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FLAT SPACETIMES WITH COMPACT HYPERBOLIC CAUCHY SURFACES

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Abstract

Given a closed hyperbolic n-manifold M, we study the flat Lorentzian structures on $M \times \mathbb{R}$ such that $M \times \{0\}$ is a Cauchy surface. We show there exist only two maximal structures sharing a fixed holonomy (one future complete and the other one past complete). We study the geometry of those maximal spacetimes in terms of cosmological time. In particular, we study the asymptotic behaviour of the level surfaces of the cosmological time. As a by-product, we get that no affine deformation of the hyperbolic holonomy $\rho : \pi_1(M) \to \mathrm{SO}(n,1)$ of M acts freely and properly on the whole Minkowski space. The present work generalizes the case $n = 2$ treated by Mess, taking from a work of Benedetti and Guadagnini the emphasis on the fundamental rôle played by the cosmological time. In the last sections, we introduce measured geodesic stratifications on M, that in a sense furnish a good generalization of measured geodesic laminations in any dimension and we investigate relationships between measured stratifications on M and Lorentzian structures on $M \times \mathbb{R}$.

1. Introduction

In this paper, we study flat $(n+1)$-spacetimes Y admitting a Cauchy surface diffeomorphic to a compact hyperbolic n-manifold M. Roughly speaking, we show how to construct a canonical future complete one, Y_ρ, among all such spacetimes sharing the same holonomy ρ. We study the geometry of Y_ρ in terms of its *canonical cosmological time* (CT). In particular, we study the asymptotic behaviour of the level surfaces of cosmological time.

The present work generalizes the case $n = 2$ treated in [**14**], taking from [**6**], the emphasis on the fundamental rôle played by canonical cosmological time. In particular, Mess showed that if F is a closed surface of genus $g \geq 2$, then the linear holonomies of the Lorentzian flat

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structures on $\mathbb{R} \times F$ such that $\{0\} \times F$ is a spacelike surface are faithful and discrete representations of $\pi_1(F)$ into $\mathrm{SO}^+(2,1)$.

Moreover, he proved that every representation $\varphi : \pi_1(F) \to \mathrm{Iso}\,(\mathbb{M}^3)$ whose linear part is faithful and discrete is the holonomy for some Lorentzian structure on $\mathbb{R} \times F$ such that $\{0\} \times F$ is a spacelike surface. In particular, he showed that there is a unique maximal future complete convex domain of $\mathbb{M}^3$, called *domain of dependence*, which is $\varphi(\pi_1(F))$-invariant such that the quotient is a globally hyperbolic manifold homeomorphic to $\mathbb{R} \times F$ with regular cosmological time T.

If we fix the linear holonomy $f : \pi_1(F) \to \mathrm{SO}^+(2,1)$, these domains (and so the affine deformations of the representation f) are parametrized by measured geodesic laminations on $\mathbb{H}^2/f(\pi_1(F))$. The link between domains of dependence and measured geodesic laminations is the *Gauss map* of the CT-level surfaces.

On the other hand, Benedetti and Guadagnini noticed in [**6**] that the *singularity in the past* of a domain of dependence is a real tree which is dual to the lamination. Moreover, they argued that the action of $\pi_1(F)$ on the CT-level surface $\widetilde{S}_a = T^{-1}(a)$ tends in the Gromov sense to the action of $\pi_1(F)$ on the singularity for $a \to 0$ and to the action of $\pi_1(F)$ on the hyperbolic plane $\mathbb{H}^2$ for $a \to \infty$. Thus, the asymptotic states of the cosmological time materialize the duality between geometric real trees (realized by the singularity in the past) and the measured geodesic laminations in the hyperbolic surface F, according to Skora Theorem [**19**].

In this paper, we try to generalize this approach in higher dimensions. By extending the Mess' method to any dimension n, we associate to each Y_ρ a Γ-invariant *geodesic stratification* of $\mathbb{H}^n$ (see Section 4 for the definition), and we discuss the duality between geodesic stratifications and singularities in the past. In particular, we recover this duality at least for an interesting class of so called spacetimes with *simplicial singularity in the past*.

We briefly describe the contents of this paper in Section 2. We first recall some basic facts about Lorentzian spacetimes and hyperbolic manifolds, then we give quite an articulated statement of the main results. Some assertions will be fully described and proved in the later sections. In the last section, we shall discuss some related questions and open problems.

2. Preliminaries and Statement of the Main Theorem

In this section, we recall a few basic facts about Lorentzian geometry, geometry of the Minkowski space, and hyperbolic space. An exhaustive treatment about Lorentzian geometry, including a careful analysis of global causality questions, can be found in [**12**] or in [**5**]. For an introduction to hyperbolic space, see [**7**]. In the last part of this section, we state the main theorem which we shall prove in the following sections.

Spacetimes. A *Lorentzian* $(n+1)$*-manifold* (M, η) is given by a smooth $(n+1)$-manifold M (this includes the topological assumption that M is metrizable and with countable basis) and a symmetric non-degenerate 2-form η with signature equal to $(n, 1)$. A basis $(e_0, \ldots, e_n)$ of a tangent space T_pM is *orthonormal* if the matrix of η_p with respect to this basis is $diag(-1, 1, \ldots, 1)$. A tangent vector v is *spacelike* (resp. *timelike*, *null*, *non-spacelike*) if $\eta(v, v)$ is positive (resp. negative, zero, non-positive). A C^1-curve in M is *chronological* (resp. *causal*) if the speed vector is timelike (resp. non-spacelike).

Let M be a Lorentzian connected $(n+1)$-manifold. Consider the set $\mathcal{C}$ in the tangent bundle of M formed by timelike tangent vectors: it turns out that either $\mathcal{C}$ is connected or it has two connected components. In the latter case, we say that M is *time-orientable*. A time-orientation is a choice of one of these components. A **spacetime** is a Lorentzian connected time-orientable manifold equipped with a time-orientation. Let M be a spacetime and $\mathcal{C}_+$ be the chosen component. A non-spacelike tangent vector is *future-directed* (resp. *past directed*) if it is not zero and lies (resp. does not lie) in the closure of $\mathcal{C}_+$. A causal curve is *future-directed* if its speed vector is future directed. For $p \in M$, *the future of* p (resp. *the past*) is the set $\mathrm{I}^+(p)$ (resp. $\mathrm{I}^-(p)$) of points in M which are future endpoints (resp. past endpoints) of chronological curves which start at p. If we replace chronological curves with causal curves, we obtain the *causal future* $\mathrm{J}^+(p)$ (resp. *causal past* $\mathrm{J}^-(p)$) of p.

Let $\gamma : I \to M$ be a causal curve. The Lorentzian length of γ is

$$\ell(\gamma) := \int_I \sqrt{-\eta(\dot\gamma(t), \dot\gamma(t))}\mathrm{d}t.$$

Given $p \in M$ and $q \in \mathrm{J}^-(p)$, the Lorentzian distance between p and q is

$$d(p, q) := \{\sup \ell(\gamma) | \gamma \text{ is a causal curve whose endpoints are } p \text{ and } q\}.$$

For every $p \in M$, we can take

$$\tau(p) := \sup\{d(p, q) | q \in \mathrm{J}^-(p)\}.$$

This defines a function

$$\tau : M \to \mathbb{R} \cup \{+\infty\}$$

which can be very degenerate (for instance $\tau \equiv +\infty$ on the Minkowski space $\mathbb{M}^{n+1}$). We are interested in those spacetimes for which the function τ is a **canonical cosmological time** (CT): this means that τ is finite and increasing on every directed causal curve (i.e., τ is a *time*) and is *regular*, that is τ tends to 0 over every inextensible past directed causal curve. Spacetimes with regular cosmological time have been pointed out and studied in [**1**]. We recall the following general result.

Theorem 2.1. *Suppose that M is a spacetime with a regular cosmological time τ. Then, τ is twice differentiable almost everywhere. Moreover, for every $p \in M$, there is an inextensible in the past timelike geodesic $\gamma : (0, \tau(p)] \to M$ which has future endpoint at p and such that $\tau(\gamma(t)) = t$. The level surfaces $S_a = \tau^{-1}(a)$ are future Cauchy surfaces.*

A **Cauchy surface** is an embedded hypersurface S of M such that every inextensible causal curve in M intersects S only in one point.

Finally, a spacetime M is **globally hyperbolic** if for every $p, q \in M$ the set $\mathrm{J}^+(p) \cap \mathrm{J}^-(q)$ is compact. It is the strongest global causality assumption and implies strict constraints on the topology of M. In particular, in [**11**], it is shown that M is globally hyperbolic if and only if there is a Cauchy surface S in M, and in this case M is homeomorphic to $\mathbb{R}\times S$.

Minkowski space. The *Minkowski $(n+1)$-spacetime* $\mathbb{M}^{n+1}$ is the flat simply connected complete Lorentzian $(n+1)$-manifold (it is unique up to isometry). Let $(x_0, \ldots, x_n)$ be the natural coordinates on $\mathbb{R}^{n+1}$, then a concrete model for $\mathbb{M}^{n+1}$ is $\mathbb{R}^{n+1}$ provided with the Lorentzian form

$$\eta = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2.$$

In what follows, we shall always use this model. Notice that the frame $\left(e_i = \frac{\partial}{\partial x_i}\right)_{i=0,\ldots,n}$ is parallel and orthonormal. Thus, we can identify in a standard way the tangent space $(T_x\mathbb{M}^{n+1}, \eta_x)$ with $\mathbb{R}^{n+1}$ provided with the inner product $\langle \cdot, \cdot \rangle$ defined by the rule

$$\langle v, w \rangle = -v_0 w_0 + v_1 w_1 + \cdots + v_n w_n.$$

Minkowski space is an orientable and time-orientable Lorentzian manifold. Let us put on it the standard orientation (such that the canonical basis $(e_0, \ldots, e_n)$ is positive) and the standard time-orientation (a timelike tangent vector v is future directed if $\langle v, e_0 \rangle < 0$). By *orthonormal affine coordinates*, we mean a set $(y_0, \ldots, y_n)$ of affine coordinates on $\mathbb{M}^{n+1}$, such that the frame $\{\frac{\partial}{\partial y_i}\}$ is *orthonormal and positive* and the vector $\frac{\partial}{\partial y_0}$ is *future directed*.

Consider the isometry group of $\mathbb{M}^{n+1}$. It is easy to see that f is an isometry of $\mathbb{M}^{n+1}$ if and only if it is affine and $\mathrm{d}f(0)$ belongs to the group $\mathrm{O}\,(n, 1)$ of linear transformations of $\mathbb{R}^{n+1}$ which preserve the inner product $\langle \cdot, \cdot \rangle$. It follows that the group of isometries of $\mathbb{M}^{n+1}$ is generated by $\mathrm{O}\,(n, 1)$ and the group of translations $\mathbb{R}^{n+1}$. Furthermore, $\mathbb{R}^{n+1}$ is a normal subgroup of $\mathrm{Iso}\,(\mathbb{M}^{n+1})$ (in fact, it is the kernel of the map $\mathrm{Iso}\,(\mathbb{M}^{n+1}) \ni f \mapsto \mathrm{d}f(0) \in \mathrm{O}\,(n,1))$, so $\mathrm{Iso}\,(\mathbb{M}^{n+1})$ is isomorphic to $\mathbb{R}^{n+1} \rtimes \mathrm{O}\,(n,1)$.

Notice that $\mathrm{O}\,(n,1)$ is the isotropy group of 0 in $\mathrm{Iso}\,(\mathbb{M}^{n+1})$. It is a semisimple Lie group and has four connected components. The connected component of the identity $\mathrm{SO}^+(n,1)$ is the group of linear transformations which preserve orientation and time-orientation. It is called the *Lorentz group*. There are two proper subgroups which contain $\mathrm{SO}^+(n,1)$: the group $\mathrm{SO}\,(n,1)$ of linear isometries which preserve orientation of $\mathbb{M}^{n+1}$ and the group $\mathrm{O}^+(n,1)$ of linear isometries which preserve time-orientation of $\mathbb{M}^{n+1}$. In each of these groups, the index of $\mathrm{SO}^+(n,1)$ is 2.

Geodesics in $\mathbb{M}^{n+1}$ are straight lines. There are three types of geodesics up to isometry: spacelike, timelike and null. Notice that they are classified by the restriction of the form η on them. Totally geodesic k-planes are affine k-planes in $\mathbb{M}^{n+1}$. Also, k-planes are classified up to isometry by the restriction of the Lorentzian form on them. Hence, a k-plane P is *spacelike* if $\eta|_P$ is a flat Riemannian form; it is *timelike* if $\eta|_P$ is a flat Lorentz form; finally, P is *null* if $\eta|_P$ is a degenerated form.

Hyperbolic space. Let $\mathbb{H}^n$ be the set of points in the future of 0 which have Lorentzian distance from 0 equal to 1. If we identify $\mathbb{M}^{n+1}$ with the tangent space $T_0\mathbb{M}^{n+1}$ via the exponential map, we get

$$\mathbb{H}^n = \{x \in \mathbb{M}^{n+1} |\, \langle x, x\rangle = -1, \quad x_0 > 0\}.$$

It follows that the tangent space $T_x\mathbb{H}^n$ is the space $x^\perp$. Since x is timelike $T_x\mathbb{H}^n$ is spacelike so $\mathbb{H}^n$ has a natural Riemannian structure. An easy computation shows that $\mathbb{H}^n$ is the simply connected complete Riemannian manifold with constant sectional curvature equal to -1.

A geodesic in $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a timelike 2-plane passing through 0. More generally, a totally geodesic k-submanifold (k-plane) of $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a timelike $(k+1)$-plane which passes through 0 (notice that such an intersection is always transverse). Thus, it follows that a subset C of $\mathbb{H}^n$ is convex if and only if it is the intersection of $\mathbb{H}^n$ with a convex cone with apex at 0.

Clearly, $\mathbb{H}^n$ is invariant for the group $\mathrm{O}^+(n,1)$ and furthermore, this group acts by isometries on $\mathbb{H}^n$. It can be shown that $\mathrm{O}^+(n,1)$ is in fact the full isometry group of $\mathbb{H}^n$. The group of orientation-preserving isometries of $\mathbb{H}^n$ is identified with $\mathrm{SO}^+(n,1)$.

Let $\mathbb{P}^n$ be the set of lines passing through 0 and $\pi : \mathbb{M}^{n+1} \to \mathbb{P}^n$ the natural projection. Then, $\pi|_{\mathbb{H}^n}$ is a diffeomorphism of $\mathbb{H}^n$ onto the set of timelike lines. The closure of this set is a closed ball and its boundary is formed by the set of null lines. Let $\partial\mathbb{H}^n$ be the set of null lines and put on $\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial\mathbb{H}^n$ the topology which makes the natural map $\pi : \overline{\mathbb{H}^n} \to \mathbb{P}^n$ a homeomorphism onto its image. Notice that every $g \in \mathrm{O}^+(n,1)$ extends uniquely to a homeomorphism of $\overline{\mathbb{H}^n}$.

Now, let us classify the elements of $\mathrm{O}^+(n,1)$. We say that $g \in \mathrm{O}^+(n,1)$ is *elliptic* if g has a timelike eigenvector (and in this case, the respective eigenvalue is 1). We say that g is *parabolic* if it is not elliptic and has a unique null eigenvector (and in this case, the respective eigenvalue is 1). Finally, we say that g is *hyperbolic* if it is not elliptic and has two null eigenvectors (in this case, there exists $\lambda > 1$ such that the respective eigenvalues are λ and λ^{-1}). If g is hyperbolic, then a unique g-invariant geodesic γ exists in $\mathbb{H}^n$. In this case, γ is called the *axis* of g.

Geometric structures. We shall consider only *oriented* manifolds or spacetimes. We shall be concerned with *hyperbolic n-manifolds* (i.e., Riemannian n-manifolds locally isometric to $\mathbb{H}^n$) and with *flat* $(n+1)$-*spacetimes* (i.e., those locally isometric to $\mathbb{M}^{n+1}$). By using the convenient setting of (X,G)-manifolds (see e.g., Chap. B of [7]), we can say that hyperbolic manifolds and flat spacetimes are, by definition, (X,G)-manifolds where (X,G) is respectively $(\mathbb{H}^n, \mathrm{SO}^+(n,1))$ and $(\mathbb{M}^{n+1}, \mathrm{Iso}\,(\mathbb{M}^{n+1}))$.

Let us summarize a few basic facts about such (X,G)-manifolds. Let N be a (X,G)-manifold and let us fix a universal covering $\pi : \widetilde{N} \to N$. Then, the (X,G)-structure on N lifts to a (X,G)-structure on $\widetilde{N}$ such that:

1) The covering map π is a local isometry;
2) The group $\pi_1(N)$ acts by isometries on $\widetilde{N}$ in such a way that $N = \widetilde{N}/\pi_1(N)$ and π is identified with the quotient map.

Let us summarize these facts by saying that $\pi : \widetilde{N} \to N$ is a (X,G)-*universal covering.*

Proposition 2.2. *Given an (X,G)-universal covering $\pi : \widetilde{N} \to N$, there exists a pair (D,ρ) such that:*

1) *$D : \widetilde{N} \to X$ is a local isometry;*
2) *$\rho : \pi_1(N) \to G$ is a representation;*
3) *The map D is $\pi_1(N)$-equivariant in the following sense*

$$D(\gamma(x)) = \rho(\gamma)D(x) \qquad \text{for all } \gamma \in \pi_1(N) \text{ and } x \in \widetilde{N}.$$

Moreover, given two such pairs (D,ρ) and (D',ρ'), there exists a unique $g \in G$ such that

$$D' = gD \qquad \text{and} \qquad \rho' = g\rho g^{-1}.$$

Definition 2.1. With the notation of Proposition 2.2, D is called a **developing map** of N and h is the holonomy representative compatible with D. The conjugacy class of h is called the **holonomy** of the (X, G)-manifold N.

Remark 2.3. Generally, D is only a local isometry neither injective nor surjective.

If D is a global isometry between $\widetilde{N}$ and X, we say that the (X, G)-manifold N is *complete*. A hyperbolic manifold is *complete* as a Riemannian manifold if and only if it is $(\mathbb{H}^n, \mathrm{SO}^+(n,1))$-complete.

If N is complete, then ρ is a faithful representation and its image Γ acts freely and properly discontinuously on X. The isometry D induces an isometry $\hat{D} : \widetilde{N}/\pi_1(N) \to X/\Gamma$.

Let $M := \mathbb{H}^n/\Gamma$ be a complete hyperbolic n-manifold. Notice that Γ acts freely and properly discontinuously on the whole $\mathrm{I}^+(0)$. The *future complete Minkowskian cone* on M is the flat spacetime $\mathcal{C}^+(M) := \mathrm{I}^+(0)/\Gamma$. Notice that $\mathrm{I}^+(0)$ has regular cosmological time $\widetilde{T}$ which is in fact a real analytic submersion with level surfaces

$$\mathbb{H}_a = \{x \in \mathrm{I}^+(0) | -x_0^2 + x_1^2 + \cdots + x_n^2 = -a^2\}.$$

For every $p \in \mathrm{I}^+(0)$, we have $\widetilde{T}(p) = d(p, 0)$ and the origin is the only point with this property. Every $\mathbb{H}_a$ is a Cauchy surface of $\mathrm{I}^+(0)$, so it is globally hyperbolic.

Since $\widetilde{T}$ is Γ-invariant, it induces the cosmological time $T : \mathcal{C}^+(M) \to \mathbb{R}_+$ with level surfaces $S_a = \mathbb{H}_a/\Gamma$. Notice that $M = S_1$ so that $\mathcal{C}^+(M)$ is diffeomorphic to $\mathbb{R}_+ \times M$.

We are interested in studying globally hyperbolic flat spacetimes Y, which admit a Cauchy surface diffeomorphic to M (hence, Y is diffeomorphic to $\mathbb{R}_+ \times M$). We shall provide a complete discussion of this problem on the assumption that M is *compact*. So, from now on, $M := \mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold.

The set of globally hyperbolic flat Lorentzian structures on $\mathbb{R}_+ \times M$ has a natural topology (induced by the C^∞-topology on symmetric forms). Let us denote this space by $Lor(M)$. We know that Diffeo $(\mathbb{R}_+ \times M)$ acts continuously on $Lor(M)$. The quotient $\mathcal{M}_{Lor}(M)$ is called *moduli space*, whereas *Teichmüller space* $\mathcal{T}_{Lor}(M)$ is the quotient of $Lor(M)$ by the action of the group of homotopically trivial diffeomorphisms. Notice that two structures which differ by a homotopically trivial diffeomorphism give the same holonomy (up to conjugacy), so that the holonomy depends only on the class of the structure in Teichmüller space.

For every group G, denote by $\mathcal{R}_G$ the set of representations

$$\pi_1(M)(= \Gamma) \to G$$

up to conjugacy. As $\pi_1(M) \cong \pi_1(\mathbb{R}_+ \times M)$, we have a continuous holonomy map

$$\rho : \mathcal{T}_{Lor}(M) \to \mathcal{R}_{\mathrm{Iso}\,(\mathbb{M}^{n+1})}$$

with linear part

$$\mathrm{d}\rho : \mathcal{T}_{Lor}(M) \to \mathcal{R}_{\mathrm{SO}^+(n,1)}.$$

In [**2**], it has been shown that every linear holonomy is faithful with discrete image (recently a more general result was given in [**4**]). So, we shall often confuse linear holonomy with its image subgroup into $\mathrm{SO}^+(n,1)$ (up to conjugacy). If $n \geq 3$, Mostow Rigidity Theorem implies that the linear holonomy group coincides with Γ (up to conjugacy). Thus, if $n \geq 3$, the image of the holonomy map $h : \mathcal{T}_{Lor}(M) \to \mathcal{R}_{\mathrm{Iso}\,(\mathbb{M}^{n+1})}$ is contained in

$$\mathcal{R}(\Gamma) = \{[\rho] \in \mathcal{R}_{\mathrm{Iso}\,(\mathbb{M}^{n+1})} |\, \mathrm{d}(\rho(\gamma))(0) = \gamma \text{ for all}\, \gamma \in \Gamma\}.$$

When $n = 2$, we have to vary the hyperbolic structure on M (i.e., the group Γ) which is now a closed surface of genus $g \geq 2$. Anyway, $\mathcal{R}(\Gamma)$ is the key object to be understood.

Let ρ be a representation of Γ into $\mathrm{Iso}\,(\mathbb{M}^{n+1})$ whose linear part is the identity. Thus, $\rho(\gamma) = \gamma + \tau_\gamma$ where $\tau_\gamma = \rho(\gamma)(0)$ is the translation part. By imposing the homomorphism condition, we obtain

$$\tau_{\alpha\beta} = \tau_\alpha + \alpha\tau_\beta \qquad \forall \alpha, \beta \in \Gamma,$$

so, $(\tau_\gamma)_{\gamma\in\Gamma}$ is a cocycle in $Z^1(\Gamma, \mathbb{R}^{n+1})$. Conversely, if $(\tau_\gamma)_{\gamma\in\Gamma}$ is a cocycle, then the map $\Gamma \ni \gamma \mapsto \gamma + \tau_\gamma \in \mathrm{Iso}\,(\mathbb{M}^{n+1})$ is a homomorphism. Hence, the homomorphisms of Γ into $\mathrm{Iso}\,(\mathbb{M}^{n+1})$ whose linear part is the identity are parametrized by cocycles in $Z^1(\Gamma, \mathbb{R}^{n+1})$.

Take two such representations ρ and ρ' and let $(\tau_\gamma)_{\gamma\in\Gamma}$ and $(\tau'_\gamma)_{\gamma\in\Gamma}$ be the respective translation parts. Suppose now that ρ and ρ' are conjugated by some element $f \in \mathrm{Iso}\,(\mathbb{M}^{n+1})$. Then, we have that the linear part of f commutes with the elements of Γ. Since the centralizer of Γ in $\mathrm{SO}^+(n,1)$ is trivial, f is a pure translation by a vector $v = f(0)$. Now, by imposing the condition $\rho'(\gamma) = f\rho(\gamma)f^{-1}$, we obtain that $\tau_\gamma - \tau'_\gamma = \gamma v - v$ so that τ_γ and τ'_γ differ by a coboundary. Conversely, if $(\tau_\gamma)_{\gamma\in\Gamma}$ and $(\tau'_\gamma)_{\gamma\in\Gamma}$ are cocycles which differ by a coboundary, then they induce representations which are conjugated. Hence, there is a natural identification between $\mathcal{R}(\Gamma)$ and the cohomology group $\mathrm{H}^1(\Gamma, \mathbb{R}^{n+1})$. In what follows, we use this identification without mentioning it. In particular, for a cocycle τ, we denote by ρ_τ and Γ_τ respectively, the homomorphism corresponding to τ and its image.

Main results. Now, we can state the main results of this paper.

Theorem 2.4. *For every* $[\tau] \in H^1(\Gamma, \mathbb{R}^{n+1})$*, there is a unique* $[Y_\tau] \in \mathcal{T}_{\mathrm{Lor}}(M)$ *represented by a maximal globally hyperbolic future complete spacetime* Y_τ *that admits a pair* (D, ρ) *of compatible developing map*

$$D : \widetilde{Y}_\tau \to \mathbb{M}^{n+1}$$

and holonomy representative

$$\rho : \pi_1(Y_\tau)(= \pi_1(M)) \to \mathrm{Iso}\,(\mathbb{M}^{n+1})$$

such that

1) $\rho = \rho_\tau$.
2) D *is injective and so, it is an isometry onto its image* $\mathcal{D}_\tau$ *which is a future complete proper convex domain of* $\mathbb{M}^{n+1}$.
3) *The action of* $\pi_1(M)$ *on* $\mathcal{D}_\tau$ *via* ρ *is free and properly discontinuous so that the developing map* D *induces an isometry between* Y_τ *and* $\mathcal{D}_\tau/\pi_1(M)$.
4) *The spacetime* $\mathcal{D}_\tau$ *has a* ***canonical cosmological time*** $\widetilde{T} : \mathcal{D}_\tau \to \mathbb{R}_+$ *which is a* C^1*-submersion. Every level surface* $\widetilde{S}_a$ *is the graph of a proper* C^1*-convex function defined over the horizontal hyperplane* $\{x_0 = 0\}$.
5) *The map* $\widetilde{T}$ *is* $\pi_1(M)$*-invariant and induces the canonical cosmological time* T *on* Y_τ*; this is a proper* C^1*-submersion and every level surface* $S_a = \widetilde{S}_a/\pi_1(M)$ *is* C^1*-diffeomorphic to* M.
6) *For every* $p \in \mathcal{D}_\tau$*, there exists a unique* $r(p) \in \mathrm{I}^-(p) \cap \partial\mathcal{D}_\tau$ *such that* $\widetilde{T}(p) = d(p, r(p))$*. The map* $r : \mathcal{D}_\tau \to \partial\mathcal{D}_\tau$ *is continuous. The image* $\Sigma_\tau := r(\mathcal{D}_\tau)$ *is called the* ***singularity in the past****.* Σ_τ *is spacelike-arc-connected, contractile and* $\pi_1(M)$*-invariant. Moreover, the map* r *is* $\pi_1(M)$*-equivariant.*

The map

$$\mathcal{R}(\Gamma) \ni [\rho_\tau] \mapsto [Y_\tau] \in \mathcal{T}_{Lor}(M)$$

is a continuous section of the holonomy map.

The same statement holds if we replace "future" with "past". Let us call Y_τ^- *and* $\mathcal{D}_\tau^-$ *the corresponding spaces.*

Every globally hyperbolic flat spacetime with compact spacelike Cauchy surface and holonomy group equal to $\rho_\tau(\pi_1(M))$ *is diffeomorphic to* $M \times \mathbb{R}_+$ *and isometrically embeds either into* Y_τ *or into* Y_τ^-.

In Section 7, we shall look at the asymptotic behaviour of the metrics properties of the action of Γ on $\widetilde{S}_a$ for $a \to +\infty$ and for $a \to 0$. In particular, we shall focus on the **Gromov convergence** when $a \to +\infty$ and on the convergence of the **marked length spectrum** when $a \to 0$

(the definition of these concepts are given in Section 7). The principal result that we get is the following.

Theorem 2.5. *Let $\mathcal{D}_\tau \subset \mathbb{M}^{n+1}$ be the universal cover of Y_τ for $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$. Let $\widetilde{S}_a$ be the CT level surface $\widetilde{T}^{-1}(a)$ of $\mathcal{D}_\tau$ and d_a be the natural distance on $\widetilde{S}_a$. We have that $\widetilde{S}_a$ is a $\pi_1(M)$-invariant spacelike surface and $\pi_1(M)$ acts by isometries on it.*

When $a \to +\infty$ the $\pi_1(M)$-action on the rescaled surfaces $(\widetilde{S}_a, \frac{d_a}{a})$ tends in the Gromov sense to the action of $\pi_1(M)$ on $\mathbb{H}^n$.

When $a \to 0$ the marked length spectrum *of the $\pi_1(M)$-action on $(\widetilde{S}_a, d_a)$ tends to the spectrum of the $\pi_1(M)$-action on the singularity in the past Σ.*

Now, let us point out some comments and corollaries.

The following statement is an immediate consequence of Theorem 2.4.

Corollary 2.6. *Let F be an n-manifold and suppose that there exists a Lorentzian flat structure on $\mathbb{R} \times F$ such that $\{0\} \times F$ is a spacelike surface. Suppose that the holonomy group for such a structure is Γ_τ for some $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$. Then, F is diffeomorphic to $M = \mathbb{H}^n/\Gamma$.*

By studying the action of $\Gamma_\tau = \rho_\tau(\pi_1(M))$ on the boundary $\partial\mathcal{D}_\tau$, we shall show that Γ_τ *does not act freely and properly discontinuously on the whole* $\mathbb{M}^{n+1}$.

On the domain $\mathcal{D}_\tau$, there is a natural field $-N$ which is the Lorentzian gradient of the cosmological time $\widetilde{T}$. We have that N is a timelike field, furthermore N is future directed by the choice of the sign. Notice that $N(x)$ is the normal vector to $\widetilde{S}_{\widetilde{T}(x)}$ at x, so that we call it the **normal field**. By the identification of $T_x\mathbb{M}^{n+1}$ with $\mathbb{M}^{n+1}$, we see that $N(x) \in \mathbb{H}^n$ (in fact, it is also called the **Gauss map** of the surfaces $\widetilde{S}_a$). The restriction $N|_{\widetilde{S}_a}$ is a surjective and proper map. The map N is $\pi_1(M)$-equivariant so it induces a map $\overline{N} : \mathcal{D}_\tau/\Gamma_\tau \to \mathbb{H}^n/\Gamma$. For all $a > 0$, the restriction of the map $\overline{N}|_{S_a}$ has degree 1.

When $n = 2$, it turns out that the singularity Σ_τ is a *real tree*. Moreover, Mess showed that the images under N of the fibres of the retraction r produce a Γ-invariant *geodesic lamination* L of $\mathbb{H}^2$. According to Skora Theorem [**19**], L is the geodesic lamination dual to the real tree Σ_τ. More precisely, the complete duality is realized by suitably equipping L with a transverse measure μ. Finally, the triple (M, L, μ) determines the spacetime Y_τ.

In Section 4, we shall see that for $n \geq 2$, the images under N of the fibres of the retraction r determine a *geodesic stratification* of $\mathbb{H}^n$ (we shall introduce this notion in Section 4 and prove that in dimension

$n = 2$, geodesic stratifications are geodesic laminations). Moreover, in Section 8, we shall introduce the notion of *measured geodesic stratification* and show that every measured geodesic stratification enables us to construct a spacetime Y_τ. For some technical reasons, we are not able, for the moment, to show that this correspondence is bijective. However, we present an interesting class of spacetimes: those with *simplicial singularity.* In dimension $n = 2$, the singularity of these spacetimes are *simplicial tree* and the corresponding geodesic lamination is a multi-curve. We shall show that for spacetimes with simplicial singularity, the complete duality between singularity and geodesic stratification is realized in a very explicit way.

3. Construction of $\mathcal{D}_\tau$

Let Γ be a torsion-free co-compact discrete subgroup of $\mathrm{SO}^+(n,1)$ and $M := \mathbb{H}^n/\Gamma$. Let us fix $[\tau] \in \mathrm{H}^1(\Gamma, \mathbb{R}^{n+1})$ and consider the image Γ_τ of the homomorphism associated with τ. Moreover, for every $\gamma \in \Gamma$, let us denote by γ_τ the affine transformation $x \mapsto \gamma(x) + \tau_\gamma$. In this section, we construct a Γ_τ-invariant future complete convex domain of $\mathbb{M}^{n+1}$. Moreover, we show that the action of Γ_τ on this domain is free and properly discontinuous and the quotient is diffeomorphic to $\mathbb{R}_+ \times M$.

First, let us show that there is a C^∞-embedded hypersurface $\widetilde{F}_\tau$ of $\mathbb{M}^{n+1}$ which is spacelike (i.e., $T_p\widetilde{F}_\tau$ is a spacelike subspace of $T_p\mathbb{M}^{n+1}$) and Γ_τ-invariant such that the quotient $\widetilde{F}_\tau/\Gamma_\tau$ is diffeomorphic to M. We start with an easy and useful lemma (see [**14**]).

Lemma 3.1. *Let S be a manifold and $f : S \to \mathbb{M}^{n+1}$ be a C^r-immersion ($r \geq 1$) such that $f^*\eta$ is a complete Riemannian metric on S . Then, f is an embedding. Moreover, by fixing orthonormal affine coordinates $(y_0, \dots, y_n)$, we get that $f(S)$ is a graph over the horizontal plane $\{y_0 = 0\}$.*

Proof. Let us set $f(s) = (i_0(s), \dots, i_n(s))$. Let $\pi : S \to \{y_0 = 0\}$ be the canonical projection (namely, $\pi(s) = (0, i_1(s), \dots, i_n(s))$). We have to show that π is a C^r-diffeomorphism.

Notice that π is a C^r-map between Riemannian manifolds. We claim that π is distance-increasing, that is

$$\langle \mathrm{d}\pi(x)[v], \mathrm{d}\pi(x)[v] \rangle \geq (f^*\eta)(v, v). \tag{1}$$

The lemma easily follows from this claim: the equation (1) implies that π is a local C^r-diffeomorphism. Furthermore, a standard argument shows that π is path-lifting and so, π is a covering map. Since the horizontal plane is simply connected, it follows that π is a C^r-diffeomorphism.

Let us prove the claim. Given $v \in T_sS$, let us set $\mathrm{d}f(s)[v] = (v_0, \dots, v_n)$. Then, we have $\mathrm{d}\pi(s)[v] = (0, v_1, \dots, v_n)$. Thus,

$$f^*\eta(v,v) = \langle \mathrm{d}\pi(x)[v], \mathrm{d}\pi(x)[v]\rangle - v_0^2.$$

q.e.d.

Remark 3.2. Let S be a Γ_τ-invariant spacelike hypersurface in $\mathbb{M}^{n+1}$, such that the action of Γ_τ on it is free and properly discontinuous. Suppose that S/Γ_τ is compact. By Hopf–Rinow Theorem, we know that S is complete and so the previous lemma applies.

Now, we want to construct a Γ_τ-invariant spacelike hypersurface $\widetilde{F}_\tau$. In fact, we shall construct $\widetilde{F}_\tau$ in a particular class of spacelike hypersurfaces.

Definition 3.1. A closed connected spacelike hypersurface S divides $\mathbb{M}^{n+1}$ into two components, the future and the past of S. We say that S is *future convex* (resp. *past convex*) if $\mathrm{I}^+(S)$ (resp. $\mathrm{I}^-(S)$) is a convex set and $S = \partial\mathrm{I}^+(S)$ (resp. $S = \partial\mathrm{I}^-(S)$). Moreover, S is *future strictly convex* (resp. *past strictly convex*) if $\mathrm{I}^+(S)$ (resp. $\mathrm{I}^-(S)$) is strictly convex.

Remark 3.3. The hyperbolic space $\mathbb{H}^n \subset \mathbb{M}^{n+1}$ is an example of spacelike future strictly convex hypersurface in $\mathbb{M}^{n+1}$.

Let N_0 be the flat Lorentzian structure on $[1/2, 3/2] \times M$ given by the standard inclusion $[1/2, 3/2] \times M \subset \mathcal{C}(M)$ (where $\mathcal{C}(M)$ is the Minkowskian cone on M). We can identify the universal covering $\widetilde{N}_0$ of N_0 with

$$\widetilde{N}_0 = \{x \in \mathbb{M}^{n+1} | x \in \mathrm{I}^+(0) \text{ and } d(0,x) \in [1/2, 3/2]\}.$$

The following theorem was stated by Mess ([**14**]), for the case $n = 2$. However, his proof runs in all dimensions. We relate it here for the sake of completeness.

Theorem 3.4. *If U is a bounded neighbourhood of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$, there exists $K > 0$ and a C^∞-map*

$$dev : U \times \widetilde{N}_0 \to \mathbb{M}^{n+1}$$

such that

1. *for every $\sigma \in U$, the restriction*

$$dev_\sigma : \widetilde{N}_0 \ni x \mapsto dev(\sigma, x) \in \mathbb{M}^{n+1}$$

is a developing map whose holonomy is the representation associated with σ;

2. *dev_0 is the multiplication by K;*
3. *$dev_\sigma(\mathbb{H}^n)$ is a strictly convex spacelike hypersurface.*

Proof. By a Thurston Theorem (see [**10**]), there exists a neighbourhood U_0 of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$ and a C$^\infty$-map

$$dev' : U_0 \times \widetilde{N}_0 \to \mathbb{M}^{n+1}$$

such that

1. For every $\sigma \in U_0$, the map $dev'_\sigma : \widetilde{N}_0 \to \mathbb{M}^{n+1}$ is a developing map whose holonomy is the representation associated with σ;
2. dev'_0 is the identity.

By using compactness of M, it is easy to show that if U_0 is chosen sufficiently small, then $dev'_\sigma(\mathbb{H}^n)$ is a spacelike future convex hypersurface.

Now, let us fix $K > 0$ so that $K \cdot U_0 \supset U$ and define $dev : U \times \widetilde{N}_0 \to \mathbb{M}^{n+1}$ by the rule

$$dev(\sigma, x) := K dev'(\sigma/K, x). \tag{2}$$

It is straightforward to see that dev_σ is a developing map whose holonomy is the representation associated with σ. Clearly, dev_0 is the multiplication by K and $dev_\sigma(\mathbb{H}^n) = K \cdot dev'_{\sigma/K}(\mathbb{H}^n)$ is a future convex spacelike surface invariant for Γ_σ. q.e.d.

Now, let us fix a bounded neighbourhood of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$ containing the cocycle τ. Consider the map dev of the previous theorem and let $\widetilde{F}_\tau$ be the hypersurface $dev_\tau(\mathbb{H}^n)$. Then, $\widetilde{F}_\tau$ is a Γ_τ-invariant future strictly convex spacelike hypersurface such that the Γ_τ-action on it is *free and properly discontinuous* and $\widetilde{F}_\tau/\Gamma_\tau \cong M$. Clearly, in the same way, we can obtain a Γ_τ-invariant spacelike hypersurface $\widetilde{F}_\tau^-$ which is *past* strictly convex such that $\widetilde{F}_\tau^-/\Gamma_\tau \cong M$.

Now, for any given hypersurface $\widetilde{F}$, we shall construct a natural domain $D(\widetilde{F})$ which includes it. Furthermore, we shall show that if $\widetilde{F}$ is Γ_τ-invariant and the action on $\widetilde{F}$ is free and properly discontinuous, then the same holds for $D(\widetilde{F})$.

Definition 3.2. Given a spacelike hypersurface $\widetilde{F}$, the **domain of dependence** of $\widetilde{F}$ is the set $D(\widetilde{F})$ of points $p \in \mathbb{M}^{n+1}$ such that all inextensible causal curves passing through p intersect $\widetilde{F}$.

The following is a well-known result (see e.g., [**11**]).

Proposition 3.5. *The domain of dependence $D(\widetilde{F})$ is open. Moreover, if $\widetilde{F}$ is complete (i.e., the natural Riemannian structure on it is* *complete), then a point $p \in \mathbb{M}^{n+1}$ lies in $D(\widetilde{F})$ if and only if each null line which passes through p intersects $\widetilde{F}$.*

Proposition 3.6. *Let $\widetilde{F}$ be a complete spacelike C^1-hypersurface. Let us fix $p \notin D(\widetilde{F})$ and a null vector v such that the line $p + \mathbb{R}v$ does not intersect $\widetilde{F}$. Then, the null plane $P = p + v^{\perp}$ does not intersect $\widetilde{F}$.*

Proof. Suppose that $S := \widetilde{F} \cap P$ is non-empty. Since this intersection is transverse, it follows that it is a closed $(n-1)$-submanifold of $\widetilde{F}$ and so, it is complete.

Let us fix a set of orthonormal affine coordinates $(y_0, \dots, y_n)$ such that p is the origin and $P = \{y_0 = y_1\}$ (i.e., $v = (1, 1, 0, \dots, 0)$). Consider the map

$$\pi : S \ni (y_0, y_1, \dots, y_n) \to (0, 0, y_2, \dots, y_n) \in \{y_0 = y_1 = 0\}.$$

Just as in the proof of Lemma 3.1, we argue that π is a diffeomorphism. Thus, $s \in \mathbb{R}$ exists such that $q = (s, s, 0, \dots, 0) \in \widetilde{F}$. But q lies on the line $p + \mathbb{R}v$ and this is a contradiction. q.e.d.

Corollary 3.7. *Let $\widetilde{F}$ be a complete spacelike hypersurface. The domain of dependence $D(\widetilde{F})$ is a convex set. Moreover, for every $p \notin D(\widetilde{F})$, a null support plane through p exists.*

Suppose that $\widetilde{F}$ is Γ_τ-invariant and $D(\widetilde{F})$ is not the whole $\mathbb{M}^{n+1}$. Then, either

$$D(\widetilde{F}) = \bigcap_{\substack{P \text{ null plane} \\ P \cap \widetilde{F} = \varnothing}} \mathrm{I}^+(P) \qquad \textit{or}$$

$$D(\widetilde{F}) = \bigcap_{\substack{P \text{ null plane} \\ P \cap \widetilde{F} = \varnothing}} \mathrm{I}^-(P).$$

Thus, $D(\widetilde{F})$ is either a future or a past set (i.e., $D(\widetilde{F}) = \mathrm{I}^+(D(\widetilde{F}))$ or $D(\widetilde{F}) = \mathrm{I}^-(D(\widetilde{F}))$).

Proof. Proposition 3.6 implies that for every $p \notin D(\widetilde{F})$, there exists a null plane P which passes through p and does not intersect $D(\widetilde{F})$. Thus, $D(\widetilde{F})$ is a convex set.

Suppose now, that $\widetilde{F}$ is Γ_τ-invariant and $D(\widetilde{F})$ is not the whole $\mathbb{M}^{n+1}$. We have to show that either $D(\widetilde{F})$ is contained in the future of its null support planes or it is contained in the past of its null support planes. Suppose, by contradiction that there exist null support planes P and Q

such that $D(\widetilde{F}_\tau) \subset \mathrm{I}^-(P) \cap \mathrm{I}^+(Q)$. First, suppose that P and Q are not parallel. Then, a timelike support plane R exists (see Fig. 1). Let us fix an affine coordinates system $(y_0, \dots, y_n)$ such that $R = \{y_n = 0\}$. By Lemma 3.1, we know that $\widetilde{F}$ is the graph of a function defined on the horizontal plane $\{y_0 = 0\}$. Then, $\widetilde{F} \cap R \neq \varnothing$ and this is a contradiction.

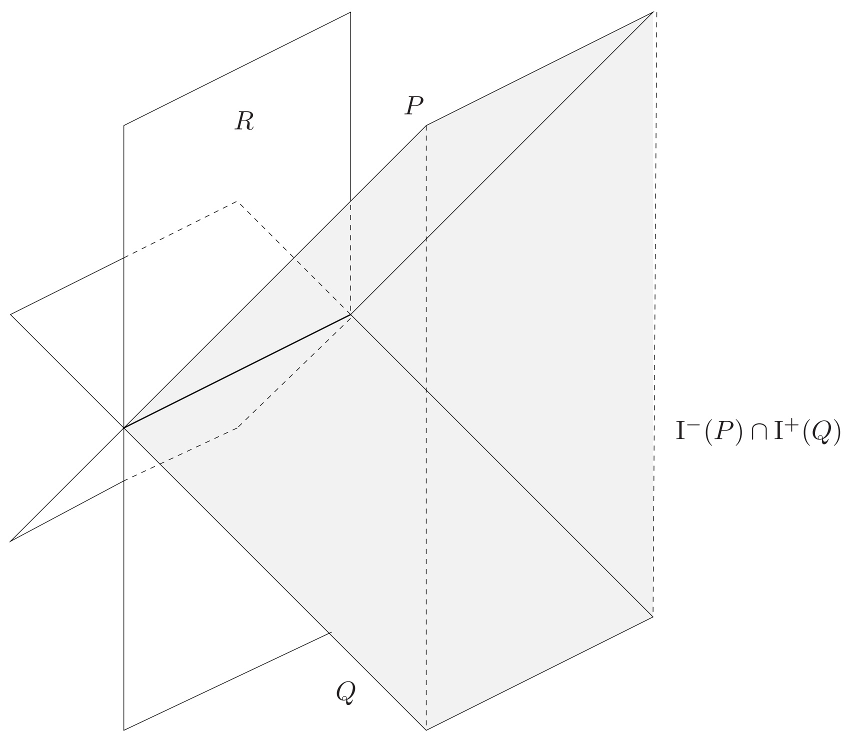

Figure 1. If P and Q are not parallel, a timelike support plane exists.

Suppose now, that we cannot choose non-parallel P and Q. Then, it follows that null support planes are all parallel. Thus, let v be the null vector orthogonal to all null support planes and $[v]$ the corresponding point on $\partial\mathbb{H}^n$. Since Γ_τ acts on $\widetilde{F}$, we have that Γ_τ permutes the null support planes of $D(\widetilde{F})$. It follows that $\Gamma \cdot [v] = [v]$. But Γ is a discrete co-compact group and so it does not fix any point in $\overline{\mathbb{H}}^n$. q.e.d.

Remark 3.8. Completeness of $\widetilde{F}$ is an essential hypothesis. For instance if a point p is removed from $\widetilde{F}$ the domain of dependence of $\widetilde{F} - \{p\}$ is no longer convex.

Remark 3.9. If $\widetilde{F}$ is future (past) convex then $D(\widetilde{F})$ is future (past) complete.

Proposition 3.10. *Let $\widetilde{F}$ be a Γ_τ-invariant spacelike hypersurface such that the Γ_τ-action on it is free and properly discontinuous. Then, Γ_τ acts freely and properly discontinuously on the whole $D(\widetilde{F})$. Moreover, $D(\widetilde{F})/\Gamma_\tau$ is diffeomorphic to $\mathbb{R}_+ \times \widetilde{F}/\Gamma_\tau$.*

Proof. Since Γ_τ is torsion-free, it is sufficient to show that the action is properly discontinuous.

Let $K, H \subset D(\widetilde{F})$ be compact sets. We have to show that the set

$$\Gamma(K,H) = \{\gamma \in \Gamma | \gamma_\tau(K) \cap H \neq \varnothing\}$$

is finite. By using Proposition 3.5, we get that the sets $C = (\mathrm{J}^+(K) \cup \mathrm{J}^-(K)) \cap \widetilde{F}$ and $D = (\mathrm{J}^+(H) \cup \mathrm{J}^-(H)) \cap \widetilde{F}$ are compact. Furthermore, $\gamma_\tau(C) = (\mathrm{J}^+(\gamma_\tau(K)) \cup \mathrm{J}^-(\gamma_\tau(K))) \cap \widetilde{F}$. Thus, $\Gamma(K,H)$ is contained in $\Gamma(C,D)$. Since the action of Γ_τ on $\widetilde{F}$ is properly discontinuous, $\Gamma(C,D)$ is finite.

Since $\widetilde{F}/\Gamma_\tau$ is a Cauchy surface in $D(\widetilde{F})/\Gamma_\tau$ we have $D(\widetilde{F})/\Gamma_\tau \cong \mathbb{R}_+ \times \widetilde{F}/\Gamma_\tau$. q.e.d.

We have constructed a Γ_τ-invariant future convex hypersurface $\widetilde{F}_\tau$ such that $\widetilde{F}_\tau/\Gamma_\tau \cong M$. Now, let us consider $D(\widetilde{F}_\tau)$: it is a Γ_τ-invariant future complete convex set and $D(\widetilde{F}_\tau)/\Gamma_\tau \cong \mathbb{R}_+ \times M$. *From now on, we shall denote $D(\widetilde{F}_\tau)$ by $\mathcal{D}_\tau$.*

Notice that, we can also consider the domain of dependence of $\widetilde{F}_\tau^-$. In the same way, we can show that $D(\widetilde{F}_\tau^-)$ is a Γ_τ-invariant, past complete convex domain of $\mathbb{M}^{n+1}$ and $D(\widetilde{F}_\tau^-)/\Gamma_\tau \cong \mathbb{R}_+ \times M$. *We shall denote it by $\mathcal{D}_\tau^-$.*

In the remaining part of this section, we shall prove that $\mathcal{D}_\tau$ is not the whole $\mathbb{M}^{n+1}$. This is a necessary condition for $\mathcal{D}_\tau$ to have regular cosmological time. In the next section, we shall see that this condition is in fact sufficient. In order to prove that $\mathcal{D}_\tau$ is a proper subset, we need some geometric properties of Γ_τ-invariant future convex sets.

Lemma 3.11. *Let Ω be a proper convex set of $\mathbb{M}^{n+1}$. If we fix a set of orthonormal affine coordinates $(y_0, \ldots, y_n)$, then Ω is a future convex set if and only if $\partial\Omega$ is the graph on the horizontal plane $\{y_0 = 0\}$ of a 1-Lipschitz convex function.*

Proof. The *if* part is quite evident. Hence, suppose that Ω is a proper future convex set. First, let us show that the projection on the horizontal plane $\pi : \partial\Omega \to \{y_0 = 0\}$ is a homeomorphism. Since $\partial\Omega$ is a topological manifold, it is sufficient to show that the projection is bijective. Since Ω is a future set, points on $\partial\Omega$ are not chronologically related and so, the projection is injective.

It remains to show that given $(a_1, \dots, a_n)$, there exists a_0 such that $(a_0, a_1, \dots, a_n) \in \partial\Omega$. Fix $p \in \partial\Omega$. It is easy to see that there exist a_+ and a_- such that $(a_+, a_1, \dots, a_n) \in \mathrm{I}^+(p)$ and $(a_-, a_1, \dots, a_n) \in \mathrm{I}^-(p)$. Since $\mathrm{I}^+(p) \subset \Omega$ and $\mathrm{I}^-(p) \cap \Omega = \varnothing$ there exists a_0 such that $(a_0, \dots, a_n) \in \partial\Omega$.

It follows that $\partial\Omega$ is the graph of a function f. Since Ω is future convex, f is convex. Since two points on $\partial\Omega$ are not chronologically related, f is 1-Lipschitz. q.e.d.

Lemma 3.12. *Let Ω be a Γ_τ-invariant proper future convex set. Then, for every $u \in \mathbb{H}^n$ there exists a plane $P = p + u^\perp$ such that $\Omega \subset \mathrm{I}^+(P)$.*

Proof. Consider, the set K of vectors v in $\mathbb{M}^{n+1}$ which are orthogonal to some support planes of Ω. Clearly, K is a convex cone with apex at 0. Since Ω is future complete it is easy to check that vectors in K are not spacelike. So, the projection $\mathbb{P}K$ of K in $\mathbb{P}^n$ is a convex subset of $\overline{\mathbb{H}}^n$. Since Ω is Γ_τ-invariant K is Γ-invariant. Then, $\mathbb{P}K$ is a Γ-invariant convex set of $\overline{\mathbb{H}}^n$. Since it is not empty (at least a support plane exists) and Γ is co-compact, then K contains the whole $\mathbb{H}^n$ and the lemma follows. q.e.d.

Lemma 3.13. *Let Ω be as in the previous lemma. For each timelike vector v, the function*

$$\partial\Omega \ni x \mapsto \langle x, v\rangle \in \mathbb{R}$$

is proper. If v is future directed, $\lim_{x\in\partial\Omega\ x\to\infty} \langle x, v\rangle = -\infty$ (we mean by ∞ the point of Alexandroff compactification of $\partial\Omega$).

Moreover, there exists a unique support plane P_v of Ω such that it is orthogonal to v and $P_v \cap \partial\Omega \neq \varnothing$.

Proof. Let us fix a timelike vector v. Clearly, we can suppose that v is future directed and $\langle v, v\rangle = -1$. Let $(y_0, \dots, y_n)$ be a set of orthonormal affine coordinates, with the origin at 0 and such that $\frac{\partial}{\partial y_0} = v$. Notice that $\langle x, v\rangle = -y_0(x)$. Now, since Ω is future complete, the boundary $\partial\Omega$ is the graph of a convex function $f : \{y_0 = 0\} \to \mathbb{R}$.

We have to show that f is proper and $\lim_{x\to\infty} f(x) = +\infty$. Thus, it is sufficient to show that the set $K_C = \{x | f(x) \leq C\}$ is compact for every $C \in \mathbb{R}$. Since f is convex K_C is a closed convex subset of $\{y_0 = 0\}$.

Suppose, by contradiction that it is not compact. It is easy to see that there exist $\overline{x} \in \{y_0 = 0\}$ and a horizontal vector w such that the ray $\{\overline{x} + tw | t \geq 0\}$ is contained in K_C.

We can suppose $\langle w, w\rangle = 1$, so that the vector $u = \sqrt{2}v + w$ is timelike and $\langle u, u\rangle = -1$. Lemma 3.12 implies that there exists $M \in \mathbb{R}$ such that $\langle p, u\rangle \leq M$ for all $p \in \partial\Omega$. On the other hand, consider $p_t = (\overline{x} + tw) + f(\overline{x} + tw)v$. We have $p_t \in \partial\Omega$ and

$$\langle p_t, u\rangle = -\sqrt{2}f(q + tw) + \langle q + tw, q + tw\rangle \geq -\sqrt{2}C + \langle q + tw, q + tw\rangle .$$

Since $\langle q + tw, q + tw\rangle \to +\infty$, we have a contradiction. q.e.d.

Proposition 3.14. *Let Ω be a Γ_τ-invariant future complete convex proper subset of $\mathbb{M}^{n+1}$. Then, a null support plane of Ω exists.*

Proof. Take $q \in \partial\Omega$ and $v \in \mathbb{H}^n$ such that $P = q + v^\perp$ is a support plane at x. Let us fix $\gamma \in \Gamma$ and consider the sequence of support planes $P_k := \gamma^k_\tau(P)$. If this sequence does not escape to infinity, there is a subsequence which converges to a support plane Q. The normal direction of Q is the limit of the normal directions of the P_k's. On the other hand, the normal direction of P_k is the direction of $\gamma^k(v)$. Since in the projective space $[\gamma^k(v)]$ tends to a null direction, we have that Q is a null support plane.

Thus, we have to prove that P_k does not escape to infinity. Let us set $v_k = |\left\langle v, \gamma^k v\right\rangle|^{-1}\gamma^k v$. We know that v_k converges to an attractor eigenvector of γ in $\mathbb{M}^{n+1}$. On the other hand, we have

$$P_k = \left\{x \in \mathbb{M}^{n+1} | \left\langle x, v_k\right\rangle \leq \left\langle \gamma^k_\tau q, v_k\right\rangle\right\}.$$

Thus, the sequence P_k does not escape to infinity if and only if the coefficients $C_k = \left\langle \gamma^k_\tau q, v_k\right\rangle$ are bounded. Since $\{v_k\}_{k\in\mathbb{N}}$ has compact closure in $\mathbb{M}^{n+1}$, it is sufficient to show that the coefficients

$$C'_k := \left\langle \gamma^k_\tau q - q, v_k\right\rangle$$

are bounded. For $\alpha \in \Gamma$, let us set $z(\alpha) = \alpha_\tau(q) - q$. It is easy to see that z is a cocycle (the difference $z(\alpha) - \tau_\alpha = \alpha q - q$ is a coboundary). Thus, we have

$$C'_k = \left|\frac{\left\langle z(\gamma^k), \gamma^k v\right\rangle}{\langle \gamma^k v, v\rangle}\right| = \left|\frac{\left\langle \gamma^{-k} z(\gamma^k), v\right\rangle}{\langle \gamma^k v, v\rangle}\right| = \left|\frac{\left\langle z(\gamma^{-k}), v\right\rangle}{\langle \gamma^k v, v\rangle}\right|. \tag{3}$$

Now, let $\lambda > 1$ be the maximum eigenvalue of γ. Denote by $\|\cdot\|$ the Euclidean norm of $\mathbb{R}^{n+1}$. Then, we have that $\|\gamma^{-1}(x)\| \leq \lambda\|x\|$ for every

$x \in \mathbb{R}^{n+1}$. Since

$$z(\gamma^{-k}) = -\sum_{i=1}^{k} \gamma^{-i}(z(\gamma))$$

it follows that $\|z(\gamma^{-k})\| \leq K\lambda^k$ for some $K > 0$. Thus, we have

$$\left| \left\langle z(\gamma^{-k}), v \right\rangle \right| \leq K'\lambda^k.$$

On the other hand, v can be decomposed into a sum $x^+ + x^- + x^0$ where x^+ is an eigenvector for λ, x^- is an eigenvector for λ^{-1} and x^0 is orthogonal to both x^+ and x^-. Since v is a future directed timelike vector, it turns out that x^+ and x^- are future directed. Thus, we have

$$\left\langle \gamma^k v, v \right\rangle = (\lambda^k + \lambda^{-k}) \left\langle x^+, x^- \right\rangle + \left\langle x^0, \gamma^k x^0 \right\rangle.$$

Now, notice that $\mathrm{Span}(x^+, x^-)^\perp$ is spacelike and γ-invariant. We deduce that $\left\langle x^0, \gamma^k x^0 \right\rangle \leq \langle x_0, x_0 \rangle$ so that there exists $M > 0$ such that $|\left\langle \gamma^k v, v \right\rangle| > M\lambda^k$. Thus, $|C'_k| \leq K'/M$ and this concludes the proof. q.e.d.

Now, we can easily prove that $\mathcal{D}_\tau$ is not the whole $\mathbb{M}^{n+1}$.

Corollary 3.15. *Let $\widetilde{F}$ be a Γ_τ-invariant future convex spacelike hypersurface. Then, there is a null support plane which does not intersect $\widetilde{F}$. Hence, $D(\widetilde{F}) \neq \mathbb{M}^{n+1}$.*

Proof. Take $\Omega = \mathrm{I}^+(\widetilde{F})$ and use Proposition 3.14. q.e.d.

In dimension $n + 1 = 4$, there is an easier argument to prove that $\mathcal{D}_\tau \neq \mathbb{M}^{n+1}$. Notice that if 1 is not an eigenvalue for some $\gamma \in \Gamma$, then the transformation γ_τ has a fixed point, namely $z := (\gamma - 1)^{-1}(\tau_\gamma)$. Generally, we say that $\gamma \in \mathrm{SO}^+(3, 1)$ is *loxodromic* if $\gamma - 1$ is invertible. So, if Γ contains a loxodromic element, then Γ_τ does not act freely on $\mathbb{M}^{3+1}$ (and in particular, $\mathbb{M}^{3+1}$ does not coincide with $\mathcal{D}_\tau$).

Lemma 3.16. *Let Γ be a discrete co-compact subgroup of $\mathrm{SO}^+(3, 1)$. Then, a loxodromic element $\gamma \in \Gamma$ exists.*

Proof. We use the identification $\mathrm{SO}^+(3, 1) \cong \mathrm{Iso}^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$. A hyperbolic element $\gamma \in PSL(2, \mathbb{C})$ is not loxodromic if and only if $\mathrm{tr}\, \gamma \in \mathbb{R}$. Hence, by contradiction, suppose that every hyperbolic $\gamma \in \Gamma$ has a real trace.

Let us fix a hyperbolic element $\gamma_0 \in \Gamma$. Up to conjugacy, we can suppose

$$\gamma_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

with $\lambda \in \mathbb{R}_+$. Moreover, we can suppose that γ_0 is a generator of the stabilizer of the axis l_0 with endpoints 0 and ∞. Now, let α be a generic element of Γ

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By general facts about Kleinian groups, we know that either α fixes the geodesic l_0 or it does fix neither 0 nor ∞. We deduce that either $b = c = 0$ or $bc \neq 0$. Suppose $\alpha \notin \operatorname{stab}(l_0)$, by imposing $\operatorname{tr}\alpha \in \mathbb{R}$ and $\operatorname{tr}(\gamma_0\alpha) \in \mathbb{R}$, we obtain

$$a + d \in \mathbb{R};$$
$$\lambda a + \lambda^{-1} d \in \mathbb{R}.$$

Thus, $a, d \in \mathbb{R}$. Since $ad - bc = 1$, we can write

$$\alpha = \begin{bmatrix} A & Be^{i\theta} \\ Ce^{-i\theta} & D \end{bmatrix}$$

with $A, D \in \mathbb{R}$, $B, C \in \mathbb{R} - \{0\}$ and $\theta \in [0, \pi)$.

Now, let $\beta \in \Gamma - \operatorname{stab}(l_0)$ be defined as follows:

$$\beta = \begin{bmatrix} A' & B'e^{i\theta'} \\ Ce^{-i\theta'} & D' \end{bmatrix}.$$

The first entry of $\alpha\beta$ is $AA' + BC'e^{i(\theta-\theta')}$. Since it is real, we can deduce that $\theta = \theta'$ (notice that $BC' \neq 0$). So, there exists a θ_0 such that for every $\gamma \in \Gamma - \operatorname{stab}(l_0)$, we have

$$\gamma = \begin{bmatrix} A & Be^{i\theta_0} \\ Ce^{-i\theta_0} & D \end{bmatrix}$$

with $A, B, C, D \in \mathbb{R}$. Hence, the rotation $R_{-\theta_0}$ conjugates Γ in $PSL(2, \mathbb{R})$ and so Γ is Fuchsian. But this is a contradiction. q.e.d.

4. Cosmological Time and Singularity in the Past

In this section, we shall see that the cosmological time on $\mathcal{D}_\tau/\Gamma_\tau$ is regular and the level surfaces are homeomorphic to M. Furthermore, we shall study the boundary of $\mathcal{D}_\tau$ and we shall see that it determines a *geodesic stratification* in M. If $n = 2$, this stratification is in fact the geodesic lamination which Mess associated with τ.

We shall study the geometry of a general class of domains of $\mathbb{M}^{n+1}$, the **regular convex domains**. We shall see that $\mathcal{D}_\tau$ is a Γ_τ-invariant regular convex domain. Most results of this section are quite general and we shall not use the action of the group Γ_τ. We shall see that every regular domain Ω is provided with a regular **cosmological time** T, a

retraction r on the **singularity in the past**, and a **normal field** $N : \Omega \to \mathbb{H}^n$ which is (up to the sign) the Lorentzian gradient of T. Moreover, if the domain is Γ_τ-invariant, then all these objects are Γ_τ-invariant. Finally, we shall see that if the normal field is surjective (this is the case, for instance, when $\Omega = \mathcal{D}_\tau$), then the images into $\mathbb{H}^n$ of the fibres of the retraction give a *geodesic stratification.* If Ω is Γ_τ-invariant, then this stratification is Γ-invariant.

Definition 4.1. Let $\Omega \subset \mathbb{M}^{n+1}$ be a non-empty convex open set. We say that Ω is a future complete (resp. past complete) regular convex domain if it is the intersection of the future (resp. the past) of at least two null support planes.

Remark 4.1. The condition that there exist at least 2 non-parallel planes excludes either that Ω is the whole $\mathbb{M}^{n+1}$ or that Ω is the future of a null plane. These domains have not regular cosmological time. On the other hand, if at least 2 non-parallel null support planes exist, then a spacelike support plane exists and we shall see that this condition ensures the existence of a regular cosmological time.

Remark 4.2. Let $\widetilde{F}$ be a Γ_τ-invariant future convex complete spacelike hypersurface. By Corollary 3.7, we know that $D(\widetilde{F})$ is the intersection of the future of its null support planes. By Proposition 3.14, it results that $D(\widetilde{F})$ is not the whole $\mathbb{M}^{n+1}$. Finally, Proposition 3.12, ensures us that spacelike support planes exist. It follows that $D(\widetilde{F})$ is a future complete regular domain. In particular, $\mathcal{D}_\tau$ is a future complete regular domain.

On the other hand, we shall see that if Ω is a Γ_τ-invariant future complete regular domain, then the cosmological time T_Ω is regular and $\Omega = D(\widetilde{S}_a)$ where $\widetilde{S}_a$ is the level surface $T_\Omega^{-1}(a)$. Moreover, $\widetilde{S}_a$ turns out to be a Γ_τ-invariant future convex spacelike complete hypersurface. Thus, we have that Γ_τ-invariant regular domains are domains of dependence of some Γ_τ-invariant future convex complete spacelike hypersurfaces.

We want to describe the cosmological time on $\mathcal{D}_\tau$ and in general on a future complete regular domain. First, we show that every future complete convex set *which has at least a spacelike support plane* has a regular cosmological time which is a C^1-function.

Proposition 4.3. *Let A be a future complete convex subset of* $\mathbb{M}^{n+1}$ *and $S = \partial A$. Suppose that a spacelike support plane exists. Then, for every $p \in A$, there exists a unique $r(p) \in S$ which maximizes the Lorentzian distance from p in $\overline{A} \cap \mathrm{J}^-(p)$. Moreover, the map $p \mapsto r(p)$ is continuous.*

The point $r = r(p)$ is the unique point in S such that, the plane $r + (p - r)^{\perp}$ is a support plane for A.

The cosmological time of A is expressed by the formula

$$T(p) = \sqrt{-\langle p - r(p), p - r(p)\rangle}.$$

It is a concave C^1 *function. The Lorentzian gradient of T is given by the formula*

$$\nabla_L T(p) = -\frac{1}{T(p)}\left(p - r(p)\right).$$

Proof. Since A is convex, the Lorentzian distance in A is the restriction of the Lorentzian distance in $\mathbb{M}^{n+1}$, that is

$$d(p,q) = \sqrt{-\langle p - q, p - q\rangle} \quad \text{for every } p \in A \text{ and } q \in \mathrm{J}^-(p) \cap A.$$

Let us fix $p \in A$ and a spacelike support plane P of A. Notice that $\mathrm{J}^-(p) \cap \mathrm{J}^+(P)$ is compact and $\mathrm{J}^-(p) \cap A \subset \mathrm{J}^-(p) \cap \mathrm{J}^+(P)$. Thus, there exists a point $r \in \overline{A} \cap \mathrm{J}^-(p)$ which maximizes the Lorentzian distance from p. Clearly, r lies on the boundary S.

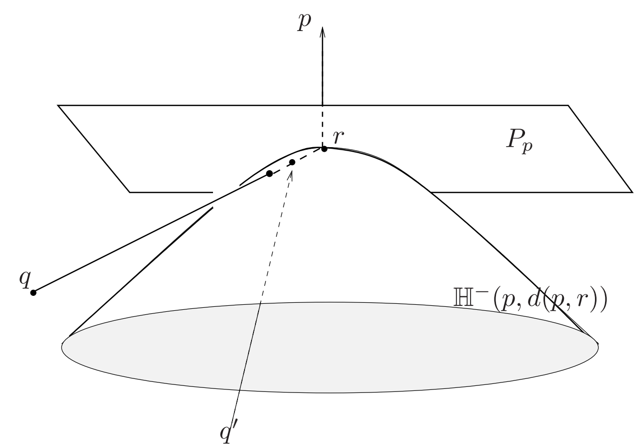

Figure 2. P_p is a support plane for A.

Now, we have to show that r is unique. Suppose by contradiction that there exists $r' \in S - \{r\}$ such that

$$d(p, r') = d(p, r).$$

Let us define

$$\mathbb{H}^-(p, \alpha) = \{x \in \mathrm{I}^-(p) |\, d(p, x) = \alpha\}.$$

Since $\mathbb{H}^-(p, d(p, r))$ is a past convex spacelike surface the segment (r, r') is contained in $\mathrm{I}^-(\mathbb{H}^-(p, d(p, r)))$. It turns out that $d(p, s) > d(p, r)$ for every $s \in (r, r')$. On the other hand, we have $(r, r') \in \overline{A}$ and this contradicts the choice of r.

We have to prove that the map $p \mapsto r(p)$ is continuous. Let $p_k \in A$ be such that $p_k \to p \in A$ and let us put $r_k = r(p_k)$. First, let us show that $\{r_k\}_{k\in\mathbb{N}}$ is bounded. Notice that for a fixed $q \in \mathrm{I}^+(p)$, there exists k_0 such that $p_k \in \mathrm{J}^-(q)$ for every $k \geq k_0$, so $r_k \in \mathrm{J}^-(q) \cap S$ for $k \geq k_0$. Since $\mathrm{J}^-(q) \cap S$ is compact, we deduce that $\{r_k\}_{k\in\mathbb{N}}$ is contained in a compact subset of S. Hence, it is sufficient to prove that if $r_k \to r$ then $r = r(p)$. If $q \in A$, then we have

$$\langle p_k - r_k, p_k - r_k \rangle \leq \langle p_k - q, p_k - q \rangle .$$

By passing to the limit, we obtain that r maximizes the Lorentzian distance.

Take $p \in A$ and let P_p be the plane $r(p) + (p - r(p))^\perp$. We claim that P_p is a support plane of A. Notice that P_p is the tangent plane $\mathbb{H}^-(p, d(p, r))$ at $r(p)$. Suppose that $q \in A \cap \mathrm{I}^-(P_p)$ exists. We have $(q, r) \subset A$. On the other hand, there exists $q' \in (q, r) \cap \mathrm{I}^-\left(\mathbb{H}^-(p, d(p, r))\right)$ (see Fig. 2). Then, $d(p, q') > d(p, r)$ and this is a contradiction. Conversely, let $s \in A$ be such that $s + (p - s)^\perp$ is a support plane for A. An analogous argument shows that s is in the past of p and maximizes the Lorentzian distance.

Now, we can prove that the cosmological time T is C^1. We shall use the following elementary fact:

Let $\Omega \subset \mathbb{R}^N$ be an open set, and $f : \Omega \to \mathbb{R}$ a continuous function. Suppose that there exist $f_1, f_2 : \Omega \to \mathbb{R}$ such that:

1. $f_1 \leq f \leq f_2$;
2. $f_1(x_0) = f_2(x_0) = f(x_0)$;
3. *f_1 and f_2 are C^1 and $\mathrm{d}f_1(x_0) = \mathrm{d}f_2(x_0)$.*

Then, f is differentiable at x_0 and $\mathrm{d}f(x_0) = \mathrm{d}f_1(x_0)$.

Let us fix $p \in A$ and consider $r = r(p)$. Let $(y_0, \ldots, y_n)$ be a set of orthonormal affine coordinates such that the origin is at $r(p)$ and P_p is the plane $\{y_0 = 0\}$ (by conseguence the coordinated of p are $(\mu, 0, \ldots, 0)$ where $\mu = T(p)$). Consider, the functions

$$\begin{aligned} &f_1 : A \ni y \mapsto +y_0^2 - \textstyle\sum_{i=1}^n y_i^2 \in \mathbb{R} \qquad \text{and} \\ &f_2 : A \ni y \mapsto y_0^2 \in \mathbb{R}. \end{aligned}$$

It is straightforward to recognize that $f_1 \leq T^2 \leq f_2$ and $f_1(p) = T^2(p) = f_2(p)$. Moreover, we have $\nabla_L f_1(p) = -2\mu \frac{\partial}{\partial y_0} = \nabla_L f_2(p)$. It turns out that T^2 is differentiable at p and $\nabla_L \left(T^2\right) = -2(p - r)$. Thus, T is differentiable at p and $\nabla_L T(p) = -\frac{1}{T(p)}(p - r)$.

Finally, let us show that T is concave. If we set $\varphi(p) := -T^2(p) = \langle p - r(p), p - r(p)\rangle$, then we have to prove

$$-\varphi\left(tp + (1-t)q\right) \geq \left(t\sqrt{-\varphi(p)} + (1-t)\sqrt{-\varphi(q)}\right)^2$$

for all $p, q \in A$ and $t \in [0,1]$. Since $r_t := t\, r(p) + (1-t)r(q) \in A$, we have by the definition of r that

$$\begin{aligned}
&-\varphi(tp + (1-t)q) \\
&\quad \geq - \left\langle (tp + (1-t)q) - r_t, (tp + (1-t)q) - r_t \right\rangle \\
&\quad = - \left\langle t\left(p - r(p)\right) + (1-t)\left(q - r(q)\right), t\left(p - r(p)\right) + (1-t)\left(q - r(q)\right)\right\rangle \\
&\quad = -\Big(t^2\varphi(p) + (1-t)^2\varphi(q) + 2t(1-t)\left\langle p - r(p), q - r(q)\right\rangle\Big).
\end{aligned}$$

Since $p - r(p)$ and $q - r(q)$ are future directed timelike vectors, we have $\langle p - r(p), q - r(q)\rangle \leq -\sqrt{\varphi(p)\varphi(q)}$ so

$$\begin{aligned}
&-\varphi(tp + (1-t)q) \\
&\quad \geq \left(t^2(-\varphi(p)) + (1-t)^2(-\varphi(q)) + 2t(1-t)\sqrt{\varphi(p)\varphi(q)}\right) \\
&\quad = \left(t\sqrt{-\varphi(p)} - (1-t)\sqrt{-\varphi(q)}\right)^2.
\end{aligned}$$

q.e.d.

Corollary 4.4. *By using the notation of Proposition 4.3, we have*

$$\lim_{k\to+\infty} T(p_k) = 0$$

for all $(p_k)_{k\in\mathbb{N}} \subset A$ such that $p_k \to \overline{p} \in \partial A$.

Proof. Let us fix $q \in \mathrm{I}^+(\overline{p})$. We have that $p_k \in \mathrm{I}^+(q)$ for all $k \gg 0$. By arguing as in Proposition 4.3, we see that $\{r(p_k)\}$ is a bounded set. Up to passing to a subsequence, we can suppose that $r(p_k) \to \overline{r}$. Since $p_k - r(p_k)$ is a timelike vector, $\overline{p} - \overline{r}$ is a non-spacelike vector. On the other hand, since S is an achronal set, it follows that $\overline{p} - \overline{r}$ is a null vector. Since $T^2(p_k) = -\langle p_k - r(p_k), p_k - r(p_k)\rangle$, we have $\lim_{k\to+\infty} T^2(p_k) = -\langle \overline{p} - \overline{r}, \overline{p} - \overline{r}\rangle = 0$. q.e.d.

Corollary 4.5. *With the above notation, let us put $\widetilde{S}_a = T^{-1}(a)$ for $a > 0$. Then, $\widetilde{S}_a$ is a future convex spacelike hypersurface and $T_p\widetilde{S}_a = (p - r(p))^\perp$ for all $p \in \widetilde{S}_a$.*

We have $\mathrm{I}^+(\widetilde{S}_a) = \bigcup_{b>a} \widetilde{S}_b$. *Moreover, let* $r_a : \mathrm{I}^+(\widetilde{S}_a) \to \widetilde{S}_a$ *be the projection and* T_a *the CT of* $\mathrm{I}^+(\widetilde{S}_a)$*, then we have:*

$$r_a(p) = \widetilde{S}_a \cap [p, r(p)]$$
$$T_a(p) = T(p) - a.$$

Let A be as in Proposition 4.3, T the CT of A and $r : A \to \partial A$ the retraction. The **normal field** on A is the map $N : A \to \mathbb{H}^n$ defined by the rule $N(p) := \frac{1}{T(p)}(p - r(p))$. It coincides up to sign with the Lorentzian gradient of T on A (we have defined $N = -\nabla_L T$ instead of $N = \nabla_L T$ because we want N to be future directed). If $\widetilde{S}_a = T^{-1}(a)$, then $N|_{\widetilde{S}_a}$ is the normal field on $\widetilde{S}_a$. Notice that the following identity holds

$$p = r(p) + T(p)N(p) \qquad \text{for all } p \in A.$$

Thus, every point in A is decomposed in a *singularity part* $r(p)$ and a *hyperbolic part* $T(p)N(p)$. We shall see that such decomposition plays an important rôle to recover the announced duality. The following inequalities are a consequence of the fact that $r(p) + N(p)^\perp$ is a support plane of A.

Corollary 4.6. *With the above notation, we have that*

$$\langle q, p - r(p)\rangle < \langle r(p), p - r(p)\rangle$$
$$\langle T(p)N(p) - T(q)N(q), r(p) - r(q)\rangle \geq 0 \qquad \textit{for all } p, q \in A. \tag{4}$$

We denote by Σ_A the image of the retraction $r : A \to \partial A$ and we refer to it as **singularity in the past**. Notice that if $r_0 = r(p)$, then the plane $r_0 + (p - r_0)^\perp$ is a *spacelike* support plane for A. Conversely, let $r_0 \in \partial A$ and suppose that there exists a future directed timelike vector v such that the spacelike plane $r_0 + v^\perp$ is a support plane. We have $p_\lambda = r_0 + \lambda v \in A$ for $\lambda > 0$ and by Proposition 4.3, $r(r_0 + \lambda v) = r_0$.

Corollary 4.7. *Let* A *be a future convex set which has a spacelike support plane. Then,* $r_0 \in \Sigma_A$ *if and only if there exists a timelike vector* v *such that the plane* $r_0 + v^\perp$ *is a support plane. Moreover,*

$$r^{-1}(r_0) = \{r_0 + v |\, r_0 + v^\perp \textit{ is a support plane}\}.$$

Remark 4.8. Notice that the map $r : A \to \Sigma_A$ continuously extends to a retraction $r : A \cup \Sigma_A \to \Sigma_A$. This map is a deformation retraction (in fact, the maps $r_t(p) = t(p - r(p)) + r(p)$ give the homotopy). Therefore, Σ_A is *contractile.*

Now, let Ω be a **future complete regular domain**. Let us use this notation:

- T is the cosmological time on Ω and $\widetilde{S}_a = T^{-1}(a)$;

- $r : \Omega \to \partial\Omega$ is the retraction and $N : \Omega \to \mathbb{H}^n$ the normal field;
- $\Sigma = r(\Omega)$ is the singularity in the past.

Lemma 4.9. *The hypersurface $\widetilde{S}_a$ is a Cauchy surface for Ω. Moreover, Ω is the domain of dependence of $\widetilde{S}_a$.*

Proof. Since Ω is a regular domain, we have $D(\widetilde{S}_a) \subset \Omega$. Now, let $p \in \Omega$ and v be a future directed non-spacelike vector. Since $T(p + \lambda v)^2 \geq -\langle p + \lambda v - r(p), p + \lambda v - r(p)\rangle$, there exists $\lambda > 0$ such that $T(p + \lambda v) > a$. On the other hand, there exists $\mu < 0$ such that $p + \mu v \in \partial\Omega$. By Corollary 4.4, we have that $\lim_{t\to\mu} T(p + tv) = 0$. So, there exists $\lambda' \in \mathbb{R}$ such that $T(p + \lambda' v) = a$. Thus, $\Omega \subset D(\widetilde{S}_a)$ and the proof is complete. q.e.d.

Remark 4.10. If Ω is a Γ_τ-*invariant* regular domain then, T is a Γ_τ-invariant function. It follows that the CT level surfaces $\widetilde{S}_a$ are Γ_τ-invariant future convex spacelike hypersurfaces. Moreover, we have $r \circ \gamma_\tau = \gamma_\tau \circ r$ and $N \circ \gamma_\tau = \gamma \circ N$. Thus, Σ is a Γ_τ-invariant subset of $\partial\Omega$.

From the previous lemma it follows that $\widetilde{S}_a/\Gamma_\tau$ is a Cauchy surface of Ω/Γ_τ. In particular if we take $\Omega = \mathcal{D}_\tau = D(\widetilde{F}_\tau)$ (see Section 3 for the definition of $\mathcal{D}_\tau$) we have that $\widetilde{S}_a/\Gamma_\tau$ is homeomorphic to $\widetilde{F}_\tau/\Gamma_\tau \cong M$ (in fact, two Cauchy surfaces are always homeomorphic).

We shall give a more precise description of the map r for a regular domain Ω. In particular, we shall describe the singularity and the fibre of a point on the singularity in terms of geometric properties of the boundary $\partial\Omega$. Let us start with a simple remark.

Lemma 4.11. *For every $p \in \partial\Omega$, there exists a future directed null vector v such that the ray $p + \mathbb{R}_+ v$ is contained in $\partial\Omega$. Furthermore, we have*

$$\Omega = \bigcap \{\mathrm{I}^+(p + v^\perp) \,|\, p \in \partial\Omega \text{ and } v \text{ is a null vector such that } p + \mathbb{R}_+ v \subset \partial\Omega\}.$$

Proof. Let us fix $p \in \partial\Omega$. There exists a null future directed vector v such that the ray $p + v^\perp$ is a support plane for Ω. Since $\mathrm{I}^+(p) \subset \Omega$ the ray $p + \mathbb{R}_+ v$ is contained in $\partial\Omega$.

Conversely, suppose that a ray $R = p + \mathbb{R}_+ v$ is contained in $\partial\Omega$. By Han–Banach Theorem, there exists a hyperplane P such that Ω and R are contained in the opposite (closed) half-spaces bounded by P. Since P is a support plane of Ω, it is not timelike. Since R is contained in $\partial\Omega$, it is contained in P. It follows that v is parallel to P so that $P = p + v^\perp$. q.e.d.

Proposition 4.12. *A point $p \in \partial\Omega$ lies in Σ if and only if there are at least 2 future directed null rays contained in $\partial\Omega$ start from p.*

Moreover, if $p \in \Sigma$, then $r^{-1}(p)$ is the intersection of Ω with the convex hull of the null rays contained in $\partial\Omega$ and starting from p.

Proof. Let us use the following elementary fact about convex sets:

Let V be a (finite dimensional) vector space and $G \subset V^*$. Consider the convex set $K = \{v \in V | g(v) \leq C_g$ for $g \in G\}$. Suppose that the following properties hold:

1. *if $g \in G$ and $\lambda > 0$, then $\lambda g \in G$ and $C_{\lambda g} = \lambda C_g$;*
2. *if $g_n \to g$ and $C_{g_n} \to C$, then $g \in G$ and $C_g \leq C$.*

Then, for all $v \in \partial K$, the set $G_v = \{g \in G | C_g = g(v)\}$ is non-empty. Moreover, the plane $v + P$ is a support plane of K if and only if there exists h in the convex hull of G_v such that $P = \ker h$.

Consider the family L of null future directed vectors which are orthogonal to some null support planes. Let $C_v = \sup_{r \in \Omega} \langle v, r\rangle$ for every $v \in L$. By Lemma 4.11, we have

$$\Omega = \{x \in \mathbb{M}^{n+1} | \langle x, v\rangle \leq C_v \text{ for all } v \in L\}.$$

Now, let us fix $p \in \partial\Omega$ and let $L(p)$ be the set of null future directed vectors v such that $p + v^\perp$ is a support plane of Ω. We can apply the remark about convex sets stated above to the family L (in fact, the inner product $\langle\cdot,\cdot\rangle$ gives an identification of $\mathbb{R}^{n+1}$ with its dual) and we obtain that $L(p)$ is non-empty. Moreover, let us fix a future directed non-spacelike vector v, then $p + v^\perp$ is a support plane if and only if v belongs to the convex hull of $L(p)$. By Corollary 4.7, we get that $r^{-1}(p)$ is the intersection of Ω with the convex hull of $p + L(p)$.

Finally, notice that a null future directed vector v lies in $L(p)$ if and only if $p + \mathbb{R}_+ v$ is contained in $\partial\Omega$. q.e.d.

Now, for $p \in \Sigma$, let us define a subset of $\mathbb{H}^n$

$$\mathcal{F}(p) := N(r^{-1}(p)).$$

In the following corollary, we point out that $\mathcal{F}(p)$ is an ideal convex set of $\mathbb{H}^n$. We recall that a convex set C of $\mathbb{H}^n$ is *ideal* if it is the convex hull of boundary points.

Corollary 4.13. *Let us fix $p \in \Sigma$ and set $L(p)$ as in the previous proposition. Let us denote by $\hat{L}(p)$ the set of points of $\partial\mathbb{H}^n$ which correspond to points in $L(p)$. Then, $\mathcal{F}(p) = N(r^{-1}(p))$ is the convex hull in $\mathbb{H}^n$ of $\hat{L}(p)$.*

Thus, we see that each point p in the singularity Σ corresponds to an ideal convex set $\mathcal{F}(p)$. Now, we shall study how the convex sets

$\{\mathcal{F}(p)\}_{p\in\Sigma}$ stay in $\mathbb{H}^n$. Given two convex sets $C, C' \subset \mathbb{H}^n$, we say that a hyperplane P *separates* C from C' if C and C' are contained in the opposite closed half-spaces bounded by P.

Proposition 4.14. *Let Ω be a future complete regular domain. For every $p, q \in \Sigma$, the plane $(p-q)^\perp$ separates $\mathcal{F}(p)$ from $\mathcal{F}(q)$. The segment $[p,q]$ is contained in Σ if and only if $\mathcal{F}(p)\cap\mathcal{F}(q) \neq \varnothing$. In this case, for all $r \in (p,q)$, we have*

$$\mathcal{F}(r) = \mathcal{F}(p)\cap(p-q)^\perp = \mathcal{F}(q)\cap(p-q)^\perp = \mathcal{F}(p)\cap\mathcal{F}(q).$$

Proof. Inequalities (4) imply that $\langle tv, p-q\rangle \leq \langle sw, p-q\rangle$ for every $v \in \mathcal{F}(q)$, $w \in \mathcal{F}(p)$ and $t, s \in \mathbb{R}_+$. This inequality can be satisfied if and only if $\langle v, p-q\rangle \leq 0$ and $\langle w, p-q\rangle \geq 0$ for all $v \in \mathcal{F}(q)$ and $w \in \mathcal{F}(p)$. This shows that $(p-q)^\perp$ separates $\mathcal{F}(p)$ from $\mathcal{F}(q)$.

Suppose, now that $\mathcal{F}(p)\cap\mathcal{F}(q) \neq \varnothing$: we have that $\mathcal{F}(p)\cap\mathcal{F}(q)$ is contained in $(p-q)^\perp$. Let $v \in \mathcal{F}(p)\cap\mathcal{F}(q)$ and P_v be the unique support plane which is orthogonal to v and intersects $\partial\Omega$. We know that P_v passes through p and q so that the segment $[p,q]$ is contained in $\partial\Omega$. Finally, since P_v is a spacelike support plane which passes through every $r \in (p,q)$, we have $[p,q] \subset \Sigma$ and $\mathcal{F}(p)\cap\mathcal{F}(q) \subset \mathcal{F}(r)$.

Conversely, suppose that $[p,q]$ is contained in Σ. Let us take $r \in (p,q)$ and $v \in \mathcal{F}(r)$. Then, we have that $\langle v, p-r\rangle \leq 0$ and $\langle v, r-q\rangle \geq 0$. Since $p-r$ and $r-q$ have the same direction, we argue that $\langle v, r-q\rangle = 0$ and $\langle v, r-p\rangle = 0$ so that $v \in \mathcal{F}(p)\cap\mathcal{F}(q)$.

In order to conclude the proof, we have to show that $\mathcal{F}(r) \supset \mathcal{F}(p)\cap(p-q)^\perp$. We know that $\mathcal{F}(p)\cap(p-q)^\perp$ is the convex hull of $\hat{L}(p)\cap(p-q)^\perp$. Thus, it is sufficient to show that $L(r) \supset L(p)\cap(p-q)^\perp$. Now, let us fix $v \in L(p)\cap(p-q)^\perp$ and consider the plane $P = p + v^\perp$. The intersection of this plane with $\overline{\Omega}$ includes the ray $p + \mathbb{R}_+ v$ and the segment $[p,q]$. Since this intersection is convex, we have that $r + \mathbb{R}_+ v$ is a subset of $P \cap \overline{\Omega}$ and thus, $v \in L(r)$. q.e.d.

Let us give a general definition.

Definition 4.2. A **geodesic stratification** of $\mathbb{H}^n$ is a family $\mathcal{C} = \{C_i\}_{i\in I}$ such that

1) C_i is an ideal convex set of $\mathbb{H}^n$;
2) $\mathbb{H}^n = \bigcup_{i\in I} C_i$;
3) For every $i, j \in I$ (with $i \neq j$), there exists a support plane $P_{i,j}$ which separates C_i from C_j. Furthermore, if $C_i \cap C_j \neq \varnothing$, then $C_i \cap C_j = C_i \cap P_{i,j} = C_j \cap P_{i,j}$.

Every C_i is called a *piece* of the stratification.

We say that the stratification is Γ-invariant if $\gamma(C_i) \in \mathcal{C}$ for all $\gamma \in \Gamma$ and for all $C_i \in \mathcal{C}$.

If Ω is a future complete regular domain of $\mathbb{M}^{n+1}$ such that *the normal field N is surjective*, we have that $\{\mathcal{F}(p)\}_{p\in\Sigma}$ is a geodesic stratification.

If Ω is a Γ_τ-*invariant* future complete regular domain by Lemma 3.12, the normal field N is surjective. In this case, it is evident that the stratification $\{\mathcal{F}(p)\}_{p\in\Sigma}$ is Γ-invariant.

Let C be an ideal convex set. Then, we say that a point p is *internal* if all the support planes passing through p contain C. Let us denote by bC the set of points of C which are not internal. Notice that, unless C has non-empty interior, bC is not the topological boundary. If $\dim C = k$, then bC has a natural decomposition into convex pieces which are ideal convex sets C_i with $\dim C_i < k$ (see [**10**]).

Now, if $\mathcal{C}$ is a geodesic stratification of $\mathbb{H}^n$, we can add to it the pieces of the decomposition of bC_i for $C_i \in \mathcal{C}$. It is easy to see that in this way, we obtain a new geodesic stratification $\overline{\mathcal{C}}$ which we call the *completion* of $\mathcal{C}$. Notice that $\overline{\overline{\mathcal{C}}} = \overline{\mathcal{C}}$. A geodesic stratification which coincides with its completion is called *complete*.

Now, let us define the k-*stratum* of $\mathcal{C}$ (for $1 \le k \le n-1$) as the set

$$X_{(k)} = \bigcup \{F \in \overline{\mathcal{C}} | \dim F \le k\}.$$

If $\mathcal{C}$ is Γ-invariant, then also $\overline{\mathcal{C}}$ is. Moreover, in this case, the strata are Γ-invariant subsets.

It is easy to see that $X_{(n-1)}$ is a closed set (in fact, $\mathbb{H}^n - X_{(n-1)}$ is the union of the interior of the n-pieces of $\mathcal{C}$). If $n = 2$, we have only the 1-stratum which in fact is a geodesic lamination of $\mathbb{H}^2$. Conversely, if L is a Γ-invariant geodesic lamination, there is a unique complete geodesic stratification $\mathcal{C}$ such that L is the 1-stratum of $\mathcal{C}$. For $n = 2$, we know that the stratification is continuous in the sense that if $r_k \in C_k$ and $r_k \to r \in \mathbb{H}^n$ then, there exists a piece $C \in \overline{\mathcal{C}}$ such that $C_k \to C$ with respect to the Hausdorff topology.

Unfortunately, in dimension $n > 2$, geodesic stratifications are more complicated: the strata $X_{(k)}$ are not closed for $k \neq n-1$ and we do not have continuity. However, as we are going to see, the stratifications which arise from complete regular domains with surjective normal field are weakly continuous in the following sense:

Definition 4.3. A geodesic stratification $\mathcal{C}$ is *weakly continuous* if the following property holds. Suppose $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence of $\mathbb{H}^n$ and $x = \lim_{k\to+\infty} x_k$. Let F_k be a piece which contains x_k and suppose that $F_k \to F$ in the Hausdorff topology. Then, there exists a piece $G \in \overline{\mathcal{C}}$ such that $F \subset G$.

Before proving the weakly continuity of stratifications associated with regular domains, let us point out a useful propriety of regular domains with surjective normal field.

Lemma 4.15. *Let Ω be a future complete regular domain of $\mathbb{M}^{n+1}$, such that the normal field $N : \Omega \to \mathbb{H}^n$ is surjective. Then, the restriction of N to the CT level surface $\widetilde{S}_a$ is a proper map.*

Proof. Let $(p_k)_{k\in\mathbb{N}}$ be a divergent sequence in $\widetilde{S}_a$ and suppose $N(p_k) \to x_\infty$. The assumption on N implies that there exists $p_\infty \in \widetilde{S}_a$ such that $N(p_\infty) = x_\infty$. Now, consider the segments $R_k = [p_\infty, p_k]$: up to passing to a subsequence, we have that R_k tends to a ray R with starting point p_∞. Now, by using that the planes $p_k + N(p_k)^\perp$ and $p_\infty + x_\infty^\perp$ are supporting planes of $\mathrm{I}^+(\widetilde{S}_a)$ it is not hard to see that the Euclidean angle between R_k and the plane $p_\infty + x_\infty^\perp$ is less than the (Euclidean) angle between the latter plane and $p_k + N(p_k)^\perp$. Since $N(p_k)^\perp$ tends to $x_\infty^\perp$, we can deduce that R is contained in the plane $p_\infty + x_\infty^\perp$.

Since this plane is a supporting plane for $\mathrm{I}^+(\widetilde{S}_a)$ and $R \subset \overline{\mathrm{I}^+(\widetilde{S}_a)}$, we have that $R \subset \widetilde{S}_a$. Moreover, we have that $R \subset N^{-1}(x_\infty)$.

Let $w \in \mathbb{R}^{n+1}$ be the spacelike direction of R: we have that $N(p_\infty + \lambda w) = x_\infty$ for all $\lambda > 0$. By inequalities (4), we can deduce $\langle w, x_\infty\rangle = 0$. Now, take $y \in \mathbb{H}^n$ such that $\langle w, y\rangle > 0$ and $q \in \widetilde{S}_a$ such that $N(q) = y$. By inequalities (4), we have that

$$\langle p_\infty + \lambda w - q, x_\infty - y\rangle \geq 0 \qquad \text{for all } \lambda > 0.$$

But $\langle w, x_\infty - y\rangle < 0$. So, we have a contradiction. q.e.d.

Now, we can prove that regular domains produce weakly continuous stratifications.

Proposition 4.16. *Let Ω be a future complete regular domain with surjective normal field N. Then, the geodesic stratification $\mathcal{C}$ associated with it is weakly continuous.*

Proof. Let $(x_k)_{k\in\mathbb{N}} \subset \mathbb{H}^n$ be a convergent sequence with $x_k \to x$. Let F_k be a piece containing x_k and suppose that $F_k \to F$. We have to prove that F is contained in a piece G.

Let us take $r_k \in \Sigma$ such that $F_k = \mathcal{F}(r_k)$. Notice that $p_k = r_k + x_k$ lies in $\widetilde{S}_1$. Since $N|_{\widetilde{S}_1}$ is a proper map, a convergent subsequence $p_{k(j)}$ exists. Let us set $p = \lim p_{k(j)}$ and $r = r(p)$. We want to show that F is contained in $\mathcal{F}(r)$. Since F is the convex hull of $\hat{L}_F = F \cap \partial\mathbb{H}^n$, it is sufficient to show that $\hat{L}_F \subset \hat{L}(r)$. Now, let us take $[v] \in \hat{L}_F$. We know that there exists a sequence $[v_n] \in \hat{L}(r_n)$ such that $[v_n] \to [v]$ in $\partial\mathbb{H}^n$.

We have that $r_n + \mathbb{R}_+ v_n \subset \partial\Omega$. Since $\partial\Omega$ is closed, this implies that $r + \mathbb{R}_+ v \subset \partial\Omega$. Thus, we can conclude that $[v] \in \hat{L}(r)$. q.e.d.

Remark 4.17. Consider the regular domain $\mathcal{D}_\tau$ constructed in Section 4. By Lemmas 3.12 and 4.5, the map $N : \widetilde{S}_1 \to \mathbb{H}^n$ is surjective. Thus, it produce a weakly continuous stratification $\mathcal{C}_\tau$ of $\mathbb{H}^n$.

Remark 4.18. We can define geodesic laminations also in dimension n higher than 2. They correspond to geodesic stratifications with empty $(n-2)$-stratum $X_{(n-2)}$. By a Zeghib result (see [**21**]), we know that Γ-invariant geodesic laminations of $\mathbb{H}^n$ are locally finite (in fact, they are lifting of totally geodesic surfaces embedded in $\mathbb{H}^n/\Gamma$).

On the other hand, Scannell showed in [**17**] that there exists a closed hyperbolic non-Haken 3-manifold M such that $\mathrm{H}^1(\pi_1(M), \mathbb{R}^{3+1})$ is non-empty. Thus, if we take $[\tau]$ in this cohomology group, it follows that $\mathcal{C}_\tau$ is not a geodesic lamination of $\mathbb{H}^3$. It follows that in order to study Lorentzian structures on $M \times \mathbb{R}$ with M a closed hyperbolic manifold, we cannot restrict ourselves to the case that $\mathcal{C}$ is a geodesic lamination (i.e., $X_{(n-2)}$ is empty).

We postpone a more careful discussion about Γ-invariant geodesic stratifications to the final sections. In the last part of this section, we shall consider the future complete regular domain $\mathcal{D}_\tau$ which we have constructed in Section 3. We shall prove that Γ_τ does not act freely and properly discontinuously on $\partial\mathcal{D}_\tau$. From this result, we shall deduce that *the action of* Γ_τ *on* $\mathbb{M}^{n+1}$ *is not free and properly discontinuous*. In particular, we shall see that the domain of dependence of *any* Γ_τ-invariant spacelike hypersurface is a regular domain (either future or past complete).

Lemma 4.19. *Let Ω be a future complete regular domain. Suppose, that Σ is closed in $\partial\Omega$. Then, the retraction $\Omega \to \Sigma$ uniquely extends to a deformation retraction $r : \overline{\Omega} \to \Sigma$.*

Proof. Since Σ is closed, it is easy to see that for every point p outside Σ, there exists a unique null ray R in $\partial\Omega$ such that p is contained in the interior of R. Thus, we can define the retraction on $\partial\Omega$ by taking $r(p)$ as the starting point of the ray R. It is easy to show that this is a continuous extension of r. q.e.d.

Let Ω be a future complete regular domain and suppose that Σ is closed. Let us set $X = \partial\Omega$. First, we construct the boundary of X. We know that $X - \Sigma$ is a C^1-manifold foliated by rays with starting points in Σ. Let $p \in X - \Sigma$ and $R(p)$ be the ray of the foliation which passes through p. The retraction on X is defined in this way: $r(p) = p$ if $p \in \Sigma$ whereas $r(p)$ is the starting point of $R(p)$ if $p \in X - \Sigma$.

The boundary of X is the leaf space of the foliation:

$$\partial X := \{R | R \text{ is a ray of the foliation}\}.$$

We want to define a topology on $\overline{X} := X \cup \partial X$ such that it agrees with the natural topology on X and makes $\overline{X}$ an n-manifold with boundary equal to ∂X. Thus, we have to define a fundamental family of neighbourhoods of a point $R \in \partial X$. Let us fix a C^1-embedded closed $(n-1)$-ball D which intersects the foliation transversely and passes through R and define

$$\begin{aligned} U(R, D) = \{p \in X - \Sigma | R(p) \cap \mathrm{int} D \neq \varnothing \text{ separates } p \text{ from } r(p)\} \cup \\ \cup \{S \in \partial X | S \cap \mathrm{int} D \neq \varnothing\}. \end{aligned}$$

Then, we can consider the topology on $\overline{X}$ which agrees with the natural topology on X and such that for each $R \in \partial X$ the sets $U(R, D)$ are a fundamental family of neighbourhoods of R. It is easy to see that $\overline{X}$ is a Hausdorff space. In order to construct an atlas on $\overline{X}$, let us consider for $p \in X - \Sigma$ a future directed timelike vector $v(p)$ that is tangent to $R(p)$ such that $y_0(v(p)) = 1$ (y_0 is a timelike affine coordinate – this is only a normalization condition). For all $(n-1)$-balls D as above, consider the maps $\mu_D : D \times (0, +\infty] \to U(R, D)$ defined by the rule

$$\mu_D(x, t) = \begin{cases} x + tv(x) & \text{if } t < +\infty \\ R(x) & \text{if } t = +\infty. \end{cases}$$

It is easy to see that these maps are local charts, so that $\overline{X}$ is a manifold with boundary. Finally, the retraction r uniquely extends to a retraction $r : \overline{X} \to \Sigma$. Notice that *this retraction is a proper map.*

Now, suppose that Γ_τ acts freely and properly discontinuously on $\overline{\Omega}$. Then, the action of Γ_τ on X uniquely extends to an action on $\overline{X}$. Moreover, the map $r : \overline{X} \to \Sigma$ is Γ_τ-equivariant. By using this remark, it follows that the action of Γ_τ on $\overline{X}$ is free and properly discontinuous. Thus, $\overline{X}/\Gamma_\tau$ is a manifold with boundary. Now, we can state the announced proposition.

Proposition 4.20. *The action of Γ_τ on $\partial\mathcal{D}_\tau$ is not free and properly discontinuous.*

Proof. By contradiction, suppose that the action is free and properly discontinuous. Let us set $X = \partial\mathcal{D}_\tau$, $M' := X/\Gamma_\tau$ and $K := \Sigma/\Gamma_\tau$. Let $\hat{r} : \mathcal{D}_\tau/\Gamma_\tau \to K$ be the surjective map which induced by the retraction $r : \mathcal{D}_\tau \to \Sigma$. Since $\widetilde{S}_1/\Gamma_\tau$ is compact (in fact, we have seen that it is homeomorphic to M), we have that K is compact. Since M' is a Hausdorff space, K is closed in M' and so, Σ is closed in $X = \partial\mathcal{D}_\tau$.

Thus, we can construct the boundary ∂X of X and M' is the interior of the manifold with boundary $\overline{M}' = \overline{X}/\Gamma_\tau$.

Consider the retraction $r : \overline{X} \to \Sigma$: since this map is Γ_τ-equivariant and proper, it induces to the quotient a proper map $\overline{r} : \overline{M}' \to K$. Since K is compact, it follows that $\overline{M}'$ is a compact manifold with boundary.

Since $\overline{r} : \overline{M}' \to K$ is a deformation retraction, we have that $\mathrm{H}_n(K) = \mathrm{H}_n(M')$ and by Poincaré duality, it follows that

$$\mathrm{H}_n(K) = \mathrm{H}_n(M') = \mathrm{H}^0(\overline{M}', \partial\overline{M}') = 0.$$

On the other hand, let us consider $Y_\tau = \mathcal{D}_\tau/\Gamma_\tau$ and $\overline{Y}_\tau = \overline{\mathcal{D}_\tau}/\Gamma_\tau$. We know that the map $r : \overline{\mathcal{D}_\tau} \to \Sigma$ induces to the quotient a deformation retraction $\overline{Y}_\tau \to K$. So

$$\mathrm{H}_n(K) = \mathrm{H}_n(\overline{Y}_\tau) = \mathrm{H}_n(Y_\tau).$$

Now, we have $Y_\tau \cong \mathbb{R} \times M$. So $\mathrm{H}_n(Y_\tau) = \mathrm{H}_n(M) = \mathbb{Z}$ and this is a contradiction. q.e.d.

Corollary 4.21. *The affine group Γ_τ does not act freely and properly discontinuously on the whole $\mathbb{M}^{n+1}$. Moreover, let $\widetilde{F}$ be a Γ_τ-invariant complete spacelike hypersurface such that Γ_τ acts freely and properly discontinuously on it. Then, $D(\widetilde{F})$ is a regular domain* (*either future or past complete*).

Proof. The first statement follows from the previous proposition. In particular, we have that $D(\widetilde{F})$ is not the whole $\mathbb{M}^{n+1}$. By Corollary 3.7, we know that $D(\widetilde{F})$ is either future or past complete and it is the intersection of the future (resp. past) of null planes. By Lemma 3.12, we know that $D(\widetilde{F})$ has spacelike support planes and so, it is a regular domain. q.e.d.

5. Uniqueness of the Domain of Dependence

Let us summarize what we have seen until now. We have fixed a free-torsion co-compact discrete subgroup Γ of $\mathrm{SO}^+(n,1)$ and we have set $M = \mathbb{H}^n/\Gamma$. Then, we have fixed a cocycle $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$ and we have studied Γ_τ-invariant regular domains of $\mathbb{M}^{n+1}$. In particular, we have constructed a Γ_τ-invariant future complete (resp. past complete) regular domain $\mathcal{D}_\tau$ (resp. $\mathcal{D}_\tau^-$) such that the action of Γ_τ on it is free and properly discontinuous and the quotient $Y_\tau = \mathcal{D}_\tau/\Gamma_\tau$ is a globally hyperbolic manifold homeomorphic to $\mathbb{R}_+ \times M$ with regular CT. Moreover, we have seen that if $\widetilde{F}$ is a Γ_τ-invariant complete spacelike hypersurface such that the action on it is free and properly discontinuous, then $D(\widetilde{F})$ is a regular domain (either future or past complete).

In this section, we want to show that $\mathcal{D}_\tau$ (resp. $\mathcal{D}^-_\tau$) is the unique Γ_τ-invariant future complete (resp. past complete) regular domain. In particular, we shall deduce that every Γ_τ-invariant spacelike hypersurface is contained either in $\mathcal{D}_\tau$ or in $\mathcal{D}^-_\tau$ and is in fact, a Cauchy surface of it.

Theorem 5.1. *With the above notation, $\mathcal{D}_\tau$ is the unique Γ_τ-invariant future complete regular domain.*

Let us give a scheme of the proof. Given a Γ_τ-invariant future complete regular domain Ω, we have to show that $\Omega = \mathcal{D}_\tau$.

Let T_Ω be the cosmological time on Ω (whereas T is the cosmological time on $\mathcal{D}_\tau$). For every $a > 0$ let $\widetilde{S}^\Omega_a = T^{-1}_\Omega(a)$ (whereas $\widetilde{S}_a = T^{-1}(a)$ is the level surface of $\mathcal{D}_\tau$). Since $\widetilde{S}^\Omega_a$ (resp. $\widetilde{S}_a$) is a Cauchy surface of Ω (resp. $\mathcal{D}_\tau$), we have that $\Omega = D(\widetilde{S}^\Omega_a)$ (resp. $\mathcal{D}_\tau = D(\widetilde{S}_a)$). Thus, it is sufficient to prove that $\widetilde{S}_a \subset \Omega$ and $\widetilde{S}^\Omega_a \subset \mathcal{D}_\tau$ for $a \gg 0$. Let us split the proof into some steps.

Step 1. $\Omega \cap \mathcal{D}_\tau \neq \varnothing$.

Step 2. Fix $p_0 \in \Omega \cap \mathcal{D}_\tau$ and let C be *the convex hull* of the Γ_τ-orbit of p_0. Then, C is a future complete convex set.

Step 3. Let $\Delta = \partial C$ be the boundary of C. Then, Δ/Γ_τ is compact.

Step 4. Let $a > \sup_{q\in\Delta} T_\Omega(q) \vee \sup_{q\in\Delta} T(q)$. Then, $\widetilde{S}^\Omega_a \subset \mathcal{D}_\tau$ and $\widetilde{S}_a \subset \Omega$.

Step 1 is quite evident. In fact, let us fix $p \in \Omega$ and $q \in \mathcal{D}_\tau$. Since Ω and $\mathcal{D}_\tau$ are future complete, we have $\mathrm{I}^+(p) \cap \mathrm{I}^+(q) \subset \Omega \cap \mathcal{D}_\tau$. On the other hand, the future sets of two points in $\mathbb{M}^{n+1}$ are not disjoint.

Step 2 is more difficult. We start with a lemma.

Lemma 5.2. *Let C be a closed convex set of $\mathbb{M}^{n+1}$ whose interior part is non-empty. Then, exactly one of the following statements holds:*

(1) *There exists a non-spacelike vector v such that $C = \{x \in \mathbb{M}^{n+1} | \alpha_1 \leq \langle x, v\rangle \leq \alpha_2\}$;*
(2) *A timelike support plane of C exists;*
(3) *Every support plane is non-timelike and C is either a future or a past convex set.*

Proof. Suppose there exist $p, q \in \partial C$ such that $q \in \mathrm{I}^+(p)$. Let us prove that C verifies either (1) or (2). More exactly, suppose that C

does not have a timelike support plane. We have to prove that C satisfies (1).

Let P_p and P_q be non-timelike support planes respectively at p and at q. Since $q \in \mathrm{I}^+(p)$, we can argue that $C \subset \mathrm{I}^+(P_p) \cap \mathrm{I}^-(P_q)$. As in the proof of Corollary 3.12, if P_p and P_q are not parallel, then a timelike support plane exists. Thus, P_p and P_q are parallel.

Since p and q are generic, it follows that for all $p', q' \in \partial C$ such that $p' \in \mathrm{I}^+(q')$ the support planes at p' and at q' are parallel. In particular, there exists a non-spacelike direction v such that the unique support plane at p' and the unique support plane at q' are orthogonal to v.

We have to show that the boundary of C coincides with $P_p \cup P_q$: in fact, it is sufficient to prove the inclusion $P_p \cup P_q \subset \partial C$. Now, let us take $e = q - p$ and define the set

$$A = \{z \in P_q | z \in \partial C \text{ and } z - e \in \partial C\}.$$

It is sufficient to prove that $A = P_q$. Clearly, A is closed and non-empty (in fact $q \in A$). Thus, it is sufficient to show that A is open. Now, let us fix $z \in A$ and set $z' = z - e$: the set

$$U = \mathrm{I}^+(z') \cap \partial C$$

is a neighbourhood of z in ∂C and for $x \in U$, there is a unique support plane P_x parallel to P_q which passes through x. Since P_q is a support plane and $P_q \cap \partial C \neq \varnothing$, it follows that $P_q = P_x$. So, U is contained in P_q. In the same way, we can deduce that $V = \mathrm{I}^-(z) \cap \partial C$ is contained in P_p. Thus, $U \cap (V + e)$ is a neighbourhood of z in ∂C which is contained in A.

We have proved that if there exist $p, q \in \partial C$ such that $p - q$ is timelike, then C verifies either (1) or (2). Suppose now that for all $p, q \in \partial C$, the vector $p - q$ is non-timelike. We have to show that C verifies (3).

Suppose that a timelike support plane exists. Then, there exists a vector u and $K \in \mathbb{R}$ such that

$$\langle u, u \rangle = 1 \qquad \text{and} \qquad \langle u, p \rangle \leq K \qquad \text{for all } p \in C.$$

Take $p_0 \in \mathrm{int} C$ and $v_+, v_- \in \mathbb{H}^n$ such that

$$\langle u, v_+ \rangle > 0 \qquad \langle u, v_- \rangle < 0.$$

Consider for $t > 0$

$$p_t = p_0 + t v_+ \qquad p_{-t} = x_0 - t v_-.$$

We have

$$\begin{aligned} \langle p_t, u \rangle &= \langle p_0, u \rangle + t \langle v_+, u \rangle \to +\infty \quad \text{as } t \to +\infty; \\ \langle p_{-t}, u \rangle &= \langle p_0, u \rangle - t \langle v_-, u \rangle \to +\infty \quad \text{as } t \to +\infty. \end{aligned}$$

Thus, there exist $t_+ > 0$ and $t_- < 0$ such that p_{t_+} and p_{t_-} lie in ∂C. But then, we have p_{t_+} in the future of p_{t_-} and this contradicts our assumption about C.

Hence, all the support planes of C are non-timelike. Let P be a support plane. We can suppose that $C \subset \overline{\mathrm{I}^+(P)}$ (the other case is analogous). We claim that $C \subset \overline{\mathrm{I}^+(Q)}$ for every support plane Q. Otherwise every timelike geodesic starting from a point $p_0 \in \mathrm{int}C$ should meet C in a compact interval I. Then, the endpoints of I should be two chronologically related points on ∂C and this contradicts our assumption on ∂C.

Thus, we have

$$C = \bigcap_{P \text{ support plane}} \overline{\mathrm{I}^+(P)}.$$

It follows that C is future complete. q.e.d.

Now, let us go back to *Step 2*. We have taken $p_0 \in \Omega \cap \mathcal{D}_\tau$ and we have considered the convex hull C of the Γ_τ orbit of p_0. Now, we have to show that C is future complete. By the previous lemma it is sufficient to prove:

a) $\mathrm{int}C \neq \varnothing$;
b) C is not of the form $\{x \in \mathbb{M}^{n+1} | \alpha_1 \leq \langle x, v\rangle \leq \alpha_2\}$;
c) C has not a timelike support plane.

a) is quite evident. In fact, if the interior of C is empty then, there exists a unique k-plane P with $0 < k < n+1$ such that $C \subset P$ and $\mathrm{int}_P(C) \neq \varnothing$. Then, since C is Γ_τ-invariant, it follows that P is Γ_τ-invariant too. So, the tangent plane of P is Γ-invariant. But, we know that Γ is co-compact and so it is irreducible. In an analogous way, we can prove (b).

It remains to prove that C does not have a timelike support plane. For this purpose, we introduce some notations. Fix a set of orthonormal affine coordinates $(y_0, \ldots, y_n)$. For every $\gamma \in \Gamma$, let us denote by $x^+(\gamma)$ (resp. $x^-(\gamma)$) the attractor null eigenvector of γ (resp. γ^{-1}) such that $y_0(x^+(\gamma)) = 1$ (resp. $y_0(x^-(\gamma)) = 1$). For every $z \in \mathbb{M}^{n+1}$, we can write

$$z = a_+(\gamma, z)x^+(\gamma) + a_-(\gamma, z)x^-(\gamma) + x^0(\gamma, z) + x^1(\gamma, z)$$

with

$$a_+, a_- \in \mathbb{R}, \quad x^0 \in \ker(\gamma - 1) \quad \text{and}$$
$$x^1 \in \ker(\gamma-1)^\perp \cap \mathrm{Span}\langle x^+(\gamma), x^-(\gamma)\rangle^\perp.$$

In order to prove (c), we now have to show the following lemma.

Lemma 5.3. *Let us fix $r \in \overline{\mathcal{D}_\tau}$ and put $z(\gamma) = \gamma_\tau r - r$ for $\gamma \in \Gamma$. Then, we have*

$$a_+(\gamma, z(\gamma)) \geq 0;$$
$$a_-(\gamma, z(\gamma)) \leq 0.$$

Furthermore, if $a_+(\gamma, z(\gamma))a_-(\gamma, z(\gamma)) = 0$, then $x^0(\gamma, z(\gamma)) = 0$.

Proof. We have seen in Proposition 3.14 that z is a cocycle. Thus, it is easy to see that

$$z(\gamma^k) = \sum_{i=0}^{k-1} \gamma^i z(\gamma)$$
$$z(\gamma^{-k}) = -\sum_{i=1}^{k} \gamma^{-i} z(\gamma).$$

Let $\lambda > 1$ be the attractive eigenvalue of γ. For simplicity, let us set $a_+ = a_+(\gamma, z(\gamma))$, $a_- = a_-(\gamma, z(\gamma))$, $x^0 = x^0(\gamma, z(\gamma))$ and $x^1 = x^1(\gamma, z(\gamma))$. By using previous formulas, we obtain

$$\begin{aligned} z(\gamma^k) &= \frac{\lambda^k - 1}{\lambda - 1} a_+ x^+(\gamma) + \frac{1}{\lambda^k} \frac{\lambda^k - 1}{\lambda - 1} a_- x^-(\gamma) + k x^0 \\ &\quad + (\gamma^{k-1} + \cdots + \gamma + 1) x^1; \\ z(\gamma^{-k}) &= -\frac{1}{\lambda^k} \frac{\lambda^k - 1}{\lambda - 1} a_+ x^+(\gamma) - \frac{\lambda^k - 1}{\lambda - 1} a_- x^-(\gamma) - k x^0 \\ &\quad - \gamma^{-k}(\gamma^{k-1} + \cdots + \gamma + 1) x^1. \end{aligned} \tag{5}$$

Now, x^1 lies in $W = \ker(1-\gamma)^\perp \cap \mathrm{Span}\langle x^+(\gamma), x^-(\gamma)\rangle^\perp$ that is a spacelike γ-invariant subspace. Moreover, the application $(1-\gamma)|_W$ is invertible. Let us denote by B_γ the map $(1-\gamma)|_W^{-1}$. Thus, it is easy to see that $(\gamma^{k-1} + \cdots + \gamma + 1)x^1 = (\gamma^k - 1)B_\gamma x^1$ and so, we have that the set $\{(\gamma^{k-1} + \cdots + \gamma + 1)x^1\}_{k \in \mathbb{N}}$ is bounded.

Let us fix a future directed timelike vector e. Since $r \in \overline{\mathcal{D}_\tau}$, there exists $K \in \mathbb{R}$ such that $\langle \alpha r, e\rangle \leq K$ for all $\alpha \in \Gamma$ and thus, $\langle z(\alpha), e\rangle \leq 2K$ for all $\alpha \in \Gamma$. Now, let us impose $\left\langle z(\gamma^k), e\right\rangle \leq 2K$ for every $k \in \mathbb{N}$. Since $\{(\gamma^{k-1} + \cdots + \gamma + 1)x^1\}_{k \in \mathbb{N}}$ is bounded, there exists K' such that

$$\frac{\lambda^k - 1}{\lambda - 1} a_+ \left\langle x^+(\gamma), e\right\rangle + \frac{1}{\lambda^k} \frac{\lambda^k - 1}{\lambda - 1} a_- \left\langle x^-(\gamma), e\right\rangle + k\left\langle x^0, e\right\rangle \leq K'. \tag{6}$$

Since $\langle x^+(\gamma), e\rangle < 0$, we can easily argue that $a_+ \geq 0$. In an analogous way, we can prove that $a_- \leq 0$.

Now, consider the case $a_+ = 0$ (the case $a_- = 0$ is analogous). Suppose by contradiction, $x^0 \neq 0$. Since x^0 is spacelike, we can choose the vector e so that $\left\langle x^0, e\right\rangle > 0$, but then the expression on the left in (6) tends to $+\infty$ as $k \to +\infty$ and this contradicts (6). q.e.d.

Now, we can prove that C does not have any timelike support plane. By contradiction, let us suppose that there exist a spacelike vector v and $K \in \mathbb{R}$ such that

$$\langle \gamma_\tau p_0, v\rangle \leq K \qquad\qquad \text{for all } \gamma \in \Gamma.$$

If we set $z(\gamma) = \gamma_\tau p_0 - p_0$, we have $\langle z(\gamma), v\rangle \leq 2K$ for all $\gamma \in \Gamma$.

Let us fix $\gamma \in \Gamma$ such that $\langle x^+(\gamma), v\rangle \geq 0$ and $\langle x^-(\gamma), v\rangle \geq 0$ (such a γ exists because the limit set of Γ is the whole $\partial\mathbb{H}^n$). Let us set $a_+ = a_+(\gamma, z(\gamma))$. Notice that $a_+ \neq 0$: in fact, if $a_+ = 0$, we have that $x^0(\gamma, z(\gamma)) = 0$. Then, from (5), it follows that $z(\gamma^k)$ runs in a compact set for $k \geq 0$. But, we know that the action of Γ_τ on C is properly discontinuous (in fact $C \subset \mathcal{D}_\tau$) and this gives a contradiction. Thus, a_+ is positive. By using (5), we can easily see that $\left\langle z(\gamma^k), v\right\rangle \to +\infty$ and this is a contradiction.

Finally, we have that C is either a future or a past convex set. Since C is contained in $\mathcal{D}_\tau \cap \Omega$ and this is future convex, we have that C is a future convex set. This concludes the proof of Step 2.

Now, let us prove *Step* 3. Since C is Γ_τ-invariant, it follows that $\Delta = \partial C$ is Γ_τ-invariant too. Furthermore, since C is a convex set with non-empty interior, then Δ is a topological n-manifold. Moreover, if $a_0 = T(p_0)$, then Δ is contained in $\overline{\mathrm{I}^+(S_{a_0})}$.

We have to show that Δ/Γ_τ *is compact*. Let $r : \mathcal{D}_\tau \to \partial\mathcal{D}_\tau$ be the retraction. Since $\Delta \subset \mathcal{D}_\tau$, we can define

$$f : \Delta \ni p \mapsto r(p) + \frac{a_0}{T(p)}(p - r(p)) \in \widetilde{S}_{a_0}$$

where $a_0 = T(p_0)$. Notice that $f(p)$ is the intersection of the timelike line $p + \mathbb{R}(p - r(p))$ with the surface $\widetilde{S}_{a_0}$. Clearly, f is Γ_τ-equivariant so induces a map $\overline{f} : \Delta/\Gamma_\tau \to \widetilde{S}_{a_0}/\Gamma_\tau$. We have seen that $\widetilde{S}_{a_0}/\Gamma_\tau$ is homeomorphic to M. So, it is sufficient to show that $\overline{f}$ is a homeomorphism.

Since C is future convex, it is easy to see that f is injective. On the other hand, let us fix $q \in \widetilde{S}_{a_0}$. We have that $q + \lambda(q - r(q)) \in C$ for $\lambda \gg 0$. Since T is concave $q + \lambda(q - r(q)) \notin C$ for $\lambda < 0$. Thus, there exists $\lambda_0 \geq 0$ such that $p = q + \lambda_0(q - r(q)) \in \Delta$.

Since $r\left(q + \lambda(q - r(q))\right) = r(q)$ for all $\lambda \in (-1, +\infty)$, we have $r(p) = r(q)$ so that the lines $p + \mathbb{R}(p - r(p))$ and $q + \mathbb{R}(q - r(q))$ coincide. Thus, q is the intersection of the line $p + \mathbb{R}(p - r(p))$ with $\widetilde{S}_{a_0}$ and so $f(p) = q$. It follows that the map f is surjective. By Theorem of the Invariance of

Domain, f is a homeomorphism and so is $\overline{f}$. This concludes the proof of *Step* 3.

Now, let us prove *Step* 4. Notice that the functions $T : \Delta \to \mathbb{R}$ and $T_\Omega : \Delta \to \mathbb{R}$ are Γ_τ-invariant. Since Δ/Γ_τ is compact, these functions are bounded on Δ so that there exists $a > 0$ such that $T(x) < a$ and $T_\Omega(x) < a$ for every $x \in \Delta$. We have to show that $\widetilde{S}_a$ and $\widetilde{S}_a^\Omega$ are contained in C. Let $y \in \mathcal{D}_\tau$ and suppose $y \notin C$ and $y \in \mathrm{I}^-(\Delta)$. There exists $y' \in \Delta \cap \mathrm{I}^+(y)$ so, we have $T(y) < T(y') < a$. It follows $\widetilde{S}_a \subset C$. An analogous argument shows that $\widetilde{S}_a^\Omega \subset C$. This concludes the proof of *Step* 4 and the proof of Theorem 5.1. q.e.d.

Clearly, an analogous theorem holds for Γ_τ-invariant past complete regular domains. So, $\mathcal{D}_\tau^-$ is the unique Γ_τ-invariant past complete regular domain.

Corollary 5.4. *If τ and σ differ by a coboundary, then $\mathcal{D}_\tau$ and $\mathcal{D}_\sigma$ differ by a translation. Moreover, $\mathcal{D}_{-\tau}$ coincides with $-(\mathcal{D}_\tau^-)$.*

Proof. Suppose that $\sigma_\gamma - \tau_\gamma = \gamma(x) - x$. Then, it is easy to see that $\mathcal{D}_\tau + x$ is a Γ_σ-invariant future complete regular domain.

On the other hand, notice that $-(\mathcal{D}_\tau^-)$ is a future complete regular domain and it is invariant by the action of $\Gamma_{-\tau}$. q.e.d.

Let us set $Y_\tau := \mathcal{D}_\tau/\Gamma_\tau$ (resp. $Y_\tau^- := \mathcal{D}_\tau^-/\Gamma_\tau$). We have that Y_τ and Y_σ are isometric if and only if τ and σ differ by a coboundary (i.e., the isometric class of Y_τ depends only on the cohomology class of τ). Notice that for $\tau = 0$, the domain $\mathcal{D}_0$ (resp. $\mathcal{D}_0^-$) coincides with $\mathrm{I}^+(0)$ (resp. $\mathrm{I}^-(0)$) so that Y_0 is the Minkowskian cone $\mathcal{C}^+(M)$.

On the other hand, a time-orientation reversing isometry between $Y_{-\tau}$ and Y_τ^- exists.

Corollary 5.5. *Let $\widetilde{F}$ be a Γ_τ-invariant complete spacelike hypersurface on which the action of Γ_τ is free and properly discontinuous. Then, $\widetilde{F}$ is contained either in $\mathcal{D}_\tau$ or in $\mathcal{D}_\tau^-$. In particular, every timelike coordinate is proper on $\widetilde{F}$ and the Gauss map has degree* 1. *Furthermore, $\widetilde{F}/\Gamma_\tau$ is homeomorphic to M.*

Proof. We know that $D(\widetilde{F})$ is a Γ_τ-invariant either future or past complete regular domain. By Theorem 5.1, we get that either $D(\widetilde{F}) = \mathcal{D}_\tau$ or $D(\widetilde{F}) = \mathcal{D}_\tau^-$. Thus, $\widetilde{F}$ is contained either in $\mathcal{D}_\tau$ or in $\mathcal{D}_\tau^-$. Notice that this implies that every timelike coordinate on $\widetilde{F}$ is proper.

Suppose $\widetilde{F} \subset \mathcal{D}_\tau$. Let us consider the map

$$\varphi : \widetilde{F} \ni x \mapsto r(x) + \frac{1}{T(x)}\,(x - r(x)) \in \widetilde{S}_1.$$

It is easy to see that this map is Γ_τ-equivariant and injective. Furthermore, since $\widetilde{F}$ is a Cauchy surface for $\mathcal{D}_\tau$ (in fact $\mathcal{D}_\tau = D(\widetilde{F})$), we can easily see that φ is surjective. Thus, it is a Γ_τ-equivariant homeomorphism. It follows that it induces a homeomorphism $\overline{\varphi} : \widetilde{F}/\Gamma_\tau \to \widetilde{S}_1/\Gamma_\tau \cong M$.

Finally, consider the Gauss map of $\widetilde{F}$. It is a Γ-equivariant map $G : \widetilde{F} \to \mathbb{H}^n$ (i.e., $G(\gamma_\tau p) = \gamma G(p)$). It is easy to see that this map induces on the quotient a map $\overline{G} : \widetilde{F}/\Gamma_\tau \to M$ which is a homotopy equivalence. Thus, it has degree 1. q.e.d.

Now, we want to prove that Y_τ and Y_τ^- are the only maximal globally hyperbolic spacetimes with a compact spacelike Cauchy surface and with holonomy group Γ_τ. We need the following remark which was stated by Mess in [**14**] for the case $n = 2$. However, his proof is valid in every dimension.

Corollary 5.6. *For every $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$, the intersection $\mathcal{D}_\tau \cap \mathcal{D}_\tau^-$ is empty.*

Proof. It is easy to see that $\mathcal{D}_\tau \cap \mathcal{D}_\tau^-$ is a Γ_τ-invariant compact set. Thus, if it is non-empty, its barycentre p is a fixed point of Γ_τ. It is straightforward to recognize that $\mathrm{I}^+(p)$ and $\mathrm{I}^-(p)$ are respectively a Γ_τ-invariant future and past complete domain of dependence (notice that the cohomology class of τ vanishes). Hence, $\mathcal{D}_\tau = \mathrm{I}^+(p)$ and $\mathcal{D}_\tau^- = \mathrm{I}^-(p)$ so, their intersection is empty. q.e.d.

Corollary 5.7. *There exists only two maximal globally hyperbolic flat spacetimes with compact spacelike Cauchy surface and with holonomy group Γ_τ.*

Proof. Let Y be a maximal globally hyperbolic flat spacetime with a compact spacelike Cauchy surface N and holonomy group equal to Γ_τ. We have to show that Y isometrically embeds in Y_τ or in Y_τ^-. It is sufficient to show that the developing map $D : \widetilde{Y} \to \mathbb{M}^{n+1}$ is an embedding with image contained either in $\mathcal{D}_\tau$ or in $\mathcal{D}_\tau^-$.

Let N be the spacelike Cauchy surface of Y. We know that $D : \widetilde{N} \to \mathbb{M}^{n+1}$ is an embedding and its image $D(\widetilde{N})$ is a Γ_τ-invariant surface such that the Γ_τ-action on it is free and properly discontinuous. Thus, $D(\widetilde{N})$ is a Cauchy surface either of $\mathcal{D}_\tau$ or $\mathcal{D}_\tau^-$. It follows that N is homeomorphic to M. In [**2**], it is shown that Y is foliated by spacelike hypersurfaces so that $D(Y) \subset \mathcal{D}_\tau \cup \mathcal{D}_\tau^-$. Since these domains are disjoint, it follows that $D(Y)$ is contained in one of them, say $\mathcal{D}_\tau$ (the other case is analogous).

Consider the map $T_D := T \circ D$ where T is the CT of $\mathcal{D}_\tau$: we have that T_D is a $\pi_1(Y)$-invariant regular map such that the level surfaces $\widetilde{N}_a$ are $\pi_1(Y)$-invariant spacelike Cauchy surfaces. Thus, $\widetilde{N}_a/\pi_1(Y) \cong N$, is compact. It follows that $D|_{\widetilde{N}_a}$ is an embedding. Moreover, let us fix $p \in \widetilde{N}_a$ and $q \in \widetilde{N}_b$ with $a \neq b$. Since, we have that $T(D(p)) = a$ and $T(D(q)) = b$, it follows $D(p) \neq D(q)$. Thus, the map D is an embedding of Y into $\mathcal{D}_\tau$. This map induces on the quotient the embedding $Y \to Y_\tau$. q.e.d.

6. Continuous Family of Domains of Dependence

We use the notation introduced in the previous sections. In particular, Γ is a torsion-free co-compact and discrete subgroup of $\mathrm{SO}^+(n,1)$ and $M = \mathbb{H}^n/\Gamma$. We have seen that a well defined correspondence exists

$$\mathrm{H}^1(\Gamma, \mathbb{R}^{n+1}) \ni [\tau] \mapsto [Y_\tau] \in \mathcal{T}_{\mathrm{Lor}}(M)$$

where Y_τ is the quotient of the unique Γ_τ-invariant future complete regular domain $\mathcal{D}_\tau$ by the action of the deformed group Γ_τ, (we recall that $\mathcal{T}_{\mathrm{Lor}}(M)$ is the Teichmüller space of globally hyperbolic flat Lorentzian structures on $\mathbb{R} \times M$ with a spacelike Cauchy surface). In this section, we shall show that this correspondence is continuous.

More precisely, we shall prove that for every bounded neighbourhood U of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$, there is a continuous map

$$dev : U \times \left(\mathbb{R}_+ \times \widetilde{M}\right) \to \mathbb{M}^{n+1}$$

such that for every $\tau \in U$, the restriction $dev_\tau = dev(\tau, \cdot)$ is a developing map of Y_τ.

Let us consider the map

$$dev^0 : U \times \widetilde{M} \to \mathbb{M}^{n+1} \tag{7}$$

constructed in Theorem 3.4. We know that dev^0_τ is an embedding onto a Γ_τ-invariant strictly convex spacelike hypersurface and the map dev^0_τ is Γ-equivariant in the following sense

$$dev^0_\tau(\gamma x) = \gamma_\tau dev^0_\tau(x).$$

For every $\tau \in U$, let $\widetilde{F}_\tau$ be the image of $\widetilde{M}$ under $dev^0_\tau(\widetilde{M})$. Now, let us fix a set of orthonormal affine coordinates $(y_0, \ldots, y_n)$. We know that there exists a convex function $\varphi_\tau : \{y_0 = 0\} \to \mathbb{R}$ such that $\widetilde{F}_\tau$ is the graph of φ_τ. The first remark is that φ_τ is a continuous function of τ. More exactly, let $(\tau_k)_{k \in \mathbb{N}}$ be a sequence in U which converges to τ in U. Then, φ_{τ_k} tends to φ_τ in the compact-open topology.

We know that for every $\tau \in U$, the domain $\mathcal{D}_\tau$ is the domain of dependence of $\widetilde{F}_\tau$. Now, let $\psi_\tau : \{y_0 = 0\} \to \mathbb{R}$ be such that $\partial\mathcal{D}_\tau$ is the graph of ψ_τ. We want to prove that ψ_τ is a continuous function of τ.

Proposition 6.1. *Let $(\tau_k)_{k\in\mathbb{N}}$ be a sequence in U which converges to $\tau \in U$. Then, ψ_{τ_k} tends to ψ_τ in the compact-open topology.*

Proof. First, let us show that the family $\{\psi_{\tau_k} : \{y_0 = 0\} \to \mathbb{R}\}_{k\in\mathbb{N}}$ is locally bounded and equicontinuous. Since two points on $\partial\mathcal{D}_{\tau_k}$ are not chronologically related, it follows that the maps ψ_{τ_k} are 1-Lipschitzian so they form an equicontinuous family. We have to prove that they are locally bounded. Since $\widetilde{F}_\tau$ is contained in $\mathcal{D}_\tau$, we have $\psi_{\tau_k} \leq \varphi_{\tau_k}$. On the other hand, we can consider a continuous family of *past* strictly convex Γ_σ-invariant spacelike hypersurfaces $\left\{\widetilde{F}^-_\sigma\right\}_{\sigma\in U}$. Let $\varphi^-_{\tau_k} : \{y_0 = 0\} \to \mathbb{R}$ be such that $\widetilde{F}^-_{\tau_k}$ is the graph of $\varphi^-_{\tau_k}$. The domain of dependence of $F^-_{\tau_k}$ is $\mathcal{D}^-_{\tau_k}$ and since it is disjoint from $\mathcal{D}_{\tau_k}$, we can deduce that

$$\varphi^-_{\tau_k} \leq \psi_{\tau_k} \leq \varphi_{\tau_k}.$$

Notice that $\{\varphi^-_{\tau_k}\}_{k\in\mathbb{N}}$ and $\{\varphi_{\tau_k}\}_{k\in\mathbb{N}}$ are convergent and hence, locally bounded. It follows that ψ_{τ_k} is locally bounded too.

Now, it remains to prove that if $\psi_{\tau_k} \to \psi_\infty$, then $\psi_\infty = \psi_\tau$. We have that ψ_∞ is a convex function and the graph S of ψ_∞ is Γ_τ-invariant. Furthermore, since ψ_∞ is a 1-Lipschitz function, then S has not timelike support planes. Hence, $\mathrm{I}^+(S)$ is the future of the graph of ψ_∞ and it is a future convex set. It is easy to see that $\mathrm{I}^+(S)$ is a future complete regular domain. Since it is Γ_τ-invariant by Theorem 5.1, we have $\mathrm{I}^+(S) = \mathcal{D}_\tau$. Thus, $\mathrm{graph}\,(\psi_\infty) = \partial\mathcal{D}_\tau$ and so $\psi_\infty = \psi_\tau$. q.e.d.

Let $(\tau_k)_{k\in\mathbb{N}}$ be a sequence in U which converges to $\tau \in U$. Let us fix K a compact subset of $\mathcal{D}_\tau$. The last proposition implies that $K \subset \mathcal{D}_{\tau_k}$ for $k \gg 0$. Thus, we can suppose that $K \subset \mathcal{D}_{\tau_k}$ for any k. Notice that the cosmological time T_{τ_k}, the normal field N_{τ_k} and the retraction r_{τ_k} of the domain $\mathcal{D}_{\tau_k}$ are maps defined over K. The following propositions show that these maps converge to the corresponding maps for $\mathcal{D}_\tau$ as $k \to +\infty$.

Proposition 6.2. *Let $(\tau_k)_{k\in\mathbb{N}}$ be a sequence as above. Let $T_k = T_{\tau_k}$ be the cosmological time of $\mathcal{D}_{\tau_k}$. Then, $T_k|_K$ uniformly tends to the restriction on K of the cosmological time $T = T_\tau$ of $\mathcal{D}_\tau$.*

In order to prove the proposition, we need the following technical lemma.

Lemma 6.3. *For $C \in \mathbb{R}$ and for every cocycle σ, let us set*

$$K_C(\sigma) = \{x \in \{y_0 = 0\} | \psi_\sigma(x) \leq C\}$$

(ψ_σ is the function defined over the horizontal plane such that $\partial\mathcal{D}_\sigma$ is the graph of such a function). Then, for every $C \in \mathbb{R}$ and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$K_{C-\varepsilon}(\tau) \subset K_C(\tau_k) \subset K_{C+\varepsilon}(\tau) \qquad \text{for all } k \geq k_0.$$

For every cocycle σ, let $M(\sigma)$ be the minimum of the function ψ_σ : $\{y_0 = 0\} \to \mathbb{R}$. Then, $M(\tau_k)$ converges to $M(\tau)$.

Proof. Notice that $K_C(\sigma)$ is a convex compact set. Moreover, if $C > M(\sigma)$, then $K_C(\sigma)$ has non-empty interior and $\partial K_C(\sigma)$ is the level set $\{x|\psi_\sigma(x) = C\}$.

Now, let us set $M = M(\tau)$. First, let us show the first statement for $C > M$. Fix $\varepsilon > 0$ and let $k_0 \in \mathbb{N}$ be such that $||\psi_\tau - \psi_{\tau_k}||_{\infty, K_{C+\varepsilon}(\tau)} < \frac{\varepsilon}{2}$ for all $k \geq k_0$. Clearly, $K_{C-\varepsilon}(\tau) \subset K_C(\tau_k)$ for all $k \geq k_0$. Now, let x be a point that does not lie in $K_{C+\varepsilon}(\tau)$. We claim that $\psi_{\tau_k}(x) \geq C + \frac{\varepsilon}{2}$ for all $k \geq k_0$ and this proves the other inclusion.

Let $k > k_0$ and $x_0 \in \{y_0 = 0\}$ be such that $\psi_\tau(x_0) = M$. Consider the map $c(t) = \psi_{\tau_k}(x_0 + t(x - x_0))$ for $t \in [0, 1]$. Let t_0 be such that $x_0 + t_0(x - x_0) \in \partial K_{C+\varepsilon}(\tau)$. We have that

$$\begin{array}{ll} c(0) \leq M + \frac{\varepsilon}{2} & \text{and} \\ c(t_0) \geq C + \frac{\varepsilon}{2}. & \end{array}$$

By imposing $c(t_0) \leq (1 - t_0)c(0) + t_0 c(1)$, we have that $\psi_{\tau_k}(x) = c(1) \geq C + \frac{\varepsilon}{2}$.

Now, suppose $C < M$. Let us fix k_0 such that

$$\begin{array}{ll} K_{M+1}(\tau_k) \subset K_{M+2}(\tau) & \text{and} \\ ||\psi_\tau - \psi_{\tau_k}||_{\infty, K_{M+2}(\tau)} < \frac{M-C}{2} & \text{for all } k > k_0. \end{array}$$

Then, it turns out that $K_C(\tau_k) = \varnothing$ for all $k > k_0$.

Thus, it turns out that $M(\tau) \leq \liminf_{k\to+\infty} M(\tau_k)$. On the other hand, since ψ_{τ_k} tends to ψ_τ, we can easily see that $M(\tau) \geq \limsup_{k\to+\infty} M(\tau_k)$. q.e.d.

Now, we can prove Proposition 6.2.

Proof. Let M be the minimum of ψ_τ. By Lemma 6.3, there exists k_0 such that $\psi_{\tau_k}(x) > M - 1$ for all $x \in \{y_0 = 0\}$ and $k \geq k_0$. Notice that the set $\mathrm{J}^-(K) \cap \{y_0 \geq M - 1\}$ is a compact set and let H be the projection of it onto the horizontal plane $\{y_0 = 0\}$. Let us fix $\varepsilon > 0$ and $k(\varepsilon)$ such that $||\psi_\tau - \psi_{\tau_k}||_{\infty, H} < \frac{\varepsilon}{2}$ for $k \geq k(\varepsilon)$.

Let $p \in K$ and r be the projection of p on $\partial\mathcal{D}_\tau$: notice that $r \in \mathrm{J}^-(K) \cap \{y_0 \geq M-1\}$. Now, if $k \geq k(\varepsilon)$, then $r + \varepsilon\frac{\partial}{\partial y_0}$ belongs to $\mathcal{D}_{\tau_k}$ so that

$$T_k(p) > \sqrt{-\left\langle p - r + \varepsilon\frac{\partial}{\partial y_0}, p - r + \varepsilon\frac{\partial}{\partial y_0}\right\rangle}.$$

Now, notice that $-\left\langle p - r + \varepsilon\frac{\partial}{\partial y_0}, p - r + \varepsilon\frac{\partial}{\partial y_0}\right\rangle = -T(p)^2 - \varepsilon^2 + 2\varepsilon(p - r)_0$. By the compactness of $\mathrm{J}^-(K) \cap \{y_0 \geq M\}$, there exists a constant C such that

$$T_k(p) > \sqrt{T(p)^2 + \varepsilon^2 - 2C\varepsilon}.$$

Thus, we can fix $\eta > 0$ such that $T_k(p) > T(p) - \eta$ for all $k \geq k(\varepsilon)$ and $p \in K$.

On the other hand, also the projection $r_k(p)$ of x on $\partial\mathcal{D}_{\tau_k}$ lies in $\mathrm{J}^-(K) \cap \{y_0 \geq M-1\}$. So, the same argument shows that $T(p) > T_k(p) - \eta$ for all $k > k(\varepsilon)$ and $p \in K$. q.e.d.

Let $\psi^a_\tau : \{y_0 = 0\} \to \mathbb{R}$ be such that graph(ψ^a_τ) is the CT level surface $\widetilde{S}_a(\tau) := T_\tau^{-1}(a)$. Proposition 6.2 implies that these maps are continuous functions of τ.

Corollary 6.4. *If $\tau_k \to \tau$, then $\psi^a_{\tau_k}$ tends to ψ^a_τ in the compact-open topology.*

Proof. Since the maps $\psi^a_{\tau_k}$ are 1-Lipschitz, we have that $\{\psi^a_{\tau_k}\}_{k\in\mathbb{N}}$ is an equicontinuous family. On the other hand, notice that

$$\psi_{\tau_k} < \psi^a_{\tau_k} < \psi_{\tau_k} + a.$$

Since $\{\psi_{\tau_k}\}$ is locally bounded, it follows that $\{\psi^a_{\tau_k}\}_{k\in\mathbb{N}}$ is also locally bounded. From Proposition 6.2, it follows that if $\psi^a_{\tau_k}$ converges and the limit is ψ^a_τ. q.e.d.

Now, let us prove that the retraction r_τ and the normal field N_τ are continuous functions of τ.

Proposition 6.5. *Let $\tau_k \to \tau$ be as above. Let r_k and N_k be respectively the retraction and the normal field of $\mathcal{D}_{\tau_k}$. Now, fix a compact subset K of $\mathcal{D}_\tau$. The maps $r_k|_K$ and $N_k|_K$ converge in the compact-open topology respectively to the retraction r and to the normal field N of the domain $\mathcal{D}_\tau$.*

Proof. Let M be the minimum of the map ψ_τ and fix k_0 such that $\psi_{\tau_k} \geq M-1$ for all $k \geq k_0$. In particular, $r_k(p) \in \mathrm{J}^-(K) \cap \{y_0 \geq M-1\}$ for all $p \in K$ and $k > k_0$. Since $\mathrm{J}^-(K) \cap \{y_0 \geq M-1\}$ is compact, we can choose C such that $||p - r_k(p)|| \leq C$ for all $p \in K$ and $k \geq k_1$ $(|| \cdot ||$

is the Euclidean norm). On the other hand, because of Proposition 6.2, we can choose $k_1 > k_0$ such that

$$\beta > T_k(p) \geq \alpha > 0 \qquad \text{for all } p \in K \text{ and } k \geq k_1.$$

(Here, T_k is the CT on $\mathcal{D}_{\tau_k}$ and T is the CT on $\mathcal{D}_\tau$.)

Thus, the image $N_k(K)$ is contained in the set $H = \{x \in \mathbb{H}^n | \, ||x|| \leq \frac{C}{\alpha}\}$ for all $k \geq k_1$. This is a compact set of $\mathbb{H}^n$ so the family of functions $\{N_k|_K\}_{k\in\mathbb{N}}$ is bounded.

In order to show that $N_k|_K \to N|_K$, it is sufficient to prove that $N_k(p_k) \to N(p)$ for all convergent sequences $p_k \to p$. Since $N_k(p_k)$ runs in a compact subset of $\mathbb{H}^n$, we can suppose that $N_k(p_k)$ tends to a timelike vector v. Let us set $a = T(p)$. In order to show that $N(p) = v$, it is sufficient to prove that $p + v^\perp$ is a support plane for the surface $\widetilde{S}_a = T^{-1}(a)$, i.e., we have to prove the following inequality

$$\langle q, v\rangle \leq \langle p, v\rangle \qquad \text{for all } q \in \widetilde{S}_a. \tag{8}$$

Let us fix $q \in \widetilde{S}_a$ and put $q = (\psi^a_\tau(y), y)$. Let us set $a_k = T_k(p_k)$ and consider the sequences

$$q_k := (\psi^{a_k}_{\tau_k}(y), y);$$
$$q'_k := (\psi^{a}_{\tau_k}(y), y).$$

By Corollary 6.4, we have that $q'_k \to q$. On the other hand, it turns out that $||q_k - q'_k|| \leq |a_k - a|$ so that $q_k \to q$. We know that $\langle q_k, N_k(p)\rangle \leq \langle p_k, N_k(p)\rangle$: by passing to the limit inequality (8) follows.

Since $r_k + T_k N_k = id$, we have that $r_k|_K \to r|_K$ uniformly. q.e.d.

Corollary 6.6. *Let K be as above. Then, the cosmological times T_{τ_k} tend to T_τ in the C^1-topology of $\mathrm{C}^1(K)$.*

Now, let us go back to the original problem.

Theorem 6.7. *For every bounded neighbourhood U of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$, there exists a continuous map*

$$dev : U \times \left(\mathbb{R}_+ \times \widetilde{M}\right) \to \mathbb{M}^{n+1}$$

such that dev_τ is a developing map of Y_τ for every $\tau \in U$.

Proof. Let $dev^0 : U \times \mathbb{H}^n \to \mathbb{M}^{n+1}$ be the map defined in (7). Now, let us fix $\tau \in U$, $x \in \mathbb{H}^n$ and $t > 0$. Consider the timelike geodesic γ in $\mathcal{D}_\tau$ which passes through $dev^0_\tau(x)$ and has the direction of the normal field at $dev^0_\tau(x)$. Now, let $dev(\tau, t, x)$ be the point on γ with CT equal to t:

$$dev(\tau, t, x) = r_\tau(dev^0_\tau(x)) \, + \, tN_\tau(dev^0_\tau(x)).$$

Clearly, *dev* satisfies the three properties required. On the other hand, by Propositions 6.2 and 6.5, we can easily see that it is continuous.

q.e.d.

Remark 6.8. With the proof of Theorem 6.7, the proof of Theorem 2.4 is complete. In the following section, we shall prove Theorem 2.5.

Remark 6.9. Notice that the coordinate t on $\mathbb{R}_+ \times \widetilde{M}$ coincides with the pull-back of the cosmological time under the map dev_τ, i.e.,

$$T_\tau(dev_\tau(t,x)) = t.$$

On the other hand, notice that $r_\tau(dev_\tau(t,x))$ and $N_\tau(dev_\tau(t,x))$ depend only on the x coordinate. Thus, there are well-defined functions

$$\mathbf{r}_\tau : \widetilde{M} \to \Sigma_\tau$$
$$\mathbf{N}_\tau : \widetilde{M} \to \mathbb{H}^n$$

such that $r_\tau(dev_\tau(t,x)) = \mathbf{r}_\tau(x)$ and $N_\tau(dev_\tau(t,x)) = \mathbf{N}_\tau(x)$.

The map *dev* is only continuous. But, we can smooth this map to obtain a $\mathbb{C}^\infty$-map dev' which verifies the properties required in Theorem 6.7. In fact, it is easy to see that, we can perturb the normal field N_τ on $\mathcal{D}_\tau$ to obtain a Γ_τ-invariant timelike smooth vector field V_τ. By considering the restriction of the flow of this vector field on the Cauchy surface $\widetilde{F}_\tau$, we obtain a smooth developing map dev'_τ which verifies Properties 1 and 2 of Theorem 6.7.

We can construct the field V_τ such that V_0 coincides with N_0 and V_τ "varies continuously" with τ in the following sense: for every convergent sequence $\tau_k \to \tau$ and for every open set $K \subset \mathcal{D}_\tau$, the fields $V_{\tau_k}|_K$ tend to $V_\tau|_K$ in the C^∞-topology. In this way, it is easy to see that $dev'(\tau, x) = dev'_\tau(x)$ is a smooth map which verifies the properties required in the theorem.

7. Gromov Convergence of the CT-Level Surfaces

Let us summarize what we have seen until now. We have fixed a closed hyperbolic n-manifold M and we have identified $\pi_1(M)$ with a torsion-free co-compact discrete subgroup of $\mathrm{SO}^+(n,1)$, say Γ, such that $M = \mathbb{H}^n/\Gamma$. Given a cocycle $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$, we have considered the deformation Γ_τ of Γ. We have proved that there exists a unique Γ_τ-invariant future complete regular domain $\mathcal{D}_\tau$ such that the action is free and properly discontinuous and the quotient is a globally hyperbolic spacetime diffeomorphic to $\mathbb{R}_+ \times M$. The domain $\mathcal{D}_\tau$ is provided with a Γ_τ-invariant regular cosmological time T which is a C^1-submersion. Moreover, there exist a retraction map $r : \mathcal{D}_\tau \to \Sigma$ onto the singularity in the past and a normal field $N : \mathcal{D}_\tau \to \mathbb{H}^n$ which is, up to sign, the Lorentzian gradient of the cosmological time T. The level surfaces $\widetilde{S}_a = T^{-1}(a)$ are spacelike C^1-hypersurfaces, so that there is a natural path-distance d_a on them. Since $\widetilde{S}_a$ is Γ_τ-invariant and $\widetilde{S}_a/\Gamma_\tau \cong M$ is compact by Hopf–Rinow Theorem, we can deduce that d_a is a complete distance. In this section, we shall study the metric properties of the surface $\widetilde{S}_a$ and in particular, the asymptotic behaviour when $a \to +\infty$ and when $a \to 0$.

In the previous section, we have constructed a developing map

$$dev_\tau : \mathbb{R}_+ \times \widetilde{M} \to \mathbb{M}^n$$

such that $dev_\tau(\{a\} \times \widetilde{M}) = \widetilde{S}_a$ and there are well-defined maps

$$\mathbf{r} : \widetilde{M} \to \Sigma$$
$$\mathbf{N} : \widetilde{M} \to \mathbb{H}^n$$

such that $r(dev_\tau(t,x)) = \mathbf{r}(x)$ and $N(dev_\tau(t,x)) = \mathbf{N}(x)$. By taking the pull-back of the distance d_a on $\widetilde{S}_a$, we obtain a family of distances δ_a on $\widetilde{M}$ such that $\pi_1(M)(=\Gamma)$ acts by isometries on $(\widetilde{M}, \delta_a)$.

The main results of this section are the following propositions.

Proposition 7.1. *For all $x, y \in \widetilde{M}$*

$$\lim_{a\to+\infty} \frac{\delta_a(x,y)}{a} = d_{\mathbb{H}}(\mathbf{N}(x), \mathbf{N}(y))$$

where $d_{\mathbb{H}}$ is the distance of $\mathbb{H}^n$. Moreover, the maps $a^{-1}\delta_a$ tend in the compact-open topology of $C(\widetilde{M}\times\widetilde{M})$ to the map $(x,y) \mapsto d_{\mathbb{H}}(\mathbf{N}(x), \mathbf{N}(y))$.

Proposition 7.2. *There exists a natural distance d_Σ on Σ such that*

$$\lim_{a\to 0} \delta_a(x,y) = d_\Sigma(\mathbf{r}(x), \mathbf{r}(y)) \qquad \text{for all } x \in \widetilde{M}.$$

Moreover, δ_a tends in the compact-open topology to the map $(x,y) \mapsto d_\Sigma(\mathbf{r}(x), \mathbf{r}(y))$.

We shall see that Proposition 7.1 implies that the action of Γ_τ on $\widetilde{S}_a$ tends in the Gromov sense to the action of Γ on $\mathbb{H}^n$ when $a \to +\infty$. On the other hand, when $a \to 0$, we can deduce by Proposition 7.2 only the convergence of the spectra of the action of Γ_τ on $\widetilde{S}_a$ to the spectrum of the action of Γ_τ on Σ.

Let us start by showing that $\{\delta_a\}_{a>0}$ and $\{a^{-1}\delta_a\}_{a>0}$ are respectively increasing and decreasing functions of a.

Lemma 7.3. *Take a Lipschitz path* $c : [0,1] \to \widetilde{S}_a$. *Then, the paths* $N(t) = N(c(t))$ *and* $r(t) = r(c(t))$ *are differentiable almost everywhere and we have*

$$N(t) = N(0) + \int_0^t \dot{N}(s)\mathrm{d}s;$$
$$r(t) = r(0) + \int_0^t \dot{r}(s)\mathrm{d}s.$$

Moreover, we have that $\dot{N}(t)$ *and* $\dot{r}(t)$ *lie in* $T_{c(t)}\widetilde{S}_a$ *(so they are space-like) and* $\left\langle \dot{N}(t), \dot{r}(t) \right\rangle > 0$ *almost everywhere.*

Proof. In order to prove the first statement, it is sufficient to show that the maps $N : \widetilde{S}_a \to \mathbb{H}^n \subset \mathbb{M}^{n+1}$ and $r : \widetilde{S}_a \to \Sigma \subset \mathbb{M}^{n+1}$ are locally Lipschitz with respect to the *Euclidean distance* d_E of $\mathbb{M}^{n+1}$. Since $p = r(p) + aN(p)$, it is sufficient to show that N is locally Lipschitz.

Let us fix a compact $K \subset \widetilde{S}_a$ and set $H = N(K)$. Since H is compact, there exists a constant C such that

$$d_E(x, y) = ||x - y|| \leq C(\langle x - y, x - y\rangle)^{1/2}.$$

On the other hand, by inequalities (4), we have that

$$(\langle N(p) - N(q), N(p) - N(q)\rangle)^{1/2} \leq \frac{1}{a}(\langle p - q, p - q\rangle)^{1/2}.$$

Since $\langle p - q, p - q\rangle \leq ||p - q||^2$, we can deduce that $||N(p) - N(q)|| \leq \frac{C}{a}||p - q||$ for all $p, q \in K$.

Now, notice that $N(t)$ is a path in $\mathbb{H}^n$ so $\dot{N}(t) \in T_{N(t)}\mathbb{H}^n = T_{c(t)}\widetilde{S}_a$. Since $\dot{r}(t) = \dot{c}(t) - a\dot{N}(t)$, we have $\dot{r}(t) \in T_{c(t)}\widetilde{S}_a$ almost everywhere. Finally, inequalities (4) show that $\langle N(t + h) - N(t), r(t + h) - r(t)\rangle \geq 0$. Thus, we can easily deduce that $\left\langle \dot{N}(t), \dot{r}(t) \right\rangle \geq 0$. q.e.d.

Lemma 7.4. *For all* $x, y \in \widetilde{M}$ *and* $a < b$, *we have*

$$\delta_a(x, y) \leq \delta_b(x, y);$$
$$d_{\mathbb{H}}(\mathbf{N}(x), \mathbf{N}(y)) \leq \frac{1}{b}\delta_b(x, y) \leq \frac{1}{a}\delta_a(x, y).$$

Proof. For $t > 0$, let us set $p_t = dev_\tau(t, x)$ and $q_t = dev_\tau(t, y)$. Let $c_b : [0,1] \to \widetilde{S}_b$ be a length-minimizing geodesic path between p_b and q_b. Consider $r(t) = r(c_b(t))$ and $N(t) = N(c_b(t))$ and let $c : [0,1] \to \widetilde{S}_a$ be the path defined by the rule $c(t) = r(t) + aN(t)$. We have that c is a rectifiable arc between p_a and q_a so that the length of c is greater than the distance $\delta_a(x, y)$. Now, notice that $\dot{c}_b(t) = \dot{c}(t) + (b - a)\dot{N}(t)$. By

Lemma 7.3, we have that $\langle \dot{c}_b(t), \dot{c}_b(t)\rangle \geq \langle \dot{c}(t), \dot{c}(t)\rangle$. This proves that the length of c_b is greater than the length of c and so, the first inequality holds. Now, we shall prove the second one.

Let $c : [0,1] \to \widetilde{S}_a$ be a Lipschitz path. For simplicity let us set $N(t) = N(c(t))$. By Lemma 7.3, we have that $a^2 \left\langle \dot{N}(t), \dot{N}(t)\right\rangle \leq \langle \dot{c}(t), \dot{c}(t)\rangle$. By this inequality, it follows that $d_{\mathbb{H}}(N(p), N(q)) \leq \frac{1}{a} d_a(p,q)$ for all $p, q \in \widetilde{S}_a$. On the other hand, for fixed $x, y \in \widetilde{M}$ let $c_a : [0,1] \to \widetilde{S}_a$ be the d_a-minimizing geodesic between $p_a = dev_\tau(a,x)$ and $q_a = dev_\tau(a,y)$. Let us define $c(t) = c_a(t) + (b-a)N(t)$ (where $N(t) = N(c_a(t))$): the endpoints of this path are p_b and q_b so that

$$\frac{\delta_b(x,y)}{b} = \frac{d_b(p_b, q_b)}{b} \leq \frac{1}{b}\int_0^1 \langle \dot{c}(t), \dot{c}(t)\rangle^{1/2}\,\mathrm{d}t.$$

By Lemma 7.3, we know that $a^2 \left\langle \dot{N}(t), \dot{N}(t)\right\rangle \leq \langle \dot{c}_a(t), \dot{c}_a(t)\rangle$ so that

$$\begin{aligned}
\langle \dot{c}(t), \dot{c}(t)\rangle^{1/2} &\leq \langle \dot{c}_a(t), \dot{c}_a(t)\rangle^{1/2} + (b-a)\left\langle \dot{N}(t), \dot{N}(t)\right\rangle^{1/2} \\
&\leq \langle \dot{c}_a(t), \dot{c}_a(t)\rangle^{1/2} + \frac{(b-a)}{a}\langle \dot{c}_a(t), \dot{c}_a(t)\rangle^{1/2} \\
&= \frac{b}{a}\langle \dot{c}_a(t), \dot{c}_a(t)\rangle^{1/2}.
\end{aligned}$$

Thus, we have

$$\frac{\delta_b(x,y)}{b} \leq \frac{1}{a}\int_0^1 (\langle \dot{c}_a(t), \dot{c}_a(t)\rangle)^{1/2}\mathrm{d}t = \frac{\delta_a(x,y)}{a}.$$

q.e.d.

Now, we can prove Proposition 7.1.

Proof of Proposition 7.1. Let us fix $x, y \in \widetilde{M}$. By Lemma 7.4, we have that $\frac{1}{a}\delta_a(x,y)$ is a decreasing function of a so there exists

$$\delta_\infty(x,y) = \lim_{a \to +\infty} \frac{1}{a}\delta_a(x,y).$$

Let us show that $a^{-1}\delta_a$ tends to δ_∞ in the compact-open topology. Since $a^{-1}\delta_a \leq \delta_1$ the family $\{a^{-1}\delta_a|_K\}_{a>1}$ is locally bounded. On the other hand, by triangular inequality, we have for $a > 1$

$$\begin{aligned}
|a^{-1}\delta_a(x,y) - a^{-1}\delta_a(x',y')| &\leq a^{-1}\delta_a(x,x') + a^{-1}\delta_a(y,y') \\
&\leq \delta_1(x,x') + \delta_1(y,y').
\end{aligned}$$

Thus, the family $\{a^{-1}\delta_a|_K\}_{a>1}$ is equicontinuous. By these remarks, we easily have that $a^{-1}\delta_a \to \delta_\infty$ in the compact-open topology of $\widetilde{M} \times \widetilde{M}$.

Clearly, δ_∞ is a pseudo-distance on $\widetilde{M}$. We claim that $\delta_\infty(x,y)=0$ if and only if $\mathbf{N}(x)=\mathbf{N}(y)$. In fact, from Lemma 7.4, we have that if $d_\infty(x,y)=0$, then $\mathbf{N}(x)=\mathbf{N}(y)$. On the other hand, if $\mathbf{N}(x)=\mathbf{N}(y)$, the segment $[\mathbf{r}(x),\mathbf{r}(y)]$ is contained in Σ and one easily see that $\frac{\delta_a(x,y)}{a}=\frac{1}{a}(\langle\mathbf{r}(y)-\mathbf{r}(x),\mathbf{r}(y)-\mathbf{r}(x)\rangle)^{1/2}$. By passing to the limit, we obtain that $\delta_\infty(x,y)=0$.

It follows that there exists a distance d on $\mathbb{H}^n$ such that

$$\delta_\infty(x,y)=d(\mathbf{N}(x),\mathbf{N}(y))$$

In the last part of this proof, we shall show that $d=d_{\mathbb{H}}$. We already know that $d_{\mathbb{H}}\le d$.

By using Theorem 5.1, it is easy to see that $\mathcal{D}_{\frac{\tau}{a}}=\frac{1}{a}\mathcal{D}_\tau$. Let us consider the map

$$f:\mathcal{D}_\tau\ni p\mapsto\frac{p}{a}\in\mathcal{D}_{\frac{\tau}{a}}.$$

For $p\in\mathcal{D}_\tau$, let c_p be the Lorentzian-length maximizing timelike geodesic of $\mathcal{D}_\tau$ with future-endpoint equal to p. Then, $f(c_p)$ is a Lorentzian-length maximizing timelike geodesic of $\mathcal{D}_{\frac{\tau}{a}}$. Since the length of $f(c_p)$ is $a^{-1}T_\tau(p)$, we have

$$T_{\frac{\tau}{a}}\left(\frac{p}{a}\right)=\frac{T_\tau(p)}{a}.$$

Thus, $\frac{1}{a}\widetilde{S}_a$ is the CT-level surface $\widetilde{S}_1(\frac{\tau}{a})=T_{\frac{\tau}{a}}^{-1}(1)$. Moreover, the distance $a^{-1}\delta_a$ is the pull-back of the natural path-distance on $\widetilde{S}_1(\frac{\tau}{a})$.

Since $p_a=\mathbf{r}(x)+a\mathbf{N}(x)$ and $q_a=\mathbf{r}(y)+a\mathbf{N}(y)$, we have $\lim_{a\to+\infty}\frac{p_a}{a}=\mathbf{N}(x)$ and $\lim_{a\to+\infty}\frac{q_a}{a}=\mathbf{N}(y)$ (recall that $p_a=dev_\tau(a,x)$ and $q_a=dev_\tau(a,y)$).

Now, let $c:[0,1]\to\mathbb{H}^n$ be a geodesic path between $\mathbf{N}(x)$ and $\mathbf{N}(y)$

$$c(t)=(\psi_0(u(t)),u(t))$$

where $\psi_0:\{y_0=0\}\to\mathbb{R}$ is the function such that $\mathbb{H}^n=\mathrm{graph}\,(\psi_0)$. Let $c_a(t)=(\psi_a(u(t)),u(t))$ be the corresponding path on the surface $\widetilde{S}_1(\frac{\tau}{a})$ and let us consider $p'_a=c_a(0)$ and $q'_a=c_a(1)$. Since $\psi_a\to\psi_0$ in C^1-topology, we have that

$$\int_0^1(\langle\dot c_a(t),\dot c_a(t)\rangle)^{1/2}\mathrm{d}t\to\int_0^1(\langle\dot c(t),\dot c(t)\rangle)^{1/2}\mathrm{d}t=d_{\mathbb{H}}(\mathbf{N}(x),\mathbf{N}(y)).$$

Let us set $a^{-1}p_a=(\psi_a(v_a),v_a)$ and $a^{-1}q_a=(\psi_a(w_a),w_a)$. It is easy to see that

$$\frac{\delta_a(x,y)}{a}\le||v_a-u(0)||+||w_a-u(1)||+\int_0^1(\langle\dot c_a(t),\dot c_a(t)\rangle)^{1/2}$$

Since $v_a \to u(0)$ and $w_a \to u(1)$ by passing to the limit, we get

$$d(\mathbf{N}(x), \mathbf{N}(y)) \leq d_{\mathbb{H}}(\mathbf{N}(x), \mathbf{N}(y)).$$

q.e.d.

We want to show that the action of Γ_τ on $(\widetilde{S}_a, a^{-1}d_a)$ tends in the Gromov sense to the action of Γ on $\mathbb{H}^n$ for $a \to +\infty$. For a complete definition of convergence in the Gromov sense of a sequence of isometric actions on metric spaces see e.g., [**15**]. However, we need only the following statement which is an immediate corollary of the definition.

Suppose that Γ acts by isometries on metric spaces (X_i, d_i) for $i \in \mathbb{N}$ and on a metric space (X_∞, d_∞). Suppose that there exists a sequence of Γ-equivariant maps $\pi_i : X_i \to X_\infty$ which verifies the following property: for every compact subset K_∞ of X_∞ and $\varepsilon > 0$ for $i \gg 0$, there exists a compact set K_i such that $\pi_i(K_i) = K_\infty$ and $|d_\infty(\pi_i(x), \pi_i(y)) - d_i(x, y)| < \varepsilon$ for all $x, y \in K_i$. Then, the action of Γ on X_i tends in the Gromov sense to the action of Γ on X_∞.

Corollary 7.5. *The action of Γ_τ on the rescaled surfaces $(\widetilde{S}_a, a^{-1}d_a)$ tends in the Gromov sense to the action of Γ on $\mathbb{H}^n$ when $a \to +\infty$.*

Proof. We want to see that the maps $N : \widetilde{S}_a \to \mathbb{H}^n$ satisfy the above condition. Let us fix a compact set $K \subset \mathbb{H}^n$ and set $H = \mathbf{N}^{-1}(K) \subset \widetilde{M}$. Since $\mathbf{N}$ is a proper map, we have that H is compact. Let $K_a = dev_\tau(a, H) \subset \widetilde{S}_a$. By Proposition 7.1, we have that for all $\varepsilon > 0$, there exists a_0 such that for all $a > a_0$

$$\left| \frac{d_a(x, y)}{a} - d_{\mathbb{H}}(N(x), N(y)) \right| \leq \varepsilon \qquad \text{for all } x, y \in K_a$$

q.e.d.

Now, we are interested in the asymptotic behaviour of the distances δ_a when $a \to 0$. The results that we obtain are similar to those that we have proved in the previous case. However, in this case, we shall see that there are some technical problems in the proofs. In particular, since $\partial\mathcal{D}_\tau$ is an achronal set, a notion of length of a curve is defined. However, the length of a non-constant curve can be zero. Taking the infimum of the lengths of the curves with fixed endpoints, yields a *pseudo-distance* on $\partial\mathcal{D}_\tau$. The first problem arises when we try to prove that this pseudo-distance restricted to Σ is in fact a distance. It seems to be more convenient to change viewpoint. First, we shall show that the distances δ_a tend to a pseudo-distance δ_0 on $\widetilde{M}$ such that

$$\delta_0(x, y) = 0 \quad \Leftrightarrow \quad \mathbf{r}(x) = \mathbf{r}(y).$$

This implies that there exists a distance d_Σ on Σ such that $\delta_0(x,y) = d_\Sigma(\mathbf{r}(x),\mathbf{r}(y))$. Later, we shall prove that this distance coincides with the natural path-distance on Σ.

Proposition 7.6. *There exists a pseudo-distance δ_0 on $\widetilde{M}$ such that $\delta_a \to \delta_0$ in the compact-open topology. Moreover, $\delta_0(x,y) = 0$ if and only if $\mathbf{r}(x) = \mathbf{r}(y)$.*

Proof. By using Lemma 7.4, we can easily argue the first statement as in the proof of Proposition 7.1.

The proof of the second statement is more difficult. We need the following technical lemma.

Lemma 7.7. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex C^1-function such that $||\nabla\varphi(x)|| < 1$ for all $x \in \mathbb{R}^n$. Let $S = \{(x_0,\ldots,x_n) \in \mathbb{M}^{n+1} | x_0 = \varphi(x_1,\ldots,x_n)\}$ be the corresponding spacelike surface in Minkowski space and suppose S to be complete. Let 0 be a minimum point of φ, then for every $y \in \mathbb{R}^n$, there exists a distance-minimizing geodesic arc $c(t) = (\varphi(x(t)), x(t))$ with starting point equal to $(\varphi(0), 0)$ and ending point equal to $(\varphi(y), y)$ such that the functions $t \mapsto ||x(t)||$ and $t \mapsto \varphi(x(t))$ are increasing ($||\cdot||$ is the Euclidean norm of $\mathbb{R}^n$).*

Proof of the lemma. First, suppose that φ is C^∞. By imposing that c is a geodesic, we can deduce that the path $x(t)$ satisfies the following equation

$$\ddot{x}(t) = \frac{\dot{x}\cdot \mathrm{H}\varphi(x)\cdot\dot{x}}{(1-||\nabla\varphi(x)||^2)^{3/2}}\nabla\varphi(x)$$

where $\mathrm{H}\varphi(x)$ is the Hessian matrix of φ at x.

If we set $f(t) = ||x(t)||^2$, we have

$$\dot{f}(t) = 2x(t)\cdot\dot{x}(t);$$
$$\ddot{f}(t) = 2\Big(\dot{x}(t)\cdot\dot{x}(t) + x(t)\cdot\ddot{x}(t)\Big).$$

Now, we have that $x(0) = 0$ so $\dot{f}(0) = 0$. Hence, it is sufficient to prove that $\ddot{f}(t) \geq 0$ for $t \geq 0$. By looking at the last expression, it follows that it is sufficient to show that $x(t)\cdot\ddot{x}(t) \geq 0$. Since φ is convex, we have that $\dot{x}\cdot\mathrm{H}\varphi(x)\cdot\dot{x} \geq 0$ so $x(t)\cdot\ddot{x}(t) \geq 0$ if and only if $x\cdot\nabla\varphi(x) \geq 0$. On the other hand, since $\varphi(0)$ is the minimum of φ, by imposing convexity of φ on the rays starting from 0, we easily deduce that this inequality holds for all $x \in \mathbb{R}^n$. An analogous computation shows that $t \mapsto \varphi(x(t))$ is increasing.

Now, suppose that φ is only C^1. Let $\{\rho_\varepsilon\}$ be a family of positive C^∞-functions on $\mathbb{R}^n$ such that:

1. $\mathrm{supp}\rho_\varepsilon = \{x \in \mathbb{R}^n | \, ||x|| \leq \varepsilon\}$;

2. $\int_{\mathbb{R}^n} \rho_\varepsilon = 1$.

Let φ_ε be the convolution $\varphi * \rho_\varepsilon$

$$\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi(x-y)\rho_\varepsilon(y)\mathrm{d}y.$$

We know that φ_ε is C^∞ and $\varphi_\varepsilon \to \varphi$ in C^1-topology. Moreover, it is easy to see that φ_ε is a convex function so that $S_\varepsilon := \operatorname{graph}(\varphi_\varepsilon)$ is a smooth future convex spacelike surface.

Let us fix $y \in \mathbb{R}^n$. By using completeness of S, we have that for $\varepsilon \ll 1$, there exists a path

$$x_\varepsilon : [0, L_\varepsilon] \to \mathbb{R}^n$$

such that

1) $c_\varepsilon(t) = (\varphi_\varepsilon(x_\varepsilon(t)), x_\varepsilon(t))$ is a parametrization of a distance-minimizing geodesic arc on the surface S_ε;
2) $x_\varepsilon(0) = x$ and $x_\varepsilon(L_\varepsilon) = y$;
3) $||\dot{x}_\varepsilon(t)|| = 1$ and L_ε is bounded.

Thus, x_ε tends to a Lipschitz arc $x(t)$ and it is easy to see that the path $t \mapsto (\varphi(x(t)), x(t))$ is a distance-minimizing geodesic between $(\varphi(0), 0)$ and $(\varphi(y), y)$.

Let $p_\varepsilon(t)$ be the orthogonal projection of $c_\varepsilon(t)$ onto $T_{(\varphi_\varepsilon(x),x)}S_\varepsilon$

$$p_\varepsilon(t) = c_\varepsilon(t) + \frac{\langle c_\varepsilon(t), (1, \nabla\varphi_\varepsilon(x))\rangle}{1 - ||\nabla\varphi_\varepsilon(x)||^2}(1, \nabla\varphi_\varepsilon(x)).$$

We know that $\langle p_\varepsilon(t), p_\varepsilon(t)\rangle$ is an increasing function of t. On the other hand, since $\nabla\varphi_\varepsilon(x) \to 0$, we have that $p_\varepsilon(t) \to x(t)$ as $\varepsilon \to 0$. Thus, $||x(t)||$ is an increasing function of t. q.e.d.

Let us go back to the proof of Proposition 7.6. We have to show $\mathbf{r}(x) = \mathbf{r}(y)$ for all $x, y \in \widetilde{M}$ such that $\delta_0(x, y) = 0$. By contradiction, suppose that there exist $x, y \in \widetilde{M}$ such that $\delta_0(x, y) = 0$ and $\mathbf{r}(x) \neq \mathbf{r}(y)$. Let us fix a set of affine coordinates $(y_0, \ldots, y_n)$ in such a way that $\frac{\partial}{\partial y_0} = \mathbf{N}(x)$ and $\mathbf{r}(x) = 0$. Let φ_a and φ be the functions defined over the horizontal plane such that $\widetilde{S}_a = \operatorname{graph}(\varphi_a)$ and $\partial\mathcal{D}_\tau = \operatorname{graph}(\varphi)$. We have $p_a = dev(a, x) = (a, 0)$ and $q_a = dev(a, y) = (\varphi_a(z_a), z_a)$. Now, for every $a > 0$, let us fix a distance-minimizing geodesic path

$$c_a(t) = (\varphi_a(x_a(t)), x_a(t)) \qquad \text{for } t \in [0, L_a]$$

between p_a and q_a such that $||x_a(t)||$ is increasing. Since $z_a \to z_0$, for $a \to 0$, there exists a constant K such that

$$||x_a(t)|| \leq K \qquad \text{for all } a \leq 1.$$

First, suppose that there exists a sequence a_k such that L_{a_k} is bounded. Then, up to passing to a subsequence, we have that x_{a_k} tends to a 1-Lipschitz path $x : [0, L] \to \mathbb{R}^n$ such that $x(0) = 0$ and $x(L) = z_0$. Let us set $\varphi_a(t) = \varphi_a(x(t))$. Then, by the hypothesis on δ_0, we have that

$$\lim_{k\to+\infty} \int_0^{L_{a_k}} \sqrt{1 - \dot{\varphi}_{a_k}(t)^2}\mathrm{d}t = 0.$$

Thus, $|\dot{\varphi}_{a_k}(t)| \to 1$ for almost all $t \in [0, L]$. By Lemma 7.7, we know that $\varphi_{a_k}(x_{a_k}(t))$ are increasing functions of t so that $\dot{\varphi}_{a_k}(t) \to 1$. Thus, it follows that $\varphi_{a_k}(t) - \varphi_{a_k}(s) \to t - s$. On the other hand, we have that $\varphi_{a_k}(t) - \varphi_{a_k}(s) \to \varphi(x(t)) - \varphi(x(s))$. So, we obtain $\varphi(x(t)) = t$. Thus, we have that the path $t \mapsto (\varphi(x(t), x(t))$ is a null path contained in $\partial\mathcal{D}_\tau$ between $\mathbf{r}(x)$ and $\mathbf{r}(y)$. But this is a contradiction (in fact, no point in Σ lies in the interior of any null ray contained in $\partial\mathcal{D}_\tau$).

Hence, suppose that $L_a \to +\infty$. Then, there exist a sequence $a_k \to 0$ and a Lipschitz path

$$x : [0, +\infty) \to \mathbb{R}^n$$

such that $x_{a_k} \to x$ in the compact-open topology. Since $||x_{a_k}(t)|| \leq K$, we have that $||x(t)|| \leq K$. On the other hand, the same argument used above shows that $\varphi(x(t)) = t$. Since φ is 1-Lipschitz, we have $||x(t)|| \geq t$ and this gives a contradiction. q.e.d.

From this proposition, it follows that there exists a distance d on Σ such that

$$d(\mathbf{r}(x), \mathbf{r}(y)) = \lim_{a\to 0} \delta_a(x, y) \qquad \text{for all } x, y \in \widetilde{M}.$$

We have to check that d coincides with the natural path-distance d_Σ. For $r, s \in \Sigma$, let us denote by $C(r, s)$ the set of Lipschitzian paths (with respect to the Euclidean distance on $\mathbb{M}^{n+1}$) in $\partial\mathcal{D}_\tau$ between r and s. Then, $d_\Sigma(r, s)$ is defined by the rule

$$d_\Sigma(r, s) := \inf_{c\in C(r,s)} \int \sqrt{\langle \dot{c}(s), \dot{c}(s)\rangle}\mathrm{d}s.$$

Proposition 7.8. *We have $d_\Sigma(r, s) = d(r, s)$ for all $r, s \in \Sigma$.*

Proof. It is easy to see that if $c : [0, 1] \to \widetilde{S}_a$ is a rectifiable path then, $r \circ c$ is a rectifiable path with a length less than the length of c. It follows that $d_\Sigma(\mathbf{r}(x), \mathbf{r}(y)) \leq \delta_a(x, y)$. Thus, $d_\Sigma(r, s) \leq d(r, s)$.

Let us show the other inequality. Let $r, s \in \Sigma$ and $x, y \in \widetilde{M}$ be such that $\mathbf{r}(x) = r$ and $\mathbf{r}(y) = s$. Moreover, let us set $p_a = dev(a, x)$ and $q_a = dev(a, y)$. Let $(y_0, \ldots, y_n)$ be a set of affine orthonormal coordinates and $\varphi : \mathbb{R}^n \to \mathbb{R}$ (resp. $\varphi_a : \mathbb{R}^n \to \mathbb{R}$) be such that $\partial\mathcal{D}_\tau = \mathrm{graph}\,(\varphi)$

(resp. $\widetilde{S}_a = \mathrm{graph}\,(\varphi_a)$). We have $r = (\varphi(u), u)$, $s = (\varphi(v), v)$, $p_a = (\varphi_a(u_a), u_a)$ and $q_a = (\varphi_a(v_a), v_a)$. Finally, let us set $q'_a = (\varphi_a(u), u)$ and $p'_a = (\varphi_a(u), u)$.

Now, consider the set E of points in $\mathbb{R}^n$ where φ is not differentiable. There exists a sequence of 1-Lipschitz path $x_k : [0, L_k] \to \mathbb{R}^n$ between u and v such that $x_k^{-1}(E)$ has null Lebesgue measure on $[0, L_k]$ and

$$\lim_{k\to+\infty} \int_0^{L_k} \sqrt{1 - (\nabla\varphi(x_k(t)) \cdot \dot{x}_k(t))^2}\mathrm{d}t = d_\Sigma(r, s).$$

Consider the path $c_a^k(t) = (\varphi_a(x_k(t)), x_k(t))$. It is a path in $\widetilde{S}_a$ between q'_a and p'_a so that

$$d_a(p_a, q_a) \leq d_a(p_a, p'_a) + \int_0^{L_k} \sqrt{1 - (\nabla\varphi_a(x_k(t)) \cdot \dot{x}_k(t))^2}\mathrm{d}t + d_a(q_a, q'_a).$$

Notice $(\varphi_a(x), x) + (1, \nabla\varphi_a(x))^\perp$ is a support plane for $\widetilde{S}_a$ at $(\varphi_a(x), x)$. So the sequence of planes $(\varphi_a(x), x) + (1, \nabla\varphi_a(x))^\perp$ converges to a support plane for $\partial\mathcal{D}_\tau$ in $(\varphi(x), x)$. Thus, it is easy to see that $\nabla\varphi_a(x) \to \nabla\varphi(x)$ for $a \to 0$ and for all $x \in \mathbb{R}^n - E$. It follows that

$$\begin{aligned}\lim_{a\to 0} &\int_0^{L_k} \sqrt{1 - (\nabla\varphi_a(x_k(t)) \cdot \dot{x}_k(t))^2}\mathrm{d}t \\ &= \int_0^{L_k} \sqrt{1 - (\nabla\varphi(x_k(t)) \cdot \dot{x}_k(t))^2}\mathrm{d}t.\end{aligned}$$

Now, we have that $d_a(p_a, p'_a) \leq ||u - u_a||$ (resp. $d_a(q_a, q'_a) \leq ||v - v_a||$) so by passing to the limit for $a \to 0$ we have

$$d(r, s) \leq \int_0^{L_k} \sqrt{1 - (\nabla\varphi(x_k(t)) \cdot \dot{x}_k(t))^2}\mathrm{d}t \qquad \text{for all } k.$$

By passing to the limit for $k \to +\infty$, we obtain $d(r, s) \leq d_\Sigma(r, s)$. q.e.d.

In order to show Gromov convergence of $\widetilde{S}_a$ to Σ, notice that we cannot use the argument of Corollary 7.5. In fact, there exists a compact set of (Σ, d_Σ) such that for every $a > 0$, no compact set in $\widetilde{S}_a$ projects onto it. For instance, consider the case $n = 2$ and let $\tau \in Z^1(\Gamma, \mathbb{R}^{2+1})$ be such that the lamination $\mathcal{L}$ associated to $\mathcal{D}_\tau$ is simplicial. In this case, the singularity is a simplicial tree such that every vertex is the endpoint of a numerable set of edges. Let us fix a vertex r_0 and consider a numeration $(e_k)_{k\in\mathbb{N}}$ of the edges with an endpoint equal to r_0. Let

$$K = \bigcup_{k\in\mathbb{N}} \{r \in e_k | d_\Sigma(r, r_0) \leq C/k\}$$

where C is the minimum of the lengths of edges in Σ. It is easy to see that K is compact. By contradiction suppose that for some $a > 0$, there exists a compact K_a such that $r(K_a) = K$. Now, let $\mathcal{F}(r_0) = N(r^{-1}(r_0))$: it is a complementary region of the lamination and $\mathcal{F}(r)$ is a component of the boundary of $\mathcal{F}(r_0)$ for all $r \in K$. Moreover, $\mathcal{F}(r)$ depends only on the edge which contains r. Let F_k be the leaf corresponding to e_k. Now, let us fix $p_0 \in K_a$ such that $r(p_0) = r_0$ and for all k, let $p_k \in K_a$ be such that $r(p_k) \in e_k$. We have that $d_a(p_k, p_0) \geq a d_{\mathbb{H}}(F_k, N(p_0))$. On the other hand, $d_{\mathbb{H}}(F_k, N(p_0)) \to +\infty$ for $k \to +\infty$ and this contradicts the compactness of K_a.

In what follows, we shall prove the convergence of the spectra of the Γ_τ-action on $\widetilde{S}_a$ to the spectrum of the Γ_τ-action on Σ. In general, let (X, d) be a metric space provided with an action of Γ. For every $\gamma \in \Gamma$, we can define the *translation length* of γ as $\ell_X(\gamma) = \inf_{x \in X} d(x, \gamma \cdot x)$. Clearly, $\ell_X(\gamma)$ depends only on the conjugation class of γ and so a function $\ell_X : \mathcal{C} \to [0, +\infty)$ is defined on the set $\mathcal{C}$ of conjugation classes of $\Gamma - \{1\}$. This function is called the *marked length spectrum* of the action. For simplicity, we denote by ℓ_a ($a > 0$) the marked length spectrum of the Γ_τ-action on the CT level surface $\widetilde{S}_a$, by ℓ_0 the marked length spectrum of the Γ_τ-action on Σ and by $\ell_{\mathbb{H}}$ the spectrum of the action on $\mathbb{H}^n$.

Corollary 7.9. *With the above notation, we have that for all $\gamma \in \Gamma$:*

$$\lim_{a \to +\infty} \ell_a(\gamma_\tau)/a = \ell_{\mathbb{H}}(\gamma);$$
$$\lim_{a \to 0} \ell_a(\gamma_\tau) = \ell_0(\gamma_\tau).$$

Proof. The first limit is a consequence of the Gromov convergence. For the second limit, notice that $\ell_a(\gamma_\tau) \geq \ell_0(\gamma_\tau)$. On the other hand, let $x \in \widetilde{M}$: then, we have

$$\ell_a(\gamma) \leq \delta_a(x, \gamma x)$$

so that $\limsup_{a \to 0} \ell_a(\gamma) \leq d_\Sigma(\mathbf{r}(x), \gamma_\tau \mathbf{r}(x))$. We may conclude

$$\limsup_{a \to 0} \ell_a(\gamma_\tau) \leq \ell_0(\gamma_\tau).$$

q.e.d.

8. Measured Geodesic Stratification

In Section 4, we have associated a geodesic stratification of $\mathbb{H}^n$ to every future complete regular domain with a surjective normal field. In this section, we define the notion of *transverse measure* on a stratification. We have seen that in dimension $n = 2$, geodesic stratifications are in fact geodesic laminations. We shall see that for $n = 2$, transverse measures on geodesic stratifications are equivalent to transverse measures on the corresponding geodesic laminations (in the classical sense). Since the behaviour of a stratification is rather more complicated than the behaviour of a lamination, the general definition of transverse measure on a stratification is more involved.

We shall see that every measured geodesic stratification gives a future complete regular domain. In his work, Mess exposed a technique to associate a future complete regular domain of $\mathbb{M}^{2+1}$ to a measured geodesic lamination. This construction is a generalization of that technique to any dimension $n \geq 2$.

Let us fix a *complete weakly continuous geodesic stratification* $\mathcal{C}$. For $p \in \mathbb{H}^n$, let us denote by $C(p)$ the piece in $\mathcal{C}$ which contains p and has minimum dimension.

The first notion that we need is the transverse measure on a piece-wise geodesic path. Let $c : [0,1] \to \mathbb{H}^n$ be a *piece-wise geodesic path*. A *transverse measure* on it is a $\mathbb{R}^{n+1}$*-valued measure* μ_c on $[0,1]$ such that

1) There exists a finite positive measure $|\mu_c|$ such that μ_c is $|\mu_c|$-absolutely continuous and $\mathrm{supp}|\mu_c|$ is the topological closure of the set $\{t \in (0,1) |\, \dot{c}(t) \notin T_{c(t)}C(c(t))\}$;
2) Let $v_c = \frac{\mathrm{d}\mu_c}{\mathrm{d}|\mu_c|}$ be the $|\mu_c|$-density of μ_c, then,

$$\begin{aligned} & v_c(t) \in T_{c(t)}\mathbb{H}^n \cap T_{c(t)}C(c(t))^\perp, \\ & \langle v_c(t), v_c(t) \rangle = 1, \\ & \langle v_c(t), \dot{c}(t) \rangle > 0 \qquad\qquad |\mu_c| - \text{ a.e.} \end{aligned} \tag{9}$$

3) The endpoints of c are not atoms of the measure $|\mu_c|$.

Let us point out a useful property of transverse measures on geodesic paths.

Lemma 8.1. *Let $c : [0,1] \to \mathbb{H}^n$ be a* geodesic *path and μ_c be a transverse measure on it. Then, for $|\mu_c|$-almost all t, we have*

$$\langle c(0), v_c(t) \rangle < 0 \qquad \langle c(1), v_c(t) \rangle > 0.$$

Thus, $v_c(t)^\perp$ separates $c(0)$ from $c(1)$.

Proof. Since c is a geodesic path, there exists $v \in T_{c(0)}\mathbb{H}^n$ such that

$$c(t) = \cosh(s(t))c(0) + \sinh(s(t))v$$

with $s(t)$ an increasing function. By (9), we have $|\mu_c|$-almost everywhere

$$0 = \langle c(t), v_c(t)\rangle = \cosh(s(t))\,\langle c(0), v_c(t)\rangle + \sinh(s(t))\,\langle v, v_c(t)\rangle\,;$$
$$0 < \langle \dot c(t), v_c(t)\rangle = \dot s(t)\big(\sinh(s(t))\,\langle c(0), v_c(t)\rangle + \cosh(s(t))\,\langle v, v_c(t)\rangle\,\big).$$

Looking at these expressions, we can easily deduce $\langle c(0), v_c(t)\rangle < 0$. An analogous computation shows the other inequality. q.e.d.

A first consequence of this lemma is that the measure μ_c determines the positive measure $|\mu_c|$.

Corollary 8.2. *Let c be a piece-wise geodesic path and μ_c a transverse measure on it. Suppose that λ is a positive measure such that μ_c is λ-absolutely continuous and the density $u = \frac{\mathrm{d}\mu_c}{\mathrm{d}\lambda}$ verifies (9). Then, $\lambda = |\mu_c|$.*

Proof. First, let us show that λ is $|\mu_c|$-absolutely continuous. Let $E \subset [0,1]$ be such that $|\mu_c|(E) = 0$. We can suppose that E is contained in an interval $I = [t_0, t_1]$ such that $c|_I$ is a geodesic path. Since $\mu_c(E) = 0$, it follows

$$0 = \left\langle c(t_0), \int_E u(t)\mathrm{d}\lambda \right\rangle = \int_E \langle c(t_0), u(t)\rangle\,\mathrm{d}\lambda.$$

The same argument of Lemma 8.1 shows that $\langle c(t_0), u(t)\rangle < 0$ for λ-almost all $t \in I$. Thus, we have $\lambda(E) = 0$.

Now, let us set $a = \frac{\mathrm{d}\lambda}{\mathrm{d}|\mu_c|}$. We have

$$v_c(t) = \frac{\mathrm{d}\mu_c}{\mathrm{d}|\mu_c|} = \frac{\mathrm{d}\mu_c}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}|\mu_c|} = a(t)u(t) \qquad |\mu_c| - \text{a.e.}$$

Since $\langle v_c(t), v_c(t)\rangle = \langle u(t), u(t)\rangle = 1$, we can deduce that $a(t) = 1$. q.e.d.

In order to define a transverse measure on a geodesic stratification, we need the following definition.

Definition 8.1. Let $\varphi_s : [0,1] \to \mathbb{H}^n$ be a homotopy between φ_0 and φ_1. We say that φ is *$\mathcal{C}$-preserving* if $C(\varphi_s(t)) = C(\varphi_0(t))$ for all $(t,s) \in [0,1]\times[0,1]$ (recall that $C(x)$ is the piece of $\mathcal{C}$ which contains x and has minimum dimension).

Now, we can give the definition of transverse measure on a geodesic stratification.

Definition 8.2. Let $\mathcal{C}$ be a complete weakly continuous stratification and let us fix a subset $Y \subset \mathbb{H}^n$ which is a union of pieces of $\mathcal{C}$ such that the Lebesgue measure of Y is 0. We mean by $(\mathcal{C}, Y)$-*admissible path* (or simply *admissible path*) any piece-wise geodesic path $c : [0,1] \to \mathbb{H}^n$ such that every maximal geodesic sub-segment has no endpoint in Y.

A **transverse measure** on $(\mathcal{C}, Y)$ is the assignment of a transverse measure μ_c to every admissible path $c : [0,1] \to \mathbb{H}^n$ such that

1) If there exists a $\mathcal{C}$-preserving homotopy between two paths c and d, then $\mu_c = \mu_d$;
2) For every admissible path c and every parametrization $s : [0,1] \to [0,1]$ of an admissible sub-arc of c, we have that $\mu_{c \circ s} = s^*(\mu_c)$;
3) The atoms of $|\mu_c|$ are contained in $c^{-1}(Y)$, and for every $y \in Y$, there exists an admissible path c such that $|\mu_c|$ has some atoms on $c^{-1}(y)$;
4) $\mu_c(c) = 0$ for every closed admissible path c;
5) For all sequences $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in \mathbb{H}^n - Y$ and $x = \lim_{k\to+\infty} x_k \in \mathbb{H}^n - Y$, we have that $\mu_{c_k}(c_k) \to 0$ where c_k is the admissible arc $[x_k, x]$.

A **measured geodesic stratification** is given by a weakly continuous geodesic stratification, a subset Y as above and a measure μ on $(\mathcal{C}, Y)$.

A measured geodesic stratification $(\mathcal{C}, Y, \mu)$ is Γ-*invariant* if $\mathcal{C}$ is Γ-invariant, Y is Γ-invariant and we have

$$\mu_{\gamma\circ c}(E) = \gamma(\mu_c(E))$$

for all admissible paths $c : [0,1] \to \mathbb{H}^n$, borelian sets $E \subset [0,1]$ and $\gamma \in \Gamma$.

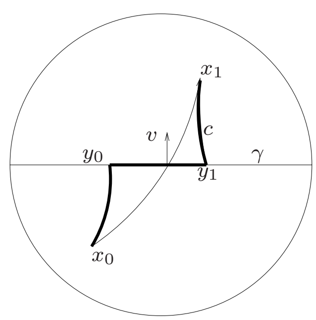

Figure 3. The figure shows a non-admissible arc.

Remark 8.3. The restriction to admissible sub-arcs is necessary for the foundation of the definition. For instance, consider the stratification of $\mathbb{H}^2$ with one geodesic γ. Let us fix a geodesic arc c: if c intersects transversely γ at a point t_0, let us put $\mu_c = v\delta_{t_0}$, where v is the normal to γ such that $\langle v, \dot{c}(t_0)\rangle > 0$. If c is contained in γ or does not intersect it, let us put $\mu_c = 0$. It is not possible to extend this definition to all piece-wise geodesic paths: in fact, consider the segment c as in Fig. 3. By applying property 4, of Definition 8.2 to the closed admissible arc $c * [x_1, x_0]$, we have $\mu_c(c) = v$. On the other hand, we have $\mu_c([x_0, y_0)) = 0$, $\mu_c([y_0, y_1]) = 0$ and $\mu_c((y_1, x_1]) = 0$ and this is a contradiction.

Given a measured geodesic stratification $(\mathcal{C}, Y, \mu)$, condition 3 in Definition 8.2, imposes a minimality property of Y.

Remark 8.4. Consider the case $n = 2$. Let Γ be a co-compact Fuchsian group. Let $\mathcal{C}$ be a Γ-invariant geodesic stratification. The 1-stratum L of $\mathcal{C}$ is a Γ-invariant geodesic lamination of $\mathbb{H}^2$. We know that $L = S \cup L_1$ where S is a simplicial lamination and L_1 projects onto $\mathbb{H}^2/\Gamma$ to a lamination with no closed leaf (see [**9**] for further details about geodesic laminations).

We shall show that maximal measured geodesic stratifications $(\mathcal{C}, Y, \mu)$ are naturally identified with the transverse measures (in the classical sense) on the lamination $L = X_{(1)}$ (notice that these concepts are quite different, in fact one is a $\mathbb{R}^{2+1}$-valued measure and the other is a positive measure).

Let us fix a transverse measure μ on $(\mathcal{C}, Y)$: we want to see that there exists a unique transverse measure λ on L such that $\lambda_c = |\mu_c|$ for all admissible paths c. First, notice that Y is a union of geodesics of L (every component of $\mathbb{H}^2 - L$ has non-empty interior). It follows that every transverse path d is a composition of paths d_i such that there exists a $\mathcal{C}$-preserving homotopy between d_i and a suitable parametrization c_i of the admissible geodesic segment $[d_i(0), d_i(1)]$. Thus, we can define λ_d such that its restriction on d_i is $|\mu_{c_i}|$. By using properties 1 and 2 of Definition 8.2, it is easy to see that this definition does not depend on the choice of the decomposition and in fact, it is the only possible one. Finally, since μ is Γ-invariant, we have that λ is Γ-invariant too.

By general facts about measured geodesic laminations (see [**9**]), it follows that for every admissible path c, the atoms of μ_c are exactly $c^{-1}(S)$. It follows that $S = Y$.

Conversely, let λ be a Γ-invariant transverse measure on L. Put $Y = S$. We want to construct a $(\mathcal{C}, Y)$-measure on $\mathbb{H}^2$. Let us fix an admissible path c and consider the function $v_c : [0, 1] \to \mathbb{R}^{2+1}$ defined in the following way: $v_c(t) = 0$ if $c(t) \notin L$, otherwise $v_c(t)$ is the normal vector to the leaf $C(c(t))$ such that $\langle v_c(t), \dot{c}(t)\rangle > 0$. Then, we can define μ_c as the $\mathbb{R}^{2+1}$-measure on $[0, 1]$ which is λ_c-absolutely continuous and has λ_c-density equal to v_c. It is easy to see that in this way, μ_c is a transverse measure on c. Furthermore, by definition, the assignment

$c \mapsto \mu_c$ verifies conditions 1 and 2 of Definition 8.2. An easy analysis of the geometry of a lamination shows that conditions 4 and 5 are satisfied.

It is easy to see that this correspondence gives an identification between Γ-invariant transverse measures on $\mathcal{C}$ and Γ-invariant transverse measures on L.

Notice that in dimension $n = 2$ condition 4 of Definition 8.2 is ensured by the geometry of the stratification. Furthermore, in this case, the set Y is determined by the lamination (i.e., it does not depend on the measure).

Before constructing a future complete regular domain with a given geodesic stratification, let us point out an easy property of measured geodesic stratifications.

Lemma 8.5. *Let μ be a transverse measure on $(\mathcal{C}, Y)$. Then, for every $x \in \mathbb{H}^n - Y$, there exists a unique maximal piece of $\mathcal{C}$ which contains x (maximal with respect to the inclusion).*

Proof. Suppose that there exist two pieces C_1, C_2 which contain x. We want to show that there exists a piece C which contains $C_1 \cup C_2$.

Let x_i be a point in C_i such that $C(x_i) = C_i$. Notice that x_i does not lie on Y (in fact, Y is a union of pieces so if $y \in Y$, then $C(y) \subset Y$). Consider the piece-wise geodesic arc $c = [x_1, x] \cup [x, x_2] \cup [x_2, x_1]$. It is closed and admissible so that

$$\mu_c([x_2, x_1]) = -\mu_c([x_1, x]) - \mu_c([x, x_2]).$$

(Notice that x, x_1 and x_2 are not atoms.) Since $[x_1, x]$ and $[x, x_2]$ are contained in C, we can easily see that $\mu_c([x_1, x_2]) = 0$. On the other hand, by Lemma 8.1, we have that $\langle v_c(t), x_2 - x_1 \rangle > 0$ for $|\mu_c|$-almost all t. Since $\langle \mu_c([x_1, x_2]), x_2 - x_1 \rangle = \int_{[x_1, x_2]} \langle v_c(t), x_2 - x_1 \rangle$, we have that $|\mu_c|([x_1, x_2]) = 0$. Thus, the segment $[x_1, x_2]$ is contained in a piece C. Clearly, C contains C_1 and C_2. q.e.d.

Given a measured geodesic stratification $(\mathcal{C}, Y, \mu)$, we are going to construct a regular domain with stratification equal to $\mathcal{C}$.

Fix a base point $x_0 \in \mathbb{H}^n - Y$ and define for $x \notin Y$

$$\rho(x) = \mu_{c_x}(c_x)$$

where c_x is an admissible path between x_0 and x. It is quite evident that this definition does not depend on the choice of the path. Furthermore, notice that

$$\rho(y) = \rho(x) + \mu_{c_{x,y}}(c_{x,y})$$

where $c_{x,y}$ is the geodesic arc between x and y. By using property 5, it follows that the map $\rho : \mathbb{H}^n - Y \to \mathbb{M}^{n+1}$ is continuous.

For $x \notin Y$, let us denote by $M(x)$ the maximal piece of $\mathcal{C}$ which contains x (by Lemma 8.5 this piece is unique). By using Lemma 8.1 it is easy to see that

$$\langle \rho(y) - \rho(x), y \rangle \geq 0 \qquad \langle \rho(y) - \rho(x), x \rangle \leq 0. \tag{10}$$

Furthermore, by arguing as in Lemma 8.5, we can easily see that the equality holds if and only if $M(x) = M(y)$. Thus, $\rho(x) = \rho(y)$ if and only if $M(x) = M(y)$.

Let us define the convex set

$$\Omega = \bigcap_{x \in \mathbb{H}^n - Y} \mathrm{I}^+(\rho(x) + x^\perp).$$

Theorem 8.6. *The convex set Ω is a future complete regular domain. For $x \in \mathbb{H}^n - Y$, we have $\rho(x) + ax \in \widetilde{S}_a$ so $r(\rho(x) + ax) = \rho(x)$ and $N(\rho(x) + ax) = x$ ($\widetilde{S}_a$ is the CT level surface $T^{-1}(a)$, whereas r and N are respectively the retraction and the normal field). In particular, $\rho(x) \in \Sigma$ for every $x \in \mathbb{H}^n - Y$.*

Moreover, $\widetilde{S}_a$ is the boundary of the convex hull of the set $\mathcal{S}_a = \{\rho(x) + ax | x \in \mathbb{H}^n - Y\}$.

Proof. First, let us show that $\rho(x) \in \partial\Omega$. Clearly, $\rho(x) \notin \Omega$. Now, let us take $v \in \mathrm{I}^+(0)$. We have to show that $\rho(x) + v \in \Omega$ i.e., $\langle \rho(x) + v - \rho(y), y \rangle < 0$ for every $y \in \mathbb{H}^n$. From inequalities (10), it follows $\langle \rho(x) - \rho(y), y \rangle \leq 0$. Since v is future directed, we have $\langle v, y \rangle < 0$ and so $\mathrm{I}^+(\rho(x)) \subset \Omega$.

Now, notice that $\rho(x) = \rho(y)$ for every $y \in M(x)$. So, by definition, the plane $\rho(x) + y^\perp$ is a support plane for Ω at $\rho(x)$ for every $y \in M(x)$. Let v be a null direction such that $[v]$ is on the boundary of $M(x)$. By taking a sequence $(y_k)_{k \in \mathbb{N}} \in M(x)$ such that $y_k \to [v]$, it is easy to see that the plane $\rho(x) + v^\perp$ is a support plane for Ω. Thus, we have

$$\Omega \subset \bigcap_{\substack{x \in \mathbb{H}^n - Y \text{ and} \\ [v] \in M(x) \cap \partial\mathbb{H}^n}} \mathrm{I}^+(\rho(x) + v^\perp).$$

If we prove the other inclusion, we obtain that Ω is a future complete regular domain. Fix $p \in \mathbb{M}^{n+1}$ and suppose that $\langle p - \rho(x), v \rangle < 0$ for every $x \in \mathbb{H}^n$ and $[v] \in M(x) \cap \partial\mathbb{H}^n$. We have to show that $p \in \Omega$. Notice that every $x \in \mathbb{H}^n - Y$ is a convex combination of a collection $v_1, \ldots, v_n$ such that $[v_i] \in M(x) \cap \partial\mathbb{H}^n$. It follows that $\langle p - \rho(x), x \rangle < 0$ and so $p \in \Omega$.

Since $\rho(x) + x^\perp$ is a support plane of Ω, we have that $\rho(x) + ax \in \widetilde{S}_a$. Moreover, it follows that $r(\rho(x) + ax) = \rho(x)$ and $N(\rho(x) + ax) = x$.

Now, let us prove the last part of this theorem. We have to show that $\mathrm{I}^+(\widetilde{S}_a)$ is the convex hull of the set $\mathcal{S}_a = \{ax + \rho(x) | x \in \mathbb{H}^n - Y\}$. The future of $\widetilde{S}_a$ contains the convex hull of $\mathcal{S}_a$. On the other hand, it is easy to see that spacelike support planes of the convex hull of $\mathcal{S}_a$ are support planes of $\mathrm{I}^+(\widetilde{S}_a)$. Thus, in order to prove the statement, it is sufficient to show that the convex hull of $\mathcal{S}_a$ has no timelike support plane. By contradiction, suppose that there exists a vector v such that $\langle v, v\rangle = 1$ and $\langle ax + \rho(x), v\rangle < C$. Up to translating Ω, we can suppose that the base point x_0 is orthogonal to v. Consider the geodesic γ such that $\gamma(0) = x_0$ and $\dot\gamma(0) = v$. We can suppose that there exists a sequence $t_k \to +\infty$ such that $x_k = \gamma(t_k) \notin Y$. By using that $\langle \rho(x_k), x_k\rangle \geq 0$ and $\langle \rho(x_k), x_0\rangle \leq 0$, we deduce that $\langle \rho(x_k), v\rangle \geq 0$. Since $\langle x_k, v\rangle \to +\infty$, we have that $\langle ax_k + \rho(x_k), v\rangle \to +\infty$ and this gives a contradiction. q.e.d.

Now, we want to point out some interesting properties of the domain Ω constructed in this way. First, notice that the image of the normal field contains $\mathbb{H}^n - Y$. Thus, it is not hard to see that for every $x \in \mathbb{H}^n$, there is a support plane of Ω orthogonal to x. It follows that the normal field N is surjective. In particular, we recall that Lemma 4.15 implies that the restriction of the normal field to the CT-level surface $\widetilde{S}_a = T^{-1}(a)$ is a proper map.

We are going to show that the stratification associated to Ω coincides with $\mathcal{C}$ at least on $\mathbb{H}^n - Y$. First, we give a technical result.

Lemma 8.7. *Let Ω be the regular domain constructed in Theorem 8.6. Then, for every $x \in \mathbb{H}^n$, we have that $N^{-1}(x) \cap \widetilde{S}_1$ is the convex hull of the limit points of the sequences $(x_k + \rho(x_k))_{k\in\mathbb{N}}$ such that $x_k \in \mathbb{H}^n - Y$ and $x_k \to x$.*

Proof. Let us take $p \in \widetilde{S}_1$ and suppose that $N(p) = x$. We have to show that for all $v \in N(p)^\perp$, there exists a sequence $(x_k) \subset \mathbb{H}^n - Y$ such that $x_k + \rho(x_k) \to p_\infty$ with $N(p_\infty) = x$ and $\langle p_\infty - p, v\rangle \geq 0$. We know that there exists a sequence $x_k \in \mathbb{H}^n - Y$ such that $x_k = \cosh d_k x + \sinh d_k v_k$ such that $d_k \to 0$ and $v_k \to v$. let us set $p_k = x_k + \rho(x_k)$. Since N is a proper map, up to passing to a subsequence, we have that $p_k \to p_\infty$ such that $N(p_\infty) = x$. Since $\langle p_k - p, x\rangle \leq 0$ and $\langle p_k - p, x_k\rangle \geq 0$, we have that $\langle p_k - p, v_k\rangle \geq 0$ and by passing to the limit, we have $\langle p_\infty - p, v\rangle \geq 0$. q.e.d.

Proposition 8.8. *Let Ω be the regular domain associated with the measured stratification $(\mathcal{C}, Y, \mu)$. Then, for every $x \in \mathbb{H}^n - Y$, we have that $r(N^{-1}(x)) = \{\rho(x)\}$. Moreover, $\mathcal{F}(\rho(x))$ is the maximal piece*

$M(x)$. (We recall that $\mathcal{F}(r_0) = N(r^{-1}(r_0))$ where $r_0 \in \Sigma$ and r is the retraction onto the singularity Σ).

Proof. Let us take $x \in \mathbb{H}^n - Y$. Since $\rho : \mathbb{H}^n - Y \to \mathbb{M}^{n+1}$ is a continuous map, Lemma 8.7 implies that $N^{-1}(x) \cap \widetilde{S}_1 = \{x + \rho(x)\}$. Thus, $r(N^{-1}(x)) = \{\rho(x)\}$. Now, let us prove that $\mathcal{F}(\rho(x)) = M(x)$. Clearly, $M(x) \subset \mathcal{F}(\rho(x))$, thus we have to prove the other inclusion. Let us take $y \in \mathcal{F}(\rho(x))$. First, suppose that $y \notin Y$. Since $y \in \mathcal{F}(\rho(y))$, we have $\langle y, \rho(y) - \rho(x)\rangle = 0$. On the other hand, by (10), we can deduce $M(x) = M(y)$.

Suppose now that $y \in Y \cap \mathcal{F}(\rho(x))$. Let us prove that y lies on the boundary $b\mathcal{F}(\rho(x))$ (see Section 4 for the definition of the boundary bK of a convex set K). Since $y \in Y$, we have that $N^{-1}(y) \cap \widetilde{S}_1$ is not only a point. So, there exists a spacelike vector v orthogonal to y such that $[\rho(x), \rho(x) + \varepsilon v]$ is contained in Σ. We have that $v^\perp$ is a support plane of $\mathcal{F}(\rho(x))$ and contains y. So, if $y \notin b\mathcal{F}(\rho(x))$, we have $x \in \mathcal{F}(\rho(x) + \varepsilon v)$ i.e., $\rho(x) + \varepsilon v \in r(N^{-1}(x))$. Since $x \notin Y$, we have a contradiction. Thus, it follows $\mathcal{F}(\rho(x)) - b\mathcal{F}(\rho(x)) \subset \mathbb{H}^n - Y$ and so $\mathcal{F}(\rho(x)) - b\mathcal{F}(\rho(x)) \subset M(x)$. Hence, we can deduce $\mathcal{F}(\rho(x)) \subset M(x)$.

q.e.d.

Remark 8.9. Notice that the stratification induced by the domain Ω coincides with $\mathcal{C}$ on $\mathbb{H}^n - Y$, so that Y is the union of pieces of the stratification associated with Ω. Moreover, by property 3 of Definition 8.2, it turns out that $Y = \{y \in \mathbb{H}^n | \# N^{-1}(y) > 1\}$.

Corollary 8.10. *Let $(\mathcal{C}, Y, \mu)$ be a Γ-invariant measured geodesic stratification of $\mathbb{H}^n$ (where Γ is a cocompact torsion-free discrete subgroup of $\mathrm{SO}^+(n,1)$). Let us fix a base point $x_0 \notin Y$ and set $\tau_\gamma = \rho(\gamma(x_0))$. Then, we have that $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$. Let Ω be a domain associated with $(\mathcal{C}, Y, \mu)$. Then, we have $\Omega = \mathcal{D}_\tau$ and $\mathcal{F}(\rho(x)) = M(x)$ for every $x \notin Y$.*

Proof. Since μ is Γ-invariant, we have

$$\rho(\gamma(x)) = \gamma\rho(x) + \rho(\gamma(x_0)).$$

Thus, $\tau_{\alpha\beta} = \alpha\tau_\beta + \tau_\alpha$, so that τ is a cocycle. The same equality shows that Ω is a Γ_τ-invariant regular domain. Thus, Theorem 5.1 implies that it is equal to $\mathcal{D}_\tau$. Finally, Proposition 8.8 implies that $\mathcal{F}(\rho(x)) = M(x)$ for every $x \notin Y$. q.e.d.

9. Simplicial Stratifications

In this section, we study future complete regular domains of $\mathbb{M}^{3+1}$ with simplicial geodesic stratifications. We have restricted ourselves to the case $n+1=4$ to make the discussion simple. However, most of the results of this section can be generalized in higher dimensions. In remarks 9.2, 9.10 and 9.13, we shall suggest how to get such a generalization.

Definition 9.1. We say that a geodesic stratification $\mathcal{C}$ of $\mathbb{H}^n$ is *simplicial* if any $p \in \mathbb{H}^n$ admits a neighbourhood U intersecting only a finite number of pieces of $\mathcal{C}$.

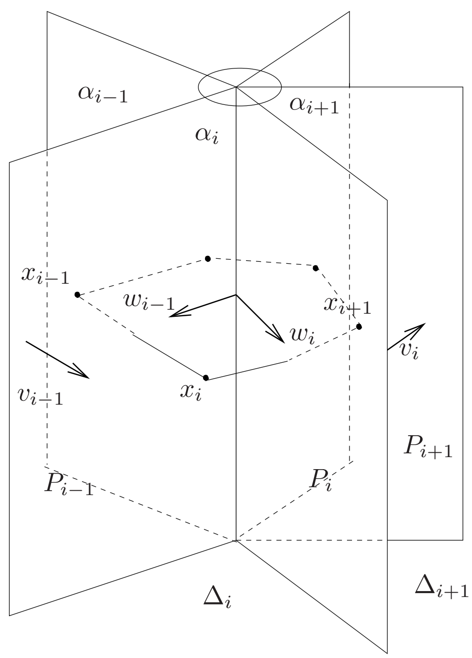

Figure 4. A neighbourhood of a point on a 1-piece of a simplicial stratification of $\mathbb{H}^3$.

We shall see that the correspondence between measured geodesic stratifications and regular domains induces an identification between measured simplicial stratifications and regular domains **with simplicial singularity**. Finally, we shall recover the duality between stratifications and singularities.

Notice that in dimension $n=2$, simplicial stratifications correspond to simplicial laminations. Moreover, in all dimensions, a simplicial stratification has closed strata: it is, in fact, a tessellation of $\mathbb{H}^n$ by locally finite ideal convex polyhedra. In Fig. 4, we show the local behaviour of a simplicial stratification of $\mathbb{H}^3$.

First, we shall describe the measures on a simplicial stratification. Let $(\mathcal{C}, Y, \mu)$ be a measured geodesic stratification of $\mathbb{H}^3$ with simplicial support $\mathcal{C}$. Let X be the 2-stratum of $\mathcal{C}$. We want to show that $X = Y$. Since Y has empty interior, we have $Y \subset X$. On the other hand, let c be a geodesic path with no endpoint in X (such a path is admissible). Notice that $\mathrm{supp}|\mu_c|$ is $c^{-1}(X)$, but this set is finite so that the measure $|\mu_c|$ has an atom on every point of $c^{-1}(X)$. Thus, $X \subset Y$.

Let us fix a 2-piece P and let Δ_1 and Δ_2 be the 3-pieces which incide on P. Let c be an admissible geodesic path which starts from Δ_1 and ends into Δ_2. Clearly, $\mu_c = a v_{1,2} \delta_{c^{-1}(P)}$ where a is a positive constant, $v_{1,2}$ is the normal vector to P which points towards Δ_2 and δ_x is the Dirac measure centred at x. By using properties 1 and 2 of Definition 8.2, we can easily see that the constant a does not depend on the path. By imposing that $\mu_c(c) + \mu_{c^{-1}}(c^{-1}) = 0$, we can deduce that the measure of a geodesic path c' which starts from Δ_2 towards Δ_1 is $\mu_\delta = a v_{2,1} \delta_{c^{-1}(P)}$. It follows that the constant a depends only on the piece P. We call it the *weight* of P and we denote it by $a_\mu(P)$.

We want to show that the set of weights $\{a_\mu(P) | P \text{ is a } 2-\text{piece}\}$ satisfies a certain set of equations and determines the measure μ.

Let us fix a geodesic l in $\mathcal{C}$ and let $P_1, \ldots, P_k$ and $\Delta_1, \ldots, \Delta_k$ be respectively the 2-pieces and the 3-pieces which incide on l. We can suppose that the numeration is such that Δ_i incides on P_{i-1} and P_i (the index $i-1$ and i are considered $\mod k$, see Fig. 4). Let us fix $x_i \in \mathrm{int}(\Delta_i)$ and consider the admissible closed path $c = [x_1, x_2] \cup [x_2, x_3] \cup \ldots \cup [x_{k-1}, x_k] \cup [x_k, x_1]$. By imposing that $\mu_c(c) = 0$, we can deduce

$$\sum_{i=1}^{k} a_\mu(P_i) v_i = 0 \tag{11}$$

where v_i is the normal vector to P_i which points towards Δ_{i+1}. Notice that v_i lies in the linear subspace of $\mathbb{M}^{3+1}$ which is orthogonal to the space generated by l (which we denote $l^\perp$). If we fix a point $x \in l$, $l^\perp$ is identified with the subspace of $T_x\mathbb{H}^3$ orthogonal to l. By performing a $\frac{\pi}{2}$-rotation on $l^\perp$, we can see that equation (11) is equivalent to the equation

$$p_l(a_\mu) = \sum_{l \subset P} a_\mu(P) w(P) = 0 \tag{12}$$

where w is the unitary vector of $l^\perp$ tangential to P and pointing inward (see Fig. 4).

Definition 9.2. A family of positive constants $a = \{a(P)\}$ parametrized by the set of 2-pieces of $\mathcal{C}$ is called a family of *weights* for the stratification if the equation $p_l(a) = 0$ is satisfied for every 1-piece l.

We have shown that there is a family of weights associated with every transverse measure μ on $\mathcal{C}$. Now, we want to prove that this correspondence is bijective.

Proposition 9.1. *For every family of weights $\{a(P)\}$, there exists a unique transverse measure μ such that $a(P) = a_\mu(P)$.*

Proof. First, let us prove uniqueness. Let μ and ν be measures such that $a_\mu(P) = a_\nu(P)$ for all 2-pieces P. If c is an admissible arc which intersects only 2-pieces, it follows that $\mu(c) = \nu(c)$. Suppose now that $c \cap X$ is a point p which lies on the 1-piece l. Consider an arc c' which has the same endpoints as c and does not intersect the 1-stratum. We have $\mu_c(c) = \mu_{c'}(c') = \nu_{c'}(c') = \nu_c(c)$. Since $\mathrm{supp}\mu_c = \mathrm{supp}\nu_c = c^{-1}(p)$, we have $\mu_c = \nu_c$.

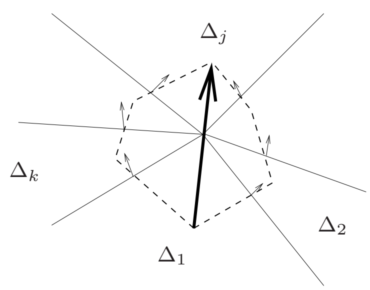

Figure 5. Definition of measure on a geodesic which passes through the 1-stratum.

Notice that every admissible path is a composition of paths $c_1 * \cdots * c_n$ such that every c_i either does not intersect the 1-stratum or intersects only one geodesic. It follows that $\mu_c = \nu_c$.

Now, let us prove existence. Let c be an admissible geodesic path: notice that $c^{-1}(P)$ is at most a point for every 2-piece P. Suppose that c does not intersect any geodesics of the stratification. Then, we can define $\mu_c = \sum_P a(P)v(P)\delta_{c^{-1}(P)}$ where $v(P)$ is the normal vector to P pointing in the direction of c (notice that this sum is finite). Suppose now that c intersects only one geodesic l. Let $P_1, \ldots, P_k$ and $\Delta_1, \ldots, \Delta_k$ be respectively the 2-pieces and the 3-pieces which incide on l. We choose the numeration as above and suppose that c comes from Δ_1 and goes into Δ_j. Thus, we can define

$$\mu_c = \left(\sum_{i=1}^{j-1} a(P_i)u(P_i)\right)\delta_{c^{-1}(l)}$$

where $u(P_i)$ is the normal vector to P_i which points towards Δ_j. Since

$$\sum_{i=1}^{j-1} a(P_i)u(P_i) = \sum_{i=j}^{k} a(P_i)u(P_i),$$

this definition does not depend on the numeration (see Fig. 5). Now, let c be an admissible path. Consider a decomposition of c in geodesic admissible paths $c_1 * \cdots * c_k$ such that every c_i intersects either only one geodesic of the stratification or only one 2-piece. Thus, we can define μ_c such that $\mu_c|c_i$ is μ_{c_i}. Notice that this definition does not depend on the decomposition of c.

Clearly, this measure satisfies properties 1, 2 and 3 of Definition 8.2. Let us show that if c is a closed admissible path, then $\mu_c(c) = 0$. First, notice that we can assume that c does not intersect any geodesic of $\mathcal{C}$. In fact, if c intersects l, we can perform the move in Fig. 6. Notice that we obtain a closed admissible arc c' such that $\mu_{c'}(c') = \mu_c(c)$ and $\#(c' \cap X_{(1)}) < \#(c \cap X_{(1)})$ (here, $X_{(1)}$ is the 1-stratum). Since these intersections are finite, we can suppose $c \cap X_{(1)} = \varnothing$.

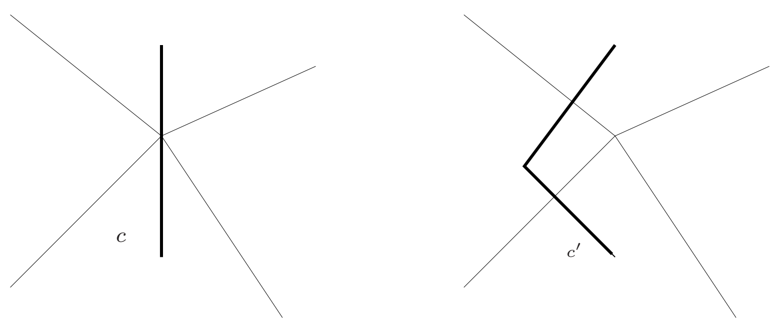

Figure 6. Performing the move in the figure, we obtain an arc which does not intersect $X_{(1)}$.

Now, $c - X$ is a union of components $m_1, \ldots, m_N$, where m_i is an oriented arc with endpoints on the faces P_i^- and P_i^+. Notice that we can suppose that the faces P_i^- and P_i^+ are different. In fact, otherwise, we can perform the move in Fig. 7. We call such a path *tight*.

Now, let $n(P, c)$ be the cardinality of the intersection of c with the face P. It is clear that if $n(P, c) = n(P, c')$ for every P, then $\mu_c(c) = \mu_{c'}(c')$.

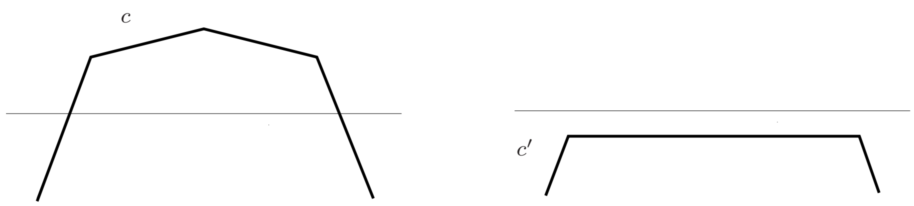

Figure 7. Performing the move in the figure we obtain a tight arc.

On the other hand, let c and c' be tight paths such that there exists a homotopy between them in $\mathbb{H}^n - X_{(1)}$: it is easy to see that $n(P, c) = n(P, c')$ for every P (in fact for every tight path c there exists $\varepsilon > 0$ such that if c' is in a ε-neighbourhood of c, then $n(P, c) = n(P, c')$).

Thus, we have that $\mu_c(c)$ depends only on the homotopy class of c in $\mathbb{H}^n - X_{(1)}$. Now, let us fix a base point $x_0 \in \mathbb{H}^n - X$. Since every $\alpha \in \pi_1(\mathbb{H}^n - X_{(1)}, x_0)$ is represented by an admissible path, we have constructed a map $\pi_1(\mathbb{H}^n - X_{(1)}) \ni [c] \to \mu_c(c) \in \mathbb{R}^{3+1}$ that turns to be a homomorphism.

For every geodesic $l \subset X_{(1)}$, let us consider an admissible path s_l which winds around l. Now, we can join s_l to x_0 with an admissible arc d. Consider the loop $c_l = d * s_l * d^{-1}$. Notice that the family $\{c_l\}$ generates $\pi_1(\mathbb{H}^n - X_{(1)}, x_0)$. On the other hand, we have $\mu_{c_l}(c_l) = \mu_{s_l}(s_l)$ and since the weights verify equation p_l, we have $\mu_{c_l}(c_l) = 0$. It follows that $\mu_c(c) = 0$. Thus, μ is a measure on $(\mathcal{C}, X)$. q.e.d.

Remark 9.2. An analogous description of measures on simplicial stratifications holds in higher dimension. In fact, for a given simplicial stratification $\mathcal{C}$ of $\mathbb{H}^n$, we can associate to every $(n-1)$-piece P a positive number $a(P)$ as in the 3-dimensional case. For every $(n-2)$-dimensional piece l, we can consider an admissible path c which winds around l. By imposing that $\mu_c(c) = 0$, we obtain that the family $\{a(P)\}$ verify the linear equation p_l. Thus, we can define a family of weights as a family of positive constants $\{a(P)\}$ parametrized by $(n-1)$-pieces and solution of equations p_l parametrized by $(n-2)$-pieces.

On the other hand, we can check that a family of weights determines a measure on $\mathcal{C}$. As in the 3-dimensional case, it is easy to prove that two measures μ and ν which produce the same family of weights are equal.

Finally, let us fix a family of weights $\{a(P)\}$. For a given admissible path c which does not intersect $X_{(n-2)}$, we can define a transverse measure μ_c as in the 3-dimensional case. Moreover, given such a path c, it turns out that $\mu_c(c)$ depends only on the free homotopy class of c as a path in $\mathbb{H}^n - X_{(n-2)}$. Thus, if we fix a base-point x_0, we have a homomorphism $\pi_1(\mathbb{H}^n - X_{(n-2)}, x_0) \to \mathbb{R}^{n+1}$.

For each $(n-2)$-piece l, let c_l be an admissible loop which starts from x_0, goes towards l, then winds around l and finally goes back to x_0. Now, $\pi_1(\mathbb{H}^n - X_{(n-2)})$ is generated by the loops c_l parametrized by the 2-pieces l. Since $\{a(P)\}$ is a family of weights, it is easy to see that $\mu_{c_l}(c_l) = 0$. Thus, for every closed admissible path c in $\mathbb{H}^n - X_{(n-2)}$, we have $\mu_c(c) = 0$. It is easy to see that we can extend this measure to a measure defined for every admissible path (we can extend the argument of Proposition 8.8 to any dimension). Thus, we can conclude that for a simplicial stratification of $\mathbb{H}^n$, the families of weights on $\mathcal{C}$ parametrize the measures on $\mathcal{C}$.

Now, let us go back to the case $n+1 = 4$. Let us consider a family of weights $a = \{a(P)\}$. We have seen that this family induces a measure μ on $\mathcal{C}$. Let Ω be the domain associated to the measured stratification $(\mathcal{C}, X, \mu)$. We want to describe the CT-level surface $S = T^{-1}(1)$ and the singularity Σ.

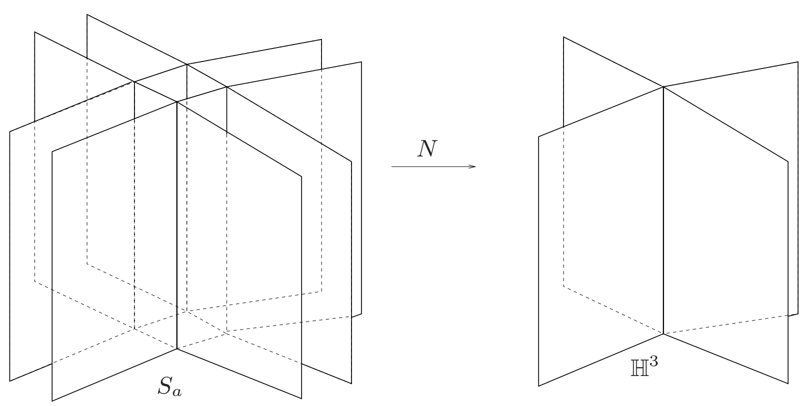

Figure 8. The inverse image of a neighbourhood of a point in $\mathbb{H}^3$ on the CT-level-surface of a regular domain with simplicial singularity.

Proposition 9.3. *Consider the decomposition of S into the sets $S_{(i)} := \{x \in S | \dim N^{-1}(N(x)) \cap S = i\}$ (for $i = 0, 1, 2$). Then, we* *have*

$$\overline{S}_{(i)} = \sqcup\{N^{-1}(C) \cap S | C \textit{ is a } 3-i \textit{ piece}\}.$$

If Δ *is a* 3*-piece* $N^{-1}(\Delta)$ *is obtained from* Δ *by translation and in particular, the normal field* N *restricted to every component of* S_0 *is an isometry.*

If P *is a* 2*-piece, then* $N^{-1}(P)$ *is isometric to* $P \times [0, a(P)]$ *and the normal field coincides with the projection onto the first factor.*

Finally, if l *is a geodesic piece, then* $N^{-1}(l)$ *is isometric to* $l \times F_l$ *where* F_l *is a Euclidean* k*-gon. More precisely, let* $P_1, \ldots, P_k$ *and* $\Delta_1, \ldots, \Delta_k$ *be respectively the* 2*-pieces and* 3*-pieces incide on* l (*numeration is chosen as above*)*. Then, there is a numeration of the edges of* F_l*, say* $e_1, \ldots, e_k$*, such that the length of* e_i *is* $a(P_i)$ *and the angle between* e_{i-1} *and* e_i *is* $\pi - \alpha_i$ *where* α_i *is the dihedral angle of* Δ_i *along* l*.*

Proof. Let us fix a base point x_0 and for every 3-piece Δ, let $\rho_\Delta = \mu_c(c)$ where c is an admissible path which joins x_0 to Δ. Clearly, ρ_Δ does not depend on the path c and so it is well defined. Moreover, we have

$$\Omega = \bigcap_{\Delta} \bigcap_{x \in \Delta} \{p \in \mathbb{M}^{3+1} | \langle p - \rho_\Delta, x\rangle < 0\}.$$

Now, for every $x \in \mathbb{H}^n$ there exists a unique support plane P_x of Ω which is orthogonal to x and such that $P_x \cap \overline{\Omega} \neq \varnothing$. On the other hand, we have that $r(N^{-1}(x)) = P_x \cap \overline{\Omega}$.

Suppose now that $x \in \Delta$: by definition of Ω, we have that P_x is the plane which passes through ρ_Δ and is orthogonal to x.

If $x \in \mathrm{int}\Delta$, then by using inequalities (10), we can see that $\Omega \cap P_x = \{\rho_\Delta\}$. Now, let us take $x \in P$ where P is a 2-piece. By Lemma 8.7, we can deduce $\Omega \cap P_x = [\rho_{\Delta_-}, \rho_{\Delta_+}]$ where Δ_- and Δ_+ are the 3-pieces which incide on P. Finally, suppose that $x \in l$ for some geodesic piece l. We have that $\Omega \cap P_x$ is the convex hull of ρ_{Δ_i} where $\Delta_1, \ldots, \Delta_k$ are the 3-pieces which incide on l. Let P_i be the 2-piece which separates Δ_i from Δ_{i+1}. Notice that $P_x \cap \overline{\Omega}$ is a k-gon with vertices $p_i = \rho_{\Delta_i}$. Moreover, we have $\rho_{\Delta_{i+1}} - \rho_{\Delta_i} = a(P_i)v_i$ (where v_i is the normal vector to P_i which point towards Δ_{i+1}). It is easy to see that the edges of $P_x \cap \overline{\Omega}$ are $e_i = [p_i, p_{i+1}]$ so that the length of e_i is $a(P_i)$. Moreover, the angle between e_{i-1} and e_i is equal to the angle between $-v_{i-1}$ and v_i. Since the angle between v_{i-1} and v_i is equal to the dihedral angle of Δ_i along l, we have that $P_x \cap \overline{\Omega}$ is isometric to the k-gon F_l defined in the proposition.

From this analysis, it follows that $N^{-1}(\Delta) \cap S = \Delta + \rho_\Delta$ and so the normal field is an isometry onto $N^{-1}(\Delta)$.

Now, let us consider a 2-piece P. We have seen that $r(N^{-1}(x)) = [\rho_{\Delta_-}, \rho_{\Delta_+}]$ for every $x \in P$. Thus, $[\rho_{\Delta_-}, \rho_{\Delta_+}] \times P \ni (p, x) \to p + x \in N^{-1}(P) \cap S$ is a parametrization: notice that this map is an isometry (in fact the segment $[\rho_{\Delta_-}, \rho_{\Delta_+}]$ is orthogonal to P), and the normal map coincides with the projection onto the second factor.

An analogous argument proves the last statement of the proposition.

q.e.d.

For $i = 0, 1, 2$, let us set $\Sigma_i = \{p \in \Sigma | \dim \mathcal{F}(p) = 3 - i\}$ (recall that $\mathcal{F}(p) = N(r^{-1}(p))$). From the last proposition, we immediately have the following corollary:

Corollary 9.4. *If Ω is as above, then Σ is naturally a cellular complex in the following sense. Σ_0 is a numerable set; every component s of Σ_1 is an open segment, moreover the closure of s is a closed segment and $\partial s = \overline{s} - s$ is contained in Σ_0; every component σ of Σ_2 is an open 2-cell, moreover, the closure of σ is a closed 2-cell and in fact it is a finite-sided polygon with vertices in Σ_0 and edges in Σ_1.*

Remark 9.5. We have not made any hypothesis about local finiteness of the cells.

Remark 9.6. If $\mathcal{C}$ is a simplicial stratification of $\mathbb{H}^3$, we can construct the dual complex. For every 3-piece Δ, there is a vertex v_Δ, for every 2-piece P, there is the segment $[v_\Delta, v'_\Delta]$ where Δ and Δ' are the 3-pieces which incide on P, and for every 1-piece l, there is the polygon with vertices $v_{\Delta_1}, \ldots, v_{\Delta_n}$ where $\Delta_1, \ldots, \Delta_n$ are the pieces which incide on l.

Notice that Σ is combinatorially equivalent to the dual complex of $\mathcal{C}$. Thus, the combinatorial structure of Σ depends only on the combinatorial structure of $\mathcal{C}$. This remark points out the duality between stratifications and singularities.

In what follows, we introduce a class of regular domains whose stratification is simplicial. However, as we are going to see, this class do not coincide with the class of all regular domains with simplicial stratification. On the other hand, a regular domain *invariant for some affine deformation* Γ_τ of a co-compact Fuchsian group Γ, belongs to this class if and only its singularity is simplicial.

Definition 9.3. Given a point p in the singularity Σ of a future complete regular domain, we say that p is a *vertex* if there exists a spacelike support plane at p which intersects only p.

We say that Σ is *simplicial* if the set of vertices Σ_0 is discrete and there exists a cellularization $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2$ such that every component of $\Sigma_1 - \Sigma_0$ is a straight segment with endpoints in Σ_0 and every component of $\Sigma_2 - \Sigma_1$ is a finite-sided spacelike polygon with vertices in Σ_0 and edges in Σ_1.

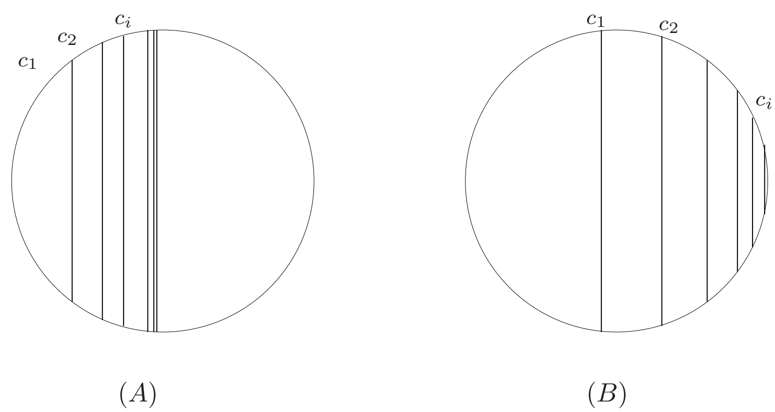

Figure 9. On the left a non-simplicial stratification and on the right a simplicial stratification.

Remark 9.7. We shall prove that a regular domain with simplicial singularity has simplicial stratification. If we do require that Σ_0 is only a numerable set this result is no longer true. Consider, for instance, the stratification of $\mathbb{H}^2$ given in Fig. 9 (A) (here, the example is given for $n = 2$, but an analogous example holds for $n = 3$). It is easy to construct a regular domain with such a stratification and singularity with cell decomposition.

On the other hand, there exist regular domains with simplicial stratification which does not have a simplicial singularity. For instance, consider the stratification of $\mathbb{H}^2$ given in Fig. 9 (B). It is easy to construct a regular domain with such a stratification whose singularity Σ is compact in $\mathbb{M}^{2+1}$. It follows that Σ_0 is not discrete.

Remark 9.8. Let Σ be a simplicial singularity of a future complete regular domain Ω. Notice that we can consider on Σ a weak topology induced by the cellularization ($A \subset \Sigma$ is open in this topology if and only if the intersection of A with every open cell is open). Since we do not require local finiteness of cells, generally, this topology is finer than the topology induced by $\mathbb{M}^{3+1}$.

Notice that every open cell has a natural distance. Thus, if c is a path in Σ, we can define the length of c as the sum of the lengths of intersections of c with the cells of Σ. Finally, we can define a path-distance on Σ such that $d_\Sigma(r, r')$ is the infimum of the lengths of paths from r to r'. It is easy to see that this distance agrees with the natural distance on Σ described in Section 7. Thus, the topology induced by $\mathbb{M}^{3+1}$ on Σ generally is finer than the topology induced by d_Σ.

Now, we can prove that regular domains with surjective normal field and simplicial singularity are given by measured simplicial stratifications of $\mathbb{H}^3$. (Notice that in order to prove that a regular domain is produced by a measured stratification, the normal field must be surjective, on the other hand, this condition ensures us that the normal field is a proper map.)

Proposition 9.9. *Let Ω be a regular domain with surjective normal field N and simplicial singularity. The stratification $\mathcal{C}$ associated to Ω is simplicial. Moreover, there exists a unique measure μ on $\mathcal{C}$ such that Ω is equal* (*up to translations*) *to the domain associated with* $(\mathcal{C}, \mu)$.

Proof. If we take $p, q \in \Sigma$ which belong to the same cell of Σ, then $\mathcal{F}(p) = \mathcal{F}(q)$. Thus, it is sufficient to show that if K is a compact set of $\mathbb{H}^3$, then $r(N^{-1}(K))$ intersects only a finite number of cells of Σ. Since N restricted to a CT level surface is a proper map, it is sufficient to prove that if H is a compact set in $\widetilde{S}_1$, then $r(H)$ intersects only a finite number of cells.

Now, let us fix such a compact set $H \subset \widetilde{S}_1$, and for every $p \in H$, let $V(p)$ be the set of vertices of the cell containing $r(p)$. Since the set of vertices is discrete, it is sufficient to show that the set $\bigcup V(p)$ is bounded in $\mathbb{M}^{3+1}$.

Suppose that there exists a divergent sequence of vertices $v_k \in V(p_k)$. Up to passing to a subsequence, we can suppose $p_k \to p_\infty$ with $p_\infty \in H$. Now, let us consider the sequence of segments $[r(p_k), v_k]$. Up to passing to a subsequence, we have that these segments tend to an infinite ray R with starting point at $r(p_\infty)$. Since $[r(p_k), v_k]$ is contained in the plane $r(p_k) + N(p_k)^\perp$, it follows that the ray R is contained in the plane $r(p_\infty) + N(p_\infty)^\perp$. Thus, R is contained in Σ, but then for all $r \in R$, we have that $N(r + N(p_k)) = N(p_k)$. Since we know that N is a proper map, we have a contradiction.

We have proved that $\mathcal{C}$ is a simplicial stratification. Now, we define a family of weights $\{a(P)\}$ on it. Given a 3-piece Δ of $\mathcal{C}$, by Proposition 4.14, there exists a vertex $v(\Delta)$ such that $\Delta = \mathcal{F}(v(\Delta))$. Now, if P is a 2-piece, there exist two 3-pieces, say Δ_0 and Δ_1, such that F is face of them. It follows that $r(N^{-1}(P))$ is the spacelike segment $[v(\Delta_0), v(\Delta_1)]$. Thus, we can define $a(P) = (\langle v(\Delta_1) - v(\Delta_0), v(\Delta_1) - v(\Delta_0)\rangle)^{1/2}$. Let us show that this is a family of weights on $\mathcal{C}$. Given a geodesic l of the stratification, let $P_1, \ldots, P_k$ and $\Delta_1, \ldots, \Delta_k$ be respectively the 2-pieces and the 3-pieces which incide on l. Let us suppose $P_i = \Delta_{i-1} \cap \Delta_i$ (take $i \mod k$). For simplicity, set $p_i = v(\Delta_i)$. We know that $p_{i+1} - p_i$ is an orthogonal vector to P_i pointing towards Δ_{i+1}. So that if v_i is the normal vector to P_i pointing towards Δ_{i+1}, we have $p_{i+1} - p_i = a(P_i)v_i$. Thus

$$\sum_i a(P_i)v_i = \sum_i p_{i+1} - p_i = 0.$$

Let μ be the measure corresponding to the family $\{a(P)\}$. We have to show that up to translations, Ω is the domain corresponding to the measure μ.

Let us fix a base point $x_0 \in \mathbb{H}^n - X$. Up to translations, we can suppose that $r(N^{-1}(x_0)) = 0$. Now, let p be a vertex of Σ. By construction, it is quite evident that $p = \mu_c(c)$ where c is an admissible path starting from x_0 and ending in the piece which corresponds to p. It follows that Ω is the regular domain which corresponds to the measure μ. q.e.d.

Remark 9.10. Let us discuss the generalization in higher dimension. It is not hard to extend to all dimensions Proposition 9.3. In fact, if Ω is a domain constructed by a measured simplicial singularity and S is the CT level surface $T^{-1}(1)$, we can consider the decomposition of S into sets $S_{(i)} = \{x \in S | \dim N^{-1}N((x)) \cap S = i\}$ for $i = 0, \ldots, n-1$. It turns out that the closure of $S_{(i)}$ is the disjoint union of inverse images under N of $(n-i)$-pieces. Moreover, if C is a $(n-i)$-piece, then $N^{-1}(C)$ is isometric to $C \times Q_C$ where Q_C is a i-dimensional finite-sided convex Euclidean polyhedron.

Clearly, the definition of domains with simplicial stratification can be extended to all dimensions. Notice that Proposition 9.9 holds in every dimension (in fact, the proof is quite general). Finally, it is easy to see that the simplicial singularity corresponding to a simplicial (measured) stratification is combinatorially equivalent to the dual complex of the stratification.

In the last part of this section, we shall study Γ-invariant simplicial geodesic stratifications where Γ is a torsion-free discrete cocompact subgroup of $\mathrm{SO}^+(3,1)$. We see that for a Γ-invariant simplicial stratification, the set of measures on it is parametrized by a finite number of positive numbers which satisfy a finite set of linear equations.

We start with some remarks about Γ-invariant simplicial stratifications.

Proposition 9.11. *Let $\mathcal{C}$ be a Γ-invariant simplicial stratification of $\mathbb{H}^3$. Then, the projection of $\pi(C)$ onto $\mathbb{H}^3/\Gamma$ is compact for every piece $C \in \mathcal{C}$. In particular, the projection of a 1-piece is a simple geodesic* *whereas the projection of a* 2*-piece is either a closed hyperbolic surface or a hyperbolic surface with geodesic boundary. Finally, there is only a finite number of pieces up to the action of* Γ*.*

Proof. Since $\mathcal{C}$ is simplicial, we can easily see that $\Gamma \cdot C$ is closed and the projection of C is compact. Thus, the projection of a 1-piece l is a compact complete geodesic and so it is closed. Since the orbit of l is formed by a disjoint union of geodesics, it follows that the projection is a simple geodesic.

An analogous argument shows that the projection of a 2-piece P is a hyperbolic surface. Notice that if P is a plane, it is closed. Otherwise, it has totally geodesic boundary.

Let K be a compact fundamental region for Γ. Since K intersects only a finite number of pieces, the last statement follows. q.e.d.

Let us fix a Γ-invariant simplicial stratification $\mathcal{C}$. We denote by $T_{\mathcal{C}}$ the projection of the 2-stratum X onto $M = \mathbb{H}^3/\Gamma$. Notice that there exists a finite set of simple geodesics $\{c_1, \ldots, c_N\}$ of M such that $c_i \subset T_{\mathcal{C}}$ and $T_{\mathcal{C}} - \bigcup c_i$ is a finite union of totally geodesic submanifolds $F_1, \ldots, F_L$ such that $\overline{F}_i = F_i \cup c_{i_1} \cup \ldots \cup c_{i_k}$. Moreover, every component of $M - T_{\mathcal{C}}$ is locally convex. The geodesics c_i are called the *edges* of the surface whereas the surfaces F_i are the *faces*. A subset $X \subset M$ equipped with such a decomposition and such that the complementary regions are locally convex is called *piece-wise geodesic surface*. Notice that piece-wise geodesic surfaces correspond bijectively to Γ-invariant geodesic stratifications.

Now, let μ be a Γ-invariant measure on $\mathcal{C}$. Notice that the corresponding family of weights $\{a(P)\}$ satisfies $a(\gamma P) = a(P)$ for every 2-piece P and every $\gamma \in \Gamma$. Thus, there exists a family of positive constants $\{\alpha(F)\}$ parametrized by the faces of $T_{\mathcal{C}}$ such that $a(P) = \alpha(\pi(P))$.

This remark suggests the following definition. Let T be a piece-wise geodesic surface in M and let $\mathcal{C}$ be the stratification associated to T. A *family of weights* on T is a family $\{\alpha(F)\}$ parametrized by the faces of T such that $\{a(P) := \alpha(\pi(P))\}$ is a family of weights on $\mathcal{C}$. Notice that for every 1-piece l, we have that the equations associated to l and $\gamma(l)$ are related by the identity

$$p_{\gamma(l)}(a) = \gamma p_l(a)$$

so solutions of equation p_l coincide with solutions of $p_{\gamma l}$. Thus, the conditions that we have to impose to ensure that $\{\alpha(F)\}$ is a family of weights can be parametrized by the edges of T.

Finally, we have that the families of weights on T correspond bijectively with Γ-invariant measures on $\mathcal{C}$. Notice that if T is a piece-wise geodesic surface with f faces and e edges, then the weights on T correspond to a subset of $\mathbb{R}_+^f$ defined by $2e$ linear equations (in fact, every p_l is equivalent to 2 linear equations). Thus, if positive solutions exist, they form a convex cone of dimension greater than $f - 2e$.

Example 9.1. Now, we exhibit some examples of piece-wise geodesic surfaces. Let us fix an hyperbolic 3-manifold with totally geodesic boundary N and consider the canonical decomposition of N in truncated polyhedra (see [**13**] for the definition). Let M be the double of N, notice that the double of the 2-skeleton of the decomposition of N gives a piece-wise geodesic surface T.

Suppose that every polyhedron of decomposition is a truncated tetrahedron. We want to estimate the number of edges and faces of T. On the boundary of N, the decomposition gives a triangulation. Let v, l, t be respectively the number of vertices, edges and faces of this triangulation. We have $v - l + t = \chi$ where χ is the Euler characteristic of ∂N ($\chi < 0$). On the other hand, we have $t = 2/3l$ and so, $v - 1/3l = \chi$. Now, let e, f be respectively the number of edges and faces of T. We have that $v = 2e$ (in fact, every edge of T intersects ∂N in two vertices) and $l = 3e$. It follows that $2e - f = \chi < 0$.

We conclude this section by proving that regular domains which are invariant for some affine deformation of Γ correspond bijectively to Γ-invariant measured simplicial stratifications.

Corollary 9.12. *There exists a bijective correspondence between* Γ*-invariant measured simplicial stratifications of* $\mathbb{H}^3$ *and future complete regular domains which are invariant for some affine deformation* Γ_τ *of* Γ *and have a simplicial singularity.*

Proof. It is sufficient to show that Γ-invariant measured simplicial stratifications give domains with simplicial singularity. Now, let us fix a Γ-invariant measured simplicial stratification $(\mathcal{C}, X, \mu)$ and let $\{a(P)\}$ be the family of weights associated with it. By Proposition 9.11, we have that there exists $a > 0$ such that $a \leq a(P)$ for all 2-faces P. Now, for a 3-piece Δ, let ρ_Δ be the corresponding point on Σ. We have that $\Sigma_0 = \{\rho_\Delta | \Delta \text{ is a 3-piece}\}$. On the other hand, we know that $\langle \rho_\Delta, -\rho_{\Delta'}, \rho_\Delta - \rho_{\Delta'} \rangle \geq a^2$ so that Σ_0 is a discrete set. q.e.d.

Remark 9.13. Clearly, there exists a similar discussion about Γ-invariant measured simplicial stratifications in every dimension (where Γ is a co-compact torsion-free discrete group). In particular, for a given Γ-invariant simplicial stratification $\mathcal{C}$, there exists only a finite number of pieces up to the action of Γ. Thus, as in the 3-dimensional case, we can deduce that the set of Γ-invariant family of weights on $\mathcal{C}$ is parametrized by the set of positive solutions of a linear system with f parameters and $2e$ equations where f is the number of $(n-1)$-pieces (modulo Γ) and e is the number of $(n-2)$-pieces (modulo Γ). Moreover, we can see that Γ-invariant simplicial geodesic stratifications of $\mathbb{H}^n$ correspond bijectively to regular domains of $\mathbb{M}^{n+1}$ invariant for the action of some affine deformation Γ_τ of Γ and with simplicial singularity.

10. Conclusions

Let Γ be a torsion-free co-compact discrete subgroup of $\mathrm{SO}^+(n,1)$. We have seen that for every cocycle $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$, there exists a unique future complete future complete regular domain $\mathcal{D}_\tau$ which is Γ_τ-invariant. So, future complete regular domains arise naturally in the study of Lorentzian flat structures on $\mathbb{R} \times M$.

In Section 4, we have associated to $\mathcal{D}_\tau$ a Γ-invariant geodesic stratification of $\mathbb{H}^n$. On the other hand, in Section 8, we have seen that given a Γ-invariant measured geodesic stratification, we can construct a future complete regular domain which is invariant for an affine deformation of Γ. Moreover, in dimension $n = 2$, this correspondence agrees with the Mess identification between measured geodesic laminations and future complete regular domains.

We can ask if this correspondence is an identification in every dimension. We have seen in Section 9 that this correspondence induces an identification between simplicial stratification and future complete regular domains with simplicial singularity.

The general case seems more difficult. Given a future complete regular domain Ω, we should construct a **measured geodesic stratification** $(\mathcal{C}, Y, \mu)$ which gives Ω. By looking at the construction of a domain Ω, we have $Y = \{x | \#(N^{-1}(x)) > 1\}$ and in fact, it is easy to see that this set has zero Lebesgue measure in $\mathbb{H}^n$. Now, suppose that for every admissible path $c : [0,1] \to \mathbb{H}^n$, there exists a *Lipschitz path* $\tilde{c} : [0,1] \to \widetilde{S}_1$ such that $N(\tilde{c}([0,1])) = c([0,1])$. On this assumption, we could define a measure μ on an admissible path in this way. Since the retraction is locally Lipschitz, the map $r(t) = r(\tilde{c}(t))$ is Lipschitz so that it is differentiable almost everywhere. Consider the $\mathbb{R}^{n+1}$-valued measure $\tilde{\mu}$ on $[0,1]$ defined by the identity

$$\tilde{\mu}(E) = \int_E r'(t)\mathrm{d}t.$$

Then, we could define the transverse measure μ_c as the image of the measure $\tilde{\mu}$:

$$\mu_c = N_*(\tilde{\mu}).$$

Notice that the assumption is always verified for regular domains Ω arising from measured geodesic stratifications. In fact, given an admissible path c, the Lipschitz path $\tilde{c}$ always exists. In fact, we can define

$$\tilde{c}(t) = c(t) + \int_0^t \mu_c.$$

Thus, the problem is: given an admissible path $c : [0,1] \to \mathbb{H}^n$, we have to find a rectifiable curve $\tilde{c} \subset \widetilde{S}_1$ such that $N(\tilde{c})$ is the curve c (notice that if $\widetilde{S}_a$ is strictly convex, there exists a unique curve such that $N(\tilde{c}) = c$, but we do not know if such an arc is rectifiable).

In dimension $n = 2$, we can see that this problem has always a solution because if v_1 and v_2 are orthogonal vectors to two leaves of the lamination, then they generate a timelike vector space so that

$$|\langle v_1, v_2\rangle| \geq \langle v_1, v_1\rangle^{1/2} \langle v_2, v_2\rangle^{1/2}.$$

By using this inequality, it can be shown [**8**] that the length of $\tilde{c}$ is less than

$$\ell(c) + \langle r(\tilde{c}(1)) - r(\tilde{c}(0)), r(\tilde{c}(1)) - r(\tilde{c}(0))\rangle$$

where $\ell(c)$ is the length of $\mathbb{H}^n$. Unfortunately, in dimension $n \geq 3$, this argument fails.

Geodesic stratifications occur in the study of conformal structures on hyperbolic closed manifold. More precisely, if N is equal to M equipped with a conformal structure (i.e., a $(S^n, \mathrm{Conf}(S^n))$-structure), then the universal covering is decomposed in a union of pieces $\{P_i\}$. The developing map takes P_i on a subset of S^n which is conformally equivalent to a hyperbolic ideal convex set. Moreover, this decomposition is intrinsic (see for instance [**3**, **18**]).

Now, suppose that we have a differentiable path N_t of conformal structures on M such that N_0 corresponds to the hyperbolic structure. Let us denote by $\mathcal{C}_t$ the stratification of N_t.

We know that the derivative of the respective holonomies ρ_t produces a cocycle $\tau \in \mathrm{H}^1_{\rho_0}(\pi_1(M), \mathfrak{so}(n+1,1))$ (in fact, the group of conformal diffeomorphisms of S^n is isomorphic to $\mathrm{SO}\,(n+1,1)$). On the other hand, by means of Weil local rigidity Theorem [**20**], it is not hard to check that if $n \geq 3$, then there exists a natural identification $\mathrm{H}^1_{Ad\circ\rho_0}(\pi_1(M), \mathfrak{so}(n+1,1)) \cong \mathrm{H}^1_{\rho_0}(\pi_1(M), \mathbb{R}^{n+1})$. Thus, we can consider the domain $\mathcal{D}_\tau$ that produces a stratification $\mathcal{C}_0$ of $\mathbb{H}^n$. So, we can ask how the stratifications $\mathcal{C}_t$ and $\mathcal{C}_0$ are related. More precisely, we can suppose that the developing maps D_t of N_t converge to the developing map D_0 of M. So, it makes sense to ask if the stratification $\mathcal{C}_t$ tends to the stratification $\mathcal{C}_0$ when $t \to 0$.

We can state this problem also in dimension 2. Consider the path of projective structures N_t corresponding, in Thurston parametrization of the space of projective structures, to points $(M, t\lambda)$. (Recall that in dimension 2, projective structures are $(S^2, \mathrm{Conf}(S^2))$-structures.). We know that N_t is projectively equivalent to the grafting of N along $t\lambda$. Thus, when $t \to 0$, the measured lamination of N_t tends to λ. Now, the derivative in 0 of the holonomies ρ_t is a cocycle $t \in \mathrm{H}^1_{\mathrm{Ad}\circ\rho_0}(\pi_1(M), \mathfrak{so}(3,1))$. Moreover, this element does not lie in $\mathrm{H}^1_{\mathrm{Ad}\circ\rho_0}(\pi_1(M), \mathfrak{so}(2,1))$. Thus, t represents a non-vanishing element

$$\tau \in \mathrm{H}^1(\pi_1(M), \mathfrak{so}(3,1))/\mathrm{H}^1(\pi_1(M), \mathfrak{so}(2,1)) \cong \mathrm{H}^1(\pi_1(M), \mathbb{R}^{2+1}).$$

By Epstein formulas of the derivative of bending deformations (see [**10**]) in [**8**], it is deduced that the lamination associated to $\mathcal{D}_\tau$ is λ. Thus, at least in dimension 2, the question has a positive answer.

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