J. DIFFERENTIAL GEOMETRY
**74** (2006) 119-154

# UNIQUENESS OF THE RICCI FLOW ON COMPLETE NONCOMPACT MANIFOLDS

Bing-Long Chen & Xi-Ping Zhu

**Abstract**

The Ricci flow is an evolution system on metrics. For a given metric as initial data, its local existence and uniqueness on compact manifolds were first established by Hamilton [**9**]. Later on, De Turck [**5**] gave a simplified proof. In the later part of 80's, Shi [**21**] generalized the local existence result to complete noncompact manifolds. However, the uniqueness of the solutions to the Ricci flow on complete noncompact manifolds is still an open question. In this paper, we give an affirmative answer for the uniqueness question. More precisely, we prove that the solution of the Ricci flow with bounded curvature on a complete noncompact manifold is unique.

## 1. Introduction

Let $(M^n, g_{ij})$ be a complete Riemannian (compact or noncompact) manifold. The Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \qquad \text{for } x \in M^n \text{ and } t \geq 0, \tag{1.1}$$

with $g_{ij}(x,0) = g_{ij}(x)$, is a weakly parabolic system on metrics. This evolution system was introduced by Hamilton in [**9**]. Now it has proved to be powerful in the research of differential geometry and lower dimensional topology (see for example Hamilton's works [**9**], [**10**], [**11**], [**14**] and the recent works of Perelman [**17**], [**18**]). The first matter for the Ricci flow (1.1) is the short time existence and uniqueness of the solutions. When the manifold $M^n$ is compact, Hamilton proved in [**9**] that the Ricci flow (1.1) has a unique solution for a short time. So the problem has been well-settled on compact manifolds. In [**5**], De Turck introduced an elegant trick to give a simplified proof. Later on, Shi [**21**] extended the short time existence result to noncompact manifolds. More precisely, Shi [**21**] proved that if $(M^n, g_{ij})$ is complete noncompact with bounded curvature, then the Ricci flow (1.1) has a solution with bounded curvature on a short time interval. In this paper, we

Received 05/30/2005.

will deal with the uniqueness of the Ricci flow on complete noncompact manifolds.

The uniqueness of the Ricci flow has been used in the theory of the Ricci flow with surgery (see for example [**17**], [**18**], [**1**] and [**3**]). When we consider the Ricci flow on a compact manifold, the Ricci flow will generally develop singularities in finite time. In the theory of the Ricci flow with surgery, one eliminates the singularities by Hamilton's geometric surgeries (cut off the high curvature part and glue back a standard cap, then run the Ricci flow again). An important question in this theory is to control the geometry of the glued cap after surgery. The uniqueness theorem of the Ricci flow insures that the solution on glued cap is sufficiently close to a (complete noncompact) standard solution, which is the evolution of capped round cylinder. Then we can apply the curvature estimate [**18**], [**3**] and the canonical neighborhood decomposition [**3**], [**1**] of the standard solutions to get the desired control. So even if we consider the Ricci flow on compact manifolds, we still have to encounter the problem of uniqueness on noncompact manifolds.

It is well-known that the uniqueness of the solution of a parabolic system on a complete noncompact manifold does not always hold if one does not impose any growth condition of the solutions. For example, even the simplest linear heat equation on $\mathbb{R}$ with zero as initial data has a nontrivial solution which might grow faster than $e^{a|x|^2}$ for any $a > 0$ whenever $t > 0$. This says, for the standard linear heat equation, the most growth rate for the uniqueness is $e^{a|x|^2}$. Note that in a Kähler manifold, the Ricci curvature is given by

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{k\bar{l}}).$$

Thus the reasonable growth rate that we can expect for the uniqueness of the Ricci flow is the solution with bounded curvature.

In this paper, we will prove the following uniqueness theorem of the Ricci flow.

**Theorem 1.1.** *Let $(M^n, g_{ij}(x))$ be a complete noncompact Riemannian manifold of dimension $n$ with bounded curvature. Let $g_{ij}(x,t)$ and $\bar{g}_{ij}(x,t)$ be two solutions to the Ricci flow on $M^n \times [0,T]$ with the same $g_{ij}(x)$ as initial data and with bounded curvatures. Then $g_{ij}(x,t) = \bar{g}_{ij}(x,t)$ for all $(x,t) \in M^n \times [0,T]$.*

Since the Ricci flow is not a strictly parabolic system, our argument will apply the De Turck trick. This is to consider the composition of the Ricci flow with a family of diffeomorphisms generated by the harmonic map flow. By pulling back the Ricci flow by this family of diffeomorphisms, the evolution equations become strictly parabolic. In order to use the uniqueness theorem of a strict parabolic system on a noncompact manifold, we have to overcome two difficulties. The first one is to establish a short time existence for the harmonic map flow between noncompact manifolds. The second one is to get a priori estimates for the harmonic map flow so that after pulling backs, the solutions to the strictly parabolic system still satisfy suitable growth conditions. To the best of our knowledge, one can only get short time existence of harmonic map flow by imposing negative curvature or convex condition on the target manifolds (see for example, [**7**] and [**6**]) or by simply assuming the image of initial data lying in a compact domain on the target manifold (see for example [**16**]).

There are two main contributions in this paper. The first one is that we observe that the condition of injectivity radius bounded from below by a positive constant ensures certain uniform (local) convexity and this is sufficient to give the short time existence and the a priori estimates for the harmonic map flow. But there are examples of complete Riemannian manifolds of bounded curvature whose injectivity radius decays to zero at infinity. Fortunately, from [**4**] or [**2**], we know that the injectivity radius of manifolds of bounded curvature decays at worst exponentially. The second contribution of this paper is to handle the case of injectivity radius decaying to zero at infinity. Our idea is to study the evolution equations coming from the composition of the Ricci flow and harmonic map flow, as well as a conformal change at infinity by an exponential factor. This new approach has the advantage of transforming the Ricci flow equation to a strictly parabolic system on a manifold with uniform geometry at infinity. This technique may potentially be used in dealing with the Ricci flow on manifolds with unbounded curvature.

As a direct consequence, we have the following result.

**Corollary 1.2.** *Suppose $(M^n, g_{ij}(x))$ is a complete Riemannian manifold, and suppose $g_{ij}(x,t)$ is a solution to the Ricci flow with bounded curvature on $M^n \times [0,T]$ and with $g_{ij}(x)$ as initial data. If $G$ is the isometry group of $(M^n, g_{ij}(x))$, then $G$ remains to be an isometric subgroup of $(M^n, g_{ij}(x,t))$ for each $t \in [0,T]$.*

This paper is organized as follows. In Section 2, we study the harmonic map flow coupled with the Ricci flow. In Section 3, we study the Ricci-De Turck flow and prove the uniqueness theorem.

We are grateful to Professor S.-T. Yau for many helpful discussions and his encouragement. The first author is supported by FANEDD 200216 and NSFC 10401042 and the second author is also partially supported by NSFC and the IMS of The Chinese University of Hong Kong.

## 2. Harmonic map flow coupled with the Ricci flow

Let $(M^n, g_{ij}(x))$ and $(N^m, h_{ij}(y))$ be two Riemannian manifolds, $f : M^n \to N^m$ be a map. The harmonic map flow is the following evolution equation for maps from $M^n$ to $N^m$,

$$\begin{cases} \frac{\partial}{\partial t} f(x,t) = \triangle f(x,t), & \text{for } x \in M^n, t > 0, \\ f(x,0) = f(x), & \text{for } x \in M^n, \end{cases} \tag{2.1}$$

where $\triangle$ is defined by using the metrics $g_{ij}(x)$ and $h_{\alpha\beta}(y)$ as follows

$$\triangle f^\alpha(x,t) = g^{ij}(x)\nabla_i\nabla_j f^\alpha(x,t),$$

and

$$\nabla_i\nabla_j f^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}. \tag{2.2}$$

Here we use $\{x^i\}$ and $\{y^\alpha\}$ to denote the local coordinates of $M^n$ and $N^m$ respectively, $\Gamma^k_{ij}$ and $\Gamma^\alpha_{\beta\gamma}$ the corresponding Christoffel symbols of $g_{ij}$ and $h_{\alpha\beta}$.

Let $g_{ij}(x,t)$ be a complete smooth solution of the Ricci flow with $g_{ij}(x)$ as initial data; then the harmonic map flow coupled with Ricci flow is the following equation:

$$\begin{cases} \frac{\partial}{\partial t} f(x,t) = \triangle_t f(x,t), & \text{for } x \in M^n, t > 0, \\ f(x,0) = f(x), & \text{for } x \in M^n, \end{cases}$$

where $\triangle_t$ is defined as above by using the metrics $g_{ij}(x,t)$ and $h_{\alpha\beta}(y)$.

Suppose $g_{ij}(x,t)$ is a solution to the Ricci flow on $M^n \times [0,T]$ with bounded curvature

$$|Rm|(x,t) \leq k_0$$

for all $(x,t) \in M^n \times [0,T]$. Let $(N^n, h_{\alpha\beta}) = (M^n, g_{ij}(\cdot,T))$ be the target manifold. The purpose of this section is to prove the following theorem:

**Theorem 2.1.** *There exists $0 < T_0 < T$, depending only on $k_0$, $T$ and $n$ such that the harmonic map flow coupled with the Ricci flow*

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = \triangle_t F(x,t), \\ F(\cdot,0) = identity, \end{cases} \tag{2.3}$$

*has a solution on $M^n \times [0,T_0]$ satisfying the following estimates:*

$$\begin{aligned} &|\nabla F| \leq \tilde{C}_1, \\ &|\nabla^k F| \leq \tilde{C}_k t^{-\frac{k-2}{2}}, \qquad \textit{for all } k \geq 2, \end{aligned} \tag{2.4}$$

*for some constants $\tilde{C}_k$ depending only on $k_0$, $T$, $k$ and $n$.*

The proof will occupy the rest of this section.

**2.1. Expanding base and target metrics at infinity.** We will construct appropriate auxiliary functions on $M^n$ and $N^n$ and do conformal deformations for the base and the target metrics. Firstly, we construct suitable positive functions on $(N^n, h_{\alpha\beta})$. These functions can be obtained by smoothing certain functions by convolution [**8**] (or by solving certain differential equations [**19**]).

**Lemma 2.2.** *Fix $p \in N^n$. Then for any $a \geqslant 1$, there exists a $C^\infty$ nonnegative function $\varphi_a$ on $N^n$ such that*

$$
\begin{cases}
\varphi_a(y) \equiv 0 & \text{on } \ B(p,a), \\
d(y,p) \leqslant \varphi_a(y) \leqslant C_0 d(y,p) & \text{on } \ N^n \backslash B(p,2a), \\
|\nabla^k \varphi_a| \leqslant C_k & \text{on } \ N^n, \quad \text{for } k \geqslant 1,
\end{cases}
\tag{2.5}
$$

*where $C_k$, $k = 0, 1, 2, \dots$, are constants depending only on $k_0$ and $T$; the distance $d(y,p)$, the covariant derivatives $\nabla^k \varphi_a$ and the norms $|\nabla^k \varphi_a|$ are computed by using the metric $h_{\alpha\beta}$.*

*Proof.* Let $\xi$ be a smooth nonnegative increasing function on $\mathbb{R}$ such that $\xi(s) = 0$ for $s \in (-\infty, \frac{5}{4}]$, and $\xi = 1$ for $s \in [\frac{7}{4}, \infty)$. For each $y \in N^n$, by averaging the functions $\xi(\frac{d(p,y)}{a})$ and $d(p,y)$ over a suitable ball of the tangent space $T_y N^n$ (see for example [**8**]), we obtain two smooth functions $\xi_a$ and $\rho$. Notice that $(N^n, h_{\alpha\beta}) = (M^n, g_{ij}(\cdot, T))$; thus all the covariant derivatives of the curvatures of $h_{\alpha\beta}$ are bounded by using Shi's gradient estimates [**21**]. Then $\varphi_a = C\xi_a\rho$, for some constant $C$ depending only on $k_0$ and $T$, is the desired function. q.e.d.

Recall from [**4**] and [**2**] that on a complete manifold with bounded curvature, the injectivity radius decays at worst exponentially; more precisely, there exists a constant $\tilde{C}(n) > 0$ depending only on the dimension, and there exists a constant $\delta > 0$ depending on $n$, $k_0$ and the injectivity radius at $p$ such that

$$
\operatorname{inj}(N^n, h_{\alpha\beta}, y) \geqslant \delta e^{-\tilde{C}(n)\sqrt{k_0}\, d(y,p)}. \tag{2.6}
$$

Fix $a \geqslant 1$, let $\varphi^a = 4\tilde{C}(n)\sqrt{k_0}\varphi_a$ and set

$$
h^a_{\alpha\beta} = e^{\varphi^a} h_{\alpha\beta}. \tag{2.7}
$$

Clearly, $h^a_{\alpha\beta} = h_{\alpha\beta}$ on $B(p,a)$. Note that $(N^n, h_{\alpha\beta}) = (M^n, g_{ij}(\cdot, T))$, so the function $\varphi_a$ is also a function on $M^n$. Let

$$
g^a_{ij}(x,t) = e^{\varphi^a} g_{ij}(x,t) \tag{2.8}
$$

be the new family of metrics on $M^n$. Instead of (2.3), we will consider a new harmonic map flow

$$
\begin{cases}
\frac{\partial}{\partial t}\overset{a}{F}(x,t) = \overset{a}{\triangle}_t \overset{a}{F}(x,t), \\
\overset{a}{F}(\cdot, 0) = \text{identity},
\end{cases}
\tag{2.3$_\mathrm{a}$}
$$

where $\overset{a}{\triangle}_t \overset{a}{F}$ is defined by using the metrics $g^a_{ij}(x,t)$ and $h^a_{\alpha\beta}(y)$.

Before we solve $(2.3)_a$, we have to discuss the geometry of the new metrics $h^a_{\alpha\beta}(y)$ and $g^a_{ij}(x,t)$. Let us first compute the curvature and its covariant derivatives and injectivity radius of $(N^n, h^a_{\alpha\beta})$ as follows.

By a direct computation, we get

$$\begin{aligned}
(2.9)\quad & \overset{a}{R}_{\alpha\beta\gamma\delta} \\
&= e^{\varphi^a} R_{\alpha\beta\gamma\delta} + \frac{e^{\varphi^a}}{4}\{|\nabla\varphi^a|^2(h_{\alpha\delta}h_{\beta\gamma} - h_{\alpha\gamma}h_{\beta\delta}) \\
&\quad + (2\nabla_\alpha\nabla_\delta\varphi^a - \nabla_\alpha\varphi^a\nabla_\delta\varphi^a)h_{\beta\gamma} + (2\nabla_\beta\nabla_\gamma\varphi^a - \nabla_\beta\varphi^a\nabla_\gamma\varphi^a)h_{\alpha\delta} \\
&\quad - (2\nabla_\beta\nabla_\delta\varphi^a - \nabla_\beta\varphi^a\nabla_\delta\varphi^a)h_{\alpha\gamma} - (2\nabla_\alpha\nabla_\gamma\varphi^a - \nabla_\alpha\varphi^a\nabla_\gamma\varphi^a)h_{\beta\delta}\}
\end{aligned}$$

where $\overset{a}{R}_{\alpha\beta\gamma\delta}$ is the curvature of $h^a_{\alpha\beta}$, $\nabla_\alpha\varphi^a$,$\nabla_\alpha\nabla_\delta\varphi^a$ and $|\nabla_\alpha\varphi^a|$ are computed by the metric $h_{\alpha\beta}$. Therefore, by combining with (2.5), we have

$$\begin{aligned}
(2.10)\qquad |\overset{a}{R}_m|_{h^a} &\leqslant e^{-\varphi^a}(k_0 + C(n)(C_2 + C_1^2)) \\
&< \infty.
\end{aligned}$$

For higher derivatives, we rewrite (2.9) in a simple form

$$\overset{a}{R}_m = e^{\varphi^a}\{R_m + \nabla\varphi^a * \nabla\varphi^a * h^2 * h^{-1} + \nabla^2\varphi^a * h\}$$

where we use $A{*}B$ to express some linear combinations of tensors formed by contractions of tensor product of $A$ and $B$. Note that

$$\begin{aligned}
\overset{a}{\Gamma}{}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} &= \frac{1}{2}[\nabla_\beta\varphi^a\delta^\alpha_\gamma + \nabla_\gamma\varphi^a\delta^\alpha_\beta - h^{\alpha\eta}h_{\beta\gamma}\nabla_\eta\varphi^a] \\
&= (\nabla\varphi^a * h * h^{-1})^\alpha_{\beta\gamma},
\end{aligned}$$

so by induction, we have

$$\begin{aligned}
(2.11)\qquad \overset{a}{\nabla}{}^k \overset{a}{R}_m &= \nabla\overset{a}{\nabla}{}^{k-1}\overset{a}{R}_m + (\overset{a}{\Gamma} - \Gamma) * \overset{a}{\nabla}{}^{k-1}\overset{a}{R}_m \\
&= e^{\varphi^a}\Bigg\{\sum_{l=0}^{k}\nabla^l R_m * \sum_{i_1+\cdots+i_p=k-l}\nabla^{i_1}\varphi^a * \cdots * \nabla^{i_p}\varphi^a \\
&\qquad + \sum_{i_1+\cdots+i_p=k+2}\nabla^{i_1}\varphi^a * \cdots * \nabla^{i_p}\varphi^a\Bigg\},
\end{aligned}$$

where we denote $\nabla^0\varphi^a = 1$. By combining with (2.5) and gradient estimate of Shi [**21**], we get

(2.12)
$$\begin{aligned}|\overset{a}{\nabla}{}^k \overset{a}{R}_m|_{h^a} &\leqslant e^{-\frac{k+2}{2}\varphi^a} C(n,k_0,k,C_1,\dots,C_{k+2}) \left(\sum_{l=0}^{k} |\nabla^l R_m| + 1\right)\\ &\leqslant C(n,k_0,T,k)e^{-\frac{k+2}{2}\varphi^a}\\ &\leqslant C(n,k_0,T,k).\end{aligned}$$

For the injectivity radius of $h^a_{\alpha\beta}$, we know from (2.5) and (2.7) that for any $y \in N^n \backslash B(p, 2a+1)$,

$$\overset{a}{B}(y,1) \supset B\left(y, e^{-2\tilde{C}(n)\sqrt{k_0}(\varphi_a + C_1)}\right)$$

and

(2.13)
$$\begin{aligned}\mathrm{Vol}_{\,h^a}(\overset{a}{B}(y,1)) &= \int_{\overset{a}{B}(y,1)} (e^{4\tilde{C}(n)\sqrt{k_0}\varphi_a})^{\frac{n}{2}}\\ &\geqslant e^{2n\tilde{C}(n)\sqrt{k_0}(\varphi_a - C_1)} \mathrm{Vol}_{\,h}(B(y, e^{-2\tilde{C}(n)\sqrt{k_0}(\varphi_a + C_1)}))\end{aligned}$$

where we denote by $\overset{a}{B}(y,1)$ the ball centered at $y$ and of radius 1 with respect to metric $h^a_{\alpha\beta}$, and $\mathrm{Vol}_{\,h^a}(\overset{a}{B}(y,1))$ its volume.

Since
$$\varphi_a(y) \geqslant d(y,p),$$
for $y \in N^n \backslash B(p, 2a+1)$, there holds

$$e^{-2\tilde{C}(n)\sqrt{k_0}(\varphi_a + C_1)} \leqslant \delta e^{-\tilde{C}(n)\sqrt{k_0}d(y,p)} \tag{2.14}$$

for $y \in N^n \backslash B(p, 2a+1+|\frac{\log \delta^{-1}}{\tilde{C}(n)\sqrt{k_0}}|)$. By (2.6), (2.10), (2.13), (2.14) and volume comparison theorem, we have

$$\begin{aligned}\mathrm{Vol}_{\,h^a}(\overset{a}{B}(y,1)) &\geqslant c(n,k_0)e^{2n\tilde{C}(n)\sqrt{k_0}(\varphi_a - C_1)}(e^{-2\tilde{C}(n)\sqrt{k_0}(\varphi_a + C_1)})^n\\ &\geqslant c(n,k_0).\end{aligned}$$

By combining this with the local injectivity radius estimate in [**4**] or [**2**], we get
$$\mathrm{inj}\,(N^n, h^a, y) \geqslant \tilde{C}(n,k_0) > 0,$$
for $y \in N^n \backslash B\left(p, 2a+1+\left|\frac{\log \delta^{-1}}{\tilde{C}(n)\sqrt{k_0}}\right|\right)$.

Consequently, we have proved the following lemma:

**Lemma 2.3.** *There exists a sequence of constants $\bar{C}_0$, $\bar{C}_1$, …, with the following property. For all $a \geqslant 1$, there exists $i_a > 0$, such that the* *metrics* $h^a_{\alpha\beta} = e^{\varphi^a} h_{\alpha\beta}$ *on* $N^n$ *satisfy*

$$\begin{aligned} &|\overset{a}{\nabla}{}^k \overset{a}{R}_m|_{h^a} \leqslant \bar{C}_k e^{-\frac{k+2}{2}\varphi^a} \leqslant \bar{C}_k \\ &\mathrm{inj}\,(N^n, h^a_{\alpha\beta}) \geqslant i_a > 0 \end{aligned} \tag{2.15}$$

*for* $k = 0, 1, \ldots$.

We next estimate the curvature and its covariant derivatives of $g^a_{ij}(x,t) = e^{\varphi^a} g_{ij}(x,t)$.

By the Ricci flow equation, we have

$$\begin{aligned} &\Gamma^l_{ij}(\cdot,T) - \Gamma^l_{ij}(\cdot,t) = \int_t^T (g^{-1} * \nabla \mathrm{Ric}\,)(\cdot,s)ds, \\ &\nabla^k_{g(\cdot,T)}(\Gamma(\cdot,T) - \Gamma(\cdot,t)) \\ &= \int_t^T \sum_{l=0}^{k} \nabla^{k+1-l}\mathrm{Ric} * \sum_{i_1+1+\cdots+i_p+1=l} \nabla^{i_1}_{g(\cdot,T)}(\Gamma(\cdot,s) - \Gamma(\cdot,T)) \\ &\qquad * \cdots * \nabla^{i_p}_{g(\cdot,T)}(\Gamma(\cdot,s) - \Gamma(\cdot,T)) * g^k * g^{-(k+1)}(\cdot,s)ds. \end{aligned} \tag{2.16}$$

By combining with the gradient estimates of Shi [**21**] and induction on $k$, we have

$$\begin{cases} |\Gamma(\cdot,T) - \Gamma(\cdot,t)| \leqslant C(n,k_0,T) \int_t^T \frac{1}{\sqrt{s}} ds, \\ |\nabla_{g(\cdot,T)}(\Gamma(\cdot,T) - \Gamma(\cdot,t))| \leqslant C(n,k_0,T)(1 + |\log t|), \\ |\nabla^k_{g(\cdot,T)}(\Gamma(\cdot,T) - \Gamma(\cdot,t))| \leqslant C(n,k_0,T,k) t^{-\frac{k-1}{2}}, \quad \text{for } k \geq 2. \end{cases} \tag{2.17}$$

Since

$$\begin{aligned} \nabla^k_{g(\cdot,t)}\varphi^a = \sum_{l=0}^{k-1} \nabla^{k-l}_{g(\cdot,T)}\varphi^a * \sum_{i_1+1+\cdots+i_p+1=l} \nabla^{i_1}_{g(\cdot,T)}(\Gamma(\cdot,t) - \Gamma(\cdot,T)) \\ * \cdots * \nabla^{i_p}_{g(\cdot,T)}(\Gamma(\cdot,t) - \Gamma(\cdot,T)) \end{aligned}$$

for $k \geqslant 1$, the combination with (2.17) and (2.5) gives

$$\begin{cases} |\nabla_{g(\cdot,t)}\varphi^a| + |\nabla^2_{g(\cdot,t)}\varphi^a| \leqslant C(n,k_0,T), \\ |\nabla^3_{g(\cdot,t)}\varphi^a| \leqslant C(n,k_0,T)(1 + |\log t|), \\ |\nabla^k_{g(\cdot,t)}\varphi^a| \leqslant C(n,k_0,T,k) t^{-\frac{k-3}{2}}, \qquad \text{for} \ \ k \geqslant 4. \end{cases} \tag{2.18}$$

Then by combining (2.11) and (2.18), the curvature and the covariant derivatives of $g^a(\cdot,t)$ can be estimated as follows

$$|\overset{a}{\nabla}{}^k \overset{a}{R}_m|_{g^a(\cdot,t)} \leqslant C(n,k_0,T,k) e^{-\frac{k+2}{2}\varphi^a} t^{-\frac{k}{2}}, \quad \text{for} \ \ k \geqslant 0.$$

Summing up, the above estimates give the following

**Lemma 2.4.** *There exists a sequence of constants $\bar{k}_0$, $\bar{k}_1$, …, with the following property. For all $a \geqslant 1$, the metrics $g^a_{ij}(\cdot,t) = e^{\varphi^a} g_{ij}(\cdot,t)$ on $M^n$ satisfy*

$$|\overset{a}{\nabla}{}^l \overset{a}{R}_m|_{g^a(\cdot,t)} \leqslant \bar{k}_l e^{-\frac{l+2}{2}\varphi^a} t^{-\frac{l}{2}}, \quad \text{for } l \geqslant 0, \tag{2.19}$$

*on $M^n \times [0,T]$.*

We remark that the fact that the curvatures of $h^a_{\alpha\beta}$ and $g^a_{ij}(\cdot,t)$ are uniformly bounded (independent of $a$) is essential in our argument, while the injectivity radius bound $i_a$ may depend on $a$.

For the new family of metrics $g^a_{ij}(\cdot,t)$, we have the following lemma.

**Lemma 2.5.**

$$\begin{aligned}
\frac{\partial}{\partial t} g^a_{ij} &= e^{\varphi^a}(-2\overset{a}{R}_{ij} + (\overset{a}{\nabla}{}^2 \varphi^a + \overset{a}{\nabla}\varphi^a * \overset{a}{\nabla}\varphi^a) * \overset{a}{g} * (\overset{a}{g})^{-1}),\\
\frac{\partial}{\partial t} \overset{a}{\Gamma}{}^k_{ij} &= e^{\varphi^a}(\overset{a}{g})^{-1} * \overset{a}{\nabla}\overset{a}{\mathrm{Ric}} + e^{\varphi^a}(\overset{a}{g})^{-2} * \overset{a}{g} * \overset{a}{\mathrm{Ric}} * \nabla\varphi^a\\
&\quad + e^{\varphi^a}(\overset{a}{g})^{-3} * (\overset{a}{g})^2 * \left[(\overset{a}{\nabla}\overset{a}{\varphi})^3 + \overset{a}{\nabla}{}^2 \varphi^a \overset{a}{\nabla}\varphi^a + \overset{a}{\nabla}{}^3 \varphi^a\right],
\end{aligned}$$

$$\begin{aligned}
& e^{\frac{\varphi^a}{2}} |\overset{a}{\nabla}\varphi^a|_{g^a(\cdot,t)} + e^{\varphi^a} |\overset{a}{\nabla}{}^2_{g^a(\cdot,t)}\varphi^a|_{g^a(\cdot,t)} \leqslant C(n,k_0,T),\\
& e^{\frac{3}{2}\varphi^a} |\overset{a}{\nabla}{}^3_{g^a(\cdot,t)}\varphi^a|_{g^a(\cdot,t)} \leqslant C(n,k_0,T)(1+|\log t|),\\
& e^{\frac{k}{2}\varphi^a} |\overset{a}{\nabla}{}^k_{g^a(\cdot,t)}\varphi^a|_{g^a(\cdot,t)} \leqslant C(n,k_0,T,k)\frac{1}{t^{\frac{k-3}{2}}}, \quad \text{for } k \geqslant 4.
\end{aligned} \tag{2.20}$$

*Proof.* Note that

$$\begin{aligned}
\overset{a}{\Gamma} - \Gamma &= g * g^{-1} * \nabla\varphi^a\\
\overset{a}{\nabla}{}^2 \varphi^a &= \nabla^2\varphi^a + (\overset{a}{\Gamma} - \Gamma) * \nabla\varphi^a\\
\overset{a}{\nabla}{}^k \varphi^a &= \sum_{i_1+\cdots+i_p=k} g^{k-1} * (g^{-1})^{k-1} * \nabla^{i_1}\varphi^a * \cdots * \nabla^{i_p}\varphi^a
\end{aligned}$$

where the summation is taken over all indices $i_j > 0$. By combining this with (2.18), we get the desired estimates for $|\overset{a}{\nabla}{}^k_{g^a(\cdot,t)}\varphi^a|_{g^a(\cdot,t)}$. On the other hand, since

$$\overset{a}{R}_{ij} = R_{ij} + (\overset{a}{\nabla}{}^2\varphi^a + \nabla\varphi^a * \nabla\varphi^a) * g * g^{-1},$$

it follows that

$$\overset{a}{\nabla}_i \overset{a}{R}_{jl} = \nabla_i R_{jl} + g^a * (g^a)^{-1} * \left( \overset{a}{\mathrm{Ric}} * \nabla\varphi^a + \overset{a}{\nabla}{}^3 \varphi^a \right) + (g^a)^2 * (g^a)^{-2} * \left( \overset{a}{\nabla}{}^2 \varphi^a * \overset{a}{\nabla} \varphi^a + (\overset{a}{\nabla}\varphi^a)^3 \right).$$

By combining this with

$$\begin{aligned} \frac{\partial}{\partial t} \overset{a}{\Gamma}{}^k_{ij} &= \frac{\partial}{\partial t}\Gamma^k_{ij} + \frac{\partial}{\partial t}(g^{-1} * g * \nabla\varphi^a) \\ &= -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{li} - \nabla_l R_{ij}) + g * g^{-2} * Ric * \nabla\varphi^a, \end{aligned}$$

we have proved the lemma. q.e.d.

**2.2. Modified harmonic map flow.** The purpose of this subsection is to solve the equation $(2.3)_{\mathrm{a}}$. More precisely, we will prove the following theorem:

**Theorem 2.6.** *There exists $0 < T_1 < T$, depending only on $k_0$, $T$ and $n$ such that for all $a \geqslant 1$ the modified harmonic map flow coupled with the Ricci flow*

$$(2.3)_{\mathrm{a}} \qquad \begin{cases} \frac{\partial}{\partial t} \overset{a}{F}(x,t) = \overset{a}{\triangle}_t \overset{a}{F}(x,t) \\ \overset{a}{F}(\cdot, 0) = identity \end{cases}$$

*has a solution on $M^n \times [0, T_1]$ satisfying the following estimates*

$$(2.21) \qquad \begin{aligned} &|\overset{a}{\nabla}\overset{a}{F}| \leq C(n, k_0, T), \\ &|\overset{a}{\nabla}{}^k \overset{a}{F}| \leq C(n, k_0, T, k) t^{-\frac{k-2}{2}}, \quad \textit{for all} \ \ k \geq 2, \end{aligned}$$

*for some constants $C(n, k_0, T, k)$ depending only on $n$, $k_0$, $T$, and $k$ but independent of $a$.*

Note that $\overset{a}{F}$ is viewed as a map from $(M^n, g^a_{ij}(x,t))$ to $(N^n, h^a_{\alpha\beta}(y))$, all the covariant derivatives and the norms in Theorem 2.6 are computed with respect to $g^a_{ij}(x,t)$ and $h^a_{\alpha\beta}(y)$. We begin with an easier short time existence of $(2.3)_a$ where the short time interval may depend on $a$.

**2.2.1. Short time existence of the modified harmonic map flows.** We consider $(2.3)_a$ with general initial data.

**Theorem 2.7.** *Let $f$ be a smooth map from $M^n$ to $N^n$ with*

$$E_0 = \sup_{x \in M^n} |\overset{a}{\nabla} f|_{g^a_{ij}(\cdot,0), h^a_{\alpha\beta}}(x) + \sup_{x \in M^n} |\overset{a}{\nabla}{}^2 f|_{g^a_{ij}(\cdot,0), h^a_{\alpha\beta}}(x) < \infty.$$

*Then there exists a $\delta_0 > 0$ such that the initial problem*

$$(2.3)_a{}' \qquad \begin{cases} \frac{\partial}{\partial t}\overset{a}{F}(x,t) = \overset{a}{\triangle}_t \overset{a}{F}(x,t), \\ \overset{a}{F}(x,0) = f(x), \end{cases}$$

*has a smooth solution on $M^n \times [0,\delta_0]$ satisfying the following estimates*

$$(2.22) \qquad \sup_{(x,t)\in M^n\times[0,\delta_0]} |\overset{a}{\nabla}\overset{a}{F}|_{g^a_{ij}(\cdot,0),h^a_{\alpha\beta}}(x,t) + \sup_{(x,t)\in M^n\times[0,\delta_0]} |\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a_{ij}(\cdot,0),h^a_{\alpha\beta}}(x,t) \leqslant C(n,k_0,T,a,E_0),$$

$$\sup_{(x,t)\in M^n\times[0,\delta_0]} |\overset{a}{\nabla}{}^k\overset{a}{F}|_{g^a_{ij}(\cdot,0),h^a_{\alpha\beta}}(x,t) \leqslant \frac{C(n,k_0,T,k,a)}{t^{\frac{k-2}{2}}},$$

*for $k \geq 3$.*

We will prove the theorem by solving the corresponding initial-boundary value problem on a sequence of exhausted bounded domains $D_1 \subseteq D_2 \subseteq \cdots$ with smooth boundaries and $D_j \supseteq B^a_{g^a(\cdot,0)}(P, j+1)$ :

$$(2.23) \qquad \begin{cases} \frac{\partial}{\partial t}\overset{a}{F^j}(x,t) = \overset{a}{\triangle}_t \overset{a}{F^j}(x,t), & \text{for } x \in D_j \text{ and } t > 0, \\ F^j(x,0) = f(x) & \text{for } x \in D_j, \\ \overset{a}{F^j}(x,t) = f(x) & \text{for } x \in \partial D_j, \end{cases}$$

and $\overset{a}{F}$ will be obtained as the limit of a convergent subsequence of $\overset{a}{F^j}$ as $j \to \infty$. Here $P$ is a fixed point on $M^n$ and $B^a_{g^a(\cdot,0)}(P, j+1)$ is the geodesic ball centered at $P$ of radius $j+1$ with respect to the metric $g^a_{ij}(\cdot,0)$

The following lemma gives the zero-order estimate of $\overset{a}{F^j}$.

**Lemma 2.8.** *There exist positive constants $0 < T_2 < T$ and $C > 0$ such that for any $j$, if (2.23) has a smooth solution $\overset{a}{F^j}$ on $\bar{D}_j \times [0,T_3]$ with $T_3 \leq T_2$, then we have*

$$(2.24) \qquad d_{(N^n,h^a)}(f(x), \overset{a}{F^j}(x,t)) \leq C\sqrt{t},$$

*for any $(x,t) \in D_j \times [0,T_3]$.*

*Proof.* For simplicity, we drop the superscripts $a$ and $j$ of $\overset{a}{F^j}$. Note that the distance function $d_{(N^n,h^a)}(y_1,y_2)$ can be regarded as a function on $N^n \times N^n$. Set $\psi(y_1,y_2) = \frac{1}{2}d^2_{(N^n,h^a)}(y_1,y_2)$ and $\rho(x,t) =$

$\psi(f(x), F(x,t))$. Then $\psi(x,t)$ is smooth when $\psi < \frac{1}{2}i_a^2$. Now we compute the equation of $\rho(x,t)$:

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)\rho = -d_{h^a}(f(x), F(x,t))\frac{\partial d_{h^a}}{\partial {y_1}^\alpha}\overset{a}{\triangle}_t f^\alpha - \mathrm{Hess}\,(\psi)(X_i, X_j)(g^a)^{ij} \tag{2.25}$$

where the vector fields $X_i$, $i = 1, 2, \ldots, n$, in local coordinates $(y_1^\alpha, y_2^\beta)$ on $N^n \times N^n$ are defined as follows

$$X_i = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial}{\partial y_1^\alpha} + \frac{\partial F^\beta}{\partial x^i}\frac{\partial}{\partial y_2^\beta}.$$

To handle the first term on the right hand side of (2.25), we use

$$\overset{a}{\Gamma}{}_{ij}^k(x,t) - \overset{a}{\Gamma}{}_{ij}^k(x,0) = \Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0) + g(\cdot,t) * g^{-1}(\cdot,t) * \nabla\varphi^a + g(\cdot,0) * g^{-1}(\cdot,0) * \nabla\varphi^a,$$

to conclude that

$$|\overset{a}{\triangle}_t f|_{g^a(\cdot,t),h^a} \leqslant C(n, k_0, T)E_0.$$

Recall from Lemma 2.3 that the curvature of the metric $h^a_{\alpha\beta}$ is bounded by $\bar{C}_0$. We claim that if $d_{h^a}(f(x), F(x,t)) \leq \min\{\frac{i_a}{4}, \frac{\pi}{4\sqrt{\bar{C}_0}}\}$, then

$$\mathrm{Hess}\,(\psi)(X_i, X_j)(g^a)^{ij} \geq \frac{1}{2}|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a,h^a} - C \tag{2.26}$$

where $C = C(E_0, \bar{C}_0)$ depends only on $E_0$ and $\bar{C}_0$.

Indeed, recall the computation of Hess$\,(\psi)$ in [**20**]. For any $(u,v) \in D = \{(u,v) : (u,v) \in N^n \times N^n, u \neq v, d_{N^n}(u,v) < \min\{\frac{i_a}{2}, \frac{\pi}{2\sqrt{\bar{C}_0}}\}\}$, let $\gamma_{uv}$ be the minimal geodesic from $u$ to $v$ and $e_1 \in T_uN^n$ be the tangent vector to $\gamma_{uv}$ at $u$. Then $e_1(u,v)$ defines a smooth vector field on $D$. Let $\{e_i\}$ be an orthonormal basis for $T_uN^n$ which defines $u$ smoothly. By parallel translation of $\{e_i\}$ along $\gamma_{uv}$, we define $\{\bar{e}_i\}$ an orthonormal basis for $T_vN^n$. Thus $\{e_1, \ldots e_n, \bar{e}_1, \ldots \bar{e}_n\}$ is a local frame on $D$. Then for any $X = X_1 + X_2 \in T_{(u,v)}D$, where

$$X_1 = \sum_{i=1}^n \xi_i e_i, \quad \text{and} \quad X_2 = \sum_{i=1}^n \eta_i \bar{e}_i,$$

by the formula (16) in [**20**], we have

$$\mathrm{Hess}\,(\psi)(X,X) = \sum_{i=1}^n(\xi_i - \eta_i)^2 + \int_0^r t\langle\nabla_{e_1}V, \nabla_{e_1}V\rangle + \int_0^r t\langle\nabla_{\bar{e}_1}V, \nabla_{\bar{e}_1}V\rangle - \int_0^r t\langle R(e_1,V)V, e_1\rangle - \int_0^r t\langle R(\bar{e}_1,V)V, \bar{e}_1\rangle$$

where $V$ is a Jacobi field on geodesic $\sigma$ (connecting $(v,v)$ to $(u,v)$) and $\bar{\sigma}$ (connecting $(u,u)$ to $(u,v)$) with $X$ as the boundary values, where $X$ is extended to be a local vector field by letting its coefficients with respect to $\{e_1,\dots e_n,\bar{e}_1,\dots\bar{e}_n\}$ be constant(see [**20**]). By the Jacobi equation, $|V|$, $r|\nabla_{e_1}V|$ and $r|\nabla_{\bar{e}_1}V|$ are bounded by $C(i_a,\bar{C}_0)|X|$. Thus we have

$$|\mathrm{Hess}\,(\psi)|_{h^a}\le C(i_a,\bar{C}_0)$$

under the assumption of the claim. So the mixed term $\frac{\partial^2\psi}{\partial y_1^\alpha\partial y_2^\beta}f_i^\alpha F_j^\beta(g^a)^{ij}$ in $\mathrm{Hess}\,(\psi)(X_i,X_j)(g^a)^{ij}$ can be bounded by $C(E_0,\bar{C}_0)E_0|\overset{a}{\nabla}\overset{a}{F}|_{g^a,h^a}$. On the other hand, the Hessian comparison theorem for the points which are not in the cut locus gives

$$\frac{\partial\psi}{\partial {y_2}^\alpha\partial {y_2}^\beta}-(\overset{a}{\Gamma}{}^\gamma_{\alpha\beta}\circ\overset{a}{F})\frac{\partial\psi}{\partial y_2^\gamma}\ge\frac{\pi}{4}h^a_{\alpha\beta},$$
$$\frac{\partial\psi}{\partial {y_1}^\alpha\partial {y_1}^\beta}-(\overset{a}{\Gamma}{}^\gamma_{\alpha\beta}\circ f)\frac{\partial\psi}{\partial y_1^\gamma}\ge\frac{\pi}{4}h^a_{\alpha\beta}.$$

Thus the claim follows.

Let

$$T_2'=\max\left\{t\le T:\sup_D d_{h^a}(f(x),F(x,t))\le\min\left\{i_a,\frac{\pi}{4\sqrt{\bar{C}_0}}\right\}\right\}.$$

If $\overset{a}{F}(x,t)$ is a smooth solution of (2.23) on $\bar{D}\times[0,T_3]$ with $T_3\leqslant T_2'$, by (2.25) and (2.26), we get

$$\left(\frac{\partial}{\partial t}-\overset{a}{\triangle}_t\right)\rho\le-\frac{1}{2}|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a,h^a}+C\sqrt{\rho}+C \tag{2.27}$$

on $D\times[0,T_3]$, for some constant $C$ depending on $E_0$, $i_a$ and $\bar{C}_0$. Note that the initial and boundary values of $\rho$ are zero, so by the maximum principle, we get

$$d_{h^a}(f(x),\overset{a}{F}(x,t))\le C\sqrt{t}.$$

This implies

$$T_2'\ge\min\left\{\frac{\min\left\{i_a,\frac{\pi}{4\sqrt{\bar{C}_0}}\right\}^2}{C^2},T_3\right\}.$$

Hence the lemma holds with

$$T_2=\min\left\{\frac{\min\left\{i_a,\frac{\pi}{4\sqrt{\bar{C}_0}}\right\}^2}{C^2},T\right\}.$$

q.e.d.

After we have the zero order estimate (2.24), we now apply the standard parabolic equation theory to get the following short time existence for (2.23).

**Lemma 2.9.** *There exists a positive constant $T_3 \leq T_2$ depending only on the dimension $n, a, T_2$ and $C$ in Lemma 2.8 such that for each $j$, the initial-boundary value problem (2.23) has a smooth solution $\overset{a}{F}{}^j$ on $\bar{D}_j \times [0, T_3]$.*

*Proof.* For an arbitrarily fixed point $x_0$ in $\bar{D}_j$, choose normal coordinates $\{x^i\}$ and $\{y^\alpha\}$ on $(M^n, g^a(\cdot, 0))$ and $(N^n, h^a)$ around $x_0$ and $f(x_0)$ respectively. The equation (2.23) can be written as

$$\begin{aligned}(2.28)\quad \frac{\partial y^\alpha}{\partial t}(x,t) = (g^a)^{ij}(x,t)\bigg\{&\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \overset{a}{\Gamma}{}^k_{ij}(x,t)\frac{\partial y^\alpha}{\partial x^k} \\ &+ \overset{a}{\Gamma}{}^\alpha_{\beta\gamma}(y^1(x,t), \ldots, y^n(x,t))\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}\bigg\}.\end{aligned}$$

Note that $\overset{a}{\Gamma}{}^\alpha_{\beta\gamma}(f(x_0)) = 0$. By applying (2.24) and a result of Hamilton (Corollary (4.12) in [**12**]), we know that the coefficients of the quadratic terms of the gradients on the RHS of (2.28) can be as small as we like provided $T_3 > 0$ sufficiently small (independent of $x_0$ and $j$).

Now for fixed $j$, we consider the corresponding parabolic system of the difference of the map $\overset{a}{F}{}^j$ and $f(x)$. Clearly the coefficients of the quadratic terms of the gradients are also very small. Thus, whenever (2.23) has a solution on a time interval $[0, T_3']$ with $T_3' \leq T_3$, we can argue exactly as in the proof of Theorem 6.1 in Chapter VII of the book [**15**] to bound the norm of $\overset{a}{\nabla}\overset{a}{F}{}^j$ over $\bar{D}_j \times [0, T_3']$ by a constant depending only on the $L^\infty$ bound of $\overset{a}{F}$ in (2.23), the map $f(x)$, the domain $D_j$, and the metrics $g^a_{ij}(\cdot, t)$ and $h^a_{\alpha\beta}$ over the domain $D_{j+1}$. Hence by the same argument as in the proof of Theorem 7.1 in Chapter VII of the book [**15**], we deduce that the initial-boundary value problem (2.23) has a smooth solution $\overset{a}{F}{}^j$ on $\bar{D}_j \times [0, T_3]$. q.e.d.

Unfortunately, the gradient estimates of $\overset{a}{F}{}^j$ in the proof of the above lemma depend also on the domain $D_j$. In order to get a convergent subsequence of $\overset{a}{F}{}^j$, we have to estimate the covariant derivatives of $\overset{a}{F}{}^j$ uniformly in each compact subset. Before we proceed, we need some preliminary estimates and notations. Note that for any $q$, we can equip the bundle $(T^*M)^{\otimes q} \otimes \overset{a}{F}{}^{-1}TN$ the metric and connection induced from $(M, \overset{a}{g})$ and $(N, \overset{a}{h})$. In fact, for any section $u \in (T^*M)^{\otimes p-1} \otimes \overset{a}{F}{}^{-1}TN$, we define the covariant derivative $\overset{a}{\nabla} u$ of $u$ as a section of $(T^*M)^{\otimes p} \otimes \overset{a}{F}{}^{-1} TN$ by the formula

$$(\overset{a}{\nabla} u)^{\alpha}_{i_1,i_2,\ldots,i_{p-1},i_p} = \frac{\partial u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}}{\partial x^{i_p}} - \overset{a}{\Gamma}{}^{l}_{i_p,i_j} u^{\alpha}_{i_1,i_2,\ldots,i_{j-1}l,i_{j+1},\ldots,i_{p-1}} + \overset{a}{\Gamma}{}^{\alpha}_{\beta\gamma} \frac{\partial \overset{a}{F}{}^{\beta}}{\partial x^{i_p}} u^{\gamma}_{i_1,i_2,\ldots,i_{p-1}},$$

where $\overset{a}{\Gamma}{}^{l}_{ij}$ and $\overset{a}{\Gamma}{}^{\alpha}_{\beta\gamma}$ are connection coefficients of $(M, \overset{a}{g})$ and $(N, \overset{a}{h})$ respectively. We can define the Laplacian of $u$ by $\overset{a}{\triangle} u = \overset{a}{g}{}^{ij} (\overset{a}{\nabla}{}^2 u)_{\ldots,ij}$. Recall the Ricci identity

$$(\overset{a}{\nabla}{}^2 u)^{\alpha}_{\ldots,i,j} - (\overset{a}{\nabla}{}^2 u)^{\alpha}_{\ldots,j,i} = -\overset{a}{R}_{iji_m l} u_{\ldots k \ldots} \overset{a}{g}{}^{kl} + \overset{a}{R}_{\beta\gamma\delta\zeta} \frac{\partial \overset{a}{F}{}^{\beta}}{\partial x^i} \frac{\partial \overset{a}{F}{}^{\gamma}}{\partial x^j} \overset{a}{h}{}^{\alpha\delta} u^{\zeta}_{\ldots}.$$

Note that the derivative $\overset{a}{\nabla} \overset{a}{F}$ ($\overset{a}{\nabla}_i \overset{a}{F}{}^{\alpha} = \frac{\partial \overset{a}{F}{}^{\alpha}}{\partial x^i}$) is a section of the bundle $T^*M \otimes \overset{a}{F}{}^{-1} TN$, the higher derivative $\overset{a}{\nabla}{}^{p} \overset{a}{F}$ is a section of $(T^*M)^{\otimes p} \otimes \overset{a}{F}{}^{-1} TN$. Since the bundle $(T^*M)^{\otimes p} \otimes \overset{a}{F}{}^{-1} TN$ changes with the time, we define a covariant time derivative $D_t$ as follows. For any section $u^{\alpha}_{i_1\ldots,i_p}$ of $(T^*M)^{\otimes p} \otimes \overset{a}{F}{}^{-1} TN$, we define

$$D_t u^{\alpha}_{i_1\ldots,i_p} = \frac{\partial}{\partial t} u^{\alpha}_{i_1\ldots,i_p} + \overset{a}{\Gamma}{}^{\alpha}_{\beta\gamma} \frac{\partial \overset{a}{F}{}^{\beta}}{\partial t} u^{\gamma}_{i_1\ldots,i_p}.$$

**Lemma 2.10.** *The covariant derivatives of $\overset{a}{F}{}^{j}$ satisfy the following equations*

$$D_t \overset{a}{\nabla} \overset{a}{F}{}^{j} = \overset{a}{\triangle}_t \overset{a}{\nabla} \overset{a}{F}{}^{j} + \overset{a}{Ric}(M^n) * \overset{a}{\nabla} \overset{a}{F}{}^{j} + \overset{a}{R}_N * (\overset{a}{\nabla} \overset{a}{F}{}^{j})^3, \tag{2.29}$$

$$D_t \overset{a}{\nabla}{}^{k} \overset{a}{F}{}^{j} = \overset{a}{\triangle}_t \overset{a}{\nabla}{}^{k} \overset{a}{F}{}^{j} + \sum_{l=0}^{k-1} \overset{a}{\nabla}{}^{l} \Big[ \Big( \overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla} \overset{a}{F}{}^{j})^2 + e^{\varphi^a} \overset{a}{R}_M + \overset{a}{\nabla}{}^2 e^{\varphi^a} \Big) * \overset{a}{\nabla}{}^{k-l} \overset{a}{F}{}^{j} \Big],$$

*where $\overset{a}{\nabla}{}^{l}(A * B)$ represents the linear combinations of $\overset{a}{\nabla}{}^{l} A * B$, $\overset{a}{\nabla}{}^{l-1} A * \overset{a}{\nabla} B$, $\ldots$, $A * \overset{a}{\nabla}{}^{l} B$, and $\overset{a}{\nabla}{}^2 e^{\varphi^a} = e^{\varphi^a} (\overset{a}{\nabla}{}^2 \varphi^a + \overset{a}{\nabla} \varphi^a * \overset{a}{\nabla} \varphi^a)$.*

*Proof.* For $k = 1$, by direct computation and Ricci formula, we have

$$D_t \overset{a}{\nabla}_i \overset{a}{F}{}^{\alpha} = \overset{a}{\triangle}_t \overset{a}{\nabla}_i \overset{a}{F}{}^{\alpha} - \overset{a}{R}{}^{l}_{i} \overset{a}{\nabla}_l \overset{a}{F}{}^{\alpha} + \overset{a}{R}{}^{\alpha}_{\beta\delta\gamma} \overset{a}{\nabla}_i \overset{a}{F}{}^{\beta} \overset{a}{\nabla}_k \overset{a}{F}{}^{\delta} \overset{a}{\nabla}_l \overset{a}{F}{}^{\gamma} (g^a)^{kl}.$$

For $k \geqslant 2$, by Ricci formula, it follows

$$\overset{a}{\nabla}\overset{a}{\triangle}\overset{a}{\nabla}^{k-1}\overset{a}{F}^j = \overset{a}{\triangle}\overset{a}{\nabla}^k\overset{a}{F}^j + \overset{a}{\nabla}[(\overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla}\overset{a}{F}^j)^2) * \overset{a}{\nabla}^{k-1}\overset{a}{F}^j].$$

Recall from (2.20) that

$$\frac{\partial}{\partial t}\overset{a}{\Gamma}{}^i_{jk} = \overset{a}{\nabla}(e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}^2 e^{\varphi^a}).$$

Then we have

$$\begin{aligned}
&D_t\overset{a}{\nabla}^k\overset{a}{F}^j - \overset{a}{\triangle}_t\overset{a}{\nabla}^k\overset{a}{F}^j \\
&= \overset{a}{\nabla}[(D_t - \overset{a}{\triangle}_t)\overset{a}{\nabla}^{k-1}\overset{a}{F}^j] + \overset{a}{\nabla}(e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}^2 e^{\varphi^a}) * \overset{a}{\nabla}^{k-1}\overset{a}{F}^j \\
&\quad + \overset{a}{R}_N * \overset{a}{\nabla}\overset{a}{F}^j * \overset{a}{\nabla}^2\overset{a}{F}^j * \overset{a}{\nabla}^{k-1}\overset{a}{F}^j \\
&\quad + \overset{a}{\nabla}[(\overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla}\overset{a}{F}^j)^2) * \overset{a}{\nabla}^{k-1}\overset{a}{F}^j] \\
&= \overset{a}{\nabla}[(D_t - \overset{a}{\triangle})\overset{a}{\nabla}^{k-1}\overset{a}{F}^j] \\
&\quad + \overset{a}{\nabla}\{(\overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla}\overset{a}{F}^j)^2 + (e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}^2 e^{\varphi^a}) * \overset{a}{\nabla}^{k-1}\overset{a}{F}^j\} \\
&= \sum_{l=0}^{k-1}\overset{a}{\nabla}^l[(\overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla}\overset{a}{F}^j)^2 + e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}^2 e^{\varphi^a}) * \overset{a}{\nabla}^{k-l}\overset{a}{F}^j].
\end{aligned}$$

This proves the lemma. q.e.d.

For each $k > 0$, let $\xi_k$ be a smooth non-increasing function from $(-\infty, +\infty)$ to $[0,1]$ so that $\xi_k(s) = 1$ for $s \in (-\infty, \frac{1}{2} + \frac{1}{2^{k+1}}]$, and $\xi_k(s) = 0$ for $s \in [\frac{1}{2} + \frac{1}{2^k})$; moreover for any $\epsilon > 0$ there exists a universal $C_{k,\epsilon} > 0$ such that

$$|\xi_k'(s)| + |\xi_k''(s)| \le C_{k,\epsilon}\xi_k(s)^{1-\epsilon}.$$

**Lemma 2.11.** *There exists a positive constant $T_4$, $0 < T_4 \le T_3$ independent of $j$ such that for any geodesic ball $B_{g^a(\cdot,0)}(x_0, \delta) \subset D_j$, there is a constant $C = C(a, \delta, E_0, \bar{C}_0, \bar{k_0})$ such that the smooth solution of (2.23) satisfies*

$$|\overset{a}{\nabla}\overset{a}{F}^j|_{g^a(\cdot,t),h^a} \leqslant C$$

*on $B_{g^a(\cdot,0)}(x_0, \frac{3\delta}{4}) \times [0, T_4]$.*

*Proof.* We compute the equation of $|\overset{a}{\nabla}\overset{a}{F}{}^j|^2_{g^a(\cdot,t),h^a}$. For simplicity, we drop the superscript $j$. By (2.20), we have

(2.30)
$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\overset{a}{\triangle}_t\right)|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}\\
&=\langle \overset{a}{\mathrm{Ric}}\,(M^n)*\overset{a}{\nabla}\overset{a}{F}+\overset{a}{R}_N*(\overset{a}{\nabla}\overset{a}{F})^3,\overset{a}{\nabla}\overset{a}{F}\rangle_{g^a,h^a}-2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a}\\
&\quad+e^{\varphi^a}(\mathrm{Ric}\,\overset{a}{(M^n)}+\overset{a}{\nabla}{}^2\varphi^a+\overset{a}{\nabla}\varphi^a*\overset{a}{\nabla}\varphi^a)*\overset{a}{\nabla}\overset{a}{F}*\overset{a}{\nabla}\overset{a}{F}\\
&\leqslant -2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a}+C(n,k_0,T)|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}+C(n)\bar{C}_0|\overset{a}{\nabla}\overset{a}{F}|^4_{g^a(\cdot,t),h^a}.
\end{aligned}$$

Setting

$$\rho_A(x,t)=(d^2_{h^a}(f(x),F(x,t))+A)|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}$$

where $A$ is determined later, and combining with (2.27) and (2.24), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_A\leqslant{}&\overset{a}{\triangle}\rho_A-2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a,h^a}(d^2_{h^a}(f(x),\overset{a}{F}(x,t))+A)-|\overset{a}{\nabla}\overset{a}{F}|^4_{g^a,h^a}\\
&+C(n)\bar{C}_0(d^2_{h^a}(f(x),\overset{a}{F}(x,t))+A)|\overset{a}{\nabla}\overset{a}{F}|^4_{g^a,h^a}\\
&+C|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a,h^a}+C(n,k_0,T)\rho_A\\
&+2|\nabla d^2_{h^a}(f(x),\overset{a}{F}(x,t))|_{g^a}|\nabla|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}|_{g^a}.
\end{aligned}$$

Since

$$\begin{aligned}
|\nabla d^2_{h^a}(f(x),\overset{a}{F}(x,t))|_{g^a}&\leqslant 2d_{h^a}(f(x),\overset{a}{F}(x,t))(|\overset{a}{\nabla}\overset{a}{F}|_{g^a,h^a}+|\overset{a}{\nabla}f|_{g^a,h^a})\\
&\leqslant C\sqrt{t}+C\sqrt{t}|\overset{a}{\nabla}\overset{a}{F}|_{g^a,h^a},\\
|\nabla|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}|_{g^a}&\leqslant 2|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a,h^a}|\overset{a}{\nabla}\overset{a}{F}|_{g^a,h^a},
\end{aligned}$$

by choosing $T_4=\min\{T_3,\frac{1}{4C(n)\bar{C}_0C^2}\}$, $A=\frac{1}{4C(n)\bar{C}_0}$, and applying Cauchy-Schwartz inequality, we have

$$\left(\frac{\partial}{\partial t}-\overset{a}{\triangle}\right)\rho_A\leqslant -C(n)\bar{C}_0\rho_A^2+C$$

where $C=C(n,k_0,T,E_0,a)$.

We compute the equation of $u = \xi_1(\frac{d_{g^a(\cdot,0)}(x_0,\cdot)}{\delta})\rho_A$ at the smooth points of function $d_{g^a(\cdot,0)}(x_0,\cdot)$,

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}\right) u \leqslant C\xi_1 - C(n)\bar{C}_0\rho_A^2\xi_1 - 2(g^a)^{ij}\nabla_i\xi_1\nabla_j\rho_A + \left(-\xi_1'\frac{\overset{a}{\triangle}d_{g^a(\cdot,0)}(x_0,\cdot)}{\delta} + e^{nk_0T}\frac{|\xi_1''|}{\delta^2}\right)\rho_A.$$

By the Hessian comparison theorem and the fact that $-\xi_1' \geqslant 0$, we have

$$\begin{aligned} \overset{a}{\nabla}_i\overset{a}{\nabla}_j d_{g^a(\cdot,0)} &\leqslant \overset{a}{\nabla}{}^0_i\overset{a}{\nabla}{}^0_j d_{g^a(\cdot,0)} + (\overset{a}{\Gamma}(\cdot,0) - \overset{a}{\Gamma}(\cdot,t)) * \nabla d_{g^a(\cdot,0)} \\ &\leqslant \left(\frac{1+\bar{k}_0 d_{g^a(\cdot,0)}}{d_{g^a(\cdot,0)}} + C\right) g^a_{ij}(\cdot,0), \\ -\xi_1'\overset{a}{\triangle}d_{g^a(\cdot,0)} &\leqslant \frac{C|\xi_1'|}{\delta}. \end{aligned}$$

These two inequalities hold on the whole manifold in the sense of support functions. Thus for any $x_1 \in M^n$, there is a function $h_{x_1}$ which is smooth on a neighborhood of $x_1$ with $h_{x_1}(\cdot) \geqslant d_{g^a(\cdot,0)}(x_0,\cdot)$, $h_{x_1}(x_1) = d_{g^a(\cdot,0)}(x_0,x_1)$ and

$$-\xi_1'\overset{a}{\triangle}h_{x_1}\,|_{x_1} \leqslant 2\frac{C|\xi_1'|}{\delta}.$$

Indeed, $h_{x_1}$ can be chosen to have the form $d_{g^a(\cdot,0)}(q,\cdot) + d_{g^a(\cdot,0)}(q,x_0)$ for some $q$, so we may require $|\overset{a}{\nabla}h_{x_1}|_{g^a(\cdot,0)} \leqslant 1$. Let $(x_1,t_0)$ be the maximum point of $u$ over $M^n \times [0,T_4]$. If $t_0 = 0$, then $\xi_1\rho_A \leqslant E_0$. Assume $t_0 > 0$. At the point $(x_1,t_0)$, we have $\frac{\partial}{\partial t}(\xi_1\rho_A)(x_1,t_0) \geqslant 0$. If $x_1$ does not lie on the cut locus of $x_0$, then

$$\begin{aligned} 0 &\leqslant -C(n)\bar{C}_0\rho_A^2\xi_1 + \frac{1}{\delta^2}\left(e^{nk_0T}\frac{|\xi_1'|^2}{\xi_1} + 2C(|\xi_1'| + |\xi_1''|)\right)\rho_A + C\xi_1 \\ &\leqslant -C(n)\bar{C}_0\rho_A^2\xi_1 + \frac{C}{\delta^2}\sqrt{\xi_1}\rho_A + C\xi_1 \\ &\leqslant -C(n)\bar{C}_0\rho_A^2\xi_1 + \frac{C}{\delta^4} \\ &\leqslant -C(n)\bar{C}_0(\rho_A\xi_1)^2 + \frac{C}{\delta^4}. \end{aligned}$$

We get

$$\xi_1\rho_A \leqslant \max\left\{E_0, \sqrt{\frac{C}{C(n)\bar{C}_0\delta^4}}\right\}$$

for all $(x,t) \in B_{g^a(\cdot,0)}(x_0,\delta) \times [0,T_4]$. If $x_1$ lies on the cut locus of $x_0$, then by applying the standard support function technique (see for example [**19**]), the above maximum principle argument still works. So by the definition of $\xi_1$ and $\rho_A$, we have

$$|\overset{a}{\nabla}\overset{a}{F}{}^j|_{g^a(\cdot,t),h^a} \leqslant \frac{C}{\delta}$$

on $B_{g^a(\cdot,0)}(x_0, \frac{3\delta}{4}) \times [0, T_4]$. The proof of the lemma is completed. q.e.d.

The next lemma estimates the higher derivatives in terms of the bound of $|\overset{a}{\nabla}\overset{a}{F}{}^j|_{g^a(\cdot,t),h^a}$.

**Lemma 2.12.** *Let $\overset{a}{F}$ be a smooth solution of equation*

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}\right)\overset{a}{F} = 0$$

*on $B_{g^a(\cdot,0)}(x_0, \delta) \times [0, \bar{T}]$, with $\bar{T} \leqslant T$. Suppose*

$$\sup_{(x,t)\in B_{g^a(\cdot,0)}(x_0,\frac{3\delta}{4})\times[0,\bar{T}]} |\overset{a}{\nabla}\overset{a}{F}|_{g^a_{ij}(\cdot,0),h^a_{\alpha\beta}}(x,t) \leqslant E_1, \tag{2.31}$$

$$\textit{and} \quad \sup_{x\in B_{g^a(\cdot,0)}(x_0,\frac{3\delta}{4})} |\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a_{ij}(\cdot,0),h^a_{\alpha\beta}}(x,0) \leqslant E_1.$$

*Then for any $k \geqslant 2$, there exists a positive constant $C = C(k, E_1, \delta, k_0, T) > 0$ such that*

$$|\overset{a}{\nabla}{}^k\overset{a}{F}|_{g^a(\cdot,t),h^a} \leq Ct^{-\frac{k-2}{2}} \tag{2.32}$$

*on $B_{g^a(\cdot,0)}(x_0, \frac{\delta}{2}) \times [0, \bar{T}]$.*

*Proof.* The proof is using the Bernstein trick. We assume $\delta < 1$ without loss of generality. For $k = 2$, from (2.15), (2.19), (2.20) and (2.29), we have

(2.33)

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\
&= \left\langle \sum_{l=0}^{1} \overset{a}{\nabla}{}^l[(\overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla}\overset{a}{F}{}^j)^2 + e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}{}^2 e^{\varphi^a}) * \overset{a}{\nabla}{}^{2-l}\overset{a}{F}], \overset{a}{\nabla}{}^2\overset{a}{F}\right\rangle_{g^a,h^a} \\
&\quad - 2|\overset{a}{\nabla}{}^3\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + e^{\varphi^a}(\mathrm{Ric}\,\overset{a}{(M^n)} + \overset{a}{\nabla}{}^2\varphi^a + \overset{a}{\nabla}\varphi^a * \overset{a}{\nabla}\varphi^a) * (\overset{a}{\nabla}{}^2\overset{a}{F})^2 \\
&\leqslant -2|\overset{a}{\nabla}{}^3\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{\sqrt{t}}|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a}.
\end{aligned}$$

In this lemma, we use $C$ to denote various constants depending only on $E_1$, $k_0$, $T$, $k$ and $\delta$. Note that by (2.30) and (2.33), we have

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \leqslant -2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C,$$

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a} \leqslant C|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a} + \frac{C}{\sqrt{t}}.$$

So by setting

$$v = |\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a} - 2C\sqrt{t} + 2C\sqrt{T} + |\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a},$$

we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)v &\leqslant -2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a} + C \\ &\leqslant -v^2 + C.\end{aligned}$$

Since at $t = 0$,

$$v \leqslant 2C\sqrt{T} + E_1 + E_1^2$$

on $B_{g^a(\cdot,0)}(x_0, \frac{3\delta}{4})$, we apply the maximum principle as in Lemma 2.11 to get

$$\xi_2\left(\frac{d_{g^a(\cdot,0)}(x_0,\cdot)}{\delta}\right)v \leqslant C$$

on $B_{g^a(\cdot,0)}(x_0, \frac{3\delta}{4}) \times [0, \bar{T}]$. This implies

$$|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a} \leq C$$

on $B_{g^a(\cdot,0)}(x_0, (\frac{1}{2} + \frac{1}{2^3})\delta) \times [0, \bar{T}]$.

Now we estimate the third-order derivatives. From Shi's gradient estimate [**21**], the estimate $|\overset{a}{\nabla}{}^2\overset{a}{F}|_{g^a(\cdot,t),h^a} \leq C$, and (2.15), (2.19), (2.20) and (2.29), we have:

(2.34)

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)|\overset{a}{\nabla}{}^3\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\ &= \left\langle \sum_{l=0}^{2}\overset{a}{\nabla}{}^l[(\overset{a}{R}_M + \overset{a}{R}_N * (\overset{a}{\nabla}\overset{a}{F})^2 + e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}{}^2 e^{\varphi^a}) * \overset{a}{\nabla}{}^{3-l}\overset{a}{F}], \overset{a}{\nabla}{}^3\overset{a}{F}\right\rangle_{g^a,h^a} \\ &\quad - 2|\overset{a}{\nabla}{}^4\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + e^{\varphi^a}(\mathrm{Ric}\,(\overset{a}{M^n}) + \overset{a}{\nabla}{}^2\varphi^a + \overset{a}{\nabla}\varphi^a * \overset{a}{\nabla}\varphi^a) * (\overset{a}{\nabla}{}^3\overset{a}{F})^2 \\ &\leqslant -2|\overset{a}{\nabla}{}^4\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^3\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{t}|\overset{a}{\nabla}{}^3\overset{a}{F}|_{g^a(\cdot,t),h^a}\end{aligned}$$

on $B_{g^a(\cdot,0)}(x_0, (\frac{1}{2} + \frac{1}{8})\delta) \times [0, \bar{T}]$. Here we used the estimates $|\overset{a}{\nabla}{}^4 e^{\varphi^a}|_{g^a} \leqslant \frac{C}{\sqrt{t}}$, $|\overset{a}{\nabla}{}^3 e^{\varphi^a}|_{g^a} \leqslant C(1 + |\log t|)$, and $|\overset{a}{\nabla}{}^2\overset{a}{R}_m| \leqslant \frac{C}{t}$.

By (2.33), it follows

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) |\overset{a}{\nabla}{}^2 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} \leqslant -2|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{\sqrt{t}} \tag{2.35}$$

on $B_{g^a(\cdot,0)}(x_0, (\frac{1}{2}+\frac{1}{8})\delta)\times[0,\bar{T}]$. Let $v=(|\overset{a}{\nabla}{}^2 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}+A)|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}$, where $A = 100 \sup\limits_{B_{g^a(\cdot,0)}(x_0,(\frac{1}{2}+\frac{1}{2^3})\delta)\times[0,\bar{T}]} |\overset{a}{\nabla}{}^2 \overset{a}{F}|_{g^a_{ij}(\cdot,t),h^a_{\alpha\beta}}(x,t) + C$. By a direct computation, it follows

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}\right) v \\
&\leq |\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}\left(-2|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{\sqrt{t}}\right) + (|\overset{a}{\nabla}{}^2 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + A) \\
&\qquad \times\left(-2|\overset{a}{\nabla}{}^4 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{t}|\overset{a}{\nabla}{}^3 \overset{a}{F}|_{g^a(\cdot,t),h^a}\right) \\
&\quad + 8|\overset{a}{\nabla}{}^2 \overset{a}{F}|_{g^a(\cdot,t),h^a}|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}|\overset{a}{\nabla}{}^4 \overset{a}{F}|_{g^a(\cdot,t),h^a}.
\end{aligned}$$

Since

$$\begin{aligned}
&8|\overset{a}{\nabla}{}^2 \overset{a}{F}|_{g^a(\cdot,t),h^a}|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}|\overset{a}{\nabla}{}^4 \overset{a}{F}|_{g^a(\cdot,t),h^a} \\
&\leqslant -|\overset{a}{\nabla}{}^3 \overset{a}{F}|^4_{g^a(\cdot,t),h^a} + 16|\overset{a}{\nabla}{}^4 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}|\overset{a}{\nabla}{}^2 \overset{a}{F}|^2_{g^a(\cdot,t),h^a},
\end{aligned}$$

we deduce

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}\right) v \leqslant -|\overset{a}{\nabla}{}^3 \overset{a}{F}|^4_{g^a(\cdot,t),h^a} + \frac{C}{t}|\overset{a}{\nabla}{}^3 \overset{a}{F}|_{g^a(\cdot,t),h^a} + \frac{C}{\sqrt{t}}|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a}$$

and

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}\right)(tv) \\
&\leqslant v - t|\overset{a}{\nabla}{}^3 \overset{a}{F}|^4_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^3 \overset{a}{F}|_{g^a(\cdot,t),h^a} + C\sqrt{t}|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\
&\leqslant -\frac{1}{t}\Big\{t^2|\overset{a}{\nabla}{}^3 \overset{a}{F}|^4_{g^a(\cdot,t),h^a} - C\sqrt{t}(\sqrt{t}|\overset{a}{\nabla}{}^3 \overset{a}{F}|_{g^a(\cdot,t),h^a}) \\
&\qquad - tv - C\sqrt{t}(t|\overset{a}{\nabla}{}^3 \overset{a}{F}|^2_{g^a(\cdot,t),h^a})\Big\} \\
&\leqslant -\frac{1}{t}\left\{\frac{(tv)^2}{10^5 C^2} - C\right\}.
\end{aligned}$$

So at the maximum point of $\xi_3(\frac{d_{g^a(\cdot,0)}(x_0,\cdot)}{\delta})(tv)$, applying the maximum principle as in Lemma 2.11, we have

$$\begin{aligned} 0 &\leqslant -\frac{1}{t}\left\{\frac{\xi_3(tv)^2}{10^5C^2} - C\xi_3\right\} + C\left(\frac{|\xi_3'|^2}{\xi_3} + |\xi_3''|\right)(tv) \\ &\leqslant -\frac{1}{t}\left\{\frac{\xi_3(tv)^2}{10^5C^2} - C\xi_3 - Ct\sqrt{\xi_3}(tv)\right\} \\ &\leqslant -\frac{1}{t}\left\{\frac{\xi_3(tv)^2}{10^6C^2} - C^4\right\}, \end{aligned}$$

which gives

$$\xi_3(tv) \leqslant \sqrt{10^6C^6}.$$

Thus by the definition of $v$ and $\xi_3$, we get

$$|\overset{a}{\nabla}{}^3\overset{a}{F}|_{g^a(\cdot,t),h^a} \leq Ct^{-\frac{1}{2}}$$

on $B_0(x_0,(\frac{1}{2}+\frac{1}{2^4})\delta)\times[0,\bar{T}]$.

Now we estimate the higher derivatives by induction. Suppose we have proved that

$$|\overset{a}{\nabla}{}^l\overset{a}{F}|_{g^a(\cdot,t),h^a} \leq Ct^{-\frac{l-2}{2}}, \qquad \text{for } l=3,\ldots,k-1,$$

on $B_0(x_0,(\frac{1}{2}+\frac{1}{2^k})\delta)\times[0,\bar{T}]$. By (2.29), we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right)|\overset{a}{\nabla}{}^k\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\ &= \left\langle \sum_{l=0}^{k-1}\overset{a}{\nabla}{}^l[(\overset{a}{R}_M + \overset{a}{R}_N*(\overset{a}{\nabla}\overset{a}{F})^2 + e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}{}^2e^{\varphi^a})*\overset{a}{\nabla}{}^{k-l}\overset{a}{F}], \overset{a}{\nabla}{}^k\overset{a}{F}\right\rangle_{g^a,h^a} \\ &\quad - 2|\overset{a}{\nabla}{}^{k+1}\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + e^{\varphi^a}(\mathrm{Ric}\,(\overset{a}{M}{}^n) + \overset{a}{\nabla}{}^2\varphi^a + \overset{a}{\nabla}\varphi^a * \overset{a}{\nabla}\varphi^a)*(\overset{a}{\nabla}{}^k\overset{a}{F})^2 \\ &\leqslant -2|\overset{a}{\nabla}{}^{k+1}\overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^k\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\ &\quad + C(n)\sum_{l=1}^{k-1}\left|\overset{a}{\nabla}{}^l\Big[\overset{a}{R}_M + \overset{a}{R}_N*(\overset{a}{\nabla}\overset{a}{F})^2\right. \\ &\qquad \left. + e^{\varphi^a}\overset{a}{R}_M + \overset{a}{\nabla}{}^2e^{\varphi^a}\Big]\right|_{g^a,h^a}|\overset{a}{\nabla}{}^{k-l}\overset{a}{F}|_{g^a,h^a}|\overset{a}{\nabla}{}^k\overset{a}{F}|_{g^a(\cdot,t),h^a}. \end{aligned}$$

By the induction hypothesis, the local derivative estimates of Shi, and (2.15), (2.19) and (2.20), it follows

$$\sum_{l=1}^{k-1} |\overset{a}{\nabla}{}^{l} \overset{a}{R}_M|_{g^a} |\overset{a}{\nabla}{}^{k-l} \overset{a}{F}|_{g^a(\cdot,t),h^a} \leqslant \frac{C}{t^{\frac{k-1}{2}}},$$

$$\sum_{l=1}^{k-1} |\overset{a}{\nabla}{}^{l} [\overset{a}{R}_N * (\overset{a}{\nabla} \overset{a}{F})^2]|_{g^a} |\overset{a}{\nabla}{}^{k-l} \overset{a}{F}|_{g^a(\cdot,t),h^a} \leqslant \frac{C}{t^{\frac{k-1}{2}}} + C|\overset{a}{\nabla}{}^{k} \overset{a}{F}|_{g^a(\cdot,t),h^a},$$

$$\sum_{l=1}^{k-1} |\overset{a}{\nabla}{}^{l+2} e^{\varphi^a}|_{g^a} |\overset{a}{\nabla}{}^{k-l} \overset{a}{F}|_{g^a(\cdot,t),h^a} \leqslant \frac{C}{t^{\frac{k-2}{2}}},$$

$$\sum_{l=1}^{k-1} |\overset{a}{\nabla}{}^{l} e^{\varphi^a} \overset{a}{R}_M|_{g^a} |\overset{a}{\nabla}{}^{k-l} \overset{a}{F}|_{g^a(\cdot,t),h^a} \leqslant \frac{C}{t^{\frac{k-1}{2}}}.$$

This gives

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) |\overset{a}{\nabla}{}^{k} \overset{a}{F}|^2_{g^a(\cdot,t),h^a} &\leqslant -2|\overset{a}{\nabla}{}^{k+1} \overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\
&\quad + C|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{t^{\frac{k-1}{2}}} |\overset{a}{\nabla}{}^{k} \overset{a}{F}|_{g^a(\cdot,t),h^a}, \\
\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) |\overset{a}{\nabla}{}^{k} \overset{a}{F}|_{g^a(\cdot,t),h^a} &\leqslant C|\overset{a}{\nabla}{}^{k} \overset{a}{F}|_{g^a(\cdot,t),h^a} + \frac{C}{t^{\frac{k-1}{2}}}, \\
\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) |\overset{a}{\nabla}{}^{k-1} \overset{a}{F}|^2_{g^a(\cdot,t),h^a} &\leqslant -2|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + \frac{C}{t^{k-\frac{5}{2}}}.
\end{aligned}$$

Let $\varepsilon = \frac{2(k-3)}{k-2} - 1$, then $0 \leq \varepsilon < 1$ for $k \geq 4$. It is clear that

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) |\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{1+\varepsilon}_{g^a(\cdot,t),h^a} \leqslant C|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{1+\varepsilon}_{g^a(\cdot,t),h^a} + \frac{C}{t^{\frac{k-1}{2}}} |\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{\varepsilon}_{g^a(\cdot,t),h^a},$$

and

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) \left(|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{1+\varepsilon}_{g^a(\cdot,t),h^a} + |\overset{a}{\nabla}{}^{k-1} \overset{a}{F}|^2_{g^a(\cdot,t),h^a}\right) \\
&\leqslant -2|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^2_{g^a(\cdot,t),h^a} + C|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{1+\varepsilon}_{g^a(\cdot,t),h^a} \\
&\quad + \frac{C}{t^{\frac{k-1}{2}}} |\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{\varepsilon}_{g^a(\cdot,t),h^a} + \frac{C}{t^{k-\frac{5}{2}}},
\end{aligned}$$

on $B_{g^a(\cdot,0)}(x_0, (\frac{1}{2} + \frac{1}{2^k})\delta) \times [0, \bar{T}]$.

Let

$$v = t^{k-3} \left(|\overset{a}{\nabla}{}^{k} \overset{a}{F}|^{1+\varepsilon}_{g^a(\cdot,t),h^a} + |\overset{a}{\nabla}{}^{k-1} \overset{a}{F}|^2_{g^a(\cdot,t),h^a}\right).$$

Then we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\overset{a}{\triangle}\right)v\\
&\leqslant (k-3)\frac{v}{t}+t^{k-3}\left(-|\overset{a}{\nabla}{}^k\overset{a}{F}|^2_{g^a(\cdot,t),h^a}+\frac{C}{t^{\frac{k-1}{2}}}|\overset{a}{\nabla}{}^k\overset{a}{F}|^{\varepsilon}_{g^a(\cdot,t),h^a}+\frac{C}{t^{k-\frac{5}{2}}}\right)\\
&\leqslant -\frac{1}{t}\{v^{\frac{2}{1+\varepsilon}}-C\sqrt{t}v^{\frac{\varepsilon}{1+\varepsilon}}-Cv-C\sqrt{t}\}\\
&\leqslant -\frac{1}{2t}\{v^{\frac{2}{1+\varepsilon}}-C\}
\end{aligned}$$

on $B_{g^a(\cdot,0)}(x_0,(\frac{1}{2}+\frac{1}{2^k})\delta)\times[0,\bar{T}]$. Similarly, at the maximum point of $\xi_k(\frac{d_{g^a(\cdot,0)}(x_0,\cdot)}{\delta})v$, we have

$$\begin{aligned}
0&\leqslant -\frac{1}{2t}\{\xi_k v^{\frac{2}{1+\varepsilon}}-C\xi_k\}+C\left(\frac{|\xi_k'|^2}{\xi_k}+|\xi_k''|\right)v\\
&\leqslant -\frac{1}{2t}\{\xi_k v^{\frac{2}{1+\varepsilon}}-C\xi_k^{\frac{1+\varepsilon}{2}}v-C\}\\
&\leqslant -\frac{1}{2t}\left\{\frac{1}{2}\xi_k v^{\frac{2}{1+\varepsilon}}-C\right\}\\
&\leqslant -\frac{1}{2t}\left\{\frac{1}{2}(\xi_k v)^{\frac{2}{1+\varepsilon}}-C\right\},
\end{aligned}$$

since $\frac{2}{1+\varepsilon}>1$. So we proved the $k$-th order estimate

$$|\overset{a}{\nabla}{}^k\overset{a}{F}|_{g^a(\cdot,t),h^a}\le Ct^{-\frac{k-2}{2}}$$

on $B_{g^a(\cdot,0)}(x_0,(\frac{1}{2}+\frac{1}{2^{k+1}})\delta)\times[0,\bar{T}]$. This completes the proof of the lemma. q.e.d.

Now we are ready to prove Theorem 2.7.

*Proof of Theorem* 2.7.

Since $D_j\supseteq B_{g^a(\cdot,0)}(P,j+1)$, by choosing $\delta=1$ and $\bar{T}=T_4$ in Lemma 2.11 and Lemma 2.12, we get a convergent subsequence of $\overset{a}{F}{}^j$ (as $j\to\infty$) on $B_{g^a(\cdot,0)}(P,j)\times[0,T_4]$. Denote the limit by $\overset{a}{F}$ (as $j\to\infty$). Then $\overset{a}{F}$ is the desired solution of $(2.3)'_a$ with estimates (2.22).

Finally we prove a uniqueness theorem for the solutions of $(2.3)'_a$ with estimates (2.22).

**Lemma 2.13.** *Let* $\overset{a}{F}$ *and* $\overset{a}{\bar{F}}$ *be two solutions of the initial value problem* $(2.3)'_{\mathrm{a}}$ *on* $[0,\bar{T}]$, $\bar{T}\leqslant T$, *with estimates* (2.22). *Then* $\overset{a}{F}=\overset{a}{\bar{F}}$ *on* $[0,\bar{T}]$.

*Proof.* Set $\psi(y_1, y_2) = \frac{1}{2} d^2_{(N^n, h^a)}(y_1, y_2)$ and $\rho(x, t) = \psi(\overset{a}{F}(x, t), \overset{a}{\bar{F}}(x,t))$. Then $\psi(x,t)$ is smooth when $\psi < \frac{1}{2} i_a^2$. Now by the same calculation as in Lemma 2.8, we have:

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) \rho = -\text{Hess}\,(\psi)(X_i, X_j)(g^a)^{ij}$$

where the vector fields $X_i$, $i = 1, 2, \ldots, n$, in local coordinates $(y_1^\alpha, y_2^\beta)$ on $N^n \times N^n$ are defined as follows

$$X_i = \frac{\partial \overset{a}{F}{}^\alpha}{\partial x^i} \frac{\partial}{\partial y_1^\alpha} + \frac{\partial \overset{a}{\bar{F}}{}^\beta}{\partial x^i} \frac{\partial}{\partial y_2^\beta}.$$

By the estimates (2.22), we know that there is a constant $0 < \bar{T}' \leqslant \bar{T}$ such that there holds

$$\rho < \min\left\{\frac{i_a^2}{8}, \frac{\pi^2}{8\bar{C}_0}\right\}$$

on $M^n \times [0, \bar{T}']$.

Similarly as in the proof of Lemma 2.8, by using the computation of Hess $(\psi)$ in [**20**] (the formula (16) in [**20**]), for any $(u,v) \in D = \{(u,v) : (u,v) \in N^n \times N^n$ with $0 < d_{N^n}(u,v) < \min\{\frac{i_a}{2}, \frac{\pi}{2\sqrt{\bar{C}_0}}\}\}$, and any $X \in T_{(u,v)}D$,

$$\text{Hess}\,(\psi)(X,X) \geqslant -\int_0^r t\langle R(e_1, V)V, e_1\rangle - \int_0^r t\langle R(\bar{e}_1, V)V, \bar{e}_1\rangle$$

where $V$ is a Jacobi field on geodesic $\sigma$ (connecting $(v,v)$ to $(u,v)$) and $\bar{\sigma}$ (connecting $(u,u)$ to $(u,v)$) with $X$ as the boundary values as before. Since $|\overset{a}{\nabla} F|_{g^a,h^a}$ and $|\overset{a}{\nabla} \bar{F}|_{g^a,h^a}$ are bounded, we know from the above formula that

$$\text{Hess}\,(\psi)(X_i, X_j)(g^a)^{ij} \geqslant -C\rho$$

on $M^n \times [0, \bar{T}']$. Thus we have

$$\left(\frac{\partial}{\partial t} - \overset{a}{\triangle}_t\right) \rho \leqslant C\rho$$

on $M^n \times [0, \bar{T}']$. By the maximum principle, it follows that $\rho = 0$ on $M^n \times [0, \bar{T}']$. Then the lemma follows by continuity method. q.e.d.

### 2.2.2. Proof of Theorem 2.6 and Theorem 2.1.

*Proof of Theorem* 2.6. Let us check the initial data. Now $f = identity$, so

$$|\overset{a}{\nabla} f|^2_{g^a(\cdot,0),h^a} = g^{ij}(\cdot, 0) g_{ij}(\cdot, T) \leqslant ne^{2nk_0T} \tag{2.36}$$

(2.37)

$$\begin{aligned}
|\overset{a}{\nabla}{}^2 f|^2_{g^a(\cdot,0),h^a} &= |\overset{a}{\Gamma}{}^k_{ij}(\cdot,0)-\overset{a}{\Gamma}{}^k_{ij}(\cdot,T)|_{g^a(\cdot,0),h^a} \\
&\leqslant C(n,k_0,T)\int_0^T e^{\varphi^a}(|\overset{a}{\nabla}\overset{a}{R}_M|_{g^a(\cdot,t)}+|\overset{a}{R}_M*\overset{a}{\nabla}\varphi^a|_{g^a(\cdot,t)} \\
&\quad +|\overset{a}{\nabla}\varphi^a|^3_{g^a(\cdot,t)}+|\overset{a}{\nabla}\varphi^a|_{g^a(\cdot,t)}|\overset{a}{\nabla}{}^2\varphi^a|_{g^a(\cdot,t)}+|\overset{a}{\nabla}{}^3\varphi^a|_{g^a(\cdot,t)})dt \\
&\leqslant C(n,k_0,T)\int_0^T \frac{1}{\sqrt{t}}+|\log t|dt \\
&\leqslant C(n,k_0,T).
\end{aligned}$$

By applying Theorem 2.7, we know that there is $\delta_0>0$ such that $(2.3)_a$ has a smooth solution $\overset{a}{F}$ on $M^n\times[0,\delta_0]$ with estimates (2.22). In views of Lemma 2.12 and Lemma 2.13, in order to prove Theorem 2.6, we only need to bound $|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}$ uniformly on a uniformly interval $[0,T_1]$ with $T_1$ independent of $a$. To this end, let

$$\tilde{T}=\sup\Big\{\tilde{T}_0 \mid \tilde{T}_0\leqslant T, (2.3)_a \text{ has a smooth solution on } M^n\times[0,\tilde{T}_0] \\ \text{with } \sup_{M^n\times[0,\tilde{T}_0]}|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}<\infty\Big\}.$$

We will estimate $\tilde{T}$ from below.

We come back to the equation (2.30) of $|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}$, where there holds

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\overset{a}{\triangle}_t\right)|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\
&\leqslant -2|\overset{a}{\nabla}{}^2\overset{a}{F}|^2_{g^a(\cdot,t),h^a}+C_1(n,k_0,T)|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \\
&\quad +C_2(n,k_0,T)|\overset{a}{\nabla}\overset{a}{F}|^4_{g^a(\cdot,t),h^a}
\end{aligned}$$

on $M^n\times[0,\tilde{T}]$. We remark that $\overset{a}{F}$ is defined on a complete manifold with bounded curvature and $\sup_{M^n\times[0,\tilde{T}_0]}|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}<\infty$, for each $\tilde{T}_0<\tilde{T}$. So by applying the maximum principle on complete manifolds, we have

$$\begin{aligned}
&\frac{d^+}{dt}(\sup_{M^n}|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}) \\
&\leqslant C_1(n,k_0,T)\sup_{M^n}|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}+C_2(n,k_0,T)\sup_{M^n}|\overset{a}{\nabla}\overset{a}{F}|^4_{g^a(\cdot,t),h^a}
\end{aligned}$$

where $\frac{d^+}{dt}$ is the upper right derivative defined by

$$\frac{d^+}{dt}u = \limsup_{\triangle t \searrow 0} \frac{u(t+\triangle t) - u(t)}{\triangle t}.$$

By combining with (2.36), we have

$$\sup_{M^n \times [0,\tilde{T}_0]} |\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a} \leqslant 2ne^{2nk_0T},$$

provided $\tilde{T}_0 \leqslant \min\{T, \frac{\log 2}{C_1(n,k_0,T)+2ne^{2nk_0T}C_2(n,k_0,T)}\}$.

By Lemma 2.12 and Lemma 2.13 and Theorem 2.7, the solution $\overset{a}{F}$ exists smoothly until $|\overset{a}{\nabla}\overset{a}{F}|^2_{g^a(\cdot,t),h^a}$ blows up, so we know

$$\tilde{T} \geqslant \min\left\{T, \frac{\log 2}{C_1(n,k_0,T)+2ne^{2nk_0T}C_2(n,k_0,T)}\right\}.$$

By choosing $T_1 = \min\{T, \frac{\log 2}{C_1(n,k_0,T)+2ne^{2nk_0T}C_2(n,k_0,T)}\}$, Theorem 2.6 follows. q.e.d.

*Proof of Theorem* 2.1. Note that $\varphi^a = 0$ on $B_{g(\cdot,T)}(P,a)$, and $g^a_{ij}(x,t) = e^{\varphi^a}g_{ij}(x,t)$, $h^a_{\alpha\beta}(y) = e^{\varphi^a}h_{\alpha\beta}$. It follows that

$$\begin{aligned} g^a_{ij}(x,t) &= g_{ij}(x,t) \quad \text{on} \;\; B_{g(\cdot,T)}(P,a), \\ h^a_{\alpha\beta}(y) &= h_{\alpha\beta}(y) \quad \text{on} \;\; B_{g(\cdot,T)}(P,a). \end{aligned}$$

By Theorem 2.6 and estimates (2.21) and letting $a \to \infty$, the solutions $\overset{a}{F}$ of $(2.3)_a$ on $M^n \times [0,T_1]$ have a convergent subsequence so that the limit is a solution of (2.3) with the estimates (2.4). q.e.d.

## 3. The uniqueness of the Ricci flow

**3.1. Preliminary estimates for the Ricci-De Turck flow.** Let $F(x,t)$ be a solution to (2.3) in Theorem 2.1 on $M^n \times [0,T_0]$. Let $\tilde{g}_{ij}(x,t) = h_{\alpha\beta}(F(x,t))\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}$ be the one-parameter family of pulled back metrics $F^*h$. We will estimate $g_{ij}(x,t)$ in terms of $\tilde{g}_{ij}(x,t)$.

**Proposition 3.1.** *There exists a constant $0 < T_5 \leqslant T_0$ depending only on $k_0$ and $T$ such that for all $(x,t) \in M^n \times [0,T_5]$, we have*

$$\frac{1}{C(n,k_0,T)}\tilde{g}_{ij}(x,t) \leqslant g_{ij}(x,t) \leqslant C(n,k_0,T)\tilde{g}_{ij}(x,t) \tag{3.1}$$

$$|\tilde{\nabla}^k g|_{\tilde{g}} \leqslant \frac{C(n,k_0,T,k)}{t^{\frac{k-1}{2}}}$$

*for $k = 1, 2, \ldots$.*

*Proof.* We first consider the zero-order estimate of $g_{ij}(x,t)$. The estimate $|\nabla F|^2 = \tilde{g}_{ij}g^{ij} \le C$ in (2.4) implies $\tilde{g}_{ij}(x,t) \le Cg_{ij}(x,t)$. For the reverse inequality, we compute the equation of $\tilde{g}_{ij}(x,t)$ by (2.4):

(3.2)
$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}_{ij} &= \triangle\tilde{g}_{ij} - R_{ik}F_l^\alpha F_j^\beta h_{\alpha\beta}g^{kl} - R_{jk}F_l^\alpha F_i^\beta h_{\alpha\beta}g^{kl} \\ &\quad + 2R_{\alpha\beta\gamma\delta}F_i^\alpha F_k^\beta F_j^\gamma F_l^\delta g^{kl} - 2h_{\alpha\beta}F_{k,i}^\alpha F_{l,j}^\beta g^{kl} \\ &\geqslant \triangle\tilde{g}_{ij} - R_{ik}\tilde{g}_{lj}g^{kl} - R_{jk}\tilde{g}_{li}g^{kl} - 2k_0|\nabla F|^2 g_{ij} - 2|\nabla^2 F|^2 g_{ij} \\ &\geqslant \triangle\tilde{g}_{ij} - R_{ik}\tilde{g}_{lj}g^{kl} - R_{jk}\tilde{g}_{li}g^{kl} - C(n,k_0,T)g_{ij}.\end{aligned}$$

Combining (3.2) with the Ricci flow equation gives

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \triangle\right)\left(\tilde{g}_{ij} + C(n,k_0,T)tg_{ij} - \frac{1}{2ne^{2nk_0T}}g_{ij}\right) \\ &\geqslant -R_{ik}\left(\tilde{g}_{lj} + C(n,k_0,T)tg_{lj} - \frac{1}{2ne^{2nk_0T}}g_{lj}\right)g^{kl} \\ &\quad - R_{jk}\left(\tilde{g}_{li} + C(n,k_0,T)tg_{li} - \frac{1}{2ne^{2nk_0T}}g_{li}\right)g^{kl}.\end{aligned}$$

Note that at $t=0$,

$$\begin{aligned}&\left(\tilde{g}_{ij} + C(n,k_0,T)tg_{ij} - \frac{1}{2ne^{2nk_0T}}g_{ij}\right)|_{t=0} \\ &= g_{ij}(\cdot,T) - \frac{1}{2ne^{2nk_0T}}g_{ij}(\cdot,0) > 0.\end{aligned}$$

By applying the maximum principle to above equation, we obtain

$$\tilde{g}_{ij} + C(n,k_0,T)tg_{ij} - \frac{1}{2ne^{2nk_0T}}g_{ij} > 0$$

on $M^n \times [0,T_0]$. Let $T_5 = \min\{T_0, \frac{1}{4ne^{2nk_0T}C(n,k_0,T)}\}$. Then we have

$$\tilde{g}_{ij} \geqslant \frac{1}{4ne^{2nk_0T}}g_{ij}, \quad \text{on} \quad M^n \times [0,T_5].$$

This gives the zero-order estimate of $g_{ij}(x,t)$.

For the first order derivative of $g_{ij}$, we compute

$$\tilde{\nabla}_k g_{ij} = (\tilde{\nabla}_k - \nabla_k)g_{ij} = (\Gamma_{ki}^l - \tilde{\Gamma}_{ki}^l)g_{lj} + (\Gamma_{kj}^l - \tilde{\Gamma}_{kj}^l)g_{li}$$

and

$$
\begin{aligned}
|(\Gamma^l_{ki} - \tilde{\Gamma}^l_{ki})|^2_{\tilde{g}} &= |(\Gamma^p_{ki} - \tilde{\Gamma}^p_{ki})\tilde{g}_{lp}|^2_{\tilde{g}} \\
&= \left| \nabla_k \nabla_i F^\alpha \frac{\partial F^\beta}{\partial x^l} h_{\alpha\beta} \right|_{\tilde{g}} \\
&\leqslant C(n, k_0, T) \left| \nabla_k \nabla_i F^\alpha \frac{\partial F^\beta}{\partial x^l} h_{\alpha\beta} \right|_g \\
&\leqslant C(n, k_0, T) |\nabla^2 F|_{g,h} |\nabla F|_{g,h} \\
&\leqslant C(n, k_0, T).
\end{aligned}
$$

This gives the first order estimate.

For higher order estimates, we prove it by induction. Suppose we have showed

$$
\begin{aligned}
|\tilde{\nabla}^l g|_{\tilde{g}} &\leqslant \frac{C}{t^{\frac{l-1}{2}}} \qquad \text{for } \ l = 1, 2, \ldots, k-1, \\
|\tilde{\nabla}^l (\Gamma - \tilde{\Gamma})|_{\tilde{g}} &\leqslant \frac{C}{t^{\frac{l}{2}}} \qquad \text{for } \ l = 0, 1, \ldots, k-2.
\end{aligned}
$$

Since by induction

$$
\begin{aligned}
&|\tilde{\nabla}^{k-1}(\Gamma - \tilde{\Gamma})|_{\tilde{g}} \\
&= |\tilde{\nabla}^{k-1}[(\Gamma - \tilde{\Gamma}) * \tilde{g}]|_{\tilde{g}} \\
&= \left| \sum_{j=0}^{k-1} \nabla^{k-1-j}[(\Gamma - \tilde{\Gamma}) * \tilde{g}] * \sum_{i_1+1+\cdots+i_q+1=j} \tilde{\nabla}^{i_1}(\Gamma - \tilde{\Gamma}) * \cdots * \tilde{\nabla}^{i_q}(\Gamma - \tilde{\Gamma}) \right|_{\tilde{g}} \\
&\leqslant C(n, k_0, T) \sum_{j=0}^{k-1} |\nabla^{k-1-j}_{g,h}(\nabla^2 F * \nabla F)|_{g,h} \\
&\qquad \cdot \sum_{i_1+1+\cdots+i_q+1=j} |\tilde{\nabla}^{i_1}(\Gamma - \tilde{\Gamma})|_{\tilde{g}} \cdots |\tilde{\nabla}^{i_q}(\Gamma - \tilde{\Gamma})|_{\tilde{g}} \\
&\leqslant \frac{C(n, k_0, T, k)}{t^{\frac{k-1}{2}}}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\nabla}^k g &= \tilde{\nabla}^{k-1}((\Gamma - \tilde{\Gamma}) * g) \\
&= \sum_{i=0}^{k-1} \tilde{\nabla}^i (\Gamma - \tilde{\Gamma}) * \tilde{\nabla}^{k-1-i} g,
\end{aligned}
$$

then we have

$$
|\tilde{\nabla}^k g|_{\tilde{g}} \leqslant \frac{C}{t^{\frac{k-1}{2}}}.
$$

This completes the induction argument and the proposition is proved. q.e.d.

**Proposition 3.2.** *Let $F(x,t)$ be the solution of (2.3) in Theorem 2.1. Then $F(\cdot,t)$ are diffeomorphisms for all $t \in [0,T_5]$; moreover, there exists a constant $C(n,k_0,T) > 0$ depending only on $n$, $k_0$ and $T$ such that*

$$d_h(F(x_1,t),F(x_2,t)) \geqslant e^{-C(n,k_0,T)} d_h(x_1,x_2)$$

*for all $x_1, x_2 \in M^n$, $t \in [0,T_5]$.*

*Proof.* Note that

$$\frac{1}{C}\tilde{g}_{ij}(x,t) \leq g_{ij}(x,t) \leq C\tilde{g}_{ij}(x,t)$$

implies that $F$ are local diffeomorphisms. So we only need to prove that $F(\cdot,t)$ is injective and proper. Suppose not. Then there exist two points $x_1 \neq x_2$, such that $F(x_1,t) = F(x_2,t)$, for some $t_0 \in (0,T_5]$. Assume $t_0 > 0$ to be the first time so that $F(x_1,t) = F(x_2,t)$. Choose small $\delta > 0$, such that there exist a neighborhood $\tilde{O}$ of $F(x_1,t_0)$ and a neighborhood $O$ of $x_1$ such that $F^{-1}(\cdot,t)$ is a diffeomorphism from $\tilde{O}$ to $O$ for all $t \in [t_0-\delta,t_0]$; moreover, letting $\tilde{\gamma}_t$ be a shortest geodesic( parametrized by arc length) on the target $(N^n, h_{\alpha\beta})$ connecting $F(x_1,t)$ and $F(x_2,t)$, we require $\tilde{\gamma} \in \tilde{O}$ for $t \in [t_0-\delta,t_0]$. We compute

$$\frac{\partial}{\partial t} d_h(F(x_1,t),F(x_2,t)) = \langle V, \tilde{\gamma}'(l)\rangle_h - \langle V, \tilde{\gamma}'(0)\rangle_h$$

where $\tilde{\gamma}(0) = F(x_1,t)$ , $\tilde{\gamma}(l) = F(x_2,t)$, and $V^\alpha = \triangle F^\alpha$. Now we pull back everything by $F^{-1}$ to $O$,

$$\begin{aligned}\frac{\partial}{\partial t} d_h(F(x_1,t),F(x_2,t)) &= \langle P_{-\tilde{\gamma}}V - V, \gamma'(0)\rangle_{F^*h} \\ &\geq -\sup_{x\in F^{-1}\tilde{\gamma}} |\tilde{\nabla} V|(x,t) d_h(F(x_1,t),F(x_2,t))\end{aligned}$$

where $P_{\tilde{\gamma}}$ is the parallel translation along $F^{-1}\tilde{\gamma}$ using the metric $F^*h$.

Since

$$\tilde{\nabla}_k V^l = \nabla_k V^\alpha \frac{\partial x^l}{\partial y^\alpha},$$

where $\nabla_k V^\alpha$ is the covariant derivative of the section $V^\alpha$ of the bundle $F^{-1}TN$, thus by (2.4),

$$|\tilde{\nabla}_k V^l| = [\nabla_k V^\alpha \nabla_l V^\beta h_{\alpha\beta}\tilde{g}^{kl}]^{\frac{1}{2}} \leqslant C|\nabla^3 F| \leqslant \frac{C}{\sqrt{t}}.$$

It follows that we have

$$d_h(F(x_1,t),F(x_2,t)) \leqslant e^{C(\sqrt{t_0}-\sqrt{t_0-\delta})} d_h(F(x_1,t_0),F(x_2,t_0)) = 0,$$

for $t \in [t_0-\delta,t_0]$, which contradicts the choice of $t_0$. So $F(\cdot,t)$ are diffeomorphisms.

By choosing $\tilde{O} = N^n$, $O = M^n$, the above computation also gives

$$d_h(F(x_1,t),F(x_2,t)) \geqslant e^{-C(n,k_0,T)} d_h(x_1,x_2).$$

This in particular implies the properness of the maps, and the proof of the proposition is completed. q.e.d.

**3.2. Ricci De-turck flow.** From the previous section, we know that the harmonic map flow coupled with Ricci flow (2.3) with identity as initial data has a short time solution $F(x,t)$ on $M^n \times [0,T_5]$, which remains being a diffeomorphism with good estimates (2.4). Let $(F^{-1})^* g$ be the one-parameter family of pulled back metrics on the target $(N^n, h_{\alpha\beta})$. Denote $g_{\alpha\beta}(y,t) = ((F^{-1})^* g)_{\alpha\beta}(y,t)$. Then $g_{\alpha\beta}(y,t)$ satisfies the so-called Ricci-De Turck flow:

$$\frac{\partial}{\partial t} g_{\alpha\beta}(y,t) = -2R_{\alpha\beta}(y,t) + \nabla_\alpha V_\beta + \nabla_\beta V_\alpha \tag{3.3}$$

where $V^\alpha = g^{\beta\gamma}(\Gamma^\alpha_{\beta\gamma}(g) - \Gamma^\alpha_{\beta\gamma}(h))$, $\Gamma^\alpha_{\beta\gamma}(g)$ and $\Gamma^\alpha_{\beta\gamma}(h)$ are the Christoffel symbols of the metrics $g_{\alpha\beta}(y,t)$ and $h_{\alpha\beta}(y)$ respectively.

By (3.1) of Proposition 3.1, we already have the following estimates for $g_{\alpha\beta}(y,t)$

$$\frac{1}{C(n,k_0,T)} h_{\alpha\beta}(y) \leqslant g_{\alpha\beta}(y,t) \leqslant C(n,k_0,T) h_{\alpha\beta}(y) \tag{3.4}$$
$$|\nabla^k_h g|_h \leqslant \frac{C(n,k_0,T,k)}{t^{\frac{k-1}{2}}}$$

on $N^n \times [0,T_5]$.

Let $g_{ij}(x,t)$ and $\bar{g}_{ij}(x,t)$ be two solutions to the Ricci flow with bounded curvature and with the same initial value as assumed in Theorem 1.1. We solve the corresponding harmonic map flow with same target $(N^n, h_{\alpha\beta}) = (M^n, g_{ij}(\cdot, T))$ by

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = \triangle F(x,t), \\ F(\cdot, 0) = \text{identity}, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \frac{\partial}{\partial t} \bar{F}(x,t) = \bar{\triangle} \bar{F}(x,t), \\ \bar{F}(\cdot, 0) = \text{identity}, \end{cases} \tag{3.6}$$

respectively. Then we obtain two solutions $F(x,t)$ and $\bar{F}(x,t)$ on $M^n \times [0,T_5]$. It is clear that $\bar{F}(x,t)$ still satisfies (2.4), Proposition 3.1 and Proposition 3.2. Let $\bar{g}_{\alpha\beta}(y,t) = (\bar{F}^{-1})^* \bar{g}(y,t)$, then $\bar{g}_{\alpha\beta}(y,t)$ still satisfies (3.4). Now we have two solutions $g_{\alpha\beta}(y,t)$ and $\bar{g}_{\alpha\beta}(y,t)$ to the Ricci De-Turck flow (3.3) with same initial data and with good estimates (3.4).

**Proposition 3.3.** *There holds*

$$g_{\alpha\beta}(y,t) = \bar{g}_{\alpha\beta}(y,t)$$

*on* $N^n \times [0,T_5]$.

*Proof.* We can write the Ricci-De Turck flow (3.3) by using the fixed metric $h_{\alpha\beta}(y)$ in the following form (see [**21**]):

$$\begin{aligned}(3.7)\quad \frac{\partial}{\partial t}g_{\alpha\beta} &= g^{\gamma\delta}\tilde{\nabla}_\gamma\tilde{\nabla}_\delta g_{\alpha\beta} - g^{\gamma\delta}g_{\alpha\xi}\tilde{g}^{\xi\eta}\tilde{R}_{\beta\gamma\eta\delta} - g^{\gamma\delta}g_{\beta\xi}\tilde{g}^{\xi\eta}\tilde{R}_{\alpha\gamma\eta\delta} \\ &\quad + \frac{1}{2}g^{\gamma\delta}g^{\xi\eta}\Big(\tilde{\nabla}_\alpha g_{\xi\gamma}\tilde{\nabla}_\beta g_{\eta\delta} + 2\tilde{\nabla}_\gamma g_{\beta\xi}\tilde{\nabla}_\eta g_{\alpha\delta} \\ &\quad - 2\tilde{\nabla}_\gamma g_{\beta\xi}\tilde{\nabla}_\delta g_{\alpha\eta} - 2\tilde{\nabla}_\beta g_{\xi\gamma}\tilde{\nabla}_\delta g_{\alpha\eta} - 2\tilde{\nabla}_\alpha g_{\xi\gamma}\tilde{\nabla}_\delta g_{\beta\eta}\Big)\end{aligned}$$

where $\tilde{g}_{\alpha\beta} = h_{\alpha\beta}$, $\tilde{\nabla}$ and $\tilde{R}$ are the covariant derivative and the curvature of $\tilde{g}_{\alpha\beta}$. Note that $\bar{g}_{\alpha\beta}$ also satisfies (3.7), and then the difference $g_{\alpha\beta} - \bar{g}_{\alpha\beta}$ satisfies the following equation:

$$\begin{aligned}&(3.8)\\ &\frac{\partial}{\partial t}(g-\bar{g}) = g^{\gamma\delta}\tilde{\nabla}_\gamma\tilde{\nabla}_\delta(g-\bar{g}) + g^{-1}*\bar{g}^{-1}*\tilde{\nabla}^2\bar{g}*(\bar{g}-g) \\ &\qquad + \bar{g}^{-1}*\tilde{g}^{-1}*\tilde{Rm}*(g-\bar{g}) \\ &\qquad + g^{-1}*\bar{g}^{-1}*g*\tilde{g}^{-1}*\tilde{Rm}*(g-\bar{g}) \\ &\qquad + g^{-1}*g^{-1}*\bar{g}^{-1}*\tilde{\nabla}g*\tilde{\nabla}g*(g-\bar{g}) \\ &\qquad + g^{-1}*\bar{g}^{-1}*\bar{g}^{-1}*\tilde{\nabla}g*\tilde{\nabla}g*(g-\bar{g}) \\ &\qquad + \bar{g}^{-1}*g^{-1}*\tilde{\nabla}g*\tilde{\nabla}(g-\bar{g}) + \bar{g}^{-1}*\bar{g}^{-1}*\tilde{\nabla}\bar{g}*\tilde{\nabla}(g-\bar{g})\end{aligned}$$

since $g^{\alpha\beta} - \bar{g}^{\alpha\beta} = g^{\alpha\xi}\bar{g}^{\eta\beta}(\bar{g}_{\eta\xi} - g_{\eta\xi})$. Let

$$|g-\bar{g}|^2 = \tilde{g}^{\alpha\gamma}\tilde{g}^{\beta\delta}(g_{\alpha\beta}-\bar{g}_{\alpha\beta})(g_{\gamma\delta}-\bar{g}_{\gamma\delta}).$$

It follows from (3.8) that:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - g^{\gamma\delta}\tilde{\nabla}_\gamma\tilde{\nabla}_\delta\right)|g-\bar{g}|^2 \\ &\leqslant -2g^{\xi\eta}\tilde{g}^{\alpha\gamma}\tilde{g}^{\beta\delta}(\tilde{\nabla}_\xi g_{\alpha\beta} - \tilde{\nabla}_\xi\bar{g}_{\alpha\beta})(\tilde{\nabla}_\eta g_{\gamma\delta} - \tilde{\nabla}_\eta\bar{g}_{\gamma\delta}) \\ &\quad + C(n)[|\tilde{Rm}|(1+|g||g^{-1}|)|\bar{g}^{-1}| + |\tilde{\nabla}^2\bar{g}||\bar{g}^{-1}||g^{-1}| \\ &\qquad + |\tilde{\nabla}g|^2(|\bar{g}^{-1}|^2|g^{-1}| + |\bar{g}^{-1}||g^{-1}\|^2)]|g-\bar{g}|^2 \\ &\quad + C(n)|\bar{g}^{-1}|^2(|\tilde{\nabla}g| + |\tilde{\nabla}\bar{g}|)|\tilde{\nabla}(g-\bar{g})||g-\bar{g}|\end{aligned}$$

where all the norms are computed with the metric $\tilde{g} = h$. By Cauchy-Schwartz inequality and (3.4), we have

$$\begin{aligned}(3.9)\qquad &\left(\frac{\partial}{\partial t} - g^{\gamma\delta}\tilde{\nabla}_\gamma\tilde{\nabla}_\delta\right)|g-\bar{g}|^2\\ &\leqslant -2g^{\xi\eta}\tilde{g}^{\alpha\gamma}\tilde{g}^{\beta\delta}(\tilde{\nabla}_\xi g_{\alpha\beta} - \tilde{\nabla}_\xi \bar{g}_{\alpha\beta})(\tilde{\nabla}_\eta g_{\gamma\delta} - \tilde{\nabla}_\eta \bar{g}_{\gamma\delta})\\ &\quad + \frac{C}{\sqrt{t}}|g-\bar{g}|^2 + C|\tilde{\nabla}(g-\bar{g})||g-\bar{g}|\\ &\leqslant \frac{C}{\sqrt{t}}|g-\bar{g}|^2\end{aligned}$$

on $N^n \times [0, T_5]$.

Let $\varphi_1$ be the nonnegative function in Lemma 2.2 with $a = 1$, then

$$\begin{aligned}&\frac{1}{C}(1+\tilde{d}(y,p)) \leqslant \varphi_1(y) \leqslant C_0\tilde{d}(y,p) \qquad \text{on } N^n\backslash B(P,2),\\ &|\tilde{\nabla}\varphi_1| + |\tilde{\nabla}^2\varphi_1| \leqslant C, \qquad \text{on } N^n.\end{aligned}$$

For any fixed $t$ and any $\varepsilon > 0$, consider the maximum of $|g-\bar{g}|^2 - \varepsilon\varphi$. Clearly, the maximum is achieved at some point $P_\varepsilon^t$ and there hold

$$\begin{aligned}|g-\bar{g}|^2(P_\varepsilon^t) &\geqslant |g-\bar{g}|^2(y) - \varepsilon\varphi(y),\\ |\tilde{\nabla}|g-\bar{g}|^2|(P_\varepsilon^t) &\leqslant C\varepsilon,\\ \tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g-\bar{g}|^2(P_\varepsilon^t) &\leqslant C\varepsilon\tilde{g}_{\alpha\beta}(P_\varepsilon^t),\end{aligned}$$

for all $y \in N^n$. This gives

$$\begin{aligned}(3.10)\qquad \limsup_{\varepsilon\to 0}|g-\bar{g}|^2(P_\varepsilon^t) &= \sup|g-\bar{g}|^2\\ g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g-\bar{g}|^2(P_\varepsilon^t) &\leqslant C\varepsilon\end{aligned}$$

by the equivalence of $g$ and $\tilde{g}$.

Define a function

$$|g-\bar{g}|^2_{\max}(t) = \sup_{y\in N^n}|g-\bar{g}|^2(y,t).$$

By (3.9) and (3.10), we have

$$\frac{d^+}{dt}|g-\bar{g}|^2_{\max}(t) \leqslant \frac{C}{\sqrt{t}}|g-\bar{g}|^2_{\max}(t),$$

and then

$$|g-\bar{g}|^2_{\max}(t) \leqslant e^{C\sqrt{T}}|g-\bar{g}|^2_{\max}(0) = 0.$$

Therefore the proof of the Proposition 3.3 is completed. q.e.d.

**3.3. Proof of the main theorem.** Let $g_{ij}(x,t)$ and $\bar{g}_{ij}(x,t)$ be two solutions to the Ricci flow (1.1) with bounded curvature and with the same initial data. We solve the corresponding harmonic map flow (3.5) and (3.6) with the same target $(N^n, h_{\alpha\beta}) = (M^n, g_{ij}(\cdot, T))$ respectively. We obtain two solutions $F(x,t)$ and $\bar{F}(x,t)$ which are diffeomorphisms for $t \in [0, T_5]$, where $T_5 > 0$ depends only on $n, k_0, T$. Then $(F^{-1})^* g$ and $(\bar{F}^{-1})^* \bar{g}$ are two solutions to the Ricci-De Turck flow with the same initial value. It follows from Proposition 3.3 that

$$(F^{-1})^* g = (\bar{F}^{-1})^* \bar{g},$$

on $N^n \times [0, T_5]$. So in order to prove $g_{ij}(x,t) \equiv \bar{g}_{ij}(x,t)$, we only need to show $F \equiv \bar{F}$. Let

$$\begin{aligned} V^\alpha(y,t) &= g^{\beta\gamma}(\tilde{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma}) = -(\triangle F \circ F^{-1})^\alpha \\ \bar{V}^\alpha(y,t) &= \bar{g}^{\beta\gamma}(\tilde{\Gamma}^\alpha_{\beta\gamma} - \bar{\Gamma}^\alpha_{\beta\gamma}) = -(\bar{\triangle} \bar{F} \circ \bar{F}^{-1})^\alpha \end{aligned}$$

be two one-parameter families of vector fields on $N^n$, where $g_{\alpha\beta}(y,t) = ((F^{-1})^* g)_{\alpha\beta}(y,t)$ and $\bar{g}_{\alpha\beta}(y,t) = ((\bar{F}^{-1})^* \bar{g})_{\alpha\beta}(y,t)$. By Proposition 3.3, we have $g_{\alpha\beta}(y,t) = \bar{g}_{\alpha\beta}(y,t)$; thus the vector fields $V \equiv \bar{V}$ on the target $N^n$. Therefore, $F$ and $\bar{F}$ satisfy the same ODE equation with the same initial value:

$$\begin{aligned} \frac{\partial}{\partial t} F &= V \circ F, \\ F(\cdot, 0) &= \text{identity}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \bar{F} &= V \circ \bar{F}, \\ \bar{F}(\cdot, 0) &= \text{identity}. \end{aligned}$$

By the same calculation as in the proof of Proposition 3.2, we have

$$\begin{aligned} \frac{\partial}{\partial t} d_{N^n}(F(x,t), \bar{F}(x,t)) &\leqslant \sup_{y \in N^n} |\tilde{\nabla} V|(y,t) d_{N^n}(F(x,t), \tilde{F}(x,t)) \\ &\leqslant \frac{C}{\sqrt{t}} d_{N^n}(F(x,t), \tilde{F}(x,t)). \end{aligned}$$

This gives

$$d_{N^n}(F(x,t), \bar{F}(x,t)) \leqslant e^{C\sqrt{T}} d_{N^n}(F(x,0), \bar{F}(x,0)) = 0,$$

which concludes that

$$F(x,t) \equiv F(\bar{x}, t).$$

Thus $g(x,t) = \bar{g}(x,t)$, for all $(x,t) \in M^n \times [0, T_5]$ and for some $T_5 > 0$. Clearly, we can extend the interval $[0, T_5]$ to the whole $[0, T]$ by continuity method.

Therefore we complete the proof of the Theorem 1.1. q.e.d.

Finally, Corollary 1.2 is a direct consequence of Theorem 1.1. Indeed, since $G$ is the isometry group of $g_{ij}(x,0)$, then for any $\sigma \in G$, $\sigma^* g(\cdot,t)$ is still a solution to the Ricci flow with bounded curvature and $\sigma^* g(\cdot,t)\ |_{t=0} = \sigma^* g(\cdot,0) = g(\cdot,0)$. By applying Theorem 1.1, we have $\sigma^* g(\cdot,t) = g(\cdot,t)$, $\forall t \in [0,T]$. So the corollary follows. q.e.d.

## References

[1] H.-D. Cao & X.P. Zhu, *A complete proof of the Poincaré and geometrization conjectures - application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. **10**(**2**) (2006) 165–492.

[2] J. Cheeger, M. Gromov, & M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982) 15–53, MR 0658471, Zbl 0493.53035.

[3] B.L. Chen & X.P. Zhu, *Ricci flow with surgery on four manifolds with positive isotropic curvature*, arXiv:math.DG/0504478 v1 April 2005, preprint.

[4] S.Y. Cheng, P. Li, & S.-T. Yau, *On the upper estimate of the heat kernel of complete Riemannian manifold*, Amer. J. Math. **103**(**5**) (1981) 1021–1063. MR 630777, Zbl 0484.53035.

[5] D. De Turck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom., **18** (1983) 157–162, MR 697987, Zbl 0517.53044.

[6] W.Y. Ding & F.H. Lin, *A generalization of Eells-Sampson's theorem*, J. Partial Differential Equations, **5**(**4**) (1992) 13–22, MR 1192714, Zbl 0765.58008.

[7] J. Eells & J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86** (1964) 109–160, MR 0164306, Zbl 0122.40102.

[8] R.E. Greene & H. Wu, *$C^\infty$ approximation of convex, subharmonic and plurisubharmonic functions*, Ann. Sci. Ec. Norm. Sup. **12** (1979) 47–84, MR 532376, Zbl 0415.31001.

[9] R.S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982) 255–306, MR 664497, Zbl 0504.53034.

[10] ———, *Four manifolds with positive curvature operator*, J. Differential Geom. **24** (1986) 153–179, MR 862046, Zbl 0628.53042.

[11] ———, *The Ricci flow on surfaces,* Contemporary Mathematics **71** (1988) 237–261, MR 954419, Zbl 0663.53031.

[12] ———, *A compactness property for solution of the Ricci flow*, Amer. J. Math. **117** (1995) 545–572, MR 1333936, Zbl 0840.53029.

[13] ———, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry (Cambridge, MA, 1993), **2**, 7–136, International Press, Combridge, MA, 1995, MR 1375255, Zbl 0867.53030.

[14] ———, *Four manifolds with positive isotropic curvature*, Comm. in Analysis and Geometry **5** (1997) 1–92; (or see, *Collected Papers on Ricci Flow*, Edited by H.-D. Cao, B. Chow, S.C. Chu and S.-T. Yau, International Press, 2002), MR 1456308, Zbl 0892.53018.

[15] O.A. Ladyzenskaja, V.A. Solonnikov, & N.N. Uralceva, *Linear and quasilinear equations of parabolic type*, Transl. Amer. Math. Soc. **23** (1968), MR 0244627.

[16] P. Li & L.F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math., **105** (1991) 1–46, MR 1109619, Zbl 0748.58006.

[17] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159 v1 November 11, 2002, preprint.

[18] ______, *Ricci flow with surgery on three manifolds*, arXiv:math.DG/0303109 v1 March 10, 2003, preprint.

[19] R. Schoen & S.-T. Yau, *Lectures on differential geometry*, in 'Conference proceedings and Lecture Notes in Geometry and Topology', **1**, International Press, 1994, MR 1336898, Zbl 0830.53001.

[20] ______, *Module space of harmonic maps, compact group actions and the topology of manifolds with non-positive curvature*, Lectures on harmonic maps, International Press, 1997, MR 1474501, Zbl 0886.53004.

[21] W.X. Shi, *Deforming the metric on complete Riemannian manifold*, J. Differential Geometry **30** (1989) 223–301, MR 1001277, Zbl 0676.53044.

Department of Mathematics
Zhongshan University
Guangzhou, P.R. China
*E-mail address*: mc88@zsu.edu.cn
*E-mail address*: stszxp@zsu.edu.cn