J. DIFFERENTIAL GEOMETRY
70 (2005) 325-328

A STRATIFIED RATIONAL HOMOLOGY MANIFOLD VERSION OF THE ATIYAH-BOTT FIXED POINT THEOREM

Dennis Sullivan

The original theorem of the title implies the rational characteristic numbers of a manifold with a circle action can be calculated from the normal bundle data of the fixed point locus of the circle action [**1**]. Thom showed how to define rational characteristic classes and characteristic numbers for rational homology manifolds [**7**]. Around 1970 Raoul Bott and I were considering a very direct proof in Thom's context that was never written by us, but will be discussed in this note. The proof works by constructing a Q-manifold bordism between the Q-manifold with circle action and a version of the projectivized normal bundle of the fixed point locus.

Setting. We assume our spaces are stratified in the sense of Whitney and Thom [**6**], circle actions respect some refined stratifications and the local homology sheaves with rational coefficients are those of a manifold or manifold with boundary. We refer to such objects as Q-manifolds.

Theorem 1.

i) *If a circle action on a Q-manifold (with or without ∂) has no totally fixed points, the quotient is a Q-manifold (with or without ∂).*
ii) *The fixed point locus of a circle action on a Q-manifold is also a Q-manifold.*
iii) *Assume a Q-manifold X (possibly with boundary ∂X) has a circle action with no totally fixed points. Then attaching different cones on each circle orbit creates a Q-manifold $C(X)$ of one dimension higher whose boundary is X if $\partial X = \emptyset$ or X union along ∂X with ∂X union disjoint cones on each of its orbits if the boundary of X is nonempty.*

The proof is described below.

Received 03/25/2005.

Definition 1. If M is a Q-manifold with circle action α and $\mathcal{V}$ is an equivariant regular neighborhood of the Q-manifold F of fixed points we call the union $\mathcal{V}\cup$ (cone on each orbit in $\partial\mathcal{V}$) the *stable projectivized normal bundle* to F and devote it $P(\mathcal{V}, F)$.

Remark. In the smooth case $P(\mathcal{V}, F)$ is a union over components of F of weighted complex projective space bundles with sections. The rational characteristic classes of weighted complex projective spaces themselves (defined by Thom [**7**]) were illuminated in the Ph.D. thesis [**8**] of Don Zagier under Atiyah (1970).

Corollary of Theorem 1 (Synthetic Atiyah Bott Fixed Point Theory)**.** *If M is a Q-manifold with circle action α, there is a canonical Q-manifold bordism W between M and $P(\mathcal{V}, F)$, the stable projectivized normal bundle to F. W carries a natural circle action and in Q-homotopy theory maps to the Borel space, $M_S = M$ x S^∞/circle action associated to α, extending the natural maps $M \xrightarrow{i} M_S$ and $P(\mathcal{V}, F) \xrightarrow{h} M_S$, where h is defined in rational homotopy theory and i is induced by the inclusion.*

Proof of Corollary.

Remove the interior of $\mathcal{V}$ from M to obtain $X = M - (int\mathcal{V})$, where $\mathcal{V}$ is any equivariant regular neighborhood of the fixed point set F. Form W equal to M x I union along X in M x1 with $C(X)$ described in part iii) of Theorem 1. Then ∂W is M x 0 disjoint union $P(\mathcal{V}, F)$.

The circle action α on M extends to W by α x I union cone (α restricted to X). In rational homotopy theory (Q-theory), the Borel construction, if there are no totally fixed points is just the niave quotient because the fibres of the natural map between them are rationally points. Thus, in Q-theory, no fixed points means the space on which we have a circle action is the total space of an oriented S^1 bundle whose base is the Borel construction. Adding a cone on each orbit describes a 2-disk bundle in the Seifert sense over the niave quotient.

These remarks explain the maps $W \to M_S$ and h, whereas i is induced by the inclusion in the first factor.

Proof of Theorem 1.

Part i) follows from the Seifert remark above.

Part ii) follows by looking at links of points and observing the fixed points of a circle action on a Q-manifold which is a Q-homology sphere is also a Q-homology sphere.

Part iii) follows by construction and the remarks above.

Remark 1. Theorem 1 and its corollary constitute an abstract form of the Atiyah Bott fixed point theory. We note here that the reader may enjoy making the above sketches into precise mathematics and developing the above in the direction of intersection homology and the associated characteristic classes as well as the refinements mentioned below in Remarks 3 and 4.

Remark 2. Actually some concreteness is possible in the abstract setting above because Don Zagier's Ph.D. thesis (1970) [**8**] calculated the rational characteristic classes of the weighted complex projective spaces which appear in the above construction.

Remark 3. Since Thom's construction qua arguments for Q-characteristic classes extends using Z/n rational homology manifolds for n odd to $KO \otimes Z[1/2]$ orientations in the context of rational homology manifolds, the above abstract result has a $KO \otimes Z[1/2]$ version. See [**2**], [**3**], [**4**] and [**5**].

Remark 4. Similarly, since the rational Hirzebruch Thom L classes have a natural Z (localized at two) enrichment defined by Thom's arguments using Z/n rational homology manifolds for n a power of two, there is a more precise homological version at the prime 2. See [**2**] and [**5**].

Reminiscences. Like most of my 60's generation, I was greatly influenced by the works of and the contacts with Raoul Bott. When I moved from Berkeley to Cambridge in the fall of 1969, Raoul and Phyliss Bott welcomed me and helped me to get settled. My daughters Lori and Amanda (6 and 4 at the time) heard me say the sequence $Z/2$ $Z/2$ 0 Z O O O Z (Bott's 8-fold periodicity leading to KO-theory) so often they made it into a song. I remember walking with Raoul near Harvard Square that fall while he compared compact complex manifolds to bronze and gold sculptures.

Raoul, in his early forties, also confided to me that he yearned to make at least one more original contribution. This was before the celebrated Bott vanishing theory for foliations, the work with Atiyah on the moment map and equivariant homology for Hamiltonian actions, the work on moduli space of stable holomorphic bundles, the work with Taubes on finite type invariants of 3-manifolds and so on.

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Mathematics Department
SUNY, Stony Brook
New York, NY 11794
E-mail address: dennis@math.sunysb.edu