

## CANONICAL METRICS ON THE MODULI SPACE OF RIEMANN SURFACES II

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### Abstract

In this paper, we continue our study of the canonical metrics on the moduli space of curves. We first prove the bounded geometry of the Ricci and perturbed Ricci metrics. By carefully choosing the perturbation constant and by studying the asymptotics, we show that the Ricci and holomorphic sectional curvatures of the perturbed Ricci metric are bounded from above and below by negative constants. Based on our understanding of the Kähler–Einstein metric, we show that the logarithmic cotangent bundle over the Deligne–Mumford moduli space is stable with respect to the canonical polarization. Finally, in the last section, we prove the strongly bounded geometry of the Kähler–Einstein metric by using the Kähler–Ricci flow and *a priori* estimates of the complex Monge–Ampere equation.

### 1. Introduction

In [11], we started the project to understand the canonical metrics on the Teichmüller and the moduli spaces of Riemann surfaces, especially the Kähler–Einstein metric. Our goal is to understand the geometry and topology of the moduli spaces from understanding those classical metrics, as well as to find new complete Kähler metrics with good curvature property and asymptotic behavior. In [11], we studied in detail two new complete Kähler metrics, the Ricci and the perturbed Ricci metric. In particular, we proved that the Ricci metric has bounded holomorphic bisectional curvature, and the perturbed Ricci metric, not only has bounded holomorphic bisectional curvature, but also has negatively pinched holomorphic sectional curvature. By using the perturbed Ricci metric as a bridge, we were able to prove the equivalence of several classical complete metrics on the Teichmüller and the moduli spaces, including the Teichmüller metric, the Kobayashi metric, the Bergman

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metric, the Carathéodory metric, the Cheng–Yau, Kähler–Einstein metric, the McMullen metric, as well as the Ricci and the perturbed Ricci metric. This also solved an old conjecture of Yau about the equivalence of the Kähler–Einstein metric and the Teichmüller metric.

In this paper, we continue our study on these metrics and other classical metrics, in particular, the Kähler–Einstein metric, and the perturbed Ricci metric. One of the main results is the good understanding of the two new metrics and the Kähler–Einstein metric, from which we will derive some important corollaries about the geometry of the moduli spaces. We first prove that the Ricci curvature of the perturbed Ricci metric has negative upper and lower bounds, and it also has bounded geometry. Recall that it also has negatively bounded holomorphic sectional curvature. The perturbed Ricci metric is the first known complete Kähler metric on the Teichmüller and the moduli spaces with such good negative curvature property. We then focus on the Kähler–Einstein metric, study in detail its boundary behaviors and prove that not only it has bounded geometry, but also all of the covariant derivatives of its curvature are uniformly bounded. It is natural to expect interesting applications of the good properties of the perturbed Ricci and the Kähler–Einstein metric. For example, as an application of our detailed understanding of these metrics, we prove that the logarithmic cotangent bundle of the moduli space is stable in the sense of Mumford.

This paper is organized as follows. In Section 2, we will prove that both the Ricci metric and the perturbed Ricci metric have bounded curvature. Especially, with a suitable choice of the perturbation constant, the holomorphic sectional curvature and the Ricci curvature of the perturbed Ricci metric are pinched by negative constants. As a simple corollary, we immediately see that the dual of the logarithmic cotangent bundle has no non-trivial holomorphic section. By using the Bers’ embedding theorem and minimal surface theory, we then prove that the Teichmüller space equipped with either of these two metrics has bounded geometry: bounded curvature and lower bound of injectivity radius. McMullen proved that the McMullen metric has bounded geometry.

Having a complete Kähler–Einstein metric puts strong restrictions on the geometric structure of the moduli space. In Section 3, we will study the cohomology classes defined by the Kähler forms and Ricci forms of the Ricci metric and the Kähler–Einstein metric. As a direct corollary, we will see easily that the moduli space is of logarithmic general type. One of the most interesting applications of this study is the proof of the stability of the logarithmic cotangent bundle of the moduli space in the sense of Mumford.

Finally, in Section 3, we will study the bounded geometry of the Kähler–Einstein metric. We set up the Monge–Amperé equation from a new metric obtained by deforming the Ricci metric along the Kähler–Ricci flow. The work in [11] provides a  $C^2$  estimate. We follow the work of Yau in [22] and do the  $C^3$  and  $C^4$  estimates. These estimates imply that the curvature of the Kähler–Einstein metric is bounded. The same method proves that all of the covariant derivatives of the curvature are bounded. This gives us a much better understanding of the Kähler–Einstein metric.

Now, we give the precise statements of the main results in this paper. We fix an integer  $g \geq 2$  and denote by  $\mathcal{T} = \mathcal{T}_g$ , the Teichmüller space, and  $\mathcal{M}_g$ , the moduli space of closed Riemann surfaces of genus  $g$ .

Our first main result, proved in Section 2, is about the curvature properties of the Ricci and the perturbed Ricci metric. We have two theorems, the first one is about the Ricci metric:

**Theorem 1.1.** *The holomorphic bisectional curvature, the holomorphic sectional curvature and the Ricci curvature of the Ricci metric  $\tau$  on the moduli space  $\mathcal{M}_g$  are bounded.*

The proof of this theorem was sketched in [11]. We give the details here.

The second theorem is about the perturbed Ricci metric:

**Theorem 1.2.** *For any constant  $C > 0$ , the bisectional curvature of the perturbed Ricci metric  $\tilde{\tau} = \tau + Ch$  is bounded. Furthermore, with a suitable choice of  $C$ , the holomorphic sectional curvature and the Ricci curvature of  $\tilde{\tau}$  are bounded from above and below by negative constants.*

Both of the above theorems are proved by a detailed analysis of the boundary behavior of these metrics and their curvature. Together with the following result, which is proved by using minimal surface theory and the Bers’ embedding theorem, they imply that Ricci metric and the perturbed Ricci metric both have bounded geometry.

**Corollary 1.1.** *The injectivity radius of the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric is bounded from below.*

Let  $\overline{\mathcal{M}}_g$  be the Deligne–Mumford compactification of the moduli space of Riemann surfaces. Let  $\overline{E}$  denote the logarithmic extension of the cotangent bundle of the moduli space. The next result is an interesting consequence of our detailed understanding of the Kähler–Einstein metric on the moduli spaces. It is proved in Section 3:

**Theorem 1.3.** *The first Chern class of  $\overline{E}$  is positive and  $\overline{E}$  is Mumford stable with respect to  $c_1(\overline{E})$ .*

We prove this theorem by using the good control of the Kähler–Einstein metric and the Ricci metric we obtained near the boundary of the moduli space. As a direct corollary, we have:

**Corollary 1.2.** *The Deligne–Mumford moduli space  $\overline{\mathcal{M}}_g$  is of logarithmic general type for any  $g \geq 2$ .*

Our final result, proved in Section 4, is the following theorem about the Kähler–Einstein metric:

**Theorem 1.4.** *The Kähler–Einstein metric on the Teichmüller space  $\mathcal{T}_g$  has bounded geometry. Furthermore, the covariant derivatives of its curvature are all uniformly bounded.*

This theorem is proved by using the Kähler–Ricci flow and the method of Yau in his proof of the Calabi conjecture to obtain higher order estimates of the curvature.

We will continue our study on the geometry of the moduli space and Teichmüller space in [12]. The topics will include the goodness of these metrics on the moduli space and the  $L^2$ -cohomology and index theory of these metrics and the Weil–Petersson metric on the Teichmüller space, and the convergence of the Kähler–Ricci flow starting from the Ricci and the perturbed Ricci metric.

## 2. The Negativity of the Ricci Curvature of the Perturbed Ricci Metric

In this section, we first study the curvature bounds of the Ricci and the perturbed Ricci metrics. By using the Bers’ embedding theorem, we then show that the injectivity radius of the Teichmüller space equipped with these metrics is bounded from below. This implies that both the Ricci metric and the perturbed Ricci metric have bounded geometry on the Teichmüller space.

The boundedness of the curvatures of these metrics was obtained by analyzing their asymptotic behavior. The proof of the negativity of the holomorphic sectional curvature and Ricci curvature of the perturbed Ricci metric is more delicate. Near the boundary of the moduli space and in the degeneration directions, these curvatures are dominated by the contribution from the Ricci metric which is negative. In the non-degeneration directions and in the interior of the moduli space, these curvatures are dominated by the contribution of the constant multiple of

the Weil–Petersson metric when the constant is large which is also negative. However, we know that the curvature of the linear combinations of two metrics is not linear, we need to handle the error terms carefully.

Let  $\mathcal{M}_g$  be the moduli space of closed oriented Riemann surfaces of genus  $g$  with  $g \geq 2$  and let  $\mathcal{T}_g$  be the corresponding Teichmüller space. Let  $\overline{\mathcal{M}}_g$  be the Deligne–Mumford compactification of  $\mathcal{M}_g$  and let  $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$  be the compactification divisor. It is well known that  $D$  is a divisor of normal crossings. In [11], we studied various metrics on  $\mathcal{M}_g$  and  $\mathcal{T}_g$ . We briefly recall the results here.

Fix a point  $p \in \mathcal{M}_g$ . Let  $X = X_p$  be a Riemann surface corresponding to  $p$ . Let  $z$  be the local holomorphic coordinate on  $X$  and let  $s_1, \dots, s_n$  be local holomorphic coordinates on  $\mathcal{M}_g$  where  $n = 3g - 3$  is the complex dimension of  $\mathcal{M}_g$ . Let  $HB(X)$  and  $Q(X)$  be the spaces of harmonic Beltrami differentials and holomorphic quadratic differentials on  $X$  respectively and let  $\lambda$  be the hyperbolic metric on  $X$ . Namely,

$$\partial_z \partial_{\bar{z}} \log \lambda = \lambda.$$

By the deformation theory of Kodaira–Spencer, we know that the tangent space  $T_p \mathcal{M}_g$  is identified with  $HB(X)$  and the map  $T_p \mathcal{M}_g \rightarrow HB(X)$  is given by

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\bar{z}$$

where  $A_i = -\partial_{\bar{z}}(\lambda^{-1} \partial_{s_i} \partial_{\bar{z}} \log \lambda)$ . Similarly, the cotangent space  $T_p^* \mathcal{M}_g$  is identified with  $Q(X)$ . For  $\mu = \mu(z) \frac{\partial}{\partial z} \otimes d\bar{z} \in HB(X)$  and  $\varphi = \varphi(z) dz^2 \in Q(X)$ , the duality between them is given by

$$\langle \mu, \varphi \rangle = \int_X \mu(z) \varphi(z) dz d\bar{z}$$

and the Teichmüller norm of  $\varphi$  is defined to be

$$\|\varphi\|_T = \int_X |\varphi(z)| dz d\bar{z}.$$

By using the above notation, the norm of the Teichmüller metric is given by

$$\|\mu\|_T = \sup_{\varphi \in Q(X)} \{\operatorname{Re} \langle \mu, \varphi \rangle \mid \|\varphi\|_T = 1\}$$

for all  $\mu \in HB(X) \cong T_p \mathcal{M}_g$ .

The Weil–Petersson metric on  $\mathcal{M}_g$  is defined by

$$h_{i\bar{j}}(p) = \int_X A_i \overline{A_j} dv$$

where  $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\bar{z}$  is the volume form of  $X$ . The Ricci metric  $\tau$  is the negative Ricci curvature of the Weil–Petersson metric

$$\tau_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det(h_{k\bar{l}}).$$

By the works in [20] and [11], we know that the Ricci metric is complete. Now, we take linear combination of the Ricci metric and the Weil–Petersson metric to define the perturbed Ricci metric

$$\tilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$$

where  $C > 0$ . In [11], we proved the following theorem:

**Theorem 2.1.** *For suitable choice of large constant  $C$ , the holomorphic sectional curvature of the perturbed Ricci metric  $\tilde{\tau}$  has negative upper bound. Furthermore, on  $\mathcal{M}_g$ , the Ricci metric, the perturbed Ricci metric, the Kähler–Einstein metric, the Asymptotic Poincaré metric are equivalent.*

This theorem was proved by using the curvature property of the perturbed Ricci metric and the estimates of its asymptotic behavior.

We recall here, our convention of the curvature of a Kähler manifold  $(M^n, g)$ . Let  $z_1, \dots, z_n$  be local holomorphic coordinates near a point  $p \in M$ . The curvature tensor of the metric  $g$  is given by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}.$$

In this case, the Ricci curvature is given by

$$R_{i\bar{j}} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}})$$

and we say the holomorphic sectional curvature of  $g$  is negative if  $R(v, \bar{v}, v, \bar{v})$  is positive for any non-zero holomorphic tangent vector.

Now, we prove several claims about the boundedness of the curvature of the Ricci metric and the perturbed Ricci metric which were stated in [11].

**Theorem 2.2.** *The holomorphic sectional curvature, bisectional curvature and the Ricci curvature of the Ricci metric  $\tau$  on the moduli space  $\mathcal{M}_g$  are bounded.*

As part of the Theorem 4.4 of [11], this theorem was roughly proved in [11]. Here, we give a detailed proof since we need the techniques later.

*Proof.* We follow the notations and computations in [11]. Let  $p \in D$  be a codimension  $m$  boundary point and let  $(t_1, \dots, t_m, s_{m+1}, \dots, s_n)$  be the pinching coordinates of  $\mathcal{M}_g$  at  $p$  where  $t_1, \dots, t_m$  represent the degeneration directions. Let  $X_{t,s}$  be the Riemann surface corresponding

to the point with coordinates  $(t_1, \dots, s_n)$  and let  $l_i$  be the length of the short geodesic loop on the  $i$ -th collar. Let  $u_i = \frac{l_i}{2\pi}$ . We fix  $\delta > 0$  and assume that  $|(t, s)| < \delta$ . When  $\delta$  is small enough, from the work of [21] and [11], we know that

$$u_i = -\frac{\pi}{\log |t_i|} \left( 1 + O \left( \left( \frac{\pi}{\log |t_i|} \right)^2 \right) \right).$$

Now, we let  $u_0 = \sum_{i=1}^m u_i + \sum_{j=m+1}^n |s_j|$ .

By Corollary 4.2 and Theorem 4.4 of [11], the work of Masur in [15] and Wolpert, if we use  $\tilde{R}_{i\bar{j}k\bar{l}}$  to denote the curvature tensor of the Ricci metric  $\tau$ , we have

- 1)  $\tau_{i\bar{i}} = \frac{3}{4\pi^2} \frac{u_i^2}{|t_i|^2} (1 + O(u_0))$ , if  $i \leq m$ ;
- 2)  $\tau_{i\bar{j}} = O \left( \frac{u_i^2 u_j^2}{|t_i t_j|} (u_i + u_j) \right)$ , if  $i, j \leq m$  and  $i \neq j$ ;
- 3)  $\tau_{i\bar{j}} = O \left( \frac{u_i^2}{|t_i|} \right)$ , if  $i \leq m$  and  $j \geq m+1$ ;
- 4)  $\tau_{i\bar{j}} = O \left( \frac{u_j^2}{|t_j|} \right)$ , if  $j \leq m$  and  $i \geq m+1$ ;
- 5)  $\tau_{i\bar{j}} = O(1)$ , if  $i, j \geq m+1$ ;
- 6) The matrix  $(\tau_{i\bar{j}})_{i, j \geq m+1}$  is positive definite and has a positive lower bound depending on  $p, n, \delta$ ;
- 7)  $\tilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4 |t_i|^4} (1 + O(u_0))$ , if  $i \leq m$ ;
- 8)  $\tilde{R}_{i\bar{i}i\bar{i}} = O(1)$ , if  $i \geq m+1$ .

Now, we let

$$\Lambda_i = \begin{cases} \frac{u_i}{|t_i|} & i \leq m \\ 1 & i \geq m+1. \end{cases}$$

We divide the index set into three parts. Let

- 1)  $A_1 = \{(i, i, i, i) \mid i \leq m\}$ ;
- 2)  $A_2 = \{(i, j, k, l) \mid \text{at least one of } i, j, k, l \leq m \text{ and they are not all equal}\}$ ;
- 3)  $A_3 = \{(i, j, k, l) \mid i, j, k, l \geq m+1\}$ .

By following the computations of [11], we know that, if  $(i, j, k, l) \in A_2$ , then

$$\tilde{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0).$$

Let  $v = a_1 \frac{\partial}{\partial t_1} + \cdots + a_n \frac{\partial}{\partial s_n}$  and  $w = b_1 \frac{\partial}{\partial t_1} + \cdots + b_n \frac{\partial}{\partial s_n}$  be two tangent vectors at  $(t, s)$ . We have

$$\begin{aligned} |\tilde{R}(v, \bar{v}, w, \bar{w})| &= \left| \sum_{i,j,k,l} a_i \bar{a}_j b_k \bar{b}_l \tilde{R}_{i\bar{j}k\bar{l}} \right| \leq \sum_{i,j,k,l} |a_i \bar{a}_j b_k \bar{b}_l \tilde{R}_{i\bar{j}k\bar{l}}| \\ &= I_1 + I_2 + I_3 \end{aligned}$$

where  $I_\alpha = \sum_{(i,j,k,l) \in A_\alpha} |a_i \bar{a}_j b_k \bar{b}_l \tilde{R}_{i\bar{j}k\bar{l}}|$ . To estimate the norms of  $v$  and  $w$ , we have

$$\begin{aligned} \tau(v, v) &= \sum_{i,j \leq m, i \neq j} a_i \bar{a}_j \tau_{i\bar{j}} + \sum_{i \leq m < j} a_i \bar{a}_j \tau_{i\bar{j}} + \sum_{j \leq m < i} a_i \bar{a}_j \tau_{i\bar{j}} \\ &\quad + \sum_{i \leq m} |a_i|^2 \tau_{i\bar{i}} + \sum_{i,j \geq m+1} a_i \bar{a}_j \tau_{i\bar{j}}. \end{aligned}$$

By using the asymptotic of  $\tau$  and the Schwarz inequality, we have

$$\left| \sum_{i,j \leq m, i \neq j} a_i \bar{a}_j \tau_{i\bar{j}} + \sum_{i \leq m < j} a_i \bar{a}_j \tau_{i\bar{j}} + \sum_{j \leq m < i} a_i \bar{a}_j \tau_{i\bar{j}} \right| \leq O(u_0) \sum_{i=1}^n |a_i|^2 \Lambda_i^2.$$

Since the matrix  $(\tau_{i\bar{j}})_{i,j \geq m+1}$  has a local positive lower bound, we know there is a positive constant  $c$  depending on  $p, n, \delta$  such that

$$\sum_{i,j \geq m+1} a_i \bar{a}_j \tau_{i\bar{j}} \geq c \sum_{i=1}^n |a_i|^2 = c \sum_{i=m+1}^n |a_i|^2 \Lambda_i^2.$$

Finally, we have

$$\sum_{i \leq m} |a_i|^2 \tau_{i\bar{i}} = \frac{3}{4\pi^2} (1 + O(u_0)) \sum_{i=1}^m |a_i|^2 \Lambda_i^2.$$

By combining the above inequalities, we know there is another positive constant  $c_1$  depending on  $p, n, \delta$  such that

$$(2.1) \quad \tau(v, v) \geq c_1 \sum_{i=1}^n |a_i|^2 \Lambda_i^2.$$

Similar estimates hold for the Ricci norm of  $w$ .

Now, for each term in  $I_2$ , by using the Schwarz inequality, we have

$$\begin{aligned} |a_i \bar{a}_j b_k \bar{b}_l \tilde{R}_{i\bar{j}k\bar{l}}| &= O(u_0) |a_i \bar{a}_j b_k \bar{b}_l \Lambda_i \Lambda_j \Lambda_k \Lambda_l| \\ &\leq O(u_0) (|a_i|^2 \Lambda_i^2 + |a_j|^2 \Lambda_j^2) (|b_k|^2 \Lambda_k^2 + |b_l|^2 \Lambda_l^2). \end{aligned}$$



So, we have

$$I_2 = O(u_0) \left( \sum_{i=1}^n |a_i|^2 \Lambda_i^2 \right) \left( \sum_{i=1}^n |b_i|^2 \Lambda_i^2 \right) \leq c_0 \tau(v, v) \tau(w, w)$$

for some positive constant  $c_0$ . By enlarging this constant, we also have

$$\begin{aligned} I_3 &\leq O(1) \sum_{(i,j,k,l) \in A_3} |a_i \bar{a}_j b_k \bar{b}_l| \leq O(1) \left( \sum_{i=m+1}^n |a_i|^2 \Lambda_i^2 \right) \left( \sum_{i=m+1}^n |b_i|^2 \Lambda_i^2 \right) \\ &\leq c_0 \tau(v, v) \tau(w, w) \end{aligned}$$

and

$$I_1 = \frac{3}{8\pi^4} (1 + O(u_0)) \sum_{i=1}^m |a_i|^2 |b_i|^2 \Lambda_i^4 \leq c_0 \tau(v, v) \tau(w, w).$$

By combining the above inequalities, we know that there is a positive constant  $\tilde{c}$  depending on  $p, n, \delta$  such that if  $\delta$  is small enough, then

$$|\tilde{R}(v, \bar{v}, w, \bar{w})| \leq \tilde{c} \tau(v, v) \tau(w, w).$$

So, we have proved that for each point  $p \in D$ , there is an open neighborhood  $U_p$  such that the bisectional curvature of the Ricci metric is bounded by a constant which depends on  $U_p$ . Since  $D$  is compact, we can find a finite cover of  $D$  by such  $U_p$ . Let  $U$  be the union of the charts  $\{U_p\}$  in this finite cover. Then, we can find a universal constant  $c$  which bounds the bisectional curvature at each point in  $U$ . Since  $\mathcal{M}_g \setminus U$  is a compact set, we know the bisectional curvature is bounded there. So, we proved that the bisectional curvature of the Ricci metric is bounded.

The boundedness of the holomorphic sectional curvature can be proved similarly if we replace  $w$  by  $v$  in the above argument. Finally, since the Ricci curvature is the average of the bisectional curvature and the holomorphic sectional curvature, it is bounded. We finish the proof.  
q.e.d.

We now investigate the curvatures of the perturbed Ricci metric. We have

**Theorem 2.3.** *For any constant  $C > 0$ , the bisectional curvature of the perturbed Ricci metric  $\tilde{\tau} = \tau + Ch$  is bounded. Furthermore, with suitable choice of  $C$ , the holomorphic sectional curvature and the Ricci curvature of  $\tilde{\tau}$  are bounded from above and below by negative constants.*

*Proof.* We use  $R_{i\bar{j}k\bar{l}}$  and  $P_{i\bar{j}k\bar{l}}$  to denote the curvature tensor of the Weil–Petersson metric and the perturbed Ricci metric respectively. We use the same notations as in the proof of the above theorem. Let  $p \in D$

be a codimension  $m$  boundary point, let  $(t_1, \dots, s_n)$  be the local pinching coordinates and let  $A_1, A_2, A_3$  be the partition of the index set. Assume  $\delta$  is a small positive constant and  $|(t, s)| < \delta$ .

By Corollaries 4.1, 4.2 and Theorem 5.2 of [11], we have

- 1)  $\tilde{\tau}_{i\bar{i}} = \frac{u_i^2}{|t_i|^2} \left( \frac{3}{4\pi^2} + \frac{1}{2}Cu_i \right) (1 + O(u_0))$ , if  $i \leq m$ ;
- 2)  $\tilde{\tau}_{i\bar{j}} = \frac{u_i^2 u_j^2}{|t_i t_j|} (O(u_i + u_j) + CO(u_i u_j))$ , if  $i, j \leq m$  and  $i \neq j$ ;
- 3)  $\tilde{\tau}_{i\bar{j}} = \frac{u_i^2}{|t_i|} (O(1) + CO(u_i))$ , if  $i \leq m$  and  $j \geq m + 1$ ;
- 4)  $\tilde{\tau}_{i\bar{j}} = \frac{u_j^2}{|t_j|} (O(1) + CO(u_j))$ , if  $j \leq m$  and  $i \geq m + 1$ ;
- 5)  $P_{i\bar{i}i\bar{i}} = \left( \left( \frac{9}{16\pi^4} - \frac{3}{16\pi^4} \left( 1 + \frac{2\pi^2 Cu_i}{3} \right)^{-1} \right) \frac{u_i^4}{|t_i|^4} + \frac{3C}{8\pi^2} \frac{u_i^5}{|t_i|^4} \right) (1 + O(u_0))$ ,  
if  $i \leq m$ ;
- 6)  $P_{i\bar{i}i\bar{i}} = O(1) + CR_{i\bar{i}i\bar{i}}$ , if  $i \geq m + 1$ ;
- 7)  $P_{i\bar{j}k\bar{l}} = O(1) + CR_{i\bar{j}k\bar{l}}$ , if  $(i, j, k, l) \in A_3$ ;
- 8)  $P_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0) + CR_{i\bar{j}k\bar{l}}$ , if  $(i, j, k, l) \in A_2$

where all the  $O$ -terms are independent of  $C$ .

Let  $v$  and  $w$  be holomorphic vectors as above. To estimate the bisectonal curvature, we have

(2.2)

$$\begin{aligned}
|P(v, \bar{v}, w, \bar{w})| &\leq \sum_{i=1}^m |a_i|^2 |b_i|^2 P_{i\bar{i}i\bar{i}} + \sum_{(i,j,k,l) \in A_2} |a_i \bar{a}_j b_k \bar{b}_l P_{i\bar{j}k\bar{l}}| \\
&\quad + \sum_{(i,j,k,l) \in A_3} |a_i \bar{a}_j b_k \bar{b}_l P_{i\bar{j}k\bar{l}}| \\
&\leq \sum_{i=1}^m |a_i|^2 |b_i|^2 P_{i\bar{i}i\bar{i}} + O(u_0) \sum_{(i,j,k,l) \in A_2} |a_i \bar{a}_j b_k \bar{b}_l \Lambda_i \Lambda_j \Lambda_k \Lambda_l| \\
&\quad + C \sum_{(i,j,k,l) \in A_2} |a_i \bar{a}_j b_k \bar{b}_l R_{i\bar{j}k\bar{l}}| + O(1) \sum_{(i,j,k,l) \in A_3} |a_i \bar{a}_j b_k \bar{b}_l| \\
&\quad + C \sum_{(i,j,k,l) \in A_3} |a_i \bar{a}_j b_k \bar{b}_l R_{i\bar{j}k\bar{l}}|.
\end{aligned}$$

Let  $c_i$  denote certain positive constants only depending on  $p, n, \delta$ . By the proof of the above theorem, since  $\tilde{\tau} \geq \tau$ , we have

$$\begin{aligned}
(2.3) \quad \left| O(u_0) \sum_{(i,j,k,l) \in A_2} |a_i \bar{a}_j b_k \bar{b}_l \Lambda_i \Lambda_j \Lambda_k \Lambda_l| \right| &\leq c_0 \tau(v, v) \tau(w, w) \\
&\leq c_0 \tilde{\tau}(v, v) \tilde{\tau}(w, w).
\end{aligned}$$

We also have

$$(2.4) \quad \left| O(1) \sum_{(i,j,k,l) \in A_3} |a_i \bar{a}_j b_k \bar{b}_l| \right| \leq c_1 \tau(v, v) \tau(w, w) \\ \leq c_2 \tilde{\tau}(v, v) \tilde{\tau}(w, w).$$

Now, we estimate  $\tilde{\tau}(v, v)$ . By using similar argument as in the above proof, we know that

$$\tilde{\tau}(v, v) \geq c_2 \sum_{i=1}^n |a_i|^2 \tilde{\tau}_{i\bar{i}}.$$

So for  $i \leq m$ , we have

$$|a_i|^2 |b_i|^2 P_{i\bar{i}\bar{i}\bar{i}} = |a_i|^2 |b_i|^2 \left( \left( \frac{9}{16\pi^4} - \frac{3}{16\pi^4} \left( 1 + \frac{2\pi^2 C u_i}{3} \right)^{-1} \right) \frac{u_i^4}{|t_i|^4} \right. \\ \left. + \frac{3C}{8\pi^2} \frac{u_i^5}{|t_i|^4} \right) (1 + O(u_0)) \\ \leq c_3 |a_i|^2 |b_i|^2 \frac{u_i^4}{|t_i|^4} \left( \frac{3}{4\pi^2} + \frac{1}{2} C u_i \right)^2 \leq c_4 |a_i|^2 |b_i|^2 \tilde{\tau}_{i\bar{i}}^2$$

which implies

$$(2.5) \quad \sum_{i=1}^m |a_i|^2 |b_i|^2 P_{i\bar{i}\bar{i}\bar{i}} \leq c_5 \sum_{i=1}^n |a_i|^2 |b_i|^2 \tilde{\tau}_{i\bar{i}}^2 \leq c_6 \tilde{\tau}(v, v) \tilde{\tau}(w, w).$$

To estimate the remaining two terms in the right-hand side of (2.2), we need the estimates of the curvature tensor of the Weil–Petersson metric which is done in the proof of Corollary 4.2 of [11]. By collecting the results there, we know that  $R_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0)$  if  $(i, j, k, l) \in A_2$  and  $R_{i\bar{j}k\bar{l}} = O(1)$  if  $(i, j, k, l) \in A_3$ . By using a similar argument as in the above proof, we know that

$$(2.6) \quad \left| C \sum_{(i,j,k,l) \in A_2} |a_i \bar{a}_j b_k \bar{b}_l R_{i\bar{j}k\bar{l}}| \right| \leq c_7 C \tilde{\tau}(v, v) \tilde{\tau}(w, w)$$

and

$$(2.7) \quad \left| C \sum_{(i,j,k,l) \in A_3} |a_i \bar{a}_j b_k \bar{b}_l R_{i\bar{j}k\bar{l}}| \right| \leq c_8 C \tilde{\tau}(v, v) \tilde{\tau}(w, w).$$

These imply that

$$|P(v, \bar{v}, w, \bar{w})| \leq (c_9 C + c_{10}) \tilde{\tau}(v, v) \tilde{\tau}(w, w).$$

By using the compactness argument as above, we proved that the bisectonal curvature of  $\tilde{\tau}$  is bounded. However, the bounds depend on the choice of  $C$ .

By using a similar method, it is easy to see that the holomorphic sectional curvature is also bounded. However, in [11], we showed that, for suitable choice of  $C$ , the holomorphic sectional curvature has a negative upper bound. So for this  $C$ , the holomorphic sectional curvature of  $\tilde{\tau}$  is pinched between negative constants.

Finally, we consider the Ricci curvature of  $\tilde{\tau} = \tau + Ch$ . We first define two new tensors. Let  $\widehat{R}_{i\bar{j}k\bar{l}} = P_{i\bar{j}k\bar{l}} - CR_{i\bar{j}k\bar{l}}$  and let  $P_{i\bar{j}} = -\text{Ric}(\omega_{\tilde{\tau}})_{i\bar{j}}$ . We only need to show that there are positive constants  $\alpha_1$  and  $\alpha_2$  which may depend on  $C$  such that

$$(2.8) \quad \alpha_1 \left( \tilde{\tau}_{i\bar{j}} \right) \leq \left( P_{i\bar{j}} \right) \leq \alpha_2 \left( \tilde{\tau}_{i\bar{j}} \right).$$

Based on Lemma 5.2 of [11] and by Corollaries 4.1 and 4.2 of [11], we can estimate the asymptotic of the perturbed Ricci metric.

**Lemma 2.1.** *Let  $p \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$  be a codimension  $m$  boundary point and let  $(t, s) = (t_1, \dots, s_n)$  be the pinching coordinates. Let  $\delta > 0$  be a small constant such that  $|(t, s)| < \delta$ . Let  $C$  be a positive constant. Let  $B_1 = \left( \tau_{i\bar{j}} \right)_{i,j \geq m+1}$ , let  $B_2 = \left( h_{i\bar{j}} \right)_{i,j \geq m+1}$  and let  $B = B_1 + CB_2$ . Let  $\left( B^{i\bar{j}} \right) = \overline{(B^{-1})}$  and let  $x_i = 2\pi^2 C u_i$  for  $i \leq m$ . Then, we have*

- 1)  $\tilde{\tau}_{i\bar{i}} = \frac{\Lambda_i^2}{4\pi^2} (3 + x_i) (1 + O(u_0))$  and  $\tilde{\tau}^{i\bar{i}} = 4\pi^2 \Lambda_i^{-2} (3 + x_i)^{-1} (1 + O(u_0))$  if  $i \leq m$ ;
- 2)  $\tilde{\tau}_{i\bar{j}} = \frac{u_i^2 u_j^2}{|t_i t_j|} (O(u_i + u_j) + CO(u_i u_j))$  and  $\tilde{\tau}^{i\bar{j}} = O(|t_i t_j|) \min\{(1 + x_i)^{-1}, (1 + x_j)^{-1}\}$  if  $i, j \leq m$  and  $i \neq j$ ;
- 3)  $\tilde{\tau}_{i\bar{j}} = \frac{u_i^2}{|t_i|} (O(1) + CO(u_i))$  and  $\tilde{\tau}^{i\bar{j}} = O(|t_i|) (1 + x_i)^{-1}$  if  $i \leq m$  and  $j \geq m + 1$ ;
- 4)  $\tilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$  and  $\tilde{\tau}^{i\bar{j}} = C^{-1} \left( h^{i\bar{j}} + O(C^{-1}) + O(u_0) \right)$  if  $i, j \geq m + 1$ ;
- 5)  $\widehat{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0)$  if  $(i, j, k, l) \in A_2$ ;
- 6)  $\widehat{R}_{i\bar{j}k\bar{l}} = O(1)$  if  $(i, j, k, l) \in A_3$ .

*Proof.* Let  $a = a(t_1, \bar{t}_1, \dots, s_n, \bar{s}_n)$  and  $b = b(t_1, \bar{t}_1, \dots, s_n, \bar{s}_n)$  be any local functions defined for  $|(t, s)| < \delta$ . Assume there is local constant  $c_1$  depending on  $p, \delta$  and  $n$  such that

$$0 < c_1 \leq a, b$$

for  $|(t, s)| < \delta$ . We first realize that there are constants  $\mu_i > 0$  depending on  $c_1, p, n$  and  $\delta$  such that

$$1 + x_i \leq \mu_i(a + Cb)$$

for  $|(t, s)| < \delta$ . In fact, we can pick  $\mu_i = \max\{\frac{1}{c_1}, \frac{2\pi^2 u_i}{c_1}\}$  since  $u_i$  is small when  $\delta$  is small.

The first four claims followed from Corollaries 4.1, 4.2, Lemmas 5.1 and 5.2 of [11]. By the proof of Lemma 5.2 of [11], we have the linear algebraic formula

$$\det(\tilde{\tau}) = \left( \prod_{i=1}^m \frac{\Lambda_i^2}{4\pi^2} (3 + x_i) \right) \det(B)(1 + O(u_0)).$$

These claims followed from similar computations of the determinants of the minor matrices.

The last two claims follow from the same techniques and computations as in the appendix of [11]. q.e.d.

Now, we estimate  $P_{i\bar{j}}$ . We first compute  $P_{i\bar{i}}$  with  $i \leq m$ . We have

$$\begin{aligned} P_{i\bar{i}} &= \tilde{\tau}^{i\bar{i}} P_{i\bar{i}i\bar{i}} + \sum_{(k,l) \neq (i,i)} \tilde{\tau}^{k\bar{l}} P_{i\bar{i}k\bar{l}} \\ &= \tilde{\tau}^{i\bar{i}} P_{i\bar{i}i\bar{i}} + \sum_{(k,l) \neq (i,i)} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{i}k\bar{l}} + C \sum_{(k,l) \neq (i,i)} \tilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}}. \end{aligned}$$

We estimate each term in the right-hand side of the above formula. We have

$$\begin{aligned} (2.9) \quad \tilde{\tau}^{i\bar{i}} P_{i\bar{i}i\bar{i}} &= 4\pi^2 \Lambda_i^{-2} (3 + x_i)^{-1} \left( \left( \frac{9}{16\pi^4} - \frac{3}{16\pi^4} \left(1 + \frac{x_i}{3}\right)^{-1} \right) \Lambda_i^4 \right. \\ &\quad \left. + \frac{3x_i}{16\pi^4} \Lambda_i^4 \right) (1 + O(u_0)) \\ &= \frac{3}{4\pi^2} \Lambda_i^2 (3 + x_i)^{-1} \left( 3 - \left(1 + \frac{x_i}{3}\right)^{-1} + x_i \right) (1 + O(u_0)) \\ &= \frac{3}{4\pi^2} \left( 1 - \frac{3}{(3 + x_i)^2} \right) \Lambda_i^2 (1 + O(u_0)). \end{aligned}$$

By the fifth claim of the above lemma, we have

$$(2.10) \quad \left| \sum_{(k,l) \neq (i,i)} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{i}k\bar{l}} \right| = O(\Lambda_i^2) O(u_0).$$

We also have

$$C \sum_{(k,l) \neq (i,i)} \tilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}} = C \sum_{k \neq i, l \neq i} \tilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}} + C \sum_{k \neq i} \tilde{\tau}^{k\bar{i}} R_{i\bar{i}k\bar{i}} + C \sum_{l \neq i} \tilde{\tau}^{i\bar{l}} R_{i\bar{i}i\bar{l}}.$$

By the proof of Corollary 4.2 of [11], we know that, if  $k \neq i$ , then  $R_{i\bar{i}k\bar{i}} = O\left(\frac{u_i^5}{|t_i|^3}\right) O(\Lambda_k)$ . By combining with the above lemma, we have

$$(2.11) \quad \left| C \sum_{k \neq i} \tilde{\tau}^{k\bar{i}} R_{i\bar{i}k\bar{i}} \right| = CO(u_i^3) \Lambda_i^2 (1+x_i)^{-1} = \frac{x_i}{1+x_i} \Lambda_i^2 O(u_i^2) = \Lambda_i^2 O(u_i^2).$$

Similarly, we have

$$(2.12) \quad \left| C \sum_{l \neq i} \tilde{\tau}^{i\bar{l}} R_{i\bar{i}i\bar{l}} \right| = \Lambda_i^2 O(u_i^2).$$

Now, we fix  $i$  and let  $v = \frac{\partial}{\partial t_i}$ ,  $w = b_1 \frac{\partial}{\partial t_1} + \dots + b_n \frac{\partial}{\partial s_n}$  with  $b_i = 0$ . Since the bisectional curvature of the Weil–Petersson metric is non-positive, we have

$$0 \leq R(v, \bar{v}, w, \bar{w}) = \sum_{k \neq i, l \neq i} b_k \bar{b}_l R_{i\bar{i}k\bar{l}}.$$

This implies that the matrix  $(R_{i\bar{i}k\bar{l}})_{k \neq i, l \neq i}$  is semi-positive definite. So, we know

$$\sum_{k \neq i, l \neq i} \tilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}} \geq 0$$

since it is the trace of the product of a positive definite matrix and a semi-positive definite matrix. Again, by using the proof of Lemma 4.2 of [11], we have

$$(2.13) \quad 0 \leq C \sum_{k \neq i, l \neq i} \tilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}} \leq x_i \Lambda_i^2 O^+(1)$$

where  $O^+(1)$  represents a positive bounded term. By combining formulas (2.9), (2.10), (2.11), (2.12) and (2.13), we have

$$P_{i\bar{i}} \geq \frac{3}{4\pi^2} \left( 1 - \frac{3}{(3+x_i)^2} \right) \Lambda_i^2 (1 + O(u_0)) + \Lambda_i^2 O(u_0) + \Lambda_i^2 O(u_i^2)$$

and

$$P_{i\bar{i}} \leq \frac{3}{4\pi^2} \left( 1 - \frac{3}{(3+x_i)^2} + x_i O^+(1) \right) \Lambda_i^2 (1 + O(u_0)) + \Lambda_i^2 O(u_0) + \Lambda_i^2 O(u_i^2)$$

which imply

$$(2.14) \quad \frac{3\Lambda_i^2}{8\pi^2} \left(1 - \frac{3}{(3+x_i)^2}\right) \leq P_{i\bar{i}} \leq \frac{3\Lambda_i^2}{2\pi^2} \left(1 - \frac{3}{(3+x_i)^2} + x_i O^+(1)\right)$$

when  $\delta$  is small. The above estimate is independent of the choice of  $C$ .

Now, we estimate  $P_{i\bar{j}}$  with  $i, j \leq m$  and  $i \neq j$ . We have

$$(2.15) \quad \begin{aligned} P_{i\bar{j}} &= \tilde{\tau}^{k\bar{l}} P_{i\bar{j}k\bar{l}} = \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + C \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \\ &= \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + C \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} + C \sum_{k, l \leq m, k \neq l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \\ &\quad + C \sum_{k \leq m < l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} + C \sum_{l \leq m < k} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} + C \sum_{k, l \geq m+1} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}}. \end{aligned}$$

By the above lemma, we have

$$(2.16) \quad \left| \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \right| = \Lambda_i \Lambda_j O(u_0).$$

We also have

$$\left| C \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} \right| \leq \left| C \sum_{k \leq m, k \neq i, j} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} \right| + \left| C \tilde{\tau}^{i\bar{i}} R_{i\bar{j}i\bar{i}} \right| + \left| C \tilde{\tau}^{j\bar{j}} R_{i\bar{j}j\bar{j}} \right|.$$

By the proof of Lemma 4.2 of [11], we have  $R_{i\bar{j}i\bar{i}} = O\left(\frac{u_i^5 u_j^3}{|t_i^3 t_j|}\right)$  which implies

$$\left| C \tilde{\tau}^{i\bar{i}} R_{i\bar{j}i\bar{i}} \right| = 4\pi^2 \Lambda_i^{-2} (3+x_i)^{-1} C O\left(\frac{u_i^5 u_j^3}{|t_i^3 t_j|}\right) = \Lambda_i \Lambda_j O(u_0).$$

Similarly, we have

$$\left| C \tilde{\tau}^{j\bar{j}} R_{i\bar{j}j\bar{j}} \right| = \Lambda_i \Lambda_j O(u_0).$$

Again, by the proof of Lemma 4.2 of [11], for  $k \leq m$  and  $k \neq i, j$ , we have

$$R_{i\bar{j}k\bar{k}} = O\left(\frac{u_i u_j u_k^3}{|t_i t_j t_k^2|} u_0\right)$$

which implies

$$\left| C \sum_{k \leq m, k \neq i, j} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} \right| = \Lambda_i \Lambda_j O(u_0).$$

By combining the above three formulas, we have

$$\left| C \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} \right| = \Lambda_i \Lambda_j O(u_0).$$

Similarly, we can show that

$$\left| C \sum_{k, l \leq m, k \neq l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| = \Lambda_i \Lambda_j O(u_0),$$

$$\left| C \sum_{k \leq m < l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| = \Lambda_i \Lambda_j O(u_0)$$

and

$$\left| C \sum_{l \leq m < k} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| = \Lambda_i \Lambda_j O(u_0).$$

Finally,

$$\begin{aligned} \left| C \sum_{k, l \geq m+1} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| &\leq \sum_{k, l \geq m+1} \left| C \tilde{\tau}^{k\bar{l}} \right| \left| R_{i\bar{j}k\bar{l}} \right| \\ &= \sum_{k, l \geq m+1} O(1) \left| R_{i\bar{j}k\bar{l}} \right| = \Lambda_i \Lambda_j O(u_0). \end{aligned}$$

By combining the above results, we have

$$(2.17) \quad P_{i\bar{j}} = \Lambda_i \Lambda_j O(u_0).$$

By using the same method, we know that, if  $i \leq m < j$ , then

$$(2.18) \quad P_{i\bar{j}} = \Lambda_i O(u_0)$$

and if  $j \leq m < i$ , then

$$(2.19) \quad P_{i\bar{j}} = \Lambda_j O(u_0).$$

The next step is to estimate the matrix  $(P_{i\bar{j}})_{i, j \geq m+1}$ . We will show that this matrix is bounded from above and below by positive constant multiples of the matrix  $B_1$  defined in the above lemma where the constants depend on  $\delta$  and  $C$ . We first estimate  $P_{i\bar{j}}$  with fixed  $i, j \geq m+1$ .



We have

$$\begin{aligned}
P_{i\bar{j}} &= \tilde{\tau}^{k\bar{l}} P_{i\bar{j}k\bar{l}} = \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + C \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \\
&= \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} \widehat{R}_{i\bar{j}k\bar{k}} + \sum_{k, l \leq m, k \neq l} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \\
&\quad + \sum_{k \leq m < l} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + \sum_{l \leq m < k} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \\
&\quad + \sum_{k, l \geq m+1} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + C \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} + C \sum_{k, l \leq m, k \neq l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \\
&\quad + C \sum_{k \leq m < l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} + C \sum_{l \leq m < k} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} + C \sum_{k, l \geq m+1} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}}.
\end{aligned}$$

By Lemma 2.1 and the proof of Corollary 4.2 of [11], we know that

$$\begin{aligned}
\left| \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} \widehat{R}_{i\bar{j}k\bar{k}} \right| &= O(u_0), & \left| \sum_{k, l \leq m, k \neq l} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \right| &= O(u_0), \\
\left| \sum_{k \leq m < l} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \right| &= O(u_0), & \left| \sum_{l \leq m < k} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \right| &= O(u_0), \\
\left| \sum_{k, l \geq m+1} \tilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \right| &= O(C^{-1}), & \left| C \sum_{k, l \leq m, k \neq l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| &= O(u_0),
\end{aligned}$$

and

$$\left| C \sum_{k \leq m < l} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| = O(u_0), \quad \left| C \sum_{l \leq m < k} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \right| = O(u_0).$$

Also, since for  $i, j, k, l \geq m+1$ ,  $R_{i\bar{j}k\bar{l}} = O(1)$ , we have

$$C \sum_{k, l \geq m+1} \tilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} = \sum_{k, l \geq m+1} h^{k\bar{l}} R_{i\bar{j}k\bar{l}} + O(C^{-1}) + O(u_0).$$

By combining the above arguments, we have

$$(2.20) \quad P_{i\bar{j}} = C \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} + \sum_{k, l \geq m+1} h^{k\bar{l}} R_{i\bar{j}k\bar{l}} + O(C^{-1}) + O(u_0).$$

The matrix  $(\sum_{k, l \geq m+1} h^{k\bar{l}} R_{i\bar{j}k\bar{l}}(0, s))_{i, j \geq m+1}$  is just the negative of the Ricci curvature matrix of the restriction of the Weil–Petersson metric to the boundary piece. So, we know it is positive definite and is bounded from below by a constant multiple of  $B_2(0, s)$ . By continuity, we know

that, when  $\delta$  is small enough, the matrix  $(\sum_{k,l \geq m+1} h^{k\bar{l}} R_{i\bar{j}k\bar{l}}(t, s))_{i,j \geq m+1}$  is bounded from below by a constant multiple of  $B_2(t, s)$ . Again, since  $h^{k\bar{l}} R_{i\bar{j}k\bar{l}} = O(1)$  when  $i, j, k, l \geq m+1$  and the fact that matrices  $B_1$  and  $B_2$  are locally equivalent, we know that  $(\sum_{k,l \geq m+1} h^{k\bar{l}} R_{i\bar{j}k\bar{l}})_{i,j \geq m+1}$  is locally bounded from above and below by positive constants multiples of  $B_1$ .

Finally, by using the fact that the bisectional curvature of the Weil–Petersson metric is non-positive and  $\tilde{\tau}^{k\bar{k}} > 0$ , we know that the matrix  $(\tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}})_{i,j \geq m+1}$  is positive semi-definite. Also, we know that  $C \sum_{k \leq m} \tilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} = O(1)$ .

Now, by using formula (2.20), we know that there are positive constants  $\beta_1 \leq \beta_2$  depending on  $\delta$ , the point  $p$  and the choice of  $C$  such that as long as  $\delta$  is small enough and  $C$  is large enough,

$$(2.21) \quad \beta_1 B_1 \leq \left( P_{i\bar{j}} \right)_{i,j \geq m+1} \leq \beta_2 B_1.$$

We know that there is a constant  $c_0 > 0$  such that  $h \leq c_0 \tau$  which implies  $\tau \leq \tilde{\tau} \leq (1 + c_0 C) \tau$ . By combining formulas (2.14), (2.17), (2.18), (2.19) and (2.21), we know that, when  $\delta$  is small enough and  $C$  is large enough, there are positive constants  $\alpha_1 \leq \alpha_2$  depending on  $p, \delta$  and  $C$  such that

$$\alpha_1 \tilde{\tau} \leq \left( P_{i\bar{j}} \right) \leq \alpha_2 \tilde{\tau}.$$

Now, by using the compactness argument as we did before, we can find an open neighborhood  $U$  of  $D$  in  $\overline{\mathcal{M}}_g$  and a  $C_0 > 0$  such that

$$\alpha_1 \tilde{\tau} \leq \left( P_{i\bar{j}} \right) \leq \alpha_2 \tilde{\tau}$$

on  $U$  for positive constants  $\alpha_1$  and  $\alpha_2$  as long as  $C \geq C_0$ .

Let  $V = \overline{\mathcal{M}}_g \setminus U$ . We know  $V$  is compact. We also know that, for  $C$  large enough,

$$\text{Ric}(\tilde{\tau}) = \text{Ric}(C^{-1}\tilde{\tau}) = \text{Ric}(h + C^{-1}\tau).$$

Since the Ricci curvature of the Weil–Petersson metric has a negative upper bound, a perturbation of the Weil–Petersson metric with a small error term still has negative Ricci curvature on a compact set  $V$ . Also, on  $V$ , the perturbed Ricci metric is bounded. So, for  $C$  large, we know that the Ricci curvature of the perturbed Ricci metric is pinched between negative constant multiples of the perturbed Ricci metric. Here, the bounds depend on the choice of  $C$ . This finished the proof.  $\square$

As a direct corollary of the above theorem, we show a vanishing theorem similar to the work of Faltings [4].

**Corollary 2.1.** *Let  $D$  be the compactification divisor of the Deligne–Mumford compactification of  $\mathcal{M}_g$ . Then,*

$$H^0(\overline{\mathcal{M}}_g, \Omega(\log D)^*) = 0.$$

*Proof.* We first pick a constant  $C > 0$  such that the Ricci curvature of the perturbed Ricci metric  $\tilde{\tau} = \tau + Ch$  is pinched by negative constants. Let  $\sigma$  be a holomorphic section of  $\Omega(\log D)^*$ .

Let  $p \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$  be a codimension  $m$  point and let  $(t_1, \dots, t_m, s_{m+1}, \dots, s_n)$  be local pinching coordinates. Then, locally, we have

$$\sigma = \sum_{i=1}^m a_i(t, s) t_i \frac{\partial}{\partial t_i} + \sum_{j=m+1}^n a_j(t, s) \frac{\partial}{\partial s_j}$$

where  $a_i$  are bounded local holomorphic functions for  $1 \leq i \leq n$ . It is clear that, restricted to  $\mathcal{M}_g$ ,  $\sigma$  is a holomorphic vector field. Now, we equip the moduli space  $\mathcal{M}_g$  with the perturbed Ricci metric  $\tilde{\tau}$ . From the above expression of  $\sigma$ , it is easy to see that

$$\|\sigma\|_{\tilde{\tau}} \in L^2(\mathcal{M}_g, \tilde{\tau})$$

since  $\tilde{\tau}$  is equivalent to the asymptotic Poincaré metric. Now, we have the Bochner formula

$$\Delta_{\tilde{\tau}} \|\sigma\|_{\tilde{\tau}}^2 = \|\nabla \sigma\|_{\tilde{\tau}}^2 - \text{Ric}_{\tilde{\tau}}(\sigma, \sigma).$$

To integrate, we need a special cut-off function. In [13], a monotone sequence of cut-off functions  $\rho_\epsilon$  with the properties that  $\Delta_{\tilde{\tau}} \rho_\epsilon$  is uniformly bounded for each  $\epsilon$  and the measure of the support of  $\Delta_{\tilde{\tau}} \rho_\epsilon$  goes to 0 as  $\epsilon$  goes to zero. We will recall the construction in the next section.

By using the cut-off function  $\rho_\epsilon$ , we have

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \Delta_{\tilde{\tau}} \|\sigma\|_{\tilde{\tau}}^2 dV_{\tilde{\tau}} = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \Delta_{\tilde{\tau}} \rho_\epsilon \|\sigma\|_{\tilde{\tau}}^2 dV_{\tilde{\tau}} = 0$$

since  $\sigma$  is an  $L^2$  section with respect to  $\tilde{\tau}$  and the measure of  $\Delta_{\tilde{\tau}} \rho_\epsilon$  goes to 0. This above formula implies  $\text{Ric}(\sigma, \sigma) = 0$ . Since  $\text{Ric}(\tilde{\tau})$  is negative, we know that  $\sigma = 0$ . Thus, we have proved the corollary. q.e.d.

Finally, we show that the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric has bounded geometry.

**Corollary 2.2.** *The injectivity radius of the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric is bounded from below.*

*Proof.* We only prove that there is a lower bound for the injectivity radius of the Ricci metric since the case of the perturbed Ricci metric can be done in the same way.

In Theorem 2.2, we showed that the curvature of the Ricci metric is bounded. We denote the sup of the curvature by  $\delta$ . If  $\delta \leq 0$ , then the injectivity radius is  $+\infty$  by the Cartan–Hadamard theorem. Now, we assume  $\delta > 0$ .

Assume the injectivity radius of  $(\mathcal{T}_g, \tau)$  is 0, then for any  $\epsilon > 0$ , there is a point  $p = p_\epsilon$  such that the injectivity radius at  $p$  is less than  $\epsilon$ .

Let  $f_p$  be the Bers' embedding map such that  $f_p(p) = 0$ . By using a similar argument as in the proof of Theorem 7.1 in [11], and by changing some constants, we know that the Ricci metric and the Euclidean metric are equivalent on the Euclidean ball  $B_1 \subset f_p(\mathcal{T}_g)$ . By using the Rauch comparison theorem to compare the Ricci metric on the ball  $B_1$  and the standard sphere of constant curvature  $\delta$ , we know that there is no conjugate point of  $p$  within distance  $\epsilon$  when  $\epsilon$  is small enough.

So, the only case we need to rule out is that there is a closed geodesic loop  $\gamma$  containing  $p$  such that  $l_\tau(\gamma) \leq 2\epsilon$ . We know that when  $\epsilon$  small enough,  $\gamma \subset B_1$  since the Ricci metric and the Euclidean metric are equivalent on  $B_1$ . This implies that the Euclidean length of  $\gamma$ , denoted by  $\tilde{l}(\gamma)$ , satisfies  $\tilde{l}(\gamma) \leq c\epsilon$  for some constant  $c$  only depending on the comparison constants of the Ricci metric and the Euclidean metric on  $B_1$ . It is clear that  $\gamma$  bounds a minimal disk  $\tilde{\Sigma}$  with respect to the Euclidean metric. By the isoperimetric inequality, we know that the Euclidean area  $A_E(\tilde{\Sigma})$  satisfies

$$A_E(\tilde{\Sigma}) \leq c_1 \tilde{l}(\gamma)^2 \leq c_1 c^2 \epsilon^2.$$

By using the equivalence of the metrics, we know the area  $A_\tau(\tilde{\Sigma})$  of the surface  $\tilde{\Sigma}$  under the Ricci metric is small if  $\epsilon$  is small enough. Thus,  $\gamma$  bounds a minimal disk  $\Sigma$  with respect to the Ricci metric.

Now, we show that the minimal disk  $\Sigma \subset B_{\frac{1}{2}}$  as long as  $\epsilon$  is small enough. Without loss of generality, we assume that the injectivity radius of the Ricci metric at 0 is  $\epsilon > 0$ . By the work of Cheng, Li and Yau [2], we know that there is a positive constant  $a$  which depends on the dimension and the curvature bound of the Ricci metric such that, when  $q \in B_{\frac{1}{2}}$ , then the injectivity radius of the Ricci metric at  $q$  is at least  $a\epsilon$ . We may assume  $a < 1$ .

If there is a point  $s \in \Sigma \setminus B_{\frac{1}{2}}$ , then we can pick a curve  $\rho \in \Sigma$  joining 0 and  $s$ . We can assume that  $\rho \subset B_1$ . Since the Ricci metric is equivalent to the Euclidean metric inside  $B_1$ , we know that there is a constant  $a_1 > 0$  such that  $d_\tau(0, s) \geq a_1$ . Thus, we can pick  $k$  points  $s_1, \dots, s_k$  in  $\rho \cap B_{\frac{1}{2}-a\epsilon}$  such that  $d_\tau(s_i, s_j) \geq 2a\epsilon$ . From the above argument, we know that  $k \sim \frac{a_2}{\epsilon}$  for some  $a_2 > 0$ . Now, we let  $B_{s_i}(a\epsilon)$  be the geodesic ball for the Ricci metric whose center is  $s_i$  and radius is  $a\epsilon$ . By our

construction, we know that  $B_{s_i}(a\epsilon) \cap B_{s_j}(a\epsilon) = \emptyset$ . Now, by using the isoperimetric inequality, we have

$$A_\tau(B_{s_i}(a\epsilon) \cap \Sigma) \geq a_3 a^2 \epsilon^2$$

where  $a_3$  only depends on  $n$  and the curvature bound of the Ricci metric. Thus, we have

$$A_\tau(\Sigma \cap B_{\frac{1}{2}}) \geq \sum_{i=1}^k A_\tau(B_{s_i}(a\epsilon) \cap \Sigma) \geq k a_3 a^2 \epsilon^2 \sim a_4 \epsilon.$$

However, we have

$$A_\tau(\Sigma \cap B_{\frac{1}{2}}) \leq A_\tau(\Sigma) \leq A_\tau(\tilde{\Sigma}) \leq a_5 A_E(\tilde{\Sigma}) \sim a_6 \epsilon^2.$$

This is a contradiction as long as  $\epsilon$  is small enough. Thus,  $\Sigma \subset B_{\frac{1}{2}}$ .

Now, by the Gauss–Codazzi equation, we know that the curvature  $R_\Sigma$  of the metric on  $\Sigma$  induced from the Ricci metric is bounded above by  $\delta$ . By using the isoperimetric inequality, we know that

$$A_\tau(\Sigma) \leq c_2 \epsilon^2.$$

However, by the Gauss–Bonnet theorem, since the geodesic  $\gamma$  has at most one vertex  $p$  and the outer angle  $\theta$  at  $p$  is at most  $\pi$ , we have

$$\int_\Sigma R_\Sigma dv_\tau + \int_\gamma \kappa_\gamma ds + \theta = 2\pi\chi(\Sigma) = 2\pi$$

where  $dv_\tau$  is the induced area form from the Ricci metric. Since  $\gamma$  is a geodesic, we see that the second term in the left-hand side of the above formula is 0. Since  $R_\Sigma \leq \delta$  and  $\theta < \pi$ , we have

$$\delta A_\tau(\Sigma) \geq \int_\Sigma R_\Sigma dv_\tau = 2\pi - \theta > \pi$$

which implies  $A_\tau(\Sigma) \geq \frac{\pi}{\delta}$ . Contribution of the curvature created by branch points will make the inequality better. Thus, by comparing the above two inequalities, we get a contradiction as long as  $\epsilon$  is small enough. This finishes the proof. q.e.d.

The technique in the proof of the above corollary is of interest by itself. We summarize two more corollaries to the above argument in the following two propositions which may be used for other problems related to injectivity radius.

**Proposition 2.1.** *Let  $(X^n, g)$  be a Riemannian manifold and let  $U \subset X$  be an open set. Assume that there is a diffeomorphism  $f : U \rightarrow B_1 \subset \mathbb{R}^n$  where  $B_1$  is the open unit ball. Let  $g_E$  be the metric on  $U$  which is the pullback via  $f$  of the Euclidean metric on  $\mathbb{R}^n$ . Assume that the curvature of  $g$  is uniformly bounded on  $U$ . If  $g$  is equivalent*

to  $g_E$  on  $U$ , then the injectivity radius of the metric  $g$  at each point in  $f^{-1}(B_{\frac{1}{2}}) \subset U$  has a lower bound which only depends on the dimension  $n$ , curvature bound of  $g$  and the equivalence bound of  $g$  and  $g_E$ .

**Proposition 2.2.** *Let  $(X^n, g)$  be a Riemannian manifold and let  $p \in X$  be a point. Let  $B_r$  be the geodesic ball in  $X$  with center  $p$  and radius  $r$ . We assume that the curvature of  $g$  is uniformly bounded on  $B_1$ . Then, there is a  $\epsilon_0 > 0$  which only depends on  $n$  and the curvature bound of  $g$  on  $B_1$  such that if any simple closed curve  $\gamma$  in  $B_1$  with length  $l(\gamma) = \epsilon \leq \epsilon_0$  bounds a disk  $\Sigma$  such that  $A_g(\Sigma) \leq C\epsilon^{1+\alpha}$  where  $C, \alpha > 0$  are fixed constants, then the injectivity radius of  $g$  at each point  $q \in B_{\frac{1}{2}}$  has a uniform lower bound which only depends on the  $n, C, \alpha$  and the curvature bound of  $g$  on  $B_1$ .*

### 3. The Stability of the Logarithmic Cotangent Bundle

In this section, we investigate the cohomology classes defined by the currents  $\omega_\tau$  and  $\omega_{KE}$ . Since both of these Kähler forms have Poincaré growth, it is natural to identify them with the first Chern class of the logarithmic cotangent bundle of  $\overline{\mathcal{M}}_g$ . This implies this bundle is positive over the compactified moduli space which directly implies that the moduli space is of log general type.

The next step is to show that the restriction of the Kähler–Einstein metric to a subbundle of the logarithmic cotangent bundle will not have growth worse than Poincaré growth. Then, we prove that the logarithmic cotangent bundle  $\overline{E}$  over  $\overline{\mathcal{M}}_g$  is stable with respect to the first Chern class of this bundle.

More precisely, we have the following theorem:

**Theorem 3.1.** *The first Chern class of  $\overline{E}$  is positive and  $\overline{E}$  is Mumford stable with respect to  $c_1(\overline{E})$ .*

**Remark 3.1.** We recall that both  $\mathcal{M}_g$  and  $\overline{\mathcal{M}}_g$  are orbifolds and the bundle  $\overline{E}$  over  $\overline{\mathcal{M}}_g$  is an orbibundle. To define the stability, we need to define the degree of a bundle which only involves the first Chern class of this bundle. We know that the singular locus (orbifold points) on  $\mathcal{M}_g$  and  $\overline{\mathcal{M}}_g$  are at least codimension two and  $\mathcal{M}_g$  has a finite cover which is smooth. Thus, when we deal with the first Chern class of a bundle  $F$  by representing it by the Ricci form of a metric on  $F$  locally near a point  $p$ , we can work on the local manifold cover (finite cover) of a neighborhood of  $p$ . In this case,  $F$  becomes a vector bundle locally, rather than an orbibundle. We then take the quotient to go back. In the following, when we work locally, we will always work on a local manifold cover without further mentioning.

We first setup our notation. On  $\mathcal{M}_g$ , let  $g_p, \tau, \tilde{\tau}, g_{WP}$  and  $g_{KE}$  be the asymptotic Poincaré metric, the Ricci metric, the perturbed Ricci metric, the Weil–Petersson metric and the Kähler–Einstein metric respectively. Let  $\omega_p, \omega_\tau$  and  $\omega_{KE}$  be the corresponding Kähler forms of these metrics. Let  $\text{Ric}(\omega_\tau)$  be the Ricci form of the Ricci metric.

Let  $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$  be the compactification divisor. In order to prove the stability, we need to control the growth of these Kähler forms near  $D$ . We fix a cover of  $\mathcal{M}_g$  by local charts.

For each point  $y \in D$ , we can pick local pinching coordinate charts  $U_y \subset \tilde{U}_y$  centered at  $y$  with  $U_y = (\Delta_{\delta_y}^*)^{m_y} \times \Delta_{\delta_y}^{n-m_y}$  and  $\tilde{U}_y = (\Delta_{2\delta_y}^*)^{m_y} \times \Delta_{2\delta_y}^{n-m_y}$  such that the estimates in Corollaries 4.1, 4.2, Theorems 4.4 and 5.2 of [11] hold on  $\tilde{U}_y$ . Here,  $\Delta_{\delta_y}$  is the disk of radius  $\delta_y > 0$  and  $\Delta_{\delta_y}^*$  is the punctured disk of radius  $\delta_y$  and  $m_y$  is the codimension of the point  $y$  and  $n = 3g - 3$  is the complex dimension of  $\mathcal{M}_g$ .

Since  $D$  is compact, we can find  $p$  such charts  $U_1 = U_{y_1}, \dots, U_p = U_{y_p}$  such that there is a neighborhood  $V_0$  of  $D$  with  $D \subset V_0 \subset \overline{V_0} \subset \cup_{i=1}^p U_i$ . Now, we choose coordinate charts  $V_1, \dots, V_q$  such that the estimate of Theorem 5.2 of [11] holds and

- 1)  $\mathcal{M}_g \subset \left(\cup_{j=1}^p U_j\right) \cup \left(\cup_{j=1}^q V_j\right)$ ;
- 2)  $\left(\cup_{j=1}^q V_j\right) \cap \overline{V_0} = \emptyset$ .

Let  $\psi_1, \dots, \psi_{p+q}$  be a partition of unity subordinate to the cover  $U_1, \dots, U_p, V_1, \dots, V_q$  such that  $\text{supp}(\psi_i) \subset U_i$  for  $1 \leq i \leq p$  and  $\text{supp}(\psi_i) \subset V_{i-p}$  for  $p+1 \leq i \leq p+q$ .

Let  $\alpha_i = m_{y_i}$  and let  $t_1^i, \dots, t_{\alpha_i}^i, s_{\alpha_i+1}^i, \dots, s_n^i$  be the pinching coordinates on  $U_i$  where  $t_1^i, \dots, t_{\alpha_i}^i$  represent the degeneration directions.

To prove the theorem, we need a special cut-off function. Such a function was used in [13]. We include a short proof here since we need to use the construction later.

**Lemma 3.1.** *For any small  $\epsilon > 0$ , there is a smooth function  $\rho_\epsilon$  such that*

- 1)  $0 \leq \rho_\epsilon \leq 1$ ;
- 2) *For any open neighborhood  $V$  of  $D$  in  $\overline{\mathcal{M}}_g$ , there is a  $\epsilon > 0$  such that  $\text{supp}(1 - \rho_\epsilon) \subset V$ ;*
- 3) *For each  $\epsilon > 0$ , there is a neighborhood  $W$  of  $D$  such that  $\rho_\epsilon|_W \equiv 0$ ;*
- 4)  $\rho_{\epsilon'} \geq \rho_\epsilon$  if  $\epsilon' \leq \epsilon$ ;
- 5) *There is a constant  $C$  which is independent of  $\epsilon$  such that*

$$-C\omega_p \leq \sqrt{-1}\partial\bar{\partial}\rho_\epsilon \leq C\omega_p.$$

*Proof.* We fix a smooth decreasing function  $\varphi \in C^\infty(\mathbb{R})$  with  $0 \leq \varphi \leq 1$  such that

$$\varphi(x) = \begin{cases} 0 & x \geq 1; \\ 1 & x \leq 0. \end{cases}$$

Now, let

$$\varphi_\epsilon(z) = \varphi \left( \frac{\left(\log \frac{1}{|z|}\right)^{-1} - \epsilon}{\epsilon} \right).$$

For  $1 \leq i \leq p$  and  $\epsilon > 0$  small, we let

$$\varphi_\epsilon^i(t_1^i, \dots, s_n^i) = \prod_{j=1}^{\alpha_i} (1 - \varphi_\epsilon(t_j^i)).$$

The cut-off function is defined by

$$\rho_\epsilon = 1 - \sum_{i=1}^p \psi_i \varphi_\epsilon^i.$$

It is easy to check that  $\rho_\epsilon$  satisfy all the conditions. q.e.d.

Now, we discuss the logarithmic cotangent bundle. Let  $U_1, \dots, V_q$  be the cover of  $\mathcal{M}_g$  as above. For each  $1 \leq i \leq p$ , let  $W_i = \Delta_{\delta_{y_i}}^{m_{y_i}} \times \Delta_{\delta_{y_i}}^{n-m_{y_i}}$ . Then,  $W_1, \dots, W_p, V_1, \dots, V_q$  is a cover of  $\overline{\mathcal{M}}_g$ . On each  $U_i$ , a local holomorphic frame of the holomorphic cotangent bundle  $T^*\mathcal{M}_g$  is  $dt_1^i, \dots, dt_{\alpha_i}^i, ds_{\alpha_i+1}^i, \dots, ds_n^i$ . Let

$$(3.1) \quad e_j^i = \begin{cases} \frac{dt_j^i}{t_j^i} & j \leq \alpha_i; \\ ds_j^i & j \geq \alpha_i + 1. \end{cases}$$

The logarithmic cotangent bundle  $\overline{E}$  is the extension of  $T^*\mathcal{M}_g$  to  $\overline{\mathcal{M}}_g$  such that on each  $U_i$ ,  $e_1^i, \dots, e_n^i$  is a local holomorphic frame of  $\overline{E}$ . It is very easy to check this fact by writing down the transition maps. In the following, we will use  $g_{WP}^*$ ,  $\tau^*$  and  $g_{KE}^*$  to represent the metrics on  $\overline{E}$  induced by the Weil–Petersson metric, the Ricci metric and the Kähler–Einstein metric respectively.

To discuss the stability of  $\overline{E}$ , we need to fix a Kähler class on  $\overline{\mathcal{M}}_g$ . It is natural to use the first Chern class of  $\overline{E}$ . We denote this class by  $\Phi$ . We first identify the current represented by the Kähler form  $\omega_{KE}$  with  $\Phi$ .



**Lemma 3.2.** *The currents  $\omega_\tau$  and  $\omega_{KE}$  are positive closed currents. Furthermore,*

$$[\omega_\tau] = [\omega_{KE}] = c_1(\overline{E}).$$

*Proof.* It is clear that  $\omega_\tau$  and  $\omega_{KE}$  are positive currents. Let  $\varphi$  be an arbitrary smooth  $(2n-3)$ -form on  $\overline{\mathcal{M}}_g$ . To show that  $\omega_{KE}$  is closed, we only need to show

$$(3.2) \quad \int_{\overline{\mathcal{M}}_g} \omega_{KE} \wedge d\varphi = 0.$$

We first check

$$(3.3) \quad \int_{\overline{\mathcal{M}}_g} |\omega_{KE} \wedge d\varphi| = \int_{\mathcal{M}_g} |\omega_{KE} \wedge d\varphi| < \infty.$$

To simplify the notations, on each  $U_i$ , we let  $t_j^i = s_j^i$  for  $\alpha_i + 1 \leq j \leq n$ . On each  $U_i$ , we assume

$$d\varphi = \sum_{\alpha, \beta} a_{\alpha\beta}^i dt_1^i \wedge \overline{dt}_1^i \cdots \wedge \widehat{dt}_\alpha^i \wedge \overline{dt}_\alpha^i \cdots \wedge dt_\beta^i \wedge \widehat{\overline{dt}}_\beta^i \cdots \wedge dt_n^i \wedge \overline{dt}_n^i$$

where  $a_{\alpha\beta}^i$  are bounded smooth functions on  $U_i$ . We denote  $dt_1^i \wedge \overline{dt}_1^i \cdots \wedge dt_n^i \wedge \overline{dt}_n^i$  by  $dt^i \wedge \overline{dt}^i$ . By [11], we know that the Kähler–Einstein metric is equivalent to the Ricci metric and the asymptotic Poincaré metric. By using Corollary 4.2 of [11], we know that, restricted to each  $U_i$ , there is a constant  $C$  depending on  $\varphi$  such that

$$\begin{aligned} |\omega_{KE} \wedge d\varphi| &\leq \left(\frac{\sqrt{-1}}{2}\right)^n \sum_{\alpha, \beta} |(g_{KE})_{\alpha\beta} a_{\alpha\beta}^i| dt^i \wedge \overline{dt}^i \\ &\leq \left(\frac{\sqrt{-1}}{2}\right)^n C \left( \sum_{j=1}^{\alpha_i} \frac{1}{|t_j^i|^2 (\log |t_j^i|)^2} + 1 \right) dt^i \wedge \overline{dt}^i \end{aligned}$$

since  $a_{\alpha\beta}^i$  is bounded on  $U_i$ . This implies

$$\begin{aligned} &\int_{U_i} \psi_i |\omega_{KE} \wedge d\varphi| \\ &\leq \int_{U_i} \psi_i \left(\frac{\sqrt{-1}}{2}\right)^n C \left( \sum_{j=1}^{\alpha_i} \frac{1}{|t_j^i|^2 (\log |t_j^i|)^2} + 1 \right) dt^i \wedge \overline{dt}^i < \infty. \end{aligned}$$

So, we have

$$\int_{\mathcal{M}_g} |\omega_{KE} \wedge d\varphi| \leq \sum_{j=1}^q \int_{V_j} \psi_{j+p} |\omega_{KE} \wedge d\varphi| + \sum_{j=1}^p \int_{U_j} \psi_j |\omega_{KE} \wedge d\varphi| < \infty.$$

Let  $\rho_\epsilon$  be the cut-off function constructed above. By the dominating convergence theorem, we have

$$(3.4) \quad \int_{\mathcal{M}_g} \omega_{KE} \wedge d\varphi = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \omega_{KE} \wedge d\varphi.$$

Let  $h$  be an Hermitian metric on  $\overline{E}$ . Let  $\text{Ric}(h) = -\partial\bar{\partial} \log \det(h)$  be its Ricci form. Clearly,

$$[\text{Ric}(h)] = c_1(\overline{E})$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \text{Ric}(h) \wedge d\varphi &= \int_{\mathcal{M}_g} \text{Ric}(h) \wedge d\varphi \\ &= \int_{\overline{\mathcal{M}}_g} \text{Ric}(h) \wedge d\varphi = - \int_{\overline{\mathcal{M}}_g} d(\text{Ric}(h)) \wedge \varphi = 0. \end{aligned}$$

Since  $\omega_{KE} = -\partial\bar{\partial} \log \det(g_{KE}^*)$ , we have

$$\begin{aligned} (3.5) \quad \int_{\mathcal{M}_g} \omega_{KE} \wedge d\varphi &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \omega_{KE} \wedge d\varphi \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \omega_{KE} \wedge d\varphi - \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \text{Ric}(h) \wedge d\varphi \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \partial\bar{\partial} \log \left( \frac{\det(h)}{\det(g_{KE}^*)} \right) \wedge d\varphi \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \partial\bar{\partial} \rho_\epsilon \wedge d\varphi. \end{aligned}$$

By using the frame in (3.1), by Theorem 1.4 and Corollary 4.2 of [11], we know that there are positive constants  $C_1$  and  $C_2$  which may depend on  $\varphi$  such that, on each  $U_i$ ,

$$C_1 \leq \frac{\det(g_{KE}^*)}{\det(h)} \leq C_2 \prod_{j=1}^{\alpha_i} (\log |t_j^i|)^2$$

which implies that there is a constant  $C_3$  such that

$$(3.6) \quad \left| \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \right| \leq C_3 + 2 \sum_{j=1}^{\alpha_i} \log \log \frac{1}{|t_j^i|}.$$

Now, by Lemma 3.1, we can pick  $\epsilon_0$  small enough such that for any  $0 < \epsilon < \epsilon_0$ ,  $\text{supp}(1 - \rho_\epsilon) \subset V_0$ . We also know that  $\text{supp}(\partial\bar{\partial}\rho_\epsilon) \subset \text{supp}(1 - \rho_\epsilon)$ . Again, by Lemma 3.1, since

$$-C\omega_p \leq \partial\bar{\partial}\rho_\epsilon \leq C\omega_p,$$

we know that there is a constant  $C_4$  which depends on  $\varphi$  such that

$$(3.7) \quad |\partial\bar{\partial}\rho_\epsilon \wedge d\varphi| \leq \left(\frac{\sqrt{-1}}{2}\right)^n C_4 \left(\sum_{j=1}^{\alpha_i} \frac{1}{|t_j^i|^2 (\log|t_j^i|)^2} + 1\right) dt^i \wedge d\bar{t}^i.$$

By combining (3.6) and (3.7), and a simple computation, we can show that

$$(3.8) \quad \int_{U_i} \left| \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\bar{\partial}\rho_\epsilon \wedge d\varphi \right| < \infty.$$

From the above argument, we know that

$$\begin{aligned} & \left| \int_{\mathcal{M}_g} \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\bar{\partial}\rho_\epsilon \wedge d\varphi \right| \\ &= \left| \int_{\mathcal{M}_g \cap \text{supp}(\partial\bar{\partial}\rho_\epsilon)} \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\bar{\partial}\rho_\epsilon \wedge d\varphi \right| \\ &\leq \sum_{j=1}^p \int_{U_j \cap \text{supp}(1-\rho_\epsilon)} \psi_j \left| \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\bar{\partial}\rho_\epsilon \wedge d\varphi \right| \rightarrow 0 \end{aligned}$$

as  $\epsilon$  goes to 0 because of (3.8) and the fact that the Lebesgue measure of  $U_i \cap \text{supp}(1 - \rho_\epsilon)$  goes to 0. By combining with (3.5), we know that

$$\int_{\mathcal{M}_g} \omega_{KE} \wedge d\varphi = 0$$

which implies  $\omega_{KE}$  is a closed current. Similarly, we can prove that  $\omega_\tau$  is a closed current by the formula

$$\omega_\tau = -\partial\bar{\partial} \log \det(g_{WP}^*)$$

and Corollary 4.1 and 4.2 of [11].

Now, we prove the second statement of the lemma. Since  $\omega_{KE}$  is a closed current, to show it represents the first Chern class of  $\bar{E}$ , we need to prove that for any closed  $(n - 1, n - 1)$ -form  $\tilde{\varphi}$  on  $\bar{\mathcal{M}}_g$ ,

$$(3.9) \quad \int_{\mathcal{M}_g} \omega_{KE} \wedge \tilde{\varphi} = \int_{\mathcal{M}_g} c_1(\bar{E}) \wedge \tilde{\varphi}.$$

However, this can be easily proved by using the above argument where we replace  $d\varphi$  by  $\tilde{\varphi}$ . The same argument works for  $\omega_\tau$ . This finishes the proof. q.e.d.

Now, we compute the degree of  $\overline{E}$ . In the following, by degree of a bundle over  $\overline{\mathcal{M}}_g$ , we always mean the  $\Phi$ -degree.

**Lemma 3.3.** *The degree of  $\overline{E}$  is given by  $\int_{\mathcal{M}_g} \omega_{KE}^n$ .*

*Proof.* Since the degree of  $\overline{E}$  is given by

$$\deg(\overline{E}) = \int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega_{KE}^{n-1},$$

we need to show that

$$(3.10) \quad \int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega_{KE}^{n-1} = \int_{\mathcal{M}_g} \omega_{KE}^n.$$

By the property of the asymptotic Poincaré metric, we know that

$$(3.11) \quad \int_{\mathcal{M}_g} \omega_p^n < \infty.$$

Since the Kähler–Einstein metric is equivalent to the asymptotic Poincaré metric, we know that  $\int_{\mathcal{M}_g} \omega_{KE}^n < \infty$ . Also, since  $c_1(\overline{E})$  is a closed  $(1, 1)$ -form on  $\overline{\mathcal{M}}_g$  which is compact, we know that any representative of  $c_1(\overline{E})$  is bounded on  $\overline{\mathcal{M}}_g$ . This implies there is a constant  $C_5$  such that  $-C_5\omega_p \leq c_1(\overline{E}) \leq C_5\omega_p$ . This implies that  $\left| \int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega_{KE}^{n-1} \right| < \infty$ . By using the notations as in the above lemma, we have

$$(3.12) \quad \begin{aligned} & \int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega_{KE}^{n-1} - \int_{\mathcal{M}_g} \omega_{KE}^n \\ &= \int_{\mathcal{M}_g} (c_1(\overline{E}) - \omega_{KE}) \wedge \omega_{KE}^{n-1} \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon (c_1(\overline{E}) - \omega_{KE}) \wedge \omega_{KE}^{n-1} \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \partial \bar{\partial} \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \wedge \omega_{KE}^{n-1} \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \partial \bar{\partial} \rho_\epsilon \wedge \omega_{KE}^{n-1} \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g \cap \text{supp}(\partial \bar{\partial} \rho_\epsilon)} \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \partial \bar{\partial} \rho_\epsilon \wedge \omega_{KE}^{n-1}. \end{aligned}$$

Now, we show that

$$(3.13) \quad \int_{\mathcal{M}_g} \left| \log \frac{\det(g_{KE}^*)}{\det(h)} \right| \omega_p^n < \infty.$$

Since

$$\begin{aligned} \int_{\mathcal{M}_g} \left| \log \frac{\det(g_{KE}^*)}{\det(h)} \right| \omega_p^n &= \sum_{i=1}^p \int_{U_i} \psi_i \left| \log \frac{\det(g_{KE}^*)}{\det(h)} \right| \omega_p^n \\ &\quad + \sum_{j=1}^q \int_{V_j} \psi_{p+j} \left| \log \frac{\det(g_{KE}^*)}{\det(h)} \right| \omega_p^n \end{aligned}$$

and  $V_j$  lies in the compact set  $\mathcal{M}_g \setminus V_0$  and  $0 \leq \varphi_i \leq 1$ , we only need to show that

$$(3.14) \quad \int_{U_i} \left| \log \frac{\det(g_{KE}^*)}{\det(h)} \right| \omega_p^n < \infty.$$

We know that there is a constant  $C_i$  such that, on  $U_i$ ,

$$\omega_p^n \leq C_i \prod_{j=1}^{\alpha_i} \frac{1}{|t_j^i|^2 (\log |t_j^i|)^2}.$$

Formula (3.14) follows from the above formula, inequality (3.6) and a simple computation.

Now, we pick  $\epsilon$  small such that  $\text{supp}(\partial\bar{\partial}\rho_\epsilon) \subset \text{supp}(1 - \rho_\epsilon) \subset V_0$ . By Lemma 3.1, we know that there is a constant  $C$  such that

$$-C\omega_p \leq \partial\bar{\partial}\rho_\epsilon \leq C\omega_p$$

and

$$0 \leq \omega_{KE} \leq C\omega_p.$$

So, we have

$$\begin{aligned} &\left| \int_{\mathcal{M}_g \cap \text{supp}(\partial\bar{\partial}\rho_\epsilon)} \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \partial\bar{\partial}\rho_\epsilon \wedge \omega_{KE}^{n-1} \right| \\ &\leq C^n \sum_{i=1}^p \int_{U_i \cap \text{supp}(1 - \rho_\epsilon)} \left| \log \left( \frac{\det(g_{KE}^*)}{\det(h)} \right) \right| \omega_p^n \rightarrow 0 \end{aligned}$$

as  $\epsilon$  goes to 0 because of inequality (3.14) and the fact that the Lebesgue measure of

$$U_i \cap \text{supp}(1 - \rho_\epsilon)$$

goes to 0. By combining with formula (3.12) we have proved this lemma.

q.e.d.

Now, we define the pointwise version of the degree. Let  $F \subset \overline{E}$  be a holomorphic subbundle of rank  $k \leq n$ . Let  $g_{KE}^*|_F$  and  $h|_F$  be the restriction to  $F$  of the metrics induced by the Kähler–Einstein metric and the metric  $h$ . Let

$$(3.15) \quad d(F) = -\partial\bar{\partial} \log \det(g_{KE}^*|_F) \wedge \omega_{KE}^{n-1}.$$

Here,  $d(F)$  is an  $(n, n)$ -form. It is clear that

$$\deg(F) = \int_{\mathcal{M}_g} d(F)$$

when the integration converges.

The following result is well-known. Please see [7] for details.

**Lemma 3.4.** *For any holomorphic subbundle  $F$  of  $\overline{E}$  with rank  $k$ , we have*

$$(3.16) \quad \frac{d(F)}{k} \leq \frac{d(\overline{E})}{n}.$$

Now, we prove the main theorem 3.1.

*Proof.* Let  $F$  be a holomorphic subbundle of  $\overline{E}$  of rank  $k$ . We first check that  $\int_{\mathcal{M}_g} d(F)$  is finite and equal to the degree of  $F$ . To prove that  $\int_{\mathcal{M}_g} d(F)$  is finite, we need to show that  $-\partial\bar{\partial} \log \det(g_{KE}^*|_F)$  has Poincaré growth. This involves the estimate of the derivatives of the Kähler–Einstein metric up to second order. Our method is to use Lemma 3.4 together with the monotone convergence theorem and integration by parts to reduce the  $C^2$  estimates of the Kähler–Einstein metric to  $C^0$  estimates.

By Lemma 3.1, we know that  $\rho_\epsilon$  is monotonically increasing when  $\epsilon$  is monotonically decreasing. Also by Lemma 3.4, we know that

$$\frac{k}{n} \omega_{KE}^n - d(F) = \frac{k}{n} d(\overline{E}) - d(F) \geq 0.$$

By the monotone convergence theorem, we have

$$(3.17) \quad \lim_{\epsilon \searrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \left( \frac{k}{n} \omega_{KE}^n - d(F) \right) = \int_{\mathcal{M}_g} \left( \frac{k}{n} \omega_{KE}^n - d(F) \right).$$

By Lemma 3.3, we have

$$(3.18) \quad \begin{aligned} \frac{k}{n} \deg(\overline{E}) - \deg(F) &= \int_{\mathcal{M}_g} \left( \frac{k}{n} \omega_{KE}^n - \text{Ric}(h|_F) \wedge \omega_{KE}^{n-1} \right) \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \left( \frac{k}{n} \omega_{KE}^n - \text{Ric}(h|_F) \wedge \omega_{KE}^{n-1} \right). \end{aligned}$$

However,

$$\begin{aligned}
(3.19) \quad & \int_{\mathcal{M}_g} \rho_\epsilon \left( \frac{k}{n} \omega_{KE}^n - d(F) \right) - \int_{\mathcal{M}_g} \rho_\epsilon \left( \frac{k}{n} \omega_{KE}^n - \text{Ric}(h|_F) \wedge \omega_{KE}^{n-1} \right) \\
&= \int_{\mathcal{M}_g} \rho_\epsilon \left( \text{Ric}(h|_F) \wedge \omega_{KE}^{n-1} - d(F) \right) \\
&= \int_{\mathcal{M}_g} \rho_\epsilon \left( \text{Ric}(h|_F) + \partial\bar{\partial} \log \det(g_{KE}^*|_F) \right) \wedge \omega_{KE}^{n-1} \\
&= \int_{\mathcal{M}_g} \rho_\epsilon \partial\bar{\partial} \log \left( \frac{\det(g_{KE}^*|_F)}{\det(h|_F)} \right) \wedge \omega_{KE}^{n-1} \\
&= \int_{\mathcal{M}_g} \log \left( \frac{\det(g_{KE}^*|_F)}{\det(h|_F)} \right) \partial\bar{\partial} \rho_\epsilon \wedge \omega_{KE}^{n-1} \\
&= \int_{\mathcal{M}_g \cap \text{supp}(1-\rho_\epsilon)} \log \left( \frac{\det(g_{KE}^*|_F)}{\det(h|_F)} \right) \partial\bar{\partial} \rho_\epsilon \wedge \omega_{KE}^{n-1}.
\end{aligned}$$

Now, we show that

$$(3.20) \quad \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}_g \cap \text{supp}(1-\rho_\epsilon)} \log \left( \frac{\det(g_{KE}^*|_F)}{\det(h|_F)} \right) \partial\bar{\partial} \rho_\epsilon \wedge \omega_{KE}^{n-1} = 0.$$

By the proof of Lemma 3.3, to prove formula (3.20), we only need to show that

$$(3.21) \quad \int_{\mathcal{M}_g} \left| \log \left( \frac{\det(g_{KE}^*|_F)}{\det(h|_F)} \right) \right| \omega_p^n < \infty$$

which is reduced to show that

$$(3.22) \quad \int_{U_i} \left| \log \left( \frac{\det(g_{KE}^*|_F)}{\det(h|_F)} \right) \right| \omega_p^n < \infty.$$

Since the Ricci metric is equivalent to the Kähler–Einstein metric, we know that  $\frac{\det(g_{KE}^*|_F)}{\det(\tau^*|_F)}$  is bounded from above and below by positive constants.

We fix a  $U_i$ . Let  $\{f_1, \dots, f_k\}$  be a local holomorphic frame of  $F$ . We know that there exists a  $k \times n$  matrix  $B = (b_{\alpha\beta})$  whose entries are holomorphic functions on  $U_i$  such that the rank of  $B$  is  $k$  and  $f_\alpha =$

$\sum_{\beta} b_{\alpha\beta} e_{\beta}^i$  where  $e_{\beta}^i$  is defined in (3.1). Now, we have

$$\frac{\det(\tau^*|_F)}{\det(h|_F)} = \frac{\det\left(B\begin{pmatrix} \tau_{i\bar{j}}^* \\ \bar{B}^T \end{pmatrix}\right)}{\det\left(B\begin{pmatrix} h_{i\bar{j}} \\ \bar{B}^T \end{pmatrix}\right)}.$$

Now, we need the following linear algebraic lemma:

**Lemma 3.5.** *For any positive Hermitian  $n \times n$  matrix  $A$ , we denote its eigenvalues by  $\lambda_1, \dots, \lambda_n$  where  $\lambda_i = \lambda_i(A)$  such that  $\lambda_1(A) \geq \dots \geq \lambda_n(A) > 0$ . Let  $A_1$  and  $A_2$  be two positive Hermitian  $n \times n$  matrices. Let  $B$  be an  $k \times n$  matrix with  $k \leq n$  such that the rank of  $B$  is  $k$ . Then, there are positive constants  $c_1$  and  $c_2$  only depending on  $n, k$  such that*

$$c_1 \frac{\lambda_{n-k+1}(A_1) \cdots \lambda_n(A_1)}{\lambda_1(A_2) \cdots \lambda_k(A_2)} \leq \frac{\det\left(BA_1\bar{B}^T\right)}{\det\left(BA_2\bar{B}^T\right)} \leq c_2 \frac{\lambda_1(A_1) \cdots \lambda_k(A_1)}{\lambda_{n-k+1}(A_2) \cdots \lambda_n(A_2)}.$$

We briefly show the proof here.

*Proof.* We fix  $A_1$  and  $A_2$  and let

$$Q(B) = \frac{\det\left(BA_1\bar{B}^T\right)}{\det\left(BA_2\bar{B}^T\right)}.$$

Let  $B_i$  be the  $k \times n$  matrix obtained by multiplying the  $i$ -th row of  $B$  by a non-zero constant  $c$  and leave other rows invariant and let  $B_{ij}$  be the  $k \times n$  matrix obtained by adding a constant multiple of the  $j$ -th row to the  $i$ -th row and leave other rows invariant. It is easy to check that  $Q(B_i) = Q(B_{ij}) = Q(B)$ .

Thus, we can assume that the row vectors of  $B$  form an orthonormal set of  $\mathbb{C}^n$ . With this assumption, it is easy to see that there are positive constants  $c_3$  and  $c_4$  only depending on  $n, k$  such that

$$c_3 \lambda_{n-k+1}(A_i) \cdots \lambda_n(A_i) \leq \det\left(BA_i\bar{B}^T\right) \leq c_4 \lambda_1(A_i) \cdots \lambda_k(A_i)$$

for  $i = 1, 2$ . The lemma follows directly. q.e.d.

Now, we go back to the proof of the theorem. By using Theorem 1.4 and corollary 4.2 of [11], we know that, under the frame (3.1),

- 1)  $\tau_{i\bar{i}}^* = u_i^{-2}(1 + O(u_0))$  if  $i \leq m$ ;
- 2)  $\tau_{i\bar{j}}^* = O(1)$  if  $i, j \leq m$  with  $i \neq j$  or  $i \leq m < j$  or  $j \leq m < i$ ;
- 3)  $\tau_{i\bar{j}}^* = \tau^{i\bar{j}}$  if  $i, j \geq m + 1$ ;



- 4) On  $U_i$ , the submatrix  $(\tau^{i\bar{j}})_{i,j \geq m+1}$  is bounded from above and below by positive constants multiple of the identity matrix where the constants depend on  $U_i$ .

It is clear that, on  $U_i$  the eigenvalues of matrix matrix of  $h$  with respect to the frame (3.1) are bounded from above and below by positive constants which depend on  $U_i$  and the choice of the metric  $h$ .

By analyzing the eigenvalues of the matrix  $(\tau^*)$  and by using Lemma 3.5, a simple computation shows that there are positive constants  $C_6$  and  $C_7$  which depend on  $F$  and  $U_i$  such that

$$(3.23) \quad \left| \log \left( \frac{\det(g_{KE}^* | F)}{\det(h | F)} \right) \right| \leq C_6 + C_7 \sum_{j=1}^{\alpha_i} \log \log \frac{1}{|t_j^i|}.$$

Now, by using a similar method to the proof of Lemma 3.3, we know that formula (3.22) and (3.21) hold which imply formula (3.20) holds. Combining (3.20) and (3.19), we have

$$(3.24) \quad \lim_{\epsilon \searrow 0} \left( \int_{\mathcal{M}_g} \rho_\epsilon \left( \frac{k}{n} \omega_{KE}^n - d(F) \right) - \int_{\mathcal{M}_g} \rho_\epsilon \left( \frac{k}{n} \omega_{KE}^n - \text{Ric}(h | F) \wedge \omega_{KE}^{n-1} \right) \right) = 0.$$

By combining (3.24), (3.18) and (3.17), we have

$$(3.25) \quad \int_{\mathcal{M}_g} \left( \frac{k}{n} \omega_{KE}^n - d(F) \right) = \frac{k}{n} \text{deg}(\bar{E}) - \text{deg}(F).$$

By Lemma 3.3, we know that  $\int_{\mathcal{M}_g} \omega_{KE}^n = \text{deg}(\bar{E})$ . From formula (3.25), we know that  $\int_{\mathcal{M}_g} d(F)$  is finite and

$$(3.26) \quad \text{deg}(F) = \int_{\mathcal{M}_g} d(F).$$

Now, by Lemma 3.4, we have

$$\frac{\text{deg}(F)}{k} - \frac{\text{deg}(\bar{E})}{n} = \int_{\mathcal{M}_g} \left( \frac{d(F)}{k} - \frac{d(\bar{E})}{n} \right) \leq 0$$

which implies

$$\frac{\text{deg}(F)}{k} \leq \frac{\text{deg}(\bar{E})}{n}.$$

This proves that the bundle  $\bar{E}$  is semi-stable in the sense of Mumford.

To prove the strict stability of the logarithm cotangent bundle, we need to show that this bundle cannot split. The following result about the modular group  $\text{Mod}_g$  and its proof is due to Luo [14].

**Proposition 3.1.** *Let  $\text{Mod}_g$  be the moduli group of closed Riemann surfaces of genus  $g$  with  $g \geq 2$ . Then, any finite index subgroup of  $\text{Mod}_g$  is not isomorphic to a product of groups.*

The proof of the above proposition is topological. For completeness, we will include Luo’s proof at the end of this section.

Now, we go back to the proof of stability. If  $\overline{E}$  is not stable, then it must split into a direct sum of holomorphic subbundles  $\overline{E} = \bigoplus_{i=1}^k E_i$  with  $k \geq 2$ . Moreover, when restricted to the moduli space, both the connection and the Kähler–Einstein metric split. It is well known that there is a finite cover  $\widetilde{\mathcal{M}}_g$  of  $\mathcal{M}_g$  which is smooth. By the decomposition theorem of de Rham, the Teichmüller space, as the universal covering space of  $\widetilde{\mathcal{M}}_g$  must split isomorphically as a product of manifolds. Furthermore, the fundamental group of  $\widetilde{\mathcal{M}}_g$  is isomorphic to a product of groups. However,  $\pi_1(\widetilde{\mathcal{M}}_g)$  is a finite index subgroup of the mapping class group. By the above proposition, this is impossible. So, we have proved the stability. q.e.d.

We remark that the positivity of  $c_1(\overline{E})$  implies that the Deligne–Mumford compactification  $\overline{\mathcal{M}}_g$  is of logarithmic general type for  $g > 1$ .

At the end of this section we give a proof by Luo of Proposition 3.1.

*Proof of Proposition 3.1.* The proof uses Thurston’s classification of elements in the mapping class group [19] and the solution of the Nielsen realization problem [6]. In the following, we will use the words “simple loops” and “subsurfaces” to denote the isotopy classes of simple loops and subsurfaces. We fix a surface  $X$ .

Suppose there is a subgroup  $G$  of  $\text{Mod}_g$  with finite index such that there are two non-trivial subgroups  $A$  and  $B$  of  $G$  so that  $G = A \times B$ . We will derive a contradiction. We need to use the following lemma.

**Lemma 3.6.** *Let  $A$ ,  $B$  and  $G$  be as above. Then*

- 1) *There are elements of infinite order in both  $A$  and  $B$ ;*
- 2) *There are no elements in  $A$  or  $B$  which is pseudo-Anosov.*

*Proof.* If the first claim is not true, then we can assume that  $A$  consists of torsion elements only. We claim that, in this case,  $A$  is a finite group.

Actually let  $\pi : \text{Mod}_g \rightarrow \text{Aut}(H_1(X))$  be the natural homomorphism. By the virtue of Theorem V.3.1 of [5], we know that the kernel of  $\pi$  contains no torsion elements. Thus,  $\pi(A)$  is isomorphic to  $A$ . Now,  $\pi(A)$  is a torsion subgroup of the general linear group  $\text{GL}(n, \mathbf{R})$ . By the well known solution of the Burnside problem for the linear groups, we see that  $\pi(A)$  must be finite and so is  $A$ .

For the finite group  $A$ , by the solution of the Nielsen realization problem of Kerckhoff [6], we know that there is a point  $d$  in the Teichmüller space of  $X$  fixed by all elements in  $A$ . Let  $\text{Fix}(A)$  be the set of all points in the Teichmüller space fixed by each element in  $A$ . Then, we see that  $\text{Fix}(A)$  is a non-empty proper subset of the Teichmüller space. Now, for each  $a \in A$ ,  $b \in B$  and  $d \in \text{Fix}(A)$ , since  $ab = ba$ , we have  $ab(d) = ba(d) = b(d)$ . This implies  $\text{Fix}(A)$  is invariant under the action of  $B$  on the Teichmüller space. Thus, we see that the finite index subgroup  $G = A \times B$  acts on the Teichmüller space leaving  $\text{Fix}(A)$  invariant. This contradicts the finite index property of  $G$  since  $\text{Fix}(A)$  is the Teichmüller space of the orbifold  $X/A$ . This proves the first claim.

Now, we check the second claim. If it is not true, then we can assume that there is an  $a \in A$  which is pseudo-Anosov. Now, we consider the action of the mapping class group on the space of all measured laminations in the surface. By Thurston's theory, there are exactly two measured laminations  $m, m'$  fixed by  $a$ . Now, for all  $b \in B$ , due to  $ab = ba$ , we see that  $b$  leaves  $\{m, m'\}$  invariant. A result of McCarthy [16] shows that the stabilizers of  $\{m, m'\}$  in the mapping class group is virtually cyclic. Thus, we see that each element  $b \in B$  has some power  $b^n$  with  $n \neq 0$  which is equal to  $a^k$  with  $k \neq 0$ . By the first claim, we know that  $B$  contains elements of infinite order. This implies that some power  $a^k$  with  $k \neq 0$  is in  $B$  which is a contradiction. This proved the second claim. q.e.d.

Now, we go back to the proof of the proposition. By the above lemma, we can take  $a \in A$  and  $b \in B$ , both are infinite order and none of them is pseudo-Anosov. Thus, by replacing  $a$  and  $b$  by a high power  $a^n$  and  $b^n$  with  $n > 0$ , we may assume that for  $a$  there is a set of disjoint simple loops  $c_1, \dots, c_k$  in  $X$  so that

- 1) Each component of  $X \setminus \cup_{i=1}^k c_i$  is invariant under  $a$ ;
- 2) The restriction of  $a$  to each component of  $X \setminus \cup_{i=1}^k c_i$  is the identity map or a pseudo-Anosov map.

Let  $N(c_i)$  be a regular neighborhood of  $c_i$  in  $X$  and let  $F(a)$  be the subsurface which is the union of all  $N(c_i)$ 's with those components of  $X \setminus \cup_{i=1}^k c_i$  on which  $a$  is the identity map. By Thurston's classification, the subsurface  $F(a)$  has the property that if  $c$  is a curve system invariant under  $a$ , then  $c$  is in  $F(a)$ . We can construct  $F(b)$  in a similar way.

Now since  $ab = ba$ , we know that if a simple loop  $c$  is invariant under  $b$ , then  $a(c)$  is also invariant under  $b$  because  $ba(c) = ab(c) = a(c)$ . Thus, for all simple loops  $c$  in  $F(b)$ ,  $a(c)$  is still in  $F(b)$ . This implies that

$a(F(b)) = F(b)$ . By using the property of  $F(a)$ , we see  $F(b) \subset F(a)$ . Similar reason implies  $F(a) \subset F(b)$ . Thus,  $F(a) = F(b)$ .

Now, we show that the subsurface  $F(a) = F(b)$  is invariant under each element in  $A$  and  $B$ . We pick  $x \in A$ . Due to  $xb = bx$ , we have  $x(F(b)) = b(x(F(b)))$  which implies  $x(F(b)) \subset F(b)$ . Since these two surfaces are homeomorphic, we have  $x(F(b)) = F(b)$ . Similarly, for all  $y \in B$ ,  $y(F(a)) = F(a)$ . Thus, all elements in  $G = A \times B$  leave the subsurface  $F(a)$  invariant.

Now, there are two cases. In the first case,  $F(a)$  is not homeomorphic to  $X$ . In the second case,  $F(a) = F(b) = X$ .

It is clear that in the first case,  $F(a)$  is not homeomorphic to  $X$  cannot occur. Otherwise, the finite index subgroup  $G$  leaves a proper subsurface  $F(a)$  invariant. This contradicts the known properties of the mapping class group. As a conclusion of this case, we see that for any element  $x \in A$ , any power  $x^n$  of  $x$  cannot contain pseudo-Anosov components in Thurston's classification.

In the second case,  $F(a) = F(b) = X$ . In this case, some power of  $a$  (and  $b$ ) is a composition of Dehn twists on disjoint simple loops. By the argument in the first case, we see that for any two elements  $x \in A$  and  $y \in B$ , some powers  $x^n, y^m$  with  $m, n \neq 0$  are either the identity map or compositions of Dehn twists on disjoint simple loops.

Let  $\text{Fix}(x)$  be the union of the disjoint simple loops so that  $x$  is the composition of Dehn twists on these simple loops. We define  $\text{Fix}(x)$  to be the empty set if  $x^n = id$  for some non-zero integer  $n$ . We need the following claim:

**Claim 1.** For any  $x \in A$  and  $y \in B$ , the geometric intersection number between  $\text{Fix}(x)$  and  $\text{Fix}(y)$  is zero.

To prove the claim, without loss of generality, we may take  $x = a$  and  $y = b$ . We may assume that  $a$  is the composition of Dehn twists on curve system  $\text{Fix}(a) = c$  and  $b$  is the composition of Dehn twists on curve system  $\text{Fix}(b) = d$ . Note that in this case, if  $s$  is a curve system invariant under  $a$ , then the geometric intersection number between  $s$  and  $c$  is zero. Namely,  $I(s, c) = 0$ . We also say that  $s$  is *disjoint* from  $c$ . Now, since  $ab = ba$  and  $a(c) = c$ , we have that  $a(b(c)) = b(c)$ . Thus, the curve system  $b(c)$  is disjoint from  $c$ . Since  $b$  is a composition of Dehn twists on disjoint simple loops  $d$ , this shows that  $I(c, d) = 0$ . Namely,  $d$  and  $c$  are disjoint curve systems.

Now, we finish the proof of the proposition as follows. First of all, by the assumption  $\text{Fix}(a)$  and  $\text{Fix}(b)$  are both non-empty. Let  $S_1$  (and  $S_2$ ) be the smallest subsurface of  $X$  which contains all curves in  $\text{Fix}(x)$  for  $x \in A$  (or  $x \in B$ ). By the above claim, we have  $S_1 \cap S_2 = \emptyset$ . Thus,

there is an essential simple loop  $c$  which is disjoint from both  $S_1$  and  $S_2$ . By the construction,  $x(c) = c$  and  $y(c) = c$  for all  $x \in A$  and  $y \in B$ . Thus, we see that for each element  $e \in G = A \times B$ , there is a power  $e^n$  with  $n \neq 0$  so that  $e^n$  leaves  $c$  invariant. This contradicts the fact that  $A \times B$  is a finite index subgroup of the mapping class group. This finishes the proof. q.e.d.

#### 4. The Bounded Geometry of the Kähler–Einstein Metric

In this section, we show that the Kähler–Einstein metric has bounded curvature and the injectivity radius of the Teichmüller space equipped with the Kähler–Einstein metric is bounded from below.

We begin with a metric that is equivalent to the Kähler–Einstein metric and its curvature as well as the covariant derivatives of the curvature are uniformly bounded. We deform the Ricci metric whose curvature is bounded to obtain this metric by using Kähler–Ricci flow. By doing this, we avoid the complicated computation of the covariant derivatives of the curvature of the Ricci metric.

We then establish the Monge–Ampère equation from this new metric. Our work in [11] implies the new metric is equivalent to the Kähler–Einstein metric which gave us  $C^2$  estimates. Based on this, we do the  $C^3$  and  $C^4$  estimates. This will give us the boundedness of the curvature of the Kähler–Einstein metric. The same method can be used to show that all the covariant derivatives of the Kähler–Einstein metric are bounded.

We slightly change our notations. We will use  $g$  to denote the Kähler–Einstein metric and use  $h$  to denote an equivalent metric whose curvature and covariant derivatives of curvature are bounded. We use  $z_1, \dots, z_n$  to denote local holomorphic coordinates on the Teichmüller space. The main result of this section is the following theorem:

**Theorem 4.1.** *Let  $g$  be the Kähler–Einstein metric on the Teichmüller space  $\mathcal{T}$ . Then, the curvature of  $g$  and all of its covariant derivatives are all bounded.*

*Proof.* We begin with the Ricci metric  $\tau$ . We first obtain a new equivalent metric  $h$  by deforming  $\tau$  with the Ricci flow. Consider the following Kähler–Ricci flow:

$$(4.1) \quad \begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -(R_{i\bar{j}} + g_{i\bar{j}}) \\ g_{i\bar{j}}(0) = \tau_{i\bar{j}} \end{cases} .$$

If we let  $s = e^t - 1$  and  $\tilde{g} = e^t g$ , we have

$$(4.2) \quad \begin{cases} \frac{\partial \tilde{g}_{i\bar{j}}}{\partial s} = -\tilde{R}_{i\bar{j}} \\ \tilde{g}_{i\bar{j}}(0) = \tau_{i\bar{j}} \end{cases}.$$

Since the initial metric has bounded curvature, by the work of Shi [18], the flow (4.2) has short time existence and for small  $s$ , the metric  $\tilde{g}(s)$  is equivalent to the initial metric  $\tau$ . Furthermore, the curvature and its covariant derivatives of  $\tilde{g}(s)$  are bounded. Hence, for small  $t$ , the metric  $g(t)$  is equivalent to the Ricci metric  $\tau$  and has bounded curvature as well as covariant derivatives of the curvature.

Now, we fix a small  $t$  and denote the metric  $g(t)$  by  $h$ . Since the Teichmüller space is contractible, there are smooth functions  $u$  and  $F$  such that

$$(4.3) \quad \omega_g = \omega_h + \partial\bar{\partial}u$$

and

$$(4.4) \quad \text{Ric}(h) + \omega_h = \partial\bar{\partial}F.$$

Since the metrics  $h$  and  $\tau$  are equivalent, we know that  $h$  and  $g$  are equivalent which implies the tensor  $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$  is bounded with respect to either metric. Also, because the curvature and its covariant derivatives of the metric  $h$  are bounded, we know that a covariant derivative of  $F$  with respect to  $h$  is bounded if this derivative is at least order 2 and has at least one holomorphic direction and one anti-holomorphic direction. So, we have  $C^2$  estimates.

By the Kähler–Einstein condition of  $g$ , we have the Monge–Ampère equation

$$(4.5) \quad \log \det(h_{i\bar{j}} + u_{i\bar{j}}) - \log \det(h_{i\bar{j}}) = u + F.$$

We use  $\Delta$ ,  $\Delta'$ ,  $\nabla$ ,  $\nabla'$ ,  $\Gamma_{ij}^k$ ,  $\tilde{\Gamma}_{ij}^k$ ,  $R_{i\bar{j}k\bar{l}}$ ,  $P_{i\bar{j}k\bar{l}}$ ,  $R_{i\bar{j}}$ ,  $P_{i\bar{j}}$ ,  $R$  and  $P$  to denote the Laplacian, gradient, Christoffel symbol, curvature tensor, Ricci curvature and scalar curvature of the metrics  $h$  and  $g$  respectively. In the following, all covariant derivatives of functions and tensors are taken with respect to the background metric  $h$ .

Inspired by Yau's work in [22], we let

$$(4.6) \quad \mathcal{F} = u + F,$$

$$(4.7) \quad S = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}}$$

and

$$(4.8) \quad V = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} u_{;i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} u_{;i\bar{n}k\bar{p}} u_{;\bar{j}m\bar{l}\bar{q}}.$$

To simplify the notation, we define the following quantities:

(4.9)

$$\begin{aligned} A_{ipkm\alpha} &= u_{;i\bar{p}k\bar{m}\alpha} - g^{\gamma\bar{\delta}} u_{;i\bar{\delta}\alpha} u_{;\gamma\bar{p}k\bar{m}} - g^{\gamma\bar{\delta}} u_{;k\bar{\delta}\alpha} u_{;i\bar{p}\gamma\bar{m}} - g^{\gamma\bar{\delta}} u_{;\gamma\bar{p}\alpha} u_{;i\bar{\delta}k\bar{m}} \\ &\quad - g^{\gamma\bar{\delta}} u_{;\gamma\bar{m}\alpha} u_{;i\bar{p}k\bar{\delta}} + g^{\gamma\bar{\delta}} u_{;i\bar{p}\gamma} u_{;\bar{\delta}k\bar{m}\alpha} \\ &\quad + g^{\gamma\bar{\delta}} u_{;i\bar{p}\gamma} u_{;k\bar{m}\alpha\bar{\delta}} + 2g^{\gamma\bar{\delta}} u_{;\bar{p}i\bar{\delta}} u_{;k\bar{m}\alpha\gamma}, \end{aligned}$$

(4.10)

$$\begin{aligned} B_{ipkm\alpha} &= u_{;i\bar{p}k\bar{m}\alpha} - g^{\gamma\bar{\delta}} u_{;\gamma\bar{i}\alpha} u_{;\bar{\delta}p\bar{k}m} - g^{\gamma\bar{\delta}} u_{;\gamma\bar{k}\alpha} u_{;i\bar{p}\bar{\delta}m} - g^{\gamma\bar{\delta}} u_{;p\bar{\delta}\alpha} u_{;i\bar{\gamma}k\bar{m}} \\ &\quad - g^{\gamma\bar{\delta}} u_{;m\bar{\delta}\alpha} u_{;i\bar{p}k\bar{\gamma}} + g^{\gamma\bar{\delta}} u_{;p\bar{i}\gamma} u_{;k\bar{m}\bar{\delta}\alpha} + g^{\gamma\bar{\delta}} u_{;i\bar{p}\bar{\delta}} u_{;\gamma\bar{k}m\alpha}, \end{aligned}$$

(4.11)

$$\begin{aligned} C_{ipkm\alpha} &= u_{;i\bar{m}k\bar{p}\alpha} - g^{\gamma\bar{\delta}} u_{;i\bar{\delta}\alpha} u_{;\gamma\bar{m}k\bar{p}} - g^{\gamma\bar{\delta}} u_{;k\bar{\delta}\alpha} u_{;i\bar{m}\gamma\bar{p}} - g^{\gamma\bar{\delta}} u_{;p\bar{\delta}\alpha} u_{;i\bar{m}k\gamma} \\ &\quad - g^{\gamma\bar{\delta}} u_{;\gamma\bar{m}\alpha} u_{;i\bar{\delta}k\bar{p}} + g^{\gamma\bar{\delta}} u_{;i\bar{m}\gamma} u_{;k\bar{\delta}p\alpha}, \end{aligned}$$

(4.12)

$$\begin{aligned} D_{ipkm\alpha} &= u_{;i\bar{m}k\bar{p}\alpha} - g^{\gamma\bar{\delta}} u_{;\gamma\bar{i}\alpha} u_{;\bar{\delta}m\bar{k}\bar{p}} - g^{\gamma\bar{\delta}} u_{;\gamma\bar{k}\alpha} u_{;i\bar{m}\bar{\delta}\bar{p}} - g^{\gamma\bar{\delta}} u_{;\gamma\bar{p}\alpha} u_{;i\bar{m}k\bar{\delta}} \\ &\quad - g^{\gamma\bar{\delta}} u_{;m\bar{\delta}\alpha} u_{;i\bar{\gamma}k\bar{p}} + g^{\gamma\bar{\delta}} u_{;i\bar{m}\bar{\delta}} u_{;\bar{k}\gamma\bar{p}\alpha}, \end{aligned}$$

(4.13)

$$\begin{aligned} W &= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;i\bar{q}k\bar{n}\alpha} u_{;\bar{j}p\bar{l}m\bar{\beta}} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;i\bar{q}k\bar{n}\bar{\beta}} u_{;\bar{j}p\bar{l}m\alpha} \\ &\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;i\bar{n}k\bar{p}\alpha} u_{;\bar{j}m\bar{l}\bar{q}\bar{\beta}} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;i\bar{n}k\bar{p}\bar{\beta}} u_{;\bar{j}m\bar{l}\bar{q}\alpha} \end{aligned}$$

and

$$\begin{aligned} \widetilde{W} &= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} (A_{iqkn\alpha} \overline{A_{jplm\beta}} + B_{jplm\alpha} \overline{B_{iqkn\beta}}) \\ &\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} (C_{ipkn\alpha} \overline{C_{jqlm\beta}} + D_{jqlm\alpha} \overline{D_{ipkn\beta}}). \end{aligned} \quad (4.14)$$

Firstly, a simple computation shows that

$$\widetilde{\Gamma}_{ik}^{\alpha} - \Gamma_{ik}^{\alpha} = g^{\alpha\bar{\beta}} u_{;i\bar{\beta}k}. \quad (4.15)$$

Now, we compute  $P_{i\bar{j}k\bar{l}}$ . We first note that

$$u_{;i\bar{j}k\bar{l}} = u_{;i\bar{j}k\bar{l}} - u_{;\bar{j}p\bar{l}} \Gamma_{ik}^p - u_{;i\bar{q}k} \overline{\Gamma_{jl}^q} + u_{p\bar{q}} \Gamma_{ik}^p \overline{\Gamma_{jl}^q} - u_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}}.$$

Since  $g_{i\bar{j}} = h_{i\bar{j}} + u_{i\bar{j}}$  we have

(4.16)

$$\begin{aligned}
P_{i\bar{j}k\bar{l}} &= \partial_k \partial_{\bar{l}} g_{i\bar{j}} - g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{l}} g_{p\bar{j}} = \partial_k \partial_{\bar{l}} h_{i\bar{j}} + u_{i\bar{j}k\bar{l}} - g_{p\bar{q}} \tilde{\Gamma}_{ik}^p \overline{\Gamma}_{jl}^q \\
&= \partial_k \partial_{\bar{l}} h_{i\bar{j}} + u_{i\bar{j}k\bar{l}} - g_{p\bar{q}} \left( \Gamma_{ik}^p + g^{p\bar{\beta}} u_{;i\bar{\beta}k} \right) \left( \overline{\Gamma}_{jl}^q + g^{\alpha\bar{q}} u_{;j\bar{\alpha}l} \right) \\
&= \left( \partial_k \partial_{\bar{l}} h_{i\bar{j}} - h_{p\bar{q}} \Gamma_{ik}^p \overline{\Gamma}_{jl}^q \right) - u_{p\bar{q}} \Gamma_{ik}^p \overline{\Gamma}_{jl}^q \\
&\quad + \left( u_{i\bar{j}k\bar{l}} - u_{j\bar{p}l} \Gamma_{ik}^p - u_{i\bar{q}k} \overline{\Gamma}_{jl}^q + 2u_{p\bar{q}} \Gamma_{ik}^p \overline{\Gamma}_{jl}^q \right) - g^{p\bar{q}} u_{;i\bar{q}k} u_{;j\bar{p}l} \\
&= R_{i\bar{j}k\bar{l}} + u_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}} + u_{;i\bar{j}k\bar{l}} - g^{p\bar{q}} u_{;i\bar{q}k} u_{;j\bar{p}l}.
\end{aligned}$$

Since the curvature of the background metric  $h$  and the tensor  $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$  are both bounded, to prove that the curvature of the metric  $g$  is bounded, we only need to show that both  $S$  and  $V$  are bounded.

We first consider the quantity  $S$ . We follow the idea of Yau in [22] and use the following notations:

**Definition 4.1.** Let  $A$  and  $B$  be two functions. We denote

- 1)  $A \stackrel{3}{\simeq} B$  if  $|A - B| \leq C_1 \sqrt{S} + C_2$ ;
- 2)  $A \stackrel{4}{\simeq} B$  if  $|A - B| \leq C_1 \sqrt{V} + C_2$ ;
- 3)  $A \stackrel{3}{\cong} B$  if  $|A - B| \leq C_1 S + C_2 \sqrt{S} + C_3$ ;
- 4)  $A \stackrel{4}{\cong} B$  if  $|A - B| \leq C_1 V + C_2 \sqrt{V} + C_3$

where  $C_1$ ,  $C_2$  and  $C_3$  are universal constants.

Also, by diagonalizing at a point  $p$ , we mean to choose local holomorphic coordinates  $z_1, \dots, z_n$  near  $p$  such that at  $p$ ,

$$h_{i\bar{j}} = \delta_{ij}$$

and

$$u_{i\bar{j}} = \delta_{ij} u_{i\bar{i}}.$$

Now, we differentiate the equation (4.5) twice and reorganize the terms. We have

$$(4.17) \quad g^{i\bar{j}} u_{;i\bar{j}k\bar{l}} = \mathcal{F}_{k\bar{l}} + g^{i\bar{j}} g^{p\bar{q}} u_{;i\bar{q}k} u_{;j\bar{p}l}.$$

By differentiating this equation once more, we have

$$\begin{aligned}
(4.18) \quad g^{i\bar{j}} u_{;i\bar{j}k\bar{l}\alpha} &= \mathcal{F}_{;k\bar{l}\alpha} + g^{i\bar{j}} g^{p\bar{q}} \left( u_{;i\bar{q}\alpha} u_{;p\bar{j}k\bar{l}} + u_{;i\bar{q}k} u_{;j\bar{p}l\alpha} + u_{;i\bar{q}k\alpha} u_{;j\bar{p}l} \right) \\
&\quad - g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left( u_{;j\bar{m}l} u_{;p\bar{n}\alpha} u_{;i\bar{q}k} + u_{;m\bar{q}k} u_{;i\bar{n}\alpha} u_{;j\bar{p}l} \right).
\end{aligned}$$

Since

$$\partial_k (\Delta u) = \partial_k \left( h^{i\bar{j}} u_{;i\bar{j}} \right) = h^{i\bar{j}} \left( u_{;i\bar{j}k} - u_{p\bar{j}} \Gamma_{ik}^p \right) = h^{i\bar{j}} u_{;i\bar{j}k},$$



by diagonalizing and the Schwarz inequality, we have

(4.19)

$$\left| \nabla'(\Delta u) \right|^2 = g^{i\bar{j}} \left( h^{k\bar{l}} u_{;k\bar{l}i} \right) \left( h^{p\bar{q}} u_{;q\bar{p}\bar{j}} \right) = \sum_i \frac{1}{1+u_{i\bar{i}}} \left| \sum_k u_{;k\bar{k}i} \right|^2 \leq C_1 S$$

since the metrics  $h$  and  $g$  are equivalent and

$$S = \sum_{i,p,k} \frac{1}{1+u_{i\bar{i}}} \frac{1}{1+u_{p\bar{p}}} \frac{1}{1+u_{k\bar{k}}} |u_{;i\bar{p}k}|^2.$$

We also have

$$\begin{aligned} \Delta'(\Delta u) &= g^{k\bar{l}} \partial_k \partial_{\bar{l}} \left( h^{i\bar{j}} u_{;i\bar{j}} \right) = g^{k\bar{l}} h^{i\bar{j}} u_{;i\bar{j}k\bar{l}} \\ &= g^{k\bar{l}} h^{i\bar{j}} \left( u_{;k\bar{l}i\bar{j}} - u_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}} + u_{p\bar{l}} h^{p\bar{q}} R_{k\bar{q}i\bar{j}} \right). \end{aligned}$$

By using equation (4.17), we have

$$\begin{aligned} (4.20) \quad \Delta'(\Delta u) &= h^{i\bar{j}} \left( \mathcal{F}_{i\bar{j}} + g^{k\bar{l}} g^{p\bar{q}} u_{;k\bar{q}i} u_{;l\bar{p}\bar{j}} \right) \\ &\quad - h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{j}} R_{i\bar{q}k\bar{l}} + h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} \\ &= \tilde{S} + \Delta \mathcal{F} - h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{j}} R_{i\bar{q}k\bar{l}} + h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} \end{aligned}$$

where  $\tilde{S} = h^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} u_{;k\bar{q}i} u_{;l\bar{p}\bar{j}}$ . Since the metrics  $h$  and  $g$  are equivalent, we know that there is a constant  $C_2$  such that

$$\tilde{S} \geq C_2 S.$$

Now, the term  $|h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{j}} R_{i\bar{q}k\bar{l}}|$  is bounded since  $h$  is equivalent to  $g$ , the curvature of  $h$  is bounded and we have  $C^2$  estimates on  $u$ . Similarly,  $|\Delta \mathcal{F}|$  is bounded. Finally, since

$$\begin{aligned} h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} &= -g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}} \\ &= -g^{k\bar{l}} h^{p\bar{q}} (g_{p\bar{l}} - h_{p\bar{l}}) R_{k\bar{q}} = g^{k\bar{q}} R_{k\bar{q}} - R, \end{aligned}$$

we know that  $|h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}}|$  is also bounded for similar reasons. By combining the above argument, we know that there is a constant  $C_3$  such that

$$(4.21) \quad \Delta'(\Delta u) \geq C_2 S - C_3.$$

Now, by the computation in [22], we know there are positive constants  $C_4$ ,  $C_5$  and  $C_6$  such that

$$(4.22) \quad \Delta'(S + C_4 \Delta u) \geq C_5 S - C_6.$$

So, for any positive  $\lambda > 0$ , we can find a positive constant  $C_7$  such that

$$(4.23) \quad S + C_4 \Delta u + C_7 \geq 0$$

and

$$(4.24) \quad \Delta'(S + C_4 \Delta u + C_7) \geq -\lambda(S + C_4 \Delta u + C_7)$$

since  $\Delta u$  is bounded. We fix  $\lambda$  and let  $f = S + C_4 \Delta u + C_7$ . Now, we know that the Ricci curvature of  $g$  is  $-1$  and  $g$  is equivalent to the Ricci metric  $\tau$  whose injectivity radius has a lower bound, by the work of Li and Schoen [10] and Li [9], we can find a positive  $r_0$  such that the mean value inequality

$$(4.25) \quad f(p) \leq C V_p^{-1}(r_0) \int_{B_p(r_0)} f \, dV$$

holds for any  $p$  in the Teichmüller space. Here,  $V_p(r_0)$  is the volume of the Kähler–Einstein ball centered at  $p$  with radius  $r_0$ ,  $dV = \omega_g^n$  is the volume element of the metric  $g$  and  $C$  is a constant depending on  $r_0$  and  $\lambda$ , but is independent of  $p$ . Let  $r(z)$  be the function on  $B_p(2r_0)$  measuring the  $g$ -distance between  $z$  and  $p$ . We fix a small  $r_0$  and let  $\rho = \rho(r)$  be a cutoff function such that  $0 \leq \rho \leq 1$ ,  $\rho(r) = 1$  for  $r \leq r_0$  and  $\rho(r) = 0$  for  $r \geq 2r_0$ . Since  $\Delta'(\Delta u) + C_3 \geq C_2 S \geq 0$ , we have

$$\begin{aligned} & C_2 \int_{B_p(2r_0)} \rho^2 S \, dV - C_3 V_p(2r_0) \\ & \leq \int_{B_p(2r_0)} \rho^2 \Delta'(\Delta u) \, dV = -2 \int_{B_p(2r_0)} \nabla' \rho \cdot (\rho \nabla'(\Delta u)) \, dV \\ & \leq C_8 \left( \int_{B_p(2r_0)} |\nabla' \rho|^2 \, dV \right)^{\frac{1}{2}} \left( \int_{B_p(2r_0)} \rho^2 |\nabla'(\Delta u)|^2 \, dV \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $|\nabla' \rho|$  is bounded and  $|\nabla'(\Delta u)|^2 \leq C_1 S$ , we have

$$C_2 \int_{B_p(2r_0)} \rho^2 S \, dV - C_3 V_p(2r_0) \leq C_9 \left( \int_{B_p(2r_0)} \rho^2 S \, dV \right)^{\frac{1}{2}} (V_p(2r_0))^{\frac{1}{2}}$$

which implies

$$(4.26) \quad \int_{B_p(2r_0)} \rho^2 S \, dV \leq C_{10} V_p(2r_0).$$

By using inequalities (4.25) and (4.26), we have

$$\begin{aligned}
(4.27) \quad f(p) &\leq CV_p^{-1}(r_0) \int_{B_p(r_0)} (S + C_4\Delta u + C_7) dV \\
&\leq CV_p^{-1}(r_0) \int_{B_p(r_0)} S dV + C_{11} \\
&\leq CV_p^{-1}(r_0) \int_{B_p(2r_0)} \rho^2 S dV + C_{11} \\
&\leq C_{12} \frac{V_p(2r_0)}{V_p(r_0)} + C_{11}.
\end{aligned}$$

For each point  $p \in \mathcal{T}_g$ , let  $f_p : \mathcal{T}_g \rightarrow \mathbb{C}^{3g-3}$  be the Bers' embedding map such that  $f_p(p) = 0$  and  $B_2 \subset f_p(\mathcal{T}_g) \subset B_6$  where  $B_r \subset \mathbb{C}^{3g-3}$  is the open Euclidean ball of radius  $r$ . Since both metrics  $h$  and  $g$  are equivalent to the Ricci metric which is equivalent to the Euclidean metric on the unit Euclidean ball  $B_1$ , we know that  $\frac{V_p(2r_0)}{V_p(r_0)}$  is uniformly bounded since both balls have Euclidean volume growth. Thus,  $f(p) \leq C_{13}$ . Since  $\Delta u$  is bounded, we conclude that  $S$  is uniformly bounded.

Now, we do the  $C^4$  estimate. Let  $\kappa$  be a large positive constant. We first compute  $\Delta'[(S + \kappa)V]$ . We have

$$\begin{aligned}
(4.28) \quad \Delta'[(S + \kappa)V] &= g^{i\bar{j}} \partial_i \partial_{\bar{j}} [(S + \kappa)V] \\
&= g^{i\bar{j}} \partial_{\bar{j}} [V \partial_i S + (S + \kappa) \partial_i V] \\
&= g^{i\bar{j}} \left[ (S + \kappa) \partial_i \partial_{\bar{j}} V + V \partial_i \partial_{\bar{j}} S + \partial_i V \partial_{\bar{j}} S + \partial_i S \partial_{\bar{j}} V \right] \\
&\geq (S + \kappa) \Delta' V + V \Delta' S - 2 \left| \nabla' S \right| \left| \nabla' V \right|.
\end{aligned}$$

The computations of the first and second derivatives of  $V$  are very long. We only list the results here. For the first derivative of  $V$ , we have

$$\begin{aligned}
(4.29) \quad \partial_\alpha V &= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;i\bar{q}k\bar{n}\alpha} u_{;\bar{j}p\bar{l}m} + u_{;i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m\alpha} \right] \\
&\quad - g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\gamma\bar{\delta}} \left[ u_{;i\bar{\delta}\alpha} u_{;\gamma\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} + u_{;k\bar{\delta}\alpha} u_{;i\bar{q}\gamma\bar{n}} u_{;\bar{j}p\bar{l}m} \right] \\
&\quad - g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\gamma\bar{\delta}} \left[ u_{;p\bar{\delta}\alpha} u_{;\bar{j}\gamma\bar{l}m} u_{;i\bar{q}k\bar{n}} + u_{;m\bar{\delta}\alpha} u_{;\bar{j}p\bar{l}\gamma} u_{;i\bar{q}k\bar{n}} \right] \\
&\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;i\bar{n}k p\alpha} u_{;\bar{j}m\bar{l}\bar{q}} + u_{;i\bar{n}k p} u_{;\bar{j}m\bar{l}\bar{q}\alpha} \right] \\
&\quad - g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\gamma\bar{\delta}} \left[ u_{;i\bar{\delta}\alpha} u_{;\gamma\bar{n}k p} u_{;\bar{j}m\bar{l}\bar{q}} + u_{;k\bar{\delta}\alpha} u_{;i\bar{n}\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} \right]
\end{aligned}$$

$$-g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\gamma\bar{\delta}}\left[u_{;p\bar{\delta}\alpha}u_{;i\bar{n}k\gamma}u_{;\bar{j}m\bar{l}\bar{q}}+u_{;m\bar{\delta}\alpha}u_{;\bar{j}\gamma\bar{l}\bar{q}}u_{;i\bar{n}kp}\right].$$

By differentiating the above formula, we have

$$(4.30) \quad \Delta'V = g^{\alpha\bar{\beta}}\partial_{\alpha}\partial_{\bar{\beta}}V = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7$$

where

$$(4.31) \quad \begin{aligned} A_1 &= g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}\left[u_{;i\bar{q}k\bar{n}\alpha}u_{;\bar{j}p\bar{l}m\bar{\beta}}+u_{;i\bar{q}k\bar{n}\beta}u_{;\bar{j}p\bar{l}m\alpha}\right] \\ &\quad + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}\left[u_{;i\bar{n}kp\alpha}u_{;\bar{j}m\bar{l}\bar{q}\bar{\beta}}+u_{;i\bar{n}kp\beta}u_{;\bar{j}m\bar{l}\bar{q}\alpha}\right] \\ &= W, \end{aligned}$$

$$(4.32) \quad \begin{aligned} A_2 &= g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}\left[u_{;i\bar{q}k\bar{n}\alpha\bar{\beta}}u_{;\bar{j}p\bar{l}m}+u_{;i\bar{q}k\bar{n}\beta}u_{;\bar{j}p\bar{l}m\alpha\bar{\beta}}\right] \\ &\quad + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}\left[u_{;i\bar{n}kp\alpha\bar{\beta}}u_{;\bar{j}m\bar{l}\bar{q}}+u_{;i\bar{n}kp\beta}u_{;\bar{j}m\bar{l}\bar{q}\alpha\bar{\beta}}\right], \end{aligned}$$

$$(4.33)$$

$$\begin{aligned} A_3 &= -g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{\delta}k\bar{n}\alpha}u_{;\bar{q}\gamma\bar{\beta}}u_{;\bar{j}p\bar{l}m}+u_{;i\bar{q}k\bar{\delta}\alpha}u_{;\bar{n}\gamma\bar{\beta}}u_{;\bar{j}p\bar{l}m}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{q}k\bar{n}\alpha}u_{;\bar{j}\gamma\bar{\beta}}u_{;\bar{\delta}p\bar{l}m}+u_{;i\bar{q}k\bar{n}\alpha}u_{;\bar{l}\gamma\bar{\beta}}u_{;\bar{j}p\bar{\delta}m}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{\delta}k\bar{n}}u_{;\bar{q}\gamma\bar{\beta}}u_{;\bar{j}p\bar{l}m\alpha}+u_{;i\bar{q}k\bar{\delta}}u_{;\bar{n}\gamma\bar{\beta}}u_{;\bar{j}p\bar{l}m\alpha}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{q}k\bar{n}}u_{;\bar{j}\gamma\bar{\beta}}u_{;\bar{\delta}p\bar{l}m\alpha}+u_{;i\bar{q}k\bar{n}}u_{;\bar{l}\gamma\bar{\beta}}u_{;\bar{j}p\bar{\delta}m\alpha}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{q}k\bar{n}\beta}u_{;i\bar{\delta}\alpha}u_{;\bar{j}p\bar{l}m}+u_{;\bar{j}p\bar{l}m\beta}u_{;i\bar{\delta}\alpha}u_{;\gamma\bar{q}k\bar{n}}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{q}\gamma\bar{n}\beta}u_{;k\bar{\delta}\alpha}u_{;\bar{j}p\bar{l}m}+u_{;\bar{j}p\bar{l}m\beta}u_{;k\bar{\delta}\alpha}u_{;i\bar{q}\gamma\bar{n}}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{q}k\bar{n}\beta}u_{;p\bar{\delta}\alpha}u_{;\bar{j}\gamma\bar{l}m}+u_{;\bar{j}\gamma\bar{l}m\beta}u_{;p\bar{\delta}\alpha}u_{;i\bar{q}k\bar{n}}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{q}k\bar{n}\beta}u_{;m\bar{\delta}\alpha}u_{;\bar{j}p\bar{l}\gamma}+u_{;\bar{j}p\bar{l}\gamma\beta}u_{;m\bar{\delta}\alpha}u_{;i\bar{q}k\bar{n}}\right], \end{aligned}$$

$$(4.34)$$

$$\begin{aligned} A_4 &= -g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{\delta}kp\alpha}u_{;\bar{n}\gamma\bar{\beta}}u_{;\bar{j}m\bar{l}\bar{q}}+u_{;i\bar{n}kp\alpha}u_{;\bar{j}\gamma\bar{\beta}}u_{;\bar{\delta}m\bar{l}\bar{q}}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{n}kp\alpha}u_{;\bar{l}\gamma\bar{\beta}}u_{;\bar{j}m\bar{\delta}\bar{q}}+u_{;i\bar{n}kp\alpha}u_{;\bar{q}\gamma\bar{\beta}}u_{;\bar{j}m\bar{l}\bar{\delta}}\right] \\ &\quad - g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left[u_{;i\bar{\delta}kp}u_{;\bar{n}\gamma\bar{\beta}}u_{;\bar{j}m\bar{l}\bar{q}\alpha}+u_{;i\bar{n}kp}u_{;\bar{j}\gamma\bar{\beta}}u_{;\bar{\delta}m\bar{l}\bar{q}\alpha}\right] \end{aligned}$$



and

(4.37)

$$\begin{aligned}
A_7 = & g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;i\bar{\alpha}} u_{;\bar{\delta}s\bar{\beta}} u_{;\gamma\bar{n}k p} u_{;\bar{j}m\bar{l}\bar{q}} + u_{;i\bar{\delta}\alpha} u_{;\bar{n}s\bar{\beta}} u_{;\gamma\bar{l}k p} u_{;\bar{j}m\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;i\bar{\delta}\alpha} u_{;\bar{j}s\bar{\beta}} u_{;\gamma\bar{n}k p} u_{;\bar{l}m\bar{l}\bar{q}} + u_{;i\bar{\delta}\alpha} u_{;\bar{l}s\bar{\beta}} u_{;\gamma\bar{n}k p} u_{;\bar{j}m\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;i\bar{\delta}\alpha} u_{;\bar{q}s\bar{\beta}} u_{;\gamma\bar{n}k p} u_{;\bar{j}m\bar{l}\bar{t}} + u_{;k\bar{t}\alpha} u_{;\bar{\delta}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;k\bar{\delta}\alpha} u_{;\bar{n}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} + u_{;k\bar{\delta}\alpha} u_{;\bar{j}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{l}m\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;k\bar{\delta}\alpha} u_{;\bar{l}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} + u_{;k\bar{\delta}\alpha} u_{;\bar{q}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{t}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;p\bar{t}\alpha} u_{;\bar{\delta}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} + u_{;p\bar{\delta}\alpha} u_{;\bar{n}s\bar{\beta}} u_{;\bar{i}t\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;p\bar{\delta}\alpha} u_{;\bar{j}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{l}m\bar{l}\bar{q}} + u_{;p\bar{\delta}\alpha} u_{;\bar{l}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;p\bar{\delta}\alpha} u_{;\bar{q}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}m\bar{l}\bar{t}} + u_{;m\bar{l}\alpha} u_{;\bar{\delta}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}\gamma\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;m\bar{\delta}\alpha} u_{;\bar{j}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{l}\gamma\bar{l}\bar{q}} + u_{;m\bar{\delta}\alpha} u_{;\bar{l}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}\gamma\bar{l}\bar{q}} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[ u_{;m\bar{\delta}\alpha} u_{;\bar{q}s\bar{\beta}} u_{;\bar{i}n\gamma p} u_{;\bar{j}\gamma\bar{l}\bar{t}} + u_{;m\bar{\delta}\alpha} u_{;\bar{n}s\bar{\beta}} u_{;\bar{i}t\gamma p} u_{;\bar{j}\gamma\bar{l}\bar{q}} \right].
\end{aligned}$$

Now, we estimate each  $A_i$  in the sum of  $\Delta'V$ . Since we have  $C^3$  estimate, that is,  $S$  is bounded, by diagonalizing, we know that each term in the sum  $A_6$  and  $A_7$  is bounded by a constant multiple of  $V$ . So, we have

$$(4.38) \quad |A_6| + |A_7| \leq C_{14}V.$$

Now, we estimate terms in  $A_5$ . We have

$$u_{;i\bar{\delta}\alpha\bar{\beta}} = u_{;\alpha\bar{\beta}i\bar{\delta}} + (u_{;i\bar{\delta}\alpha\bar{\beta}} - u_{;\alpha\bar{\beta}i\bar{\delta}}) = u_{;\alpha\bar{\beta}i\bar{\delta}} + h^{s\bar{t}}(u_{;s\bar{\beta}} R_{\alpha\bar{t}i\bar{\delta}} - u_{;s\bar{\delta}} R_{i\bar{t}\alpha\bar{\beta}}).$$

By using equation (4.17), we have

$$\begin{aligned}
g^{\alpha\bar{\beta}} u_{;i\bar{\delta}\alpha\bar{\beta}} &= g^{\alpha\bar{\beta}} u_{;\alpha\bar{\beta}i\bar{\delta}} + g^{\alpha\bar{\beta}} h^{s\bar{t}}(u_{;s\bar{\beta}} R_{\alpha\bar{t}i\bar{\delta}} - u_{;s\bar{\delta}} R_{i\bar{t}\alpha\bar{\beta}}) \\
&= \mathcal{F}_{i\bar{\delta}} + g^{\alpha\bar{\beta}} g^{s\bar{t}} u_{;\alpha\bar{t}i} u_{;\bar{\beta}s\bar{\delta}} + g^{\alpha\bar{\beta}} h^{s\bar{t}}(u_{;s\bar{\beta}} R_{\alpha\bar{t}i\bar{\delta}} - u_{;s\bar{\delta}} R_{i\bar{t}\alpha\bar{\beta}}).
\end{aligned}$$

Since the curvature of the metric  $h$  is bounded and we have  $C^2$  and  $C^3$  estimate, we know that

$$\left| -g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} u_{;i\bar{\delta}\alpha\bar{\beta}} u_{;\gamma\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} \right| \leq CV.$$

Similarly, we can compute other terms in the sum  $A_5$ . So, we have

$$(4.39) \quad |A_5| \leq C_{15}V.$$

By combining formulas (4.30), (4.38) and (4.39), we have

$$(4.40) \quad \Delta' V \geq A_1 + A_2 + A_3 + A_4 - C_{16}V.$$

Now, we deal with terms in  $A_2$ . By differentiating formula (4.18) in a holomorphic direction or an anti-holomorphic direction, we have

$$(4.41) \quad \begin{aligned} & g^{i\bar{j}} u_{;i\bar{j}k\bar{l}\alpha\bar{\beta}} \\ &= g^{i\bar{j}} g^{p\bar{q}} \left[ u_{;i\bar{q}k\bar{l}\alpha} u_{;\bar{j}p\bar{\beta}} + u_{;i\bar{q}k\alpha\bar{\beta}} u_{;\bar{j}p\bar{l}} + u_{;p\bar{j}k\bar{l}\bar{\beta}} u_{;i\bar{q}\alpha} + u_{;\bar{j}p\bar{l}\alpha\bar{\beta}} u_{;i\bar{q}k} \right] \\ &+ g^{i\bar{j}} g^{p\bar{q}} \left[ u_{;p\bar{j}k\bar{l}} u_{;i\bar{q}\alpha\bar{\beta}} + u_{;\bar{j}p\bar{l}\alpha} u_{;i\bar{q}k\bar{\beta}} + u_{;i\bar{q}k\alpha} u_{;\bar{j}p\bar{l}\bar{\beta}} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;p\bar{n}k\bar{l}} u_{;\bar{j}m\bar{\beta}} u_{;i\bar{q}\alpha} + u_{;p\bar{j}k\bar{l}} u_{;\bar{q}m\bar{\beta}} u_{;i\bar{n}\alpha} + u_{;\bar{n}p\bar{l}\alpha} u_{;\bar{j}m\bar{\beta}} u_{;i\bar{q}k} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;\bar{j}p\bar{l}\alpha} u_{;\bar{q}m\bar{\beta}} u_{;i\bar{n}k} + u_{;i\bar{n}k\alpha} u_{;\bar{q}m\bar{\beta}} u_{;\bar{j}p\bar{l}} + u_{;i\bar{q}k\alpha} u_{;\bar{j}m\bar{\beta}} u_{;\bar{n}p\bar{l}} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;\bar{j}m\bar{l}\bar{\beta}} u_{;p\bar{n}\alpha} u_{;i\bar{q}k} + u_{;\bar{j}p\bar{l}\bar{\beta}} u_{;i\bar{n}\alpha} u_{;m\bar{q}k} + u_{;p\bar{n}\alpha\bar{\beta}} u_{;\bar{j}m\bar{l}} u_{;i\bar{q}k} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;i\bar{q}k\bar{\beta}} u_{;\bar{j}m\bar{l}} u_{;p\bar{n}\alpha} + u_{;m\bar{q}k\bar{\beta}} u_{;\bar{j}p\bar{l}} u_{;i\bar{n}\alpha} + u_{;i\bar{n}\alpha\bar{\beta}} u_{;\bar{j}p\bar{l}} u_{;m\bar{q}k} \right] \\ &+ g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} g^{s\bar{t}} \left[ u_{;i\bar{q}k} u_{;\bar{j}s\bar{\beta}} u_{;p\bar{n}\alpha} u_{;\bar{t}m\bar{l}} + u_{;i\bar{q}k} u_{;\bar{n}s\bar{\beta}} u_{;p\bar{t}\alpha} u_{;\bar{j}m\bar{l}} \right] \\ &+ g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} g^{s\bar{t}} \left[ u_{;i\bar{t}k} u_{;\bar{q}s\bar{\beta}} u_{;p\bar{n}\alpha} u_{;\bar{j}m\bar{l}} + u_{;m\bar{t}k} u_{;\bar{q}s\bar{\beta}} u_{;i\bar{n}\alpha} u_{;\bar{j}p\bar{l}} \right] \\ &+ g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} g^{s\bar{t}} \left[ u_{;m\bar{q}k} u_{;\bar{n}s\bar{\beta}} u_{;i\bar{t}\alpha} u_{;\bar{j}p\bar{l}} + u_{;m\bar{q}k} u_{;\bar{j}s\bar{\beta}} u_{;i\bar{n}\alpha} u_{;\bar{t}p\bar{l}} \right] \\ &+ u_{;k\bar{l}\alpha\bar{\beta}} + F_{;k\bar{l}\alpha\bar{\beta}} \end{aligned}$$

and

$$(4.42) \quad \begin{aligned} & g^{i\bar{j}} u_{;i\bar{j}k\bar{l}\alpha\gamma} \\ &= g^{i\bar{j}} g^{p\bar{q}} \left[ u_{;p\bar{j}k\bar{l}\alpha} u_{;i\bar{q}\gamma} + u_{;p\bar{j}k\bar{l}\gamma} u_{;i\bar{q}\alpha} + u_{;i\bar{q}k\alpha\gamma} u_{;\bar{j}p\bar{l}} + u_{;\bar{j}p\bar{l}\alpha\gamma} u_{;i\bar{q}k} \right] \\ &+ g^{i\bar{j}} g^{p\bar{q}} \left[ u_{;p\bar{j}k\bar{l}} u_{;i\bar{q}\alpha\gamma} + u_{;\bar{j}p\bar{l}\alpha} u_{;i\bar{q}k\gamma} + u_{;i\bar{q}k\alpha} u_{;\bar{j}p\bar{l}\gamma} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;m\bar{j}k\bar{l}} u_{;p\bar{n}\gamma} u_{;i\bar{q}\alpha} + u_{;p\bar{j}k\bar{l}} u_{;i\bar{n}\gamma} u_{;m\bar{q}\alpha} + u_{;\bar{j}m\bar{l}\alpha} u_{;p\bar{n}\gamma} u_{;i\bar{q}k} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;\bar{j}p\bar{l}\alpha} u_{;i\bar{n}\gamma} u_{;m\bar{q}k} + u_{;m\bar{q}k\alpha} u_{;i\bar{n}\gamma} u_{;\bar{j}p\bar{l}} + u_{;i\bar{q}k\alpha} u_{;p\bar{n}\gamma} u_{;\bar{j}m\bar{l}} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;i\bar{q}k\gamma} u_{;p\bar{n}\alpha} u_{;\bar{j}m\bar{l}} + u_{;p\bar{n}\alpha\gamma} u_{;i\bar{q}k} u_{;\bar{j}m\bar{l}} + u_{;\bar{j}m\bar{l}\gamma} u_{;i\bar{q}k} u_{;p\bar{n}\alpha} \right] \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left[ u_{;m\bar{q}k\gamma} u_{;i\bar{n}\alpha} u_{;\bar{j}p\bar{l}} + u_{;i\bar{n}\alpha\gamma} u_{;m\bar{q}k} u_{;\bar{j}p\bar{l}} + u_{;\bar{j}p\bar{l}\gamma} u_{;m\bar{q}k} u_{;i\bar{n}\alpha} \right] \\ &+ g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} g^{s\bar{t}} \left[ u_{;i\bar{t}\gamma} u_{;s\bar{q}k} u_{;p\bar{n}\alpha} u_{;\bar{j}m\bar{l}} + u_{;p\bar{t}\gamma} u_{;i\bar{q}k} u_{;s\bar{n}\alpha} u_{;\bar{j}m\bar{l}} \right] \end{aligned}$$

$$\begin{aligned}
& + g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} g^{s\bar{t}} \left[ u_{;m\bar{t}\gamma} u_{;i\bar{q}k} u_{;p\bar{n}\alpha} u_{;\bar{j}s\bar{l}} + u_{;m\bar{t}\gamma} u_{;s\bar{q}k} u_{;i\bar{n}\alpha} u_{;\bar{j}p\bar{l}} \right] \\
& + g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} g^{s\bar{t}} \left[ u_{;i\bar{t}\gamma} u_{;m\bar{q}k} u_{;s\bar{n}\alpha} u_{;\bar{j}p\bar{l}} + u_{;p\bar{t}\gamma} u_{;m\bar{q}k} u_{;i\bar{n}\alpha} u_{;\bar{j}s\bar{l}} \right] \\
& + u_{;k\bar{l}\alpha\gamma} + F_{;k\bar{l}\alpha\gamma}.
\end{aligned}$$

By using a similar computation as in [22], we have

$$(4.43) \quad u_{;i\bar{q}k\bar{n}\alpha\bar{\beta}} \stackrel{4}{\simeq} u_{;\alpha\bar{\beta}i\bar{q}k\bar{n}},$$

$$(4.44) \quad u_{;\bar{j}p\bar{l}m\alpha\bar{\beta}} \stackrel{4}{\simeq} u_{;\beta\alpha\bar{j}p\bar{l}m},$$

$$(4.45) \quad u_{;i\bar{n}k p\alpha\bar{\beta}} \stackrel{4}{\simeq} u_{;\alpha\bar{\beta}i\bar{n}k p}$$

and

$$(4.46) \quad u_{;\bar{j}m\bar{l}q\alpha\bar{\beta}} \stackrel{4}{\simeq} u_{;\beta\alpha\bar{j}m\bar{l}q}$$

which imply that

$$(4.47) \quad g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;i\bar{q}k\bar{n}\alpha\bar{\beta}} u_{;\bar{j}p\bar{l}m} \stackrel{4}{\cong} g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\alpha\bar{\beta}i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m},$$

$$(4.48) \quad g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\bar{j}p\bar{l}m\alpha\bar{\beta}} u_{;i\bar{q}k\bar{n}} \stackrel{4}{\cong} g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\beta\alpha\bar{j}p\bar{l}m} u_{;i\bar{q}k\bar{n}},$$

$$(4.49) \quad g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;i\bar{n}k p\alpha\bar{\beta}} u_{;\bar{j}m\bar{l}q} \stackrel{4}{\cong} g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\alpha\bar{\beta}i\bar{n}k p} u_{;\bar{j}m\bar{l}q}$$

and

$$(4.50) \quad g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\bar{j}m\bar{l}q\alpha\bar{\beta}} u_{;i\bar{n}k p} \stackrel{4}{\cong} g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\beta\alpha\bar{j}m\bar{l}q} u_{;i\bar{n}k p}.$$

By using equations (4.41), (4.42) and their conjugations, we have

$$(4.51) \quad \left| g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\alpha\bar{\beta}i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} - T_1 \right| \leq C_{17} V^{\frac{3}{2}} + C_{18} V + C_{19} V^{\frac{1}{2}},$$

$$(4.52) \quad \left| g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\beta\alpha\bar{j}p\bar{l}m} u_{;i\bar{q}k\bar{n}} - T_2 \right| \leq C_{17} V^{\frac{3}{2}} + C_{18} V + C_{19} V^{\frac{1}{2}},$$

$$(4.53) \quad \left| g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\alpha\bar{\beta}i\bar{n}k p} u_{;\bar{j}m\bar{l}q} - T_3 \right| \leq C_{17} V^{\frac{3}{2}} + C_{18} V + C_{19} V^{\frac{1}{2}}$$

and

$$(4.54) \quad \left| g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} u_{;\beta\alpha\bar{j}m\bar{l}q} u_{;i\bar{n}k p} - T_4 \right| \leq C_{17} V^{\frac{3}{2}} + C_{18} V + C_{19} V^{\frac{1}{2}}$$



where

(4.55)

$$T_1 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;i\bar{q}k\bar{n}\alpha} u_{;\bar{j}p\bar{\delta}} u_{;\bar{l}m\bar{\beta}\gamma} + u_{;i\bar{n}kp\bar{\beta}} u_{;\bar{j}m\bar{\delta}} u_{;\bar{l}\gamma\bar{q}\alpha} \right] \\ + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;i\bar{q}k\bar{n}\beta} u_{;p\bar{j}\gamma} u_{;\bar{l}m\bar{\delta}\alpha} + u_{;i\bar{q}k\bar{n}\beta} u_{;\bar{j}p\bar{l}m\bar{\beta}} u_{;i\bar{q}\gamma} u_{;\bar{\delta}k\bar{n}\alpha} \right],$$

(4.56)

$$T_2 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;\bar{j}p\bar{l}m\bar{\beta}} u_{;i\bar{q}\gamma} u_{;k\bar{n}\alpha\bar{\delta}} + u_{;\bar{j}m\bar{l}\bar{q}\alpha} u_{;i\bar{n}\gamma} u_{;k\bar{\delta}p\bar{\beta}} \right] \\ + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;\bar{j}p\bar{l}m\alpha} u_{;\bar{q}i\bar{\delta}} u_{;k\bar{n}\gamma\bar{\beta}} + u_{;i\bar{q}k\bar{n}\alpha} u_{;\bar{j}p\bar{\delta}} u_{;\gamma\bar{l}m\bar{\beta}} \right],$$

(4.57)

$$T_3 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;i\bar{q}k\bar{n}\alpha} u_{;p\bar{j}\gamma} u_{;\bar{l}m\bar{\beta}\bar{\delta}} + u_{;i\bar{n}kp\alpha} u_{;\bar{j}m\bar{\delta}} u_{;\bar{l}\gamma\bar{q}\bar{\beta}} \right] \\ + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;i\bar{q}k\bar{n}\alpha} u_{;p\bar{j}\gamma} u_{;\bar{l}m\bar{\delta}\bar{\beta}} + u_{;\bar{j}p\bar{l}m\alpha} u_{;i\bar{q}\gamma} u_{;\bar{\delta}k\bar{n}\bar{\beta}} \right]$$

and

(4.58)

$$T_4 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;\bar{j}p\bar{l}m\bar{\beta}} u_{;\bar{q}i\bar{\delta}} u_{;k\bar{n}\alpha\gamma} + u_{;\bar{j}m\bar{l}\bar{q}\bar{\beta}} u_{;i\bar{n}\gamma} u_{;k\bar{\delta}p\alpha} \right] \\ + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left[ u_{;\bar{j}p\bar{l}m\bar{\beta}} u_{;\bar{q}i\bar{\delta}} u_{;k\bar{n}\gamma\alpha} + u_{;i\bar{q}k\bar{n}\beta} u_{;\bar{j}p\bar{\delta}} u_{;\gamma\bar{l}m\alpha} \right].$$

Now, we choose local coordinates such that  $g_{i\bar{j}} = \delta_{ij}$ . By combining formulas (4.40), (4.33), (4.34) and (4.47)–(4.58), we have

$$(4.59) \quad \Delta' V \geq \sum_{i,p,k,m,\alpha} \left[ |A_{ipk m\alpha}|^2 + |B_{ipk m\alpha}|^2 + |C_{ipk m\alpha}|^2 + |D_{ipk m\alpha}|^2 \right] \\ - C_{20} V^{\frac{3}{2}} - C_{21} V - C_{22} V^{\frac{1}{2}} \\ = \widetilde{W} - C_{20} V^{\frac{3}{2}} - C_{21} V - C_{22} V^{\frac{1}{2}}.$$

Now, we estimate  $|\nabla' V|$ . For each fixed  $\alpha$ , by (4.29), we have

$$(4.60) \quad \partial_\alpha V = \sum_{i,p,k,m} \left[ A_{ipk m\alpha} u_{;i\bar{p}k\bar{m}} + B_{ipk m\alpha} u_{;i\bar{p}k\bar{m}} \right. \\ \left. + C_{ipk m\alpha} u_{;i\bar{m}k\bar{p}} + D_{ipk m\alpha} u_{;i\bar{m}k\bar{p}} \right] + X_\alpha$$

where

$$|X_\alpha| \leq CV.$$

By using the Schwarz inequality, it is easy to see that

$$|\partial_\alpha V| \leq \sqrt{2} V^{\frac{1}{2}} \widetilde{W}^{\frac{1}{2}} + CV$$

which implies

$$(4.61) \quad \left| \nabla' V \right| \leq C_{23} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} + C_{24} V.$$

Now, we estimate the derivatives of  $S$ . By using a similar computation as above, we have

$$(4.62) \quad \begin{aligned} \partial_\alpha S = & g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \left( u_{;i\bar{q}k\alpha} u_{;\bar{j}p\bar{l}} + u_{;\bar{j}p\bar{l}\alpha} u_{;i\bar{q}k} \right) \\ & - g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} \left( u_{;m\bar{q}k} u_{;i\bar{n}\alpha} u_{;\bar{j}p\bar{l}} + u_{;i\bar{q}m} u_{;k\bar{n}\alpha} u_{;\bar{j}p\bar{l}} + u_{;i\bar{q}k} u_{;p\bar{n}\alpha} u_{;\bar{j}m\bar{l}} \right). \end{aligned}$$

Since  $S$  is bounded, we have

$$|\partial_\alpha S| \leq C V^{\frac{1}{2}} + \tilde{C}$$

which implies

$$(4.63) \quad \left| \nabla' S \right| \leq C_{25} V^{\frac{1}{2}} + C_{26}.$$

By Yau's work in [22], we have

$$(4.64) \quad \Delta' S \geq V - C_{27} V^{\frac{1}{2}} - C_{28}.$$

From (4.59), we have

$$(4.65) \quad \begin{aligned} (S + \kappa) \Delta' V & \geq (S + \kappa) \left( \widetilde{W} - C_{20} V^{\frac{3}{2}} - C_{21} V - C_{22} V^{\frac{1}{2}} \right) \\ & \geq \kappa \widetilde{W} - C_{29} V^{\frac{3}{2}} - C_{30} V - C_{31} V^{\frac{1}{2}} \end{aligned}$$

where the constants  $C_{29}$ ,  $C_{30}$  and  $C_{31}$  depend on  $\kappa$ . By (4.64), (4.61) and (4.64), we also have

$$(4.66) \quad V \Delta' S \geq V^2 - C_{27} V^{\frac{3}{2}} - C_{28} V$$

and

$$(4.67) \quad 2 \left| \nabla' S \right| \left| \nabla' V \right| \leq C_{32} \widetilde{W}^{\frac{1}{2}} V + C_{33} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} + C_{34} V^{\frac{3}{2}} + C_{35} V.$$

Combining (4.28), (4.65), (4.66) and (4.67), we have

$$(4.68) \quad \begin{aligned} \Delta' [(S + \kappa) V] & \geq \kappa \widetilde{W} + V^2 - C_{32} \widetilde{W}^{\frac{1}{2}} V - C_{33} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} \\ & \quad - C_{36} V^{\frac{3}{2}} - C_{37} V - C_{38} V^{\frac{1}{2}} \end{aligned}$$

where constants  $C_{36}$ ,  $C_{37}$  and  $C_{38}$  depend on  $\kappa$ . Now, we fix a  $\kappa$  such that

$$\kappa \geq \max\{3C_{32}^2, 3C_{33}^2, 1\}.$$

We have

$$\frac{\kappa}{3} \widetilde{W} - C_{32} \widetilde{W}^{\frac{1}{2}} V + \frac{1}{4} V^2 \geq 0$$

and

$$\frac{\kappa}{3}\widetilde{W} - C_{33}\widetilde{W}^{\frac{1}{2}}V^{\frac{1}{2}} \geq -\frac{1}{4}V.$$

With this choice of  $\kappa$ , by formula (4.68), we have

$$\begin{aligned} (4.69) \quad \Delta' [(S + \kappa)V] &\geq \frac{\kappa}{3}\widetilde{W} + \frac{3}{4}V^2 - C_{39}V^{\frac{3}{2}} - C_{40}V - C_{41}V^{\frac{1}{2}} \\ &\geq C_{42}[(S + \kappa)V]^2 - C_{43}[(S + \kappa)V]^{\frac{3}{2}} \\ &\quad - C_{44}[(S + \kappa)V] - C_{45}[(S + \kappa)V]^{\frac{1}{2}} \end{aligned}$$

since  $S + \kappa \geq \kappa \geq 1$  and  $S + \kappa$  is bounded from above uniformly.

By the work of Cheng and Yau in [3], we know inequality (4.69) implies that  $(S + \kappa)V$  is bounded. This implies  $V$  is bounded since  $V \leq (S + \kappa)V$ . Thus, we obtain the  $C^4$  estimate. By formula (4.16), we know that the curvature of the Kähler–Einstein metric  $g$  is bounded. This finishes the proof of Theorem 4.1.

Now, we briefly describe how to control the covariant derivatives of the curvature of the Kähler–Einstein metric.

Firstly, by differentiating equation (4.16), we see that the boundedness of the derivatives of the  $P_{i\bar{j}k\bar{l}}$  is equivalent to the boundedness of the covariant derivatives of  $u$  with respect to the background metric. Furthermore, the derivatives involved are at least order 2 and were taken in at least one holomorphic direction and one anti-holomorphic direction.

To bound such  $k$ -th order derivatives of  $u$ , we form the quantity  $S_k$  such that  $S_3 = S$ ,  $S_4 = V$  and  $S_5 = W$ . In general, if we fix normal coordinates with respect to the Kähler–Einstein metric at one point, then  $S_k$  is a sum of square of terms where each term is a covariant derivative of  $u$  and the derivative is described above. All terms are obtained in the following way:

For each covariant derivative of  $u$  whose square appeared in the sum  $S_{k-1}$ , we take covariant derivative of this term with respect to the background metric in  $z_\alpha$  and  $\bar{z}_\beta$  respectively. Then, we obtain two terms whose square appear in the sum  $S_k$ . It is easy to see that  $S_k$  is a sum of  $2^{n-3}$  squares of certain covariant derivatives of  $u$  where the derivatives are of the type described above.

It is clear that the covariant derivatives of the curvature of the Kähler–Einstein metric is bounded is equivalent to the fact that the quantities  $S_k$  is bounded.

We now estimate  $S_k$  inductively. Assume  $S_l$  is bounded for any  $l \leq k - 1$ , we compute

$$\Delta' ((S_{k-1} + \kappa)S_k)$$

where  $\kappa$  is a large constant. Similar to inequality (4.28), we have

$$\Delta'((S_{k-1} + \kappa)S_k) \geq (S_{k-1} + \kappa)\Delta'S_k + S_k\Delta'S_{k-1} - 2\left|\nabla'S_{k-1}\right|\left|\nabla'S_k\right|.$$

In the above formula, the leading term of  $\Delta'S_{k-1}$  is  $S_k$  as we did in formula (4.59) and the term  $\nabla'S_{k-1}$  is of order  $S_k^{\frac{1}{2}}$  as we know in formula (4.63). Similarly, the term  $\nabla'S_k$  is of order  $S_{k+1}^{\frac{1}{2}}S_k^{\frac{1}{2}}$  as we did in formula (4.61). When we compute  $\Delta'S_k$ , the leading term is  $S_{k+1}$ . However, there will be products of  $(k+2)$ -th order derivatives of  $u$  and  $k$ -th order derivatives of  $u$ . We can reduce the  $(k+2)$ -th order derivatives of  $u$  by using the Monge–Ampère equation as we did in formulas (4.41)–(4.58). That is, by differentiating equation (4.17) successively and by switching the order of derivatives, we see that these products are of order at most  $S_{k+1}^{\frac{1}{2}}S_k^{\frac{1}{2}} + S_k^{\frac{3}{2}}$ .

By using similar argument as above, finally, we can derive an inequality of form (4.69) when  $\kappa$  is large enough. By using Cheng–Yau’s work, we conclude that  $S_k$  is bounded. The computation is very long but straightforward. We omit it here for simplicity. q.e.d.

As a direct corollary, we have

**Corollary 4.1.** *The injectivity radius of the Kähler–Einstein metric on the Teichmüller space is bounded from below. Thus, the Teichmüller space equipped with this metric has bounded geometry.*

This corollary can be proved in the same way as Corollary 2.2 by using the above theorem.

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