

**NON-ABELIAN LOCALIZATION FOR CHERN-SIMONS
THEORY**

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Abstract

We reconsider Chern-Simons gauge theory on a Seifert manifold M (the total space of a nontrivial circle bundle over a Riemann surface Σ). When M is a Seifert manifold, Lawrence and Rozansky have shown from the exact solution of Chern-Simons theory that the partition function has a remarkably simple structure and can be rewritten entirely as a sum of local contributions from the flat connections on M . We explain how this empirical fact follows from the technique of non-abelian localization as applied to the Chern-Simons path integral. In the process, we show that the partition function of Chern-Simons theory on M admits a topological interpretation in terms of the equivariant cohomology of the moduli space of flat connections on M .

1. Introduction

Chern-Simons gauge theory is remarkable for the deep connections it bears to an array of otherwise disparate topics in mathematics and physics. For instance, Chern-Simons theory is intimately related to the theory of knot invariants and the topology of three-manifolds [1, 2], to two-dimensional rational conformal field theory [3] via a holographic correspondence, to three-dimensional quantum gravity [4, 5, 6, 7], to the open string field theory of the topological A-model [8], and via a large N duality to the Gromov-Witten theory of non-compact Calabi-Yau threefolds [9, 10, 11, 12, 13].

Of course, Chern-Simons theory is also a topological gauge theory, though of a very exotic sort. In the case of a more conventional topological gauge theory such as topological Yang-Mills theory on a Riemann surface or on a four-manifold (for a review of both topics, see [14]), the theory can be fundamentally interpreted in terms of the cohomology ring

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of some classical moduli space of connections. In this sense, such gauge theories are themselves essentially classical. In contrast, Chern-Simons theory is intrinsically a quantum theory, and it is exotic precisely because it does not admit a general mathematical interpretation in terms of the cohomology of some classical moduli space of connections.

Yet if we consider Chern-Simons theory not on a general three-manifold M but only on three-manifolds which are of a simple sort and which perhaps carry additional geometric structure, then we might expect Chern-Simons theory itself to simplify. In particular, we might hope that the theory in this case admits a more conventional mathematical interpretation in terms of the cohomology of some classical moduli space of connections.

For instance, in the very special case that M is just the product of S^1 and a Riemann surface Σ , so that $M = S^1 \times \Sigma$, then the partition function Z of Chern-Simons theory on M does have a well-known topological interpretation. In this case, Z is the dimension of the Chern-Simons Hilbert space, obtained from canonical quantization on $\mathbb{R} \times \Sigma$. In turn, this Hilbert space can be interpreted geometrically as the space of global holomorphic sections of a certain line bundle over the moduli space \mathcal{M}_0 of flat connections on Σ .

If we consider for simplicity Chern-Simons theory with gauge group $G = SU(r + 1)$ at level k , then the relevant line bundle over \mathcal{M}_0 is the k -th power of a universal determinant line \mathcal{L} on \mathcal{M}_0 . Of course, the moduli space \mathcal{M}_0 is singular at the points corresponding to the reducible flat connections on Σ . However, suitably interpreted, the index theorem in combination with the Kodaira vanishing theorem for the higher cohomology of \mathcal{L}^k still yields a topological expression for Z ,

$$(1.1) \quad Z(k) = \dim H^0(\mathcal{M}_0, \mathcal{L}^k) = \chi(\mathcal{M}_0, \mathcal{L}^k) = \int_{\mathcal{M}_0} \exp(k \Omega') \operatorname{Td}(\mathcal{M}_0),$$

where $\Omega' = c_1(\mathcal{L})$ is the first Chern class of \mathcal{L} and $\operatorname{Td}(\mathcal{M}_0)$ is the Todd class of \mathcal{M}_0 .

In this paper, we show that the Chern-Simons partition function has an analogous topological interpretation on a related but much broader class of three-manifolds. Specifically, we consider the case that M is a Seifert manifold, so that M can be succinctly described as the total space of a *nontrivial* circle bundle over a Riemann surface Σ ,

$$(1.2) \quad S^1 \longrightarrow M \xrightarrow{\pi} \Sigma,$$

where, as we later explain, Σ is generally allowed to have orbifold points and the circle bundle is allowed to be a corresponding orbifold bundle.

In this case, our fundamental result is to reinterpret the Chern-Simons partition function as a topological quantity determined entirely by a suitable equivariant cohomology ring on the moduli space of flat connections on M . Because the moduli space of flat connections on M is directly related to the moduli space of solutions of the Yang-Mills equation on Σ , our result implies that Chern-Simons theory on M can be also be interpreted as a two-dimensional topological theory on Σ akin, in a way which we make precise, to two-dimensional Yang-Mills theory. This two-dimensional interpretation of Chern-Simons theory on M has also been noted recently by Aganagic and collaborators in [15], where the theory is identified with a q -deformed version of two-dimensional Yang-Mills theory. For other work on relations between Chern-Simons theory and two-dimensional Yang-Mills theory, see [16, 17, 18, 19].

Of course, physical Yang-Mills theory on a Riemann surface Σ also has a well-known topological interpretation in terms of intersection theory on the moduli space \mathcal{M}_0 of flat connections on Σ . This interpretation follows from the technique of non-abelian localization, as applied to the Yang-Mills path integral [20]. In an analogous fashion, we arrive at our new interpretation of Chern-Simons theory by applying non-abelian localization to the Chern-Simons path integral,

$$(1.3) \quad Z(k) = \int \mathcal{D}A \exp \left[i \frac{k}{4\pi} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right].$$

As we recall in Section 4, non-abelian localization provides a method for computing symplectic integrals of the canonical form

$$(1.4) \quad Z(\epsilon) = \int_X \exp \left[\Omega - \frac{1}{2\epsilon} (\mu, \mu) \right].$$

Here X is an arbitrary symplectic manifold with symplectic form Ω . We assume that a Lie group H acts on X in a Hamiltonian fashion, with moment map $\mu : X \rightarrow \mathfrak{h}^*$, where \mathfrak{h}^* is the dual of the Lie algebra \mathfrak{h} of H . Finally, (\cdot, \cdot) is an invariant quadratic form on \mathfrak{h} and dually on \mathfrak{h}^* which we use to define the action $S = \frac{1}{2}(\mu, \mu)$, and ϵ is a coupling parameter.

As we briefly review in Section 2, the path integral of Yang-Mills theory on a Riemann surface immediately takes the canonical form in (1.4), where the affine space of all connections on a fixed principal bundle plays the role of X and where the group of gauge transformations plays the role of H . In contrast, the path integral (1.3) of Chern-Simons

theory on a Seifert manifold is not manifestly of this required form. Nonetheless, in Section 3 we show that this path integral can be cast into the form (1.4) for which non-abelian localization applies. More abstractly, we show that Chern-Simons theory on a Seifert manifold has a symplectic interpretation generalizing the classic interpretation due to Atiyah and Bott [21] of two-dimensional Yang-Mills theory.

Because the path integral of Chern-Simons theory on a Seifert manifold M assumes the canonical form (1.4), we deduce as an immediate corollary that the path integral localizes on critical points of the Chern-Simons action, which are the flat connections on M . In fact, this observation has been made previously by Lawrence and Rozansky [22, 23] (and later generalized by Mariño in [24]) as an entirely empirical statement deduced from the known formula for the exact partition function. For a selection of explicit computations of the Chern-Simons partition function, see for instance [25, 26, 27, 28, 29, 30, 31, 32].

Considering $SU(2)$ Chern-Simons theory on a Seifert homology sphere M , Lawrence and Rozansky managed to recast the known formula for $Z(k)$, which initially involves an unwieldy sum over the integrable representations of an $SU(2)$ WZW model at level k , into a simple sum of contour integrals and residues which can be formally identified with the contributions from the flat connections on M in the stationary phase approximation to the path integral.

A very simple example of a Seifert manifold is S^3 , by virtue of the Hopf fibration over $\mathbb{C}P^1$. The result of Lawrence and Rozansky in the case of $SU(2)$ Chern-Simons theory on S^3 then amounts to rewriting the well-known expression for $Z(k)$ as below,

$$(1.5) \quad Z(k) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \\ = \frac{1}{2\pi i} e^{-\frac{i\pi}{k+2}} \int_{-\infty}^{+\infty} dx \sinh^2\left(\frac{1}{2} e^{\frac{i\pi}{4}} x\right) \exp\left(-\frac{(k+2)}{8\pi} x^2\right).$$

We note that, when the hyperbolic sine is expressed as a sum of exponentials, the integral in (1.5) becomes a sum of elementary Gaussian integrals which conspire to produce the standard expression for $Z(k)$. Because the only flat connection on S^3 is the trivial connection, the integral over x in (1.5) is to be identified with the stationary phase contribution from the trivial connection to the path integral.

So one immediate application of our work here is to provide an underlying mathematical explanation for the phenomenological results in

[22, 23, 24]. In fact, we will apply localization to the Chern-Simons path integral to derive directly the expression of Lawrence and Rozansky in (1.5) for the partition function on S^3 . One amusing aspect of this computation is that we will see the famous shift in the level $k \rightarrow k + 2$.

In order to perform concrete computations in Chern-Simons theory using localization, we must have a thorough understanding of the local symplectic geometry near each flat connection. As we will see, this local geometry shares important features with the local geometry near the higher, unstable critical points of Yang-Mills theory on a Riemann surface.

Thus, as a warmup for our computations in Chern-Simons theory, we begin in Section 4 by discussing localization for Yang-Mills theory. We first review the computation in [20] of the contribution to the path integral from flat Yang-Mills connections, corresponding to the stable minima of the Yang-Mills action, and then we extend this result to compute precisely the contributions from the higher, unstable critical points as well. Localization at the unstable critical points of Yang-Mills theory has been studied previously in the physics literature by Blau and Thompson [33] and (most recently) in the mathematics literature by Woodward and Teleman [34, 35], but we find it useful to supplement these references with another discussion more along the lines of [20]. Of course, the roots of our work on localization trace back to the beautiful equivariant interpretation by Atiyah and Bott [36] of the Duistermaat-Heckman formula [37].

In Section 5 we then apply localization to perform path integral computations in Chern-Simons theory on a Seifert manifold. As mentioned above, these computations depend on the nature of the local symplectic geometry near each critical point, and for illustration we consider two extreme cases.

First, we consider localization at the trivial connection on a Seifert homology sphere. In this case, the first homology group of M is zero, $H_1(M, \mathbb{Z}) = 0$, and the trivial connection is an isolated flat connection. On the other hand, all constant gauge transformations on M fix the trivial connection, and this large isotropy group, isomorphic to the gauge group G itself, plays an important role in the localization. Here we directly derive a formula found by Lawrence and Rozansky in [22] and generalized by Mariño in [24].

Second, we consider localization on a smooth component of the moduli space of flat connections. Such a component consists of irreducible

connections, for which the isotropy group arises solely from the center of G . In this case, we derive a formula originally obtained by Rozansky in [23] by again working empirically from the known formula for the partition function.

Finally, although we will not elaborate on this perspective here, one of the original motivations for our study of localization in Chern-Simons theory was to place computations in this theory into a theoretical framework analogous to the framework of abelian localization in the topological A -model of open and closed strings (see Chapter 9 of [38] for a nice mathematical review of abelian localization in the closed string A -model).

Special Note

We would like to thank Raoul Bott for his inspiration. Many of us learned much of our differential topology from the book by Bott and Tu [39]. The second author first learned of equivariant cohomology from Bott, in 1983. This was in the context of Bott explaining the mathematical context for certain results that had been suggested in [40], following an earlier lecture given by Bott at a physics conference [41] where the second author and many other physicists had heard of Morse theory for the first time.

2. The Symplectic Geometry of Yang-Mills Theory on a Riemann Surface

A central theme of this paper is the close relationship between Chern-Simons theory on a Seifert manifold M and Yang-Mills theory on the associated Riemann surface Σ . Thus, as a prelude to our discussion of the path integral of Chern-Simons theory on M , we begin by recalling how the path integral of Yang-Mills theory on Σ can be understood as a symplectic integral of the canonical form (1.4).

In fact, we start by considering the path integral of Yang-Mills theory on a compact Riemannian manifold Σ of *arbitrary* dimension, so that

$$(2.1) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G}(P))} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta_{\mathcal{G}(P)}/2} \int_{\mathcal{A}(P)} \mathcal{D}A \exp \left[\frac{1}{2\epsilon} \int_{\Sigma} \text{Tr} (F_A \wedge \star F_A) \right],$$

$$\Delta_{\mathcal{G}(P)} = \dim \mathcal{G}(P).$$

Here $F_A = dA + A \wedge A$ is the curvature of the connection A . We assume that the Yang-Mills gauge group G is compact, connected, and simple. If $G = SU(r + 1)$, then “Tr” in (2.1) denotes the trace in the fundamental representation. With our conventions, A is an anti-hermitian element of the Lie algebra of $SU(r + 1)$, so that the trace determines a negative-definite quadratic form. For more general G , “Tr” denotes the unique invariant, negative-definite quadratic form on the Lie algebra \mathfrak{g} of G which is normalized so that, for simply-connected G , the Chern-Simons level k in (1.3) obeys the conventional integral quantization. Of course, the parameter ϵ is related to the Yang-Mills coupling g via $\epsilon = g^2$.

In order to define Z formally, we fix a principal G -bundle P over Σ . Then the space $\mathcal{A}(P)$ over which we integrate is the space of connections on P . The group $\mathcal{G}(P)$ of gauge transformations acts on $\mathcal{A}(P)$, and we have normalized Z in (2.1) by dividing by the volume of $\mathcal{G}(P)$ and a formal power of ϵ . As we review in Section 4, this normalization of Z is the natural normalization when Σ is a Riemann surface and we apply non-abelian localization to compute Z .

The space $\mathcal{A}(P)$ is an affine space, which means that, if we choose a particular basepoint A_0 in $\mathcal{A}(P)$, then we can identify $\mathcal{A}(P)$ with its tangent space at A_0 . This tangent space is the vector space of sections of the bundle $\Omega_\Sigma^1 \otimes \text{ad}(P)$ of one-forms on Σ taking values in the adjoint bundle associated to P . In other words, an arbitrary connection A on P can be written as $A = A_0 + \eta$ for some section η of $\Omega_\Sigma^1 \otimes \text{ad}(P)$.

Of course, to discuss an integral over $\mathcal{A}(P)$ even formally, we must also discuss the measure $\mathcal{D}A$ that appears in (2.1). Because the space $\mathcal{A}(P)$ is affine, we can define $\mathcal{D}A$ up to an overall multiplicative constant by taking any translation-invariant measure on $\mathcal{A}(P)$.

In general, the Yang-Mills action is only defined once we choose a metric on Σ , which induces a corresponding duality operator \star , as appears in (2.1). This duality operator \star induces a metric on $\mathcal{A}(P)$ such that if η is any tangent vector to $\mathcal{A}(P)$, then the norm of η is defined by

$$(2.2) \quad (\eta, \eta) = - \int_\Sigma \text{Tr} (\eta \wedge \star \eta).$$

Thus, a convenient way to represent the path integral measure and to fix its normalization is to take $\mathcal{D}A$ to be the Riemannian measure induced by the metric (2.2) on $\mathcal{A}(P)$. We also use the operator \star to define a similar invariant metric on $\mathcal{G}(P)$, which formally determines the volume of $\mathcal{G}(P)$.

Although we generally require a metric on Σ to define physical Yang-Mills theory, when Σ is a Riemann surface we actually need much less geometric structure to define the theory. In this case, to define the Yang-Mills action in (2.1) we only require a duality operator \star which relates the zero-forms and the two-forms on Σ . In turn, to define such an operator we require only a symplectic structure with associated symplectic form ω on Σ , so that \star is defined by $\star 1 = \omega$.

The symplectic form ω is invariant under all area-preserving diffeomorphisms of Σ , and this large group acts as a symmetry of two-dimensional Yang-Mills theory. More precisely, this symmetry group is “large” in the sense that its complexification is the full group of orientation-preserving diffeomorphisms of Σ [42]. This fact is fundamentally responsible for the topological nature of two-dimensional Yang-Mills theory.

Furthermore, when Σ is a Riemann surface, the affine space $\mathcal{A}(P)$ acquires additional geometric structure. First, $\mathcal{A}(P)$ has a natural symplectic form Ω . If η and ξ are any two tangent vectors to $\mathcal{A}(P)$, then Ω is defined by

$$(2.3) \quad \Omega(\eta, \xi) = - \int_{\Sigma} \text{Tr} (\eta \wedge \xi).$$

Clearly Ω is closed and non-degenerate. Second, $\mathcal{A}(P)$ has a natural complex structure. This complex structure is associated to the duality operator \star itself, since $\star^2 = -1$ when acting on the tangent space of $\mathcal{A}(P)$. Finally, the metric on $\mathcal{A}(P)$ is manifestly Kahler with respect to this symplectic form and complex structure, since we see that the metric defined by (2.2) can be rewritten as $\Omega(\cdot, \star \cdot)$.

An important consequence of the fact that the metric on $\mathcal{A}(P)$ is Kahler when Σ is a Riemann surface is that the Riemannian measure $\mathcal{D}A$ on $\mathcal{A}(P)$ is actually the same as the symplectic measure defined by Ω . If X is a symplectic manifold of dimension $2n$ with symplectic form Ω , then the symplectic measure on X is given by the top-form $\Omega^n/n!$. This measure can be represented uniformly for X of arbitrary dimension by the expression $\exp(\Omega)$, where we implicitly pick out from the series expansion of the exponential the term which is of top degree on X . Consequently, because the Riemannian and the symplectic measures on $\mathcal{A}(P)$ agree, we can formally replace $\mathcal{D}A$ in the Yang-Mills path integral (2.1) by the expression $\exp(\Omega)$, as in the canonical symplectic integral (1.4). This natural symplectic measure on $\mathcal{A}(P)$ makes no reference to the metric on Σ .

The Yang-Mills Action as the Square of the Moment Map

Of course, as an affine space, $\mathcal{A}(P)$ is pretty boring. What makes Yang-Mills theory interesting is the fact that $\mathcal{A}(P)$ is acted on by the group $\mathcal{G}(P)$ of gauge transformations. In fact, another special consequence of considering Yang-Mills theory on a Riemann surface is that the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is Hamiltonian with respect to the symplectic form Ω .

To recall what the Hamiltonian condition implies, we consider the general situation that a connected Lie group H with Lie algebra \mathfrak{h} acts on a symplectic manifold X preserving the symplectic form Ω . The action of H on X is then Hamiltonian when there exists an algebra homomorphism from \mathfrak{h} to the algebra of functions on X under the Poisson bracket. The Poisson bracket of functions f and g on X is given by $\{f, g\} = -V_f(g)$, where V_f is the Hamiltonian vector field associated to f . This vector field is determined by the relation $df = \iota_{V_f}\Omega$, where ι_{V_f} is the interior product with V_f . More explicitly, in local canonical coordinates on X , the components of V_f are determined by f as $V_f^m = -(\Omega^{-1})^{mn} \partial_n f$, where Ω^{-1} is an “inverse” to Ω that arises by considering the symplectic form as an isomorphism $\Omega : TM \rightarrow T^*M$ with inverse $\Omega^{-1} : T^*M \rightarrow TM$. In coordinates, Ω^{-1} is defined by $(\Omega^{-1})^{lm} \Omega_{mn} = \delta_n^l$, and $\{f, g\} = \Omega_{mn} V_f^m V_g^n$. The algebra homomorphism from the Lie algebra \mathfrak{h} to the algebra of functions on X under the Poisson bracket is then specified by a moment map $\mu : X \rightarrow \mathfrak{h}^*$, under which an element ϕ of \mathfrak{h} is sent to the function $\langle \mu, \phi \rangle$ on X , where $\langle \cdot, \cdot \rangle$ is the dual pairing between \mathfrak{h} and \mathfrak{h}^* .

The moment map by definition satisfies the relation

$$(2.4) \quad d\langle \mu, \phi \rangle = \iota_{V(\phi)}\Omega,$$

where $V(\phi)$ is the vector field on X which is generated by the infinitesimal action of ϕ . In terms of μ , the Hamiltonian condition then becomes the condition that μ also satisfy

$$(2.5) \quad \{\langle \mu, \phi \rangle, \langle \mu, \psi \rangle\} = \langle \mu, [\phi, \psi] \rangle.$$

Geometrically, the equation (2.5) is an infinitesimal expression of the condition that the moment map μ commute with the action of H on X and the coadjoint action of H on \mathfrak{h}^* .

Returning from this abstract discussion to the case of Yang-Mills theory on Σ , we first consider the moment map for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$, as originally discussed in [21]. Elements of the Lie algebra of $\mathcal{G}(P)$ are represented by sections of the adjoint bundle $\text{ad}(P)$ on Σ , so

if ϕ is such a section then the corresponding vector field $V(\phi)$ on $\mathcal{A}(P)$ is given as usual by

$$(2.6) \quad V(\phi) = d_A \phi = d\phi + [A, \phi].$$

We then compute directly using (2.3),

$$(2.7) \quad \iota_{V(\phi)} \Omega = - \int_{\Sigma} \text{Tr} (d_A \phi \wedge \delta A) = \int_{\Sigma} \text{Tr} (\phi d_A \delta A) = \delta \int_{\Sigma} \text{Tr} (F_A \phi).$$

Here we write δ for the exterior derivative acting on $\mathcal{A}(P)$, so that, for instance, δA is regarded as a one form on $\mathcal{A}(P)$. Thus, the relation (2.4) determines, up to an additive constant, that the moment map μ for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is

$$(2.8) \quad \mu = F_A.$$

Here we regard F_A , being a section of $\Omega_{\Sigma}^2 \otimes \text{ad}(P)$, as an element of the dual of the Lie algebra of $\mathcal{G}(P)$.

One can then check directly that μ in (2.8) satisfies the condition (2.5) that it arise from a Lie algebra homomorphism, and this condition fixes the arbitrary additive constant that could otherwise appear in μ to be zero. Thus, $\mathcal{G}(P)$ acts in a Hamiltonian fashion on $\mathcal{A}(P)$ with moment map given by $\mu = F_A$. In particular, if we introduce the obvious positive-definite, invariant quadratic form on the Lie algebra of $\mathcal{G}(P)$, defined by

$$(2.9) \quad (\phi, \phi) = - \int_{\Sigma} \text{Tr} (\phi \wedge \star \phi),$$

then the Yang-Mills action S is proportional to the square of the moment map,

$$(2.10) \quad S = -\frac{1}{2} \int_{\Sigma} \text{Tr} (F_A \wedge \star F_A) = \frac{1}{2} (\mu, \mu).$$

As a result, the path integral of Yang-Mills theory on Σ can be recast completely in terms of the symplectic data associated to the Hamiltonian action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$,

$$(2.11) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G}(P))} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta_{\mathcal{G}(P)}/2} \int_{\mathcal{A}(P)} \exp \left[\Omega - \frac{1}{2\epsilon} (\mu, \mu) \right],$$

precisely as in (1.4).

3. The Symplectic Geometry of Chern-Simons Theory on a Seifert Manifold

In this section, we explain how the path integral of Chern-Simons theory on a Seifert manifold can be recast as a symplectic integral of the canonical form (1.4) which is suitable for non-abelian localization. More generally, we explain some beautiful facts about the symplectic geometry of Chern-Simons theory on a Seifert manifold.

To set up notation, we consider Chern-Simons theory on a three-manifold M with compact, connected, simply-connected, and simple gauge group G . With these assumptions, any principal G -bundle P on M is necessarily trivial, and we denote by \mathcal{A} the affine space of connections on the trivial bundle. We denote by \mathcal{G} the group of gauge transformations acting on \mathcal{A} .

We begin with the Chern-Simons path integral,

$$(3.1) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta_{\mathcal{G}}} \int_{\mathcal{A}} \mathcal{D}A \exp \left[\frac{i}{2\epsilon} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right],$$

$$\epsilon = \frac{2\pi}{k}, \quad \Delta_{\mathcal{G}} = \dim \mathcal{G}.$$

We have introduced a coupling parameter ϵ by analogy to the canonical integral in (1.4), and we have included a number of formal factors in Z . First, we have the measure $\mathcal{D}A$ on \mathcal{A} , which we define up to norm as a translation-invariant measure on \mathcal{A} . As is standard, we have also divided the path integral by the volume of the gauge group \mathcal{G} . Finally, to be fastidious, we have normalized Z by a formal power of ϵ which, as in (2.1), will be natural in defining Z by localization.

3.1. A New Formulation of Chern-Simons Theory. At the moment, we make no assumption about the three-manifold M . However, if M is an S^1 bundle over a Riemann surface Σ , or an orbifold thereof, then to reduce Chern-Simons theory on M to a topological theory on Σ we must eventually decouple one of the three components of the gauge field A . This observation motivates the following general reformulation of Chern-Simons theory, which proves to be key to the rest of the paper.

In order to decouple one of the components of A , we begin by choosing a one-dimensional subbundle of the cotangent bundle T^*M of M . Locally on M , this choice can be represented by the choice of an everywhere non-zero one-form κ , so that the subbundle of T^*M consists of all one-forms proportional to κ . However, if t is any non-zero function,

then clearly κ and $t\kappa$ generate the same subbundle in T^*M . Thus, our choice of a one-dimensional subbundle of T^*M corresponds locally to the choice of an equivalence class of one-forms under the relation

$$(3.2) \quad \kappa \sim t\kappa.$$

We note that the representative one-form κ which generates the subbundle need only be defined locally on M . Globally, the subbundle might or might not be generated by a non-zero one-form which is defined everywhere on M ; this condition depends upon whether the sign of κ can be consistently defined under (3.2) and thus whether the subbundle is orientable or not.

We now attempt to decouple one of the three components of A . Specifically, our goal is to reformulate Chern-Simons theory on M as a theory which respects a new local symmetry under which A varies as

$$(3.3) \quad \delta A = \sigma\kappa.$$

Here σ is an *arbitrary* section of the bundle $\Omega_M^0 \otimes \mathfrak{g}$ of Lie algebra-valued functions on M .

The Chern-Simons action certainly does not respect the local “shift” symmetry in (3.3). However, we can trivially introduce this shift symmetry into Chern-Simons theory if we simultaneously introduce a new scalar field Φ on M which transforms like A in the adjoint representation of the gauge group. Under the shift symmetry, Φ transforms as

$$(3.4) \quad \delta\Phi = \sigma.$$

Now, if κ in (3.3) is scaled by a non-zero function t so that $\kappa \rightarrow t\kappa$, then this rescaling can be absorbed into the arbitrary section σ which also appears in (3.3) so that the transformation law for A is well-defined. However, from the transformation (3.4) of Φ under the same symmetry, we see that because we absorb t into σ we must postulate an inverse scaling of Φ , so that $\Phi \rightarrow t^{-1}\Phi$. As a result, although κ is only locally defined up to scale, the product $\kappa\Phi$ is well-defined on M .

The only extension of the Chern-Simons action which now incorporates both Φ and the shift symmetry is the Chern-Simons functional $CS(\cdot)$ of the shift invariant combination $A - \kappa\Phi$. Thus, we consider the theory with action

$$(3.5) \quad S(A, \Phi) = CS(A - \kappa\Phi),$$

or more explicitly,

$$(3.6) \quad S(A, \Phi) = CS(A) - \int_M \left[2\kappa \wedge \text{Tr}(\Phi F_A) - \kappa \wedge d\kappa \text{Tr}(\Phi^2) \right].$$

To proceed, we play the usual game used to derive field theory dualities by path integral manipulations, as for T -duality in two dimensions [43, 44] or abelian S -duality in four dimensions [45]. We have introduced a new degree of freedom, namely Φ , into Chern-Simons theory, and we have simultaneously enlarged the symmetry group of the theory so that this degree of freedom is completely gauge trivial. As a result, we can either use the shift symmetry (3.4) to gauge Φ away, in which case we recover the usual description of Chern-Simons theory, or we can integrate Φ out, in which case we obtain a new description of Chern-Simons theory which respects the action of the shift symmetry (3.3) on A .

A Contact Structure on M

Hitherto, we have supposed that the one-dimensional subbundle of T^*M represented by κ is arbitrary, but at this point we must impose an important geometric condition on this subbundle. From the action $S(A, \Phi)$ in (3.6), we see that the term quadratic in Φ is multiplied by the local three-form $\kappa \wedge d\kappa$. In order for this quadratic term to be everywhere non-degenerate on M , so that we can easily perform the path integral over Φ , we require that $\kappa \wedge d\kappa$ is also everywhere non-zero on M .

Although κ itself is only defined locally and up to rescaling by a non-zero function t , the condition that $\kappa \wedge d\kappa \neq 0$ pointwise on M is a globally well-defined condition on the subbundle generated by κ . For when κ scales as $\kappa \rightarrow t\kappa$ for any non-zero function t , we easily see that $\kappa \wedge d\kappa$ also scales as $\kappa \wedge d\kappa \rightarrow t^2 \kappa \wedge d\kappa$. Thus, the condition that $\kappa \wedge d\kappa \neq 0$ is preserved under arbitrary rescalings of κ .

The structure which we thus introduce on M is the choice of a one-dimensional subbundle of T^*M for which any local generator κ satisfies $\kappa \wedge d\kappa \neq 0$ at each point of M . This geometric structure, which appears so naturally here, is known as a contact structure [46, 47, 48]. More generally, on an arbitrary manifold M of odd dimension $2n+1$, a contact structure on M is defined as a one-dimensional subbundle of T^*M for which the local generator κ satisfies $\kappa \wedge (d\kappa)^n \neq 0$ everywhere on M .

In many ways, a contact structure is the analogue of a symplectic structure for manifolds of odd dimension. The fact that we must choose a contact structure on M for our reformulation of Chern-Simons theory

is thus closely related to the fact, mentioned previously, that we must choose a symplectic structure on the Riemann surface Σ in order to define Yang-Mills theory on Σ .

We will say a bit more about contact structures on Seifert manifolds later, but for now, we just observe that, by a classic theorem of Martinet [49], any compact, orientable¹ three-manifold possesses a contact structure.

Path Integral Manipulations

Thus, we choose a contact structure on the three-manifold M , and we consider the theory defined by the path integral

$$(3.7) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G})} \frac{1}{\text{Vol}(\mathcal{S})} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta\mathcal{G}} \times \\ \times \int \mathcal{D}A \mathcal{D}\Phi \exp \left[\frac{i}{2\epsilon} \left(CS(A) - \int_M 2\kappa \wedge \text{Tr}(\Phi F_A) + \int_M \kappa \wedge d\kappa \text{Tr}(\Phi^2) \right) \right].$$

Here the measure $\mathcal{D}\Phi$ is defined independently of any metric on M by the invariant, positive-definite quadratic form

$$(3.8) \quad (\Phi, \Phi) = - \int_M \kappa \wedge d\kappa \text{Tr}(\Phi^2),$$

which is invariant under the scaling $\kappa \rightarrow t\kappa$, $\Phi \rightarrow t^{-1}\Phi$. We similarly use this quadratic form to define formally the volume of the group \mathcal{S} of shift symmetries, as appears in the normalization of (3.7).

Using the shift symmetry (3.4), we can fix $\Phi = 0$ trivially, with unit Jacobian, and the resulting group integral over \mathcal{S} produces a factor of $\text{Vol}(\mathcal{S})$ to cancel the corresponding factor in the normalization of $Z(\epsilon)$. Hence, the new theory defined by (3.7) is fully equivalent to Chern-Simons theory.

On the other hand, because the field Φ appears only quadratically in the action (3.6), we can also perform the path integral over Φ directly. Upon integrating out Φ , the new action $S(A)$ for the gauge field becomes

$$(3.9) \quad S(A) = \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \int_M \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left[(\kappa \wedge F_A)^2 \right].$$

We find it convenient to abuse notation slightly by writing “ $1/\kappa \wedge d\kappa$ ” in (3.9). To explain this notation precisely, we observe that, as $\kappa \wedge d\kappa$ is

¹We note that, because $\kappa \wedge d\kappa \rightarrow t^2 \kappa \wedge d\kappa$ under a local rescaling of κ and because t^2 is always positive, the sign of the local three-form $\kappa \wedge d\kappa$ is well-defined. So any three-manifold with a contact structure is necessarily orientable.

nonvanishing, we can always write $\kappa \wedge F_A = \varphi \kappa \wedge d\kappa$ for some function φ on M taking values in the Lie algebra \mathfrak{g} . Thus, we set $\kappa \wedge F_A / \kappa \wedge d\kappa = \varphi$, and the second term in $S(A)$ becomes $\int_M \kappa \wedge \text{Tr} (F_A \varphi)$. As our notation in (3.9) suggests, this term is invariant under the transformation $\kappa \rightarrow t \kappa$, since φ transforms as $\varphi \rightarrow t^{-1} \varphi$.

By construction, the new action $S(A)$ in (3.9) is invariant under the action of the shift symmetry (3.3) on A . We can directly check this invariance once we note that, under the shift symmetry, the expression $\kappa \wedge F_A$ transforms as

$$(3.10) \quad \kappa \wedge F_A \longrightarrow \kappa \wedge F_A + \sigma \kappa \wedge d\kappa.$$

The partition function $Z(\epsilon)$ now takes the form

$$(3.11) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G})} \frac{1}{\text{Vol}(\mathcal{S})} \left(\frac{-i}{2\pi\epsilon} \right)^{\Delta_{\mathcal{G}}/2} \times \\ \times \int_{\mathcal{A}} \mathcal{D}A \exp \left[\frac{i}{2\epsilon} \left(\int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \int_M \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left[(\kappa \wedge F_A)^2 \right] \right) \right],$$

where the Gaussian integral over Φ cancels some factors of $2\pi\epsilon$ in the normalization of Z . As is standard, in integrating over Φ we assume that the integration contour has been slightly rotated off the real axis, effectively giving ϵ a small imaginary part, to regulate the oscillatory Gaussian integral. Thus, the theory described by the path integral (3.11) is fully equivalent to Chern-Simons theory, but now one component of A manifestly decouples.

3.2. Contact Structures on Seifert Manifolds. Our reformulation of Chern-Simons theory in (3.11) applies to any three-manifold M with a specified contact structure. However, in order to apply non-abelian localization to Chern-Simons theory on M , we require that M has additional symmetry.

Specifically, we require that M admits a locally-free $U(1)$ action, which means that the generating vector field on M associated to the infinitesimal action of $U(1)$ is nowhere vanishing. A free $U(1)$ action on M clearly satisfies this condition, but more generally it is satisfied by any $U(1)$ action such that no point on M is fixed by all of $U(1)$ (at such a point the generating vector field would vanish). Such an action need not be free, since some points on M could be fixed by a cyclic subgroup

of $U(1)$. The class of three-manifolds which admit a $U(1)$ action of this sort are precisely the Seifert manifolds [50].

To proceed further to a symplectic description of Chern-Simons theory, we now restrict attention to the case that M is a Seifert manifold. We first review a few basic facts about such manifolds, for which a complete reference is [50].

M Admits a Free $U(1)$ Action

For simplicity, we begin by assuming that the three-manifold M admits a free $U(1)$ action. In this case, M is the total space of a circle bundle over a Riemann surface Σ ,

$$(3.12) \quad S^1 \longrightarrow M \xrightarrow{\pi} \Sigma,$$

and the free $U(1)$ action simply arises from rotations in the fiber of (3.12). The topology of M is completely determined by the genus g of Σ and the degree n of the bundle. Assuming that the bundle is nontrivial, we can always arrange by a suitable choice of orientation for M that $n \geq 1$.

At this point, one might wonder why we restrict attention to the case of nontrivial bundles over Σ . As we now explain, in this case M admits a natural contact structure which is invariant under the action of $U(1)$. As a result, our reformulation of Chern-Simons theory in (3.11) still respects this crucial symmetry of M .

To describe this $U(1)$ invariant contact structure on M , we simply exhibit an invariant one-form κ , defined globally on M , which satisfies the contact condition that $\kappa \wedge d\kappa$ is nowhere vanishing. To describe κ , we begin by choosing a symplectic form ω on Σ which is normalized so that

$$(3.13) \quad \int_{\Sigma} \omega = 1.$$

Regarding M as the total space of a principal $U(1)$ -bundle, we take κ to be a connection on this bundle (and hence a real-valued one-form on M) whose curvature satisfies

$$(3.14) \quad d\kappa = n\pi^*\omega,$$

where we recall that $n \geq 1$ is the degree of the bundle. For a nice, explicit description of κ in this situation, see the description of the angular form in [39, §6].

We let R (for “rotation”) be the non-vanishing vector field on M which generates the $U(1)$ action and which is normalized so that its

orbits have unit period. By the fundamental properties of a connection, κ is invariant under the $U(1)$ action and satisfies $\langle \kappa, R \rangle = 1$. Here we use $\langle \cdot, \cdot \rangle$ generally to denote the canonical dual pairing. Thus, κ pulls back to a non-zero one-form which generates the integral cohomology of each S^1 fiber of M , and we immediately see from (3.14) that $\kappa \wedge d\kappa$ is everywhere non-vanishing on M so long as the bundle is nontrivial.

For future reference, we note that the integral of $\kappa \wedge d\kappa$ over M is determined as follows. Because κ satisfies $\langle \kappa, R \rangle = 1$, where R is the generator of the $U(1)$ action whose orbits correspond to the S^1 fibers over Σ in (3.12), the integral of κ over any such fiber is given by

$$(3.15) \quad \int_{S^1} \kappa = 1.$$

Upon integrating over the S^1 fiber of M , we see from (3.13), (3.14), and (3.15) that

$$(3.16) \quad \int_M \kappa \wedge d\kappa = n \int_M \kappa \wedge \pi^* \omega = n \int_\Sigma \omega = n.$$

Orbifold Generalization

Of course, in the above construction we have assumed that M admits a free $U(1)$ action, which is a more stringent requirement than the condition that no point of M is completely fixed by the $U(1)$ action. However, an arbitrary Seifert manifold does admit an orbifold description precisely analogous to the description of M as a principal $U(1)$ -bundle over a Riemann surface. This point of view is taken in a nice paper by Furuta and Steer [51] for an application somewhat related to ours, and we follow their basic exposition below.

To generalize our previous discussion to the case of an arbitrary Seifert manifold, we simply replace the Riemann surface Σ with an orbifold, and we replace the principal $U(1)$ -bundle over Σ with its orbifold counterpart. Concretely, the orbifold base $\widehat{\Sigma}$ of M is now described by a Riemann surface of genus g with N marked points p_j , $j = 1, \dots, N$, at which the coordinate neighborhoods are modeled not on \mathbb{C} but on $\mathbb{C}/\mathbb{Z}_{\alpha_j}$ for some cyclic group \mathbb{Z}_{α_j} , which acts on the local coordinate z at p_j as

$$(3.17) \quad z \mapsto \zeta \cdot z, \quad \zeta = e^{2\pi i/\alpha_j}.$$

The choice of the particular orbifold points p_j is topologically irrelevant, and the orbifold base $\widehat{\Sigma}$ can be completely specified by the genus g and the set of integers $\{\alpha_1, \dots, \alpha_N\}$.

We now consider a line V -bundle over $\widehat{\Sigma}$. Such an object is precisely analogous to a complex line bundle, except that the local trivialization over each orbifold point p_j of $\widehat{\Sigma}$ is now modeled on $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_{\alpha_j}$, where \mathbb{Z}_{α_j} acts on the local coordinates (z, s) of the base and fiber as

$$(3.18) \quad z \mapsto \zeta \cdot z, \quad s \mapsto \zeta^{\beta_j} \cdot s, \quad \zeta = e^{2\pi i/\alpha_j},$$

for some integers $0 \leq \beta_j < \alpha_j$.

Given such a line V -bundle over $\widehat{\Sigma}$, an arbitrary Seifert manifold M can be described as the total space of the associated S^1 fibration. Of course, we require that M itself is smooth. This condition implies that each pair of integers (α_j, β_j) above must be relatively prime so that the local action (3.18) of the orbifold group \mathbb{Z}_{α_j} on $\mathbb{C} \times S^1$ is free (in particular, we require $\beta_j \neq 0$ above).

The $U(1)$ action on M again arises from rotations in the fibers over $\widehat{\Sigma}$, but this action is no longer free. Rather, the points in the S^1 fiber over each ramification point p_j of $\widehat{\Sigma}$ are fixed by the cyclic subgroup \mathbb{Z}_{α_j} of $U(1)$, due to the orbifold identification in (3.18).

Once the integers $\{\beta_1, \dots, \beta_N\}$ are fixed, the topological isomorphism class of a line V -bundle on $\widehat{\Sigma}$ is specified by a single integer n , the degree. Thus, in total, the description of an arbitrary Seifert manifold M is given by the Seifert invariants

$$(3.19) \quad \left[g; n; (\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N) \right], \quad \gcd(\alpha_j, \beta_j) = 1.$$

Because the basic notions of bundles, connections, curvatures, and (rational) characteristic classes generalize immediately from smooth manifolds to orbifolds [52, 53], our previous construction of an invariant contact form κ as a connection on a principal $U(1)$ -bundle immediately generalizes to the orbifold situation here. In this case, if $\widehat{\mathcal{L}}$ denotes the line V -bundle over $\widehat{\Sigma}$ which describes M , with Seifert invariants (3.19), then $\widehat{\mathcal{L}}$ is nontrivial so long as its Chern class is non-zero (and positive by convention),

$$(3.20) \quad c_1(\widehat{\mathcal{L}}) = n + \sum_{j=1}^N \frac{\beta_j}{\alpha_j} > 0,$$

which generalizes our previous condition that $n \geq 1$. In particular, n can now be any integer such that the expression in (3.20) is positive.

In the Chern-Weil description of the Chern class, $c_1(\widehat{\mathcal{L}})$ is represented by smooth curvature in the bulk of the orbifold $\widehat{\Sigma}$. In contrast, the

degree n receives contributions from both the bulk curvature in $\widehat{\Sigma}$ and from local, delta-function curvatures at the orbifold points of $\widehat{\Sigma}$. That is why n is an integer but the orbifold first Chern class $c_1(\widehat{\mathcal{L}})$ is not. The delta-function contributions to n are cancelled by the rational numbers β_j/α_j appearing explicitly in the formula (3.20) for $c_1(\widehat{\mathcal{L}})$.

From (3.20), to define a contact structure on M we choose the connection κ so that its curvature is given by

$$(3.21) \quad d\kappa = \left(n + \sum_{j=1}^N \frac{\beta_j}{\alpha_j} \right) \pi^* \widehat{\omega},$$

where $\widehat{\omega}$ is a symplectic form on $\widehat{\Sigma}$ of unit volume, as in (3.13). Then, exactly as in (3.16), the integral of $\kappa \wedge d\kappa$ over M is determined by the Chern class of $\widehat{\mathcal{L}}$,

$$(3.22) \quad \int_M \kappa \wedge d\kappa = n + \sum_{j=1}^N \frac{\beta_j}{\alpha_j}.$$

For future reference, we also note that the Riemann-Roch formula for a line bundle on a Riemann surface has a direct generalization to the case of a line V -bundle on an orbifold [54], so that

$$(3.23) \quad \chi(\widehat{\mathcal{L}}) = \dim_{\mathbb{C}} H^0(\widehat{\Sigma}, \widehat{\mathcal{L}}) - \dim_{\mathbb{C}} H^1(\widehat{\Sigma}, \widehat{\mathcal{L}}) = n + 1 - g,$$

which justifies calling n the degree of $\widehat{\mathcal{L}}$.

In this discussion, we have used the notation $\widehat{\Sigma}$ and $\widehat{\mathcal{L}}$ to distinguish these orbifold quantities from their smooth counterparts Σ and \mathcal{L} . In the future, we will not make this artificial distinction, and in our discussion of Chern-Simons theory we will use Σ and \mathcal{L} to denote general orbifold quantities.

3.3. A Symplectic Structure For Chern-Simons Theory. We now specialize to the case of Chern-Simons theory on a Seifert manifold M , which carries a distinguished $U(1)$ action and an invariant contact form κ . Initially, the path integral of Chern-Simons theory on M is an integral over the affine space \mathcal{A} of all connections on M . Unlike the case of two-dimensional Yang-Mills theory, \mathcal{A} is not naturally symplectic and cannot play the role of the symplectic manifold X that appears in the canonical symplectic integral (1.4).

However, we now reap the reward of our reformulation of Chern-Simons theory to decouple one component of A . Specifically, we consider

the following two-form Ω on \mathcal{A} . If η and ξ are any two tangent vectors to \mathcal{A} , and hence are represented by sections of the bundle $\Omega_M^1 \otimes \mathfrak{g}$ on M , then we define Ω by

$$(3.24) \quad \Omega(\eta, \xi) = - \int_M \kappa \wedge \text{Tr} (\eta \wedge \xi).$$

Because κ is a globally-defined one-form on M , this expression is well-defined. Further, Ω is closed and invariant under all the symmetries. In particular, Ω is invariant under the group \mathcal{S} of shift symmetries, and by virtue of this shift invariance Ω is degenerate along tangent vectors to \mathcal{A} of the form $\sigma\kappa$, where σ is an arbitrary section of $\Omega_M^0 \otimes \mathfrak{g}$. However, unlike the gauge symmetry \mathcal{G} , which acts nonlinearly on \mathcal{A} , the shift symmetry \mathcal{S} acts in a simple, linear fashion on \mathcal{A} . Thus, we can trivially take the quotient of \mathcal{A} by the action of \mathcal{S} , which we denote as $\overline{\mathcal{A}}$,

$$(3.25) \quad \overline{\mathcal{A}} = \mathcal{A}/\mathcal{S}.$$

Under this quotient, the presymplectic form Ω on \mathcal{A} descends immediately to a symplectic form on $\overline{\mathcal{A}}$, which becomes a symplectic space naturally associated to Chern-Simons theory on M . In the following, $\overline{\mathcal{A}}$ plays the role of the abstract symplectic manifold X in (1.4).

More About the Path Integral Measure

Our reformulation of the Chern-Simons action $S(A)$ in (3.9) is invariant under the shift symmetry \mathcal{S} , so $S(A)$ descends to the quotient $\overline{\mathcal{A}}$ of \mathcal{A} by \mathcal{S} . But we should also think (at least formally) about the path integral measure $\mathcal{D}A$. As in Yang-Mills theory, we define $\mathcal{D}A$ up to norm as a translation-invariant measure on \mathcal{A} , and a convenient way both to describe $\mathcal{D}A$ and to fix its normalization is to consider this measure as induced from a Riemannian metric on \mathcal{A} . In turn, we describe this metric on \mathcal{A} as induced from a corresponding metric on M , so that a tangent vector η to \mathcal{A} has norm

$$(3.26) \quad (\eta, \eta) = - \int_M \text{Tr} (\eta \wedge \star \eta).$$

We normalize the volume of \mathcal{G} in (3.1) using the similarly induced, invariant metric on \mathcal{G} .

We assume that $U(1)$ acts on M by isometries, so that the metric on M associated to the operator \star in (3.26) takes the form

$$(3.27) \quad ds_M^2 = \pi^* ds_\Sigma^2 + \kappa \otimes \kappa.$$

Here ds_Σ^2 represents any Kahler metric on Σ which is normalized so that the corresponding Kahler form pulls back to $d\kappa$. As a result of this normalization convention, the duality operator \star defined by the metric (3.27) satisfies $\star 1 = \kappa \wedge d\kappa$.

Tangent vectors to the orbits of the shift symmetry \mathcal{S} are described by sections of $\Omega_M^1 \otimes \mathfrak{g}$ which take the form $\sigma\kappa$, where σ is any function taking values in \mathfrak{g} on M . Similarly, tangent vectors to the quotient $\overline{\mathcal{A}}$ are naturally represented by sections of $\Omega_M^1 \otimes \mathfrak{g}$ which are annihilated by the interior product ι_R with the vector field R , the generator of the $U(1)$ action on M . When the metric on M takes the form in (3.27), the one-forms annihilated by ι_R are orthogonal to the one-forms proportional to κ . Thus, the tangent space to \mathcal{S} is orthogonal to the tangent space to $\overline{\mathcal{A}}$ in the corresponding metric (3.26) on \mathcal{A} .

We can exhibit the orthogonal decomposition of the metric in (3.26) explicitly as

$$(3.28) \quad (\eta, \eta) = - \int_M \kappa \wedge d\kappa \operatorname{Tr} \left[(\iota_R \eta)^2 \right] - \int_M \kappa \wedge \operatorname{Tr} \left[\Pi(\eta) \wedge \star_2 \Pi(\eta) \right].$$

The first term in (3.28) describes the metric on \mathcal{S} which we have already introduced in (3.8), and the second term describes the induced metric on $\overline{\mathcal{A}}$. The form of the first term follows immediately from the fact that $\star\kappa = d\kappa$.

In the second term of (3.28), we have introduced two natural operators. First, we introduce the operator Π which projects from the tangent space of \mathcal{A} to the tangent space of $\overline{\mathcal{A}}$, so that Π is given by

$$(3.29) \quad \Pi(\eta) = \eta - (\iota_R \eta) \kappa.$$

Trivially, $\iota_R \circ \Pi = 0$.

Second, we introduce an effective “two-dimensional” duality operator \star_2 on M which induces a corresponding complex structure on $\overline{\mathcal{A}}$. This operator is defined globally on M by

$$(3.30) \quad \star_2 = -\iota_R \circ \star.$$

Using that $\star\kappa = d\kappa$ and $\star 1 = \kappa \wedge d\kappa$, we see immediately that $\star_2 \kappa = \star_2 (\kappa \wedge d\kappa) = 0$ and that $\star_2 1 = -d\kappa$. Also, one can easily check (for instance by considering local coordinates) that \star_2 satisfies $(\star_2)^2 = -1$ when acting on one-forms in the image of Π , representing tangent vectors to $\overline{\mathcal{A}}$. This latter property is important, since it implies that \star_2 defines a complex structure on $\overline{\mathcal{A}}$ exactly as in two-dimensional Yang-Mills theory.

With this notation in place, the form of the second term in (3.28) follows immediately from the simple computation below,

$$\begin{aligned}
 (3.31) \quad \Pi(\eta) \wedge \star \Pi(\eta) &= \iota_R \left(\kappa \wedge \Pi(\eta) \right) \wedge \star \Pi(\eta), \\
 &= -\kappa \wedge \Pi(\eta) \wedge \iota_R \left(\star \Pi(\eta) \right), \\
 &= \kappa \wedge \Pi(\eta) \wedge \star_2 \Pi(\eta).
 \end{aligned}$$

In passing from the first to the second line of (3.31), we have “integrated by parts” with respect to the operator ι_R , as $\iota_R(\kappa \wedge \Pi(\eta) \wedge \star \Pi(\eta))$ is trivially zero on the three-manifold M by dimensional reasons.

We thus see from the second term in (3.28) that the induced metric on $\overline{\mathcal{A}}$ is Kahler with respect to the symplectic form Ω in (3.24) and the complex structure \star_2 . Hence the Riemannian measure induced on $\overline{\mathcal{A}}$ from (3.28) is identical to the symplectic measure induced by Ω .

Finally, because the measure along the orbits of \mathcal{S} in \mathcal{A} is the same as the invariant measure (3.8) which we defined on \mathcal{S} itself, we can trivially integrate over these orbits, which simply contribute a factor of the volume $\text{Vol}(\mathcal{S})$ to the path integral. Consequently, the Chern-Simons path integral in (3.11) reduces to an integral over $\overline{\mathcal{A}}$ with its symplectic measure,

$$\begin{aligned}
 (3.32) \quad Z(\epsilon) &= \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{-i}{2\pi\epsilon} \right)^{\Delta_{\mathcal{G}}/2} \int_{\overline{\mathcal{A}}} \exp \left[\Omega + \frac{i}{2\epsilon} S(A) \right], \\
 S(A) &= \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \int_M \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left[(\kappa \wedge F_A)^2 \right].
 \end{aligned}$$

3.4. Hamiltonian Symmetries of Chern–Simons Theory . To complete our symplectic description of the Chern-Simons path integral on M , we must show that the action $S(A)$ in (3.32) is the square of a moment map μ for the Hamiltonian action of some symmetry group \mathcal{H} on the symplectic space $\overline{\mathcal{A}}$.

By analogy to the case of Yang-Mills theory on Σ , one might naively guess that the relevant symmetry group for Chern-Simons theory would also be the group \mathcal{G} of gauge transformations. One can easily check that the action of \mathcal{G} on \mathcal{A} descends under the quotient to a well-defined action on $\overline{\mathcal{A}}$, and clearly the symplectic form Ω on $\overline{\mathcal{A}}$ is invariant under \mathcal{G} . However, one interesting aspect of non-abelian localization for Chern-Simons theory is the fact that the group \mathcal{H} which we use for localization must be somewhat more complicated than \mathcal{G} itself.

A trivial objection to using \mathcal{G} for localization is that, by construction, the square of the moment map $\overline{\mu}$ for any Hamiltonian action on $\overline{\mathcal{A}}$ defines an invariant function on $\overline{\mathcal{A}}$, but the action $S(A)$ is not invariant under the group \mathcal{G} . Instead, the action $S(A)$ is the sum of a manifestly gauge invariant term and the usual Chern-Simons action, and the Chern-Simons action shifts by integer multiples of 2π under “large” gauge transformations, those not continuously connected to the identity in \mathcal{G} .

This trivial objection is easily overcome. We consider not the disconnected group \mathcal{G} of all gauge transformations but only the identity component \mathcal{G}_0 of this group, under which $S(A)$ is invariant.

We now consider the action of \mathcal{G}_0 on $\overline{\mathcal{A}}$, and our first task is to determine the corresponding moment map μ . If ϕ is an element of the Lie algebra of \mathcal{G}_0 , described by a section of the bundle $\Omega_M^0 \otimes \mathfrak{g}$ on M , then the corresponding vector field $V(\phi)$ generated by ϕ on \mathcal{A} is given by $V(\phi) = d_A\phi$. Thus, from our expression for the symplectic form Ω in (3.24) we see that

$$(3.33) \quad \iota_{V(\phi)}\Omega = - \int_M \kappa \wedge \text{Tr} (d_A\phi \wedge \delta A).$$

Integrating by parts with respect to d_A , we can rewrite (3.33) in the form $\delta\langle\mu, \phi\rangle$, where

$$(3.34) \quad \langle\mu, \phi\rangle = \int_M \kappa \wedge \text{Tr} (\phi F_A) - \int_M d\kappa \wedge \text{Tr} (\phi(A - A_0)).$$

Here A_0 is an arbitrary connection, corresponding to a basepoint in \mathcal{A} , which we must choose so that the second term in (3.34) can be honestly interpreted as the integral of a differential form on M . In the case that the gauge group G is simply-connected, so that the principal G -bundle over M is necessarily trivial, the choice of a basepoint connection A_0 corresponds geometrically to the choice of a trivialization for the bundle on M . We will say more about this choice momentarily, but we first observe that the expression for μ in (3.34) is invariant under the shift symmetry and immediately descends to a moment map for the action of \mathcal{G} on $\overline{\mathcal{A}}$.

The fact that we must choose a basepoint A_0 in \mathcal{A} to define the moment map is very important in the following, and it is fundamentally a reflection of the affine structure of \mathcal{A} . In general, an affine space is a space which can be identified with a vector space only after some basepoint is chosen to represent the origin. In the case at hand, once A_0 is chosen, we can identify \mathcal{A} with the vector space of sections η of the

bundle $\Omega_M^1 \otimes \mathfrak{g}$ on M , via $A = A_0 + \eta$, as we used in (3.34). However, \mathcal{A} is not naturally itself a vector space, since \mathcal{A} does not intrinsically possess a distinguished origin. This statement corresponds to the geometric statement that, though our principal G -bundle on M is trivial, it does not possess a canonical trivialization.

In terms of the moment map μ , the choice of A_0 simply represents the possibility of adding an arbitrary constant to μ . In general, our ability to add a constant to μ means that μ need *not* determine a Hamiltonian action of \mathcal{G}_0 on $\overline{\mathcal{A}}$. Indeed, as we show below, the action of \mathcal{G}_0 on $\overline{\mathcal{A}}$ is not Hamiltonian and we cannot simply use \mathcal{G}_0 to perform localization.

In order not to clutter the expressions below, we assume henceforth that we have fixed a trivialization of the G -bundle on M and we simply set $A_0 = 0$.

To determine whether the action of \mathcal{G}_0 on $\overline{\mathcal{A}}$ is Hamiltonian, we must check the condition (2.5) that μ determine a homomorphism from the Lie algebra of \mathcal{G}_0 to the algebra of functions on $\overline{\mathcal{A}}$ under the Poisson bracket. So we directly compute

$$\begin{aligned}
 (3.35) \quad \left\{ \langle \mu, \phi \rangle, \langle \mu, \psi \rangle \right\} &= \Omega(d_A \phi, d_A \psi) \\
 &= - \int_M \kappa \wedge \text{Tr} (d_A \phi \wedge d_A \psi), \\
 &= \int_M \kappa \wedge \text{Tr} ([\phi, \psi] F_A) - \int_M d\kappa \wedge \text{Tr} (\phi d_A \psi), \\
 &= \langle \mu, [\phi, \psi] \rangle - \int_M d\kappa \wedge \text{Tr} (\phi d\psi).
 \end{aligned}$$

Thus, the failure of μ to determine an algebra homomorphism is measured by the cohomology class of the Lie algebra cocycle

$$\begin{aligned}
 (3.36) \quad c(\phi, \psi) &= \left\{ \langle \mu, \phi \rangle, \langle \mu, \psi \rangle \right\} - \langle \mu, [\phi, \psi] \rangle, \\
 &= - \int_M d\kappa \wedge \text{Tr} (\phi d\psi) \\
 &= - \int_M \kappa \wedge d\kappa \wedge \text{Tr} (\phi \mathcal{L}_R \psi).
 \end{aligned}$$

In the second line of (3.36), we have rewritten the cocycle more suggestively by using the Lie derivative \mathcal{L}_R along the vector field R on M which generates the $U(1)$ action. The class of this cocycle is not zero, and no Hamiltonian action on $\overline{\mathcal{A}}$ exists for the group \mathcal{G}_0 .

Some Facts About Loop Groups

The cocycle appearing in (3.36) has a very close relationship to a similar cocycle that arises in the theory of loop groups, and some well-known loop group constructions feature heavily in our study of Chern-Simons theory. We briefly review these ideas, for which a general reference is [55].

When G is a finite-dimensional Lie group, we recall that the loop group LG is defined as the group of smooth maps $\text{Map}(S^1, G)$ from S^1 to G . Similarly, the Lie algebra $L\mathfrak{g}$ of LG is the algebra $\text{Map}(S^1, \mathfrak{g})$ of smooth maps from S^1 to \mathfrak{g} . When \mathfrak{g} is simple, then the Lie algebra $L\mathfrak{g}$ admits a unique, G -invariant cocycle up to scale, and this cocycle is directly analogous to the cocycle we discovered in (3.36). If ϕ and ψ are elements in the Lie algebra $L\mathfrak{g}$, then this cocycle is defined by

$$(3.37) \quad c(\phi, \psi) = - \int_{S^1} \text{Tr}(\phi d\psi) = - \int_{S^1} dt \text{Tr}(\phi \mathcal{L}_R \psi).$$

In passing to the last expression, we have by analogy to (3.36) introduced a unit-length parameter t on S^1 , so that $\int_{S^1} dt = 1$, and we have introduced the dual vector field $R = \partial/\partial t$ which generates rotations of S^1 .

In general, if \mathfrak{g} is any Lie algebra and c is a nontrivial cocycle, then c determines a corresponding central extension $\tilde{\mathfrak{g}}$ of \mathfrak{g} ,

$$(3.38) \quad \mathbb{R} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}.$$

As a vector space, $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, and the Lie algebra of $\tilde{\mathfrak{g}}$ is given by the bracket

$$(3.39) \quad [(\phi, a), (\psi, b)] = ([\phi, \psi], c(\phi, \psi)),$$

where ϕ and ψ are elements of \mathfrak{g} , and a and b are elements of \mathbb{R} .

In the case of the Lie algebra $L\mathfrak{g}$, the cocycle c appearing in (3.37) consequently determines a central extension $\widetilde{L\mathfrak{g}}$ of $L\mathfrak{g}$. When G is simply connected, the extension determined by c or any integral multiple of c lifts to a corresponding extension of LG by $U(1)$,

$$(3.40) \quad U(1) \longrightarrow \widetilde{LG} \longrightarrow LG.$$

Topologically, the extension \widetilde{LG} is the total space of the S^1 bundle over LG whose Euler class is represented by the cocycle of the extension, interpreted as an invariant two-form on LG . The fact that the Euler class must be integral is responsible for the corresponding quantization condition on the cocycle of the extension.

When \mathfrak{g} is simple, the algebra $L\mathfrak{g}$ has a non-degenerate, invariant inner product which is unique up to scale and is given by

$$(3.41) \quad (\phi, \psi) = - \int_{S^1} dt \operatorname{Tr} (\phi\psi).$$

On the other hand, the corresponding extension $\widetilde{L\mathfrak{g}}$ does *not* possess a non-degenerate, invariant inner product, since any element of $\widetilde{L\mathfrak{g}}$ can be expressed as a commutator, so that $[\widetilde{L\mathfrak{g}}, \widetilde{L\mathfrak{g}}] = \widetilde{L\mathfrak{g}}$, and the center of $\widetilde{L\mathfrak{g}}$ is necessarily orthogonal to every commutator under an invariant inner product.

However, we can also consider the semidirect product $U(1) \ltimes \widetilde{LG}$. Here, the rigid $U(1)$ action on S^1 induces a natural $U(1)$ action on \widetilde{LG} by which we define the product, and the important observation about this group $U(1) \ltimes \widetilde{LG}$ is that it does admit an invariant, non-degenerate inner product on its Lie algebra.

Explicitly, the Lie algebra of $S^1 \ltimes \widetilde{LG}$ is identified with $\mathbb{R} \oplus \widetilde{L\mathfrak{g}} = \mathbb{R} \oplus L\mathfrak{g} \oplus \mathbb{R}$ as a vector space, and the Lie algebra is given by the bracket

$$(3.42) \quad \left[(p, \phi, a), (q, \psi, b) \right] = \left(0, [\phi, \psi] + p\mathcal{L}_R\psi - q\mathcal{L}_R\phi, c(\phi, \psi) \right),$$

where \mathcal{L}_R is the Lie derivative with respect to the vector field R generating rotations of S^1 . We then consider the manifestly non-degenerate inner product on $\mathbb{R} \oplus \widetilde{L\mathfrak{g}}$ which is given by

$$(3.43) \quad \left((p, \phi, a), (q, \psi, b) \right) = - \int_M dt \operatorname{Tr} (\phi\psi) - pb - qa.$$

One can directly check that this inner product is invariant under the adjoint action determined by (3.42). We note that although this inner product is non-degenerate, it is not positive-definite because of the last two terms in (3.43).

Extension To Chern-Simons Theory

We now return to our original problem, which is to find a Hamiltonian action of a group \mathcal{H} on $\overline{\mathcal{A}}$ to use for localization. The natural guess to consider the identity component \mathcal{G}_0 of the gauge group does not work, because the cocycle c in (3.36) obstructs the action of \mathcal{G}_0 on $\overline{\mathcal{A}}$ from being Hamiltonian.

However, motivated by the loop group constructions, we consider now the central extension $\widetilde{\mathcal{G}}_0$ of \mathcal{G}_0 by $U(1)$ which is determined by the

cocycle c in (3.36),

$$(3.44) \quad U(1) \longrightarrow \widetilde{\mathcal{G}}_0 \longrightarrow \mathcal{G}_0.$$

We assume that the central $U(1)$ subgroup of $\widetilde{\mathcal{G}}_0$ acts trivially on $\overline{\mathcal{A}}$, so that the moment map for the central generator $(0, a)$ of the Lie algebra is constant. Then, by construction, we see from (3.36) and (3.39) that the new moment map for the action of $\widetilde{\mathcal{G}}_0$ on $\overline{\mathcal{A}}$, which is given by

$$(3.45) \quad \langle \mu, (\phi, a) \rangle = \int_M \kappa \wedge \text{Tr} (\phi F_A) - \int_M d\kappa \wedge \text{Tr} (\phi A) + a,$$

satisfies the Hamiltonian condition

$$(3.46) \quad \left\{ \langle \mu, (\phi, a) \rangle, \langle \mu, (\psi, b) \rangle \right\} = \left\langle \mu, [(\phi, a), (\psi, b)] \right\rangle.$$

The action of the extended group $\widetilde{\mathcal{G}}_0$ on $\overline{\mathcal{A}}$ is thus Hamiltonian with moment map in (3.45).

But $\widetilde{\mathcal{G}}_0$ is still not the group \mathcal{H} which we must use to perform non-abelian localization in Chern-Simons theory! In order to realize the action $S(A)$ as the square of the moment map μ for some Hamiltonian group action on $\overline{\mathcal{A}}$, the Lie algebra of the group must first possess a non-degenerate, invariant inner product. Just as for the loop group extension \widetilde{LG} , the group $\widetilde{\mathcal{G}}_0$ does not possess such an inner product.

However, we can elegantly remedy this problem, just as it was remedied for the loop group, by also considering the action of $U(1)$ on M . The $U(1)$ action on M induces an action of $U(1)$ on $\widetilde{\mathcal{G}}_0$, so we consider the associated semidirect product $U(1) \ltimes \widetilde{\mathcal{G}}_0$. Then a non-degenerate, invariant inner product on the Lie algebra of $U(1) \ltimes \widetilde{\mathcal{G}}_0$ is given by

$$(3.47) \quad \left((p, \phi, a), (q, \psi, b) \right) = - \int_M \kappa \wedge d\kappa \text{Tr} (\phi \psi) - pb - qa,$$

in direct correspondence with (3.43). As for the loop group, this quadratic form is of indefinite signature, due to the hyperbolic form of the last two terms in (3.47).

Finally, the $U(1)$ action on M immediately induces a corresponding $U(1)$ action on \mathcal{A} . Since the contact form κ is invariant under this action, the induced $U(1)$ action on \mathcal{A} descends to a corresponding action on the quotient $\overline{\mathcal{A}}$. In general, the vector field upstairs on \mathcal{A} which is generated by an arbitrary element (p, ϕ, a) of the Lie algebra of $U(1) \ltimes \widetilde{\mathcal{G}}_0$ is then given by

$$(3.48) \quad \delta A = d_A \phi + p \mathcal{L}_R A,$$

where R is the vector field on M generating the action of $U(1)$. Clearly the moment for the new generator $(p, 0, 0)$ is given by

$$(3.49) \quad \langle \mu, (p, 0, 0) \rangle = -\frac{1}{2} p \int_M \kappa \wedge \text{Tr} (\mathcal{L}_R A \wedge A).$$

This moment is manifestly invariant under the shift symmetry and descends to $\overline{\mathcal{A}}$.

In fact, the action of $U(1) \times \tilde{\mathcal{G}}_0$ on $\overline{\mathcal{A}}$ is Hamiltonian, with moment map

$$(3.50) \quad \langle \mu, (p, \phi, a) \rangle = -\frac{1}{2} p \int_M \kappa \wedge \text{Tr} (\mathcal{L}_R A \wedge A) + \int_M \kappa \wedge \text{Tr} (\phi F_A) - \int_M d\kappa \wedge \text{Tr} (\phi A) + a.$$

To check this statement, it suffices to compute

$$(3.51) \quad \left\{ \langle \mu, (p, 0, 0) \rangle, \langle \mu, (0, \psi, 0) \rangle \right\},$$

which is the only nontrivial Poisson bracket that we have not already computed. Thus,

$$(3.52) \quad \begin{aligned} \left\{ \langle \mu, (p, 0, 0) \rangle, \langle \mu, (0, \psi, 0) \rangle \right\} &= \\ &= \Omega \left(p \mathcal{L}_R A, d_A \psi \right) \\ &= -p \int_M \kappa \wedge \text{Tr} (\mathcal{L}_R A \wedge d_A \psi), \\ &= p \int_M \kappa \wedge \text{Tr} (\mathcal{L}_R \psi F_A) - p \int_M d\kappa \wedge \text{Tr} (\mathcal{L}_R \psi A), \\ &= \langle \mu, (0, p \mathcal{L}_R \psi, 0) \rangle, \end{aligned}$$

as required by the Lie bracket (3.42).

Thus, we identify $\mathcal{H} = U(1) \times \tilde{\mathcal{G}}_0$ as the relevant group of Hamiltonian symmetries which we use for localization in Chern-Simons theory.

3.5. The Action $S(A)$ as the Square of the Moment Map. By construction, the square (μ, μ) of the moment map μ in (3.50) for the Hamiltonian action of \mathcal{H} on $\overline{\mathcal{A}}$ is a function on $\overline{\mathcal{A}}$ invariant under \mathcal{H} . The new Chern-Simons action $S(A)$ in (3.9) is also a function on $\overline{\mathcal{A}}$ invariant under \mathcal{H} . Given the high degree of symmetry, we certainly expect that (μ, μ) and $S(A)$ agree up to normalization. We now check this fact directly and fix the relative normalization.

We first observe that, in terms of the invariant form (\cdot, \cdot) in (3.47) on the Lie algebra of \mathcal{H} , we can express the moment map dually as determined by the inner product with the vector

$$(3.53) \quad \left(-1, - \left(\frac{\kappa \wedge F_A - d\kappa \wedge A}{\kappa \wedge d\kappa} \right), \frac{1}{2} \int_M \kappa \wedge \text{Tr}(\mathcal{L}_R A \wedge A) \right)$$

in the Lie algebra of \mathcal{H} , so that

$$(3.54) \quad \begin{aligned} \langle \mu, (p, \phi, a) \rangle &= \\ &= \left\langle \left(-1, - \left(\frac{\kappa \wedge F_A - d\kappa \wedge A}{\kappa \wedge d\kappa} \right), \frac{1}{2} \int_M \kappa \wedge \text{Tr}(\mathcal{L}_R A \wedge A) \right), (p, \phi, a) \right\rangle. \end{aligned}$$

Thus, by duality, the square of μ is determined to be

$$(3.55) \quad \begin{aligned} (\mu, \mu) &= \left\langle \mu, \left(-1, - \left(\frac{\kappa \wedge F_A - d\kappa \wedge A}{\kappa \wedge d\kappa} \right), \frac{1}{2} \int_M \kappa \wedge \text{Tr}(\mathcal{L}_R A \wedge A) \right) \right\rangle, \\ &= \int_M \kappa \wedge \text{Tr}(\mathcal{L}_R A \wedge A) - \int_M \kappa \wedge d\kappa \text{Tr} \left(\left(\frac{\kappa \wedge F_A - d\kappa \wedge A}{\kappa \wedge d\kappa} \right)^2 \right). \end{aligned}$$

To simplify the first term of (3.55), we use the fact that the Lie derivative \mathcal{L}_R can be expressed as an anti-commutator $\mathcal{L}_R = \{\iota_R, d\}$, so that

$$(3.56) \quad \int_M \kappa \wedge \text{Tr}(\mathcal{L}_R A \wedge A) = \int_M \kappa \wedge \text{Tr}(\{\iota_R, d\} A \wedge A).$$

We now observe that $\iota_R A$ can be expressed as

$$(3.57) \quad \iota_R A = \frac{A \wedge d\kappa}{\kappa \wedge d\kappa}.$$

Using this fact and integrating by parts² with respect to the outermost operator d or ι_R in both of the two terms from the anti-commutator

²We observe that trivially $\iota_R(\kappa \wedge \text{Tr}(dA \wedge A)) = 0$.

(3.56), we find that

(3.58)

$$\begin{aligned} \int_M \kappa \wedge \text{Tr} \left(\mathcal{L}_R A \wedge A \right) &= \int_M \left[\iota_R \kappa \wedge \text{Tr} (dA \wedge A) - \kappa \wedge \text{Tr} (dA \iota_R A) + \right. \\ &\quad \left. + d\kappa \wedge \text{Tr} (\iota_R A \wedge A) - \kappa \wedge \text{Tr} (\iota_R A \wedge dA) \right], \\ &= \int_M \left[\text{Tr} (A \wedge dA) - 2\kappa \wedge \text{Tr} \left(\frac{d\kappa \wedge A}{\kappa \wedge d\kappa} dA \right) + \right. \\ &\quad \left. + d\kappa \wedge \text{Tr} \left(\frac{d\kappa \wedge A}{\kappa \wedge d\kappa} A \right) \right]. \end{aligned}$$

Consequently, after some algebra, we find that (3.55) becomes

$$(3.59) \quad (\mu, \mu) = - \int_M \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left((\kappa \wedge F_A)^2 \right) + \\ + \int_M \text{Tr} (A \wedge dA) + 2 \int_M \kappa \wedge \text{Tr} \left((\iota_R A) A \wedge A \right).$$

In arriving at (3.59), we have observed that the terms involving κ in (3.58) are cancelled by corresponding terms from the second term in (3.55), arising from the perfect square $((\kappa \wedge F_A - d\kappa \wedge A) / \kappa \wedge d\kappa)^2$, after expanding $F_A = dA + A \wedge A$. The last term in (3.59), cubic in A , arises from the cross-term in this perfect square when we express $F_A = dA + A \wedge A$ and we apply the identity (3.57).

To simplify the last term of (3.59), we observe that

(3.60)

$$0 = \iota_R \left(\kappa \wedge \text{Tr} (A \wedge A \wedge A) \right) = -3\kappa \wedge \text{Tr} \left((\iota_R A) A \wedge A \right) + \text{Tr} (A \wedge A \wedge A),$$

so that

(3.61)

$$(\mu, \mu) = - \int_M \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left((\kappa \wedge F_A)^2 \right) + \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

We thus find the beautiful result,

(3.62)

$$S(A) = (\mu, \mu).$$

We finally write the Chern-Simons path integral as a symplectic integral over $\overline{\mathcal{A}}$ of the canonical form,

$$(3.63) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{-i}{2\pi\epsilon} \right)^{\Delta_{\mathcal{G}}/2} \int_{\overline{\mathcal{A}}} \exp \left[\Omega + \frac{i}{2\epsilon} (\mu, \mu) \right].$$

4. Non-Abelian Localization and Two-Dimensional Yang-Mills Theory

In this section, we recall following [20] how the technique of non-abelian localization can be generally applied to study a symplectic integral of the canonical form

$$(4.1) \quad Z(\epsilon) = \frac{1}{\text{Vol}(H)} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta_H/2} \int_X \exp \left[\Omega - \frac{1}{2\epsilon} (\mu, \mu) \right], \quad \Delta_H = \dim H.$$

Here X is a symplectic manifold with symplectic form Ω , and H is a Lie group which acts on X in a Hamiltonian fashion with moment map μ . Finally, (\cdot, \cdot) is an invariant, positive-definite³ quadratic form on the Lie algebra \mathfrak{h} of H and dually on \mathfrak{h}^* which we use to define the “action” $S = \frac{1}{2}(\mu, \mu)$ and the volume $\text{Vol}(H)$ of H that appear in (4.1).

Later in this section, we also review and extend the ideas of [20] to apply non-abelian localization to Yang-Mills theory on a Riemann surface.

4.1. General Aspects of Non-Abelian Localization. To apply non-abelian localization to an integral of the form (4.1), we first observe that $Z(\epsilon)$ can be rewritten as

$$(4.2) \quad Z(\epsilon) = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times X} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2} (\phi, \phi) \right].$$

Here ϕ is an element of the Lie algebra \mathfrak{h} of H , and $[d\phi]$ is the Euclidean measure on \mathfrak{h} that is determined by the same invariant form (\cdot, \cdot) which we use to define the volume $\text{Vol}(H)$ of H . The Gaussian integral over ϕ in (4.2) leads immediately to the expression in (4.1). The measure $[d\phi/2\pi]$ includes a factor of $1/2\pi$ for each real component of ϕ .

³In the case of Chern-Simons theory, the corresponding quadratic form (3.47) on \mathfrak{h} has indefinite signature, due to the hyperbolic summand associated to the two extra $U(1)$ generators of \mathcal{H} relative to the group of gauge transformations \mathcal{G}_0 . Also, invariance under large gauge transformations requires the Chern-Simons symplectic integral (3.63) to be oscillatory, instead of exponentially damped. These features do not essentially change our discussion of localization below, and we reserve further comment until Section 5.

A BRST Symmetry

The advantage of writing Z in the form (4.2) is that, once we introduce ϕ , then Z becomes invariant under a BRST symmetry, and this BRST symmetry leads directly to a localization formula for (4.2).

To describe this BRST symmetry, we recall that the moment map satisfies

$$(4.3) \quad d\langle\mu, \phi\rangle = \iota_{V(\phi)}\Omega,$$

where $V(\phi)$ is the vector field on X associated to the infinitesimal action of ϕ . Because of the relation (4.3), the argument of the exponential in (4.2) is immediately annihilated by the BRST operator D defined by

$$(4.4) \quad D = d + i \iota_{V(\phi)}.$$

To exhibit the action of D locally, we choose a basis ϕ^a for \mathfrak{h} , and we introduce local coordinates x^m on X . We also introduce the notation $\chi^m \equiv dx^m$ for the corresponding basis of local one-forms on X , and we expand the vector field $V(\phi)$ into components as $V(\phi) = \phi^a V_a^m \partial/\partial x^m$. Then the action of D in (4.4) is described in terms of these local coordinates by

$$(4.5) \quad \begin{aligned} Dx^m &= \chi^m, \\ D\chi^m &= i \phi^a V_a^m, \\ D\phi^a &= 0. \end{aligned}$$

From this local description (4.5), we see that the action of D preserves a ghost number, or grading, under which x carries charge 0, χ carries charge +1, ϕ carries charge +2, and D itself carries charge +1.

The most important property of a BRST operator is that it squares to zero. In this case, either from (4.4) or from (4.5), we see that D squares to the Lie derivative along the vector field $V(\phi)$,

$$(4.6) \quad D^2 = i \{d, \iota_{V(\phi)}\} = i \mathcal{L}_{V(\phi)}.$$

Thus, $D^2 = 0$ exactly when D acts on the subspace of H -invariant functions $\mathcal{O}(x, \chi, \phi)$ of x , χ , and ϕ .

For simplicity, we restrict attention to functions $\mathcal{O}(x, \chi, \phi)$ which are polynomial in ϕ . Then an arbitrary function of this form can be expanded as a sum of terms

$$(4.7) \quad \mathcal{O}(x)_{m_1 \dots m_p a_1 \dots a_q} \chi^{m_1} \dots \chi^{m_p} \phi^{a_1} \dots \phi^{a_q},$$

for some $0 \leq p \leq \dim X$ and $q \geq 0$. (The restriction on p arises from the fact that χ satisfies Fermi statistics, whereas ϕ satisfies Bose statistics.)

Globally, each term of the form (4.7) is specified by a section of the bundle $\Omega_X^p \otimes \text{Sym}^q(\mathfrak{h}^*)$ of p -forms on X which take values in the q -th symmetric tensor product of the dual \mathfrak{h}^* of the Lie algebra of H . Thus, if we consider the complex $(\Omega_X^* \otimes \text{Sym}^*(\mathfrak{h}^*))^H$ of all H -invariant differential forms on X which take values in the ring of polynomial functions on \mathfrak{h} , then we see that D defines a cohomology theory associated to the action of H on X . This cohomology theory is known as the Cartan model of the H -equivariant cohomology of X . With the exception of the last computation in Section 5.3, our applications will not require a greater familiarity with equivariant cohomology than what we have described here. However, in Section 5.3 we will need to use a few additional properties of equivariant cohomology that we discuss in Appendix C, and we recommend [36, 56] as basic references.

Localization for Z

Because the argument of the exponential in (4.2) is annihilated by D and because this argument is manifestly invariant under H , the integrand of the symplectic integral Z determines an equivariant cohomology class on X . Furthermore, by the usual arguments, Z is formally unchanged by the addition of any D -exact invariant form to its integrand. This formal statement can fail if X is not compact and Z suffers from divergences, as we analyze in great detail in Appendix A, but for the moment we ignore this issue and assume X is compact. Thus, Z depends only on the equivariant cohomology class of its integrand.

We now explain how this fact leads immediately to a localization formula for Z . We first observe that we can add to the argument of the exponential in (4.2) an arbitrary term of the form $t D\lambda$, where λ is any H -invariant one-form on X and t is a real parameter, so that

$$(4.8) \quad Z(\epsilon) = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times X} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2} (\phi, \phi) + t D\lambda \right].$$

This deformation of the integrand of (4.2) is D -exact and does not change Z . In particular, Z does not depend on t .

By definition, $D\lambda$ is given explicitly by

$$(4.9) \quad D\lambda = d\lambda + i \langle \lambda, V(\phi) \rangle.$$

As before, $\langle \cdot, \cdot \rangle$ denotes the canonical dual pairing, so that in components the last term of (4.9) is given by $\lambda_m V_a^m \phi^a$.

Thus, apart from a polynomial in t that arises from expanding the term $\exp(t d\lambda)$, all of the dependence on t in the integrand of Z arises

from the factor $\exp[it \langle \lambda, V(\phi) \rangle]$ that now appears in (4.8). So if we consider the limit $t \rightarrow \infty$, then the stationary phase approximation to the integral is valid, and all contributions to Z localize around the critical points of the function $\langle \lambda, V(\phi) \rangle$.

We expand this function in the basis ϕ^a for \mathfrak{h} which we introduced previously,

$$(4.10) \quad \langle \lambda, V(\phi) \rangle = \phi^a \langle \lambda, V_a \rangle.$$

Thus, the critical points of $\langle \lambda, V(\phi) \rangle$ arise from the simultaneous solutions in $\mathfrak{h} \times X$ of the equations

$$(4.11) \quad \begin{aligned} \langle \lambda, V_a \rangle &= 0, \\ \phi^a d\langle \lambda, V_a \rangle &= 0. \end{aligned}$$

The first equation in (4.11) implies that Z necessarily localizes on points in $\mathfrak{h} \times X$ for which $\langle \lambda, V_a \rangle$ vanishes. As for the second equation in (4.11), we see that it is invariant under an overall scaling of ϕ in the vector space \mathfrak{h} . Consequently, upon integrating over ϕ in (4.8), we see that the critical locus of the function $\langle \lambda, V(\phi) \rangle$ in $\mathfrak{h} \times X$ projects onto the vanishing locus of $\langle \lambda, V_a \rangle$ in X . So Z localizes on the subset of X where $\langle \lambda, V_a \rangle = 0$.

By making a specific choice of the one-form λ , we can describe the localization of Z more precisely. In particular, we now show that Z localizes on the set of critical points of the function $S = \frac{1}{2}(\mu, \mu)$ on X .

We begin by choosing an almost complex structure J on X . That is, $J : TX \rightarrow TX$ is a linear map from TX to itself such that $J^2 = -1$. We assume that J is compatible with the symplectic form Ω in the sense that Ω is of type $(1, 1)$ with respect to J and the associated metric $G(\cdot, \cdot) = \Omega(\cdot, J\cdot)$ on X is positive-definite. Such an almost complex structure always exists.

Using J and S , we now introduce the invariant one-form

$$(4.12) \quad \lambda = J dS = (\mu, J d\mu).$$

In components, $\lambda = dx^m J_m^n \partial_n S = dx^m \mu^a J_m^n \partial_n \mu_a$.

The integral Z now localizes on the subset of X where $\langle \lambda, V_a \rangle = 0$. Comparing to (4.12), we see that this subset certainly includes all critical points of S , since by definition $dS = 0$ at these points.

Conversely, we now show that if $\langle \lambda, V_a \rangle = 0$ at some point on X , then this point is a critical point of S . To prove this assertion, we use the inverse Ω^{-1} to Ω , which arises by considering the symplectic form as

an isomorphism $\Omega : TM \rightarrow T^*M$ with inverse $\Omega^{-1} : T^*M \rightarrow TM$. In components, Ω^{-1} is defined by $(\Omega^{-1})^{lm} \Omega_{mn} = \delta_n^l$.

In terms of Ω^{-1} , the moment map equation (4.3) is equivalent to the relation

$$(4.13) \quad V = \Omega^{-1} d\mu,$$

or $V_a^m = (\Omega^{-1})^{mn} \partial_n \mu_a$. Thus,

$$(4.14) \quad \Omega^{-1} dS = (\mu, \Omega^{-1} d\mu) = (\mu, V),$$

or $(\Omega^{-1})^{mn} \partial_n S = \mu^a V_a^m$.

In particular, the condition that $\langle \lambda, V_a \rangle = 0$ implies that

$$(4.15) \quad 0 = (\mu, \langle \lambda, V \rangle) = \langle \lambda, \Omega^{-1} dS \rangle = \langle J dS, \Omega^{-1} dS \rangle,$$

or more explicitly,

$$(4.16) \quad 0 = \mu^a \lambda_m V_a^m = \lambda_m (\Omega^{-1})^{mn} \partial_n S = (\Omega^{-1})^{mn} J_m^l \partial_l S \partial_n S.$$

We recognize the last expression in (4.15) as the norm of the one-form dS with respect to the metric G on X . As G is positive-definite, we conclude that the condition $\langle \lambda, V_a \rangle = 0$ implies the vanishing of dS . Thus, the symplectic integral Z localizes precisely on the critical set of S .

4.2. Non-Abelian Localization For Yang-Mills Theory, Part I.

In the rest of this section, we apply non-abelian localization to perform path integral computations in two-dimensional Yang-Mills theory on a smooth Riemann surface Σ . These computations are an essential warmup for our later computations in Chern-Simons theory.

As we discussed in Section 2, the Yang-Mills path integral is naturally a symplectic integral of the canonical form (4.1), where the abstract symplectic manifold X is now the affine space $\mathcal{A}(P)$ of connections on a fixed principal G -bundle P over Σ , and where the abstract group H is now the group $\mathcal{G}(P)$ of gauge transformations. Also, the moment map for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is simply the curvature of the connection, $\mu = F_A$.

As a result of our general discussion above, the Yang-Mills path integral localizes on critical points of the Yang-Mills action. These critical points fall into two qualitatively different sorts. Because the action $S = \frac{1}{2}(\mu, \mu)$ is quadratic in the moment map μ , so that $dS = (\mu, d\mu)$, we see that the critical locus of S includes all points where μ vanishes, as well as other points where μ is generally non-zero. The points at which $\mu = 0$ are clearly stable minima of S , and any other critical points at

which $\mu \neq 0$ are higher extrema of S , which in our applications are unstable. In the case of Yang-Mills theory, the stable minima of the action are the flat connections on Σ , and the higher extrema are connections with non-zero curvature which represent classical solutions of Yang-Mills theory, so that $d_A \star F_A = 0$ with $F_A \neq 0$.

For our application to Chern-Simons theory, we must understand localization at both the flat and the non-flat solutions of classical Yang-Mills theory. So in the rest of Section 4.2, we review following [20] how non-abelian localization works for flat connections, and then in Section 4.3 we discuss the generalization for solutions of Yang-Mills theory with curvature.

Localization on a Smooth Component of the Moduli Space of Flat Connections

We assume that \mathcal{M}_0 is a smooth component of the moduli space of flat connections on Σ . For ease of future notation, we make the identifications

$$(4.17) \quad \begin{aligned} X &= \mathcal{A}(P), \\ H &= \mathcal{G}(P), \\ \mu &= F_A. \end{aligned}$$

We now identify \mathcal{M}_0 abstractly as a symplectic quotient of the zero locus $\mu^{-1}(0) \subset X$ by the free action of the group H , so that $\mathcal{M}_0 = \mu^{-1}(0)/H$.

The fundamental result of [20], whose derivation we now recall, is that the local contribution $Z(\epsilon)|_{\mathcal{M}_0}$ to the path integral from \mathcal{M}_0 is given by the topological expression

$$(4.18) \quad Z(\epsilon)|_{\mathcal{M}_0} = \int_{\mathcal{M}_0} \exp(\Omega + \epsilon \Theta).$$

Here Ω is the symplectic form on \mathcal{M}_0 induced from the corresponding symplectic form on X (also denoted previously by Ω), and Θ is a characteristic class of degree four on \mathcal{M}_0 which appears explicitly as part of the derivation of (4.18). In particular, when the coupling ϵ is zero, then $Z(0)|_{\mathcal{M}_0}$ is the symplectic volume of \mathcal{M}_0 .

To derive (4.18) by localization, we start by considering the local geometry of the zero set $\mu^{-1}(0)$ in X . Thus, we let N be a small open neighborhood of $\mu^{-1}(0)$ in X , so that $\mu^{-1}(0) \subset N \subset X$. We assume that this neighborhood is chosen so that N is preserved by the action of H and so that N retracts equivariantly onto $\mu^{-1}(0)$. By composing this retraction with the quotient by the action of H , we define a projection

$pr : N \rightarrow \mathcal{M}_0$. Thus, denoting the fiber of pr by F , we have the following equivariant bundle

$$(4.19) \quad F \longrightarrow N \xrightarrow{pr} \mathcal{M}_0.$$

The symplectic integral which describes the local contribution of \mathcal{M}_0 to Z is now given by

$$(4.20) \quad Z(\epsilon)|_{\mathcal{M}_0} = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times N} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2}(\phi, \phi) + t D\lambda \right],$$

where λ is the invariant one-form that we introduced in (4.12) to localize Z . Because N is noncompact, this integral in (4.20) is only defined by localization, so that we require $t \neq 0$.

As explained in detail in [20], because N retracts equivariantly onto \mathcal{M}_0 and because the action of H is free near $\mu^{-1}(0)$, the equivariant cohomology class of degree two⁴ represented by the expression $\Omega - i \langle \mu, \phi \rangle$ in (4.20) is simply the pullback by pr of the induced symplectic form on \mathcal{M}_0 . Similarly, the equivariant cohomology class of degree four represented by $-\frac{1}{2}(\phi, \phi)$ in (4.20) is the pullback by pr of an ordinary cohomology class Θ of degree four on \mathcal{M}_0 . Since H acts freely on $\mu^{-1}(0)$, Θ represents a degree four characteristic class of $\mu^{-1}(0)$ regarded as a principal H -bundle over \mathcal{M}_0 .

Thus, as the only term appearing in the argument of the exponential in (4.20) which does not pull back from \mathcal{M}_0 is $t D\lambda$ itself, to derive (4.18) from (4.20) we must only show that the integral of $\exp(t D\lambda)$ over the fiber F of (4.19) produces a trivial factor of 1,

$$(4.21) \quad \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times F} \left[\frac{d\phi}{2\pi} \right] \exp [t D\lambda] = 1.$$

This computation is what we must essentially generalize to discuss localization at non-flat Yang-mills solutions, so we review it in detail.

A Local Model For F From Hodge Theory

In order to perform the direct computation of the integral in (4.21), we first identify the correct local model for the geometry of F . By assumption, the group H acts freely on F , so F must contain a copy of H . Since F must also be symplectic, the simplest local model for F is just the cotangent bundle T^*H of H , with its canonical symplectic structure.

⁴We recall that ϕ carries degree +2 with respect to equivariant cohomology.

In fact, the simple guess that $F = T^*H$ is precisely correct, and it has an important infinite-dimensional interpretation in the context of Yang-Mills theory. To explain this interpretation, we consider the tangent space to $\mathcal{A}(P)$ at a point corresponding to a flat connection A . As we have discussed, the tangent space to $\mathcal{A}(P)$ at A can be identified with the space of smooth sections $\Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P))$ of the bundle of one-forms on Σ taking values in the adjoint bundle $\text{ad}(P)$.

By definition, the flatness of A implies that the covariant derivative d_A satisfies $d_A^2 = 0$. Because of this fact, d_A has many of the same properties as the de Rham exterior derivative d , and the usual Hodge decomposition for d has an immediate analogue for d_A .

In the case of the covariant derivative d_A , the Hodge decomposition implies that the vector space $\Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P))$ decomposes into three subspaces, orthogonal with respect to the metric induced by \star on $\mathcal{A}(P)$, of the form

$$(4.22) \quad \Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P)) = \mathcal{H}_1 \oplus \text{Im}(d_A) \oplus \text{Im}(d_A^\dagger).$$

Here $d_A^\dagger = -\star d_A \star$ is the standard adjoint to d_A with respect to the metric on $\mathcal{A}(P)$. Also, \mathcal{H}_1 denotes the finite-dimensional subspace of harmonic one-forms taking values in $\text{ad}(P)$, so that elements of \mathcal{H}_1 are annihilated by the Laplacian $\Delta_A = d_A d_A^\dagger + d_A^\dagger d_A$. Finally, $\text{Im}(d_A)$ and $\text{Im}(d_A^\dagger)$ denote the images of d_A and d_A^\dagger when these operators act respectively on sections of the bundles $\text{ad}(P)$ and $\Omega_\Sigma^2 \otimes \text{ad}(P)$ on Σ .

Concretely, the Hodge decomposition implies that, if η is any section of $\Omega_\Sigma^1 \otimes \text{ad}(P)$, then η can be uniquely written as a sum of three terms, all orthogonal,

$$(4.23) \quad \eta = \xi + d_A \phi + d_A^\dagger \Psi,$$

where ξ satisfies $\Delta_A \xi = 0$ and where ϕ and Ψ are respectively sections of the bundles $\text{ad}(P)$ and $\Omega_\Sigma^2 \otimes \text{ad}(P)$.

To interpret the Hodge decomposition (4.22) as a geometric statement, we note that the finite-dimensional vector space \mathcal{H}_1 of harmonic one-forms can be identified with the tangent space to the moduli space \mathcal{M}_0 of flat connections at A . For instance, since $d_A^2 = 0$, we can consider the cohomology of d_A . As usual, we identify the harmonic forms in \mathcal{H}_1 as representatives of cohomology classes in $H^1(\Sigma, \text{ad}(P))$. These cohomology classes describe infinitesimal deformations of the flat connection A .

On the other hand, since we assume that the gauge group $\mathcal{G}(P)$ acts freely at A , d_A has no kernel when acting on sections of $\text{ad}(P)$. Otherwise, if a section ϕ of $\text{ad}(P)$ did satisfy $d_A\phi = 0$, then the gauge transformation generated by ϕ would fix A . Equivalently, we have that $H^0(\Sigma, \text{ad}(P)) = 0$.

Because d_A has no kernel when acting on sections of $\text{ad}(P)$, d_A can be formally inverted and the image of d_A in $\Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P))$ identified with the space of sections of $\text{ad}(P)$ itself. Of course, a section ϕ of $\text{ad}(P)$, as appears in (4.23), is interpreted geometrically as a tangent vector to the gauge group $\mathcal{G}(P)$.

Similarly, we can also identify the image of the adjoint d_A^\dagger with the space of sections of the bundle $\Omega_\Sigma^2 \otimes \text{ad}(P)$. Such a section Ψ , as in (4.23), is interpreted geometrically as a cotangent vector to the gauge group $\mathcal{G}(P)$.

Furthermore, if we recall the natural symplectic form Ω on $\mathcal{A}(P)$ in (2.3), we see that $\text{Im}(d_A)$ is isotropic with respect to Ω . For if ϕ and ψ are any two sections of the bundle $\text{ad}(P)$ on Σ , then

$$(4.24) \quad \Omega(d_A\phi, d_A\psi) = - \int_\Sigma \text{Tr}(d_A\phi \wedge d_A\psi) = \int_\Sigma \text{Tr}(\phi d_A^2\psi) = 0.$$

This fact crucially relies on the flatness of A , since we use that $d_A^2 = 0$ in deducing the last equality of (4.24). Of course, the fact that $\text{Im}(d_A)$ is isotropic with respect to Ω is mirrored by the fact that H is a Lagrangian submanifold of T^*H .

Thus, the Hodge decomposition (4.22) applied to $\Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P))$ locally reflects the geometric statement that F is modeled on the cotangent bundle T^*H . In this example, it may seem perverse to translate the simple statement that $F = T^*H$ into the infinite-dimensional statement of the Hodge decomposition. However, when we consider the corresponding local geometry for higher critical points, this infinite-dimensional perspective allows us to deduce directly how the simple symplectic model based on T^*H must be modified to describe higher critical points of Yang-Mills theory.

*Computing a Symplectic Integral on T^*H*

Having identified the symplectic model for F as the cotangent bundle T^*H , we compute in the remainder of this subsection the symplectic integral

$$(4.25) \quad \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times T^*H} \left[\frac{d\phi}{2\pi} \right] \exp [t D\lambda].$$

We review this short computation from [20] simply because we must generalize it to discuss localization at non-flat Yang-Mills connections.

Thus, we consider the symplectic manifold T^*H with its canonical symplectic structure. By convention, the action of H on T^*H is induced from the right action of H on itself. By passing to a basis of right-invariant one-forms and using the invariant metric (\cdot, \cdot) on H , we identify $T^*H \cong H \times \mathfrak{h}$. Under this identification, we introduce coordinates (g, γ) on $H \times \mathfrak{h}$.

In these coordinates, the canonical right-invariant one-form on H which takes values in \mathfrak{h} is given by

$$(4.26) \quad \theta = dg g^{-1}.$$

In terms of θ , the canonical symplectic structure on T^*H is given by the invariant two-form

$$(4.27) \quad \begin{aligned} \Omega &= d(\gamma, \theta) = (d\gamma, \theta) + (\gamma, d\theta), \\ &= \left(d\gamma + \frac{1}{2}[\gamma, \theta], \theta \right), \end{aligned}$$

where in passing to the second line of (4.27) we recall that $d\theta = \theta \wedge \theta = \frac{1}{2}[\theta, \theta]$. Also, if ϕ is an element of \mathfrak{h} , then the corresponding vector field $V(\phi)$ on T^*H which is generated by the infinitesimal right-action of ϕ is given by

$$(4.28) \quad \delta g = -g\phi, \quad \delta \gamma = 0.$$

To proceed, we require an explicit formula for the invariant one-form λ appearing in (4.25). Abstractly, $\lambda = (\mu, J d\mu)$ is determined by the moment map μ for the H -action on T^*H and an almost complex structure J compatible with Ω in (4.27), both of which are easy to determine. A convenient formula for λ was obtained in [20]. In brief, one has $\langle \mu, \phi \rangle = -(\gamma, g\phi g^{-1})$, and one defines a G -invariant almost complex structure compatible with Ω by

$$(4.29) \quad J(\theta) = - \left(d\gamma + \frac{1}{2}[\gamma, \theta] \right), \quad J \left(d\gamma + \frac{1}{2}[\gamma, \theta] \right) = \theta.$$

One then finds that $(\mu, J d\mu) = (\gamma, \theta)$ after using the fact that $[\gamma, \gamma] = 0$. So finally

$$(4.30) \quad \lambda = (\mu, J d\mu) = (\gamma, \theta).$$

Thus, from (4.28), (4.30), and the definition of D in (4.4), we see that

$$(4.31) \quad D\lambda = \Omega - i(\gamma, g\phi g^{-1}).$$

Without loss, we set $t = 1$ in (4.25) and we change variables from ϕ to $g\phi g^{-1}$, under which the measure $[d\phi]$ on \mathfrak{h} is invariant. Then the symplectic integral takes the simple form

$$(4.32) \quad \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times T^*H} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i (\gamma, \phi) \right].$$

The integral over γ can be done using the fact that

$$(4.33) \quad \int_{-\infty}^{+\infty} dy \exp(-ixy) = 2\pi \delta(x),$$

and the resulting multi-dimensional delta function can be used to perform the integral over ϕ . We note that the factors of 2π from (4.33) nicely cancel the factors of 2π in the measure for ϕ . Finally, the remaining integral over g in H produces a factor of the volume $\text{Vol}(H)$ which cancels the prefactor in (4.32). Thus, assuming T^*H is suitably oriented, the symplectic integral over T^*H is indeed 1, as claimed in (4.21).

4.3. Non-Abelian Localization For Yang-Mills Theory, Part II.

We now study localization at the higher, unstable critical points of the Yang-Mills action, which correspond to non-flat connections which solve the Yang-Mills equation on Σ . Localization at the higher critical points of two-dimensional Yang-Mills theory has recently been discussed from a mathematical perspective by Woodward and Teleman [34, 35], but we find it useful to proceed with a more naive discussion along the lines of [20]. We begin with some generalities about non-flat connections which solve the Yang-Mills equation on Σ .

We first introduce the notation f for the section of $\text{ad}(P)$ dual to the curvature F_A ,

$$(4.34) \quad f = \star F_A.$$

Then, by definition, any Yang-Mills solution on Σ satisfies the classical equation of motion

$$(4.35) \quad d_A f = 0.$$

This equation simply expresses the geometric condition that f be a covariantly constant section of $\text{ad}(P)$, and we can consequently regard f as an element of the Lie algebra \mathfrak{g} of G .

Because f is constant, f yields a reduction of the structure group G of the bundle to the subgroup $G_f \subset G$ which commutes with f . In

physical terms, the background curvature breaks the gauge group from G to G_f .

As a result of the reduction from G to G_f , any non-flat Yang-Mills solution for gauge group G can be succinctly described as a flat connection for gauge group G_f which is twisted by a constant curvature line bundle associated to the $U(1)$ subgroup of G generated by f .

In general, we denote by \mathcal{M}_f the moduli space of Yang-Mills connections whose curvature lies in the conjugacy class of f . We have already discussed localization on the moduli space \mathcal{M}_0 of flat connections, for which $G_0 = G$. At the opposite extreme, f breaks G to a maximal torus G_f commuting with f . We refer to such a Yang-Mills solution as “maximally reducible,” and one basic goal in this section is to obtain an explicit formula, as in (4.18), for the contribution to the path integral from the corresponding moduli space \mathcal{M}_f of maximally reducible Yang-Mills solutions. Of course, we could also consider the local contributions from Yang-Mills solutions between the extremes of the flat and maximally reducible connections, but this further generalization is not necessary for our discussion of Chern-Simons theory.

Because f is constant, the adjoint action of f determines a bundle map from $\text{ad}(P)$ to itself, and a good idea is to decompose $\text{ad}(P)$ under this action. With our conventions, f is anti-hermitian, so following [21] we introduce a hermitian operator Λ ,

$$(4.36) \quad \Lambda = i[f, \cdot],$$

which acts on a section ϕ of $\text{ad}(P)$ as $\Lambda \phi = i[f, \phi]$.

When we consider the action of Λ , it is natural to work with complex, as opposed to real, quantities. So we now consider in place of the real bundle $\text{ad}(P)$ the complex bundle $\text{ad}_{\mathbb{C}}(P) = \text{ad}(P) \otimes \mathbb{C}$. When we complexify $\text{ad}(P)$, the $(1,0)$ and $(0,1)$ parts of an $\text{ad}(P)$ -valued connection become independent complex variables. After picking a local complex coordinate z on Σ , these can be written locally as A_z and $A_{\bar{z}}$.

Under the action of Λ , the bundle $\text{ad}_{\mathbb{C}}(P)$ decomposes into a direct sum of subbundles, each associated to a distinct eigenvalue of Λ . For our purposes, we need only consider the decomposition of $\text{ad}_{\mathbb{C}}(P)$ into the positive, zero, and negative eigenspaces of Λ ,

$$(4.37) \quad \text{ad}_{\mathbb{C}}(P) = \text{ad}_+(P) \oplus \text{ad}_0(P) \oplus \text{ad}_-(P),$$

where $\text{ad}_{\pm}(P)$ and $\text{ad}_0(P)$ denote respectively the subbundles of $\text{ad}_{\mathbb{C}}(P)$ associated to these eigenspaces. The eigenspace decomposition of $\text{ad}_{\mathbb{C}}(P)$ in (4.37) will play an important role shortly.

Example: $G = SU(2)$

As a simple example of these ideas, we consider the higher Yang-Mills critical points when the gauge group G is $SU(2)$. In this case, all non-flat Yang-Mills solutions are maximally reducible, since any $f \neq 0$ reduces the structure group to a maximal torus $U(1) \subset SU(2)$.

The rank-one case $G = SU(2)$ of Yang-Mills theory is also the essential case to understand for our application to Chern-Simons gauge theory, with gauge group of arbitrary rank. As we explain in Section 5, near a flat Chern-Simons connection on the three-manifold M , the local geometry in the symplectic manifold $\overline{\mathcal{A}}$ of (3.25) can be modeled on the geometry of infinitely-many copies of the geometry near a higher $SU(2)$ Yang-Mills critical point. This correspondence arises heuristically by identifying the background Yang-Mills curvature f , which generates the torus $U(1) \subset SU(2)$, with the geometric curvature of M regarded as a principal $U(1)$ -bundle over the surface Σ .

In the case of Yang-Mills theory, since f reduces the structure group of the $SU(2)$ bundle to $U(1)$, the $SU(2)$ bundle on Σ splits as a direct sum of line bundles. As f itself is associated to a constant curvature line bundle on Σ , up to conjugacy f takes the form

$$(4.38) \quad f = 2\pi i \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix},$$

for some integer $n \neq 0$. Because the Weyl group of $SU(2)$ acts on f by sending $n \rightarrow -n$, without loss we can assume that $n > 0$.

Introducing the standard generators of $\mathfrak{su}(2)$ regarded as a complex Lie algebra,

$$(4.39) \quad \sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we see that Λ acts on $\mathfrak{su}(2)$, and hence on $\text{ad}_{\mathbb{C}}(P)$, with eigenvalues 0 and $\pm 4\pi n$. Thus, in this case the general decomposition of $\text{ad}_{\mathbb{C}}(P)$ in (4.37) takes the simple form

$$(4.40) \quad \text{ad}_{\mathbb{C}}(P) = \mathcal{L}^{-1}(-2n) \oplus \mathcal{O} \oplus \mathcal{L}(2n).$$

Here \mathcal{O} is the trivial line bundle on Σ , \mathcal{L} is an arbitrary flat line bundle on Σ , and we use the standard notation $\mathcal{L}(2n) = \mathcal{L} \otimes \mathcal{O}(2n)$, where $\mathcal{O}(2n)$ is the $2n$ -th tensor power of a fixed line bundle $\mathcal{O}(1)$ of degree one on Σ .

Thus, for each $n > 0$, the choice of a non-flat connection solving the Yang-Mills equation on Σ is determined by the choice of the flat line

bundle \mathcal{L} on Σ . Such a line bundle is specified by the $U(1)$ holonomy of its connection, and hence the moduli space of flat line bundles on Σ is parametrized by a complex torus, the Jacobian of Σ . If Σ has genus g , with $2g$ periods, then the Jacobian has complex dimension g . Thus, for fixed $f \neq 0$, the moduli space \mathcal{M}_f of higher critical points of $SU(2)$ Yang-Mills theory on Σ is simply a complex torus of dimension g .

More generally, if we consider an arbitrary gauge group G of rank r such that f breaks G to a maximal torus, then the corresponding moduli space \mathcal{M}_f is again a complex torus of dimension gr which describes the holonomy in $U(1)^r$.

The Partition Function of $SU(2)$ Yang-Mills Theory

One of our basic goals in the rest of this section is to compute directly the contributions from higher critical points to the partition function Z of $SU(2)$ Yang-Mills theory. Of course, Z can be computed exactly [57], and we can readily extract from the known expression for Z a formula for the local contributions from the higher critical points. So before we delve into our path integral computation, we present now the answer which we expect to reproduce and we preview its most interesting features.

In general, if the gauge group G is simply-connected, then the partition function of Yang-Mills theory on a unit area Riemann surface of genus g is given by a sum over representations \mathcal{R} of G of the form

$$(4.41) \quad Z(\epsilon) = (\text{Vol}(G))^{2g-2} \sum_{\mathcal{R}} \frac{1}{\dim(\mathcal{R})^{2g-2}} \exp\left(-\frac{1}{2}\epsilon \tilde{C}_2(\mathcal{R})\right).$$

Here $\tilde{C}_2(\mathcal{R})$ is a renormalized⁵ version of the quadratic Casimir associated to the representation \mathcal{R} , and the volume $\text{Vol}(G)$ of G is determined in our conventions by the invariant form $-\text{Tr}$ on the Lie algebra \mathfrak{g} . We recall that for $G = SU(r+1)$, “Tr” denotes the trace in the fundamental representation.

Finally, because of the possibility of weighting the Yang-Mills path integral on Σ by a purely topological factor $\exp(c(2g-2))$ for an arbitrary constant c , we have fixed the prefactor in (4.41) so that $Z(0)$ agrees, at least up to a sign which we will not try to fix, with the symplectic volume of the moduli space \mathcal{M}_0 of flat connections on Σ

⁵The renormalized quadratic Casimir $\tilde{C}_2(\mathcal{R})$ differs from the usual quadratic Casimir solely by an additive constant.

as computed in [58] from the theory of Reidemeister-Ray-Singer torsion. Our choice of c differs from the choice in [58] simply because the symplectic form Ω in (2.3) which we use here is related to the integral symplectic form Ω' used in [58] by $\Omega = 4\pi^2 \Omega'$.

We now evaluate (4.41) in the case $G = SU(2)$. In this case, each representation is labelled by its dimension, so we denote by \mathcal{R}_n the $SU(2)$ representation of dimension n . The renormalized quadratic Casimir of \mathcal{R}_n , which is just the usual quadratic Casimir with an additive constant, is then

$$(4.42) \quad \tilde{C}_2(\mathcal{R}_n) = \frac{1}{2} n^2.$$

Finally, using the metric on $SU(2)$ determined by $-\text{Tr}$, the volume of $SU(2)$ is given⁶ by $\text{Vol}(SU(2)) = 2^{5/2}\pi^2$. Thus, the partition function (4.41) of $SU(2)$ Yang-Mills theory on Σ becomes

$$(4.43) \quad Z(\epsilon) = (32\pi^4)^{g-1} \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}} \exp\left(-\frac{\epsilon n^2}{4}\right).$$

In order to extract the contributions of the higher critical points from (4.43), we first differentiate $Z(\epsilon)$ with respect to ϵ to cancel the prefactor $n^{-2(g-1)}$ in the summand of (4.43),

$$(4.44) \quad \begin{aligned} \frac{\partial^{g-1} Z(\epsilon)}{\partial \epsilon^{g-1}} &= (-8\pi^4)^{g-1} \sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon n^2}{4}\right) \\ &= \frac{1}{2} (-8\pi^4)^{g-1} \left(-1 + \sum_{n \in \mathbb{Z}} \exp\left(-\frac{\epsilon n^2}{4}\right)\right). \end{aligned}$$

To obtain a manifestly convergent expression in the weak coupling regime of small ϵ , we apply Poisson summation to the last term of (4.44) to obtain

$$(4.45) \quad \frac{\partial^{g-1} Z(\epsilon)}{\partial \epsilon^{g-1}} = \frac{1}{2} (-8\pi^4)^{g-1} \left(-1 + \sqrt{\frac{4\pi}{\epsilon}} \sum_{n \in \mathbb{Z}} \exp\left(-\frac{(2\pi n)^2}{\epsilon}\right)\right).$$

Finally, to identify the contribution in (4.45) from higher Yang-Mills critical points, we observe that at a higher critical point of degree n , the

⁶This fact follows immediately if we recall that the volume of an S^3 of unit radius is $2\pi^2$. However, in our metric on $SU(2)$, the $U(1)$ subgroup associated to the normalized generator $T_z = \frac{1}{\sqrt{2}} \sigma_z$, as in (4.39), has length $2\pi\sqrt{2}$, so $SU(2)$ has radius $r = \sqrt{2}$ in our metric.

classical Yang-Mills action S_n determined by f in (4.38) is given by $S_n = (2\pi n)^2/\epsilon$ (assuming Σ has unit area). The semiclassical contribution to Z from such a critical point is weighted by the usual exponential factor $\exp(-S_n)$, which we see directly in the last term of (4.45). Thus, the locus \mathcal{M}_n of higher critical points of degree n contributes to the sum in (4.45) as

$$(4.46) \quad \left. \frac{\partial^{g-1} Z(\epsilon)}{\partial \epsilon^{g-1}} \right|_{\mathcal{M}_n} = (-8\pi^4)^{g-1} \sqrt{\frac{4\pi}{\epsilon}} \exp\left(-\frac{(2\pi n)^2}{\epsilon}\right).$$

We note that a trivial factor of two in (4.46) arises from the action of the Weyl group, since the two terms in (4.45) for both $\pm n$ arise from the higher critical points of degree n .

This expression (4.46) is what we compute using localization, and it has a number of interesting features. Most fundamentally, we see that the natural quantity to consider is not Z but its derivative $\partial^{g-1} Z(\epsilon)/\partial \epsilon^{g-1}$. In discussing the higher critical points, we lose nothing by considering this derivative, since any terms in Z that are polynomial in ϵ , and hence are annihilated by the derivative, arise as contributions from the moduli space \mathcal{M}_0 of flat connections. Moreover, although the formula in (4.46) is expressed in terms of elementary functions, its integral with respect to ϵ cannot be expressed so simply.

We also see from (4.46) that the local contributions from the higher critical points to $\partial^{g-1} Z(\epsilon)/\partial \epsilon^{g-1}$ are essentially independent of g and n , apart from a numerical prefactor and the usual exponential dependence on the classical action S_n .

Finally, we see that the only dependence on ϵ in (4.46) besides the classical dependence on S_n is through the prefactor proportional to $\epsilon^{-1/2}$. As we will see, this prefactor reflects the geometric fact that the gauge group does not act freely on the locus of non-flat Yang-Mills solutions. To explain this fact, we note that for any Yang-Mills solution the section f of $\text{ad}(P)$ satisfies $d_A f = 0$, so that $f \neq 0$ generates a $U(1)$ subgroup of the gauge group $\mathcal{G}(P)$ that fixes the corresponding point of $\mathcal{A}(P)$.

This geometric observation about higher critical points of Yang-Mills theory is actually a general property of any higher critical points of the abstract symplectic model with quadratic action $S = \frac{1}{2}(\mu, \mu)$. Namely, the abstract Hamiltonian group H can never act freely at a higher critical point of S .

By definition, such a higher critical point x_0 in the symplectic manifold X is described by the conditions $dS = (\mu, d\mu) = 0$ with $\mu \neq 0$ at x_0 . To show that H does not act freely at x_0 , we now exhibit a Hamiltonian vector field which vanishes at x_0 . We first recall the quantity $V = \Omega^{-1}d\mu$ which we introduced in Section 4.1. Geometrically V , or $V_a^m = (\Omega^{-1})^{mn}\partial_n\mu_a$ in components, is a linear map from the Lie algebra \mathfrak{h} of H to the space of Hamiltonian vector fields on X . From (4.13) and (4.14), we see that V trivially satisfies $(\mu, V) = \mu^a V_a^m = 0$ at x_0 . But since $\mu(x_0)$ is non-zero, we can consider on X the Hamiltonian vector field generated by $\mu(x_0)$ itself. This vector field is given by $(\mu(x_0), V) = \mu(x_0)^a V_a^m$, and by our observations above it vanishes at x_0 .

The Hodge Decomposition at a Higher Yang-Mills Critical Point

In many respects, localization at an irreducible, flat Yang-Mills solution is precisely opposite to localization at a maximally reducible, non-flat Yang-Mills solution. In both cases, the local geometry in $\mathcal{A}(P)$ near these critical points can be described as the total space N of an equivariant bundle with infinite-dimensional fiber F over a finite-dimensional moduli space \mathcal{M}_f ,

$$(4.47) \quad F \longrightarrow N \xrightarrow{pr} \mathcal{M}_f.$$

However, in the case of a flat connection the interesting contributions to the integral over N arise from the moduli space \mathcal{M}_0 itself, and the integral over the infinite-dimensional fiber $F = T^*H$ contributes a trivial factor of 1. In contrast, for a maximally reducible Yang-Mills solution, the integral over \mathcal{M}_f is essentially trivial, and the interesting contributions arise from the fiber F . Therefore, the most important aspect of our discussion of non-abelian localization at higher critical points in Yang-Mills theory is to identify the correct symplectic model for F , analogous to the identification $F = T^*H$ used previously.

At this point, we can immediately see that a local symplectic model for F based on T^*H does not correctly describe the geometry near \mathcal{M}_f if $f \neq 0$. First, as we have already observed, the gauge group does not act freely at points on \mathcal{M}_f , as we used in identifying F with T^*H when we considered the geometry near \mathcal{M}_0 . Second, if ϕ and ψ are any two sections of $\text{ad}(P)$ representing tangent vectors to $\mathcal{G}(P)$, then the computation in (4.24) shows that the symplectic form Ω at a point on

\mathcal{M}_f satisfies

$$(4.48) \quad \begin{aligned} \Omega(d_A\phi, d_A\psi) &= - \int_{\Sigma} \text{Tr}(d_A\phi \wedge d_A\psi) \\ &= \int_{\Sigma} \text{Tr}(\phi d_A^2\psi) = \int_{\Sigma} \text{Tr}(\phi [F_A, \psi]). \end{aligned}$$

Here we just use the fact that $d_A^2 = F_A$ is nonzero, and we observe that the last expression in (4.48) need not vanish for suitable ϕ and ψ . Thus, the orbit of $\mathcal{G}(P)$ through any point on \mathcal{M}_f is no longer an isotropic submanifold of $\mathcal{A}(P)$, as would be required to model this orbit on H embedded in the cotangent bundle T^*H with its canonical symplectic structure.

Now, the fact that F is not modelled on T^*H at a higher critical point of Yang-Mills theory must be reflected in a breakdown of the naive Hodge decomposition for the corresponding covariant derivative d_A , so that

$$(4.49) \quad \Gamma(\Sigma, \Omega_{\Sigma}^1 \otimes \text{ad}(P)) \neq \mathcal{H}_1 \oplus \text{Im}(d_A) \oplus \text{Im}(d_A^{\dagger}).$$

Thus, a natural strategy to determine the correct symplectic model for F is just to consider how the Hodge decomposition is modified when A is a non-flat solution of the Yang-Mills equation.

In expanding around a flat connection, the tangent space to the moduli space \mathcal{M}_0 of flat connections is given by $H_{d_A}^1(\Sigma, \text{ad}(P))$. For a non-flat Yang-Mills connection, d_A only squares to zero when restricted to $\text{ad}_0(P)$, the subspace of $\text{ad}(P)$ that commutes with f . On the other hand, deformations of a Yang-Mills solution automatically preserve f up to gauge transformation, simply because f automatically has integral eigenvalues. So tangent vectors to \mathcal{M}_f can always be represented by $\text{ad}_0(P)$ -valued one-forms, which represent deformations of the Yang-Mills solution by flat connections valued in the subgroup of G that commutes with f . So the tangent space to \mathcal{M}_f is $\mathcal{H}_1 = H_{d_A}^1(\Sigma, \text{ad}_0(P))$. By standard Hodge theory, this can also be defined as

$$(4.50) \quad \mathcal{H}_1 = H_{\bar{\partial}}^1(\Sigma, \text{ad}_0(P)).$$

Similarly, the Lie algebra of the unbroken subgroup G_f , which leaves fixed the given Yang-Mills connection, is

$$(4.51) \quad \mathcal{H}_0 = H_{d_A}^0(\Sigma, \text{ad}_0(P)) = H_{\bar{\partial}}^0(\Sigma, \text{ad}_0(P)).$$

What we have said so far is a fairly direct generalization of the usual statements in the flat case. However, if A is a non-flat Yang-Mills solution, then the usual Hodge theory needs to be modified from the flat case in two essential ways. First, once we get out of $\text{ad}_0(P)$, the image of d_A and the image of d_A^\dagger are no longer transverse. They have a nonzero, finite-dimensional intersection that we will call \mathcal{E}_0 :

$$(4.52) \quad \text{Im}(d_A) \cap \text{Im}(d_A^\dagger) = \mathcal{E}_0.$$

Second, the image of d_A plus the image of d_A^\dagger plus the tangent space \mathcal{H}_1 to the moduli space no longer generates $T_P = \Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P))$. The quotient $T_P / (\text{Im}(d_A) \oplus \text{Im}(d_A^\dagger))$ is another finite-dimensional vector space \mathcal{E}_1 . The bundles \mathcal{E}_0 and \mathcal{E}_1 both have natural complex structures. They will turn out to be

$$(4.53) \quad \begin{aligned} \mathcal{E}_0 &= H_{\bar{\partial}}^0(\Sigma, \text{ad}_+(P)), \\ \mathcal{E}_1 &= H_{\bar{\partial}}^1(\Sigma, \text{ad}_+(P)) \oplus H_{\bar{\partial}}^1(\Sigma, \text{ad}_-(P)). \end{aligned}$$

We will often regard these complex vector spaces as real vector spaces of twice the dimension.

Thus, the correct generalization of (4.49) is informally

$$(4.54) \quad \Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P)) = \mathcal{H}_1 \oplus \text{Im}(d_A) \oplus \text{Im}(d_A^\dagger) \ominus \mathcal{E}_0 \oplus \mathcal{E}_1.$$

As indicated by our use of “ \ominus ”, the expression in (4.54) is to be interpreted somewhat in the sense of K -theory. Since $\text{Im}(d_A)$ and $\text{Im}(d_A^\dagger)$ have a non-trivial intersection \mathcal{E}_0 , this extra copy of \mathcal{E}_0 must be removed to get the right description of $\Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P))$.

The definition of the Dolbeault cohomology groups in (4.53) requires a complex structure on Σ . Abstractly, this complex structure is induced from the duality operator \star on Σ . Because $\star^2 = -1$ when \star acts on any one-form on Σ , we can define the bundles $\Omega^{0,1}$ and $\Omega^{1,0}$ of complex one-forms of either type on Σ by the respective $+i$ and $-i$ eigenspaces of \star . This decomposition by type determines the complex structure and hence the Dolbeault $\bar{\partial}$ operator appearing in (4.53).

However, for the following we find it useful to give an explicit formula for the operator $\bar{\partial}$, acting on the bundle $\text{ad}_\mathbb{C}(P)$, in terms of \star and the covariant derivative d_A . We define the operators $\bar{\partial}^{(p)}$ acting on complex

p -forms on Σ taking values in $\text{ad}_{\mathbb{C}}(P)$ by

$$(4.55) \quad \begin{aligned} \bar{\partial}^{(0)} &= d_A - i\star d_A, \\ \bar{\partial}^{(1)} &= -i d_A + d_A\star, \\ \bar{\partial}^{(2)} &= 0. \end{aligned}$$

Again because $\star^2 = -1$ when acting on one-forms on Σ , one can easily check the essential requirement that $\bar{\partial}^{(1)} \circ \bar{\partial}^{(0)} = 0$. From the expression for $\bar{\partial}^{(1)}$ in (4.55), we also see that $\bar{\partial}^{(1)}$ annihilates all one-forms in the $+i$ eigenspace of \star , which we have identified with the space of one-forms of type $(0, 1)$.

The subbundle $\text{ad}_0(P)$ has a de Rham cohomology (with respect to d_A) that we have already encountered. The subbundles $\text{ad}_+(P)$ and $\text{ad}_-(P)$ do not have de Rham cohomology, but they have Dolbeault cohomology groups

$$(4.56) \quad H_{\bar{\partial}}^0(\Sigma, \text{ad}_+(P)), \quad H_{\bar{\partial}}^0(\Sigma, \text{ad}_-(P)), \quad H_{\bar{\partial}}^1(\Sigma, \text{ad}_+(P)), \quad H_{\bar{\partial}}^1(\Sigma, \text{ad}_-(P))$$

that we should expect will enter somehow. Of these cohomology groups, $H_{\bar{\partial}}^0(\Sigma, \text{ad}_-(P))$ is zero by the Kodaira vanishing theorem [21], which is the reason that \mathcal{E}_0 in (4.53) only involves $\text{ad}_+(P)$. (We also note parenthetically that $H_{\bar{\partial}}^1(\Sigma, \text{ad}_+(P))$ is similarly zero for critical points associated to line bundles of sufficiently high degree.) So we are left to show that \mathcal{E}_0 corresponds to the finite-dimensional intersection of $\text{Im}(d_A)$ and $\text{Im}(d_A^\dagger)$ and \mathcal{E}_1 describes the tangent vectors to $\mathcal{A}(P)$ not contained in $\text{Im}(d_A) \oplus \text{Im}(d_A^\dagger) \oplus \mathcal{H}_1$.

We identify \mathcal{E}_0 as described in (4.52) immediately from our formula for $\bar{\partial}^{(0)}$ in (4.55). It is convenient to write $\text{ad}(P) = \text{ad}_0(P) \oplus \text{ad}_\perp(P)$, with $\text{ad}_\perp(P)$ (whose complexification is $\text{ad}_+(P) \oplus \text{ad}_-(P)$) the orthocomplement of $\text{ad}_0(P)$. By standard Hodge theory, if we restrict to $\text{ad}_0(P)$, $\text{Im}(d_A) \cap \text{Im}(d_A^\dagger) = 0$. So the nontrivial intersection of $\text{Im}(d_A)$ and $\text{Im}(d_A^\dagger)$ occurs in $\text{ad}_\perp(P)$. Such an intersection arises if there is $\phi \in \Gamma(\Sigma, \text{ad}_\perp(P))$ and $\Psi \in \Omega^2(\Sigma, \text{ad}_\perp(P))$ such that $d_A\phi = d_A^\dagger\Psi$. If so, let $\psi = \star\Psi$, whereupon, since $d_A^\dagger = -\star d_A\star$ and $\star^2 = -1$, we have $d_A\phi = -\star d_A\psi$. So if $\varphi = \phi + i\psi$, we have $\bar{\partial}^{(0)}\varphi = (d_A - i\star d_A)\varphi = 0$. Hence $\varphi \in H_{\bar{\partial}}^0(\Sigma, \text{ad}_+(P) \oplus \text{ad}_-(P))$. But by Kodaira vanishing, $\text{ad}_-(P)$ does not contribute, and $\varphi \in H_{\bar{\partial}}^0(\Sigma, \text{ad}_+(P))$. This argument can also

be run backwards, to map $H_{\bar{\partial}}^0(\Sigma, \text{ad}_+(P))$ to \mathcal{E}_0 . This explains the claim that $\mathcal{E}_0 = H_{\bar{\partial}}^0(\Sigma, \text{ad}_+(P))$.

Finally, we can identify \mathcal{E}_1 , the subspace of $\Gamma(\Sigma, \text{ad}_\perp(P))$ that is orthogonal to the image of d_A and the image of d_A^\dagger . We begin with the tautological observation that the orthocomplement of the image of d_A is precisely the kernel of d_A^\dagger , and similarly the orthocomplement of the image of d_A^\dagger is precisely the kernel of d_A . Thus, \mathcal{E}_1 , the orthocomplement to the image of d_A and d_A^\dagger , consists of forms annihilated by both d_A^\dagger and d_A .⁷ Given the formula $\bar{\partial}^{(1)} = -id_A + d_A\star$, it follows that $\bar{\partial}^{(1)}$ annihilates \mathcal{E}_1 . Moreover, $\bar{\partial}^{\dagger(1)}$, the $\bar{\partial}^\dagger$ operator acting on one-forms, is $\bar{\partial}^{\dagger(1)} = d_A^\dagger - id_A^\dagger\star$, and so annihilates \mathcal{E}_1 . This reasoning can also be read backwards to show that a form annihilated by $\bar{\partial}^{(1)}$ and its adjoint $\bar{\partial}^{\dagger(1)}$ is annihilated by d_A and d_A^\dagger and hence is contained in \mathcal{E}_1 . By Hodge theory, the joint kernel of $\bar{\partial}$ and $\bar{\partial}^\dagger$ is the same as the cohomology of $\bar{\partial}$. So finally, $\mathcal{E}_1 = H_{\bar{\partial}}^1(\Sigma, \text{ad}_+(P) \oplus \text{ad}_-(P))$, as we have claimed.

A New Symplectic Model For Localization

The Hodge decomposition (4.54) implicitly describes the local symplectic model to use at a higher Yang-Mills critical point. We now present this model and compute via localization the canonical symplectic integral in this case.

Abstractly, our local model for F now differs in two ways from the model based on the cotangent bundle T^*H . First, H no longer acts freely at the given critical point. We let $H_0 \subset H$ denote the subgroup of H which fixes the critical point. Thus, the orbit of H through the critical point can be identified with H/H_0 . In the case of Yang-Mills theory, the vector space \mathcal{H}_0 of harmonic sections of $\text{ad}_0(P)$ is abstractly identified with the Lie algebra \mathfrak{h}_0 of H_0 .

Second, because of the appearance of \mathcal{E}_0 and \mathcal{E}_1 in the Hodge decomposition in (4.54), the naive model based on the cotangent bundle of the orbit H/H_0 must be modified in the following way. If we simply wanted to discuss the cotangent bundle of the orbit H/H_0 , then we could again pass to a basis of right-invariant forms and use the invariant metric (\cdot, \cdot)

⁷Notice that although d_A^2 and $d_A^{\dagger 2}$ are nonzero in general, they annihilate $\Omega^1(\Sigma, \text{ad}_\perp(P))$ for dimensional reasons, as a result of which d_A and d_A^\dagger can have a kernel!

on \mathfrak{h} to present $T^*(H/H_0)$ as a homogeneous bundle

$$(4.57) \quad T^*(H/H_0) \cong H \times_{H_0} (\mathfrak{h} \ominus \mathfrak{h}_0).$$

Here $\mathfrak{h} \ominus \mathfrak{h}_0$ denotes the orthogonal complement to \mathfrak{h}_0 in \mathfrak{h} , and “ \times_{H_0} ” indicates that we identify points (g, γ) in the product $H \times (\mathfrak{h} \ominus \mathfrak{h}_0)$ under the following action of H_0 ,

$$(4.58) \quad h \cdot (g, \gamma) = (hg, h\gamma h^{-1}), \quad h \in H_0.$$

To incorporate the appearance of \mathcal{E}_0 and \mathcal{E}_1 in (4.54), we now introduce abstractly a subspace E_0 of the Lie algebra \mathfrak{h} which has a trivial intersection with \mathfrak{h}_0 and is preserved under the adjoint action of H_0 , so that infinitesimally $[\mathfrak{h}_0, E_0] \subseteq E_0$. This condition certainly holds in Yang-Mills theory for the vector space \mathcal{E}_0 . Similarly, we introduce another vector space E_1 on which H_0 acts in some representation. We assume that, like the subspace E_0 , the representation E_1 admits a metric invariant under the action of H_0 .

We now describe our model for F as a homogeneous bundle over the orbit H/H_0 which generalizes (4.57). To describe this bundle, we need only specify the fiber of F over the identity coset of H/H_0 and the action of H_0 on the fiber. Thus, as in the modified Hodge decomposition (4.54), we subtract E_0 from the cotangent fiber of H/H_0 in (4.57), meaning that we take the orthogonal complement to E_0 in $\mathfrak{h} \ominus \mathfrak{h}_0$, and we also add E_1 to the cotangent fiber of H/H_0 . So the resulting fiber of F over the identity is given by $\mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 \oplus E_1$. By our assumptions on E_0 and E_1 , this vector space transforms as a representation of H_0 .

In summary, the local model for F is given abstractly by the following homogeneous bundle over H/H_0 ,

$$(4.59) \quad F = H \times_{H_0} (\mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 \oplus E_1).$$

We now use γ to denote an element of the orthogonal complement \mathfrak{h}^\perp to $\mathfrak{h}_0 \oplus E_0$ in \mathfrak{h} ,

$$(4.60) \quad \gamma \in \mathfrak{h}^\perp = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0,$$

and we use v to denote a vector in E_1 . So in (4.59), we identify points (g, γ, v) in the product $H \times (\mathfrak{h}^\perp \oplus E_1)$ under the following action of H_0 ,

$$(4.61) \quad h \cdot (g, \gamma, v) = (hg, h\gamma h^{-1}, h \cdot v), \quad h \in H_0.$$

To specify completely our local model, we must also discuss the symplectic structure and the Hamiltonian H -action on F . We will be somewhat brief, since we are just applying standard techniques to construct symplectic bundles, as explained for instance in Ch. 35–41 of [59].

In order to construct a symplectic structure on F , we must make some additional assumptions about the representations E_0 and E_1 of H_0 . We first introduce an element γ_0 of \mathfrak{h}_0 . Abstractly, γ_0 corresponds to the value of the moment map at the given critical point, and in the Yang-Mills context γ_0 is identified with f .

As in Yang-Mills theory, we assume that the hermitian operator Λ ,

$$(4.62) \quad \Lambda = i [\gamma_0, \cdot],$$

annihilates \mathfrak{h}_0 and acts on the vector spaces E_0 and E_1 with strictly non-zero eigenvalues. The first assumption implies that γ_0 is central in \mathfrak{h}_0 and is invariant under the adjoint action of H_0 ,

$$(4.63) \quad H_0 \gamma_0 H_0^{-1} = \gamma_0.$$

Because the action of γ_0 preserves the invariant metrics on E_0 and E_1 , the action of γ_0 is represented by a real, anti-symmetric matrix. By our second assumption above, this matrix is non-degenerate. Consequently, the decomposition of E_0 , and similarly E_1 , into the positive and negative eigenspaces of Λ defines a complex structure which is invariant under the action of H_0 and for which the invariant metric (\cdot, \cdot) is hermitian.

Having introduced γ_0 , we now describe the symplectic structure on F . As in Section 4.2, we let θ be the canonical right-invariant one-form on H taking values in \mathfrak{h} ,

$$(4.64) \quad \theta = dgg^{-1}.$$

We recall that in the case of the cotangent bundle T^*H or $T^*(H/H_0)$, we can immediately describe the symplectic structure with the manifestly closed and non-degenerate two-form Ω_0 ,

$$(4.65) \quad \Omega_0 = d(\gamma, \theta),$$

which reduces on the orbit H/H_0 , where $\gamma = 0$, to the canonical form $(d\gamma, \theta)$.

Similarly, when we consider the homogeneous bundle F in (4.59), Ω_0 in (4.65) still descends to a closed two-form on F . However, because γ now takes values in \mathfrak{h}^\perp as in (4.60), the restriction of Ω_0 to the orbit H/H_0 is degenerate on the subspace E_0 of the tangent space to the orbit. Thus, if we ignore the vector space E_1 for the moment, then to construct a symplectic structure on the homogeneous bundle with fiber \mathfrak{h}^\perp over H/H_0 we must supplement the canonical two-form Ω_0 with an additional two-form which is non-degenerate on E_0 .

What other two-form should we consider? For motivation, while keeping $E_1 = 0$, let us consider the opposite case from the cotangent bundle.

As the cotangent bundle has $E_0 = 0$, the other extreme is for E_0 to be all of $\mathfrak{h} \ominus \mathfrak{h}_0$, so that $\mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 = 0$ and $F = H/H_0$. Since we have postulated that γ_0 acts non-degenerately on E_0 , while commuting with \mathfrak{h}_0 , it follows in this case that \mathfrak{h}_0 is precisely the subalgebra of \mathfrak{h} that commutes with γ_0 . Therefore, H/H_0 is precisely the orbit of γ_0 in the Lie algebra of H . Such an orbit is called a coadjoint orbit (for compact Lie groups the difference between the adjoint representation and its dual is not important here) and has a natural symplectic structure, namely

$$(4.66) \quad \Omega_1 = d(\gamma_0, \theta) = \frac{1}{2} (\theta, [\gamma_0, \theta]),$$

where we observe that $d\theta = \theta \wedge \theta = \frac{1}{2}[\theta, \theta]$ in deducing the second equality of (4.66). Because γ_0 is invariant under the adjoint action of H_0 in (4.63), Ω_1 is also invariant under the action of H_0 in (4.61) and descends to a manifestly closed and nondegenerate two-form on H/H_0 . Indeed, coadjoint orbits are the basic examples of homogeneous symplectic manifolds.

In fact, we have already seen the coadjoint form Ω_1 arise in the context of Yang-Mills theory. We recall from (4.48) that the restriction of the Yang-Mills symplectic form Ω on the affine space $\mathcal{A}(P)$ to the orbit of $\mathcal{G}(P)$ through a non-flat Yang-Mills solution is given by

$$(4.67) \quad \Omega(d_A\phi, d_A\psi) = \int_{\Sigma} \text{Tr} (\phi [F_A, \psi]).$$

Upon identifying the abstract element γ_0 with f , we see that Ω_1 in (4.66) precisely represents (4.67).

The general case, still with $E_1 = 0$, is a mixture of the cotangent bundle and the coadjoint orbit. We thus naturally add the two two-forms that arise in those two cases and consider the sum

$$(4.68) \quad \Omega_0 + \Omega_1 = d(\gamma + \gamma_0, \theta),$$

which restricts on the orbit H/H_0 , where $\gamma = 0$, to the simple expression

$$(4.69) \quad (\Omega_0 + \Omega_1)|_{H/H_0} = (d\gamma, \theta) + \frac{1}{2} (\theta, [\gamma_0, \theta]).$$

We see immediately from (4.69) that $\Omega_0 + \Omega_1$ defines a symplectic form on a neighborhood of H/H_0 in the homogeneous bundle with fiber \mathfrak{h}^\perp . For instance, since the expression in (4.68) is manifestly invariant under the right action of H on H/H_0 , we need only consider (4.69) as restricted to the tangent space $(\mathfrak{h} \ominus \mathfrak{h}_0) \oplus \mathfrak{h}^\perp$ of the bundle at the identity coset on H/H_0 . The top power of (4.69) on this tangent space is then

manifestly non-zero, since all tangent vectors in \mathfrak{h}^\perp are paired by Ω_0 and the remaining tangent vectors to the orbit in E_0 are paired by Ω_1 .

Finally, we need to include E_1 . By assumption, E_1 has a metric and a complex structure invariant under the action of H_0 , so that E_1 has an associated symplectic form $\tilde{\Omega}$ invariant under H_0 .

In order to pass from the symplectic form $\tilde{\Omega}$ on E_1 to a closed two-form on F which is non-degenerate on the E_1 fiber at the identity coset of H/H_0 and compatible with the bundle structure of F , we must further suppose that H_0 acts on E_1 in a Hamiltonian fashion with moment map $\tilde{\mu}$. We can always choose $\tilde{\mu}$ to vanish at the origin of E_1 . We also observe that since the action of H_0 on E_1 is linear, of the form $\delta v = \psi \cdot v$ for v in E_1 and ψ in \mathfrak{h}_0 , the moment map $\tilde{\mu}$ depends quadratically on v and satisfies $d\tilde{\mu} = 0$ at the origin of E_1 .

With these observations in hand, we consider the two-form Ω_2 defined below,

$$(4.70) \quad \Omega_2 = \tilde{\Omega} + d\langle \tilde{\mu}, \theta \rangle.$$

This two-form is manifestly closed, as $\tilde{\Omega}$ is closed. It also is clearly invariant under the action of H_0 in (4.61).

Finally, to explain the appearance of the second term in (4.70), we note that the action of \mathfrak{h}_0 on F can be described as follows. For $\psi \in \mathfrak{h}_0$, the corresponding vector field $V(\psi)$ on F acts by

$$(4.71) \quad \delta g = \psi g, \quad \delta \gamma = [\psi, \gamma], \quad \delta v = \psi \cdot v.$$

In order that Ω_2 descend under the quotient by H_0 which defines the bundle, we require that Ω_2 be invariant under H_0 (as we have already seen) and that Ω_2 be annihilated by contraction with $V(\psi)$. By the defining moment map relation, the contraction of $V(\psi)$ with $\tilde{\Omega}$ is $\iota_{V(\psi)}\tilde{\Omega} = d\langle \tilde{\mu}, \psi \rangle$. As for the second term in (4.70), the one-form $\langle \tilde{\mu}, \theta \rangle$ is invariant under the action of H_0 and hence annihilated by the Lie derivative $\mathcal{L}_{V(\psi)} = \{d, \iota_{V(\psi)}\}$. Thus we see that $\iota_{V(\psi)} d\langle \tilde{\mu}, \theta \rangle = -d\iota_{V(\psi)}\langle \tilde{\mu}, \theta \rangle = -d\langle \tilde{\mu}, \psi \rangle$, which cancels the contraction of $\iota_{V(\psi)}$ with $\tilde{\Omega}$.

Because $\tilde{\mu} = d\tilde{\mu} = 0$ at the origin of E_1 , the restriction of Ω_2 to the orbit H/H_0 in F is simply the symplectic form $\tilde{\Omega}$ on E_1 . Thus, the sum of Ω_0 , Ω_1 , and Ω_2 defines a symplectic form Ω on a neighborhood of the orbit H/H_0 in F ,

$$(4.72) \quad \begin{aligned} \Omega &= \Omega_0 + \Omega_1 + \Omega_2, \\ &= d(\gamma + \gamma_0, \theta) + d\langle \tilde{\mu}, \theta \rangle + \tilde{\Omega}. \end{aligned}$$

Having placed a symplectic structure on F , we are left to consider the action of H on F . As in the model based on the cotangent bundle, we assume that H acts from the right on the orbit H/H_0 in F , so that

$$(4.73) \quad h \cdot (g, \gamma, v) = (gh^{-1}, \gamma, v), \quad h \in H.$$

The corresponding element ϕ in \mathfrak{h} generates the vector field

$$(4.74) \quad \delta g = -g\phi, \quad \delta\gamma = 0, \quad \delta v = 0.$$

Since the one-form θ appearing in Ω is right-invariant, the symplectic form Ω is manifestly invariant under H .

Finally, using (4.72) and (4.74), one can easily check that the action of H on F is Hamiltonian with moment map μ given by

$$(4.75) \quad \langle \mu, \phi \rangle = \langle \gamma + \gamma_0, g\phi g^{-1} \rangle + \langle \tilde{\mu}, g\phi g^{-1} \rangle.$$

In particular, we see that the value of μ at the point corresponding to the identity coset on the orbit H/H_0 is just the dual of γ_0 in \mathfrak{h}^* , as we have claimed.

Computing the Symplectic Integral over F

For our applications to both Yang-Mills theory and Chern-Simons theory, we now compute the canonical symplectic integral over F ,

$$(4.76) \quad Z(\epsilon) = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times F} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2} (\phi, \phi) + t D\lambda \right].$$

In this expression, λ is the canonical one-form defined as in (4.12) by $\lambda = J dS$, where $S = \frac{1}{2}(\mu, \mu)$ and J is a compatible almost-complex structure, and t is a non-zero parameter.

Before we delve into computations, let us make a few remarks about how this symplectic integral over F is to be interpreted. We start by considering the canonical symplectic integral (4.8) of the same form as (4.76) but defined as an integral over a compact symplectic manifold X instead of F . Because X is compact, this integral is convergent for arbitrary t , including $t = 0$, and does not depend on either t or λ .

By our general analysis of Section 4.1, in the limit $t \rightarrow \infty$ and for λ of the canonical form, the integral over X localizes on the critical set of S and reduces to a finite sum of contributions from the components of this set. Although the global integral over X is perfectly defined, independent of t and λ , the contributions from the critical locus of S are only defined via localization, with $t \neq 0$ and λ of the canonical form. For instance, at a higher critical point of S , for which we model the normal symplectic geometry on F , the unstable modes of S make the

integral over the non-compact fibers of F ill-defined when $t = 0$. Thus, the symplectic integral $Z(\epsilon)$ over F as in (4.76) represents a *definition* of the local contribution from an unstable critical point of S in X .

Although we use the canonical one-form $\lambda = J dS$ to define via localization the integral over F in (4.76), we are free to compute $Z(\epsilon)$ using any other invariant form λ' which is homotopic to λ on F . In particular, though λ is defined globally on X , λ' need only be defined locally on F .

The reason that we might want to compute $Z(\epsilon)$ using some alternative form λ' instead of the canonical one-form λ is just that generically the integral over F defined by λ is not Gaussian even in the limit $t \rightarrow \infty$ and cannot be easily evaluated in closed form. See the appendix of [20] for a simple example of this behavior. However, by making a convenient choice for λ' , we can greatly simplify our computation and essentially reduce it to the evaluation of Gaussian integrals.

So in order to compute $Z(\epsilon)$ in (4.76), we first make a convenient choice for λ' . Since the motivation for our choice is fundamentally to simplify the evaluation of $Z(\epsilon)$, we next evaluate (4.76) using λ' in place of λ . Finally, in Appendix A, we perform the analysis required to show that $Z(\epsilon)$ as defined using the canonical one-form λ can be equivalently evaluated using λ' .

To describe our choice for λ' , we introduce a projection $\Pi_{\mathfrak{h}_0}$ onto \mathfrak{h}_0 and a projection Π_{E_0} onto E_0 in the Lie algebra \mathfrak{h} of H . We define these projections using the invariant metric on \mathfrak{h} , so that they are invariant under the adjoint action of H_0 on \mathfrak{h} . We then introduce the quantities

$$(4.77) \quad \begin{aligned} \theta_{\mathfrak{h}_0} &= \Pi_{\mathfrak{h}_0}(\theta), & (g\phi g^{-1})_{\mathfrak{h}_0} &= \Pi_{\mathfrak{h}_0}(g\phi g^{-1}), \\ \theta_{E_0} &= \Pi_{E_0}(\theta), & (g\phi g^{-1})_{E_0} &= \Pi_{E_0}(g\phi g^{-1}). \end{aligned}$$

We now define λ' as

$$(4.78) \quad \begin{aligned} \lambda' &= (\gamma, \theta) - i(\theta_{E_0}, g\phi g^{-1}) + \\ &\quad + i\left((g\phi g^{-1})_{\mathfrak{h}_0} \cdot v, dv\right) - i\left((g\phi g^{-1})_{\mathfrak{h}_0} \cdot v, \theta_{\mathfrak{h}_0} \cdot v\right). \end{aligned}$$

The first term in (4.78) has the same form as the canonical one-form which we used for localization on T^*H . However, we recall that now γ takes values not in \mathfrak{h} but in $\mathfrak{h}^\perp = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0$. As before, this first term has degree one under the grading on equivariant cohomology. The other three terms are associated to the new vector spaces E_0 and E_1 that appear at a higher critical point. Since ϕ carries charge +2 under

the grading on equivariant cohomology, these terms are all of degree three.

The most basic requirement that λ' must satisfy is that it descends to an invariant form on F under the quotient by H_0 which defines the homogeneous bundle. So we first observe that λ' is manifestly invariant under the action of H_0 in (4.61). Furthermore, if $V(\psi)$ denotes the vector field on the product $H \times (\mathfrak{h}^\perp \oplus E_1)$ generated by ψ in \mathfrak{h}_0 as in (4.71), then the first two terms in λ' are trivially annihilated upon contraction with $V(\psi)$ since both γ and θ_{E_0} take values in the orthocomplement to \mathfrak{h}_0 . Because of the identity

$$(4.79) \quad \iota_{V(\psi)} dv = \psi \cdot v = (\iota_{V(\psi)} \theta_{\mathfrak{h}_0}) \cdot v,$$

the last two terms in λ' are also annihilated upon contraction with $V(\psi)$. So λ' descends to a well-defined form on F .

Finally, to check that λ' is invariant under the action of H on F in (4.73), we simply note that ϕ transforms under the adjoint action of H so that the quantity $g\phi g^{-1}$ is invariant. Since θ is also invariant under the action of H , λ' is manifestly invariant.

To motivate our definition (4.78), we now use λ' to compute the symplectic integral over F . We first compute $D\lambda'$. As we saw when we considered localization on T^*H , the final expression for $D\lambda'$ will only involve ϕ in the invariant combination $g\phi g^{-1}$. Thus, even before presenting our formula for $D\lambda'$, we make the change of variables from ϕ to $g\phi g^{-1}$ in the symplectic integral in order to simplify slightly our result. If we recall that $D = d + i \iota_{V(\phi)}$ and we use the formula in (4.74) for $V(\phi)$, we find by a straightforward computation that

$$(4.80) \quad D\lambda' = (d\gamma, \theta) - i(\gamma, \phi) - i(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}]) - (\phi_{E_0}, \phi_{E_0}) + \\ + i(\phi_{\mathfrak{h}_0} \cdot dv, dv) - (\phi_{\mathfrak{h}_0} \cdot v, \phi_{\mathfrak{h}_0} \cdot v) + \mathcal{X}.$$

Here \mathcal{X} consists of extra terms in $D\lambda'$ that will not actually contribute to the symplectic integral in the limit $t \rightarrow \infty$. Explicitly,

$$(4.81) \quad \mathcal{X} = \left(\gamma, \frac{1}{2}[\theta, \theta] \right) - i \left(\frac{1}{2} [\theta^\perp, \theta^\perp], \phi_{E_0} \right) - i \left([\theta^\perp, \theta_{E_0}], \phi^\perp \right) - \\ - i \left(\frac{1}{2} [\theta_{E_0}, \theta_{E_0}], \phi^\perp \right) - i \left(\phi_{\mathfrak{h}_0} \cdot v, \frac{1}{2} [\theta, \theta]_{\mathfrak{h}_0} \cdot v \right) \pmod{\theta_{\mathfrak{h}_0}}.$$

(Terms involving $\theta_{\mathfrak{h}_0}$ in $D\lambda'$, some of which are omitted here, actually cancel since $D\lambda'$ is a pullback from F .) We use the fact that $d\theta = \frac{1}{2}[\theta, \theta]$

to simplify somewhat the form of \mathcal{X} , and we use the natural notation θ^\perp and ϕ^\perp to denote the projections of θ and ϕ onto \mathfrak{h}^\perp .

In (4.80), the first two terms arise from the action of D on the first term in λ' , the next two arise from the action of D on the second term in λ' , and the final two terms arise from the action of D on the last two terms in λ' . We remark that our choice of the i 's that appear in the definition (4.78) of λ' was made to ensure that the quadratic terms in (4.80) involving ϕ_{E_0} and $\phi_{\mathfrak{h}_0} \cdot v$ are both negative-definite.

We now consider the canonical symplectic integral in (4.76) with λ' in place of λ and in the limit $t \rightarrow \infty$. This symplectic integral is an integral over the product $\mathfrak{h} \times F$. We can perform this integral over $\mathfrak{h} \times F$ in two steps. First, we hold the projection $\phi_{\mathfrak{h}_0}$ of the variable ϕ in $\mathfrak{h}_0 \subset \mathfrak{h}$ fixed, and we perform the integral over the remaining variables in $\tilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F$. This integral produces a measure on \mathfrak{h}_0 , which we then use to perform the remaining integral over \mathfrak{h}_0 . The utility of this way of performing the symplectic integral is that, with our ansatz for λ' , we will see that the first integral over $(\mathfrak{h} \ominus \mathfrak{h}_0) \times F$ can be performed directly as a Gaussian integral in the limit $t \rightarrow \infty$ and under the assumption that $\phi_{\mathfrak{h}_0}$ acts in a non-degenerate fashion on E_0 and E_1 .

To prove this fact, we first consider the symplectic integral over $\tilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F$ which arises if \mathcal{X} is omitted from $D\lambda'$. So we consider the integral

$$\begin{aligned}
 (4.82) \quad I(\phi_{\mathfrak{h}_0}) &= \frac{1}{\text{Vol}(H)} \int_{\tilde{F}} \left[\frac{d\phi}{2\pi} \right] \exp \left[t(d\gamma, \theta) - it(\gamma, \phi) - \right. \\
 &\quad \left. - it(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}]) - t(\phi_{E_0}, \phi_{E_0}) \right] \times \\
 &\quad \times \exp \left[it(\phi_{\mathfrak{h}_0} \cdot dv, dv) - t(\phi_{\mathfrak{h}_0} \cdot v, \phi_{\mathfrak{h}_0} \cdot v) \right].
 \end{aligned}$$

For fixed $\phi_{\mathfrak{h}_0}$ acting non-degenerately on E_0 and E_1 , this integral (4.82) is a Gaussian integral, which we now evaluate. In performing this integral, we recall that the vector spaces E_0 and E_1 carry a complex structure, invariant under the action of $\phi_{\mathfrak{h}_0}$, for which the metric (\cdot, \cdot) is hermitian.

Assuming E_1 is suitably oriented, the Gaussian integral over v in E_1 first produces a factor

$$(4.83) \quad \det \left(\frac{\phi_{\mathfrak{h}_0}}{2\pi} \Big|_{E_1} \right)^{-1}.$$

This expression does not depend on t , due to a cancellation between the factors of t that arise from the Gaussian integral over v and the factors of t that appear in the measure on E_1 .

The remainder of the integration is similar, but is actually perhaps more easily explained if we adopt a physicist's notation rather than the mathematical notation in which (4.82) has been written. In mathematical notation, $\theta = dg g^{-1}$ is a one-form; we are supposed to expand the exponential to produce a top-form which is then integrated. In physics notation, θ is understood as a fermionic variable, and (4.82) must be reexpressed to contain an extra factor $dg d\theta$ in the measure.

In the physics notation, we now perform the Gaussian integrals over ϕ_{E_0} and θ_{E_0} . The powers of t cancel, just as in the integration over v (which in physics notation would have been an integral over v and an independent fermionic variable $\hat{v} = dv$), and we are left with a determinantal factor

$$(4.84) \quad \det \left(\frac{\phi_{\mathfrak{h}_0}}{2\pi} \Big|_{E_0} \right),$$

which now appears in the numerator as it comes from a fermionic integration. The factors of 2π come from the Gaussian integral together with the measure $[d\phi/2\pi]$.

Similarly, in physics notation, γ and $\hat{\gamma} = d\gamma$ are treated as independent bosonic and fermion variables and the measure contains an extra factor $d\gamma d\hat{\gamma}$. Likewise, we integrate separately over H/H_0 and over fermionic variables θ . In fact, we have already performed the integration over θ_{E_0} , so we are only left with the component of θ in \mathfrak{h}^\perp . The integral over γ gives a delta function setting to zero the projection of ϕ to \mathfrak{h}^\perp . The integral over $\hat{\gamma}$ gives a delta function setting to zero the component of θ in \mathfrak{h}^\perp , and canceling the power of t generated by the γ integral. Finally, the integration over H/H_0 produces a factor of $\text{Vol}(H)/\text{Vol}(H_0)$.

So finally, simplifying the notation by setting $\psi = \phi_{\mathfrak{h}_0}$, the result arising from the Gaussian integration is

$$(4.85) \quad I(\psi) = \frac{1}{\text{Vol}(H_0)} \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right) \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right)^{-1}, \quad \psi \in \mathfrak{h}_0.$$

Of course, a conventional mathematical exposition of the calculation would arrive at the same result after grouping the factors a little differently.

The result (4.85) for the integral (4.82) is independent of t . We now observe that the terms in \mathcal{X} which we omitted from $D\lambda'$ when computing (4.85) are all of at least third order in the integration variables on $\tilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F$ (which do *not* include the constant $\phi_{\mathfrak{h}_0}$). Thus, upon rescaling all the integration variables by $t^{-\frac{1}{2}}$ so that the quadratic terms in (4.82) become independent of t , we see that any contributions from terms in \mathcal{X} to the symplectic integral fall off at least as fast as $t^{-\frac{1}{2}}$ for large t . Thus, our Gaussian evaluation of the symplectic integral over \tilde{F} is exact as $t \rightarrow \infty$.

So we are left to consider the remaining integral over \mathfrak{h}_0 , which is now given formally by

$$(4.86) \quad Z'(\epsilon) = \frac{1}{\text{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi} \right] \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right) \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right)^{-1} \times \exp \left[-i(\gamma_0, \psi) - \frac{\epsilon}{2}(\psi, \psi) \right].$$

In obtaining this expression, we recall from (4.75) that the value of the moment map μ at the identity coset on the orbit H/H_0 is γ_0 . Also, we denote this quantity as $Z'(\epsilon)$, instead of $Z(\epsilon)$, to emphasize that we compute it with λ' instead of the canonical form λ that defines the local contributions to $Z(\epsilon)$.

Now, this formal integral over \mathfrak{h}_0 in (4.86) might or might not actually be defined. Due to the exponential factor in the integrand of (4.86), the integral is certainly convergent at large ψ . However, on the locus in \mathfrak{h}_0 where the determinant of ψ acting on E_1 vanishes (for instance at the origin of \mathfrak{h}_0), the measure $I(\psi)$ in (4.85) might be singular if there is no compensating zero from the determinant of ψ acting on E_0 . If $I(\psi)$ is singular, then the integral in (4.86) could fail to be convergent at the singularity. Since $Z(\epsilon)$ as defined using the canonical one-form λ is always finite, our computation using λ' cannot generally be valid.

On the other hand, because E_0 and E_1 are both finite-dimensional vector spaces, with

$$(4.87) \quad \dim_{\mathbb{C}} E_0 = d_0, \quad \dim_{\mathbb{C}} E_1 = d_1,$$

the determinants appearing in $I(\psi)$ in (4.85) are just invariant polynomials, homogeneous of degrees d_0 and d_1 , of ψ in \mathfrak{h}_0 . For our application to $SU(2)$ Yang-Mills theory, for which $H_0 = U(1)$, we need only consider the simplest case that $\mathfrak{h}_0 = \mathbb{R}$ is one-dimensional. In this case, the

invariant polynomials are just monomials

$$(4.88) \quad \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right) = c_0 \psi^{d_0}, \quad \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right) = c_1 \psi^{d_1},$$

for some constants c_0 and c_1 .

Assuming (4.88), we see that (4.86) becomes

$$(4.89) \quad Z'(\epsilon) = \frac{1}{\text{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi} \right] \left(\frac{c_0}{c_1} \right) \psi^{d_0-d_1} \exp \left[-i(\gamma_0, \psi) - \frac{\epsilon}{2}(\psi, \psi) \right].$$

Although this expression in (4.89) is ill-defined if $d_1 > d_0$, we can still apply our previous work to compute using λ' a completely well-defined integral. Namely, instead of considering the symplectic integral $Z'(\epsilon)$, we introduce the differential operator Q ,

$$(4.90) \quad Q = \left(-2 \frac{\partial}{\partial \epsilon} \right)^{\frac{1}{2}(d_1-d_0)},$$

and we consider instead the quantity

$$(4.91) \quad Q \cdot Z'(\epsilon) = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times F} \left[\frac{d\phi}{2\pi} \right] (\phi, \phi)^{\frac{1}{2}(d_1-d_0)} \times \\ \times \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2}(\phi, \phi) + t D\lambda' \right].$$

Using the same definition for λ' and proceeding exactly as before, we compute

$$(4.92) \quad Q \cdot Z'(\epsilon) = \frac{1}{\text{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi} \right] \left(\frac{c_0}{c_1} \right) \exp \left[-i(\gamma_0, \psi) - \frac{\epsilon}{2}(\psi, \psi) \right], \\ = \frac{1}{\text{Vol}(H_0)} \left(\frac{c_0}{c_1} \right) \frac{1}{\sqrt{2\pi\epsilon}} \exp \left[-\frac{(\gamma_0, \gamma_0)}{2\epsilon} \right].$$

The fact that the differential operator Q in (4.90) can be used to cancel the determinants of ψ in (4.88) that arise from localization is a special consequence of our assumption that $\dim \mathfrak{h}_0 = 1$. For an arbitrary Lie algebra \mathfrak{h}_0 , we cannot generally express these determinants as functions of only the quadratic invariant (ψ, ψ) that appears in the canonical symplectic integral. As a result, in the general case we cannot cancel such determinants simply by differentiating $Z(\epsilon)$ with respect to the coupling ϵ . Though we will not require the generalization for this paper, we explain in Appendix B how to extend the discussion above to the case of general \mathfrak{h}_0 .

We see from (4.92) that, although our computation using λ' does not always give a sensible answer for $Z'(\epsilon)$, it does give a sensible answer for the derivative $Q \cdot Z'(\epsilon)$. Knowledge of this derivative implicitly determines the contribution of a higher critical point to $Z'(\epsilon)$, as the only ambiguity in integrating (4.92) is a polynomial in ϵ which cannot arise from a higher critical point. Finally, as we show in Appendix A, the quantity $Q \cdot Z'(\epsilon)$ in (4.92) defined using λ' agrees with the corresponding quantity $Q \cdot Z(\epsilon)$ defined using the canonical one-form λ . Hence, provided we take derivatives when necessary, we can use λ' for localization computations on F .

Our computation also shows that it may be easier to consider the contributions of higher critical points not to $Z(\epsilon)$ but to the derivative $Q \cdot Z(\epsilon)$. We have already seen an example of this phenomenon in our discussion of $SU(2)$ Yang-Mills theory. In that case, we found it more natural to compute the contributions of higher Yang-Mills critical points to the derivative $\partial^{g-1}Z(\epsilon)/\partial\epsilon^{g-1}$ in (4.46) as opposed to $Z(\epsilon)$ itself.

Application to Higher Critical Points of Yang-Mills Theory

To finish this section, we apply our abstract study of localization on F to compute the path integral contributions from maximally reducible Yang-Mills solutions. We focus on the specific case of $SU(2)$ Yang-Mills theory, for which we reproduce the explicit expression in (4.46) for the contributions from the locus \mathcal{M}_n of degree n critical points.

As we have discussed, if $f = \star F_A$ is the curvature of a maximally reducible Yang-Mills solution for gauge group G of rank r , then f breaks the gauge group to a maximal torus $G_f = U(1)^r$. In terms of our abstract model, we thus identify the stabilizer group H_0 with the subgroup $U(1)^r \subset \mathcal{G}(P)$ of constant gauge transformations in this maximal torus. As we have also discussed, this fact implies that the corresponding moduli space \mathcal{M}_f of maximally reducible Yang-Mills solutions is just a complex torus of dimension gr .

Now, our description of the local symplectic model F for the normal geometry over a higher Yang-Mills critical point is completely general, since in deriving the model for F we did not make any assumptions about the reducibility of the connection. However, if we wish to use this local model to compute contributions from arbitrary higher Yang-Mills critical points, we will generally find that both the integral over F and the integral over the associated moduli space \mathcal{M}_f make nontrivial contributions to $Z(\epsilon)$ which depend on ϵ .

In contrast, if we restrict to the special case that \mathcal{M}_f describes maximally reducible Yang-Mills solutions, then only the integral over F is nontrivial, and the integral over the torus \mathcal{M}_f contributes a multiplicative factor $\text{Vol}(\mathcal{M}_f)$ independent of ϵ , where

$$(4.93) \quad \text{Vol}(\mathcal{M}_f) = \int_{\mathcal{M}_f} \exp(\Omega).$$

From a physical perspective, the contribution from \mathcal{M}_f to $Z(\epsilon)$ does not involve the coupling ϵ because abelian gauge theory is free. From a mathematical perspective, the Donaldson theory of $U(1)$ bundles is simple, as the corresponding universal bundle is a line bundle having only a first Chern class, which is proportional to Ω .

In the case of $SU(2)$ Yang-Mills theory, the stabilizer group H_0 is just $U(1)$, and \mathfrak{h}_0 has dimension one. Thus, we can apply our computation of the integral over F in (4.92) to conclude that the local contribution from the moduli space \mathcal{M}_n of higher critical points of degree n is described by

$$(4.94) \quad \left(-2 \frac{\partial}{\partial \epsilon}\right)^{\frac{1}{2}(d_1-d_0)} \cdot Z(\epsilon) \Big|_{\mathcal{M}_n} = \frac{\text{Vol}(\mathcal{M}_n)}{\text{Vol}(H_0)} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left[-\frac{(2\pi n)^2}{\epsilon}\right].$$

We immediately see that this expression has the same form as the expression that appeared earlier in (4.46).

To make a precise comparison of our formula (4.94) to (4.46), we must compute the various constants appearing in (4.94). To start, we introduce the normalized generator T_0 of H_0 ,

$$(4.95) \quad T_0 = \frac{1}{\sqrt{2}} \sigma_z = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which satisfies $\text{Tr}(T_0^2) = -1$. From (4.95), we immediately see that the volume of H_0 in our metric on \mathfrak{h}_0 is

$$(4.96) \quad \text{Vol}(H_0) = 2\pi\sqrt{2}.$$

In the case of $SU(2)$ Yang-Mills theory, we have already identified in (4.40) the bundles $\text{ad}_\pm(P)$ with the line bundles $\mathcal{L}(+2n)$ and $\mathcal{L}^{-1}(-2n)$. Thus, from (4.53), the complex vector spaces \mathcal{E}_0 and \mathcal{E}_1 , abstractly identified with E_0 and E_1 , are now given by the following Dolbeault

cohomology groups,

$$(4.97) \quad \begin{aligned} E_0 &= H_{\bar{\partial}}^0(\Sigma, \mathcal{L}(2n)), \\ E_1 &= H_{\bar{\partial}}^1(\Sigma, \mathcal{L}(2n)) \oplus H_{\bar{\partial}}^1(\Sigma, \mathcal{L}^{-1}(-2n)). \end{aligned}$$

The index theorem, in combining with the vanishing of $H_{\bar{\partial}}^0(\Sigma, \mathcal{L}^{-1}(-2n))$, implies that

$$(4.98) \quad \begin{aligned} \chi(\mathcal{L}(2n)) &= \dim_{\mathbb{C}} H_{\bar{\partial}}^0(\Sigma, \mathcal{L}(2n)) - \dim_{\mathbb{C}} H_{\bar{\partial}}^1(\Sigma, \mathcal{L}(2n)) = 2n + 1 - g, \\ \chi(\mathcal{L}^{-1}(-2n)) &= \dim_{\mathbb{C}} H_{\bar{\partial}}^1(\Sigma, \mathcal{L}^{-1}(-2n)) = 2n - 1 + g. \end{aligned}$$

Thus, from (4.98) we determine the exponent $\frac{1}{2}(d_1 - d_0)$ appearing in (4.94) to be

$$(4.99) \quad \frac{1}{2}(d_1 - d_0) = \frac{1}{2} \left[\chi(\mathcal{L}^{-1}(-2n)) - \chi(\mathcal{L}(2n)) \right] = g - 1.$$

To fix the ratio c_0/c_1 appearing in (4.94), which is determined by the determinant of $\psi/2\pi$ acting on E_0 and E_1 as in (4.88), we recall that $\mathcal{L}(2n)$ and $\mathcal{L}^{-1}(-2n)$ arise from the standard generators σ_{\pm} of the complex Lie algebra of $SU(2)$, as in (4.39). Since σ_z in (4.95) acts with eigenvalues $\pm 2i$ on σ_{\pm} , we see that $\psi \equiv \psi \cdot T_0$ acts on sections of $\mathcal{L}(2n)$ and $\mathcal{L}^{-1}(-2n)$ with eigenvalues $\pm i\sqrt{2}\psi$. Thus, in this case,

$$(4.100) \quad \begin{aligned} \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right) \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right)^{-1} &= \left(\frac{i\sqrt{2}\psi}{2\pi} \right)^{2n+1-g} \left(\frac{-i\sqrt{2}\psi}{2\pi} \right)^{-2n+1-g}, \\ &= \left(\frac{\psi^2}{2\pi^2} \right)^{1-g}. \end{aligned}$$

So

$$(4.101) \quad \left(\frac{c_0}{c_1} \right) = (2\pi^2)^{g-1}.$$

Finally, we must compute the symplectic volume $\text{Vol}(\mathcal{M}_n)$. This is equivalent to the moduli space of flat connections for the group $U(1)$, and appears with the same symplectic structure as if we were doing $U(1)$ gauge theory. The symplectic form is hence equivalent to $\Omega = \sum_{i=1}^g dx_i \wedge dy_i$, where our normalization is such that each of dx_i and dy_i have period $2\pi\sqrt{2}$ on the appropriate one-cycle. (This is the same

factor that appeared in (4.96).) Thus,

$$(4.102) \quad \text{Vol}(\mathcal{M}_n) = (8\pi^2)^g.$$

So from (4.96), (4.99), (4.101), and (4.102), we evaluate (4.94) as

$$(4.103) \quad \left. \frac{\partial^{g-1} Z(\epsilon)}{\partial \epsilon^{g-1}} \right|_{\mathcal{M}_n} = (-8\pi^4)^{g-1} \sqrt{\frac{4\pi}{\epsilon}} \exp\left(-\frac{(2\pi n)^2}{\epsilon}\right),$$

which agrees with (4.46).

5. Non-Abelian Localization For Chern-Simons Theory

We now discuss non-abelian localization for Chern-Simons theory on a Seifert manifold M . As we recall from Section 3, the Chern-Simons path integral then takes the symplectic form

$$(5.1) \quad Z(\epsilon) = \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{1}{2\pi i \epsilon}\right)^{\Delta_{\mathcal{G}}/2} \int_{\mathcal{A}} \exp\left[\Omega - \frac{1}{2i\epsilon}(\mu, \mu)\right].$$

Our general discussion in Section 4 implies that $Z(\epsilon)$ localizes on critical points of the action $S = \frac{1}{2}(\mu, \mu)$. Explicitly,

$$(5.2) \quad S = \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \int_M \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left[(\kappa \wedge F_A)^2 \right].$$

Our first task is thus to classify the critical points of S . We claim that, up to the action of the shift symmetry, the critical points of S correspond precisely to the flat connections on M . To prove this statement, we simply observe that the critical points of S satisfy the equation of motion

$$(5.3) \quad F_A - \left(\frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right) d\kappa - \kappa \wedge d_A \left(\frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right) = 0,$$

where the first term of (5.3) arises from the variation of the Chern-Simons functional and the last two terms arise from the variation of the last term in (5.2). To classify solutions of (5.3), we recall that S is invariant under the shift symmetry $\delta A = \sigma \kappa$, where σ is an arbitrary function on M taking values in the Lie algebra \mathfrak{g} of the gauge group G . Under the shift symmetry, the quantity $\kappa \wedge F_A$ transforms as

$$(5.4) \quad \kappa \wedge F_A \longrightarrow \kappa \wedge F_A + \sigma \kappa \wedge d\kappa.$$

Thus, since $\kappa \wedge d\kappa$ is everywhere non-zero on M , we can unambiguously fix a gauge for the shift symmetry by the condition

$$(5.5) \quad \kappa \wedge F_A = 0.$$

In this gauge, any solution of the equation of motion (5.3) is precisely a flat connection on M . So, as we certainly expect, the Chern-Simons path integral localizes around points of $\overline{\mathcal{A}}$ which represent flat connections on M .

It is interesting to contrast this situation to the case of Yang-Mills theory on a Riemann surface Σ . In that case, the path integral receives contributions from two qualitatively different kinds of critical points, for which the moment map $\mu = F_A$ satisfies either $\mu = 0$ or $\mu \neq 0$, and the critical point is respectively stable or unstable. Since the critical points of Chern-Simons theory are described by flat connections on M , one might naively suppose that these critical points are analogous to the stable critical points of Yang-Mills theory, which are also described by flat connections. However, let us recall our expression from Section 3 for the Chern-Simons moment map,

$$(5.6) \quad \langle \mu, (p, \phi, a) \rangle = -\frac{1}{2}p \int_M \kappa \wedge \text{Tr}(\mathcal{L}_R A \wedge A) + \int_M \kappa \wedge \text{Tr}(\phi F_A) - \int_M d\kappa \wedge \text{Tr}(\phi A) + a.$$

The last term of (5.6) is simply a constant piece of μ dual to the generator a of the central extension of the group \mathcal{G}_0 , and this generator acts trivially on $\overline{\mathcal{A}}$. As a result of this term, the Chern-Simons moment map is everywhere non-zero, and the critical points of Chern-Simons theory are actually of the same kind as the higher, unstable critical points of Yang-Mills theory.

Our goal in the rest of the paper is now to compute the local contributions to $Z(\epsilon)$ from two especially simple sorts of flat connections on M . First, we compute the contribution to $Z(\epsilon)$ from the trivial connection when M is a Seifert homology sphere. Second, we compute the contribution to $Z(\epsilon)$ from a smooth component in the moduli space of irreducible flat connections when M is a principal $U(1)$ -bundle over a Riemann surface. As we will see, these local computations in Chern-Simons theory are direct generalizations of the local computation at a higher critical point of two-dimensional Yang-Mills theory. The two cases we consider are the extreme cases in which the connection is either trivial or irreducible. Other cases are intermediate between these.

The Normalization of $Z(\epsilon)$

Before we perform any detailed computations, we must make a few general remarks about the normalization of $Z(\epsilon)$. As we see from (5.1),

we have normalized the Chern-Simons path integral with the formal prefactor

$$(5.7) \quad \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{1}{2\pi i \epsilon} \right)^{\Delta_{\mathcal{G}}/2}, \quad \Delta_{\mathcal{G}} = \dim \mathcal{G},$$

which is defined in terms of the group \mathcal{G} of gauge transformations.

On the other hand, as we discussed in Section 3, the Hamiltonian group which we use for localization in Chern-Simons theory is *not* \mathcal{G} but rather the group $\mathcal{H} = U(1) \times \tilde{\mathcal{G}}_0$, where $\tilde{\mathcal{G}}_0$ is a central extension by $U(1)$ of the identity component \mathcal{G}_0 of \mathcal{G} . We also introduce the group $\mathcal{H}' = U(1) \times \tilde{\mathcal{G}}$, which arises from the corresponding central extension $\tilde{\mathcal{G}}$ of the full group \mathcal{G} of all gauge transformations.

When we apply non-abelian localization to Chern-Simons theory, the path integral which we compute most directly is not given by (5.1) but by the canonically normalized symplectic integral

$$(5.8) \quad Z_0(\epsilon) = \frac{1}{\text{Vol}(\mathcal{H}')} \int_{\mathfrak{h} \times \bar{\mathcal{A}}} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{i\epsilon}{2} (\phi, \phi) \right],$$

as we computed abstractly in Section 4. The appearance of the volume of the disconnected group \mathcal{H}' in (5.8), as opposed to the connected group \mathcal{H} , accounts for the action of gauge transformations in the disconnected components of \mathcal{G} on critical points in $\bar{\mathcal{A}}$. Also, because the Chern-Simons path integral is oscillatory, an imaginary coupling $i\epsilon$ now appears in (5.8).

If we perform the Gaussian integral over ϕ in (5.8), then $Z_0(\epsilon)$ becomes

$$(5.9) \quad Z_0(\epsilon) = \frac{i}{\text{Vol}(\mathcal{H}')} \left(\frac{1}{2\pi i \epsilon} \right)^{\Delta_{\mathcal{H}}/2} \int_{\bar{\mathcal{A}}} \exp \left[\Omega - \frac{1}{2i\epsilon} (\mu, \mu) \right], \quad \Delta_{\mathcal{H}} = \dim \mathcal{H}.$$

In computing this integral over ϕ , we must be careful to remember that the quadratic form (\cdot, \cdot) on the Lie algebra \mathfrak{h} of \mathcal{H} is the direct sum of a positive-definite form on the Lie algebra of the gauge group \mathcal{G} and a hyperbolic form (with signature $(+, -)$) on the two additional generators in \mathcal{H} relative to \mathcal{G} . Had the form on \mathfrak{h} been positive-definite, the Gaussian integral over each generator in \mathfrak{h} would have contributed an identical factor $(2\pi i \epsilon)^{-\frac{1}{2}}$ to the prefactor in front of (5.9). However, due to the hyperbolic summand in (\cdot, \cdot) , the phases that result from the Gaussian integral over the two generators in the hyperbolic subspace

of \mathfrak{h} actually cancel. To account for this cancellation, we include the extra factor of ‘ i ’ appearing in (5.9).

Although $Z_0(\epsilon)$ in (5.9) takes the same form as the physical Chern-Simons path integral $Z(\epsilon)$ in (5.1), evidently the prefactor (5.7) which fixes the normalization of $Z(\epsilon)$ differs from the corresponding prefactor in $Z_0(\epsilon)$ by the ratio

$$(5.10) \quad \frac{\text{Vol}(\mathcal{H}')}{i \text{Vol}(\mathcal{G})} \cdot \left(\frac{1}{2\pi i \epsilon} \right)^{\frac{1}{2}(\Delta_{\mathcal{G}} - \Delta_{\mathcal{H}'})} = \text{Vol}(U(1)^2) \cdot 2\pi\epsilon.$$

The finite factors $\text{Vol}(U(1)^2)$ and $2\pi\epsilon$ arise in the obvious way from the two extra generators in \mathcal{H} relative to \mathcal{G} .

When we perform localization computations in Chern-Simons theory, we apply our abstract localization computations in Section 4 to compute $Z_0(\epsilon)$. By our observation above, for the purpose of computing the physical Chern-Simons path integral $Z(\epsilon)$, we must multiply the results from our abstract local computations by the finite factor in (5.10). As we will see, this expression turns out to cancel nicely against corresponding factors from the local computation.

5.1. A Two-Dimensional Interpretation of Chern-Simons Theory on M . Our symplectic interpretation of Chern-Simons theory on M fundamentally relies on the fact that the shift symmetry decouples one component of the gauge field A . As a result, we can essentially perform Kaluza-Klein reduction over the S^1 fiber of M to the base Σ to express Chern-Simons theory as a two-dimensional topological theory on Σ . From this two-dimensional perspective, we can immediately apply our localization computations in Section 4 to Chern-Simons theory.

In fact, the two-dimensional topological theory on Σ arising from Chern-Simons theory on M is closely related to Yang-Mills theory on Σ , a point also recently emphasized in [15]. At the level of the classical moduli spaces, the relationship between Chern-Simons theory on M and Yang-Mills theory on Σ was noted long ago by Furuta and Steer in [51]. These authors identify a correspondence between the moduli space of flat connections on M and certain components of the moduli space of Yang-Mills solutions on Σ . Since the relationship between flat connections on M and Yang-Mills solutions on Σ underlies our study of Chern-Simons theory, we now explain the fundamental aspects of this correspondence.

Flat Connections on M From Yang-Mills Solutions on Σ

We start by considering the moduli space of flat connections on M . As before, we suppose that the gauge group G is compact, connected, simply-connected, and simple.

A flat connection on M is determined by its holonomies, and the moduli space of flat connections on M , up to gauge equivalence, can be concretely described as the space of group homomorphisms from the fundamental group $\pi_1(M)$ to G , up to conjugacy. Hence the structure of the moduli space of flat connections on M is determined by $\pi_1(M)$.

On the other hand, because M is a Seifert manifold, and hence generally a $U(1)$ V -bundle over an orbifold Σ , the structure of $\pi_1(M)$ is closely tied to the structure of the orbifold fundamental group $\pi_1(\Sigma)$. This topological fact underlies the close relationship between flat connections on M and Yang-Mills solutions on Σ , and to explain it we now present the group $\pi_1(M)$.

As in Section 3, we describe M using the Seifert invariants

$$(5.11) \quad \left[g; n; (\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N) \right], \quad \gcd(\alpha_j, \beta_j) = 1.$$

We recall that g is the genus of Σ , n is the degree of the $U(1)$ V -bundle over Σ , and the relatively prime integers (α_j, β_j) for $j = 1, \dots, N$ specify the local geometry of M near the N orbifold points on Σ .

To present $\pi_1(M)$, we introduce elements

$$(5.12) \quad \begin{aligned} a_p, b_p, \quad p = 1, \dots, g, \\ c_j, \quad j = 1, \dots, N, \\ h. \end{aligned}$$

Then $\pi_1(M)$ is generated by these elements in (5.12) subject to the following relations,

$$(5.13) \quad \begin{aligned} [a_p, h] = [b_p, h] = [c_j, h] = 1, \\ c_j^{\alpha_j} h^{\beta_j} = 1, \\ \prod_{p=1}^g [a_p, b_p] \prod_{j=1}^N c_j = h^n. \end{aligned}$$

We will not give a formal proof of this presentation of $\pi_1(M)$, which follows from the standard surgery construction of M and which can be found in [50], but we will describe the geometric interpretation of the generators in (5.12). The generator h , which is a central element of $\pi_1(M)$ by the first line of (5.13), arises from the generic S^1 fiber over

Σ . Since Σ has genus g , the generators a_p and b_p for $p = 1, \dots, g$ arise from the $2g$ non-contractible cycles on Σ . Finally, the generators c_p for $p = 1, \dots, N$ arise from small one-cycles in Σ about each of the orbifold points. We note that from the presentation of $\pi_1(M)$ in (5.12) and (5.13) one can immediately compute the corresponding homology group $H_1(M, \mathbb{Z})$ as the abelianization of $\pi_1(M)$.

For example, with a view to our application below, let us determine the condition to have $H_1(M) = 0$. This requires $g = 0$ (or the homology of Σ will appear in $H_1(M)$). So $\pi_1(M)$ has generators c_j , $j = 1, \dots, N$, and $c_0 = h$. There are $N + 1$ relations, namely $c_j^{\alpha_j} c_0^{\beta_j} = 1$, $j = 1 \dots, N$, and $\prod_{j=1}^N c_j \cdot c_0^{-n} = 1$. So we can write the relations in the general form $\prod_{j=0}^N c_j^{K_{j,l}} = 1$ in terms of an $(N + 1) \times (N + 1)$ matrix K . A general element of $H_1(M)$ of the form $\prod_{j=0}^N c_j^{v_j}$ is trivial if and only if one can write $v_j = \sum_{j'} K_{jj'} w_{j'}$ for some integer-valued vector w . So $H_1(M)$ is trivial if and only if $\det(K) = \pm 1$. With the actual form of K , one can work out this determinant and find that the condition is that

$$(5.14) \quad n + \sum_{j=1}^N \frac{\beta_j}{\alpha_j} = \pm \prod_{j=1}^n \frac{1}{\alpha_j}.$$

The left hand side is also equal to the orbifold first Chern class $c_1(\mathcal{L})$ of the line V -bundle \mathcal{L} discussed in Section 3.2.

With the presentation of $\pi_1(M)$ in (5.12) and (5.13), we can immediately present $\pi_1(\Sigma)$ as well. Thus, $\pi_1(\Sigma)$ is generated by the elements a_p , b_p , and c_j in (5.12), omitting the generator h which arises from the S^1 fiber, and the relations in $\pi_1(\Sigma)$ are given by the relations in (5.13) upon setting $h = 1$. A very succinct description of this relation between $\pi_1(M)$ and $\pi_1(\Sigma)$ is to recognize $\pi_1(M)$ as a central extension of $\pi_1(\Sigma)$,

$$(5.15) \quad 1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1,$$

where h is the generator of \mathbb{Z} above.

Given the close relationship between the groups $\pi_1(M)$ and $\pi_1(\Sigma)$ expressed in (5.15), we can immediately deduce a relationship between flat connections on M and Yang-Mills solutions on Σ . To describe this relationship, we consider a homomorphism ρ ,

$$(5.16) \quad \rho : \pi_1(M) \longrightarrow G,$$

which describes the holonomies of a given flat connection on M .

Because h is central in $\pi_1(M)$, the image of ρ must lie in the centralizer $G_{\rho(h)}$ of the element $\rho(h)$ in G . To simplify the following discussion, we suppose that $\rho(h)$ actually lies in the center Γ of G , implying that $G_{\rho(h)} = G$. This condition is necessary whenever the connection described by ρ is irreducible, and it certainly holds also when the connection is trivial, which are the two main cases we consider when we perform computations in Chern-Simons theory. We refer to [51] for a discussion of the general case.

Clearly if $\rho(h) = 1$, so that the corresponding flat connection on M has trivial holonomy around the S^1 fiber over Σ , then ρ factors through the extension (5.15) to induce a homomorphism from $\pi_1(\Sigma)$ to G . Hence ρ describes a flat connection on M that pulls back from a flat Yang-Mills connection on Σ .

More generally, when $\rho(h)$ is non-trivial in Γ , then the corresponding flat connection on M has non-trivial holonomy around the S^1 fiber of M and is not the pull back of a flat G -connection on Σ . However, if we pass from G to the quotient group $\overline{G} = G/\Gamma$, so that we consider the connection on M as a flat connection on the trivial \overline{G} -bundle, then the holonomy of this connection around the S^1 fiber of M becomes trivial.

As a result, the homomorphism ρ can be interpreted as describing a flat connection on M which arises from the pull back of a flat Yang-Mills connection on a generally non-trivial V -bundle over Σ whose structure group is now \overline{G} , as opposed to G . In general, a flat connection on a non-trivial \overline{G} -bundle over Σ can be described as a flat connection on the trivial G -bundle over Σ such that the connection has non-trivial monodromies in Γ around the orbifold points as well as around one additional, arbitrarily chosen smooth point of Σ . These monodromies represent the obstruction to smoothly extending the given flat connection to the trivial G -bundle over all of Σ , and hence they describe the non-trivial \overline{G} -structure on the bundle.

In the case at hand, we see from the relations (5.13) which describe $\pi_1(M)$ as an extension of $\pi_1(\Sigma)$ that the relevant monodromies are determined by the holonomies of the connection on M associated to the elements h^{β_j} and h^n , so that these holonomies determine the topology of the corresponding \overline{G} -bundle on Σ . For instance, if we consider the simplest case that the gauge group G is $SU(2)$ and M arises from a principal $U(1)$ -bundle over a smooth Riemann surface Σ such that the degree n is odd, then flat connections on M whose holonomies satisfy $\rho(h) = \rho(h)^n = -1$ correspond bijectively to flat $SU(2)$ connections on

Σ which have monodromy -1 around a specified puncture. Such flat $SU(2)$ connections can then be identified with flat connections on the topologically non-trivial principal $SO(3)$ -bundle over Σ .

On the other hand, if the degree n of the principal $U(1)$ -bundle is even, then $\rho(h)^n = 1$ for both $\rho(h) = \pm 1$, so points in both of these components of the moduli space of flat connections on M are identified with flat $SU(2)$ connections on Σ .

The Local Symplectic Geometry Near a Critical Point of Chern-Simons Theory

The discussion above shows that irreducible flat connections on M can be identified with corresponding flat Yang-Mills connections on Σ . We now extend this observation to give a “two-dimensional” description of the local symplectic geometry in $\overline{\mathcal{A}}$ around such a critical point of Chern-Simons theory.

Because $\overline{\mathcal{A}}$ is the quotient of the affine space \mathcal{A} by the shift symmetry \mathcal{S} , we are free to work in any convenient gauge for \mathcal{S} . For instance, in order to identify the critical points of the new Chern-Simons action S in (5.2), we found it convenient to impose the gauge condition (5.5).

However, in order to describe the local geometry in $\overline{\mathcal{A}}$ in terms of geometric quantities on Σ , we make a new gauge choice for \mathcal{S} , corresponding to the gauge condition

$$(5.17) \quad \iota_R A = 0.$$

Because A transforms under the shift symmetry as $\delta A = \sigma \kappa$, the quantity $\iota_R A$ transforms as $\iota_R A \rightarrow \iota_R A + \sigma$, and the gauge condition in (5.17) is unambiguous.

To describe a critical point of the action S in the gauge (5.17), we consider as above a flat Yang-Mills connection B_0 on a generally non-trivial V -bundle with structure group \overline{G} over Σ . Then, in the gauge (5.17), the full tangent space to the symplectic manifold $\overline{\mathcal{A}}$ at B_0 is described by the space of sections ξ of the bundle $\Omega_M^1 \otimes \mathfrak{g}$ which satisfy the gauge condition

$$(5.18) \quad \iota_R \xi = 0.$$

Because our symplectic description of Chern-Simons theory respects the geometric $U(1)$ action on M , we naturally consider the decomposition of the tangent space to $\overline{\mathcal{A}}$ under the action of this $U(1)$. In terms of the section ξ , this statement simply means that we consider the Fourier

decomposition of ξ into eigenmodes of the operator \mathcal{L}_R . Thus we write

$$(5.19) \quad \xi = \sum_{t=-\infty}^{+\infty} \xi_t,$$

where, in addition to the gauge condition (5.18), each eigenmode ξ_t satisfies

$$(5.20) \quad \mathcal{L}_R \xi_t = -2\pi i t \cdot \xi_t.$$

We can similarly perform this Fourier decomposition on the tangent space to the group of gauge transformations \mathcal{G} . Thus, if ϕ is a section of $\Omega_M^0 \otimes \mathfrak{g}$, we write

$$(5.21) \quad \phi = \sum_{t=-\infty}^{+\infty} \phi_t,$$

where

$$(5.22) \quad \mathcal{L}_R \phi_t = -2\pi i t \cdot \phi_t.$$

To describe these eigenmodes ξ_t and ϕ_t geometrically on Σ , we recall that \mathcal{L} denotes the line V -bundle over Σ associated to the Seifert manifold M . Since non-trivial representations of the $U(1)$ action on M are associated to non-zero powers of \mathcal{L} on Σ , we can describe the modes ξ_t and ϕ_t geometrically on Σ as being respectively sections of the bundles $\Omega_\Sigma^1 \otimes \text{ad}(P) \otimes \mathcal{L}^t$ and $\Omega_\Sigma^0 \otimes \text{ad}(P) \otimes \mathcal{L}^t$. Here we have also replaced the trivial bundle \mathfrak{g} on M by the possibly nontrivial \overline{G} -bundle $\text{ad}(P)$ on Σ .

So, at least formally, the tangent space to $\overline{\mathcal{A}}$ at B_0 decomposes into the following sum of spaces of sections on Σ ,

$$(5.23) \quad T\overline{\mathcal{A}} = \bigoplus_{t=-\infty}^{+\infty} \Gamma(\Sigma, \Omega_\Sigma^1 \otimes \text{ad}(P) \otimes \mathcal{L}^t),$$

and similarly for the Lie algebra of \mathcal{G} ,

$$(5.24) \quad T\mathcal{G} = \bigoplus_{t=-\infty}^{+\infty} \Gamma(\Sigma, \Omega_\Sigma^0 \otimes \text{ad}(P) \otimes \mathcal{L}^t).$$

By assumption, the covariant derivative d_{B_0} commutes with the Lie derivative \mathcal{L}_R , $[d_{B_0}, \mathcal{L}_R] = 0$, so these decompositions are compatible with the action of d_{B_0} .

As in Section 4.2, the local structure of the space of fields over which we integrate near a given component \mathcal{M} of the moduli space of critical points is a fibration

$$(5.25) \quad F \longrightarrow N \xrightarrow{pr} \mathcal{M}.$$

As before, F is given by a symplectic bundle

$$(5.26) \quad F = \mathcal{H} \times_{H_0} (\mathfrak{h} \oplus \mathfrak{h}_0 \oplus \mathcal{E}_0 \oplus \mathcal{E}_1),$$

where the invariance group H_0 and the exceptional bundles \mathcal{E}_0 and \mathcal{E}_1 must be identified. As we observed at the start of this section, because the Chern-Simons moment map is non-vanishing, the local model is analogous to the geometry near a higher critical point of Yang-Mills theory, with some \mathcal{E}_0 and \mathcal{E}_1 .

In the model (5.26) for F , $\mathcal{H} = U(1) \times \tilde{\mathcal{G}}_0$ is the Hamiltonian group which we use for localization, and H_0 is the subgroup of \mathcal{H} which fixes B_0 . In general, H_0 is a finite-dimensional group of the form

$$(5.27) \quad H_0 = U(1)^2 \times K_0.$$

One $U(1)$ factor in H_0 arises from the action of \mathcal{L}_R on $\overline{\mathcal{A}}$, which fixes B_0 by assumption, and the other $U(1)$ factor arises from the central $U(1)$ in $\tilde{\mathcal{G}}_0$. This $U(1)$ acts trivially on all of $\overline{\mathcal{A}}$. Finally, K_0 denotes the group of gauge transformations acting on $\text{ad}(P)$ which fix B_0 . These gauge transformations are generated by covariantly constant sections ϕ of $\text{ad}(P) \otimes \mathcal{L}^0$, so that ϕ is annihilated by \mathcal{L}_R , and consequently K_0 commutes with both $U(1)$ factors in H_0 .

To identify \mathcal{E}_0 and \mathcal{E}_1 , we must look at the images of d_{B_0} and of $\star_2 d_{B_0}$ mapping $T\mathcal{G}$ to $T\overline{\mathcal{A}}$. The bundle $\text{ad}(P) \otimes \mathcal{L}^t$ has connection $C = B_0 + t\kappa$ (κ is the constant curvature connection on \mathcal{L} introduced in Section 3.2). For fixed t , the three-dimensional operators d_{B_0} and $\star_2 d_{B_0}$ reduce to two-dimensional operators d_C and $\star d_C$. As B_0 is flat, the connection C has curvature equal to t times a positive two-form. So the analysis of the intersection and unions of the images of d_C and $\star d_C$ precisely follows Section 4.3, with the following dictionary between quantities in the two-dimensional analysis of that section and quantities in the

present three-dimensional problem:

$$\begin{aligned}
 (5.28) \quad \text{ad}_0(P) &\longleftrightarrow \text{ad}(P) \\
 \text{ad}_+(P) &\longleftrightarrow \bigoplus_{t>0} \text{ad}(P) \otimes \mathcal{L}^t \\
 \text{ad}_-(P) &\longleftrightarrow \bigoplus_{t<0} \text{ad}(P) \otimes \mathcal{L}^t.
 \end{aligned}$$

In two dimensions, we decomposed $\text{ad}(P)$ into $\text{ad}_0(P)$, $\text{ad}_+(P)$, and $\text{ad}_-(P)$ according to the sign of the curvature. Here, curvature comes only from \mathcal{L} . So finally, we get

$$\begin{aligned}
 (5.29) \quad \mathcal{E}_0 &= \bigoplus_{t \neq 0} H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})), \\
 \mathcal{E}_1 &= \bigoplus_{t \neq 0} H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})).
 \end{aligned}$$

Unlike in the case of Yang-Mills theory, these exceptional bundles \mathcal{E}_0 and \mathcal{E}_1 now have infinite dimension, since the cohomology groups in (5.29) are non-zero for infinitely many t 's.

5.2. Localization at the Trivial Connection on a Seifert Homology Sphere. We are finally prepared to carry out a computation in Chern-Simons theory using non-abelian localization. We consider localization at the trivial connection when M is a Seifert manifold that also is a homology sphere, that is, it has $H_1 = 0$. We start by stating some necessary facts about the topology of M in this case.

Seifert Homology Spheres and a Slight Generalization

We recall that we generally characterize M with the Seifert invariants

$$(5.30) \quad \left[g; n; (\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N) \right], \quad \text{gcd}(\alpha_i, \beta_i) = 1.$$

As we have explained above, M is a homology sphere, with $H_1(M, \mathbb{Z}) = 0$, if and only if the invariants in (5.30) satisfy

$$(5.31) \quad g = 0, \quad c_1(\mathcal{L}_0) = n + \sum_{j=1}^N \frac{\beta_j}{\alpha_j} = \pm \prod_{j=1}^N \frac{1}{\alpha_j}.$$

Here \mathcal{L}_0 denotes the line V -bundle over the orbifold Σ which describes M .

To interpret geometrically the condition on \mathcal{L}_0 in (5.31), we note that this condition implies the arithmetic condition that the numbers α_j be pairwise relatively prime, so that

$$(5.32) \quad \gcd(\alpha_j, \alpha_{j'}) = 1, \quad j \neq j'.$$

In turn, as explained in Section 1 of [51], this arithmetic condition on the orders of the orbifold points of Σ implies that the Picard group of line V -bundles on Σ is isomorphic to \mathbb{Z} , just as for $\mathbb{C}\mathbb{P}^1$. In analogy to the case of S^3 , which arises from a generator of the Picard group of $\mathbb{C}\mathbb{P}^1$, the condition on $c_1(\mathcal{L}_0)$ in (5.31) is then precisely the condition that \mathcal{L}_0 generate the Picard group of Σ .

As previously, we orient M so that $c_1(\mathcal{L}_0)$ is positive, and we introduce the notation β_j^0 to distinguish the orbifold invariants of this fundamental line V -bundle \mathcal{L}_0 on Σ ,

$$(5.33) \quad c_1(\mathcal{L}_0) = n + \sum_{j=1}^N \frac{\beta_j^0}{\alpha_j} = \prod_{j=1}^N \frac{1}{\alpha_j}.$$

The reason that we distinguish the invariants β_j^0 of \mathcal{L}_0 is that, more generally, we will also consider the case that M arises not from the fundamental line V -bundle \mathcal{L}_0 on Σ but from some multiple \mathcal{L}_0^d for $d \geq 1$. In this case, we simply require that $g = 0$ in (5.31) and that the invariants α_j be relatively prime to each β_j and also pairwise relatively prime, as in (5.32). The Seifert manifold arising from \mathcal{L}_0^d is a quotient by the cyclic group \mathbb{Z}_d of the Seifert manifold associated to \mathcal{L}_0 , and in this case $H_1(M, \mathbb{Z}) = \mathbb{Z}_d$. So the integer d can be characterized topologically as the order of $H_1(M, \mathbb{Z})$,

$$(5.34) \quad d = |H_1(M, \mathbb{Z})|.$$

These Seifert manifolds are still rational homology spheres, with $H_1(M, \mathbb{R}) = 0$, and the trivial connection on M is an isolated flat connection.

We note that when the Seifert manifold M is described by a smooth, degree n line-bundle over $\mathbb{C}\mathbb{P}^1$, then M is a lens space, and the Seifert invariant n coincides with d in (5.34).

The Result of Lawrence and Rozansky

Our basic results on localization for Chern-Simons theory imply that the Chern-Simons partition function Z can be expressed as a sum of local contributions from the flat connections on M . In the case $G = SU(2)$ and with M as above, Lawrence and Rozansky [22] have already made this simple structure of Z explicit by working backwards from the previously known formula for Z . Our goal here is to compute directly one term in their formula, the local contribution from the trivial connection. However, because the general result in [22] is both very elegant and very suggestive, we now pause to present it.

To express⁸ Z as in [22], we find it useful to introduce the numerical quantities

$$(5.35) \quad \begin{aligned} \epsilon_r &= \frac{2\pi}{k+2}, \\ P &= \prod_{j=1}^N \alpha_j \quad \text{if } N \geq 1, \quad P = 1 \quad \text{otherwise,} \\ \theta_0 &= 3 - \frac{d}{P} + 12 \sum_{j=1}^N s(\beta_j, \alpha_j). \end{aligned}$$

Here ϵ_r is the renormalized coupling incorporating the famous shift $k \rightarrow k+2$ in the level in the case $G = SU(2)$, and $s(\beta, \alpha)$ is the Dedekind sum,

$$(5.36) \quad s(\beta, \alpha) = \frac{1}{4\alpha} \sum_{l=1}^{\alpha-1} \cot\left(\frac{\pi l}{\alpha}\right) \cot\left(\frac{\pi l \beta}{\alpha}\right).$$

For brevity, we also introduce the analytic functions

$$(5.37) \quad \begin{aligned} F(z) &= \left(2 \sinh\left(\frac{z}{2}\right)\right)^{2-N} \cdot \prod_{j=1}^N \left(2 \sinh\left(\frac{z}{2\alpha_j}\right)\right), \\ G^{(l)}(z) &= \frac{i}{4\epsilon_r} \left(\frac{d}{P}\right) z^2 - \frac{2\pi l}{\epsilon_r} z. \end{aligned}$$

⁸Our notation differs somewhat from [22], and we have normalized $Z(\epsilon)$ so that the partition function on $S^2 \times S^1$ is 1, whereas the authors of [22] normalize the partition function on S^3 to be 1.

Then, from the results of [22], the partition function $Z(\epsilon)$ of Chern-Simons theory on M can be written as

$$\begin{aligned}
 (5.38) \quad Z(\epsilon) = & (-1) \frac{\exp\left(\frac{3\pi i}{4} - \frac{i}{4}\theta_0\epsilon_r\right)}{4\sqrt{P}} \left\{ \sum_{l=0}^{d-1} \frac{1}{2\pi i} \int_{\mathcal{C}^{(l)}} dz F(z) \exp\left[G^{(l)}(z)\right] - \right. \\
 & - \sum_{m=1}^{2P-1} \text{Res} \left(\frac{F(z) \exp\left[G^{(0)}(z)\right]}{1 - \exp\left(-\frac{2\pi}{\epsilon_r} z\right)} \right) \Big|_{z=2\pi i m} - \\
 & \left. - \sum_{l=1}^{d-1} \sum_{m=1}^{\lfloor \frac{2Pl}{d} \rfloor} \text{Res} \left(F(z) \exp\left[G^{(l)}(z)\right] \right) \Big|_{z=-2\pi i m} \right\}.
 \end{aligned}$$

Here $\mathcal{C}^{(l)}$ for $l = 0, \dots, d - 1$ denote a set of contours in the complex plane over which we evaluate the integrals in the first line of (5.38). In particular, $\mathcal{C}^{(0)}$ is the diagonal line contour through the origin,

$$(5.39) \quad \mathcal{C}^{(0)} = e^{\frac{i\pi}{4}} \times \mathbb{R},$$

and the other contours $\mathcal{C}^{(l)}$ for $l > 0$ are diagonal line contours parallel to $\mathcal{C}^{(0)}$ running through the stationary phase point of the integrand, given by $z = -4\pi i l (P/d)$. Also, ‘‘Res’’ denotes the residue of the given analytic function evaluated at the given point.

We now wish to point out a few general features of this result (5.38) from the perspective of non-abelian localization.

First, the d contour integrals in the first term of (5.38) are identified in [22] with the local contributions from the d reducible flat connections on M . In particular, the integral arising from $l = 0$ above is the local contribution from the trivial connection, which takes the form

$$\begin{aligned}
 (5.40) \quad Z(\epsilon) \Big|_{\{0\}} = & (-1) \frac{\exp\left(\frac{3\pi i}{4} - \frac{i}{4}\theta_0\epsilon_r\right)}{4\sqrt{P}} \times \\
 & \times \frac{1}{2\pi i} \int_{\mathcal{C}^{(0)}} dz \exp\left[\frac{i}{4\epsilon_r} \left(\frac{d}{P}\right) z^2\right] \times \\
 & \times \left(2 \sinh\left(\frac{z}{2}\right)\right)^{2-N} \cdot \prod_{j=1}^N \left(2 \sinh\left(\frac{z}{2\alpha_j}\right)\right).
 \end{aligned}$$

For instance, one can directly check that, in the case $M = S^3$, the integral in (5.40) reduces to our much simpler expression for $Z(\epsilon)$ in (1.5).

Similarly, the integrals for $l > 0$ arise from reducible flat connections whose holonomies lie in a maximal torus of $SU(2)$, and hence these connections are fixed by a $U(1)$ subgroup of the gauge group. As we generally saw in Section 4 when we considered higher critical points of Yang-Mills theory, non-abelian localization at a reducible connection leads to an integral over the Lie algebra \mathfrak{h}_0 of the stabilizer group H_0 . This integral over \mathfrak{h}_0 is represented by the contour integrals above.

In contrast, the residues in the remaining terms of (5.38) are identified in [22] with the local contributions from the irreducible flat connections on M . As we show later, at least in the non-orbifold case $N = 0$ and $g > 0$, the local path integral contribution from a smooth component \mathcal{M} in the moduli space of irreducible flat connections on M is given by a computation in the cohomology ring of \mathcal{M} . In the context of two-dimensional Yang-Mills theory, cohomology computations on \mathcal{M} are often expressed in the form of residues, and we expect the residues in (5.38) to arise in this fashion.

Finally, the phase of $Z(\epsilon)$ in (5.38) is quite subtle. As explained in [60], this phase can be defined given the choice of a 2-framing on M , meaning a trivialization of $TM \oplus TM$, and for each three-manifold M a canonical choice of 2-framing exists. The partition function can thus be presented with a canonical phase, as originally computed in [25, 27] and as given in (5.38). The phase of $Z(\epsilon)$ which arises naturally when we define Chern-Simons theory via localization differs from this canonical phase, and we discuss this fact at the end of the section.

Localization at the Trivial Connection

We now compute using localization the contribution from the trivial connection to $Z(\epsilon)$ when M is a Seifert homology sphere. Although the results of Lawrence and Rozansky in (5.38) hold for gauge group $G = SU(2)$, Mariño has presented in [24] an expression for the contribution from the trivial connection for an arbitrary simply-laced gauge group G . With our methods, the generalization from $G = SU(2)$ to arbitrary simply-laced G is immediate, so we also consider the general case.

At the trivial connection, the moduli space \mathcal{M} is trivial, so the local geometry in $\bar{\mathcal{A}}$ is entirely described by the normal symplectic fiber F in (5.26), with the appropriate \mathfrak{h}_0 , E_0 , and E_1 . So we need only evaluate the canonical symplectic integral over F for this case.

We first observe that the stabilizer subgroup $H_0 \subset \mathcal{H}$ for the trivial connection is given as in (5.27) by

$$(5.41) \quad H_0 = U(1)^2 \times G,$$

where the factor G arises from the constant gauge transformations on M . Because H_0 decomposes as a product, we decompose an arbitrary element ψ of its Lie algebra $\mathfrak{h}_0 = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$ as

$$(5.42) \quad \psi = p + \phi + a,$$

where p and a generate the $U(1)$ factors of H_0 and ϕ is an element of \mathfrak{g} , according to the notation of Section 3.

As in (5.29), the exceptional bundles \mathcal{E}_0 and \mathcal{E}_1 at the trivial connection are now given by

$$(5.43) \quad \begin{aligned} \mathcal{E}_0 &= \bigoplus_{t \geq 1} H_{\bar{\partial}}^0(\Sigma, \mathfrak{g} \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})), \\ \mathcal{E}_1 &= \bigoplus_{t \geq 1} H_{\bar{\partial}}^1(\Sigma, \mathfrak{g} \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})). \end{aligned}$$

Here $\mathcal{L} = \mathcal{L}_0^d$ is the line V -bundle on Σ which describes M .

From our localization formula (4.86) in Section 4, the contribution of the trivial connection to $Z(\epsilon)$ is now given formally by the following integral over \mathfrak{h}_0 ,

$$(5.44) \quad Z(\epsilon) \Big|_{\{0\}} = \frac{(2\pi\epsilon)}{\text{Vol}(G)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi} \right] e(\psi) \exp \left[-i(\gamma_0, \psi) - \frac{i\epsilon}{2} (\psi, \psi) \right],$$

where $e(\psi)$ is an infinite-dimensional determinant,

$$(5.45) \quad e(\psi) = \det \left(\frac{\psi}{2\pi} \Big|_{\mathcal{E}_0} \right) \det \left(\frac{\psi}{2\pi} \Big|_{\mathcal{E}_1} \right)^{-1}.$$

In normalizing (5.44), we have cancelled the factor $\text{Vol}(U(1)^2)$ that appears in the relative normalization (5.10) against a corresponding factor in $1/\text{Vol}(H_0)$ from the localization formula (4.86), leaving the factor $1/\text{Vol}(G)$. We have also included the factor $(2\pi\epsilon)$ from (5.10).

Evaluating $e(\psi)$

We first evaluate $e(\psi)$, which turns out to be the only non-trivial piece of our computation. From (5.45), we see that $e(\psi)$ is described formally by the determinant of the operator ψ acting on the infinite-dimensional vector spaces \mathcal{E}_0 and \mathcal{E}_1 . So to evaluate $e(\psi)$, we will have to decide how to define such a determinant.

Here we employ the standard analytic technique of zeta/eta-function regularization to define the various infinite products that represent the determinant $e(\psi)$. This choice is somewhat *ad hoc*, and our best justification for it is the fact that it eventually leads to agreement with the results of Lawrence and Rozansky. However, this method of regularization does feature in the usual perturbative approach to Chern-Simons gauge theory, for instance in the one-loop computation in [1]. So, optimistically, one might be able to better justify the use of zeta/eta-function regularization here by comparing the localization computation with conventional perturbation theory. We make a few further remarks in Section 5.3.

Since the general element of \mathcal{H} acts on $\overline{\mathcal{A}}$ as

$$(5.46) \quad \delta A = d_A \phi + p \mathcal{L}_R A,$$

we see that the determinants in $e(\psi)$ can be written concretely in terms of p and ϕ in (5.42) as

$$(5.47) \quad e(\psi) = e(p, \phi) = \det \left[\frac{1}{2\pi} (p \mathcal{L}_R - [\phi, \cdot]) \Big|_{\mathcal{E}_0} \right] \det \left[\frac{1}{2\pi} (p \mathcal{L}_R - [\phi, \cdot]) \Big|_{\mathcal{E}_1} \right]^{-1}.$$

In particular, $e(p, \phi)$ does not depend on a in \mathfrak{h}_0 , since this generator acts trivially. This fact is important later.

As \mathcal{L}_R acts on sections of \mathcal{L}^t with eigenvalue $-2\pi i t$, we rewrite $e(p, \phi)$ as a product over the non-zero eigenvalues of \mathcal{L}_R as

$$(5.48) \quad e(p, \phi) = \prod_{t \neq 0} \det \left[\left(-itp - \frac{[\phi, \cdot]}{2\pi} \right) \Big|_{\mathfrak{g}} \right]^{\chi(\mathcal{L}^t)}.$$

Here $\chi(\mathcal{L}^t)$ is the Euler character of \mathcal{L}^t , so that we incorporate the cancellation between the action of ψ on elements of \mathcal{E}_0 and \mathcal{E}_1 , and the determinant in (5.48) indicates the determinant with respect to the action on \mathfrak{g} .

We now evaluate this finite-dimensional determinant on \mathfrak{g} . This determinant is invariant under the adjoint action on \mathfrak{g} , and without loss we assume that ϕ lies in the Lie algebra \mathfrak{t} of a maximal torus T of G . In this case, if β denotes a root of \mathfrak{g} and g_β the corresponding generator of \mathfrak{g} , then the adjoint action of ϕ on g_β is given by $[\phi, g_\beta] = i \langle \beta, \phi \rangle g_\beta$.

Thus diagonalizing the adjoint action of ϕ , we see that

$$\begin{aligned}
 (5.49) \quad \det \left(-itp - \frac{[\phi, \cdot]}{2\pi} \right) \Big|_{\mathfrak{g}} &= (-itp)^{\Delta_G} \prod_{\beta} \left(1 + \frac{\langle \beta, \phi \rangle}{2\pi tp} \right), \\
 &= (-itp)^{\Delta_G} \prod_{\beta > 0} \left(1 - \left(\frac{\langle \beta, \phi \rangle}{2\pi tp} \right)^2 \right).
 \end{aligned}$$

Here Δ_G denotes the dimension of G . In the first line of (5.49), the product runs over all the roots β of \mathfrak{g} , whereas in the second line of (5.49), we have grouped together the two terms arising from the roots $\pm\beta$ and rewritten the product over a set of positive roots.

Now from (5.48) and (5.49), we rewrite $e(p, \phi)$ as

$$(5.50) \quad e(p, \phi) = \exp \left(-\frac{i\pi}{2} \eta \right) \cdot \prod_{t \geq 1} \left| (tp)^{\Delta_G} \prod_{\beta > 0} \left(1 - \left(\frac{\langle \beta, \phi \rangle}{2\pi tp} \right)^2 \right) \right|^{\chi(\mathcal{L}^t) + \chi(\mathcal{L}^{-t})}.$$

Here $\exp \left(-\frac{i\pi}{2} \eta \right)$ represents the phase of $e(p, \phi)$, which involves an infinite product of factors $\pm i$, and the product written explicitly in (5.50) represents the norm. We first evaluate this norm, as the quantity η is much more delicate to determine.

To start, we evaluate the exponent that appears in (5.50). By the Riemann-Roch theorem in (3.23),

$$(5.51) \quad \chi(\mathcal{L}^t) + \chi(\mathcal{L}^{-t}) = \deg(\mathcal{L}^t) + \deg(\mathcal{L}^{-t}) + 2.$$

In general, the degree of a line V -bundle is *not* multiplicative, so that $\deg(\mathcal{L}^t) \neq t \deg(\mathcal{L})$, and the first two terms on the right of (5.51) do not necessarily cancel as they do for ordinary line bundles.

So we must work a little bit to simplify (5.51). As we now show, this exponent can be simplified as

$$(5.52) \quad \chi(\mathcal{L}^t) + \chi(\mathcal{L}^{-t}) = 2 - N + \sum_{j=1}^N \varphi_{\alpha_j}(t),$$

where $\varphi_{\alpha_j}(t)$ is an arithmetic function which takes the value 1 if α_j divides t and is 0 otherwise,

$$(5.53) \quad \begin{aligned}
 \varphi_{\alpha_j}(t) &= 1 && \text{if } \alpha_j \mid t, \\
 &= 0 && \text{otherwise.}
 \end{aligned}$$

To deduce (5.52), we suppose that the line V -bundle \mathcal{L}^t is characterized on Σ by isotropy invariants γ_j , where

$$(5.54) \quad \gamma_j \equiv t\beta_j \pmod{\alpha_j}, \quad 0 \leq \gamma_j < \alpha_j,$$

and, as before, the isotropy invariants β_j characterize the line V -bundle \mathcal{L} itself. From (5.14), the degree of \mathcal{L}^t is given in terms of the first Chern class, which is multiplicative, and γ_j as

$$(5.55) \quad \deg(\mathcal{L}^t) = t c_1(\mathcal{L}) - \sum_{j=1}^N \frac{\gamma_j}{\alpha_j}.$$

On the other hand, the isotropy invariants $\bar{\gamma}_j$ for the inverse line V -bundle \mathcal{L}^{-t} are given by

$$(5.56) \quad \bar{\gamma}_j \equiv -t\beta_j \pmod{\alpha_j}, \quad 0 \leq \bar{\gamma}_j < \alpha_j,$$

so that in terms of γ_j ,

$$(5.57) \quad \begin{aligned} \bar{\gamma}_j &= \alpha_j - \gamma_j && \text{if } \gamma_j \neq 0, \\ &= \gamma_j = 0 && \text{otherwise.} \end{aligned}$$

We note from (5.54) that γ_j vanishes whenever $t\beta_j \equiv 0 \pmod{\alpha_j}$. Because β_j is relatively prime to α_j by assumption, the vanishing of γ_j is then equivalent to the condition that α_j divide t , so that

$$(5.58) \quad \gamma_j = 0 \iff \alpha_j \mid t.$$

Thus, using the arithmetic function $\varphi_{\alpha_j}(t)$ defined in (5.53) in conjunction with (5.57) and (5.58), we see that the degree of \mathcal{L}^{-t} can be written as

$$(5.59) \quad \begin{aligned} \deg(\mathcal{L}^{-t}) &= -t c_1(\mathcal{L}) - \sum_{j=1}^N \frac{\bar{\gamma}_j}{\alpha_j}, \\ &= -t c_1(\mathcal{L}) - \sum_{j=1}^N \left(1 - \frac{\gamma_j}{\alpha_j} - \varphi_{\alpha_j}(t) \right). \end{aligned}$$

From (5.51), (5.55), and (5.59), we immediately deduce (5.52).

Consequently, $e(p, \phi)$ now becomes

$$\begin{aligned}
 (5.60) \quad e(p, \phi) &= \exp\left(-\frac{i\pi}{2}\eta\right) \times \\
 &\quad \times \prod_{t \geq 1} \left| (tp)^{\Delta_G} \prod_{\beta > 0} \left(1 - \left(\frac{\langle \beta, \phi \rangle}{2\pi tp}\right)^2\right) \right|^{2-N+\sum_{j=1}^N \varphi_{\alpha_j}(t)}, \\
 &= \exp\left(-\frac{i\pi}{2}\eta\right) \cdot f_0(p, \phi)^2 \cdot \prod_{j=1}^N \left| \frac{f_{\alpha_j}(p, \phi)}{f_0(p, \phi)} \right|,
 \end{aligned}$$

where

$$(5.61) \quad f_0(p, \phi) = \prod_{t \geq 1} \left[(tp)^{\Delta_G} \prod_{\beta > 0} \left(1 - \left(\frac{\langle \beta, \phi \rangle}{2\pi tp}\right)^2\right) \right],$$

and f_{α_j} is related to f_0 by

$$(5.62) \quad f_{\alpha_j}(p, \phi) = f_0(\alpha_j \cdot p, \phi).$$

In deducing (5.60) from (5.61) and (5.62), we apply the following arithmetic identity, which holds for an arbitrary function $f(t)$,

$$(5.63) \quad \prod_{t \geq 1} f(t)^{\varphi_{\alpha_j}(t)} = \prod_{t \geq 1} f(\alpha_j \cdot t).$$

We finally evaluate the infinite product which defines $f_0(p, \phi)$. We use the well known identity below,

$$(5.64) \quad \frac{\sin(x)}{x} = \prod_{t \geq 1} \left(1 - \frac{x^2}{\pi^2 t^2}\right),$$

and we use the Riemann zeta-function ζ to define trivial, but infinite, products

$$\begin{aligned}
 (5.65) \quad \prod_{t \geq 1} p^{\Delta_G} &= \exp(\Delta_G \ln p \cdot \zeta(0)) = p^{-\Delta_G/2}, \\
 \prod_{t \geq 1} t^{\Delta_G} &= \exp(-\Delta_G \cdot \zeta'(0)) = (2\pi)^{\Delta_G/2}.
 \end{aligned}$$

So from (5.64) and (5.65), we evaluate $f_0(p, \phi)$ to be

$$\begin{aligned}
 (5.66) \quad f_0(p, \phi) &= \left(\frac{p}{2\pi}\right)^{-\Delta_G/2} \prod_{\beta>0} \left[\frac{2p}{\langle\beta, \phi\rangle} \sin\left(\frac{\langle\beta, \phi\rangle}{2p}\right) \right], \\
 &= (2\pi)^{\Delta_G/2} p^{-\Delta_T/2} \prod_{\beta>0} \left[\frac{2}{\langle\beta, \phi\rangle} \sin\left(\frac{\langle\beta, \phi\rangle}{2p}\right) \right].
 \end{aligned}$$

Here Δ_T denotes the dimension of the maximal torus T of G (hence the rank of G), and in passing to the second line of (5.66) we just pull the factors of p outside the product over the positive roots of G .

From (5.60), (5.62), and (5.66), we finally evaluate $e(p, \phi)$ to be

$$\begin{aligned}
 (5.67) \quad e(p, \phi) &= \exp\left(-\frac{i\pi}{2}\eta\right) \cdot \frac{(2\pi)^{\Delta_G}}{(p\sqrt{P})^{\Delta_T}} \times \\
 &\quad \times \prod_{\beta>0} \langle\beta, \phi\rangle^{-2} \left| 2 \sin\left(\frac{\langle\beta, \phi\rangle}{2p}\right) \right|^{2-N} \prod_{j=1}^N \left| 2 \sin\left(\frac{\langle\beta, \phi\rangle}{2\alpha_j p}\right) \right|,
 \end{aligned}$$

where P is defined in (5.35) as the product of all the α_j .

Evaluating η and the Quantum Shift in the Chern-Simons Level

We now evaluate the phase factor $\exp\left(-\frac{i\pi}{2}\eta\right)$, from which we will find the famous quantum shift in the Chern-Simons level $k \rightarrow k + \check{c}_{\mathfrak{g}}$, where $\check{c}_{\mathfrak{g}}$ is the dual Coxeter number of \mathfrak{g} . For instance, we recall that in the case $G = SU(r + 1)$, $\check{c}_{\mathfrak{g}} = r + 1$.

To start, we consider the operator

$$(5.68) \quad \frac{i}{2\pi} (p\mathcal{L}_R - [\phi, \cdot]),$$

acting on the vector spaces \mathcal{E}_0 and \mathcal{E}_1 in (5.43). The spectrum of this operator is real, so at least formally, we see from the definition of $e(p, \phi)$ in (5.47) that the phase η is given by

$$(5.69) \quad \eta \approx \sum_{\lambda_{(0)} \neq 0} \text{sign}(\lambda_{(0)}) - \sum_{\lambda_{(1)} \neq 0} \text{sign}(\lambda_{(1)}),$$

where $\lambda_{(0)}$ and $\lambda_{(1)}$ range, respectively, over the eigenvalues of the operator in (5.68) acting on \mathcal{E}_0 and \mathcal{E}_1 .

We have not written (5.69) with an equality because the sums on the right of (5.69) are ill-defined without a regulator. To regulate these

sums, we follow the philosophy of [61] and introduce the eta-function

$$(5.70) \quad \eta_{(p,\phi)}(s) = \sum_{\lambda_{(0)} \neq 0} \text{sign}(\lambda_{(0)}) |\lambda_{(0)}|^{-s} - \sum_{\lambda_{(1)} \neq 0} \text{sign}(\lambda_{(1)}) |\lambda_{(1)}|^{-s}.$$

Here s is a complex variable. When the real part of s is sufficiently large, the sums in (5.70) are absolutely convergent so that $\eta_{(p,\phi)}(s)$ is defined in this case. Otherwise, $\eta_{(p,\phi)}(s)$ is defined by analytic continuation in the s -plane. Assuming that the limit $s \rightarrow 0$ exists, we then set

$$(5.71) \quad \eta = \eta_{(p,\phi)}(0).$$

Thus, η is basically the classic eta-invariant of [61] which is here associated to the operator in (5.68) acting on the virtual vector space $\mathcal{E}_0 \ominus \mathcal{E}_1$, where the “ \ominus ” simply indicates the relative sign in (5.70).

In our problem, because we explicitly know the spectrum of the operator in (5.68), we can directly evaluate $\eta_{(p,\phi)}(0)$ without too much work. One advantage of this direct approach is that it very concretely displays the origin of the finite shift in the Chern-Simons level k , a very subtle quantum effect to understand otherwise.

Ultimately this shift in k arises because, despite what might be one’s naive expectation from (5.69), η depends nontrivially on p and ϕ . To isolate this interesting functional dependence of $\eta_{(p,\phi)}(0)$ on p and ϕ , we observe that, for $s = 0$, the sum in (5.70) is invariant under an overall scaling of the eigenvalues $\lambda_{(0)}$ and $\lambda_{(1)}$, so that $\eta_{(p,\phi)}(0)$ is invariant under an overall scaling of the operator itself in (5.68). In particular, so long as $p > 0$ (as holds when we later set $p = 1/\epsilon$), we are free to rescale the operator in (5.68) by $1/p$ without changing η .

As a technical convenience, we thus introduce another eta-function $\eta'_{(p,\phi)}(s)$ which is defined as in (5.70) but is associated to the rescaled operator

$$(5.72) \quad \frac{i}{2\pi} \left(\mathcal{L}_R - \left[\frac{\phi}{p}, \cdot \right] \right).$$

Because $\eta = \eta_{(p,\phi)}(0) = \eta'_{(p,\phi)}(0)$, we see from (5.72) that η can only depend on p and ϕ in the combination ϕ/p .

We also introduce the eta-function $\eta_0(s)$ which is associated to the constant operator $i\mathcal{L}_R/2\pi$, and to isolate the functional dependence of η on p and ϕ we define

$$(5.73) \quad \delta\eta(p, \phi) = \eta'_{(p,\phi)}(0) - \eta_0(0).$$

As we now compute directly,

$$(5.74) \quad \delta\eta(p, \phi) = -\frac{\check{c}_g}{2(\pi p)^2} \left(\frac{d}{P} \right) \text{Tr}(\phi^2) \pmod{2}.$$

The role of the mod 2 terms is to remove the absolute value bars $|\cdot|$ that appear in (5.67), so that $e(p, \phi)$ depends analytically on p and ϕ as its definition suggests.

Of course, η itself is given by $\eta = \delta\eta(p, \phi) + \eta_0(0)$. We also discuss $\eta_0(0)$, though this constant is much less interesting than $\delta\eta(p, \phi)$.

A Warmup Computation on S^1

Before we directly evaluate $\delta\eta$, $\eta_0(0)$, and η for the case at hand, we find it useful to warm up with a simpler example, originally presented in [61, II]. Thus, we consider the eta-function $\eta_\nu(s)$ which is associated to the operator D_ν acting on functions on S^1 ,

$$(5.75) \quad D_\nu = \frac{i}{2\pi} \frac{d}{dx} + \nu.$$

Here ν is a real parameter in the interval $0 < \nu < 1$, and x is a coordinate on S^1 with period 2π . If we wish, we can equivalently consider D_ν as the covariant derivative acting on sections of a flat $U(1)$ bundle over S^1 whose connection has holonomy parametrized by ν .

Clearly the eigenvalues λ of D_ν are given by $\lambda = t + \nu$ as t runs over all integers. So we compute

$$(5.76) \quad \begin{aligned} \eta_\nu(s) &= \sum_{\lambda} \text{sign}(\lambda) |\lambda|^{-s}, \\ &= \sum_{t \geq 0} \frac{1}{(t + \nu)^s} - \sum_{t \geq 1} \frac{1}{(t - \nu)^s}, \\ &= \frac{1}{\nu^s} - \sum_{t \geq 1} \frac{2\nu s}{t^{s+1}} + \sum_{t \geq 1} s \cdot \mathcal{O}\left(\frac{1}{t^{s+2}}\right). \end{aligned}$$

In passing from the second to the third lines of (5.76), we apply the binomial expansion, and we collect into $\mathcal{O}(1/t^{s+2})$ the terms in this expansion for which the sum over t is absolutely convergent near $s = 0$. Thus, when we evaluate $\eta_\nu(s)$ at $s = 0$, the last term of (5.76) vanishes.

On the other hand, for the term involving the sum over $1/t^{s+1}$, we have

$$(5.77) \quad \sum_{t \geq 1} \frac{2\nu s}{t^{s+1}} = 2\nu s \zeta(1 + s).$$

Because $\zeta(1 + s)$ has a simple pole with residue 1 at $s = 0$, we see that (5.77) makes a non-zero contribution to $\eta_\nu(0)$, and

$$(5.78) \quad \eta_\nu(0) = 1 - 2\nu.$$

Physically the term involving ν arises as a finite renormalization effect, due to the divergence in the sum over eigenvalues in (5.77).

The Computation of η on M

Given the formal similarity of the operators in (5.72) and (5.75), we now evaluate $\eta_{(p,\phi)}(0)$ just as in our warmup computation on S^1 . In the case at hand, we must consider the eigenvalue multiplicities which are associated to the dimensions of the Dolbeault cohomology groups $H_{\bar{\partial}}^0(\Sigma, \mathcal{L}^t)$ and $H_{\bar{\partial}}^1(\Sigma, \mathcal{L}^t)$, and as in our earlier computation we must also consider the eigenvalues of the adjoint action of ϕ on \mathfrak{g} . Taking these considerations into account, we find the following compact expression for $\eta'_{(p,\phi)}(s)$,

$$(5.79) \quad \eta'_{(p,\phi)}(s) = \sum_{t=-\infty}^{+\infty} \sum_{\beta} \chi(\mathcal{L}^t) \operatorname{sign}(\lambda(t, \beta)) |\lambda(t, \beta)|^{-s},$$

$$\lambda(t, \beta) = t + \frac{\langle \beta, \phi \rangle}{2\pi p}.$$

Here the sum over β is again a sum over the roots of \mathfrak{g} , including the roots $\beta = 0$ from the Cartan subalgebra. We note that the appearance of the Euler character $\chi(\mathcal{L}^t)$ in (5.79) accounts both for the multiplicities and the relative signs of the eigenvalue contributions from \mathcal{E}_0 and \mathcal{E}_1 in (5.70).

We can give a similar, simpler expression for $\eta_0(s)$,

$$(5.80) \quad \eta_0(s) = \sum_{t \neq 0} \sum_{\beta} \chi(\mathcal{L}^t) \operatorname{sign}(t) |t|^{-s},$$

$$= \sum_{t \geq 1} \sum_{\beta} \frac{\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})}{t^s}.$$

In the general orbifold case, the index difference $\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})$ that arises in (5.80) appears to be a somewhat complicated arithmetic function of t , in contrast to our simple expression for the index sum in (5.52), and we will not evaluate $\eta_0(0)$ in complete generality here.

However, if we consider the special case of a degree d line-bundle \mathcal{L} over a smooth Riemann surface Σ , then the Riemann-Roch theorem

immediately implies that

$$(5.81) \quad \chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t}) = 2dt,$$

independent of the genus of Σ . So in this special case, we have from (5.80) that

$$(5.82) \quad \begin{aligned} \eta_0(s) &= \Delta_G \sum_{t \geq 1} \frac{2dt}{t^s}, \\ &= 2d\Delta_G \zeta(s-1). \end{aligned}$$

Thus,

$$(5.83) \quad \eta_0(0) = 2d\Delta_G \zeta(-1) = -\frac{d\Delta_G}{6}.$$

Having discussed $\eta_0(0)$, we now compute the more interesting quantity $\delta\eta(p, \phi)$ in (5.73). Upon expressing (5.79) as in (5.80) and collecting terms, we find that

$$(5.84) \quad \begin{aligned} \eta'_{(p, \phi)}(s) - \eta_0(s) &= \sum_{t \geq 0} \sum_{\beta > 0} (\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})) \cdot \left[\frac{1}{\left(t + \frac{\langle \beta, \phi \rangle}{2\pi p}\right)^s} - \frac{1}{t^s} \right] + \\ &+ \sum_{t \geq 1} \sum_{\beta > 0} (\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})) \cdot \left[\frac{1}{\left(t - \frac{\langle \beta, \phi \rangle}{2\pi p}\right)^s} - \frac{1}{t^s} \right]. \end{aligned}$$

In writing this expression, we assume without loss that the condition below holds for each positive root β ,

$$(5.85) \quad 0 < \frac{\langle \beta, \phi \rangle}{2\pi p} < 1.$$

Otherwise, when the quantity in (5.85) undergoes an integral shift, then the overall phase $\exp(-i\pi\eta/2)$ of $e(p, \phi)$ simply picks up a sign so as to effectively remove the absolute value bars $|\cdot|$ appearing in (5.67). Hence $e(p, \phi)$ depends analytically on p and ϕ .

We now observe from our general expressions (5.55) and (5.59) for $\deg(\mathcal{L}^t)$ and $\deg(\mathcal{L}^{-t})$ that the index difference in (5.84) depends generally on t as

$$(5.86) \quad \chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t}) = 2t \left(\frac{d}{P} \right) + \mathcal{O}(t^0).$$

We have used the fact that $c_1(\mathcal{L}) = d/P$, since $\mathcal{L} = \mathcal{L}_0^d$, and $c_1(\mathcal{L}_0) = \prod_j 1/\alpha_j = 1/P$.

If we now consider the binomial expansion of the denominators in (5.84), we see immediately that no contribution at $s = 0$ can arise from the terms of order t^0 in (5.86). The leading terms in the expansion which arise from these $\mathcal{O}(t^0)$ terms are proportional to $\pm \langle \beta, \phi \rangle / (2\pi p) \cdot t^{-(s+1)}$, and such terms linear in ϕ cancel between the two sums in (5.84). The same cancellation occurs between the leading expansion terms which arise from the term linear in t in (5.86), and fundamentally these cancellations reflect the fact that no invariant linear function of ϕ exists.

Thus, expanding the denominators in (5.84) to second order, we find

$$(5.87) \quad \eta'_{(p,\phi)}(s) - \eta_0(s) = 2 \left(\frac{d}{P} \right) \sum_{t \geq 1} \sum_{\beta > 0} \left(\frac{\langle \beta, \phi \rangle}{2\pi p} \right)^2 \cdot \frac{s(s+1)}{t^{s+1}} + \sum_{t \geq 1} \sum_{\beta > 0} s \cdot \mathcal{O} \left(\frac{1}{t^{s+2}} \right).$$

We evaluate (5.87) at $s = 0$ to determine $\delta\eta(p, \phi)$, which is thus given by

$$(5.88) \quad \delta\eta(p, \phi) = 2 \left(\frac{d}{P} \right) \sum_{\beta > 0} \left(\frac{\langle \beta, \phi \rangle}{2\pi p} \right)^2.$$

To simplify the sum over roots on the right side of (5.88), we note that this sum defines an invariant quadratic polynomial of ϕ and hence must be proportional to $\text{Tr}(\phi^2)$. When \mathfrak{g} is simply-laced, we have the following identity, as shown for instance in [62, VI],

$$(5.89) \quad \sum_{\beta > 0} \langle \beta, \phi \rangle^2 = -\check{c}_{\mathfrak{g}} \text{Tr}(\phi^2).$$

Together, (5.88) and (5.89) imply the main result in (5.74).

Thus the full determinant $e(p, \phi)$ is now given by

$$(5.90) \quad e(p, \phi) = \exp \left(-\frac{i\pi}{2} \eta_0(0) \right) \cdot \frac{(2\pi)^{\Delta_G}}{(p\sqrt{P})^{\Delta_T}} \exp \left[\frac{i\check{c}_{\mathfrak{g}}}{4\pi p^2} \left(\frac{d}{P} \right) \text{Tr}(\phi^2) \right] \times \prod_{\beta > 0} \langle \beta, \phi \rangle^{-2} \left[2 \sin \left(\frac{\langle \beta, \phi \rangle}{2p} \right) \right]^{2-N} \prod_{j=1}^N \left[2 \sin \left(\frac{\langle \beta, \phi \rangle}{2\alpha_j p} \right) \right].$$

As we will see directly, the exponential term involving $\text{Tr}(\phi^2)$ in $e(p, \phi)$ describes the quantum shift in the Chern-Simons level k .

Evaluating the Integral over \mathfrak{h}_0

We are finally left to consider the integral over \mathfrak{h}_0 in (5.44). We first observe that the norm (ψ, ψ) appearing in the exponent of the integrand there is given explicitly by

$$(5.91) \quad \begin{aligned} (\psi, \psi) &= - \int_M \kappa \wedge d\kappa \text{Tr}(\phi^2) - 2pa, \\ &= - \left(\frac{d}{P} \right) \text{Tr}(\phi^2) - 2pa. \end{aligned}$$

In passing to the second line of (5.91), we use the fact that ϕ is constant so that the integral over M simply evaluates to $c_1(\mathcal{L}) = d/P$. Second, we recall from Section 3 that the moment map at the trivial connection satisfies

$$(5.92) \quad \langle \mu, \psi \rangle = (\gamma_0, \psi) = a.$$

Hence the integral over \mathfrak{h}_0 takes the explicit form

$$(5.93) \quad \begin{aligned} Z(\epsilon) \Big|_{\{0\}} &= \frac{(2\pi\epsilon)}{\text{Vol}(G)} \int_{\mathfrak{h}_0} \left[\frac{dp}{2\pi} \right] \left[\frac{da}{2\pi} \right] \left[\frac{d\phi}{2\pi} \right] e(p, \phi) \times \\ &\quad \times \exp \left[-ia + i\epsilon pa + \frac{i\epsilon}{2} \left(\frac{d}{P} \right) \text{Tr}(\phi^2) \right]. \end{aligned}$$

We now evaluate the integral over a , which is easy since a only appears in the exponent of the integrand in (5.93). From a previous identity (4.33), this integral produces the delta function $2\pi \delta(1 - \epsilon p)$.

In turn, we use the delta function to perform the integral over p , setting $p = 1/\epsilon$. In the process, we cancel the explicit factor of $2\pi\epsilon$ which appears in the normalization of (5.93), and the integral over \mathfrak{h}_0 simplifies to an integral over \mathfrak{g} ,

$$(5.94) \quad Z(\epsilon) \Big|_{\{0\}} = \frac{1}{\text{Vol}(G)} \int_{\mathfrak{g}} \left[\frac{d\phi}{2\pi} \right] e(\epsilon^{-1}, \phi) \exp \left[\frac{i\epsilon}{2} \left(\frac{d}{P} \right) \text{Tr}(\phi^2) \right].$$

Because the integrand of (5.94) is invariant under the adjoint action on \mathfrak{g} , we can apply the classical Weyl integral formula to reduce the integral over \mathfrak{g} to an integral over the Cartan subalgebra \mathfrak{t} , in which form we make contact with the results in [22, 24]. In its infinitesimal

version, the Weyl integral formula states that, if f is a function on \mathfrak{g} invariant under the adjoint action, then

$$(5.95) \quad \int_{\mathfrak{g}} [d\phi] f(\phi) = \frac{1}{|W|} \frac{\text{Vol}(G)}{\text{Vol}(T)} \int_{\mathfrak{t}} [d\phi] \prod_{\beta>0} \langle \beta, \phi \rangle^2 f(\phi).$$

Here $|W|$ is the order of the Weyl group of G , and the product over the positive roots β of G appearing on the right of (5.95) is a Jacobian factor.

Applying (5.95) and recalling the form of E in (5.90), we rewrite (5.94) explicitly as

$$(5.96) \quad \begin{aligned} Z(\epsilon) \Big|_{\{0\}} &= e^{(-\frac{i\pi}{2} \eta_0(0))} \frac{1}{|W|} \frac{1}{\text{Vol}(T)} \left(\frac{\epsilon}{\sqrt{P}} \right)^{\Delta_T} \times \\ &\times \int_{\mathfrak{t}} [d\phi] \exp \left[\frac{i\epsilon}{2} \left(\frac{d}{P} \right) \left(1 + \frac{\epsilon \check{c}_{\mathfrak{g}}}{2\pi} \right) \text{Tr}(\phi^2) \right] \times \\ &\times \prod_{\beta>0} \left[2 \sin \left(\frac{\epsilon \langle \beta, \phi \rangle}{2} \right) \right]^{2-N} \prod_{j=1}^N \left[2 \sin \left(\frac{\epsilon \langle \beta, \phi \rangle}{2\alpha_j} \right) \right]. \end{aligned}$$

We finally make the change of variables $\phi \rightarrow \epsilon\phi$ to remove some of the extraneous factors of ϵ in front of (5.96), so that

$$(5.97) \quad \begin{aligned} Z(\epsilon) \Big|_{\{0\}} &= \exp \left(-\frac{i\pi}{2} \eta_0(0) \right) \frac{1}{|W|} \frac{1}{\text{Vol}(T)} \left(\frac{1}{\sqrt{P}} \right)^{\Delta_T} \times \\ &\times \int_{\mathfrak{t}} [d\phi] \exp \left[\frac{i}{2\epsilon_r} \left(\frac{d}{P} \right) \text{Tr}(\phi^2) \right] \times \\ &\times \prod_{\beta>0} \left[2 \sin \left(\frac{\langle \beta, \phi \rangle}{2} \right) \right]^{2-N} \prod_{j=1}^N \left[2 \sin \left(\frac{\langle \beta, \phi \rangle}{2\alpha_j} \right) \right]. \end{aligned}$$

Here we introduce the usual renormalized coupling ϵ_r ,

$$(5.98) \quad \epsilon_r = \frac{2\pi}{k + \check{c}_{\mathfrak{g}}},$$

to absorb the explicit shift in the coefficient of $\text{Tr}(\phi^2)$ that arises from the phase $\delta\eta$ and that appears in (5.96).

As it stands, the integral over \mathfrak{t} in (5.97) has oscillatory, as opposed to exponentially damped, behavior at infinity due to purely imaginary Gaussian factor involving $\text{Tr}(\phi^2)$. Such oscillatory Gaussian integrals typically arise in quantum field theory. For instance, we saw an earlier

example in our path integral manipulations at the end of Section 3.1, when we integrated out the auxiliary scalar field Φ that appeared there.

Exactly as in Section 3.1, the standard analytic prescription to define such an oscillatory integral is to shift the integration contour slightly off the real axis. That is, in the context of (5.97) we consider the complexification $\mathfrak{t} \otimes \mathbb{C}$ of the real Lie algebra \mathfrak{t} , and we define (5.97) by integrating over $\mathfrak{t} \times (1 - i\varepsilon)$ for a small real parameter ε . This $i\varepsilon$ prescription has the added virtue that the new contour avoids any poles of the integrand on the real axis that generally occur for $N > 2$.

Once we define (5.97) with the $i\varepsilon$ prescription, we are free to analytically continue the contour to lie along the diagonal $\mathfrak{t} \times e^{-i\pi/4}$, so that the Gaussian factor in (5.97) becomes purely real and negative-definite.⁹

To make contact with the result of Lawrence and Rozansky in (5.38), we finally make another change of variables $\phi \rightarrow i\phi$, so that

$$\begin{aligned}
 (5.99) \quad Z(\epsilon) \Big|_{\{0\}} &= \exp\left(-\frac{i\pi}{2} \eta_0(0)\right) \frac{1}{|W|} \frac{(-1)^{(\Delta_G - \Delta_T)/2}}{\text{Vol}(T)} \left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T} \times \\
 &\times \int_{\mathcal{C} \times \mathfrak{t}} [d\phi] \exp\left[-\frac{i}{2\epsilon_r} \left(\frac{d}{P}\right) \text{Tr}(\phi^2)\right] \times \\
 &\times \prod_{\beta > 0} \left[2 \sinh\left(\frac{\langle \beta, \phi \rangle}{2}\right)\right]^{2-N} \prod_{j=1}^N \left[2 \sinh\left(\frac{\langle \beta, \phi \rangle}{2\alpha_j}\right)\right],
 \end{aligned}$$

where \mathcal{C} is the diagonal contour $\mathbb{R} \times e^{\frac{i\pi}{4}}$, as in (5.39).

We immediately see that (5.99) has the same form as our earlier expression in (5.40) for the contribution from the trivial connection in the case $G = SU(2)$, and with a suitable choice of generator for \mathfrak{t} one can see that (5.99) agrees, up to the overall phase, with the result of Lawrence and Rozansky. For general G , our expression takes the same form as that found by Mariño in [24].

The Phase of $Z(\epsilon)$

We now discuss the phase of our result (5.99) for the contribution of the trivial connection to the Chern-Simons path integral. In the simplest case that M is described by a smooth line-bundle of degree $d = n$ over $\mathbb{C}\mathbb{P}^1$, we have computed this phase explicitly, as determined

⁹We recall that Tr is a negative-definite form.

by the constant

$$(5.100) \quad \eta_0(0) = -\frac{d\Delta_G}{6}.$$

Since we have not performed a careful analysis of the path integral phases that arise from the η invariant when M is an orbifold, we restrict attention to the smooth case in the following.

If we compare our result to the result (5.40) of Lawrence and Rozansky for gauge group $SU(2)$, we see that the overall phase of $Z(\epsilon)$ which arises naturally from localization does not agree with the canonical phase. To be more precise, the result of Mariño [24] in the case of a general gauge group G shows that the ratio $\exp(i\delta\Psi)$ between the canonical phase of $Z(\epsilon)$ and the phase we determine via (5.100) is given by

$$(5.101) \quad \begin{aligned} \exp(i\delta\Psi) &= \exp\left(\frac{i\pi\Delta_G}{4} - \frac{i\pi\Delta_G\check{c}_{\mathfrak{g}}}{12(k+\check{c}_{\mathfrak{g}})}\theta_0 + \frac{i\pi}{2}\eta_0(0)\right), \\ &= \exp\left(\frac{i\pi\Delta_G}{12}(3-d) - \frac{i\pi\Delta_G\check{c}_{\mathfrak{g}}}{12(k+\check{c}_{\mathfrak{g}})}\theta_0\right). \end{aligned}$$

Here k is the Chern-Simons level. The quantity θ_0 is defined in general in (5.35), and in the smooth case we see that θ_0 is given by

$$(5.102) \quad \theta_0 = 3 - d.$$

Hence the expression in (5.101) simplifies greatly to

$$(5.103) \quad \exp(i\delta\Psi) = \exp\left(\frac{i\pi k\Delta_G}{12(k+\check{c}_{\mathfrak{g}})}(3-d)\right).$$

As we now explain, the phase discrepancy in (5.103) is not really a discrepancy at all, and it merely reflects the fact that our path integral computation is effectively performed in a framing of M which differs from the canonical two-framing of Atiyah [60], which has been used by Lawrence and Rozansky. We first recall from [1] that the partition function of Chern-Simons theory generally transforms under a change in the framing of M by

$$(5.104) \quad Z \longrightarrow \exp\left(\frac{i\pi c}{12}s\right)Z, \quad c = \frac{k\Delta_G}{k+\check{c}_{\mathfrak{g}}}, \quad s \in \mathbb{Z}.$$

Here c arises as the central charge of the two-dimensional WZW model associated to the group G , and s is an integer that labels the shift in the frame. As a result, we see immediately from (5.104) that the phase

discrepancy (5.103) can be eliminated by a shift in $s = (3 - d)$ units from the canonical framing of M .

Of course, in evaluating the Chern-Simons path integral by localization, we did not explicitly specify any framing of M . Given the framing ambiguity (5.104) in Z , one might naturally wonder how we managed to obtain a definite answer for the phase of Z in the first place.

To answer this question, we observe generally that if M is an integral homology sphere, then the choice of a locally-free $U(1)$ action on M implies a canonical choice, up to homotopy, of a framing of M . Concretely, a framing of M amounts to the choice of three linearly independent, non-vanishing vector fields on M , and the $U(1)$ action on M immediately supplies us with one such vector field, the generating vector field R of $U(1)$. We decompose the tangent bundle to M as $TM = L \oplus W$, where L is a one-dimensional bundle generated by R and W is the complement. We are left to make a choice for the other two vector fields, which must span the rank two sub-bundle W of TM which lies in the kernel of the contact form κ . The choice of these two vector fields amounts to a trivialization of W , so if the Euler class of W is non-zero, W is non-trivial and our construction fails. However, since the Euler class of W lies in the cohomology group $H^2(M, \mathbb{Z})$, which vanishes for an integral homology sphere, W is automatically trivial in this case. Finally, because W has rank two, possible changes of trivialization of W are classified by homotopy classes of maps of M to $SO(2)$. But for a homology sphere M (or even a rational homology sphere), the space of maps to $SO(2)$ is connected.¹⁰ So, given the choice of the original $U(1)$ action, we produce a unique framing of M up to homotopy.

More generally, if M is not assumed to be a homology sphere, then W might be nontrivial. To define the Chern-Simons invariant of a three-manifold M , however, it is not quite necessary to have a framing of TM . It is enough to have a “two-framing,” a trivialization of $TM \oplus TM$. We claim that every Seifert fibration $\pi : M \rightarrow \Sigma$ determines a natural two-framing on M (which might depend on the choice of π , as a given M may admit more than one Seifert fibration). As $TM \oplus TM = L \oplus L \oplus W \oplus W$, it suffices to trivialize $W \oplus W$. First of all, $W \oplus W$ has a natural spin

¹⁰Let $w = du$ be an angular form on $SO(2) \cong S^1$ and let $v : M \rightarrow SO(2)$ be any map. As M is a rational homology sphere, $v^*(w)$ vanishes in de Rham cohomology, so $v^*(w) = df$ where $f : M \rightarrow \mathbb{R}$ is some real-valued function. Because \mathbb{R} is contractible, we can define a homotopy from f to a constant map from M to \mathbb{R} by simply setting $f_t = tf$, $0 \leq t \leq 1$. Now let $\pi : \mathbb{R} \rightarrow S^1 \cong \mathbb{R}/2\pi$ be the projection. Then setting $v_t = \pi \circ f_t$, we get the desired homotopy from v to a constant map from M to S^1 .

structure, the spin bundle being the sum of exterior powers of W . A trivial bundle, which is a product $M \times V$ for some fixed vector space V , also has a natural spin bundle, namely $M \times C(V)$, where $C(V)$ is a Clifford module for V , which is unique up to isomorphism. Any trivialization of W determines a spin structure, since a trivialization of W identifies it with a trivial bundle, which as we just noted has a natural spin structure. One condition we want to put on a trivialization of $W \oplus W$ is that the spin structure of $W \oplus W$ that it determines should coincide with the natural one. The second condition we want is that the trivialization of $W \oplus W$ should be invariant under the $U(1)$ action on the Seifert manifold. W is a pullback from some $SO(2)$ bundle W_0 over Σ , so $W \oplus W$ is the pullback of $U = W_0 \oplus W_0$. The rank four real bundle U has vanishing w_1 and w_2 (they are killed by taking two copies of W_0), so it is trivial. Compatibility with a given spin structure of a rank k real bundle U – in our application $k = 4$ – means that changes of trivialization really come from maps to $Spin(k)$ rather than $SO(k)$. As $\pi_i(Spin(k)) = 0$ for $i \leq 2$, $k \geq 3$, a trivial $SO(k)$ bundle U over Σ of rank $k \geq 3$ has up to homotopy only one trivialization compatible with a given spin structure. So finally the Seifert fibration $\pi : M \rightarrow \Sigma$ endows M with a natural two-framing (which may differ from its canonical two-framing [60], which is determined by a different construction).

In sum, then, a Seifert fibration of a homology sphere M determines a natural trivialization of the tangent bundle TM , which we will call the Seifert framing, and any Seifert fibration $\pi : M \rightarrow \Sigma$ (even if M is not a homology sphere) determines a natural trivialization of $TM \oplus TM$, which we will call the Seifert two-framing. If M is a Seifert homology sphere, the Seifert two-framing just arises by applying the Seifert framing to each copy of TM .

Now we consider in detail the illustrative example $M = S^3$. S^3 has no one natural framing. However, if we identify it with the Lie group $SU(2)$, then it has two equally natural framings, one which is left-invariant and one which is right-invariant. They are exchanged by an orientation-reversing reflection of S^3 , so neither one is preferred. In regarding S^3 as a Seifert fibration over $\mathbb{C}P^1$, we write $\mathbb{C}P^1 = S^3/U(1)$, where $U(1)$ is either part of the left action of $SU(2)$ on itself or part of the right action. For either choice of $U(1)$, our construction produces a framing that is canonically determined by the choice of $U(1)$ generator and so is invariant under any symmetry that commutes with $U(1)$. If

the $U(1)$ is part of the left $SU(2)$, then it commutes with the right $SU(2)$ and so we get the right-invariant framing; and likewise if the $U(1)$ is part of the right $SU(2)$, we get the left-invariant framing.

We naturally expect that the phase of Z in our computation of the Chern-Simons path integral is based on the Seifert framing. In view of our direct computation of the phase of Z , the Seifert two-framing of M must differ from the canonical two-framing of [60] by $s = (3 - d)$ units. We now give a simple proof of this fact in the case $M = S^3$ and $d = 1$ (though we will not be careful about the sign of the shift).

When $M = S^3$, the canonical two-framing of [60] can be described as follows. It is the trivialization of $TM \oplus TM$ that comes from the left-invariant framing on, say, the first copy of TM and the right-invariant framing on the second. (This is the unique reflection-invariant two-framing of S^3 , so it must be the canonical two-framing.) On the other hand, the Seifert framing of M is (for a suitable choice of fibration $\pi : S^3 \rightarrow \mathbb{C}P^1$) the left-invariant framing of TM , so the Seifert two-framing comes by applying the left-invariant framing to each of the two copies of TM . Hence the comparison between the Seifert two-framing and the canonical one is the same as the comparison between the left-invariant two-framing and the right-invariant two-framing for a single copy of TM .

The right-invariant framing of S^3 is determined by the basis of right-invariant one-forms $\theta = dg g^{-1}$, while the left-invariant framing is determined by the basis of left-invariant one-forms $\hat{\theta} = g^{-1}dg$. We are supposed to compare them by writing $\theta = T\hat{\theta}T^{-1}$, where T is a map from M to $SO(3)$. Such a map has a “degree,” an integer which measures by how many units the two framings differ. Clearly, in this case, $T = g$, so T is the “identity” map from $S^3 \cong SU(2)$ to itself. This map is of degree 1 as a map to $SU(2)$, but as a map from S^3 to $SO(3) = SU(2)/\mathbb{Z}_2$, it is of degree 2. This shows, as expected, that the Seifert two-framing of S^3 differs from the canonical two-framing by $3 - d = 2$ units.

The degrees are appropriately counted for maps to $SO(3)$, rather than $SU(2)$, because this is the structure group of the tangent bundle of M . To illustrate the role of $SO(3)$, let us consider one more simple example, which is $M = SO(3) = S^3/\mathbb{Z}_2$. This is the case $d = 2$ of the lens space considered above, so we expect the Seifert two-framing and the canonical two-framing to differ by $3 - d = 1$ unit. The comparison again reduces to comparing the right-invariant framing of TM with the left-invariant one. So again we have to compare $\theta = dg g^{-1}$ with $\hat{\theta} = g^{-1}dg$. We have

again $\theta = g\widehat{\theta}g^{-1}$, where now g is the identity map from $SO(3)$ to itself, which is of degree 1, showing that the two two-framings differ by one unit.

For any d , the general analysis of framings by Freed and Gompf in [25] can be used to check that the canonical two-framing and the Seifert two-framing on M differ by $s = (3 - d)$ units.

5.3. Localization on a Smooth Component of the Moduli Space of Irreducible Flat Connections. We now extend our work in the previous section to describe the local contribution to the Chern-Simons path integral from a smooth component \mathcal{M} of the moduli space of irreducible flat connections on a Seifert manifold M . We assume here for simplicity that M is described by a line bundle \mathcal{L} of degree n over a smooth Riemann surface Σ of genus $g \geq 1$. The orbifold case is also discussed by Rozansky in [23] but is somewhat more involved.

As we recall from Section 5.1, \mathcal{M} is literally the moduli space of flat connections on the trivial G -bundle over M such that the holonomy $\rho(h)$ around the S^1 fiber of M is a fixed element of the center Γ of G . This moduli space is not smooth for arbitrary $\rho(h)$ in Γ , but it is smooth in certain cases. The main such case, and the case we consider here, arises when the gauge group G is $SU(r + 1)$, $\rho(h)$ is a generator of $\Gamma = \mathbb{Z}_{r+1}$, and n and $r + 1$ are relatively prime. Under these conditions, $\rho(h)^n$ also generates Γ , and \mathcal{M} is smooth and can be identified with an unramified $(r + 1)^{2g}$ -fold cover of the moduli space \mathcal{M}_0 of flat Yang-Mills connections on an associated principal bundle P over Σ with structure group $\overline{G} = G/\Gamma$. (\overline{G} enters because when we project to \overline{G} , $\rho(h)$ projects to 1 and the representation ρ becomes a pullback from Σ . But as the three-dimensional gauge group is really G , the holonomies of ρ around one-cycles in Σ are defined as elements of G , not \overline{G} ; this leads to the unramified cover.)

Our general discussion of non-abelian localization in Section 4 implies that the path integral contribution from \mathcal{M} can be expressed entirely in terms of the cohomology ring of \mathcal{M} , or equivalently \mathcal{M}_0 . One of the reasons that localization on \mathcal{M} is interesting is that we find in Chern-Simons theory a natural generalization of the cohomological formula (4.18) for the path integral contribution from \mathcal{M}_0 in two-dimensional Yang-Mills theory.

We recall from our discussion in Section 5.1 that a local symplectic neighborhood N near \mathcal{M} in $\overline{\mathcal{A}}$ is described by an equivariant bundle

$$(5.105) \quad F \longrightarrow N \xrightarrow{pr} \mathcal{M},$$

where the normal fiber F takes the (by now familiar) form

$$(5.106) \quad F = \mathcal{H} \times_{H_0} (\mathfrak{h} \ominus \mathfrak{h}_0 \ominus \mathcal{E}_0 \oplus \mathcal{E}_1).$$

By assumption, the only gauge transformations which fix the irreducible flat connections associated to points in \mathcal{M} are constant gauge transformations by elements in the center Γ of G , since the center of G always acts trivially in the adjoint representation. So the stabilizer subgroup H_0 in \mathcal{H} is now given by

$$(5.107) \quad H_0 = U(1)^2 \times \Gamma,$$

where we recall that the torus $U(1)^2$ arises from the two extra generators in \mathcal{H} relative to \mathcal{G} .

Also, we recall that the vector spaces \mathcal{E}_0 and \mathcal{E}_1 are now given over a point of \mathcal{M} by

$$(5.108) \quad \begin{aligned} \mathcal{E}_0 &= \bigoplus_{t \neq 0} H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})), \\ \mathcal{E}_1 &= \bigoplus_{t \neq 0} H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})). \end{aligned}$$

The Canonical Symplectic Integral Over N

Having described the local geometry near \mathcal{M} in $\bar{\mathcal{A}}$, we next consider the canonical symplectic integral over N . This integral takes the form

$$(5.109) \quad Z(\epsilon) \Big|_{\mathcal{M}} = \frac{2\pi\epsilon \cdot \text{Vol}(U(1)^2)}{\text{Vol}(\mathcal{H})} \times \int_{\mathfrak{h} \times N} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{i\epsilon}{2} (\phi, \phi) + tD\lambda \right],$$

where we include in the normalization of (5.109) the prefactor from (5.10). To define the integral over the non-compact directions in N , we also include in (5.109) the localization form $tD\lambda$.

Our goal now is to reduce the integral over $\mathfrak{h} \times N$ in (5.109) to an integral over the moduli space \mathcal{M} itself. We have already discussed a problem of this sort in Section 4.2, when we considered the path integral contribution from irreducible flat connections in two-dimensional Yang-Mills theory. As we briefly recall, in the case of Yang-Mills theory the fiber F in (5.105) is modelled on the cotangent bundle T^*H (with H being the group of gauge transformations in that case), so that N

retracts equivariantly onto a principal H -bundle P_H over the the moduli space \mathcal{M}_0 . Because H acts freely on P_H , the H -equivariant cohomology of the total space P_H can be identified with the ordinary cohomology of the quotient $P_H/H = \mathcal{M}_0$, so $H_H^*(P_H) \cong H^*(\mathcal{M}_0)$. In particular, the H -equivariant cohomology classes of $[\Omega - i \langle \mu, \phi \rangle]$ and $[-\frac{1}{2}(\phi, \phi)]$ on P_H pull back from ordinary cohomology classes Ω and Θ of degrees two and four on \mathcal{M}_0 , and we apply this fundamental fact to reduce the symplectic integral in Yang-Mills theory to an integral over \mathcal{M}_0 .

In the case of Chern-Simons theory, the group $H \equiv \mathcal{H}$ no longer acts freely on N , but we can still apply much the same logic as for the case of Yang-Mills theory. Here a subgroup H_0 of H acts with fixed points on N , so N equivariantly retracts onto a bundle with fiber H/H_0 over \mathcal{M} . We denote the total space of this bundle by N_0 , so that $H/H_0 \rightarrow N_0 \rightarrow \mathcal{M}$.

Because N_0 is an equivariant retraction from N , the H -equivariant cohomology ring of N is the same as that of N_0 . As we explain in Appendix C, the formal properties of equivariant cohomology further imply that the H -equivariant cohomology ring of N_0 is identified under pullback with the H_0 -equivariant cohomology ring of \mathcal{M} itself. So in total, we have the relation $H_H^*(N) \cong H_{H_0}^*(\mathcal{M})$.

As a result, in precise analogy to the case of two-dimensional Yang-Mills theory, the H -equivariant cohomology classes of $[\Omega - i \langle \mu, \phi \rangle]$ and $[-\frac{1}{2}(\phi, \phi)]$ which appear in the symplectic integral over N can be identified as the pullbacks from \mathcal{M} of elements in the ring $H_{H_0}^*(\mathcal{M})$.

To identify the elements of $H_{H_0}^*(\mathcal{M})$ which pull back to these classes appearing in the symplectic integral over N , we note that $H_{H_0}^*(\mathcal{M})$ has a very simple structure. As we also explain in Appendix C, because H_0 acts trivially on \mathcal{M} , $H_{H_0}^*(\mathcal{M})$ is given by the tensor product of the ordinary cohomology ring $H^*(\mathcal{M})$ of \mathcal{M} with the H_0 -equivariant cohomology ring $H_{H_0}^*(pt)$ of a point. Thus, $H_{H_0}^*(\mathcal{M}) = H^*(\mathcal{M}) \otimes H_{H_0}^*(pt)$.

Finally, our previous discussion of the Cartan model of equivariant cohomology explicitly identifies the H_0 -equivariant cohomology ring of a point with the ring of invariant functions on the Lie algebra \mathfrak{h}_0 . Thus, all elements of $H_{H_0}^*(\mathcal{M})$ can be written as sums of terms having the form $x \cdot f(\psi)$, where x is an ordinary cohomology class on \mathcal{M} and $f(\psi)$ is an invariant function of ψ in \mathfrak{h}_0 .

With our concrete description of $H_{H_0}^*(\mathcal{M})$, we can immediately identify the elements of this ring which pull back to the H -equivariant classes $[\Omega - i \langle \mu, \phi \rangle]$ and $[-\frac{1}{2}(\phi, \phi)]$ on N . Let us decompose the Lie algebra \mathfrak{h}

of \mathcal{H} as a sum $\mathfrak{h} = (\mathfrak{h} \ominus \mathfrak{h}_0) \oplus \mathfrak{h}_0$. As a result, we write $\phi = \varphi + p + a$, where φ is an element of $\mathfrak{h} \ominus \mathfrak{h}_0$, which can be identified as the Lie algebra of \mathcal{G} , and, in the same notation from Section 3.4, p and a are elements of the Lie algebra \mathfrak{h}_0 of H_0 .

We then identify the H -equivariant classes on N appearing in (5.109) with corresponding H_0 -equivariant classes on \mathcal{M} via

$$(5.110) \quad \begin{aligned} \Omega - i \langle \mu, \phi \rangle &\longleftrightarrow \Omega - i a, \\ -\frac{1}{2}(\phi, \phi) &\longleftrightarrow n \Theta + p a. \end{aligned}$$

We abuse notation slightly in the first line of (5.110). On the left, Ω is the symplectic form on $\overline{\mathcal{A}}$ restricted to N , and on the right Ω is the induced symplectic form on \mathcal{M} (or equivalently \mathcal{M}_0), exactly as in our discussion of two-dimensional Yang-Mills theory. In identifying the dependence of this degree two class in $H_{H_0}^*(\mathcal{M})$ on p and a , we use the fact, evident from the formula for μ in (3.50), that the value of the moment map $\langle \mu, \phi \rangle$ evaluated at a flat connection which pulls back from Σ is just the constant a appearing on the right of the first line in (5.110).

Similarly, in the second line of (5.110), the degree four class Θ on \mathcal{M} is the same degree four class that appeared in our discussion of Yang-Mills theory. The identification in (5.110) arises by writing the degree four invariant $-\frac{1}{2}(\phi, \phi)$ in terms of φ , p , and a as

$$(5.111) \quad -\frac{1}{2}(\phi, \phi) = \frac{1}{2} \int_M \kappa \wedge d\kappa \operatorname{Tr}(\varphi^2) + p a = \frac{n}{2} \int_\Sigma \omega \operatorname{Tr}(\varphi^2) + p a,$$

where we recall that n is the degree of the line-bundle \mathcal{L} over Σ which defines M and ω is a unit-volume symplectic form on Σ . As in the case of two-dimensional Yang-Mills theory, the term quadratic in the generators φ of the gauge symmetry is associated by the Chern-Weil homomorphism to the degree four class Θ .

With the identifications in (5.110), we can rewrite the symplectic integral over N as

$$(5.112) \quad \begin{aligned} Z(\epsilon) \Big|_{\mathcal{M}} &= \frac{2\pi\epsilon \cdot \operatorname{Vol}(U(1)^2)}{\operatorname{Vol}(\mathcal{H})} \times \\ &\times \int_{\mathfrak{h} \times N} \left[\frac{d\phi}{2\pi} \right] \exp [(pr^*\Omega) - ia(1 - \epsilon p) + i\epsilon n(pr^*\Theta) + tD\lambda]. \end{aligned}$$

As in the case of localization at the trivial connection, the generator a acts trivially on all of N and so does not appear in the localization form $tD\lambda$. So we can perform the integrals over a and p exactly as before,

and the integral over a produces a delta-function that sets $p = 1/\epsilon$. As a result, the symplectic integral reduces to the form

$$(5.113) \quad Z(\epsilon)\Big|_{\mathcal{M}} = \frac{\text{Vol}(U(1)^2)}{\text{Vol}(\mathcal{H})} \times \int_{(\mathfrak{h} \ominus \mathfrak{h}_0) \times N} \left[\frac{d\phi}{2\pi} \right] \exp \left[(pr^*\Omega) + i\epsilon n (pr^*\Theta) + tD\lambda \Big|_{p=1/\epsilon} \right].$$

The only term in (5.113) which does not pull back from \mathcal{M} is the localization term $tD\lambda$, so we are left to integrate $tD\lambda$ over the fiber F of N . In the case of two-dimensional Yang-Mills theory, with $F = T^*H$, this integral gave a trivial factor of unity. In Chern-Simons theory, the result is much more interesting.

An Equivariant Euler Class From F

To evaluate (5.113), we consider the following integral,
 (5.114)

$$I(\psi) = \frac{1}{\text{Vol}(\mathcal{H})} \int_{\tilde{F}} \left[\frac{d\phi}{2\pi} \right] \exp [tD\lambda], \quad \tilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F, \quad \psi \in \mathfrak{h}_0.$$

Here we let $\psi = p + a$ be an arbitrary element of \mathfrak{h}_0 , though in general the generator a will not appear in (5.114) since a acts trivially on N , and we set $p = 1/\epsilon$ at the end of the discussion, as in (5.113).

Of course, in Section 4.3 we computed this integral over the abstract model for F . There we assumed \mathcal{M} to be a point, and we found the result

$$(5.115) \quad I(\psi) = \frac{1}{\text{Vol}(H_0)} \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right) \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right)^{-1}, \quad \psi \in \mathfrak{h}_0.$$

Unfortunately, we cannot apply this result directly to the case at hand. When F is fibered over a non-trivial moduli space \mathcal{M} , then $I(\psi)$ will generally involve cohomology classes on \mathcal{M} which are associated to the twisting of the bundle and which our previous computation did not detect.

To compute $I(\psi)$ in (5.114), one approach is simply to generalize the abstract localization computation in Section 4.3 to allow for a non-trivial moduli space \mathcal{M} . We perform this computation in Appendix D. However, we can also make an immediate guess, on the basis of mathematical naturality, for what the generalization of the formula (5.115) must be when \mathcal{M} is non-trivial. This guess relies on a more intrinsic topological interpretation of the result (5.115) even in the case that \mathcal{M} is a point. For this reason, it turns out to be much more illuminating

to “guess” the generalization of (5.115) rather than simply to compute, so we pursue this approach now.

Let us think about what our result for $I(\psi)$ really *means* in the case that $\mathcal{M} = pt$. Abstractly, the data which enter the formula (5.115) are the group H_0 , which acts trivially on \mathcal{M} , and the finite-dimensional unitary representations E_0 and E_1 of H_0 . In general, to say that E is a representation of H_0 is the same thing as to say that E is an H_0 -equivariant bundle over a point, so if we like, we can consider E_0 and E_1 as H_0 -equivariant bundles over $\mathcal{M} = pt$.

This language is useful, since whenever we have a vector bundle (even a vector bundle over a point!) an extremely natural set of topological invariants to consider are the characteristic classes of the bundle. In our context, we naturally consider the H_0 -equivariant characteristic classes¹¹ of E_0 and E_1 as H_0 -equivariant bundles over $\mathcal{M} = pt$. These characteristic classes are valued in the H_0 -equivariant cohomology ring of \mathcal{M} — since \mathcal{M} is a point, this ring is the ring of invariant functions on the Lie algebra \mathfrak{h}_0 of H_0 .

If E is a unitary representation of H_0 and we consider E as an H_0 -equivariant bundle over a point, then the H_0 -equivariant characteristic classes of E have a simple description. We let $U(E)$ be the unitary group acting on E . Since H_0 acts in a unitary fashion on E , the relevant characteristic classes of E to consider are the equivariant Chern classes. As is well known, the ordinary Chern classes of a vector bundle are associated via the Chern-Weil homomorphism to the generators c_i of the ring of invariant polynomials on the Lie algebra of the unitary group. To describe the corresponding H_0 -equivariant Chern classes of E , we observe that, since E is a unitary representation of H_0 , we have an induced map $H_0 \rightarrow U(E)$. Consequently, any invariant polynomial on the Lie algebra of $U(E)$ pulls back to an invariant polynomial on the Lie algebra \mathfrak{h}_0 of H_0 . The pullbacks of the generators c_i to invariant polynomials on \mathfrak{h}_0 are then the H_0 -equivariant Chern classes of E . In particular, if the action of H_0 on E is non-trivial, then the equivariant Chern classes of E can also be non-trivial, despite the fact that E is a bundle over only a point.

The invariant polynomials appearing in $I(\psi)$, namely

$$(5.116) \quad e_{H_0}(pt, E_0) \equiv \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right), \quad e_{H_0}(pt, E_1) \equiv \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right),$$

¹¹Although we will not require the generalization here, we refer the reader to Chapter 8.5 of [56] for a general discussion of equivariant characteristic classes.

arise from determinants. The Chern-Weil homomorphism associates the determinant to the top Chern class, so by our discussion above the invariant polynomials in (5.116) can be characterized intrinsically as the H_0 -equivariant top Chern classes, or equivalently Euler classes, of E_0 and E_1 as equivariant bundles over a point. Thus, when \mathcal{M} is a point, we write $I(\psi)$ in (5.115) intrinsically as

$$(5.117) \quad I(\psi) = \frac{1}{\text{Vol}(H_0)} \frac{e_{H_0}(pt, E_0)}{e_{H_0}(pt, E_1)}.$$

More generally, if E is an H_0 -equivariant vector bundle over a complex manifold \mathcal{M} , then we can still consider the H_0 -equivariant Euler class $e_{H_0}(\mathcal{M}, E)$ of E , which takes values in the H_0 -equivariant cohomology ring of \mathcal{M} . If H_0 acts trivially on \mathcal{M} (but not necessarily trivially on E), we have already identified this cohomology ring as a product $H_{H_0}^*(\mathcal{M}) \cong H^*(\mathcal{M}) \otimes H_{H_0}^*(pt)$. We describe $e_{H_0}(\mathcal{M}, E)$ in this case explicitly below.

In our application to Chern-Simons theory, the infinite-dimensional vector spaces \mathcal{E}_0 and \mathcal{E}_1 in (5.108) determine associated H_0 -equivariant bundles over the moduli space \mathcal{M} , on which H_0 in (5.107) acts trivially. Given our intrinsic interpretation of $I(\psi)$ when \mathcal{M} is a point, we certainly expect that the integral over F in (5.114) produces the natural generalization of (5.117), involving the H_0 -equivariant Euler classes of the bundles associated to \mathcal{E}_0 and \mathcal{E}_1 over \mathcal{M} . That is,

$$(5.118) \quad I(\psi) = \frac{1}{\text{Vol}(H_0)} \frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0)}{e_{H_0}(\mathcal{M}, \mathcal{E}_1)}.$$

As our direct computation in Appendix D shows, this formula is correct.

We remark that the appearance of the equivariant Euler class of the bundle \mathcal{E}_1 in the denominator of (5.118) is quite standard. This class appears in precisely the same way in the classic Duistermaat-Heckman formula [37] for abelian localization, as was explained in [36]. The essentially new feature of the formula (5.118) is the appearance of a corresponding Euler class from \mathcal{E}_0 in the numerator.

We set

$$(5.119) \quad e(\psi) = \frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0)}{e_{H_0}(\mathcal{M}, \mathcal{E}_1)}.$$

Then from (5.113), (5.114), and (5.118), the local contribution from \mathcal{M} in Chern-Simons theory is given abstractly by

$$(5.120) \quad Z(\epsilon)\Big|_{\mathcal{M}} = \frac{1}{|\Gamma|} \int_{\mathcal{M}} e(p)\Big|_{p=1/\epsilon} \exp(\Omega + i\epsilon n\Theta).$$

In arriving at (5.120), we note that the prefactor $\text{Vol}(U(1)^2)$ in (5.113) cancels against a corresponding factor in $\text{Vol}(H_0)$ from $I(\psi)$. This cancellation leaves the factor $1/|\Gamma|$ in (5.114), where $|\Gamma|$ is the order of the center Γ of G .

As we recall in writing (5.120), since the generator a in \mathfrak{h}_0 acts trivially on N , $e(\psi) \equiv e(p)$ depends only on p in \mathfrak{h}_0 . Once we set $p = 1/\epsilon$ in (5.120), $e(\epsilon^{-1})$ will become an ordinary cohomology class on \mathcal{M} . As in the case of localization at the trivial connection, our computation now reduces to determining explicitly this class.

More About the Equivariant Euler Class

Before we evaluate the equivariant Euler classes of the infinite-dimensional bundles corresponding to \mathcal{E}_0 and \mathcal{E}_1 , we first give a more explicit description of the equivariant Euler class in a simpler, finite-dimensional situation. To make contact with Chern-Simons theory, we assume abstractly that H_0 is a torus which acts trivially on a complex manifold \mathcal{M} , and we assume that E is a complex representation of H_0 which is fibered over \mathcal{M} to determine an associated H_0 -equivariant bundle. Our goal is now to give a concrete topological formula for $e_{H_0}(\mathcal{M}, E)$, which we will then apply to evaluate $e(\psi)$ in (5.119) for Chern-Simons theory.

In general, $e_{H_0}(\mathcal{M}, E)$ incorporates both the algebraic data associated to the action of H_0 on E as well as the topological data that describes the twisting of E over \mathcal{M} . To encode the data related to the action of H_0 on E , we decompose E under the action of H_0 into a sum of one-dimensional complex eigenspaces

$$(5.121) \quad E = \bigoplus_{j=1}^{\dim E} E_{\beta_j},$$

where each β_j is a weight in \mathfrak{h}_0^* which describes the action of H_0 on the eigenspace E_{β_j} .

To encode the topological data associated to the vector bundle determined by E over \mathcal{M} , we apply the splitting principle in topology, as explained for instance in Chapter 21 of [39]. By this principle, we can assume that the vector bundle determined by E over \mathcal{M} splits equivariantly into a sum of line-bundles associated to each of the eigenspaces

E_{β_j} for the action of H_0 . Under this assumption, we let $x_j = c_1(E_{\beta_j})$ be the first Chern class of the corresponding line-bundle. These virtual Chern roots x_j determine the total Chern class of E as

$$(5.122) \quad c(E) = \prod_{j=1}^{\dim E} (1 + x_j).$$

In particular, the ordinary Euler class of E over \mathcal{M} is then given by

$$(5.123) \quad e(\mathcal{M}, E) = \prod_{j=1}^{\dim E} x_j.$$

The equivariant Euler class $e_{H_0}(\mathcal{M}, E)$ is now determined in terms of the weights β_j and the Chern roots x_j . We recall that $e_{H_0}(\mathcal{M}, E)$ is defined as an element of $H_{H_0}^*(\mathcal{M}, E) = H^*(\mathcal{M}) \otimes H_{H_0}^*(pt)$ since H_0 acts trivially on \mathcal{M} . Thus, $e_{H_0}(\mathcal{M}, E)$ will be a function of $\psi \in \mathfrak{h}_0$ with values in the cohomology of \mathcal{M} . Explicitly, the H_0 -equivariant Euler class of E over \mathcal{M} is given by

$$(5.124) \quad e_{H_0}(\mathcal{M}, E) = \prod_{j=1}^{\dim E} \left(\frac{i \langle \beta_j, \psi \rangle}{2\pi} + x_j \right).$$

We see that this expression is a natural generalization of the ordinary Euler class (5.123) of E . Also, when \mathcal{M} is only a point, the Chern roots x_j do not appear in (5.124) for dimensional reasons, and the product over the weights β_j in (5.124) reproduces the determinant of ψ acting on E as in (5.116).

Evaluating $e(p)$

We now evaluate $e(p)$ for Chern-Simons theory¹². First we recall that the complex vector spaces \mathcal{E}_0 and \mathcal{E}_1 appearing in (5.119) arise from the Dolbeault cohomology groups of the bundles $\text{ad}(P) \otimes \mathcal{L}^t$ over Σ , with

$$(5.125) \quad \begin{aligned} \mathcal{E}_0 &= \bigoplus_{t \neq 0} H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})), \\ \mathcal{E}_1 &= \bigoplus_{t \neq 0} H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t})). \end{aligned}$$

¹²We set $p = 1/\epsilon$ only at the very end of the computation.

We also recall that the action of H_0 on \mathcal{E}_0 and \mathcal{E}_1 is determined by the operator $p\mathcal{L}_R$, whose action in turn only depends on the grading t in (5.125). We naturally decompose \mathcal{E}_0 and \mathcal{E}_1 under the action of H_0 , and we consider the finite-dimensional eigenspaces

$$(5.126) \quad \mathcal{E}_0^{(t)} = H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t), \quad \mathcal{E}_1^{(t)} = H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t).$$

The abelian group H_0 acts canonically on both $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ with eigenvalue $-2\pi it$.

In terms of this decomposition, the quantity $e(p)$ is given by the following infinite product,

$$(5.127) \quad e(p) = \prod_{t \neq 0} \left[\frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(t)})}{e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(t)})} \right] = \prod_{t \geq 1} \left[\frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(t)}) \cdot e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(-t)})}{e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(t)}) \cdot e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(-t)})} \right].$$

Here $e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(t)})$ and $e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(t)})$ denote the H_0 -equivariant Euler classes of the finite-dimensional bundles determined by $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ over \mathcal{M} .

Our basic strategy to evaluate the product in (5.127) is to deduce a recursive relation between the equivariant Euler classes of $\mathcal{E}_0^{(t)}$, $\mathcal{E}_0^{(t-1)}$, $\mathcal{E}_1^{(t)}$, and $\mathcal{E}_1^{(t-1)}$. So far, we have only specified the line-bundle \mathcal{L} topologically, by specifying its degree n . The holomorphic structure of \mathcal{L} really was not important. Now we want to pick a convenient holomorphic structure on \mathcal{L} to simplify our computation. We pick n arbitrary points $\sigma_1, \dots, \sigma_n$ on Σ and we take \mathcal{L} to be $\mathcal{O}(\sigma_1 + \dots + \sigma_n)$.

With this choice of \mathcal{L} , we have the following short exact sequence of coherent sheaves on Σ ,

$$(5.128) \quad 0 \longrightarrow \text{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t-1} \longrightarrow \text{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^t \longrightarrow \bigoplus_{i=1}^n \text{ad}_{\mathbb{C}}(P)|_{\sigma_i} \longrightarrow 0.$$

Here t is any integer, and $\text{ad}_{\mathbb{C}}(P)|_{\sigma_i}$ denotes the skyscraper sheaf associated to the fiber of $\text{ad}_{\mathbb{C}}(P)$ over the point σ_i . The appearance of this skyscraper sheaf explains our need to work a bit more generally with coherent sheaves, as opposed to more innocuous bundles.

Associated to this short exact sequence we have the usual long exact sequence in sheaf cohomology,

$$\begin{aligned}
 (5.129) \quad 0 \longrightarrow H^0(\Sigma, \text{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t-1}) &\longrightarrow H^0(\Sigma, \text{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^t) \longrightarrow \\
 &\longrightarrow \bigoplus_{i=1}^n H^0(\Sigma, \text{ad}_{\mathbb{C}}(P)|_{\sigma_i}) \longrightarrow \\
 &\longrightarrow H^1(\Sigma, \text{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t-1}) \longrightarrow H^1(\Sigma, \text{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^t) \longrightarrow 0.
 \end{aligned}$$

Since a skyscraper sheaf has no higher cohomology, we observe that $H^1(\Sigma, \text{ad}_{\mathbb{C}}(P)|_{\sigma_i}) = 0$ for the last term of (5.129).

Each cohomology group appearing in (5.129) can be considered as the fiber of an associated holomorphic bundle over the moduli space \mathcal{M} , and the exactness of the sequence (5.129) implies the exactness of the corresponding sequence of bundles on \mathcal{M} . Except for the single term involving the skyscraper sheaf, we see that the bundles which appear in (5.129) are those associated to $\mathcal{E}_0^{(t-1)}$, $\mathcal{E}_0^{(t)}$, $\mathcal{E}_1^{(t-1)}$, and $\mathcal{E}_1^{(t)}$. In analogy to (5.126), we set

$$(5.130) \quad \mathcal{V}_{(i)} = H^0(\Sigma, \text{ad}_{\mathbb{C}}(P)|_{\sigma_i}).$$

Over \mathcal{M} , $\mathcal{V}_{(i)}$ also fibers as a holomorphic bundle. Although the holomorphic structure of $\mathcal{V}_{(i)}$ depends on σ_i , its topology, which is all we will care about, does not (as is clear from the fact that the points σ_i can be moved continuously), so we just write \mathcal{V} for any of the $\mathcal{V}_{(i)}$. Thus, the exact sequence in (5.129) implies the following exact sequence of associated bundles on \mathcal{M} ,

$$(5.131) \quad 0 \longrightarrow \mathcal{E}_0^{(t-1)} \longrightarrow \mathcal{E}_0^{(t)} \longrightarrow \mathcal{V}^{\oplus n} \longrightarrow \mathcal{E}_1^{(t-1)} \longrightarrow \mathcal{E}_1^{(t)} \longrightarrow 0.$$

This sequence is an exact sequence of bundles on \mathcal{M} , but we need an exact sequence of H_0 -equivariant bundles on \mathcal{M} , such that the maps in the sequence are compatible with the action of H_0 . Because H_0 acts with different eigenvalues on the equivariant bundles $\mathcal{E}_0^{(t-1)}$ and $\mathcal{E}_0^{(t)}$, and similarly on $\mathcal{E}_1^{(t-1)}$ and $\mathcal{E}_1^{(t)}$, the canonical action of H_0 is not compatible with the maps in (5.131).

To fix this problem, we note that we are free to consider actions of H_0 on $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ other than the canonical action. That is, we consider H_0 -equivariant bundles over \mathcal{M} whose fibers are still given by the cohomology groups $H_{\bar{\partial}}^0(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t)$ and $H_{\bar{\partial}}^1(\Sigma, \text{ad}(P) \otimes \mathcal{L}^t)$ but where the action of H_0 is not the canonical action fixed by t . In fact,

so long as H_0 acts uniformly on the fiber, we can take H_0 to act with any eigenvalue.

Thus we let $\mathcal{E}_{0,m}^{(t)}$ and $\mathcal{E}_{1,m}^{(t)}$ denote the H_0 -equivariant bundles over \mathcal{M} whose fibers are determined by t as before but where H_0 acts with eigenvalue $-2\pi im$ for some integer m . In this notation, the bundles $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ with the canonical action of H_0 are $\mathcal{E}_{0,t}^{(t)}$ and $\mathcal{E}_{1,t}^{(t)}$. We similarly denote by \mathcal{V}_m the H_0 -equivariant bundle associated to \mathcal{V} for which H_0 acts uniformly on the fiber with eigenvalue $-2\pi im$.

The exact sequence in (5.131) on \mathcal{M} now determines a corresponding exact sequence of H_0 -equivariant bundles,

$$(5.132) \quad 0 \longrightarrow \mathcal{E}_{0,m}^{(t-1)} \longrightarrow \mathcal{E}_{0,m}^{(t)} \longrightarrow (\mathcal{V}_m)^{\oplus n} \longrightarrow \mathcal{E}_{1,m}^{(t-1)} \longrightarrow \mathcal{E}_{1,m}^{(t)} \longrightarrow 0.$$

Since the action of H_0 is the same on every term in this sequence, the maps are trivially compatible with the group action.

We now recall that a fundamental property of the equivariant Euler class is that it behaves multiplicatively with respect to an exact sequence of equivariant bundles, just like the ordinary Euler class. Thus, if E_1 , E_2 , and E_3 are H_0 -equivariant bundles on \mathcal{M} which fit into an exact sequence whose maps respect the action of H_0 ,

$$(5.133) \quad 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0,$$

then the H_0 -equivariant Euler classes of these bundles satisfy the relation

$$(5.134) \quad e_{H_0}(\mathcal{M}, E_2) = e_{H_0}(\mathcal{M}, E_1) \cdot e_{H_0}(\mathcal{M}, E_3).$$

More generally, given an exact sequence of arbitrary length,

$$(5.135) \quad 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{2N} \longrightarrow E_{2N+1} \longrightarrow 0,$$

the relation (5.134) generalizes in the natural way, with

$$(5.136) \quad e_{H_0}(\mathcal{M}, E_2) \cdots e_{H_0}(\mathcal{M}, E_{2N}) = e_{H_0}(\mathcal{M}, E_1) \cdots e_{H_0}(\mathcal{M}, E_{2N+1}).$$

We apply this multiplicative property of the equivariant Euler class to the exact sequence in (5.132). For the following, it is very natural to introduce the ratio of equivariant Euler classes,

$$(5.137) \quad \mathcal{Q}_m^{(t)} \equiv \left[\frac{e_{H_0}(\mathcal{M}, \mathcal{E}_{0,m}^{(t)})}{e_{H_0}(\mathcal{M}, \mathcal{E}_{1,m}^{(t)})} \right],$$

so that $e(p)$ is given by

$$(5.138) \quad e(p) = \prod_{t \neq 0} \mathcal{Q}_t^{(t)}.$$

In terms of $\mathcal{Q}_m^{(t)}$, the multiplicative relation (5.134) applied to (5.132) implies that

$$(5.139) \quad \mathcal{Q}_m^{(t)} = \mathcal{Q}_m^{(t-1)} \cdot [e_{H_0}(\mathcal{M}, \mathcal{V}_m)]^n.$$

Expanding the recursive relation (5.139), we find

$$(5.140) \quad \mathcal{Q}_m^{(t)} = \mathcal{Q}_m^{(0)} \cdot [e_{H_0}(\mathcal{M}, \mathcal{V}_m)]^{nt}.$$

What has this work gained us? As we now explain, we can give a very concrete expression for the quantity on the right of (5.140). By definition, the bundles over \mathcal{M} which determine the ratios $\mathcal{Q}_{\pm t}^{(0)}$ have fibers

$$(5.141) \quad \mathcal{E}_0^{(0)} = H_{\mathbb{R}}^0(\Sigma, \text{ad}_{\mathbb{C}}(P)), \quad \mathcal{E}_1^{(0)} = H_{\mathbb{R}}^1(\Sigma, \text{ad}_{\mathbb{C}}(P)).$$

By our assumption that all points in the moduli space \mathcal{M} correspond to irreducible connections, $\mathcal{E}_0^{(0)} = 0$. Further, as we mentioned in Section 4.3, $\mathcal{E}_1^{(0)}$ is naturally identified with the holomorphic tangent bundle $T\mathcal{M}$ of the moduli space itself, so $\mathcal{E}_1^{(0)} = T\mathcal{M}$. We introduce the convenient notation $\mathcal{E}_{1,t}^{(0)} \equiv T\mathcal{M}_t$ to indicate the H_0 -equivariant version of $T\mathcal{M}$. Because of this observation, we can apply the relations (5.138) and (5.140) to rewrite $e(p)$ entirely in terms of the equivariant bundles $T\mathcal{M}_t$ and \mathcal{V}_t ,

$$(5.142) \quad e(p) = \prod_{t \neq 0} \frac{1}{e_{H_0}(\mathcal{M}, T\mathcal{M}_t)} \cdot [e_{H_0}(\mathcal{M}, \mathcal{V}_t)]^{nt}.$$

Let us make the factors appearing on the right in (5.142) more explicit. To this end, we introduce the Chern roots ϖ_j of $T\mathcal{M}$, where $j = 1, \dots, \dim \mathcal{M}$, and the Chern roots ν_l of \mathcal{V} , where $l = 1, \dots, \text{rk } \mathcal{V}$. Since \mathcal{V} arises from the fiber of the adjoint bundle $\text{ad}_{\mathbb{C}}(P)$, the rank of \mathcal{V} is simply $\text{rk } \mathcal{V} = \dim G \equiv \Delta_G$. As in our general discussion of the equivariant Euler class, the Chern roots ϖ_j and ν_l are “virtual” degree two classes in $H^*(\mathcal{M})$ which are defined in terms of the total Chern classes of $T\mathcal{M}$ and \mathcal{V} as

$$(5.143) \quad c(T\mathcal{M}) = \prod_{j=1}^{\dim \mathcal{M}} (1 + \varpi_j), \quad c(\mathcal{V}) = \prod_{l=1}^{\Delta_G} (1 + \nu_l).$$

In terms of these Chern roots, our general description of the equivariant Euler class in (5.124) implies that

$$(5.144) \quad e_{H_0}(\mathcal{M}, T\mathcal{M}_t) = \prod_{j=1}^{\dim \mathcal{M}} (-itp + \varpi_j), \quad e_{H_0}(\mathcal{M}, \mathcal{V}_t) = \prod_{l=1}^{\Delta_G} (-itp + \nu_l).$$

The terms in (5.144) which involve p arise via the infinitesimal action of H_0 on the fibers of $T\mathcal{M}_t$ and \mathcal{V}_t . We recall that H_0 acts infinitesimally as $p\mathcal{L}_R = -2\pi itp$.

Together, (5.142) and (5.144) imply the following formal expression for $e(p)$,

$$(5.145) \quad e(p) = \prod_{t \neq 0} \left[\prod_{j=1}^{\dim \mathcal{M}} \frac{1}{(-itp + \varpi_j)} \right] \left[\prod_{l=1}^{\Delta_G} (-itp + \nu_l)^{nt} \right].$$

This infinite product represents the determinant of a first-order operator \mathcal{D} acting on $\mathcal{E}_0 \ominus \mathcal{E}_1$, where

$$(5.146) \quad \mathcal{D} = \frac{1}{2\pi} (p\mathcal{L}_R + i\mathcal{R}).$$

Here \mathcal{R} is the curvature operator acting on \mathcal{E}_0 and \mathcal{E}_1 as bundles over \mathcal{M} , as appears in the computation in Appendix D, and “ \ominus ” indicates that we actually take the inverse of the determinant of \mathcal{D} acting on \mathcal{E}_1 .

The determinant in (5.145) is only a formal expression, and to define it we must choose some regularization procedure. For instance, we considered the determinant of a similar operator \mathcal{D}_0 in our computation at the trivial connection in Section 5.2,

$$(5.147) \quad \mathcal{D}_0 = \frac{1}{2\pi} (p\mathcal{L}_R - [\phi, \cdot]).$$

In that case, we defined the determinant of \mathcal{D}_0 analytically, using the zeta-function to define its absolute value and the eta-function to define its phase.

We follow a similar strategy to define the determinant of \mathcal{D} , or more explicitly the infinite product in (5.145). To start, we find it useful to rewrite the product in (5.145) by pulling out an overall factor of p ,

$$(5.148) \quad e(p) = p^{\dim \mathcal{M}} \prod_{t \neq 0} \left[\prod_{j=1}^{\dim \mathcal{M}} \left(-it + \left(\frac{\varpi_j}{p} \right) \right)^{-1} \right] \left[\prod_{l=1}^{\Delta_G} \left(-it + \left(\frac{\nu_l}{p} \right) \right)^{nt} \right].$$

In passing from (5.145) to (5.148), we use as in Section 5.2 the classical Riemann zeta-function to define the trivial, but infinite, product over p which arises from (5.145),

$$(5.149) \quad \prod_{t \geq 1} p^{-2 \dim \mathcal{M}} = \exp(-2 \dim \mathcal{M} \cdot \ln p \cdot \zeta(0)) = p^{\dim \mathcal{M}}.$$

(There is no contribution from the factors in (5.145) which are associated to \mathcal{V} due to a cancellation between the terms for $\pm t$.) Thus, we are left to consider the determinant of the rescaled operator \mathcal{D}' ,

$$(5.150) \quad \mathcal{D}' = \frac{1}{2\pi} \left(\mathcal{L}_R + i \frac{\mathcal{R}}{p} \right),$$

which represents the infinite product appearing in (5.148) and which depends on p and the Chern roots only in the combinations ϖ_j/p and ν_l/p .

One interesting distinction between the operator \mathcal{D} , or equivalently \mathcal{D}' , and the operator \mathcal{D}_0 which appeared previously is that whereas \mathcal{D}_0 is an anti-hermitian operator, with a purely imaginary spectrum, the operator \mathcal{D} has no particular hermiticity properties and its spectrum has no particular phase. This is manifest in the product (5.148), since $-it$ is imaginary but both the Chern roots and p are real. In terms of (5.150), both \mathcal{L}_R and \mathcal{R} are anti-hermitian operators, but we have an explicit factor of ‘ i ’ in front of \mathcal{R} . Because \mathcal{D}' is neither hermitian nor anti-hermitian, we will have to generalize the zeta/eta-function regularization technique which we applied to define the determinant of \mathcal{D}_0 in Section 5.2.

Before we supply a definition for the determinant of \mathcal{D}' , or equivalently for the products in (5.148), let us consider what general properties our definition should possess. To start, we factorize the product in (5.148) into the two infinite products below,

$$(5.151) \quad f_{\mathcal{M}}(z) = \prod_{t \neq 0} \prod_{j=1}^{\dim \mathcal{M}} (-it + z\varpi_j)^{-1},$$

$$f_{\mathcal{V}}(z) = \prod_{t \neq 0} \prod_{l=1}^{\Delta_G} (-it + z\nu_l)^{nt},$$

where $z = 1/p$ is now a formal parameter.

The expressions in (5.151) are ill-defined as they stand. However, if we formally differentiate $\log f_{\mathcal{M}}(z)$ and $\log f_{\mathcal{V}}(z)$ with respect to z a

sufficient number of times, we eventually obtain well-defined, absolutely convergent sums. For instance, in the case of $f_{\mathcal{M}}(z)$, we see that

$$\begin{aligned}
 (5.152) \quad \frac{d^2}{dz^2} \log f_{\mathcal{M}}(z) &= \sum_{j=1}^{\dim \mathcal{M}} \sum_{t \neq 0} \frac{\varpi_j^2}{(-it + z\varpi_j)^2} \\
 &= \sum_{j=1}^{\dim \mathcal{M}} \frac{d^2}{dz^2} \log \left[\frac{(\pi z \varpi_j)}{\sinh(\pi z \varpi_j)} \right].
 \end{aligned}$$

The second equality in (5.152) follows from the same product identity (5.64) for $\sin(x)/x$ as we applied in Section 5.2.

So any reasonable definition for $f_{\mathcal{M}}(z)$ in (5.151) must be compatible with the relation (5.152). In particular, upon integrating (5.152), we see that $\log f_{\mathcal{M}}(z)$ is determined up to a linear function of z , and hence $f_{\mathcal{M}}(z)$ is determined up to two arbitrary real constants a_0 and a_1 ,

$$(5.153) \quad f_{\mathcal{M}}(z) = \exp[a_0 + a_1 z c_1(T\mathcal{M})] \prod_{j=1}^{\dim \mathcal{M}} \frac{(\pi z \varpi_j)}{\sinh(\pi z \varpi_j)}.$$

Here $c_1(T\mathcal{M}) = \sum_j \varpi_j$ is the first Chern class of \mathcal{M} . In deducing the form (5.153), we have applied the fact, manifest from (5.151), that $f_{\mathcal{M}}(z)$ can only depend on z and the Chern roots ϖ_j in the combinations $z\varpi_j$, and we have also used the fact that only symmetric combinations of the Chern roots have any real meaning — hence each Chern root ϖ_j must appear with the same coefficient a_1 in the exponential factor of (5.153). Comparing to the product (5.151), we also note that $f_{\mathcal{M}}(z)$ is formally real (for real z), so a_0 and a_1 must be real.

We can also apply this general analysis to $f_{\mathcal{V}}(z)$ in (5.151). Here we observe that $\log f_{\mathcal{V}}(z)$ should satisfy

$$\begin{aligned}
 (5.154) \quad \frac{d^3}{dz^3} \log f_{\mathcal{V}}(z) &= \sum_{t \neq 0} \sum_{l=1}^{\Delta_G} \frac{2nt\nu_l^3}{(-it + z\nu_l)^3}, \\
 &= \sum_{t \geq 1} \sum_{l=1}^{\Delta_G} \left[\frac{2nt\nu_l^3}{(-it + z\nu_l)^3} + \frac{2nt\nu_l^3}{(-it - z\nu_l)^3} \right], \\
 &= 0.
 \end{aligned}$$

In contrast to the case of $f_{\mathcal{M}}(z)$, we must take three derivatives of $\log f_{\mathcal{V}}(z)$ to get a convergent sum, due to the exponent nt appearing in (5.151). In passing to the second equality of (5.154), we have simply

paired terms for $\pm t$. However, to deduce the cancellation in the third line of (5.154), we must use some topological facts about the bundle \mathcal{V} .

We recall that \mathcal{V} is the bundle over \mathcal{M} whose fibers are given by $H^0(\Sigma, \text{ad}_{\mathbb{C}}(P)|_{\sigma})$ for some point σ on Σ . This bundle is naturally the complexification of a real bundle over \mathcal{M} , namely the bundle whose fibers are $H^0(\Sigma, \text{ad}(P)|_{\sigma})$. Consequently, the non-zero Chern roots of \mathcal{V} are paired such that for each root ν there is a corresponding root ν' with $\nu' = -\nu$. This fact implies that any odd, symmetric function of the Chern roots vanishes. In particular, all odd Chern classes of \mathcal{V} vanish.

We now consider a series expansion of the denominators in the second line of (5.154) in terms of the nilpotent quantities $z\nu_l$. Because of the relative signs in these denominators, and because of the explicit cubic factor ν_l^3 in the numerators, all terms of even degree in the Chern roots ν_l automatically cancel. However, by our observation about \mathcal{V} above, the remaining terms of odd degree in the ν_l cancel when we sum over roots.

From (5.154), we see that $\log f_{\mathcal{V}}(z)$ is determined up to a quadratic function of z . Hence $f_{\mathcal{V}}(z)$ is determined up to two real constants b_0 and b_2 ,

$$(5.155) \quad f_{\mathcal{V}}(z) = \exp [ib_0 + ib_2z^2 \Theta].$$

A term linear in z would necessarily appear with the first Chern class $c_1(\mathcal{V})$, which vanishes by our observation above. Since $c_1(\mathcal{V}) = 0$, the only degree two class that can appear in (5.155) is the characteristic class Θ . We also observe from the product (5.151) that $f_{\mathcal{V}}(z)$ must be simply a phase (for real z), since under complex conjugation $f_{\mathcal{V}}(z)$ goes to $f_{\mathcal{V}}^{-1}(z)$. This observation fixes the factors of ‘ i ’ in (5.155).

Having fixed the general forms (5.153) and (5.155) of $f_{\mathcal{M}}$ and $f_{\mathcal{V}}$, we now compute the undetermined constants. To do this, we must still decide how to define the determinant of the operator

$$(5.156) \quad \mathcal{D}' = (1/2\pi)[\mathcal{L}_R + i(\mathcal{R}/p)].$$

Motivated by our work in Section 5.2, we proceed as follows. First, although p is a positive, real variable in our problem, we will define the determinant of \mathcal{D}' more generally for complex p . Second, once we allow p to be complex, we impose the requirement that the determinant of \mathcal{D}' depend analytically on p . In particular, if we evaluate the determinant for purely *imaginary* p , of the form $p = i/y$ for real $y > 0$ (the fact that we use $1/y$ is just for notational convenience later), then the determinant is defined for real $p > 0$ by analytic continuation. Finally, when $p = i/y$,

we see that $\mathcal{D}' = (1/2\pi) [\mathcal{L}_R + y\mathcal{R}]$ is an anti-hermitian operator exactly like \mathcal{D}_0 , and we can use zeta/eta-function regularization to define the determinant of \mathcal{D}' for these values of p as we did in Section 5.2.

In terms of $f_{\mathcal{M}}$ and $f_{\mathcal{V}}$ in (5.151), this definition of the determinant of \mathcal{D}' amounts to the prescription to use zeta/eta-function regularization to define the products

$$(5.157) \quad \begin{aligned} f_{\mathcal{M}}(z = -iy) &= \prod_{t \neq 0} \prod_{j=1}^{\dim \mathcal{M}} \frac{i}{(t + y\varpi_j)}, \\ f_{\mathcal{V}}(z = -iy) &= \prod_{t \neq 0} \prod_{l=1}^{\Delta_G} (-i)^{nt} (t + y\nu_l)^{nt}. \end{aligned}$$

We first ignore the factors of ‘ i ’ in (5.157) and we compute the absolute values of $f_{\mathcal{M}}$ and $f_{\mathcal{V}}$.

For instance,

$$(5.158) \quad \begin{aligned} |f_{\mathcal{M}}(-iy)| &= \prod_{t \geq 1} \prod_{j=1}^{\dim \mathcal{M}} [t^2 - (y\varpi_j)^2]^{-1} \\ &= \left(\frac{1}{2\pi}\right)^{\dim \mathcal{M}} \cdot \prod_{j=1}^{\dim \mathcal{M}} \frac{(\pi y \varpi_j)}{\sin(\pi y \varpi_j)}. \end{aligned}$$

Since the Chern roots ϖ_j are nilpotent, the terms in the first product in (5.158) are manifestly positive. In passing to the second equality, we apply the same identities (5.64) and (5.65) from Section 5.2. This form of $|f_{\mathcal{M}}(-iy)|$ is clearly compatible with our general expression (5.153).

On the other hand, one can easily check that zeta-function regularization defines the absolute value of $f_{\mathcal{V}}$ to be trivial, for the same topological reason that we explained following (5.154), so

$$(5.159) \quad |f_{\mathcal{V}}(-iy)| = \prod_{t \geq 1} \prod_{l=1}^{\Delta_G} \left[\frac{t + y\nu_l}{t - y\nu_l} \right]^{nt} = 1.$$

We are left to compute the phases of $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$. We define these using the eta-function, as in Section 5.2. More precisely, we write

$$(5.160) \quad f_{\mathcal{M}}(-iy) = \exp\left(-\frac{i\pi}{2} \eta_{\mathcal{M}}\right) \cdot |f_{\mathcal{M}}|, \quad f_{\mathcal{V}}(-iy) = \exp\left(-\frac{i\pi}{2} \eta_{\mathcal{V}}\right).$$

Here $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{V}}$ denote the eta-invariants which arise as the values at $s = 0$ of the eta-functions $\eta_{\mathcal{M}}(s)$ and $\eta_{\mathcal{V}}(s)$ abstractly associated to the hermitian operator $i\mathcal{D}'$ as it acts on $\mathcal{E}_0 \ominus \mathcal{E}_1$,

$$(5.161) \quad i\mathcal{D}' = \frac{i}{2\pi} (\mathcal{L}_R + y\mathcal{R}).$$

This operator should be compared to the corresponding operator which we considered when computing the phase of $e(p, \phi)$ at the trivial connection,

$$(5.162) \quad \frac{i}{2\pi} \left(\mathcal{L}_R - \left[\frac{\phi}{p}, \cdot \right] \right).$$

We recall from Section 5.2 that the eta-invariant associated to the operator in (5.162) acquires an anomalous dependence on (ϕ/p) which produces the finite shift in the Chern-Simons level. In the case at hand, a similar anomalous dependence of $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{V}}$ on $y\mathcal{R}$ gives rise to the same shift in the level.

Concretely, the eta-functions $\eta_{\mathcal{M}}(s)$ and $\eta_{\mathcal{V}}(s)$ are given by the following regularized sums over the factors which appear in $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$ in (5.157) and which represent the eigenvalues λ of $i\mathcal{D}'$,

$$(5.163) \quad \begin{aligned} \eta_{\mathcal{M}}(s) &= \sum_{t \neq 0} \sum_{j=1}^{\dim \mathcal{M}} -\text{sign}(\lambda(t, \varpi_j)) \cdot |\lambda(t, \varpi_j)|^{-s}, \quad \lambda(t, \varpi_j) = t + y\varpi_j, \\ \eta_{\mathcal{V}}(s) &= \sum_{t \neq 0} \sum_{l=1}^{\Delta_G} nt \cdot \text{sign}(\lambda(t, \nu_l)) \cdot |\lambda(t, \nu_l)|^{-s}, \quad \lambda(t, \nu_l) = t + y\nu_l. \end{aligned}$$

The various constants appearing in (5.163) are perhaps most clear if we compare to the formal expressions for $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$ in (5.157). Thus, the overall minus sign in $\eta_{\mathcal{M}}(s)$ arises because i as opposed to $-i$ appears in $f_{\mathcal{M}}(-iy)$, which is in turn associated to the fact that we consider $\mathcal{E}_0 \ominus \mathcal{E}_1$ as opposed to $\mathcal{E}_0 \oplus \mathcal{E}_1$. Similarly, the multiplicity factor nt appears in $\eta_{\mathcal{V}}(s)$ because of the factor $(-i)^{nt}$ in $f_{\mathcal{V}}(-iy)$.

Because the Chern roots are nilpotent, we note that $\text{sign}(\lambda(t, x)) = \text{sign}(t)$, where $x = \varpi_j$ or $x = \nu_l$ as the case may be. Thus, we write the

regularized sums in (5.163) explicitly as

$$(5.164) \quad \eta_{\mathcal{M}}(s) = \sum_{t \geq 1} \sum_{j=1}^{\dim \mathcal{M}} \frac{-1}{(t + y\varpi_j)^s} + \sum_{t \geq 1} \sum_{j=1}^{\dim \mathcal{M}} \frac{1}{(t - y\varpi_j)^s},$$

$$\eta_{\mathcal{N}}(s) = \sum_{t \geq 1} \sum_{l=1}^{\Delta_G} \frac{nt}{(t + y\nu_l)^s} + \sum_{t \geq 1} \sum_{l=1}^{\Delta_G} \frac{nt}{(t - y\nu_l)^s}.$$

As in Section 5.2, we are left to evaluate these sums at $s = 0$.

In fact, we have already done all of the required computation. The sum which defines $\eta_{\mathcal{M}}(s)$ is the same as the sum (5.76) which we evaluated in the warmup computation on S^1 in Section 5.2. Thus we find

$$(5.165) \quad \eta_{\mathcal{M}}(0) = 2y \sum_{j=1}^{\dim \mathcal{M}} \varpi_j = 2y c_1(T\mathcal{M}).$$

In deducing the second equality, we note that the trace over all Chern roots of $T\mathcal{M}$ is the first Chern class of $T\mathcal{M}$.

To evaluate $\eta_{\mathcal{N}}(0)$, we perform a computation precisely isomorphic to our computation of $e(p, \phi)$ in Section 5.2. Applying our earlier results, we find

$$(5.166) \quad \eta_{\mathcal{N}}(0) = \eta_0 + ny^2 \sum_{l=1}^{\Delta_G} \nu_l^2, \quad \eta_0 = -\frac{n\Delta_G}{6}.$$

Here η_0 is the same constant that appeared in our localization computation at the trivial connection. As for the term quadratic in ν_l , this term arises in the same way as the term quadratic in ϕ in (5.88).

We now recall from Section 5.2 that we applied a Lie algebra identity (5.89) involving $\check{c}_{\mathfrak{g}}$ to rewrite the term quadratic in ϕ in (5.88) in terms of the natural quadratic invariant $\frac{1}{2}\text{Tr}(\phi^2)$. Under the Chern-Weil homomorphism, by which we identify the Chern roots ν_l with the eigenvalues of the curvature operator $i\mathcal{R}/2\pi$, we can apply the same Lie algebra identity to rewrite the degree four class $\sum_l \nu_l^2$ in terms of the class Θ that already appears in the integral over \mathcal{M} . We find from the identity (5.89) that

$$(5.167) \quad \sum_{l=1}^{\Delta_G} \nu_l^2 = \frac{\check{c}_{\mathfrak{g}}\Theta}{\pi^2},$$

and $\eta_{\mathcal{V}}(0)$ becomes

$$(5.168) \quad \eta_{\mathcal{V}}(0) = \eta_0 + \frac{n\check{c}_{\mathfrak{g}}}{\pi^2} y^2 \Theta.$$

With these results (5.165) and (5.168), we evaluate $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$ to be

$$(5.169) \quad \begin{aligned} f_{\mathcal{M}}(-iy) &= \exp(-i\pi y c_1(T\mathcal{M})) \cdot \left(\frac{1}{2\pi}\right)^{\dim \mathcal{M}} \cdot \prod_{j=1}^{\dim \mathcal{M}} \frac{(\pi y \varpi_j)}{\sin(\pi y \varpi_j)}, \\ f_{\mathcal{V}}(-iy) &= \exp\left(-\frac{i\pi}{2} \eta_0 - \frac{in\check{c}_{\mathfrak{g}}}{2\pi} y^2 \Theta\right). \end{aligned}$$

Upon setting $z = -iy$, these expressions assume the same form as the general expressions in (5.153) and (5.155).

We recall that p is related to y via $p = i/y$. So $e(p)$, as determined by the analytic continuation of (5.169), is finally given by

$$(5.170) \quad \begin{aligned} e(p) &= p^{\dim \mathcal{M}} \cdot f_{\mathcal{M}}(p) \cdot f_{\mathcal{V}}(p), \\ &= \exp\left(-\frac{i\pi}{2} \eta_0 + \frac{\pi}{p} c_1(T\mathcal{M}) + \frac{in\check{c}_{\mathfrak{g}}}{2\pi p^2} \Theta\right) \times \\ &\quad \times \left(\frac{p}{2\pi}\right)^{\dim \mathcal{M}} \prod_{j=1}^{\dim \mathcal{M}} \frac{(\pi \varpi_j/p)}{\sinh(\pi \varpi_j/p)}. \end{aligned}$$

As we will see, this formula incorporates the famous shift in the Chern-Simons level k , and leads to agreement with the results of Rozansky.

Some Further Remarks

Our use of zeta/eta-function regularization to define $e(p)$, and especially the analytic continuation we performed in p , is somewhat *ad hoc*. The need for analytic continuation would have been avoided if at the very beginning of this paper, we had introduced the Cartan model of equivariant cohomology with the differential $D = d + \iota_V$ rather than the choice we actually made, $D = d + i\iota_V$. This would have resulted in the basic symplectic volume integral on a symplectic manifold \mathcal{M} being $\int_{\mathcal{M}} \exp(i\Omega)$ rather than the more standard $\int_{\mathcal{M}} \exp(\Omega)$; it also would clash with some conventions of physicists about reality conditions for fermions. However, it would clarify our discussion of the determinants, since if all factors of i are omitted from the localization form λ , then

the operator $i\mathcal{D}'$ would come out to be hermitian. Hence, the zeta/eta-function definition of determinants could be implemented with no need for artificial analytic continuation.

That definition is really most natural for oscillatory bosonic integrals such as appear in Chern-Simons theory. If a bosonic integral

$$(5.171) \quad Z = \int D\Phi \exp(i(\Phi, M\Phi)),$$

for some indefinite real symmetric operator M , is regularized by $M \rightarrow M + i\varepsilon$, for small positive ε , then the phase of Z is naturally $\exp(i\pi\eta(M)/2)$. This is really why, in Chern-Simons theory, eta-invariants appear in the one-loop corrections. If we take $D = d + \iota_V$, and take the localization form λ to be purely imaginary rather than purely real, then all integrals in Appendix D are oscillatory Gaussian integrals rather than real Gaussians. This gives a natural framework for zeta/eta-function regularization of the determinants in our localization computation.

Our general analysis of $d^2 \log f_{\mathcal{M}}(z)/dz^2$ and $d^3 \log f_{\mathcal{V}}(z)/dz^3$ showed that any reasonable definition of these determinants would differ from the zeta/eta-function approach by adding a constant to η_0 and changing the coefficients of $c_1(T\mathcal{M})$ and Θ in (5.170). We will see shortly that the coefficients as written in (5.170) do agree with Chern-Simons theory; in fact, they show up in Chern-Simons theory at the one-loop level. Ultimately, to justify the coefficients in (5.170) on an *a priori* basis requires a more rigorous comparison between the localization procedure and Chern-Simons theory.

The Contribution From \mathcal{M} in Chern-Simons Theory

Having evaluated $e(p)$, we now set $p = 1/\varepsilon$ and substitute (5.170) into our expression (5.120) for the contribution from \mathcal{M} to the Chern-Simons path integral. Thus,

$$(5.172) \quad \begin{aligned} Z(\varepsilon)|_{\mathcal{M}} &= \frac{1}{|\Gamma|} \exp\left(-\frac{i\pi}{2}\eta_0\right) \left(\frac{1}{2\pi\varepsilon}\right)^{\dim \mathcal{M}} \times \\ &\quad \times \int_{\mathcal{M}} \exp\left[\Omega + \pi\varepsilon c_1(T\mathcal{M}) + i\varepsilon n \left(1 + \frac{\varepsilon\check{c}_{\mathfrak{g}}}{2\pi}\right) \Theta\right] \times \\ &\quad \times \prod_{j=1}^{\dim \mathcal{M}} \left[\frac{\pi\varepsilon\varpi_j}{\sinh(\pi\varepsilon\varpi_j)}\right]. \end{aligned}$$

Since we are dealing with an integral, by making changes of variables we can rewrite the integrand of (5.172) in different ways which illuminate different features of this result. In the form at hand, we note that one can define a non-trivial scaling limit of (5.172) such that the Chern-Simons coupling ϵ goes to zero (so that the level k goes to ∞) and the degree n of \mathcal{L} goes to ∞ with ϵn held fixed. In this limit, which physically decouples all the higher Kaluza-Klein modes of the gauge field, we see directly that the contribution from \mathcal{M} in Chern-Simons theory has the same form as the simple expression (4.18) for the corresponding contribution from \mathcal{M}_0 in two-dimensional Yang-Mills theory.

To express (5.172) more compactly, we now rescale all elements of the cohomology ring of \mathcal{M} by a factor $(2\pi\epsilon)^{q/2}$, where q is the degree of the given class. So for instance, the degree two Chern roots ϖ_j scale as $\varpi_j \rightarrow 2\pi\epsilon\varpi_j$. This trivial change of variables cancels the prefactor involving ϵ in (5.172) and reduces the product over Chern roots in (5.172) to a well-known characteristic class, the \widehat{A} -genus of \mathcal{M} .

We recall that the \widehat{A} -genus of \mathcal{M} is given in terms of the Chern roots of $T\mathcal{M}$ as

$$(5.173) \quad \widehat{A}(\mathcal{M}) = \prod_{j=1}^{\dim \mathcal{M}} \frac{\varpi_j/2}{\sinh(\varpi_j/2)}.$$

In a sense, the appearance of the \widehat{A} -genus in our problem is not so surprising, since it appears in roughly the same way as in the standard path integral derivations of the index theorem. See [63] for a derivation of the index theorem that applies abelian localization to a sigma model path integral; at least formally, that computation shares many features of our computation here.

In terms of the \widehat{A} -genus, our expression in (5.172) simplifies to

$$(5.174) \quad Z(\epsilon)\Big|_{\mathcal{M}} = \frac{1}{|\Gamma|} \exp\left(-\frac{i\pi}{2}\eta_0\right) \times \\ \times \int_{\mathcal{M}} \widehat{A}(\mathcal{M}) \exp\left[\frac{1}{2\pi\epsilon}\Omega + \frac{1}{2}c_1(T\mathcal{M}) + \frac{in}{4\pi^2\epsilon_r}\Theta\right].$$

Here we have absorbed the contribution from $\eta_{\mathcal{V}}(0)$ into a renormalization of the coupling $\epsilon_r = 2\pi/(k + \check{c}_{\mathfrak{g}})$ that appears in front of Θ .

Of course, we would like to write (5.172) entirely in terms of the renormalized coupling ϵ_r . To do so, we apply a theorem of [64] which relates the first Chern class $c_1(T\mathcal{M})$ to the symplectic form Ω in the

case of gauge group $G = SU(r + 1)$. In this case,

$$(5.175) \quad c_1(T\mathcal{M}) = 2(r + 1) \Omega',$$

where $\Omega' = \Omega/(2\pi)^2$ is the standard, integral symplectic form on \mathcal{M} . Happily, the dual Coxeter number $\check{c}_{\mathfrak{g}}$ of $G = SU(r + 1)$ is also given by $\check{c}_{\mathfrak{g}} = r + 1$, so we see that (5.174) can be expressed very simply using ϵ_r ,

$$(5.176) \quad Z(\epsilon)|_{\mathcal{M}} = \frac{1}{|\Gamma|} \exp\left(-\frac{i\pi}{2}\eta_0\right) \int_{\mathcal{M}} \widehat{A}(\mathcal{M}) \exp\left[\frac{1}{2\pi\epsilon_r} \left(\Omega + \frac{in}{2\pi} \Theta\right)\right].$$

This expression is of the same form as the corresponding result of Rozansky in [23].

We close with the following amusing observation. On general grounds, the \widehat{A} -genus of \mathcal{M} is related to the Todd class $\text{Td}(\mathcal{M})$ of \mathcal{M} by

$$(5.177) \quad \text{Td}(\mathcal{M}) = \exp\left(\frac{1}{2}c_1(T\mathcal{M})\right) \widehat{A}(\mathcal{M}).$$

So from (5.174), we see that an alternative expression for the path integral contribution from \mathcal{M} is

$$(5.178) \quad Z(\epsilon)|_{\mathcal{M}} = \frac{1}{|\Gamma|} \exp\left(-\frac{i\pi}{2}\eta_0\right) \int_{\mathcal{M}} \text{Td}(\mathcal{M}) \exp\left[k\Omega' + \frac{in}{4\pi^2\epsilon_r} \Theta\right].$$

Although our derivation of (5.178) is not valid for the trivial case $M = S^1 \times \Sigma$, we see that, upon setting $n = 0$, our result (5.178) takes the same form as the index formula (1.1) for $Z(\epsilon)$ in the trivial case. It is satisfying to see that both the index formula (1.1) and the two-dimensional Yang-Mills formula (4.18) are reproduced as special limits of our general result.

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Appendix A. Brief Analysis to Justify the Localization Computation in Section 4

In this appendix, we show that the quantity $Q \cdot Z'(\epsilon)$ computed using λ' in (4.92) of Section 4.3 agrees with the same quantity defined using λ , so that $Z'(\epsilon)$ as defined by integrating (4.92) agrees with $Z(\epsilon)$. Thus we consider the following one-parameter family of invariant forms, interpolating from λ to λ' on F ,

$$(A.1) \quad \Lambda(s) = s \lambda + (1 - s) \lambda', \quad s \in [0, 1],$$

and to start we consider the corresponding family $Z(\epsilon, s)$ of integrals over F ,

$$(A.2) \quad Z(\epsilon, s) = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times F} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2} (\phi, \phi) + t D\Lambda(s) \right].$$

If this integral is convergent for all s and also continuous as a function of s , then $Z(\epsilon, s)$ is independent of s , so that $Z(\epsilon) = Z(\epsilon, 1) = Z(\epsilon, 0) = Z'(\epsilon)$. This fact follows by differentiating the integrand of (A.2) with respect to s , which produces a total derivative on F .

We thus need to consider the basic convergence and continuity of $Z(\epsilon, s)$. Very broadly, divergences in the integral over F in (A.2) can only arise from integration over the non-compact fibers \mathfrak{h}^\perp and E_1 which sit over the compact orbit H/H_0 . However, the first, degree one term of λ' in (4.78) is precisely of the canonical form to define localization on the fiber \mathfrak{h}^\perp , exactly as in our computation on T^*H . Thus, no divergence arises from the integral over \mathfrak{h}^\perp , and we need only analyze the integral over the complex vector space E_1 . As we have already seen, precisely this integral over E_1 leads to the dangerous, possibly singular factor in $I(\psi)$ in (4.85). Furthermore, in our application to Yang-Mills theory, the corresponding vector space \mathcal{E}_1 describes the set of gauge-equivalence classes of unstable modes of the Yang-Mills action, and we expect the integral over these modes to be the most delicate.

We now analyze directly the symplectic integral over E_1 that arises from (A.2). To set up notation, we recall that E_1 is a complex vector space, $\dim_{\mathbb{C}} E_1 = d_1$, with an invariant, hermitian metric (\cdot, \cdot) and

an invariant symplectic form $\tilde{\Omega}$. In terms of holomorphic and anti-holomorphic coordinates v^n and \bar{v}^n on E_1 , $\tilde{\Omega}$ is given by

$$(A.3) \quad \tilde{\Omega} = -\frac{i}{2} (dv, dv) = -\frac{i}{2} d\bar{v}_n \wedge dv^n.$$

If ψ is an element of \mathfrak{h}_0 , then the corresponding vector field $V(\psi)$ on E_1 is described by

$$(A.4) \quad \delta v = \psi \cdot v,$$

or in coordinates, $\delta v^n = \psi_m^n v^m$, and similarly for the conjugate components of $V(\psi)$.

From (A.3) and (A.4), we see that the moment map $\tilde{\mu}$ for the action of H_0 on E_1 is explicitly given by

$$(A.5) \quad \langle \tilde{\mu}, \psi \rangle = \frac{i}{2} (v, \psi \cdot v).$$

By our assumption that (\cdot, \cdot) is invariant under (A.4), ψ is anti-hermitian and the expression in (A.5) is real.

Of course, the complex structure J acts on E_1 as $J(dv) = -i dv$ and $J(d\bar{v}) = +i d\bar{v}$. Thus, since

$$(A.6) \quad S = \frac{1}{2} (\tilde{\mu}, \tilde{\mu}) = \frac{1}{8} (v, v)^2,$$

we see that the canonical one-form $\lambda = J dS$ is given by

$$(A.7) \quad \lambda = -\frac{i}{4} (v, v) ((v, dv) - (dv, v)).$$

On the other hand, from (4.78) we see that λ' on E_1 reduces to

$$(A.8) \quad \lambda' = i(\psi \cdot v, dv).$$

Thus, if we restrict the integral in (A.2) to E_1 and keep only the terms relevant in the limit of large t (after which we set $t = 1$), we just consider the reduced integral

$$(A.9) \quad Z_{red}(\epsilon, s) = \int_{\mathfrak{h}_0 \times E_1} \left[\frac{d\psi}{2\pi} \right] \exp \left[-i(\gamma_0, \psi) - \frac{\epsilon}{2} (\psi, \psi) \right] \times \\ \times \exp [s D\lambda + (1-s) D\lambda'].$$

Of the original integral over the full Lie algebra \mathfrak{h} of H , only the integral over the subalgebra \mathfrak{h}_0 is relevant to the integral over E_1 .

We first perform integral over ψ in \mathfrak{h}_0 . To illustrate the essential behavior of the integral over E_1 , we assume as before that $\mathfrak{h}_0 = \mathbb{R}$ has dimension one. Explicitly, $D\lambda$ and $D\lambda'$ depend on ψ as

$$(A.10) \quad D\lambda = d\lambda + \frac{1}{2} (v, v) (v, \psi \cdot v),$$

and

$$(A.11) \quad D\lambda' = i (\psi \cdot dv, dv) - (\psi \cdot v, \psi \cdot v),$$

so the integral over ψ is purely Gaussian. Upon performing this integral over ψ , we find that Z_{red} is formally given by

$$(A.12) \quad Z_{red}(\epsilon, s) = \int_{E_1} (4\pi A)^{-\frac{1}{2}} \exp \left[s d\lambda + \frac{1}{4} (J, A^{-1} J) \right],$$

where A is defined in terms of the normalized generator T_0 of \mathfrak{h}_0 by

$$(A.13) \quad A = \frac{\epsilon}{2} + (1 - s) (T_0 \cdot v, T_0 \cdot v),$$

and J in \mathfrak{h}_0 is defined by

$$(A.14) \quad J = -i \gamma_0 + \frac{s}{2} (v, v) (v, T_0 \cdot v) T_0 + i (1 - s) (T_0 \cdot dv, dv) T_0.$$

We are now interested in the behavior of the integral in (A.12) for large $|v|$, where the non-compactness of E_1 is essential. So long as $s \neq 0$, then the integrand of (A.12) falls off at least as fast as $\exp[-(v, v)^3]$ for large v , due to the term quartic in v in (A.14) that arises from λ and the term quadratic in v in (A.13) that arises from λ' . Thus, the integral over E_1 is strongly convergent for $s \neq 0$ and depends smoothly on s away from 0. Of course, this integral is also non-Gaussian and cannot be simply expressed using elementary functions.

However, when $s = 0$, the integrand of (A.12) is no longer suppressed exponentially and decays only as a power law at infinity. This behavior arises because the bosonic term of $D\lambda'$ is quadratic in ψ , whereas the bosonic term of $D\lambda$ is linear in ψ . Because the integrand of (A.12) decays only as a power law for $s = 0$, the integral over E_1 does not generally converge. The prefactor proportional to $A^{-1/2}$ decays like $1/|v|$, and for $s = 0$ the measure arising from the quadratic term $(J, A^{-1} J)$ in the exponential of (A.12) is of the form $1/|v|^{d_1} d^{2d_1} v$. Consequently, the integral over E_1 behaves as $\int d^{2d_1} v 1/|v|^{(d_1+1)}$ for large $|v|$ and diverges.

However, we now consider the same analysis as applied to $Q \cdot Z(\epsilon, s)$. By our analysis above, we are only concerned with the potentially dangerous behavior near $s = 0$ and for large $|v|$, for which we must consider

the following integral over E_1 ,

$$(A.15) \quad \left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}d_1} \cdot Z_{red}(\epsilon, s) = \\ = \int_{E_1} \left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}d_1} \left((4\pi A)^{-\frac{1}{2}} \exp \left[s d\lambda + \frac{1}{4} (J, A^{-1} J) \right] \right).$$

To analyze (A.15), we first note that ϵ only appears in the quantity A in (A.13), and A satisfies

$$(A.16) \quad \left(-2\frac{\partial}{\partial\epsilon} + \frac{1}{(1-s)} \frac{\partial^2}{\partial\bar{v}_i \partial v^i}\right) A = 0.$$

Thus, we can rewrite (A.15) as

$$(A.17) \quad \left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}d_1} \cdot Z_{red}(\epsilon, s) = \int_{E_1} \left(-\frac{1}{(1-s)} \frac{\partial^2}{\partial\bar{v}_i \partial v^i}\right)^{\frac{1}{2}d_1} \times \\ \times \left((4\pi A)^{-\frac{1}{2}} \exp \left[s d\lambda + \frac{1}{4} (J, A^{-1} J) \right] \right).$$

We now apply simple scaling arguments to (A.17) to show that this integral is convergent at $s = 0$ and behaves continuously as $s \rightarrow 0$. First, at $s = 0$, we immediately see that this integral behaves for large $|v|$ as $\int d^{2d_1} v \, 1/|v|^{(2d_1+1)}$ and hence is convergent, though just barely.

To discuss the limit $s \rightarrow 0$, we assume s is fixed at a small, non-zero value. All terms involving s which we previously dropped for $s = 0$ now appear in the argument of the exponential in (A.17). For large $|v|$, this argument behaves schematically as

$$(A.18) \quad s |v|^2 (dv, dv) + \frac{(\gamma_0, \gamma_0)}{|v|^2} + s |v|^2 (\gamma_0, T_0) + \\ + \frac{(dv, dv)}{|v|^2} (\gamma_0, T_0) + s^2 |v|^6 + \frac{(dv, dv)^2}{|v|^2}.$$

Since our argument is only a scaling argument, we ignore all signs and constants in writing (A.18), though we do recall that the dominant term $s^2 |v|^6$ leads to an exponential decay of the integrand at large v .

We see three terms in (A.18) which vanish in the limit $s \rightarrow 0$. Of these terms, we can ignore the quadratic term $s|v|^2(\gamma_0, T_0)$, since it is subleading compared to $s^2|v|^6$ for fixed s and large $|v|$.

However, we need to consider the effect of the measure $s^2|v|^4 (dv, dv)^2$, which dominates the measure $(dv, dv)^2/|v|^2$ at $s = 0$ by a relative factor of $s^2|v|^6$. We also need to consider the terms which arise when the derivative $\partial^2/\partial\bar{v}_i \partial v^i$ in (A.17) acts on $\exp(-s^2|v|^6)$ to bring down the term $s^2|v|^4$, which dominates $1/|v|^2$ by the same relative factor $s^2|v|^6$.

These terms lead to contributions depending on s in (A.17) which behave for large $|v|$ as

$$(A.19) \quad \int_{E_1} d^{2d_1}v \frac{1}{|v|^{2d_1+1}} s^{2n} |v|^{6n} \exp(-s^2|v|^6), \quad n = 1, \dots, d_1.$$

Since these integrals only converge for $s \neq 0$, when the integrand is exponentially damped, one might have worried that these terms could cause the limit $s \rightarrow 0$ to be singular. However, we see by scaling that the expression in (A.19) behaves as $s^{+1/3}$ for all n and hence the asymptotic contributions to (A.17) from these terms still go continuously to zero as $s \rightarrow 0$.

Finally, apart from the terms in (A.19) with $n \geq 1$, the integrand of (A.17) is a smooth function $F(v, s)$ of v and s which behaves asymptotically for large $|v|$ as

$$(A.20) \quad F(v, s) \sim \frac{1}{|v|^{2d_1+1}} \exp(-s^2|v|^6).$$

Thus, $F(v, s)$ decays exponentially for $s \neq 0$, is integrable for all s , and is dominated by $F(v, 0)$, which has a pure power law decay at infinity. On general grounds, the integral of $F(v, s)$ over E_1 then depends continuously on s , and, for the purpose of computing $Q \cdot Z(\epsilon)$, we can validly interpolate from λ to λ' on F .

Appendix B. More About Localization at Higher Critical Points: Higher Casimirs

In this appendix, we continue from Section 4.3 our general discussion of non-abelian localization at higher critical points. We recall that we obtained a formal expression for the canonical symplectic integral over F in terms of an integral over the Lie algebra \mathfrak{h}_0 of the stabilizer group

H_0 ,

$$(B.1) \quad Z(\epsilon) = \frac{1}{\text{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi} \right] \det \left(\frac{\psi}{2\pi} \Big|_{E_0} \right) \det \left(\frac{\psi}{2\pi} \Big|_{E_1} \right)^{-1} \times \\ \times \exp \left[-i \langle \gamma_0, \psi \rangle - \frac{\epsilon}{2} (\psi, \psi) \right].$$

As we discussed in Section 4.3, this integral generally fails to converge when the ratio of determinants in the integrand has singularities in \mathfrak{h}_0 . In the special case $H_0 = U(1)$, relevant for higher critical points of $SU(2)$ Yang-Mills theory, we deal with this problem by computing not $Z(\epsilon)$ itself but a higher derivative $Q \cdot Z(\epsilon)$, where $Q \equiv Q(\partial/\partial\epsilon)$ is a differential operator which we choose so that the action of Q on the integrand of (B.1) brings down sufficient powers of (ψ, ψ) to cancel any poles that would otherwise appear.

However, if we consider higher critical points of Yang-Mills theory with general gauge group G , then the rank of H_0 can be arbitrary, and the determinants in (B.1) cannot generally be expressed as a functions of only the quadratic invariant (ψ, ψ) . Consequently, we cannot simply differentiate $Z(\epsilon)$ with respect to ϵ to cancel the poles in (B.1).

Nevertheless, by applying some simple ideas about the localization construction, we can generalize our discussion in Section 4.3 to the case that H_0 has higher rank. As in Section 4.1, we recall the form of the localization integral:

$$(B.2) \quad Z(\epsilon) = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times X} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - \frac{\epsilon}{2} (\phi, \phi) \right].$$

In the case of Yang-Mills theory, $H = \mathcal{G}(P)$ and $X = \mathcal{A}(P)$ in the notation of Section 2.

Let us consider what natural generalizations of (B.2) exist. Of the terms appearing in (B.2), the quantity $\Omega - i \langle \mu, \phi \rangle$ is distinguished as an element of the equivariant cohomology ring of X , since it represents the equivariant extension of the symplectic form on X . However, nothing really distinguishes the quadratic function $-\frac{1}{2}(\phi, \phi)$ among all invariant polynomials of ϕ , and we are free to consider a general symplectic integral over $\mathfrak{h} \times X$ of the form

$$(B.3) \quad Z[V] = \frac{1}{\text{Vol}(H)} \int_{\mathfrak{h} \times X} \left[\frac{d\phi}{2\pi} \right] \exp \left[\Omega - i \langle \mu, \phi \rangle - V(\phi) \right].$$

Here $V(\phi)$ is any invariant polynomial on \mathfrak{h} such that the integral over \mathfrak{h} remains convergent at large ϕ . We can take

$$(B.4) \quad V(\phi) = \sum_j \epsilon_j C_j(\phi),$$

where C_j are the Casimirs of H – the homogeneous generators of the ring of invariant polynomials on \mathfrak{h} – and ϵ_j are parameters. The standard localization technique can be applied to evaluate this integral. The fact that V is not quadratic in ϕ leads to no special complications.

In the case of Yang-Mills theory on a Riemann surface Σ with symplectic form ω , we would write

$$(B.5) \quad V(\phi) = \sum_{j=1}^r \epsilon_j \int_{\Sigma} \omega \cdot C_j(\phi).$$

We assume that the gauge group G has rank r , and now $C_j(\phi)$ are the Casimirs of G . We associate to each generator a corresponding coupling ϵ_j . If we want to compare to standard methods of studying two-dimensional Yang-Mills theory by cut and paste methods, we should integrate over ϕ to express the theory in terms of the gauge field (and noninteracting fermions) alone. Of course, if $V(\phi)$ is not quadratic, we can no longer perform the integral over ϕ in (B.3) as a Gaussian integral. Instead, if we abstractly introduce the Fourier transform

$$(B.6) \quad \exp[-\widehat{V}(\phi^*)] \equiv \int_{\mathfrak{h}} \left[\frac{d\phi}{2\pi} \right] \exp[-i \langle \phi^*, \phi \rangle - V(\phi)],$$

which is an invariant function of ϕ^* in the dual algebra \mathfrak{h}^* , then the generalized symplectic integral over X takes the form

$$(B.7) \quad Z[V] = \frac{1}{\text{Vol}(H)} \int_X \exp(\Omega - \widehat{V}(\mu)).$$

In the case of Yang-Mills theory, we recall that $\mu = F_A$. So in that case, (B.7) corresponds to a generalization of Yang-Mills theory in which the action is not the usual $\text{Tr } f^2$ (with $f = \star F$) but $\text{Tr } \widehat{V}(f)$, for some more general function \widehat{V} . The partition function of this generalized Yang-Mills theory can be computed by the usual cut and paste methods [57]. If G is simply-connected and we apply the same normalization conventions as we used in (4.41) for the case $G = SU(2)$, the generalized partition function is

$$(B.8) \quad Z[V] = (\text{Vol}(G))^{2g-2} \sum_{\mathcal{R}} \frac{1}{\dim(\mathcal{R})^{2g-2}} \exp(-V'(\mathcal{R})),$$

where $V'(\mathcal{R})$ is the energy of the representation \mathcal{R} . (We are taking the area of Σ to be 1; for a general area α , the exponential factor would be $\exp(-\alpha V'(\mathcal{R}))$.) To compute the energy $V'(\mathcal{R})$, we start with the action $\widehat{V}(f)$ and compute the canonical momentum $\Pi = \partial \widehat{V} / \partial f$. The Hamiltonian, whose eigenvalue is the energy, is then $H = f\Pi - \widehat{V}(f)$, which must be extremized with respect to f and regarded as a function of Π . Thus, $H(\Pi)$ is the Legendre transform of $\widehat{V}(f)$. After computing $H(\Pi)$, Π is interpreted as the generator of the group G and taken to act on the representation \mathcal{R} to get the energy $V'(\mathcal{R})$. Since the Legendre transform is a semiclassical approximation to the Fourier transform, the Legendre transform approximately undoes the Fourier transform, and hence $H(\Pi) = V(\Pi) + \text{lower order terms}$. As discussed in [20], if the representation \mathcal{R} has highest weight h , the precise formula needed to match with the localization computation is $V'(\mathcal{R}) = V(h + \delta)$, where the constant δ is half the sum of positive roots of the Lie algebra of G . This formula incorporates the difference between the Legendre transform and the Fourier transform and other possible quantum corrections.

To generalize what we said in Section 4.3, we want to find a polynomial $F(C_j)$ of the Casimirs of H which when restricted to \mathfrak{h}_0 is divisible by the troublesome factor in the denominator, namely

$$(B.9) \quad w(\psi) = \det \left(\psi / 2\pi \Big|_{E_1} \right).$$

Then $Q = F(-\partial / \partial \epsilon_j)$ is a differential operator that when acting on $\exp(-V)$ will produce the factor F and cancel the denominator. Thus, Q generalizes the operator $\partial^{g-1} / \partial \epsilon^{g-1}$ that we used in Section 4.3 for two-dimensional $SU(2)$ gauge theory in genus g .

The troublesome factor w is an invariant polynomial on the Lie algebra of \mathfrak{h}_0 or equivalently, a polynomial on the maximal torus of H_0 that is invariant under the Weyl group of H_0 . This polynomial can be extended, though not canonically, to a polynomial w' on the maximal torus of H . We can pick the extension to be invariant under the Weyl group of H_0 but not necessarily under the Weyl group of H . However, by multiplying w' by all its conjugates under the Weyl group of H , we make a polynomial \tilde{w} on the maximal torus of H that is invariant under the Weyl group of H , and whose restriction to H_0 is divisible by w . The

Weyl-invariant polynomial \tilde{w} corresponds to the polynomial $F(C_j)$ of the Casimirs that was used in the last paragraph.

Finally, let us make this more explicit for Yang-Mills theory. The denominator factor in (B.8) that we need to cancel is $\dim(\mathcal{R})^{2g-2}$, so it suffices to know that $\dim(\mathcal{R})^2$ is a polynomial of the Casimirs. This can be proved using the Weyl character formula, discussed in [65, §123], which provides a general formula for $\dim(\mathcal{R})$. Parametrizing the representation \mathcal{R}_h by a highest weight h ,

$$(B.10) \quad \dim(\mathcal{R}_h) = \prod_{\beta > 0} \frac{(\beta, h + \delta)}{(\beta, \delta)}.$$

The product in (B.10) runs over the positive roots β , and we recall that δ is a constant, equal to half the sum of the positive roots. We regard this as a function of $h' = h + \delta$.

The formula (B.10) exhibits a polynomial function d on the Cartan subalgebra of the Lie algebra \mathfrak{g} of G such that $\dim(\mathcal{R}_h) = d(h')$. The polynomial d is not strictly invariant under the action of the Weyl group on h' , but is invariant up to sign, so d^2 is Weyl invariant. As such, d^2 extends to an invariant polynomial on all of \mathfrak{g} , and thus a polynomial in the Casimirs. Finally, we observe that the shift $h \rightarrow h' = h + \delta$ is the same renormalization that we introduced for the potential $V'(\mathcal{R}_h) = V(h + \delta)$, so that by differentiating with respect to the couplings of each Casimir in V' we can cancel the denominator $\dim(\mathcal{R})^{2g-2}$.

Appendix C. A Few Additional Generalities About Equivariant Cohomology

Following the discussion in Section 5.3, we discuss in this appendix the identification of the H -equivariant cohomology of N_0 with the H_0 -equivariant cohomology of \mathcal{M} , a fact which fundamentally leads to the correspondence (5.110).

To start, we find it useful to employ another topological model of equivariant cohomology, explained for instance in Chapter 1 of [56]. In this model, if X is any topological space on which a group H acts, the H -equivariant cohomology ring of X is defined as the ordinary cohomology ring of the fiber product $X_H = X \times_H EH$, where EH is any contractible space on which H acts freely. Such an EH always exists, and the choice of EH does not matter, since EH is unique up to H -equivariant homotopies. Thus, $H_H^*(X) = H^*(X_H)$.

As a simple example, if H acts freely on X , implying that X is a principal H -bundle over X/H , then X_H is equivalent to a product $X_H = (X/H) \times EH$. Since EH is contractible, we see that $H_H^*(X) = H^*(X/H)$, a fact we applied in our discussion of two-dimensional Yang-Mills theory.

At the opposite extreme, if H acts trivially on X , then X_H is also a product $X_H = X \times BH$, where $BH = EH/H$ is the classifying space associated to the group H . In this case, $H_H^*(X) = H^*(X) \otimes H^*(BH)$. However, by the definition of equivariant cohomology above, the ordinary cohomology of BH is the H -equivariant cohomology of a point, so that $H_H^*(X) = H^*(X) \otimes H_H^*(pt)$. For the latter factor, our description of the Cartan model in Section 4.1 clearly identifies $H_H^*(pt)$ with the ring of invariant functions on the Lie algebra \mathfrak{h} of H .

We want the case in which X is a fiber bundle over \mathcal{M} with fiber H/H_0 for some H . H acts on the fibers, with fixed subgroup H_0 . Now suppose that there exists a principal bundle $Y \rightarrow \mathcal{M}$, with fibers H , and the following properties. We suppose that $H \times H_0$ acts on Y , with H acting on the fibers on the left and H_0 on the right. We also suppose that $Y/H_0 = X$.

In this situation, H and H_0 both act freely on Y , the quotient Y/H being \mathcal{M} and the quotient Y/H_0 being X . Moreover, H_0 acts trivially on X .

We can now argue as follows. First, $H_{H \times H_0}^*(Y) = H_H^*(X)$, as H_0 acts freely on Y with quotient X . On the other hand $H_{H \times H_0}^*(Y) = H_{H_0}^*(\mathcal{M})$ because H acts freely on Y with quotient \mathcal{M} . Finally, as H_0 acts trivially on \mathcal{M} , $H_{H_0}^*(\mathcal{M}) = H^*(\mathcal{M}) \otimes H_{H_0}^*(pt)$. Putting these facts together, we have our desired result that $H_H^*(X) = H^*(\mathcal{M}) \otimes H_{H_0}^*(pt)$.

In general such a Y only exists rationally (which is good enough for de Rham cohomology), but for our problem with Chern-Simons theory on a Seifert manifold, a natural Y can be constructed as follows.

First of all, over any symplectic manifold $\overline{\mathcal{A}}$, a “prequantum line bundle” \mathcal{L} is a unitary line bundle with connection whose curvature is the symplectic form. For Chern-Simons theory, \mathcal{L} exists and is unique up to isomorphism as $\overline{\mathcal{A}}$ is an affine space. We let \mathcal{L}_0 be the bundle of unit vectors in \mathcal{L} . So \mathcal{L}_0 is a circle bundle over $\overline{\mathcal{A}}$.

In general, any connected Lie group of symplectomorphisms of a symplectic manifold that has an invariant moment map lifts to an action on the prequantum line bundle. For Chern-Simons theory on a Seifert

manifold, the group \mathcal{G} of gauge transformations does not have a moment map, but its central extension $\tilde{\mathcal{G}}$ does. We recall that $\tilde{\mathcal{G}}$ is an extension of \mathcal{G} by a subgroup $U(1)_Z$ that acts trivially on $\overline{\mathcal{A}}$ but has moment map 1. Having a moment map, $\tilde{\mathcal{G}}$ acts on \mathcal{L} , and hence on \mathcal{L}_0 . $U(1)_Z$ acts by rotating the fibers of the fibration $\mathcal{L}_0 \rightarrow \overline{\mathcal{A}}$. This action is free.

Finally, the Hamiltonian group \mathcal{H} that we really use for our quantization is a semidirect product of $\tilde{\mathcal{G}}$ with a $U(1)_R$ that rotates the fibers of the Seifert fibration. $U(1)_R$ acts on \mathcal{L} and \mathcal{L}_0 , but not freely. To get the desired space Y on which $U(1)_R$ acts freely, we simply take $Y = U(1) \times \mathcal{L}_0$, where $U(1)_R$ acts by rotation on $U(1)$ together with its natural action on \mathcal{L}_0 . So in fact $H_0 = U(1)_R \times U(1)_Z$ acts freely on Y .

We now want to restrict this construction from $\overline{\mathcal{A}}$, the space of all connections, to N_0 , the space of flat connections, whose quotient N_0/H is \mathcal{M} , the moduli space of gauge-equivalence classes of flat connections. We let Y_0 be the restriction to N_0 of the fibration $Y \rightarrow \overline{\mathcal{A}}$. So $H \times H_0$ acts on Y_0 ; H_0 acts freely on Y_0 with quotient N_0 , and H acts freely on Y_0 with quotient \mathcal{M} . Finally, H_0 acts trivially on \mathcal{M} . With these facts at hand, the general argument presented above shows that $H_H^*(N_0) = H^*(\mathcal{M}) \otimes H_{H_0}^*(pt)$.

Appendix D. More About Localization at Higher Critical Points: Localization Over a Nontrivial Moduli Space

In this appendix, we consider the general case that our abstract model for F is fibered over a non-trivial moduli space \mathcal{M} . Our goal is to compute the equivariant cohomology class on \mathcal{M} which is produced by the canonical symplectic integral over F ,

$$(D.1) \quad I(\psi) = \frac{1}{\text{Vol}(H)} \int_{\tilde{F}} \left[\frac{d\phi}{2\pi} \right] \exp [tD\lambda], \quad \tilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F, \quad \psi \in \mathfrak{h}_0.$$

We begin with some geometric preliminaries. Very briefly, we recall that we model F as a vector bundle with fiber $\mathfrak{h}^\perp \oplus E_1$ over a homogeneous base H/H_0 . Here $\mathfrak{h}^\perp = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0$, and explicitly,

$$(D.2) \quad F = H \times_{H_0} (\mathfrak{h}^\perp \oplus E_1).$$

To describe the total space N of the fiber bundle $F \rightarrow N \rightarrow \mathcal{M}$, we introduce a principal H -bundle P_H over \mathcal{M} . Besides the given action

of H on P_H , we assume that P_H also admits a free action of H_0 which commutes with the action of H . As a result, we can describe the bundle N concretely in terms of P_H as

$$(D.3) \quad N = P_H \times_{H_0} (\mathfrak{h}^\perp \oplus E_1).$$

Upon setting $P_H = H$, where H acts on the right and H_0 acts on the left, this model for N reduces to the model for F itself, with \mathcal{M} being a point.

Of course, the key ingredient in our localization computation is to choose a good representative of the canonical localization form λ on N . As in Section 4.3, we introduce another localization form λ' which (under the same caveats as in Section 4.3 and Appendix A) is homotopic to λ on N and takes the form

$$(D.4) \quad \lambda' = \lambda'_\perp + \lambda'_{E_0} + \lambda'_{E_1},$$

with

$$(D.5) \quad \begin{aligned} \lambda'_\perp &= (\gamma, \theta), \\ \lambda'_{E_0} &= -i (\theta_{E_0}, g\phi g^{-1} + i\mathcal{R}(\theta)), \quad \mathcal{R}(\theta) = d\theta - \frac{1}{2}[\theta, \theta], \\ \lambda'_{E_1} &= i \left((g\phi g^{-1})_{\mathfrak{h}_0} \cdot v, dv - \theta_{\mathfrak{h}_0} \cdot v \right). \end{aligned}$$

In these expressions, we recall that γ is an element of \mathfrak{h}^\perp , g is an element of H , ϕ is an element of \mathfrak{h} , and v is an element of the vector space E_1 . Finally, θ is now a connection on the principal H -bundle P_H . In particular, θ is a globally-defined one-form on P_H . As usual, we let $\mathcal{R}(\theta)$ denote the curvature of θ .

Our choice for λ' is precisely analogous to the choice we made in Section 4.3 in the case that $P_H = H$, and in (D.4) we have simply grouped the terms in λ' in a natural way for the localization computation. The only term present in (D.5) which was not present in Section 4.3 is the term involving the curvature $\mathcal{R}(\theta)$ in λ'_{E_0} . The curvature of θ is a horizontal form on P_H , meaning that it is annihilated by contraction with the vector fields $V(\phi)$ which generate the action of H on P_H , so this curvature term could not appear when \mathcal{M} was only a point. Equivalently, if the connection θ takes the global form $\theta = dg g^{-1}$ as in Section 4.3, then $\mathcal{R}(\theta)$ vanishes identically.

In (D.4) and (D.5) we have written λ' as an invariant form on the direct product $P_H \times (\mathfrak{h}^\perp \oplus E_1)$, but one can check exactly as in Section

4.3 that λ' descends under the quotient by H_0 to an invariant form on N .

Although λ' is globally defined on N , we have written λ' in coordinates on P_H with respect to a local trivialization of this bundle about some point m on the base \mathcal{M} . The integral we perform will be an integral over the fiber F_m above this point m , and since m is arbitrary, this local computation suffices to determine the cohomology class on \mathcal{M} that arises after we perform the integral over all the fibers of $F \rightarrow N \rightarrow \mathcal{M}$. In particular, upon pulling θ back to the fiber F_m , θ takes the canonical form,

$$(D.6) \quad \theta|_{F_m} = dg g^{-1}.$$

However, since the curvature $\mathcal{R}(\theta)$ can be non-zero, in general $d\theta \neq \frac{1}{2}[\theta, \theta]$ at points in the fiber over m .

At this point, we repeat our earlier computation of $D\lambda'$, allowing for the presence of the curvature $\mathcal{R}(\theta)$. We find

$$(D.7) \quad \begin{aligned} D\lambda'_\perp &= (d\gamma, \theta) - i(\gamma, \phi + i d\theta), \\ D\lambda'_{E_0} &= -i(d\theta_{E_0}, \phi + i \mathcal{R}(\theta)) + i(\theta_{E_0}, [\theta, \phi + i \mathcal{R}(\theta)]) - \\ &\quad - (\phi_{E_0}, \phi + i \mathcal{R}(\theta)), \\ D\lambda'_{E_1} &= i(\phi_{\mathfrak{h}_0} \cdot dv, dv) - \left(\phi_{\mathfrak{h}_0} \cdot v, (\phi + i \mathcal{R}(\theta))_{\mathfrak{h}_0} \cdot v \right) + \mathcal{X}, \end{aligned}$$

with

$$(D.8) \quad \mathcal{X} = i \left([\theta, \phi]_{\mathfrak{h}_0} \cdot v, dv \right) + i \left(\phi_{\mathfrak{h}_0} \cdot v, \frac{1}{2} [\theta, \theta]_{\mathfrak{h}_0} \cdot v \right) \pmod{\theta_{\mathfrak{h}_0}}.$$

As before, in writing these expressions we make the change of variable from ϕ to $g\phi g^{-1}$ at the end of the calculation to simplify the result. Also, the terms appearing in \mathcal{X} are at least of cubic order in the “massive” variables θ , v , and dv and so are irrelevant in the limit $t \rightarrow \infty$. Finally, we are free to work modulo terms involving $\theta_{\mathfrak{h}_0}$ since $D\lambda'$ is a pullback from the quotient $P_H \times_{H_0} (\mathfrak{h}^\perp \oplus E_1)$.

We now compute directly the integral below in the limit $t \rightarrow \infty$,

$$(D.9) \quad I(\phi_{\mathfrak{h}_0}) = \frac{1}{\text{Vol}(H)} \int_{\tilde{F}_m} \left[\frac{d\phi}{2\pi} \right] \exp [tD\lambda'_\perp + tD\lambda'_{E_0} + tD\lambda'_{E_1}],$$

$$\tilde{F}_m = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F_m.$$

This integral behaves essentially the same as the integral in Section 4.3, so we will be brief.

We first consider the integral over E_1 , which we perform as a Gaussian integral using the terms from $tD\lambda'_{E_1}$ in the large t limit. Explicitly, the integral over E_1 is given by

$$(D.10) \quad \int_{E_1} \exp \left[it (\phi_{\mathfrak{h}_0} \cdot dv, dv) - t (\phi_{\mathfrak{h}_0} \cdot v, (\phi + i\mathcal{R}(\theta))_{\mathfrak{h}_0} \cdot v) + t\mathcal{X} \right].$$

Since \mathcal{X} is of at least cubic order in the massive variables θ , v , and dv , this term can be dropped from the integrand when t is large. Keeping the other terms quadratic in v and dv in (D.10), the Gaussian integral over E_1 immediately produces

$$(D.11) \quad \det \left(\frac{1}{2\pi} (\phi_{\mathfrak{h}_0} + i\mathcal{R}(\theta)_{\mathfrak{h}_0}) \Big|_{E_1} \right)^{-1}.$$

We now integrate over both γ and ϕ in $\mathfrak{h}^\perp = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0$. We see from (D.7) that γ still appears only linearly in $tD\lambda'$, so the integral over γ produces a delta-function of ϕ_\perp , where ϕ_\perp denotes the component of ϕ in \mathfrak{h}^\perp . As is evident from the form of $tD\lambda'_\perp$, this delta-function sets $\phi_\perp = -id\theta_\perp$. (As in Section 4.3, the factors of t cancel between the integral over γ and the integral over ϕ_\perp .)

We are left to integrate over ϕ_{E_0} and over the base H/H_0 of F_m . Of course, upon Taylor expanding the exponential $\exp(d\gamma, \theta)$ from $D\lambda'_\perp$ to produce the measure for γ , we also produce the canonical measure on the tangent directions to H/H_0 lying in \mathfrak{h}^\perp . So infinitesimally we have only to integrate over the remaining tangent directions to H/H_0 which lie in E_0 in addition to ϕ_{E_0} .

So we are left to integrate over E_0 using the terms in $tD\lambda'_{E_0}$. This integral takes the form

$$(D.12) \quad \int_{E_0} \exp \left[-it (\theta_{E_0}, [\phi_{\mathfrak{h}_0} + i\mathcal{R}(\theta)_{\mathfrak{h}_0}, \theta_{E_0}]) + t (\mathcal{R}(\theta)_{E_0}, \mathcal{R}(\theta)_{E_0}) \right] \times \\ \times \exp \left[-2it (\mathcal{R}(\theta)_{E_0}, \phi_{E_0}) - t (\phi_{E_0}, \phi_{E_0}) \right].$$

In deducing (D.12), we have expanded and simplified various terms in $D\lambda'_{E_0}$ in (D.7). For instance, the curvature term $(\mathcal{R}(\theta)_{E_0}, \mathcal{R}(\theta)_{E_0})$ arises from the linear combination of terms $(d\theta_{E_0}, \mathcal{R}(\theta)) - (\theta_{E_0}, [\theta, \mathcal{R}(\theta)])$ in $D\lambda'_{E_0}$. To see this, we rewrite this expression as $(d\theta_{E_0} - [\theta, \theta_{E_0}], \mathcal{R}(\theta)_{E_0}) \equiv (\mathcal{R}(\theta)_{E_0}, \mathcal{R}(\theta)_{E_0})$, where “ \equiv ” indicates that the equality holds modulo $\theta_{\mathfrak{h}_0}$ and θ_\perp , which is good enough since these forms do not contribute to the integral over E_0 .

In writing (D.12), we also note that when we set $\phi_\perp = -id\theta_\perp$ in $D\lambda'_{E_0}$, we effectively cancel similar terms in $D\lambda'_{E_0}$ which involve the

components of the curvature $\mathcal{R}(\theta)$ in \mathfrak{h}^\perp . So $\mathcal{R}(\theta)_\perp$ does not appear in (D.12).

We first perform the Gaussian integral over ϕ_{E_0} in (D.12). The result of this integral produces a term proportional to $\exp[-t(\mathcal{R}(\theta)_{E_0}, \mathcal{R}(\theta)_{E_0})]$ which precisely cancels the term quadratic in the curvature $\mathcal{R}(\theta)_{E_0}$ in the first line of (D.12). Consequently, once we collect factors of t and 2π exactly as in Section 4.3, the term quadratic in θ_{E_0} in (D.12) produces another determinant,

$$(D.13) \quad \det \left(\frac{1}{2\pi} (\phi_{\mathfrak{h}_0} + i \mathcal{R}(\theta)_{\mathfrak{h}_0}) \Big|_{E_0} \right).$$

Including the factor $\text{Vol}(H)/\text{Vol}(H_0)$ that arises from the integral over H/H_0 and setting $\phi_{\mathfrak{h}_0} \equiv \psi$ for notational simplicity, we find our final result for the integral in (D.9),

$$(D.14) \quad I(\psi) = \frac{1}{\text{Vol}(H_0)} \det \left(\frac{1}{2\pi} (\psi + i \mathcal{R}(\theta)_{\mathfrak{h}_0}) \Big|_{E_0} \right) \times \\ \times \det \left(\frac{1}{2\pi} (\psi + i \mathcal{R}(\theta)_{\mathfrak{h}_0}) \Big|_{E_1} \right)^{-1}.$$

Since both E_0 and E_1 are representations of H_0 , the associated bundles $P_H \times_{H_0} E_0$ and $P_H \times_{H_0} E_1$ determine H_0 -equivariant bundles over \mathcal{M} once we divide by the action of H on P_H . The determinants appearing in (D.14) are then the Chern-Weil representatives of the H_0 -equivariant Euler classes of these bundles.

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