathbb{R}}} & C^*_{\mathrm{red},\mathbb{R}}\pi \\ \downarrow & & \downarrow \\ C^*_{\max}\pi & \xrightarrow{\omega} & C^*_{\mathrm{red}}\pi \end{array}$$

and a commutative diagram

$$\begin{array}{ccccc} KO^{\pi}_*(\underline{E}\pi) & \xrightarrow{\mu^{\mathbb{R}}_{\max}} & KO_*(C^*_{\max,\mathbb{R}}\pi) & \xrightarrow{\omega^{\mathbb{R}}} & KO_*(C^*_{\mathrm{red},\mathbb{R}}\pi) \\ \downarrow & & \downarrow & & \downarrow \\ K^{\pi}_*(\underline{E}\pi) & \xrightarrow{\mu_{\max}} & K_*(C^*_{\max}\pi) & \xrightarrow{\omega} & K_*(C^*_{red}\pi) \end{array} \tag{1.3}$$

where the vertical maps are given by complexification and the horizontal compositions are the reduced analytic assembly maps μ or $\mu^{\mathbb{R}}$. If $\pi = \pi_1(M)$, the map $\omega^?_*$ sends $\alpha^?_{\max}(M)$ to $\alpha^?_{\mathrm{red}}(M)$ (this is true for the real and the complex version), and the complexification maps send $\alpha^{\mathbb{R}}_?(M)$ to $\alpha_?(M)$ (for the max and red version). In particular, vanishing of $\alpha^{\mathbb{R}}_{\max}(M)$ implies vanishing of all the other index invariants. Consequently, following Rosenberg, one should formulate the Gromov-Lawson-Rosenberg conjecture using $\alpha^{\mathbb{R}}_{\max}$. We point out that this conjecture holds stably by the result of Stephan Stolz cited above, if the Baum-Connes map $\mu^{\mathbb{R}}_{\max}$ is injective. It could possibly happen that $\alpha^{\mathbb{R}}_{\max}(M) \neq 0$ whereas $\alpha^{\mathbb{R}}_{\mathrm{red}}(M) = 0$. However, this would imply that a number of important conjectures are wrong, most notably that the (reduced) Baum-Connes assembly map is not always injective.

The maximal C^*-algebra has much better functorial properties than the reduced one, a fact that we are using in the construction of the homomorphism ϕ alluded to in (1.2).

In view of these considerations and the results presented in this paper, we propose to always use the obstruction $\alpha^{\mathbb{R}}_{\mathrm{max}}(M)$ to the existence of positive scalar curvature metrics, instead of $\alpha^{\mathbb{R}}_{\mathrm{red}}(M)$. Note that the two obstructions are equivalent, if the Baum-Connes map is injective. Note also that the two obstructions coincide if the fundamental group π is K-amenable (in particular if it is amenable) because in this case the map $K_*(C^*_{\mathrm{max}}\pi) \to K_*(C^*_{\mathrm{red}}\pi)$ is an isomorphism.

In this paper, we show non-vanishing of the complex version $\alpha_{\mathrm{max}}(M)$ under an enlargeability assumption. The diagram (1.3) implies non-vanishing of $\alpha^{\mathbb{R}}_{\mathrm{max}}(M)$, as well.

1.5. Enlargeability and the fundamental class. Turning to another application of our methods, we show

Theorem 1.4. *Let M be an enlargeable or area-enlargeable manifold, $f\colon M \to B\pi_1(M)$ classify the universal cover of M and $[M] \in H_n(M;\mathbb{Q})$ be the fundamental class. Then*

$$f_*([M]) \neq 0 \in H_n(B\pi_1(M);\mathbb{Q}).$$

This theorem implies an affirmative answer to a question of Burghelea [**19**, Problem 11.1]. We emphasize that contrary to the original formulation of Burghelea's question, no spin assumption on M or its universal cover is required. It is somewhat remarkable that we prove Theorem 1.4 by making a detour through the K-theory of C^*-algebras and an assembly map.

For (length)-enlargeable manifolds, one may use coarse geometry methods to get a shorter proof of this result. We will address this elsewhere. It would be interesting to extend these coarse methods to the area-enlargeable case.

For $n \geq 4$, we will construct closed oriented manifolds M^n (that may be chosen to be spin) whose fundamental classes are sent to nontrivial classes in $H_n(B\pi_1(M);\mathbb{Q})$, but which are not area-enlargeable. In this respect, a converse of Theorem 1.4 does not hold to be true.

Because the proof of Theorem 1.4 is based on an analysis of general Dirac type operators on M, the necessary index theory in Section 3 will be developed in the required generality.

Theorem 1.4 implies

Corollary 1.5. *Every (area)-enlargeable manifold M is essential in the sense of Gromov [**5**], and therefore its 1-systole satisfies Gromov's main inequality*

$$sys_1(M) \leq c(n)\,\mathrm{vol}(M)^{1/n}.$$

Therefore, such an M has a non-contractible closed geodesic of length at most $c(n)\operatorname{vol}(M)^{1/n}$. Here, $n = \dim(M)$ and $c(n) > 0$ is a constant which depends only on n.

2. Assembling almost flat bundles

Let M be a closed smooth n-dimensional Riemannian manifold, let d_i, $1 \le i < \infty$, be a sequence of natural numbers and let $(P_i, \nabla_i)_{i\in\mathbb{N}}$ be an *almost flat* sequence of principal $\mathrm{U}(d_i)$ bundles over M equipped with $\mathrm{U}(d_i)$-connections ∇_i. By definition, this means that the curvature 2-forms

$$\Omega_i \in \Omega^2(M; \mathfrak{u}(d_i))$$

associated to ∇_i vanish asymptotically with respect to the maximum norm on the unit sphere bundle in $\Lambda^2 M$ and the operator norm on each $\mathfrak{u}(d_i) \subset \operatorname{Mat}(d_i) := \mathbb{C}^{d_i \times d_i}$, i.e.

$$\lim_{i\to\infty} \|\Omega_i\| = 0.$$

Let $\mathbb{K}$ denote the C^*-algebra of compact operators on $l^2(\mathbb{N})$. We choose embeddings of complex C^*-algebras

$$\gamma_i\colon \operatorname{Mat}(\mathbb{C}, d_i) \hookrightarrow \mathbb{K}.$$

Hence, each P_i has an associated bundle

$$F_i := P_i \times_{U(d_i)} \mathbb{K}$$

consisting of projective right $\mathbb{K}$-modules with one generator and which is equipped with a $\mathbb{K}$-linear connection

$$\nabla_i : \Gamma(F_i) \to \Gamma(T^*M \otimes F_i).$$

Note that, since the map $U(d_i) \to \mathbb{K}$ is not unital, the fibers of F_i are *not* free, but are isomorphic as right $\mathbb{K}$-modules to $q_i\mathbb{K}$, where $q_i = \gamma_i(1)$. Because the structure group of F_i is $\mathrm{U}(d_i)$, each bundle F_i has the structure of a $\mathbb{K}$-Hilbert bundle induced by the inner product

$$\mathbb{K} \times \mathbb{K} \to \mathbb{K}, (X, Y) \mapsto X^*Y$$

on each fiber. The connections ∇_i are compatible with these inner products. Let A be the complex (non-unital) C^*-algebra of norm bounded sequences

$$(a_i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} \mathbb{K}.$$

For $i \in \mathbb{N}$, we denote by $A_i \subset A$ the sub-algebra of sequences such that all but the ith entry vanish. The algebra A_i can be identified with $\mathbb{K}$. Define the element (being a projection)

$$q := (q_i)_{i\in\mathbb{N}} \in A; \qquad q_i = \gamma_i(1).$$

The following theorem says that the bundles F_i can be assembled to a smooth bundle of right Hilbert A-modules in a particularly nice way. For the necessary background concerning Hilbert module bundles, we refer to [**18**].

Theorem 2.1. *There is a smooth Hilbert A-module bundle $V \to M$, each fiber of V being a finitely generated projective right A-module, together with an A-linear metric connection*

$$\nabla^V : \Gamma(V) \to \Gamma(T^*M \otimes V)$$

such that the following holds:

- *For $i \in \mathbb{N}$, let V_i be the subbundle $V \cdot A_i \subset V$. Then V_i ("the ith block of V") is isomorphic to F_i (as a $\mathbb{K}$-Hilbert bundle).*
- *The connection ∇ preserves the subbundles V_i.*
- *Let ∇_i^V be the connection induced on V_i by ∇^V and let Ω_i^V be the corresponding curvature form in $\Gamma(\Lambda^2 M \otimes \mathrm{End}_{A_i}(V_i))$. Then*

$$\lim_{i\to\infty} \|\Omega_i^V\| = 0\,.$$

The remainder of this section is devoted to the construction of V. At first, we obtain a Lipschitz-Hilbert-A-module bundle $L \to M$ that will be approximated by a smooth bundle $V \to M$. After this has been done, the bundle V will be equipped with a connection ∇^V as stated in Theorem 2.1.

Let

$$D^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1\} \subset \mathbb{R}^n$$

be the standard n-dimensional cube and let $(\phi_j)_{j\in J}$ be a finite family of diffeomorphisms [1]

$$M \supset W_j \overset{\phi_j}{\to} D^n$$

such that

$$M \subset \bigcup_{j\in J} \mathring{W}_j\ .$$

Identify each of the W_j with D^n, using ϕ_j. In order to obtain the bundle L, we construct trivializations

$$\psi_{i,j} \colon F_i|_{W_j} \cong D^n \times q_i\mathbb{K}$$

for all $i \in \mathbb{N}$ and $j \in J$ as follows: Choose a $\mathbb{K}$-linear unitary isomorphism

$$\psi_{i,j} \colon F_i|_{(0,0,\ldots,0)} \cong q_i\mathbb{K}.$$

The map $\psi_{i,j}$ can be extended to a unique isomorphism of smooth $\mathbb{K}$-module bundles

$$\psi_{i,j} \colon F_i|_{[0,1]\times 0\times\ldots\times 0} \cong ([0,1] \times 0 \times \ldots \times 0) \times q_i\mathbb{K}$$

[1]In the sense that each ϕ_j extends to a diffeomorphism from an open neighborhood of $W_j \subset M$ to an open neighborhood of $D^n \subset \mathbb{R}^n$.

such that the constant sections

$$[0,1] \times 0 \times \ldots \times 0 \to ([0,1] \times 0 \times \ldots \times 0) \times q_i\mathbb{K}$$

are parallel with respect to ∇. Inductively, we assume that $\psi_{i,j}$ has already been defined on

$$F_i|_{D^k \times 0 \times \ldots \times 0}.$$

Then $\psi_{i,j}$ can be extended to a unique isomorphism of smooth $\mathbb{K}$-module bundles

$$F_i|_{D^{k+1} \times 0 \times \ldots \times 0} \cong D^{k+1} \times q_i\mathbb{K}$$

such that the covariant derivative along the tangent vector field

$$\frac{\partial}{\partial x_{k+1}} \in \Gamma(TD^{k+1})$$

of each constant section $D^{k+1} \to D^{k+1} \times q_i\mathbb{K}$ vanishes.

Definition 2.2. We denote by

$$\omega_{i,j} \in \Gamma(T^*D^n \otimes \mathrm{End}_{\mathbb{K}}(q_i\mathbb{K})) \cong \Omega^1(D^n; q_i\mathbb{K}q_i)$$

the connection 1-form induced on $D^n \times q_i\mathbb{K}$ by ∇_i and $\psi_{i,j}$. Note that the right $\mathbb{K}$-module endomorphisms of $q_i\mathbb{K}$ are canonically isomorphic to the unital C^*-algebra $q_i\mathbb{K}q_i$.

Furthermore, we denote by $\|\omega_{i,j}\|$ the L^∞-norm of $\omega_{i,j}$ induced by the usual Euclidean metric on D^n and the operator norm on $\mathrm{End}_{\mathbb{K}}(q_i\mathbb{K})$.

We will show now that the special construction of the trivializations $\psi_{i,j}$ ensures that we have upper bounds for $\|\omega_{i,j}\|$. Let

$$\eta_{i,j} = d\omega_{i,j} - \omega_{i,j} \wedge \omega_{i,j} \in \Gamma(\Lambda^2 D^n \otimes \mathrm{End}_{\mathbb{K}}(q_i\mathbb{K})) = \Omega^2(D^n; q_i\mathbb{K}q_i)$$

be the curvature 2-form on D^n induced by $\psi_{i,j}$ and ∇_i.

Lemma 2.3. *For each i and j, we have*

$$\|\omega_{i,j}\| \leq n \cdot \|\eta_{i,j}\|.$$

Proof. For brevity, we drop the indices i and j and abbreviate $\frac{\partial}{\partial x_\nu}$ by ∂_ν. By construction of the trivialization ψ, we have

$$\omega_{(x_1,\ldots,x_k,0\ldots,0)}(\partial_\nu) = 0,$$

if $\nu \geq k$. Now, if $k > \nu$, we get

$$\begin{aligned}
&\|\omega_{(x_1,\ldots,x_k,0,\ldots,0)}(\partial_\nu)\| \\
&= \|\omega_{(x_1,\ldots,x_{k-1},0,\ldots,0)}(\partial_\nu) + \int_0^{x_k} d\omega_{(x_1,\ldots,x_{k-1},t,0,\ldots,0)}(\partial_k,\partial_\nu)\,dt\| \\
&\leq \|\omega_{(x_1,\ldots,x_{k-1},0,\ldots,0)}(\partial_\nu)\| + \int_0^{x_k} \|\eta_{(x_1,\ldots,t,0,\ldots,0)}(\partial_k,\partial_\nu)\|\,dt \\
&\leq \|\omega_{(x_1,\ldots,x_{k-1},0,\ldots,0)}(\partial_\nu)\| + \|\eta_{i,j}\| \cdot |x_k|.
\end{aligned}$$

The second inequality uses the fact that

$$(\omega \wedge \omega)_{(x_1,\ldots,x_{k-1},t,0,\ldots,0)}(\partial_k, \partial_\nu) = 0$$

by construction of the trivialization $\psi_{i,j}$. Because $\omega_{(x_1,\ldots,x_\nu,0,\ldots,0)}(\partial_\nu) = 0$, we see inductively that

$$\|\omega_{(x_1,\ldots,x_k,0,\ldots,0)}\| \leq \|\eta_{i,j}\| \cdot (|x_{\nu+1}| + \ldots + |x_k|) \leq n \cdot \|\eta_{i,j}\|.$$

q.e.d.

Remark 2.4. By assumption, the bundles F_i form an almost flat sequence of bundles. Consequently, the supremum norms $\|\eta_{i,j}\|$ (which do not depend on the particular trivializations) have an upper bound, and by Lemma 2.3 the same is true for the $\|\omega_{i,j}\|$. The following lemma shows an application of this fact.

Lemma 2.5. *Let $l \geq 0$. Then there is a constant $C(l)$ (independent of i, j) such that if*

$$\phi\colon [0,1] \to D^n \times q_i\mathbb{K}$$

is a parallel vector field (with respect to $\omega_{i,j}$) along a piecewise smooth path $\gamma\colon [0,1] \to D^n$ of length $l(\gamma) \leq l$ (measured with respect to the usual metric on D^n), then

$$\|\phi(1) - \phi(0)\| \leq C(l) \cdot \|\omega_{i,j}\| \cdot l(\gamma) \cdot \|\phi(0)\|$$

for all i, j.

Proof. Since the bundle $D^n \times q_i\mathbb{K}$ is trivial, we consider the section ϕ as a path $[0,1] \to q_i\mathbb{K}$. It satisfies the differential equation

$$\phi'(t) + \big((\omega_{i,j})_{\gamma(t)}(\dot\gamma(t))\big) \cdot \phi(t) = 0.$$

It follows that

$$\|\phi(1) - \phi(0)\| \leq \exp\left(2l(\gamma)\,\|\omega_{i,j}\|\right) \cdot \|\phi(0)\|\,.$$

The function $\exp\colon q_i\mathbb{K}q_i \to q_i\mathbb{K}q_i$ is uniformly Lipschitz continuous on each bounded neighborhood of 0. Hence, the proof is complete. q.e.d.

These estimates allow for the following important implication. For $\alpha, \beta \in J$, $i \in \mathbb{N}$, we denote by

$$\phi_{\alpha,\beta}\colon \phi_\alpha(W_\alpha \cap W_\beta) \to \phi_\beta(W_\alpha \cap W_\beta)$$

the transition function for the charts ϕ_i of our manifold M, and by

$$\psi_{\alpha,\beta,i}\colon \psi_\alpha(F_i|_{W_\alpha \cap W_\beta}) \to \psi_\beta(F_i|_{W_\alpha \cap W_\beta})$$

the transition function for the trivializations of the bundles F_i.

Proposition 2.6. *There is a constant $C \in \mathbb{R}$ such that (independently of the particular smooth bundle in the almost flat sequence) the following holds for all α and β: Considering $\psi_{\alpha,\beta,i}$ (i.e., the (smooth) transition function for the i-th vector bundle) as a function*

$$\psi\colon W_\alpha \cap W_\beta \to q_i\mathbb{K}q_i,$$

we have $\|D\psi(x)\| \leq C$ *for all* $x \in W_\alpha \cap W_\beta$.

Proof. Let

$$x = (x_1, \ldots, x_n) \in \phi_\alpha(W_\alpha \cap W_\beta) \cap \overset{\circ}{D}{}^n$$

and let $1 \leq \nu \leq n$. We have to study the function

$$f \colon (-\epsilon, \epsilon) \to q_i \mathbb{K} q_i$$

defined by the property that

$$\psi_\beta \psi_\alpha^{-1}(x + te_\nu, v) = (\phi_{\alpha,\beta}(x + te_\nu), f(t) \cdot v)$$

for all $v \in q_i\mathbb{K}$ and all $t \in (-\epsilon, \epsilon)$ (where ϵ is sufficiently small). For $v \in q_i\mathbb{K}$, the element $f(t)(v) \in q_i\mathbb{K}$ can be constructed as follows: Consider the path

$$\gamma \colon [0, t] \to D^n\,,\ \xi \mapsto x + \xi e_\nu.$$

Now parallel transport the element v along γ^{-1} using the connection ω_α to get $w \in q_i\mathbb{K}$ and then parallel transport the element $f(0)w$ along $\phi_{\alpha,\beta} \circ \gamma$ using the connection ω_β. This works since, in terms of the bundle F_i, this means we use the same parallel transport (given by ω_α and ω_β in the two trivializations) to transport a given vector in the fiber of $\gamma(t)$ to the fiber of $\gamma(0)$, where, in terms of the trivialization, they are identified using $f(0)$.

By Lemmas 2.5 and 2.3, together with the facts that $f(0)$ is an isometry and the curvatures of the bundles F_i are universally bounded, the norms

$$\|f(0)v - f(0)w\| = \|v - w\| \text{ and } \|f(0)w - f(t)v\|$$

are bounded up to a universal constant by the length of γ or $\phi_{\alpha,\beta} \circ \gamma$, respectively, and hence are bounded by $C' \cdot t$, where C' is a constant independent of i, α and β (note that $\|D\phi_{\alpha,\beta}\|$ is uniformly bounded as the supremum of finitely many compactly supported continuous functions). This implies the assertion of the proposition with $C := 2C'$.

q.e.d.

We call a continuous Banach-space bundle

$$F \hookrightarrow L \to M$$

(where the typical fiber F is a complex Banach space) a *Lipschitz bundle* if the following holds: There is an open covering $(U_j)_{j \in J}$ of M and there are trivializations

$$L|_{U_j} \cong U_j \times F$$

so that the associated transition functions

$$U_\alpha \cap U_\beta \to \operatorname{End}(F)$$

are locally Lipschitz continuous, $\operatorname{End}(F, F)$ being equipped with the operator norm.

In this context, Proposition 2.6 can be summarized as follows:

Theorem 2.7. *The bundles $F_i \to M$ can be assembled to a Lipschitz Hilbert A-module bundle*

$$L \to M$$

with typical fiber qA. The bundles $L \cdot A_i$ all have a smooth structure compatible with the induced Lipschitz structure and are isomorphic to F_i.

In the following, we will use results about Hilbert A-module bundles as explained in [**18**] where the role of smooth structures of Hilbert A-module bundles is quite carefully explained. Here, we will frequently use Lipschitz structures of such bundles (i.e. the transition functions of a Lipschitz atlas are (locally) Lipschitz continuous). It is straightforward to check that all the results described in [**18**] we are using here carry over immediately to the Lipschitz category.

In order to construct the bundle V described in Theorem 2.1, we use [**18**, Theorem 3.14] and write the bundle $L \to M$ as a subbundle of a trivial Hilbert A-module bundle

$$M \times A^k \to M.$$

Hence, L is the image of a projection valued *Lipschitz continuous* section ϕ of this bundle. The section ϕ can be approximated arbitrarily close (in the operator norm of $\mathrm{Hom}_A(A^k, A^k) = A^{k\times k}$ and the maximum norm on M) by a smooth projection valued section. The resulting bundle V (consisting of the images of these projections) is a smooth Hilbert A-module bundle. We choose the approximation close enough such that V is Lipschitz isomorphic as a Hilbert A-module bundle to the bundle L; in particular, it also has typical fiber isomorphic to qA (cf. [**18**, Lemma 3.12.]).

By the algebraic structure of A, also V has "blocks" $V \cdot A_i$ and, being an A-module bundle isomorphism, the isomorphism between V and L maps the blocks $V \cdot A_i$ to $L \cdot A_i$. By construction of V, this way we obtain Lipschitz Hilbert A_i-module bundle isomorphisms between the smooth bundles $V \cdot A_i$ and F_i which are therefore also smoothly isomorphic.

The trivializations $\psi_{i,j}$ assemble to Lipschitz continuous trivializations

$$L|_{W_j} \cong W_j \times qA.$$

On the other hand, we can choose smooth Hilbert A-module bundle trivializations

$$V|_{W_j} \cong W_j \times qA;$$

because the W_j are contractible and the typical fiber of V is isomorphic to qA.

Observe that $\mathrm{End}_A(qA) \cong qAq$, where $\mathrm{End}_A(qA)$ denotes the right A-module endomorphisms, and qAq acts by left multiplication. The isomorphism $L \cong V$ can hence be described by a Lipschitz continuous map

$$\tau_j \colon D^n \to qAq$$

with values in the unitary group of qAq.

The desired connection on V is now constructed as follows. Let $(W_j)_{j\in J}$ be the open covering of M from above and recall the trivializations

$$\psi_{i,j} \colon F_i|_{W_j} \cong D^n \times q_i\mathbb{K}\,.$$

For each i and j, the connection ∇_i induces a smooth connection 1-form in $\Gamma(T^*D^n \otimes \mathrm{End}_{\mathbb{K}}(F_i)) \cong \Omega^1(D^n; q_i\mathbb{K}q_i)$ using the trivialization $\psi_{i,j}$. Since the connection is a connection of Hilbert $\mathbb{K}$-modules, the values consist actually of skew-adjoint elements of q_iKq_i. Using the canonical chart on D^n, we consider these connections as smooth functions

$$\omega_{i,j} \colon D^n \to (q_i\mathbb{K}q_i)^n$$

and as such they have C^1-norms which are uniformly bounded in i and j. This follows from Lemma 2.3 and the curvature assumption on the sequence (∇_i). In particular, the functions $\omega_{i,j}$ can be assembled to Lipschitz continuous functions

$$\omega_j^L \colon D^n \to (qAq)^n.$$

The above isomorphism $L \cong V$ gives rise to induced connection forms

$$\omega_j^V \colon D^n \to (qAq)^n$$

equal to

$$((\omega_j^V)(x))_\nu = \tau_j(x) \circ ((\omega_j^L)(x))_\nu \circ \tau_j(x)^*$$

with $\nu = 1, \dots, n$. Unfortunately, the functions ω_j^V need not be smooth.

We choose $\epsilon > 0$ so small that the ϵ-neighborhood of D^n in $\mathbb{R}^n$ is still mapped diffeomorphically to an open subset of M by ϕ_j. Now, define a smooth function $\widetilde{\omega}_j^V \colon D^n \to (qAq)^n$ by

$$\widetilde{\omega}_j^V(x) := \int_{D^n} \delta_\epsilon(x-t)\omega_j^V(t)dt$$

using the Bochner integral and a smooth nonnegative bump function $\delta_\epsilon \colon D^n \to \mathbb{R}$ of total integral 1 whose support is contained in the ϵ-ball around 0. We consider ω_j^V as a smooth connection 1-form in $\Gamma(T^*D^n; \mathrm{End}_A(V)) = \Omega^1(D^n; qAq)$ and hence as a smooth A-linear connection

$$\nabla^{V,j} \in \Omega^1(W_j; qAq).$$

Let $\rho_j \colon M \to \mathbb{R}$ be a smooth partition of unity subordinate to the covering (W_j) and set

$$\nabla^V := \sum_{j\in J} \rho_j \cdot \nabla^{V,j}.$$

Then ∇^V is a Hilbert A-module connection on V, since the forms $\widetilde{\omega}^V_j$ are still skew-adjoint. Since it preserves the A-module structure, it also preserves the blocks $V \cdot A_i$. We claim that it has the asymptotic curvature properties stated in Theorem 2.1. We denote by

$$\omega^L_{i,j}\colon D^n \to (q_i \mathbb{K} q_i)^n,\ \omega^V_{i,j}\colon D^n \to (q_i \mathbb{K} q_i)^n\ ,\ \widetilde{\omega}^V_{i,j}\colon D^n \to (q_i \mathbb{K} q_i)^n$$

the connection forms that are induced by the projection $p_i\colon A \to A_i = \mathbb{K}$, i.e., $\omega^L_{i,j} = p_i \omega^L_j p_i$, etc. By construction,

$$\omega^L_{i,j} = \omega_{i,j}.$$

Because L and V are Lipschitz isomorphic with a global Lipschitz constant on M (with respect to the covering of M by the subsets W_j), there is a constant C such that we have an estimate of L^1-norms

$$\|\omega^V_{i,j}\|_1 \le C \cdot \|\omega^L_{i,j}\|_1 = C \cdot \|\omega_{i,j}\|_1.$$

Furthermore,

$$\widetilde{\omega}^V_{i,j} = \int_{D^n} \delta_\epsilon(x-t)\omega^V_{i,j}(t)dt.$$

The formula shows that we get pointwise bounds on $\widetilde{\omega}^V_{i,j}$ and its derivatives up to order d in terms of the sup-norm of the fixed function δ_ϵ and its derivatives up to order d and of the L^1-norm of $\omega^V_{i,j}$. Since the curvature of $\widetilde{\omega}^V_{i,j}$ is in local coordinates given by certain derivatives up to order 1 of $\widetilde{\omega}^V_{i,j}$ and because the derivatives of the functions ρ_j, the derivatives of the transition functions for the bundle V, and the derivatives of the chart transition functions (with respect to the cover $M \subset \bigcup W_j$) are globally bounded, the claim about the asymptotic behavior of the connection ∇^V follows.

Remark 2.8. An alternative construction of ∇^V consists of assembling the given connections ∇_i to a Lipschitz connection on L inducing a Lipschitz connection on V. This is then smoothed to yield the desired connection ∇^V. Our argument given before avoids the discussion of Lipschitz connections.

Remark 2.9. We have been careful to write down the rather explicit connection ∇^V with its curvature properties, because this is used in Proposition 3.4 to show that a suitable quotient bundle is *flat*, which is the main ingredient in the proof of our main result Theorem 3.8.

Alternatively, one could use a different argument to show directly that this quotient bundle admits a flat connection. We thank Ulrich Bunke for pointing out that this could be done by studying the parallel transport on the path groupoid of our manifold.

3. Almost flat bundles and index theory

This section provides a link between the construction from the last section and the index theory for Dirac operators.

Let M^{2n} be a closed oriented Riemannian manifold of even dimension and let $S \to M$ be a complex Dirac bundle equipped with a hermitian metric and a compatible connection (cf. [**13**, Definition 5.2]). As usual, Clifford multiplication with the complex volume element $i^n\omega_{\mathbb{C}}$ induces a splitting $S^{\pm} \to M$ into ± 1 eigenspaces. The corresponding Dirac type operator

$$D\colon \Gamma(S^+) \to \Gamma(S^-)$$

has an index in $K_0(\mathbb{C}) \cong \mathbb{Z}$. Denoting the universal cover of M by $\widetilde{M}$ and using the usual representation of $\pi_1(M)$ on $C^*_{\max}\pi_1(M)$, the maximal real C^*-algebra of $\pi_1(M)$, we obtain the flat Mishchenko line bundle

$$E := \widetilde{M} \times_{\pi_1(M)} C^*_{\max}\pi_1(M) \to M.$$

The twisted Dirac type operator

$$D \otimes \mathrm{id}\colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

has an index (cf. [**16**])

$$\alpha_S(M) \in K_0(C^*_{\max}\pi_1(M)).$$

In order to detect the non-triviality of $\alpha_S(M)$ in certain cases, we use a C^*-algebra morphism

$$C^*_{\max}(\pi_1(M)) \to Q$$

where Q is another C^*-algebra whose K-theory can be understood explicitly and study the image of $\alpha_S(M)$ under the induced map in K-theory.

First, we recall the following universal property of $C^*_{\max}\pi$ for a discrete group π: Each involutive multiplicative map

$$\pi \to C$$

with values in the unitaries of some unital C^*-algebra C can be extended to a unique C^*-algebra morphism

$$C^*_{\max}(\pi) \to C.$$

We now prove a naturality property of indices of twisted Dirac operators. In the following, we always use the maximal tensor product.

Lemma 3.1. *Let M be a compact oriented manifold of even dimension, $S \to M$ be a Dirac bundle, F and G be C^*-algebras and $\psi : F \to G$ be a C^*-algebra morphism. Further, let X be a Hilbert F-module bundle on M. We define the Hilbert G-module bundle*

$$Y := X \otimes_{\psi} G.$$

Let $[D_X] \in K_0(F)$ and $[D_Y] \in K_0(G)$ be the indices of the twisted Dirac type operators

$$\begin{aligned} D_X \colon \Gamma(S^+ \otimes X) &\to \Gamma(S^- \otimes X) \\ D_Y \colon \Gamma(S^+ \otimes Y) &\to \Gamma(S^- \otimes Y) \end{aligned}$$

(with an arbitrary F-module connection on X and G-module connection on Y). Then we have

$$[D_Y] = \psi_*([D_X]).$$

Proof. This follows from the functoriality of Kasparov's KK-machinery. We denote by $[D] \in KK(C(M), \mathbb{C})$ the KK-element defined by the Dirac operator $D \colon \Gamma(S^+) \to \Gamma(S^-)$ and by

$$[X] \in KK(\mathbb{C}, C(M) \otimes F)$$

the KK-element given by the Kasparov triple $(\Gamma(X), \mu_X, 0)$, where $\mu_X \colon C(M) \otimes F \to \mathbb{B}(\Gamma(X))$ is the map induced by the right F-module structure on X. Using the Kasparov intersection product, we get

$$[D_X] = [X] \otimes_{C(M)} [D] \in KK(\mathbb{C}, F)$$

and $[D_Y] \in KK(\mathbb{C}, G)$ is equal to

$$[(\Gamma(X \otimes_\psi G), \mu_Y, 0)] \otimes_{C(M)} [D] = [(\Gamma(X) \otimes_\psi G, \mu_Y, 0)] \otimes_{C(M)} [D].$$

By definition,

$$\psi_*[(\Gamma(X), \mu_F, 0)] = [\Gamma(X) \otimes_\psi G, \mu_Y, 0]$$

and our claim follows from the naturality of the Kasparov intersection product. For more details on the connection between the KK-description of the index and the usual definition in terms of kernel and cokernel, compare e.g., [**18**]. q.e.d.

Remark 3.2. Of course, in the situation of Lemma 3.1 there is also a corresponding index for odd dimensional manifolds, taking values in $K_1(F)$, with the corresponding properties.

Corollary 3.3. *Let*

$$\pi_1(M) \to \mathrm{U}(d)$$

be a finite dimensional and unitary representation with induced C^*-morphism $\psi \colon C^*_{\max}\pi_1(M) \to \mathrm{Mat}(\mathbb{C}, d)$. Let

$$\psi_* \colon K_0(C^*_{\max}\pi_1(M)) \to K_0(\mathrm{Mat}(\mathbb{C}, d)) \cong K_0(\mathbb{C}) = \mathbb{Z}$$

be the map induced by ψ. Then $\psi_*(\alpha_S(M))$ coincides with the index of the Dirac type operator D twisted by the bundle

$$\widetilde{M} \times_{\pi_1(M)} \mathbb{C}^d \to M.$$

Here we used the following well known instance of Morita equivalence: The index of D twisted with the Hilbert $\mathbb{C}$-module bundle (i.e., vector bundle) $\widetilde{M} \times_{\pi_1(M)} \mathbb{C}^d$ is equal to the $\mathrm{Mat}(\mathbb{C}, d)$-index of D twisted with the Hilbert $\mathrm{Mat}(\mathbb{C}, d)$-module bundle $\widetilde{M} \times_{\pi_1(M)} \mathrm{Mat}(\mathbb{C}, d)$.

Unfortunately, because the higher Chern classes of finite dimensional flat bundles vanish (using Chern-Weil theory), the element $\psi(\alpha_S(M))$ is simply equal to $d \cdot \mathrm{ind}(D)$.

We will now use the construction of Section 2 in order to get a useful *infinite dimensional* holonomy representation of $\pi_1(M)$.

Let $(P_i, \nabla_i)_{i \in \mathbb{N}}$ be a sequence of almost flat vector bundles on M and let $V \to M$ be the smooth Hilbert A-module bundle constructed in Theorem 2.1. Let

$$A' = \overline{\bigoplus_{i=1}^{\infty} \mathbb{K}} \subset A$$

be the closed two sided ideal consisting of sequences of elements in $\mathbb{K}$ that converge to 0 and let

$$Q := A/A'$$

be the quotient C^*-algebra. The bundle

$$W := V/(V \cdot A') \to M$$

is a smooth Hilbert Q-module bundle with fiber $\overline{q}Q$. Here $\overline{q}$ is the image of the projection $q \in A$ in Q. The connection ∇^V induces a connection on W. The following fact follows immediately from the construction of the bundle V.

Proposition 3.4. *The curvature form*

$$\Omega_V \in \Gamma(\Lambda^2 M \otimes \mathrm{End}_A(V))$$

can be considered as a form in

$$\Gamma(\Lambda^2(M) \otimes \mathrm{Hom}_A(V, V \cdot A')).$$

As a consequence, the induced connection on W is flat.

Fixing a base point $x \in M$ and an isomorphism of the fiber $W_x \cong \overline{q}Q$, the holonomy around loops based at x gives rise to a multiplicative involutive map

$$\pi_1(M, x) \to \mathrm{Hom}_Q(W_x, W_x) \cong \mathrm{Hom}_Q(\overline{q}Q, \overline{q}Q) = \overline{q}Q\overline{q}$$

with values in the unitaries of the subalgebra $\overline{q}Q\overline{q}$ of Q and hence to a map of C^*-algebras

$$\phi_1 \colon C^*_{\max}\pi_1(M) \to \overline{q}Q\overline{q} \to Q.$$

Let $\phi_2\colon C^*_{\max}\pi_1(M) \to Q$ be the homomorphism obtained by the same construction, but now applied to the sequence $P_i' := M \times U(d_i)$ of trivial bundles, with trivial (and hence flat) connections. Define

$$\phi_* := (\phi_1)_* - (\phi_2)_*\colon K_0(C^*_{\max}\pi_1(M)) \to K_0(Q). \tag{3.5}$$

It is not difficult to compute the K-theory of A and Q. Throughout the following argument, we work with the usual fixed isomorphism

$$\mathbb{Z} = K_0(\mathbb{K}).$$

Recall also that $K_1(\mathbb{K}) = 0$. Since K-theory commutes with direct limits, we obtain an isomorphism $K_0(A') \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}$ and $K_1(A') = 0$ (recall that A' is the ideal $\overline{\bigoplus_{i=1}^{\infty} \mathbb{K}}$ in A).

Proposition 3.6. *Let*

$$J \subset \prod_{i\in\mathbb{N}} \mathbb{Z}$$

be the subgroup consisting of sequences with only finitely many nonzero elements, i.e., $J = \bigoplus_{i=1}^{\infty} \mathbb{Z}$. *Then we have*

$$\begin{aligned} K_0(A) &\cong \prod_{i\in\mathbb{N}} \mathbb{Z}, \\ K_0(Q) &\cong \left(\prod \mathbb{Z}\right)/J. \end{aligned}$$

Under the above isomorphisms, the natural map $K_0(A) \to K_0(Q)$ *corresponds to the projection*

$$\prod \mathbb{Z} \to \left(\prod \mathbb{Z}\right)/J.$$

Proof. Observe that the projections $\{(p_{n_i})_{i\in\mathbb{N}} \mid n_i \in \mathbb{N}\}$ form an (uncountable) approximate unit of A, where $p_n \in \mathbb{K}$ is the standard projection of rank n. Consequently, A is stably unital in the sense of [**2**, Definition 5.5.4]. By [**2**, Proposition 5.5.5], elements in $K_0(A)$ are represented by formal differences of projections in

$$\mathrm{Mat}_\infty(A) = \mathrm{Mat}_\infty\left(\prod \mathbb{K}\right),$$

where Mat_∞ is the union of all the Mat_r. The main point of this stable unitality is that we don't have to adjoin a unit to A in order to compute K_0. By projecting to the different "coefficients" A_i we get an induced map

$$\chi\colon K_0(A) \to \prod_{i=1}^{\infty} K_0(\mathbb{K}) = \prod \mathbb{Z}.$$

Writing down appropriate projections, we see that χ is surjective. For the injectivity of χ, consider two projections $P, Q \in \mathrm{Mat}_r(A)$ such that for all $i \in \mathbb{N}$ the components $P_i, Q_i \in \mathrm{Mat}_r(\mathbb{K})$ are equivalent, where the subscript i indicates application of the projection $A = \prod_i \mathbb{K} \to A_i = \mathbb{K}$

onto the ith factor. We get a family of partial isometries $V_i \in \mathrm{Mat}_r(\mathbb{K})$ such that

$$V_i V_i^* = P_i \qquad V_i^* V_i = Q_i.$$

Because all the matrices V_i have norm 1, they can be assembled to a partial isometry $V \in \mathrm{Mat}_r(A)$ such that $VV^* = P$ and $V^*V = Q$.

The calculation of $K_0(Q)$ uses the exact sequence

$$K_0(A') \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(Q) \to K_1(A')$$

induced by the short exact sequence

$$0 \to A' \xrightarrow{\iota} A \xrightarrow{\pi} Q \to 0,$$

where $\iota\colon A' \to A$ and $\pi\colon A \to Q$ are the obvious maps. Since $K_1(A') = 0$, π_* is surjective.

The inclusion of the first k summands

$$\mathbb{K} \oplus \ldots \oplus \mathbb{K} \to A$$

induces an injective map

$$K_0(\mathbb{K} \oplus \ldots \oplus \mathbb{K}) \to K_0(A)$$

that can be identified with the inclusion

$$\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \to \prod \mathbb{Z}$$

onto the first k factors. The map ι_* is now given by passing to the direct limit of the last map, and this finishes our calculation of $K_0(Q)$. q.e.d.

Remark 3.7. In a similar way, it can be shown that

$$K_1(A) = K_1(Q) = 0.$$

Now we can formulate the following important fact which shows that the asymptotic index theoretic information of the sequence of almost flat bundles (P_i) is completely contained in $\alpha_S(M)$.

Theorem 3.8. *For all $i \in \mathbb{Z}$, define*

$$z_i := \mathrm{ind}(D_{E_i}) - d_i \,\mathrm{ind}(D) \in K_0(\mathbb{C}) = \mathbb{Z},$$

the index of the Dirac type operator D twisted by the virtual bundle $E_i - \underline{\mathbb{C}}^{d_i}$ where $E_i \to M$ is the d_i-dimensional unitary vector bundle with connection induced by the connection ∇_i on P_i, $S^{\pm} \otimes E_i$ is equipped with the product connection and $\underline{\mathbb{C}}$ is the trivial bundle. Then the element

$$\phi_*(\alpha_S(M)) \in K_0(Q)$$

is represented by

$$(z_i) \in \prod \mathbb{Z}.$$

Proof. The idea of the proof is to study the image of a (computable) index of the Dirac operator twisted with a *non-flat* bundle of A-modules over M under the canonical map

$$K_0(A) \to K_0(Q).$$

Using Lemma 3.1, this index turns out to be equal to the index of D twisted with a *flat* bundle which is induced by the given holonomy representation of $\pi_1(M)$ on Q. Therefore it is equal to $\phi_*(\alpha_S(M))$.

In order to make this idea precise, we consider the bundle of A-Hilbert modules $V \to M$ constructed in Theorem 2.1 and the element

$$[D_V] \in KK(\mathbb{C}, A) \cong K_0(A) = \prod \mathbb{Z}$$

represented by the Dirac operator

$$D_V : \Gamma(S^+ \otimes V) \to \Gamma(S^- \otimes V)$$

on M. For $i \in \mathbb{N}$ let

$$p : A \to \mathbb{K}$$

be the projection onto the ith factor. By Lemma 3.1, the induced map

$$p_* : K_0(A) \to K_0(\mathbb{K}) \cong \mathbb{Z}$$

sends $[D_V]$ to the index of $D_{P_i \times_{\mathrm{U}(d_i)} \mathbb{K}}$. Hence,

$$p_*([D_V]) = \mathrm{ind}(D_{E_i}).$$

If we carry out the same construction with the trivial $U(d_i)$-bundle P_i' we obtain

$$p_*([D_{V'}]) = d_i \, \mathrm{ind}(D),$$

and it follows that

$$[D_V] - [D_{V'}] = (z_1, z_2, z_3, \ldots).$$

Under the canonical map

$$K_0(A) \to K_0(Q),$$

the element $[D_V]$ is mapped to the element represented by the index of the Dirac operator D twisted with the flat bundle

$$W = \widetilde{M} \times_{\pi_1(M)} Q$$

using the holonomy representation ϕ_1 constructed from the (P_i) of $\pi_1(M)$ on Q. This element coincides with $(\phi_1)_*(\alpha_S(M))$. In a similar way, $(\phi_2)_*(\alpha_S(M))$ is the image of $[D_{V'}]$ under the canonical map $K_0(A) \to K_0(Q)$ and it remains to take the difference in order to finish the proof of Theorem 3.8. q.e.d.

The reason for using the virtual bundles $E_i - \underline{\mathbb{C}}^{d_i}$ will become apparent in the applications described in the next sections.

4. Enlargeability and universal index

For a closed spin manifold M^{2n} of even dimension, we consider the Dirac bundle $S \to M$ given by the complex spinor bundle on M. In this case, we define

$$\alpha_{\max}(M) \in K_{2n}(C^*_{\max}\pi_1(M)) = K_0(C^*_{\max}\pi_1(M))$$

to be equal to $\alpha_S(M) \in K_0(C^*_{\max}\pi_1(M))$ (cf. Section 3). If the dimension of M is odd, note that

$$K_0(C^*_{\max}(\pi_1(M) \times \mathbb{Z})) = K_0(C^*_{\max}\pi_1(M)) \otimes 1 \oplus K_1(C^*_{\max}\pi_1(M)) \otimes e,$$

using the exterior Kasparov product

$$K_*(C^*_{\max}\pi_1(M)) \otimes K_*(C^*\mathbb{Z}) \to K_*(C^*_{\max}(\pi_1(M) \times \mathbb{Z}))$$

with the canonical generators $1 \in K_0(C^*\mathbb{Z})$ and $e \in K_1(C^*\mathbb{Z})$. Using this splitting, we define $\alpha_{\max}(M) \in K_1(C^*_{\max}\pi_1(M))$ by requiring that

$$\alpha_{\max}(M) \otimes e = \alpha_{\max}(M \times S^1).$$

This is consistent with the direct definition of $\alpha_{\max}(M)$ alluded to in Remark 3.2 and the product formula [**21**, Theorem 9.20]

$$\alpha_{\max}(M \times S^1) = \alpha_{\max}(M) \otimes e, \text{ with } e = \alpha_{\max}(S^1).$$

The following fact is well known and can be proven in the usual way by an appropriate Weitzenböck formula.

Proposition 4.1. *Let M be a closed spin manifold. If M admits a metric of positive scalar curvature, then*

$$\alpha_{\max}(M) = 0.$$

Theorem 4.2. *Let M^m be an enlargeable or area-enlargeable spin manifold. Then*

$$\alpha_{\max}(M) \neq 0 \in K_m(C^*_{\max}(\pi_1(M))).$$

Proof. We first show how we can reduce to the case that M has even dimension. If not, consider the commutative diagram

$$\begin{array}{ccc}
K_m(M) & \xrightarrow{\times[S^1]} & K_{m+1}(M \times S^1) \\
\downarrow & & \downarrow \\
K_m(B\pi_1(M)) & \xrightarrow{\times[S^1]} & K_{m+1}(B\pi_1(M) \times B\mathbb{Z}) \\
\downarrow & & \downarrow \\
K_m(C^*_{\max}\pi_1(M)) & \xrightarrow{\times\alpha_{\max}(S^1)} & K_{m+1}(C^*_{\max}(\pi_1(M) \times \mathbb{Z})).
\end{array}$$

Here we use the fact that the α-index is multiplicative with respect to the exterior Kasparov product (note that $B\mathbb{Z} = S^1$ and $C^*_{\max}(\pi_1(M) \times \mathbb{Z}) = C^*_{\max}\pi_1(M) \otimes C^*\mathbb{Z}$, compare [**21**, Theorem 9.20]). Since $M \times S^1$ is (area)-enlargeable if M is, and the image of $\alpha_{\max}(M)$ under the bottom horizontal arrow is $\alpha_{\max}(M \times S^1)$, it suffices to treat non-vanishing of this invariant for even dimensional area-enlargeable spin manifolds.

Therefore, we assume that M has even dimension $2n$ so that $\alpha_{\max}(M)$ can be considered as an element in $K_0(C^*_{\max}\pi_1(M))$.

Because M is area-enlargeable, there is a sequence of almost flat principal unitary bundles (P_i) on M such that the Chern classes in $H^*(M;\mathbb{Z})$ of the associated (finite dimensional) complex vector bundles E_i satisfy

$$\begin{array}{rcl} c_\nu(E_i) & = & 0, \text{ if } 0 < \nu < n \\ \langle c_n(E_i), [M]\rangle & \neq & 0, \text{ if } \nu = n. \end{array}$$

Such a sequence can be constructed as follows: Because the Chern character

$$K^0(S^{2n}) \otimes \mathbb{Q} \to H^{\mathrm{even}}(S^{2n};\mathbb{Q})$$

is an isomorphism, there is a vector bundle

$$E \to S^{2n}$$

with

$$c_n(E) \neq 0 \in H^{2n}(S^{2n};\mathbb{Z}).$$

Pick a connection on E. Now let $i \in \mathbb{N}$ and choose a finite covering $\overline{M} \to M$ with covering group G such that there is a $\frac{1}{i}$-area contracting map $\psi\colon \overline{M} \to S^{2n}$ of nonzero degree. Passing to a finite cover of $\overline{M}$ if necessary, we can assume without loss of generality that the covering $\overline{M} \to M$ is regular. The G-action on $\overline{M}$ can be extended to an action of this group on

$$\bigoplus_{g\in G} g^*(\psi^*(E))$$

by vector bundle automorphisms. Note that the norm of the curvature of this direct sum of bundles is equal to the norm of the curvature of $\psi^*(E)$ and this is bounded by $\frac{1}{i}$ times the norm of the curvature of E as ψ is $\frac{1}{i}$-area contracting. (If the map was $\frac{1}{i}$-contracting, we would get a factor $\frac{1}{i^2}$). Let $E_i \to M$ be the quotient vector bundle. By naturality of Chern classes we have $c_n(E_i) \neq 0$ and $c_\nu(E_i) = 0$, if $0 < \nu < n$. The last statement is true, because the canonical map

$$H^*(M;\mathbb{Q}) \to H^*(\overline{M};\mathbb{Q})$$

is injective, transfer followed by division by n giving a splitting.

By construction, the Chern character of the virtual vector bundle $E_i - \underline{\mathbb{C}}^{d_i}$ is

$$\mathrm{ch}(E_i - \underline{\mathbb{C}}^{d_i}) = C \cdot c_n(E_i) \neq 0$$

with some non-zero constant C (dependent of n). In particular, $\mathrm{ch}(E_i - \underline{\mathbb{C}}^{d_i})$ is concentrated in degree $2n$.

The Atiyah-Singer index formula implies that the integer valued index in $K_0(\mathbb{C}) \cong \mathbb{Z}$ of the Dirac operator

$$D_{E_i - \underline{\mathbb{C}}^{d_i}} : \Gamma(S^+ \otimes (E_i - \underline{\mathbb{C}}^{d_i})) \to \Gamma(S^- \otimes (E_i - \underline{\mathbb{C}}^{d_i}))$$

is equal to

$$\langle \hat{\mathcal{A}}(TM) \cup \mathrm{ch}(E_i - \underline{\mathbb{C}}^{d_i}), [M] \rangle = C \cdot \langle c_n(E_i), [M] \rangle \neq 0$$

where $\hat{\mathcal{A}}$ denotes the total $\hat{A}$-class. Now, Theorem 3.8 implies our assertion. q.e.d.

5. On a question by Burghelea

Question ([**19**, Problem 11.1])**.** "If M^n is an enlargeable manifold and

$$f\colon M \to B\pi_1(M)$$

induces an isomorphism on the fundamental groups, does f_* map the fundamental class of $H_n(M;\mathbb{Q})$ non-trivially? Is the converse statement true?"

The next theorem answers the first question affirmatively. We can even drop any spin assumption on M or its universal cover. At the end of this section, we will show by an example that the converse of Burghelea's question in its stated form must be answered in the negative.

Theorem 5.1. *Let M be an enlargeable or area-enlargeable manifold of dimension m. Then*

$$f_*([M]) \neq 0 \in H_m(B\pi_1(M);\mathbb{Q}).$$

We first reduce to the case that M has even dimension $2n$. Else, observe that $M \times S^1$ also is enlargeable and we have the commutative diagram

$$\begin{array}{ccc} H_m(M;\mathbb{Q}) & \xrightarrow{\times[S^1]} & H_{m+1}(M\times S^1;\mathbb{Q}) \\ \downarrow & & \downarrow \\ H_m(B\pi_1(M);\mathbb{Q}) & \xrightarrow{\times[S^1]} & H_{m+1}(B(\pi_1(M)\times\mathbb{Z});\mathbb{Q})\,, \end{array}$$

where the image of the fundamental class of M is mapped to the image of the fundamental class of $M \times S^1$ under the bottom horizontal map.

Given M of dimension $2n$, we choose a sequence of almost flat bundles (E_i) as in the proof of Theorem 4.2. Now consider the commutative diagram

$$(5.2)\qquad \begin{array}{ccccc} K_0(M)\otimes\mathbb{Q} & \xrightarrow{f_*} & K_0(B\pi_1(M))\otimes\mathbb{Q} & \xrightarrow{\beta} & K_0(Q)\otimes\mathbb{Q} \\ {\scriptstyle\mathrm{ch}}\downarrow & & {\scriptstyle\mathrm{ch}}\downarrow & & {\scriptstyle=}\downarrow \\ H_{\mathrm{even}}(M;\mathbb{Q}) & \xrightarrow{f_*} & H_{\mathrm{even}}(B\pi_1(M);\mathbb{Q}) & \longrightarrow & K_0(Q)\otimes\mathbb{Q} \end{array}$$

In this diagram, the map β is induced by the composition of the assembly map

$$\mu\colon K_0(B\pi_1(M)) \to K_0(C^*_{\max}\pi_1(M))$$

with the map $\phi_*\colon K_0(C^*_{\max}\pi_1(M)) \to K_0(Q)$ which was defined in equation (3.5). Furthermore, ch denotes the homological Chern character.

We need the following description of the K-homology $K_0(M)$ from [**12**], Definition 2.6 and Lemma 2.8.

Proposition 5.3. *Let X be a connected CW-complex. Elements in $K_0(X)$ are represented by triples (N, S, u), where N is a closed oriented Riemannian manifold of even dimension (consisting of components of possibly different dimension), $S \to M$ is a Dirac bundle on M and $u\colon N \to X$ is a continuous map. Two such triples are identified, if they are equivalent under the equivalence relation generated by direct sum/disjoint union, bordism and vector bundle modification.*

Using vector bundle modification and the bordism relation introduced above, we can assume that in the triple (N, S, u) above, the manifold N is connected.

Now let M be the given manifold and let (N, S, u) represent an element $k \in K_0(M)$. Let $D\colon \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac type operator associated to the Dirac bundle S.

Lemma 5.4. *The element $\beta \circ f_*(k)$ (compare (5.2)) is represented by*

$$(z_1, z_2, \ldots) \in \prod \mathbb{Z} = K_0(A)$$

where

$$z_i = \operatorname{ind}(D_{u^*(E_i) - \underline{\mathbb{C}}^{d_i}})$$

is the index of D twisted by the virtual bundle $u^*(E_i) - \underline{\mathbb{C}}^{d_i}$.

The proof of this statement is analogous to the proof of Theorem 4.2. Note that the Kasparov KK-theory element in $K_0(M)$ represented by (N, S, u) is equal to

$$u_*([D])$$

where $[D] \in KK(\mathbb{C}(N), \mathbb{C})$ is the KK-element induced by D (cf. the explanations before Example 2.9 in [**12**]). In particular, the element

$$\mu \circ f_*(k) \in K_0(C^*_{\max}\pi_1(M))$$

is given as the index of the Dirac operator D twisted by $u^*(E)$, where $E \to M$ is the Mishchenko-Fomenko line bundle with fiber $C^*_{\max}\pi_1(M)$.

Lemma 5.5. *The element* $\operatorname{ch}(k) \in H_{\text{even}}(M; \mathbb{Q})$ *is equal to*

$$(-1)^n \cdot u_*\Big(\big(p_! \operatorname{ch}(\sigma(D)) \cup \mathcal{T}(TN \otimes \mathbb{C})\big) \cap [N]\Big),$$

where $\sigma(D)$ is the K-theoretic symbol class, $\mathcal{T}$ is the total Todd class and

$$p_!\colon H^*_c(TN;\mathbb{Q}) \to H^{*-\dim(N)}(N;\mathbb{Q})$$

is the Gysin map induced by the canonical projection $p\colon TN \to N$.

Proof. In a first step, one shows that the assignment

$$\omega\colon (N,S,u) \mapsto (-1)^n \cdot u_*\Big(\big(p_!\operatorname{ch}(\sigma(D)) \cup \mathcal{T}(TN\otimes\mathbb{C})\big) \cap [N]\Big)$$

is compatible with the equivalence relation on the set of triples (N,S,u) used in the definition of $K_0(M)$. For disjoint union and bordism, this is straightforward. The invariance under vector bundle modification uses the same calculation as in Section 7 of [**8**], p. 64. Consequently, ω induces an additive map

$$K_0(M)\otimes\mathbb{Q} \to H_{\mathrm{even}}(M;\mathbb{Q}).$$

In order to prove that this map is indeed equal to the homological Chern character, it is enough to consider triples (N,S,u), where N is a Spin^c-manifold and S is the canonical spinor bundle on N (cf. [**12**, 2.3]). But in this special case an explicit calculation shows that

$$(-1)^n \cdot p_!\operatorname{ch}(\sigma(D)\big) \cup \mathcal{T}(TN\otimes\mathbb{C}) = e^{\frac{1}{2}c}\cdot\hat{\mathcal{A}}(TN)$$

where $c \in H^2(N;\mathbb{Q})$ is the first Chern class of the complex line bundle associated to the Spin^c-structure on N. Now one uses the calculation of the homological Chern character in [**11**, 4.2]. q.e.d.

We continue the proof of Theorem 5.1. Let (N,S,u) be a triple (with connected N) representing an element in $K_0(M)\otimes\mathbb{Q}$ which under the homological Chern character is mapped to $q\cdot[M] \in H_{2n}(M;\mathbb{Q})$. Here, q denotes an appropriate nonzero rational number. As before, let $D\colon \Gamma(S^+) \to \Gamma(S^-)$ be the associated Dirac type operator. We will show that

$$\beta\circ f_*([N,S,u]) \neq 0 \in K_0(Q)$$

which implies the assertion of Theorem 5.1 by the commutativity of diagram (5.2).

Using Lemma 5.4,

$$\beta\circ f_*([N,S,u]) \in \prod \mathbb{Z}/\bigoplus\mathbb{Z} = K_0(Q)$$

is represented by the sequence $(z_1,z_2,\dots) \in \prod\mathbb{Z}$ where

$$z_i = \operatorname{ind}(D_{u^*(E_i)-\underline{\mathbb{C}}^{d_i}}).$$

By the Atiyah-Singer index theorem, this index is given by the zero dimensional component of the homology class

$$(-1)^n\Big(p_!\operatorname{ch}(\sigma(D))\cup\mathcal{T}(TN\otimes\mathbb{C})\cup\operatorname{ch}(u^*(E_i)-\underline{\mathbb{C}}^{d_i})\Big)\cap[N] \in H_*(N;\mathbb{Q}).$$

On the other hand, because

$$u_*\colon H_*(N;\mathbb{Q}) \to H_*(M;\mathbb{Q})$$

induces an isomorphism in degree 0, the number z_i is equal to the zero dimensional component of

$$\begin{aligned}
&u_*\Big((-1)^n\Big(p_!\,\mathrm{ch}(\sigma(D)) \cup \mathcal{T}(TN\otimes\mathbb{C}) \cup \mathrm{ch}\,\big(u^*(E_i)-\underline{\mathbb{C}}^{d_i}\big)\Big)\cap[N]\Big)\\
&= \mathrm{ch}(E_i-\underline{\mathbb{C}}^{d_i})\cap u_*\Bigg((-1)^n\Big(p_!\,\mathrm{ch}(\sigma(D))\cup\mathcal{T}(TN\otimes\mathbb{C})\Big)\cap[N]\Bigg)\\
&= q\cdot\big(\,\mathrm{ch}(E_i-\underline{\mathbb{C}}^{d_i})\cap[M]\big).
\end{aligned}$$

The last equality uses Lemma 5.5. Hence,

$$z_i = q\cdot C\cdot\langle c_n(E_i),[M]\rangle \in q\cdot(\mathbb{Z}\setminus\{0\})$$

by the construction of the sequence E_i (the constant C was introduced at the end of Section 4). It follows that $\beta\circ f_*([N,S,u])\neq 0$ and the proof of Theorem 5.1 is complete.

The following lemma prepares the construction of an example showing that the converse of Burghelea's question must be answered in the negative.

Lemma 5.6. *For every natural number $n>0$, there is a finitely presented group G without proper subgroups of finite index and such that*

$$H_n(G;\mathbb{Q})\neq 0.$$

Proof. The proof is modeled on a similar construction in [**1**] (cf. Theorem 6.1 and the following remarks in this reference). Let

$$K_1 := \langle a,b,c,d \mid a^{-1}ba=b^2, b^{-1}cb=c^2, c^{-1}dc=d^2, d^{-1}ad=a^2\rangle$$

be the Higman group [**10**]. This is a finitely presented infinite group without nontrivial finite quotients (and hence without proper subgroups of finite index). By [**1**], K_1 is acyclic and in particular

$$\widetilde{H}_*(K_1;\mathbb{Q})=0.$$

There is an element $z\in K_1$ generating a subgroup $G_1<K_1$ of infinite order. The amalgamated product

$$G_2 := K_1 *_{G_1} K_1$$

is finitely presented and still has no nontrivial finite quotients as one checks directly with help of the universal property of push-outs. The Mayer-Vietoris sequence shows that

$$H_*(G_2;\mathbb{Q})\cong H_*(S^2;\mathbb{Q}).$$

By [**1**, Theorem 6.1] the group G_2 embeds in the rationally acyclic group without nontrivial finite quotients

$$K_2 := (K_1 \times G_1) *_{G_1} K_1.$$

Here we identify G_1 on the left with the second factor of $K_1 \times G_1$. We set

$$G_3 := K_2 *_{G_2} K_2.$$

Again using the Meyer-Vietoris sequence,

$$H_*(G_3; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q}).$$

This process is now carried out inductively by embedding G_i in the rationally acyclic finitely presented group without nontrivial finite quotients

$$K_i := (K_{i-1} \times G_{i-1}) *_{G_{i-1}} K_{i-1}$$

and defining

$$G_{i+1} := K_i *_{G_i} K_i.$$

The group $G := G_n$ then has the desired properties. q.e.d.

The following theorem provides a negative answer to the converse of the question by Burghelea.

Theorem 5.7. *Let $n \geq 4$ be a natural number. Then there exists a closed connected n-dimensional spin manifold M which is not area-enlargeable, but whose classifying map $M \to B\pi_1(M)$ sends the fundamental class of M to a nonzero class in $H_n(B\pi_1(M); \mathbb{Q})$.*

Proof. Let G be the group constructed in Lemma 5.6 for the number n. The Atiyah-Hirzebruch spectral sequence computing the rational spin bordism of BG (or any other space) collapses at E_2, and hence there is a closed n-dimensional spin manifold N together with a map $f\colon N \to BG$ such that

$$f_*([N]) \neq 0 \in H_n(BG, \mathbb{Q}).$$

Because $n \geq 4$ and because G is finitely presented, there is also an n-dimensional closed spin manifold A with fundamental group G. Let $g\colon A \to BG$ denote the classifying map and consider the map

$$N \sharp A \xrightarrow{f \sharp g} BG.$$

This map is surjective on π_1 and sends the fundamental class of $N\sharp A$ to a nontrivial class in $H_n(BG; \mathbb{Q})$ (if this is not the case, simply take the connected sum with more copies of A). Carrying out spin surgery on $N\sharp A$ over BG in order to kill the kernel of $\pi_1(f\sharp g)$, we obtain a spin manifold M with fundamental group G and such that the classifying map $M \to BG$ sends the fundamental class of M to a nontrivial class in $H_n(BG; \mathbb{Q})$. However, G does not have any proper subgroups of finite index and therefore M does not have any nontrivial finite covers whatsoever. Consequently, M is not area-enlargeable. q.e.d.

Remark 5.8. It is not clear if the converse of Burghelea's question has an affirmative answer when working with a notion of enlargeability allowing infinite covers.

We address the relation between the corresponding Gromov-Lawson obstruction to positive scalar curvature [**6**] and $\alpha_{\max}$ at [**9**].

6. Concluding remarks

In [**6**], it is shown that enlargeability is a homotopy invariant and is preserved under some natural geometric constructions such as taking connected sums or taking the Cartesian product of two enlargeable manifolds. If all manifolds under consideration are spin, then using the universal property of $C^*_{\max}$, one can show by purely formal arguments that the manifolds resulting from these constructions have non-vanishing $\alpha_{\max}$. On the other hand, this reasoning can be used to prove the non-vanishing of $\alpha_{\max}$ in some cases that do not seem to be accessible to the classical geometric arguments by Gromov and Lawson. For example, using the methods developed in this paper, one can show

Proposition 6.1. *Let F and M be connected enlargeable spin manifolds of even dimension. Let*

$$F \hookrightarrow E \to M$$

be a smooth fiber bundle admitting a spin structure and inducing a split short exact sequence

$$1 \to \pi_1(F) \to \pi_1(E) \to \pi_1(M) \to 1$$

which is equivalent to the canonical split sequence

$$1 \to \pi_1(F) \to \pi_1(F) \times \pi_1(M) \to \pi_1(M) \to 1.$$

Then $\alpha_{\max}(E) \neq 0$ and in particular E does not admit a metric of positive scalar curvature.

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