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HIGHER DIRECT IMAGES OF LOG CANONICAL DIVISORS

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Abstract

In this paper, we investigate higher direct images of log canonical divisors. After we reformulate Kollár's torsion-free theorem, we treat the relationship between higher direct images of log canonical divisors and the canonical extensions of Hodge filtration of gradedly polarized variations of mixed Hodge structures. As a corollary, we obtain a logarithmic version of Fujita–Kawamata's semi-positivity theorem. The final section is an appendix, which is a result of Morihiko Saito.

1. Introduction

In this paper, we investigate higher direct images of log canonical divisors.

First, we reformulate Kollár's torsion-free theorem and vanishing theorem. This part is more or less known to experts. See [**4**] and [**1**, Section 3]. However, we explain the details since there are no appropriate references for our purposes and torsion-freeness will play an important role in this paper.

Next, we treat the relationship between higher direct images of log canonical divisors and the canonical extensions of Hodge filtration of gradedly polarized variations of mixed Hodge structures.

Let $f : X \longrightarrow Y$ be a surjective morphism between non-singular projective varieties and D a simple normal crossing divisor on X. We assume that D is *strongly horizontal* (see Definition 1.6) with respect to f. Then, under some suitable assumptions, $R^i f_* \omega_{X/Y}(D)$ is characterized as the (upper) canonical extension of the bottom Hodge filtration of the suitable polarized variation of mixed Hodge structures. When $D = 0$, it is the theorem of Kollár and Nakayama (see [**15**, Theorem

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2.6] and [**20**, Theorem 1]). If Y is a curve, then the above theorem immediately follows from the study of the gradedly polarized variation of mixed Hodge structures by Steenbrink and Zucker (see [**30**, Section 5 The geometric case]). By this characterization, it is not difficult to see that $R^i f_* \omega_{X/Y}(D)$ is semi-positive on some monodromy conditions. It is a logarithmic version of Fujita–Kawamata's semi-positivity theorem.

We treat no applications to make this paper short and readable. We hope that this paper will provide us with the fundamental techniques for the study of log canonical pairs. One of the main purposes of this paper is to update the old, but important results from the recent studies of the log Minimal Model Program (cf. [**29**]). We believe that this paper is indispensable for the log Minimal Model Program.

Finally, Section 4 is an appendix, which is a result of Morihiko Saito.

Remark 1.1. I removed Sections 4 and 5 from the preprint version of this paper [**6**], where we treated applications of our logarithmic generalization of semi-positivity theorem (Theorem 3.9), to shorten this paper according to the editor's recommendation. I will publish Sections 4 and 5 in [**6**] elsewhere.

1.1. Main results. Let us explain the results of this paper more precisely. We will work over $\mathbb{C}$, the complex number field, throughout this paper.

1.1.1. In Section 2, we reformulate Kollár's torsion-free theorem.

Theorem (cf. Theorems 2.1 and 2.2)**.** *Let $f : X \longrightarrow Y$ be a surjective morphism between projective varieties. Assume that X is non-singular and $D := \sum_{i \in I} D_i$ a simple normal crossing divisor on X. We assume that D is* strongly horizontal *with respect to f, that is, every irreducible component of $D_{i_1} \cap D_{i_2} \cap \ldots \cap D_{i_k}$, where $\{i_1, \ldots, i_k\} \subset I$, is dominant onto Y (see* Definition 1.6*). Then, $R^i f_* \omega_{X/Y}(D)$ is torsion-free.*

This is a special case of [Theorem 3.2(i)] in [**1**]. That theorem is much more general than this one. We explain in detail and give a precise proof. Our proof is a modification of Arapura's argument [**2**, Theorem 1] and relies on the theory of (geometric) variation of mixed Hodge structures over curves. So, it is a warm-up to the rest of section 2.

In Section 2.2, we treat a slight generalization of Kollár's vanishing theorem (see Theorem 2.6). We note that we do not use it later. Thus, we omit it here.

1.1.2. Section 3 is the main part of this paper. It is a logarithmic generalization of the theorem of Kollár and Nakayama. As a corollary, we obtain a logarithmic generalization of Fujita–Kawamata's semi-positivity theorem.

Theorem (cf. Theorems 3.1 and 3.4)**.** *Let $f : X \longrightarrow Y$ be a surjective morphism between non-singular projective varieties and D a simple normal crossing divisor on X, which is strongly horizontal with respect to f. Let Σ be a simple normal crossing divisor on Y. We put $Y_0 := Y \setminus \Sigma$. If f is smooth and D is relatively normal crossing over Y_0, then $R^i f_* \omega_{X/Y}(D)$ is the upper canonical extension of the bottom Hodge filtration. In particular, it is locally free.*

Note that on the above assumptions, we have a (geometric) variation of mixed Hodge structures on Y_0. Our theorem is a direct consequence of [**30**, Section 5] when Y is a curve. If $D = 0$, then it is the theorem of Kollár and Nakayama (see [**15**, Theorem 2.6] and [**20**, Theorem 1]). A key point of our proof is the torsion-freeness of $R^i f_* \omega_{X/Y}(D)$ that is obtained in Section 2.

We put $X_0 := f^{-1}(Y_0)$, $D_0 := D \cap X_0$, $f_0 := f|_{X_0}$, and $d := \dim X - \dim Y$.

Theorem (cf. Theorem 3.9)**.** *We further assume that all the local monodromies on the local system $R^{d+i} f_{0*} \mathbb{C}_{X_0 - D_0}$ around every irreducible component of Σ are unipotent, then $R^i f_* \omega_{X/Y}(D)$ is a semi-positive vector bundle.*

As stated above, it is a logarithmic version of Fujita–Kawamata's semi-positivity theorem. This theorem will play crucial roles in Section 4 in [**6**]. We note that Kawamata obtained another logarithmic generalization of Fujita–Kawamata's semi-positivity theorem (see [**10**, Theorem 32]). For the relationships between his result and Theorem 3.9, see Remark 3.10.

1.1.3. Section 4 is an appendix, remarking on Section 3. After I finished the preliminary version of this paper, I asked Professor Morihiko Saito about the topic in Section 3. I received an e-mail [**25**] from him, where he gave a different proof (Proposition 2 in 4.1) to Theorems 3.1 and 3.4. It depends on the theory of mixed Hodge Modules [**26**], [**27**]. I added it in this paper as an appendix. Note that I made no contribution to Section 4.

1.1.4. Section 1.2 recollects some basic definitions and fixes our notation. We also recall some vanishing theorems. We recommend the readers to skip Section 1.2 for the first reading.

1.2. Preliminaries. Let us recall the basic definitions and fix our notation (cf. [**12**], [**17**], and [**18**]). We also recall some vanishing theorems. We note that we will work over $\mathbb{C}$, the complex number field, throughout this paper.

Definition 1.2 (Canonical divisor)**.** Let X be a normal variety. The *canonical divisor* K_X is defined so that its restriction to the regular part of X is a divisor of a regular n-form. The reflexive sheaf of rank one $\omega_X := \mathcal{O}_X(K_X)$ corresponding to K_X is called the *canonical sheaf*.

The following is the definition of singularities of pairs. Note that the definitions in [**12**] or [**17**] are slightly different from ours (see [**7**]).

Definition 1.3 (Discrepancies and singularities for pairs)**.** Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a proper birational morphism from a normal variety Y. Then, we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, D) \in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define

$$\operatorname{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

On the assumption that $d_i \leq 1$ for every i, we say that (X, D) is

$$\begin{cases} \text{sub klt} \\ \text{sub lc} \end{cases} \quad \text{if} \quad \operatorname{discrep}(X, D) \quad \begin{cases} > -1 \quad \text{and} \quad \llcorner D \lrcorner \leq 0, \\ \geq -1. \end{cases}$$

If (X, D) is sub klt (resp. sub lc) and D is effective, then we say that (X, D) is *klt* (resp. *lc*). Here, klt (resp. lc) is short for *Kawamata log terminal* (resp. *log canonical*).

Definition 1.4 (Center of lc singularities)**.** Let X be a normal variety and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there exists a proper birational morphism from a normal variety $\mu : Y \longrightarrow X$ and a prime divisor E on Y with the discrepancy coefficient $a(E, X, D) \leq -1$ such that $\mu(E) = W$.

Remark 1.5 (cf. [**17**, Lemmas 2.29, 2.30, and 2.45])**.** Let X be a non-singular variety and D a simple normal crossing divisor on X. Then (X, D) is lc. More precisely, (X, D) is a typical example of dlt pairs (see Remark 1.13 and [**7**]). Let $D = \sum_{i \in I} D_i$ be the irreducible decomposition of D. Then, W is a center of log canonical singularities for the pair (X, D) if and only if W is an irreducible component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ for some $\{i_1, i_2, \ldots, i_k\} \subset I$.

We introduce the following new notion, which will play an important role in this paper. It is a birationally harmless assumption for applications (see Sections 4 and 5 in [**6**]).

Definition 1.6 (Strongly horizontal)**.** Let (X, D) be a sub lc pair and $f : X \longrightarrow Y$ a surjective morphism. If all the centers of log canonical singularities for the pair (X, D) are dominant onto Y, then we call D *strongly horizontal with respect to* f.

Remark 1.7. Let $f : X \longrightarrow Y$ and D be as in Definition 1.6. If (X, D) is sub klt, then D is strongly horizontal with respect to f. It is obvious by the definition. We note that $D = 0$ is strongly horizontal when X is non-singular.

We note the following easy fact.

Lemma 1.8. *Let $f : X \longrightarrow Y$ and D be as in* Definition 1.6*, that is, D is strongly horizontal with respect to f. Let Λ be a free linear system on Y and $V \in \Lambda$ a general member. We put $W := f^{-1}(V)$. Then $D|_W$ is strongly horizontal with respect to $f_W := f|_W : W \longrightarrow V$.*

The following notion was introduced by Reid.

Definition 1.9 (Nef and log big divisors)**.** Let (X, D) be lc and L a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. The divisor L is called *nef and log big* on (X, D) if L is nef and big, and $(L^{\dim W} \cdot W) > 0$ for every center of log canonical singularities W for the pair (X, D). We note that an ample divisor is nef and log big.

We prepare some vanishing theorems. The following lemma is a special case of [**8**, Lemma], which is a variant of the Kawamata–Viehweg vanishing theorem (cf. [**21**]).

Lemma 1.10. *Let X be a non-singular complete variety and D a simple normal crossing divisor on X. Let H be a nef and log big divisor on X. Then, $H^i(X, K_X + D + H) = 0$ for every $i > 0$.*

Proof. If $D = 0$, then $H^i(X, K_X + H) = 0$ for every $i > 0$ by the Kawamata–Viehweg vanishing theorem. So, we can assume that $D \neq 0$. Let D_0 be an irreducible component of D. We consider the following exact sequence:

$$\begin{aligned}\cdots &\to H^i(X, K_X + D - D_0 + H) \to H^i(X, K_X + D + H)\\ &\to H^i(D_0, K_{D_0} + (D - D_0)|_{D_0} + H|_{D_0}) \to \cdots\end{aligned}$$

By the inductions on the number of the irreducible components of D and on $\dim X$, the first and the last terms are zero. Therefore, we obtain that $H^i(X, K_X + D + H) = 0$ for every $i > 0$. q.e.d.

The following proposition is a slight generalization of the Grauert–Riemenschneider vanishing theorem.

Proposition 1.11. *Let $f : X \longrightarrow Y$ be a proper birational morphism from a non-singular variety X. Let D be a simple normal crossing divisor on X, where D may be zero. Assume that f is an isomorphism at every generic point of center of log canonical singularities for the pair (X, D). Then, $R^i f_* \omega_X(D) = 0$ for $i > 0$.*

Proof. If $D = 0$, then it is nothing but the Grauert–Riemenschneider vanishing theorem. So, we can assume that $D \neq 0$. Let D_0 be an irreducible component of D. We consider the following exact sequence:

$$\cdots \to R^i f_* \omega_X(D - D_0) \to R^i f_* \omega_X(D) \to R^i f_* \omega_{D_0}((D - D_0)|_{D_0}) \to \cdots$$

Then $R^i f_* \omega_X(D) = 0$ for every positive i by the same argument as in the proof of Lemma 1.10. q.e.d.

The following corollary is an easy consequence of Proposition 1.11, which will be used in Sections 2 and 3.

Corollary 1.12. *Let $f : Z \longrightarrow X$ be a proper birational morphism from a non-singular variety Z. Let D be a reduced Weil divisor on X such that (X, D) is lc. Assume that D' is the strict transform of D and f is an isomorphism over a Zariski open set U of X such that U contains every generic point of center of log canonical singularities for the pair (X, D). Then, $f_* \omega_Z(D') \simeq \omega_X(D)$.*

We further assume that the f-exceptional locus $\mathrm{Exc}(f)$ *(see* 1.14 (e) *below) and $D' \cup \mathrm{Exc}(f)$ are both simple normal crossing divisors on Z. Then, f is an isomorphism at every generic point of center of log canonical singularities for the pair (Z, D') and $R^i f_* \omega_Z(D') = 0$ for $i > 0$. In particular, f induces a one to one correspondence between the* *generic points of center of log canonical singularities for the pair* (Z, D') *and those for the pair* (X, D).

Proof. First, we write

$$K_Z + D' = f^*(K_X + D) + \sum_i a_i E_i,$$

where E_i is an f-exceptional irreducible Cartier divisor on Z for every i. Since f is an isomorphism over U that contains every generic point of center of log canonical singularities for the pair (X, D), we have that $a_i > -1$ for every i. Thus, we obtain that $f_*\omega_Z(D') \simeq \omega_X(D)$.

Next, let W be a center of log canonical singularities for the pair (Z, D'). Then, $W \not\subset \mathrm{Exc}(f)$ since $\mathrm{Exc}(f)$ and $D' \cup \mathrm{Exc}(f)$ are both simple normal crossing divisors. Therefore, f is an isomorphism at every generic point of center of log canonical singularities for the pair (Z, D'). So, we can apply Proposition 1.11. Thus, we obtain that $R^i f_*\omega_Z(D') = 0$ for $i > 0$. The final statement is obvious by the above arguments. q.e.d.

Remark 1.13 (Divisorial log terminal)**.** The notion of *dlt* pairs may help the readers to understand Corollary 1.12. Here, dlt is short for *divisorial log terminal*. It is one of the most useful variants of *log terminal singularities*. In this paper, however, we do not use it explicitly. So, we omit the details. For the precise definition and the basic properties of dlt pairs, see [**31**], [**17**, Section 2.3], and [**7**].

Finally, we fix the following notation and convention.

Notation and Convention 1.14. Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the set of positive (resp. non-negative) integers.

(a) An *algebraic fiber space* $f : X \longrightarrow Y$ is a proper surjective morphism between non-singular projective varieties X and Y with connected fibers.
(b) Let $f : X \longrightarrow Y$ be a dominant morphism between varieties. We put $\dim f := \dim X - \dim Y$.
(c) The words *locally free sheaf* and *vector bundle* are used interchangeably.
(d) A Cartier (resp. Weil) divisor D on a normal variety X and the associated line bundle (resp. rank one reflexive sheaf) $\mathcal{O}_X(D)$ are used interchangeably if there is no danger of confusion.

(e) Let $f : X \longrightarrow Y$ be a proper birational morphism between normal varieties. By the *exceptional locus* of f, we mean the subset $\{x \in X \mid \dim f^{-1}f(x) \geq 1\}$ of X, and denote it by $\mathrm{Exc}(f)$. We note that $\mathrm{Exc}(f)$ is of pure codimension one in X if f is birational and Y is $\mathbb{Q}$-factorial.

(f) When we use the desingularization theorem, we often forbid unnecessary blow-ups implicitly, that is, we do not use the weak Hironaka theorem, but use the original Hironaka theorem. Unnecessary blow-ups sometimes make the proof more difficult. We recommend the readers to see [**31**, Resolution Lemma], and [**3**, Corollary 7.9, Definition 7.10, and Theorem 7.11], in particular, [**3**, 7.12 The motivation]. Note that the readers have to check the definition of *normal crossing* in [**3**, Definition 2.1]. See also Remark 2.3. My paper [**7**] may help the readers to understand the subtleties of the desingularization theorem and various kinds of log terminal singularities.

(g) Let X be a normal variety and D a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a proper birational morphism $g : Y \longrightarrow X$ such that Y is non-singular, $\mathrm{Exc}(g)$ and $\mathrm{Exc}(g) \cup g^{-1}(\mathrm{Supp}D)$ are both simple normal crossing divisors. See [**17**, Notation 0.4 (10)] and [**7**].

2. Torsion-freeness and Vanishing theorem

In this section, we generalize Kollár's torsion-free theorem and vanishing theorem: [**14**, Theorem 2.1]. The following is the main theorem of this section (see also Theorems 2.2 and 2.6).

Theorem. *Let X be a non-singular projective variety, D a simple normal crossing divisor on X, Y an arbitrary* (*reduced*) *projective variety, and $f : X \longrightarrow Y$ a surjective morphism. Then*

(i) *If H is an ample divisor on Y, then $H^j(Y, H \otimes R^i f_*\omega_{X/Y}(D)) = 0$ for $j > 0$.*

Assume, furthermore, that D is strongly horizontal with respect to f (*see* Definition 1.6). *Then*

(ii) *$R^i f_*\omega_X(D)$ is torsion-free for $i \geq 0$.*

(iii) *$R^i f_*\omega_X(D) = 0$ if $i > \dim f$. Furthermore, if $D \neq 0$, then $R^i f_*\omega_X(D) = 0$ for $i \geq \dim f$.*

The statement (iii) is obvious by (ii). So, it is sufficient to prove (i) and (ii). In Theorem 2.1, we treat (ii) in a more general setting. We will prove (i), which is not used later, in Section 2.2.

2.1. Torsion-free theorem. The following is the main theorem of this subsection. It is a special case of [**1**, Theorem 3.2 (i)]. We adopt Arapura's proof of torsion-freeness (see the proof of Theorem 1 in [**2**]). This proof is suitable for our paper.

Theorem 2.1 (Torsion-freeness)**.** *Let $f : X \longrightarrow Y$ be a surjective morphism from a non-singular projective variety X to a* (*possibly singular*) *projective variety Y. Let D be a simple normal crossing divisor on X. Assume that D is strongly horizontal with respect to f. Let L be a semi-ample line bundle on X. Then, for all i, the sheaves $R^if_*(\omega_X(D)\otimes L)$ are torsion-free. In particular, $R^if_*\omega_X(D)$ is torsion-free for every i.*

Proof. In order to explain the plan of the proof, let us introduce the following notation, where $f : X \longrightarrow Y$ and the divisor D are as in the statement.

$$P_n^{\log} \quad : \quad \begin{cases} \text{If } \dim \operatorname{Supp}(R^if_*\omega_X(D)_{\text{tor}}) \le n \text{ for all } i, \text{ then the} \\ \text{sheaves } R^if_*\omega_X(D) \text{ are torsion-free for all } i, \text{ where} \\ R^if_*\omega_X(D)_{\text{tor}} \text{ is the torsion part of } R^if_*\omega_X(D). \end{cases}$$

$$P^{\log}(Y) \quad : \quad \begin{cases} \text{The sheaf } R^if_*\omega_X(D) \text{ is torsion-free for every } i. \end{cases}$$

$$Q^{\log}(Y) \quad : \quad \begin{cases} \text{Then the sheaf } R^if_*(\omega_X(D)\otimes L) \text{ is torsion-free for} \\ \text{every } i. \end{cases}$$

It is enough to prove the following four claims:

- $P^{\log}(Y)$ implies $Q^{\log}(Y)$.
- $P^{\log}(Y)$ when Y is a curve.
- $Q^{\log}(\mathbb{P}^1)$ implies $P_0^{\log}$.
- $P_{n-1}^{\log}$ implies $P_n^{\log}$.

Step 1 ($P^{\log}(Y)$ implies $Q^{\log}(Y)$)**.** Since L is semi-ample, there exists an $m > 0$ for which L^m is generated by global sections. Hence, by Bertini's theorem, we can find $B \in |L^m|$ such that B is smooth and $B+D$ is a simple normal crossing divisor on X. Let $\pi : Z \longrightarrow X$ be the m-fold cyclic covering branched along B. Then, $\pi_*\omega_Z \simeq \bigoplus_{j=0}^{m-1} \omega_X \otimes L^j$. Therefore, $\omega_X(D)\otimes L$ is a direct summand of $\pi_*\omega_Z(\pi^*D)$. Since π is finite, we have

$$R^i(f\circ\pi)_*\omega_Z(\pi^*D)=R^if_*(\pi_*\omega_Z(\pi^*D))=\bigoplus_{j=0}^{m-1}R^if_*(\omega_X(D)\otimes L^j).$$

By $P^{\log}(Y)$, the left hand side is torsion-free. We note that π^*D is a simple normal crossing divisor on Z and strongly horizontal with respect to $f\circ\pi$. Then, so is $R^if_*(\omega_X(D)\otimes L)$.

Step 2 ($P^{\log}(Y)$ when Y is a curve)**.** Now suppose that Y is a curve. By the Stein factorization, we can assume that Y is smooth. Let Y_0 be a non-empty Zariski open set of Y such that f is smooth and D is relatively simple normal crossing over Y_0. By blowing up X, we can assume that $\mathrm{Supp}(f^{-1}P\cup D)$ is simple normal crossing for $P\in Y\setminus Y_0$ (cf. Corollary 1.12). If $f:X\longrightarrow Y$ is semi-stable, then the theorem follows from [**30**, (5.7)] (see also Theorem 3.2 and Step 1 in the proof of Theorem 3.1). If $f:X\longrightarrow Y$ is not semi-stable, then we apply the semi-stable reduction theorem (cf. [**13**, Chapter II] and [**24**, I.9]). We consider the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\ \alpha\ } & X' & \xleftarrow{\ \beta\ } & \widetilde{X} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ Y & \xleftarrow[\ \tau\ ]{} & Y' & = & Y', \end{array}$$

where $\tau:Y'\to Y$ is a finite morphism from a non-singular projective curve and X' is the normalization of $X\times_Y Y'$, and β is a birational morphism such that $\widetilde{f}:\widetilde{X}\to Y'$ is semi-stable. We note that we can assume that β is an isomorphism over a non-empty Zariski open set of Y' (cf. 1.14 (f)). Then, $R^if_*\omega_X(D)$ is a direct summand of $\tau_*R^if'_*\omega_{X'}(\alpha^*D)$. So, it is sufficient to check the local freeness of $R^if'_*\omega_{X'}(\alpha^*D)$. We note that $R^if'_*\omega_{X'/Y'}(\alpha^*D)\simeq R^i\widetilde{f}_*\omega_{\widetilde{X}/Y'}(D')$, where D' is the strict transform of α^*D (cf. Corollary 1.12). We can assume that $\mathrm{Supp}(\widetilde{f}^{-1}P\cup D')$ is simple normal crossing for every $P\in Y'$ (cf. [**24**, I.9]). Since $\widetilde{f}$ is semi-stable, $R^i\widetilde{f}_*\omega_{\widetilde{X}/Y'}(D')$ is locally free by the above argument. Thus, we obtain that $R^if_*\omega_X(D)$ is locally free for every i.

Step 3 ($Q^{\log}(\mathbb{P}^1)$ implies $P_0^{\log}$)**.** We assume that the sheaf $R^if_*\omega_X(D)$ has torsion supported on a finite set of points $S:=\{p_1,\dots,p_r\}$ for some

i. Now, take a pencil of hyperplane sections of Y. We can assume that the base locus is disjoint from S and that the preimage of the base locus in X is smooth and meets all the centers of log canonical singularities of (X, D) transversely. Blow up the base locus and its preimage in X to get a diagram.

$$\begin{array}{ccccc} f': & X' & \longrightarrow & Y' & \\ & & \searrow & & \swarrow g \\ & & & \mathbb{P}^1 & \end{array}$$

Let $\mathcal{H}$ be an ample line bundle on Y'. Replacing $\mathcal{H}$ by $\mathcal{H}^k$ for some $k \gg 0$, we can assume that $R^p g_*(\mathcal{H} \otimes R^q f'_* \omega_{X'}(D')) = 0$ for all $p > 0$ and for all q, where D' is the strict transform of D on X'. Therefore, the spectral sequence collapses to give isomorphisms $g_*(\mathcal{H} \otimes R^q f'_* \omega_{X'}(D')) = R^q (g \circ f')_*(f'^* \mathcal{H} \otimes \omega_{X'}(D'))$. By $Q^{\log}(\mathbb{P}^1)$, the right-hand side is torsion-free. However, by the assumption, the sheaf $\mathcal{H} \otimes R^i f'_* \omega_{X'}(D')$ has torsion supported at the points $p_j \in Y'$. Therefore, $g_*(\mathcal{H} \otimes R^i f'_* \omega_{X'}(D'))$ has torsion at the points $g(p_j)$. This is a contradiction. Thus, the sheaf $R^i f_* \omega_X(D)$ must be torsion-free.

Step 4 ($P^{\log}_{n-1}$ implies $P^{\log}_n$)**.** Assume that $\dim \mathrm{Supp}(R^i f_* \omega_X(D)_{\mathrm{tor}}) \leq n$ for all i. We suppose that for some i the sheaf $R^i f_* \omega_X(D)$ is not torsion-free. Then, there must be a positive dimensional component of $\mathrm{Supp}(R^i f_* \omega_X(D)_{\mathrm{tor}})$ by $P^{\log}_{n-1}$.

Let $\mathcal{H}$ be a very ample line bundle on Y and let $B \in |\mathcal{H}|$ be a general member such that f^*B is smooth and $f^*B + D$ is a simple normal crossing divisor. Then, $R^i f_* \omega_{f^*B}(D|_{f^*B}) \simeq R^i f_* \omega_X(D) \otimes \mathcal{O}_B(B)$. Applying $P^{\log}_{n-1}$ to $(f^*B, D|_{f^*B}) \longrightarrow B$, we obtain that the left-hand side is torsion-free. This contradicts the assumption that $R^i f_* \omega_X(D)$ has torsion. So, we obtain the required result.

Therefore, we complete the proof. q.e.d.

We can omit the assumption that X and Y are projective in Theorem 2.1. We will use Theorem 2.2 in the proof of Theorem 3.1.

Theorem 2.2. *Let $f : X \longrightarrow Y$ be a projective surjective morphism from a non-singular variety to a (possibly singular) variety. Let D be a simple normal crossing divisor on X. Assume that D is strongly horizontal with respect to f. Let L be a semi-ample line bundle on X. Then, for all i, the sheaves $R^i f_*(\omega_X(D) \otimes L)$ are torsion-free. In particular, $R^i f_* \omega_X(D)$ is torsion-free for every i.*

Proof. By Step 1 in the proof of Theorem 2.1, it is enough to prove that $R^if_*\omega_X(D)$ is torsion-free for every i. Since the statement is local, we can shrink Y and assume that Y is quasi-projective. We can take a suitable compactification and assume that X and Y are both projective (see Remark 2.3). Thus, by Theorem 2.1, we obtain the required result. q.e.d.

Remark 2.3. Here, we had better use Szabó's resolution lemma: [**31**, Resolution Lemma]. See also [**3**, Corollary 7.9, Theorem 7.11]. We recommend the readers to see [**3**, 7.12, The motivation]. Note that their definition of *normal crossing* is slightly different from the usual one (see [**3**, Definition 2.1]). The paper [**7**] may help the readers understand subtleties of desingularization theorems.

The following example implies that if D is not strongly horizontal, then the torsion-freeness is not necessarily true.

Example 2.4. Let Y be a non-singular projective surface and $f : X \longrightarrow Y$ be a blow-up at a point $P \in Y$. We put $D := f^{-1}(P)$. We consider the following exact sequence:

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(D) \longrightarrow \omega_D \longrightarrow 0.$$

Then, we obtain that $R^1f_*\omega_X(D) \simeq R^1f_*\omega_D \simeq H^1(D, \omega_D) \simeq \mathbb{C}_P$ by the Grauert–Riemenschneider vanishing theorem. So, $R^1f_*\omega_X(D)$ is not torsion-free.

Corollary 2.5. *Let $f : X \longrightarrow Y$ and D be as in Theorem 2.2. Assume that L is a relatively ample line bundle on X. Then $R^if_*(\omega_X(D) \otimes L) = 0$ for $i > 0$.*

Proof. It is essentially the same as [**2**, Corollary 2 (i)]. It is sufficient to use Lemma 1.10, instead of the Kodaira vanishing theorem. q.e.d.

2.2. Vanishing theorem. The following is a slight generalization of Kollár's vanishing theorem: [**14**, Theorem 2.1 (iii)]. It is also a special case of [**1**, Theorem 3.2 (ii)]. We adopt the proof of [**16**, Theorem 9.14]. We will not use this vanishing theorem later. So, most readers can skip this subsection.

Theorem 2.6 (Vanishing theorem)**.** *Let $f : X \longrightarrow Y$ be a morphism from a non-singular projective variety X onto a variety Y. Let D be a simple normal crossing divisor on X. Let H be an ample line bundle on Y. Then*

$$H^i(Y, H \otimes R^j f_* \omega_X(D)) = 0$$

for $i > 0$ and $j \geq 0$.

Proof. Let n be a positive integer such that $n \geq 2$ and the linear system $|H^n|$ is base point free. Take a general member $E \in |H^n|$ such that $Z := f^{-1}(E)$ is smooth and $Z \cup D$ is a simple normal crossing divisor. By [4, 5.1 b)],

$$H^i(X, \omega_X(D) \otimes f^* H) \longrightarrow H^i(X, \omega_X(D) \otimes f^* H^{1+kn}) \tag{1}$$

is injective for $i \geq 0$. We prove the theorem by induction on $\dim Y$. The assertion is evident if $\dim Y = 0$. We have an exact sequence:

$$0 \to \omega_X(D) \otimes f^* H^t \to \omega_X(D) \otimes f^* H^{t+n} \to \omega_Z(D|_Z) \otimes (f^* H^t|_Z) \to 0.$$

Using induction and the corresponding long exact sequence, we obtain that

$$H^i(Y, H^t \otimes R^j f_* \omega_X(D)) \simeq H^i(Y, H^{t+n} \otimes R^j f_* \omega_X(D))$$

for $i \geq 2$. By the Serre vanishing theorem,

$$H^i(Y, H^{t+kn} \otimes R^j f_* \omega_X(D)) = 0$$

for $k \gg 0$. Thus,

$$H^i(Y, H^t \otimes R^j f_* \omega_X(D)) = 0$$

for $t \geq 1$ and $i \geq 2$. Once this much of the theorem is established, the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, H^t \otimes R^q f_* \omega_X(D)) \Longrightarrow E^{p+q} = H^{p+q}(X, \omega_X(D) \otimes f^* H^t)$$

has only two columns, and therefore it degenerates. This means that

$$0 \longrightarrow E_2^{1,j} \longrightarrow E^{j+1} \longrightarrow E_2^{0,j+1} \longrightarrow 0.$$

Then, we have

$$\begin{array}{ccccc} 0 \longrightarrow & H^1(Y, H \otimes R^i f_* \omega_X(D)) & \longrightarrow & H^{j+1}(X, \omega_X(D) \otimes f^* H) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & H^1(Y, H^{1+kn} \otimes R^i f_* \omega_X(D)) & \longrightarrow & H^{j+1}(X, \omega_X(D) \otimes f^* H^{1+kn}). \end{array}$$

Using (1), this implies that

$$H^1(Y, H \otimes R^j f_*\omega_X(D)) \longrightarrow H^1(Y, H^{1+kn} \otimes R^j f_*\omega_X(D))$$

is injective for every k. As before, by the Serre vanishing theorem, this implies that $H^1(Y, H \otimes R^j f_*\omega_X(D)) = 0$. We complete the proof. q.e.d.

3. Variation of mixed Hodge structures

To investigate $R^i f_*\omega_X(D)$, we use the notion of gradedly polarized variation of mixed Hodge structures. We note that all the variations of mixed Hodge structures which we treat in this section are geometric.

3.1. Canonical Extension. In this section, we generalize [**15**, Theorem 2.6] and [**20**, Theorem 1].

3.1.1. Let $f : X \longrightarrow Y$ be a projective surjective morphism between non-singular varieties over $\mathbb{C}$. Let D be a simple normal crossing divisor on X such that D is strongly horizontal. Assume that there is a non-empty Zariski open set Y_0 of Y such that $\Sigma := Y \setminus Y_0$ is a simple normal crossing divisor on Y and that $f_0 : X_0 \longrightarrow Y_0$ is smooth and D_0 is relatively simple normal crossing over Y_0, where $X_0 := f^{-1}(Y_0)$, $f_0 := f|_{X_0}$ and $D_0 := D \cap X_0$. The local system $R^i f_{0*}\mathbb{C}_{X_0 - D_0}$ on Y_0 forms a gradedly polarized variation of mixed Hodge structure (see [**23**]).

3.1.2. Assume that all the local monodromies of the local system $R^k f_{0*}\mathbb{C}_{X_0-D_0}$ around Σ are unipotent. Put $\mathcal{H}^k_0 := (R^k f_{0*}\mathbb{C}_{X_0-D_0}) \otimes \mathcal{O}_{Y_0}$ and let $F^p(\mathcal{H}^k_0)$ be the p-th Hodge filtration of $\mathcal{H}^k_0$. Let $\mathcal{H}^k_Y$ be the canonical extension of $\mathcal{H}^k_0$ to Y. Then, there exists an extension $F^p(\mathcal{H}^k_Y)$ of $F^p(\mathcal{H}^k_0)$, which is locally free. See [**30**, Section 5 The geometric case], [**24**, I.10], and [**9**, Lemma 1.11.2]. We note that $F^p(\mathcal{H}^k_Y) = j_*F^p(\mathcal{H}^k_0) \cap \mathcal{H}^k_Y$, where $j : Y_0 \longrightarrow Y$ is the natural inclusion. As stated above, in this paper, we only treat *geometric* gradedly polarized variation of mixed Hodge structures.

The following is the main theorem of this subsection (see also Theorem 3.4). The proof is essentially the same as the proof of [**20**, Theorem 1].

Theorem 3.1. *Under the same notation as in* 3.1.1, *let* $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$ *and* $d := \dim f$. *Assume that all the local monodromies of* *the local system* $R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$ *around* Σ *are unipotent. Then, there exists an isomorphism*

$$R^i f_*\omega_{X/Y}(D) \simeq F^d(\mathcal{H}_Y^{d+i}).$$

In particular, $R^i f_*\omega_{X/Y}(D)$ *is locally free.*

Our proof of Theorem 3.1 relies on the following theorem. We can take it out from ([**30**, Section 5 The geometric case]) with a little effort. See also [**24**, I.10].

Theorem 3.2 ([**30**, Section 5]). *Let* $f : X \longrightarrow Y$ *be a projective surjective morphism from a non-singular variety* X *onto a non-singular curve* Y*. Let* D *be a simple normal crossing divisor on* X*. Assume that there is a divisor* Σ *on* Y *such that* f *is smooth over* $Y_0 := Y \setminus \Sigma$ *and* D *is relatively simple normal crossing over* Y_0 *and that* $C \cup D$ *is a simple normal crossing divisor, where* $C := (f^*\Sigma)_{\mathrm{red}}$*. Assume that all the local monodromies on* $R^i f_{0*}\mathbb{C}_{X_0-D_0}$ *around* Σ *are unipotent. Then*

$$\mathcal{H}_Y^i \simeq \mathbf{R}^i f_*\Omega^\bullet_{X/Y}(\log(C\cup D))$$

and

$$F^p(\mathcal{H}_Y^i) \simeq \mathbf{R}^i f_* F^p(\Omega^\bullet_{X/Y}(\log(C\cup D)))$$

for all p.

Here, $\Omega^\bullet_{X/Y}(\log(C\cup D))$ *is the* relative log complex:

$$\Omega^\bullet_{X/Y}(\log(C\cup D)) := \Omega^\bullet_X(\log(C\cup D))/f^*\Omega^1_Y(\log\Sigma)\wedge\Omega^\bullet_X(\log(C\cup D))[-1],$$

and $K^\bullet = F^p(\Omega^\bullet_{X/Y}(\log(C\cup D)))$ *is a complex such that*

$$K^q = \begin{cases} 0 & \text{if} \quad q<p, \\ \Omega^q_{X/Y}(\log(C\cup D)) & \text{otherwise.}\end{cases}$$

Proof of Theorem 3.1*.* By Corollaries 1.12 and 1.14 (f), we can assume that $C \cup D$ is a simple normal crossing divisor on X without loss of generality, where $C := (f^*\Sigma)_{\mathrm{red}}$.

Step 1 (The case when $\dim Y = 1$)**.** By Theorem 3.2, we have

$$F^d(\mathcal{H}_Y^{d+i}) \simeq \mathbf{R}^i f_*\Omega^d_{X/Y}(\log(C\cup D)).$$

On the other hand, $\Omega^d_{X/Y}(\log(C\cup D)) \simeq \omega_{X/Y}(C - f^*\Sigma + D)$. Therefore, if f is semi-stable, then $R^i f_*\omega_{X/Y}(D) \simeq F^d(\mathcal{H}_Y^{d+i})$. If f is not semi-stable, then we use the covering arguments in [**15**, Lemma 2.11] and [**9**, Lemma 1.9.1]. Thus, we obtain $F^d(\mathcal{H}_Y^{d+i}) \simeq R^i f_*(\Omega^d_{X/Y}(\log(C \cup D)) \otimes \mathcal{O}_X(\sum(a_i-1)C_i)) \simeq R^i f_*\omega_{X/Y}(D)$, where $f^*\Sigma := \sum a_iC_i$. Note that the middle term is the *upper canonical extension* (cf. [**15**, Definition 2.3]) and there is no difference between the *canonical extension* and the upper canonical extension (the *right canonical extension* in the proof of [**9**, Lemma 1.9.1]), since all the local monodromies are unipotent. See Remark 4.1.

Step 2 (The case when $l := \dim Y \geq 2$)**.** We shall prove by induction on l.

By Step 1, there is an open subset Y_1 of Y such that $\operatorname{codim}(Y \setminus Y_1) \geq 2$ and that

$$R^i f_*\omega_{X/Y}(D)|_{Y_1} \simeq F^d(\mathcal{H}_Y^{d+i})|_{Y_1}.$$

Since $F^d(\mathcal{H}_Y)$ is locally free, we obtain a homomorphism

$$\varphi^i_Y : R^i f_*\omega_{X/Y}(D) \longrightarrow F^d(\mathcal{H}_Y^{d+i}).$$

By Theorem 2.2, $\mathrm{Ker}\varphi^i_Y = 0$. We put $G^i_Y := \mathrm{Coker}\varphi^i_Y$. Taking a general hyperplane cut, we see that $\mathrm{Supp}G^i_Y$ is a finite set by the induction hypothesis. Assume that $G^i_Y \neq 0$. Take a point $P \in G^i_Y$. Let $\mu : W \longrightarrow Y$ be the blowing up at P and set $E = \mu^{-1}(P)$. Then $E \simeq \mathbb{P}^{l-1}$. By 1.14 (f), we can take a projective birational morphism $p : X' \longrightarrow X$ from a non-singular variety X' with the following properties:

- (i) the composition $X' \longrightarrow X \longrightarrow Y \dashrightarrow W$ is a morphism.
- (ii) p is an isomorphism over X_0.
- (iii) $\mathrm{Exc}(p)$ and $\mathrm{Exc}(p) \cup D'$ are simple normal crossing divisors on X', where D' is the strict transform of D.

By Corollary 1.12, we obtain that $R^i f_*\omega_{X/Y}(D) \simeq R^i(f \circ p)_*\omega_{X'/Y}(D')$ for every i. We note that D' is strongly horizontal with respect to $f \circ p$. By replacing (X, D) with (X', D'), we can assume that there is a morphism $g : X \longrightarrow W$ such that $f = \mu \circ g$. Since $g : X \longrightarrow W$ is in the same situation as f, we obtain the exact sequence:

$$0 \to R^i g_*\omega_{X/W}(D) \to F^d(\mathcal{H}_W^{d+i}) \to G^i_W \to 0.$$

Tensoring $\mathcal{O}_W(\nu E)$ for $0 \le \nu \le l-1$ and applying $R^j\mu_*$ for $j \ge 0$ to each ν, we have a exact sequence

$$\begin{aligned} 0 &\to \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to \mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \\ &\to \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \to R^1\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \\ &\to R^1\mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \to 0 \end{aligned}$$

and $R^q\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \simeq R^q\mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E))$ for $q \ge 2$.

By Lemma 3.3, $F^d(\mathcal{H}_W^{d+i}) \simeq \mu^* F^d(\mathcal{H}_Y^{d+i})$. We have

$$\mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \simeq F^d(\mathcal{H}_Y^{d+i})$$

and

$$R^q\mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) = 0$$

for $q \ge 1$. Therefore, $R^q\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) = 0$ for $q \ge 2$ and

$$\begin{aligned} 0 &\to \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to \mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \\ &\to \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \to R^1\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to 0 \end{aligned}$$

is exact. Since $\omega_W = \mu^*\omega_Y \otimes \mathcal{O}_W((l-1)E)$, we have a spectral sequence

$$E_2^{p,q} = R^p\mu_*(R^q g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \Longrightarrow R^{p+q} f_*\omega_{X/Y}(D).$$

However, $E_2^{p,q} = 0$ for $p \ge 2$ by the above argument; thus

$$\begin{aligned} 0 &\to R^1\mu_* R^{i-1} g_*\omega_{X/Y}(D) \to R^i f_*\omega_{X/Y}(D) \\ &\to \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \to 0. \end{aligned}$$

By Theorem 2.2, $R^1\mu_* R^{i-1} g_*\omega_{X/Y}(D) = 0$. Therefore, for $q \ge 1$, we obtain

(a) $R^i f_*\omega_{X/Y}(D) \simeq \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E))$ and
(b) $R^q\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) = 0$.

Next, we shall consider the following commutative diagram:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-2)E) & \longrightarrow & R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E) \\ \downarrow & & \downarrow \\ F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W((l-2)E) & \longrightarrow & F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W((l-1)E) \\ \downarrow & & \downarrow \\ G_W^i \otimes \mathcal{O}_W((l-2)E) & \longrightarrow & G_W^i \otimes \mathcal{O}_W((l-1)E) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

By applying μ_*, we have

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-2)E)) & \longrightarrow & \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \\ \downarrow & & \downarrow \\ F^d(\mathcal{H}_Y^{d+i}) & \simeq & F^d(\mathcal{H}_Y^{d+i}) \\ \downarrow & & \downarrow \\ \mu_*(G_W^i \otimes \mathcal{O}_W((l-2)E)) & \longrightarrow & \mu_*(G_W^i \otimes \mathcal{O}_W((l-1)E)) \\ & & \downarrow \\ & & 0 \end{array}$$

By (a) and (b), $G_Y^i \simeq \mu_*(G_W^i \otimes \mathcal{O}_W((l-1)E))$ and

$$\mu_*(G_W^i \otimes \mathcal{O}_W((l-2)E)) \to \mu_*(G_W^i \otimes \mathcal{O}_W((l-1)E))$$

is surjective. Since $\dim \mathrm{Supp} G_W^i = 0$ and $E \cap \mathrm{Supp} G_W^i \neq \emptyset$, it follows that $G_W^i = 0$ by Nakayama's lemma. Therefore, $G_Y^i = 0$. This proves the theorem.

q.e.d.

The following lemma played an essential role in the proof of Theorem 3.1.

Lemma 3.3. *Let $f : X \longrightarrow Y$ and D be as in 3.1.1. Let $\pi : V \longrightarrow Y$ be a morphism from a non-singular variety such that $\mathrm{Supp}\pi^{-1}(\Sigma)$ is a simple normal crossing divisor on V. Then, we obtain the gradedly polarized variation of mixed Hodge structures on $V_0 := V \setminus \pi^{-1}(\Sigma)$ by the base change. Assume that all the local monodromies of the local system $R^k f_{0*}\mathbb{C}_{X_0 - D_0}$ around Σ are unipotent. Then, $\pi^* F^p(\mathcal{H}_Y^k) \simeq F^p(\mathcal{H}_V^k)$, where $\mathcal{H}_V^k$ is the canonical extension of $\mathcal{H}_{V_0} := \pi^* \mathcal{H}_0^k$ to V.*

Proof. Note that $F^p(\mathcal{H}^k_Y) = j_*F^p(\mathcal{H}^k_0) \cap \mathcal{H}^k_Y$, where $j : Y_0 \longrightarrow Y$ is the natural inclusion. See, for example, [**24**, I.10] and [**9**, Lemma 1.11.2]. Then, we have $\pi^*F^p(\mathcal{H}^k_Y) \simeq F^p(\mathcal{H}^k_V)$. See [**11**, Proposition 1]. q.e.d.

By using the unipotent reduction theorem, we obtain the following theorem.

Theorem 3.4. *We use the same notation and assumptions as in 3.1.1. We put $\omega_{X/Y} := \omega_X \otimes \omega_Y^{-1}$ and $d := \dim f$. Then $R^if_*\omega_{X/Y}(D)$ is locally free. More precisely, $R^if_*\omega_{X/Y}(D)$ is the upper canonical extension of $R^if_{0*}\omega_{X_0/Y_0}(D_0)$ (see* [**15**, Definition 2.3] *and Remark 4.1).*

Proof. (cf. [**15**, Reduction 2.12]) It is sufficient to prove the local freeness of $R^if_*\omega_{X/Y}(D)$. So, by shrinking Y, we can assume that Y is quasi-projective. We already checked that $R^if_*\omega_{X/Y}(D)$ is the upper canonical extension in codimension one (see Step 1 in the proof of Theorem 3.1). We use the unipotent reduction theorem with respect to the local system $R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$. First, we can assume that $\mathrm{Supp}(D\cup f^{-1}(\Sigma))$ is a simple normal crossing divisor (cf. Corollary 1.12 and 1.14 (f)). We consider the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\ \alpha\ } & X' & \xleftarrow{\ \beta\ } & \widetilde{X} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ Y & \xleftarrow[\ \tau\ ]{} & Y' & = & Y', \end{array}$$

where $\tau : Y' \to Y$ is a finite morphism from a non-singular variety obtained by Kawamata's covering trick, X' is the normalization of $X\times_Y Y'$, β is a projective birational morphism from a non-singular variety $\widetilde{X}$, and D' is a simple normal crossing divisor on $\widetilde{X}$ that is the strict transform of α^*D. We note that we can use Kawamata's covering trick since Y is quasi-projective. We can assume that $\widetilde{f} : \widetilde{X} \longrightarrow Y'$ and D' satisfy the conditions and the assumptions in 3.1.1 and Theorem 3.1 for a suitable simple normal crossing divisor Σ' on Y'. By Proposition 1.11 and Corollary 1.12, we can check that $R^i\widetilde{f}_*\omega_{\widetilde{X}}(D') \simeq R^if'_*\omega_{X'}(\alpha^*D)$ for $i \geq 0$. We note that (X', α^*D) is lc and every center of log canonical singularities for the pair (X', α^*D) is dominant onto Y'. Thus, $R^if'_*\omega_{X'}(\alpha^*D)$ is locally free for $i \geq 0$. Since $R^if_*\omega_X(D)$ is a direct summand of

$\tau_* R^i f'_* \omega_{X'}(\alpha^* D)$, we obtain that $R^i f_* \omega_X(D)$ is locally free for $i \geq 0$. So, we complete the proof. q.e.d.

3.2. Semi-positivity theorem. In this subsection, we prove the semi-positivity of $R^i f_* \omega_{X/Y}(D)$ on suitable assumptions. It is a generalization of Fujita–Kawamata's semi-positivity theorem and related to [**10**, Theorem 32]. See Remark 3.10.

First, let us recall the definition of semi-positive vector bundles.

Definition 3.5 (Semi-positivity)**.** Let V be a complete variety and $\mathcal{E}$ a locally free sheaf on V. We say that $\mathcal{E}$ is *semi-positive* if and only if the tautological line bundle $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef on $\mathbb{P}_V(\mathcal{E})$. We note that $\mathcal{E}$ is semi-positive if and only if for every complete curve C and morphism $g : C \longrightarrow V$ every quotient line bundle of $g^*\mathcal{E}$ has non-negative degree.

We collect the basic properties of semi-positive vector bundles, for the readers' convenience. We omit the proof here. Details are left to the readers. See, for example, the corresponding part of [**19**].

Proposition 3.6 (Properties of semi-positive vector bundles)**.** *Let V be a complete variety. Then we have the following properties*:

(i) *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles on V. Then, the direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semi-positive if and only if both $\mathcal{E}_1$ and $\mathcal{E}_2$ are semi-positive;*

(ii) *A vector bundle $\mathcal{E}$ on V is semi-positive if and only if so is $S^k\mathcal{E}$ for every k, where $S^k\mathcal{E}$ is the k-th symmetric product of $\mathcal{E}$;*

(iii) *If $\mathcal{E}$ is semi-positive and $\mathcal{F}$ is a semi-positive* (*resp. an ample*) *vector bundle on V, then $\mathcal{E} \otimes \mathcal{F}$ is semi-positive* (*resp. ample*)*;*

(iv) *Any tensor product or exterior product of semi-positive vector bundles is semi-positive.*

Remark 3.7. It is obvious that a line bundle $\mathcal{L}$ is semi-positive if and only if $\mathcal{L}$ is nef. We note that, in some literatures (for example, [**19**]), semi-positive vector bundles are called *nef vector bundles*.

The following lemma, which is not difficult to prove, will be used in the proof of Theorem 3.9. We leave the details to the readers.

Lemma 3.8 (Extension of semi-positive vector bundles)**.** *Let Y be a complete variety. Assume that there exists a short exact sequence on Y*:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

such that both $\mathcal{E}'$ and $\mathcal{E}''$ are semi-positive vector bundles. Then, so is $\mathcal{E}$.

The next theorem is the main theorem of this subsection.

Theorem 3.9 (Semi-positivity theorem)**.** *Let $f : X \longrightarrow Y$ be a projective surjective morphism between non-singular varieties over $\mathbb{C}$. Let D be a simple normal crossing divisor on X such that D is strongly horizontal. Assume that there is a non-empty Zariski open set Y_0 of Y such that $\Sigma := Y \setminus Y_0$ is a simple normal crossing divisor on Y and that $f_0 : X_0 \longrightarrow Y_0$ is smooth and D_0 is relatively simple normal crossing over Y_0, where $X_0 := f^{-1}(Y_0)$, $f_0 := f|_{X_0}$ and $D_0 := D \cap X_0$. Let $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$ and $d := \dim f$. Assume that all the local monodromies of the local system $R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$ around Σ are unipotent. Let Z be a complete subvariety of Y. Then, the restriction $R^if_*\omega_{X/Y}(D)|_Z$ is semi-positive. In particular, if Y is complete, then $R^if_*\omega_{X/Y}(D)$ is semi-positive.*

Proof. Let

$$0 \subset \cdots \subset W_k \subset W_{k+1} \subset \cdots \subset \mathcal{H}_0^{d+i} := R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$$

be the weight filtration of the gradedly polarized variation of mixed Hodge structures and

$$0 \subset \cdots \subset \widetilde{W}_k \subset \widetilde{W}_{k+1} \subset \cdots \subset \mathcal{H}_Y^{d+i}$$

be the canonical extension of the above weight filtration. Then, the vector bundle $F^d(\mathcal{H}_Y^{d+i})$ induces the canonical extension of the bottom Hodge filtration on each $\mathrm{Gr}_k^{\widetilde{W}}(\mathcal{H}_Y^{d+i})$. Therefore, Lemma 3.8, [**11**, Theorem 2] and [**10**, Section 4] imply that $F^d(\mathcal{H}_Y^{d+i})|_Z$ is semi-positive. (Note that Kawamata's proof of semi-positivity theorem heavily relies on the asymptotic behavior of the Hodge metric near a puncture. It is not so easy for the non-expert to take it out from [**28**, Section 6]. We recommend the readers to see [**22**, Sections 2, 3] or [**33**]. Section 4 of [**5**] is an exposition of Fujita–Kawamata's semi-positivity theorem.) On the other hand, by Theorem 3.1, we have the following isomorphism

$$R^if_*\omega_{X/Y}(D) \simeq F^d(\mathcal{H}_Y^{d+i}).$$

Thus, we obtain that $R^if_*\omega_{X/Y}(D)|_Z$ is semi-positive. q.e.d.

Remark 3.10. The semi-positivity of $f_*\omega_{X/Y}(D)$ in Theorem 3.9 is very similar to [**10**, Theorem 32]. Unfortunately, our theorem: Theorem 3.9, does not contain Theorem 32 in [**10**]. On the other hand, [**10**, Theorem 32] and Step 1 in the proof of Theorem 3.1 recover the semi-positivity of $f_*\omega_{X/Y}(D)$ quickly. The following argument was inspired by the referee. In Theorem 3.9, without loss of generality, we can assume that $C \cup D$ is a simple normal crossing divisor, where $C = (f^*\Sigma)_{red}$. Then [**10**, Theorem 32] implies that $f_*\omega_{X/Y}(D + C - f^*\Sigma) \simeq F^d(\mathcal{H}^d_Y)$ and it is semi-positive. Note that there exists a natural inclusion $f_*\omega_{X/Y}(D + C - f^*\Sigma) \subset f_*\omega_{X/Y}(D)$. By Step 1 in the proof of Theorem 3.1, the above inclusion is an isomorphism in codimension one. We note that the left-hand side is locally free by [**10**, Theorem 32] and the right-hand side is torsion-free. Thus, we have $f_*\omega_{X/Y}(D + C - f^*\Sigma) \simeq f_*\omega_{X/Y}(D)$ and it is a semi-positive vector bundle.

4. Appendix: A remark on Section 3 by M. Saito

In this section, we give a different proof to Theorems 3.1 and 3.4. It is based on the theory of mixed Hodge Modules [**26**], [**27**]. As I explained in 1.1.3, the following 4.1 is [**25**]. I made no contribution in this section.

4.1. ([25]). Let X be a smooth complex algebraic variety, and D a divisor with normal crossings whose irreducible components D_i are smooth. Let $U = X \setminus D$ with the inclusion $j : U \to X$. Let $(M; F, W)$ be a bifiltered (left) $\mathcal{O}_X$-Module underlying a mixed Hodge Module. Assume that $L := M|_U$ is a locally free $\mathcal{O}_U$-Module, i.e., it underlies an admissible variation of mixed Hodge structure on U.

By the definition of pure Hodge Modules, we have the strict support decomposition

$$\mathrm{Gr}^W_k(M, F) = \bigoplus_Z (M_{k,Z}, F),$$

where Z is either X or a closed irreducible variety of D (by the assumption on $M|_U$), and the $M_{k,Z}$ have no non-trivial subobject or quotient object with strictly smaller support.

Proposition 1. *Let $p_0 = \min\{p : F_pM \neq 0\}$, and assume*

$$F_{p_0}M_{k,Z} = 0 \quad \text{if } Z \subset D. \tag{1}$$

Then we have the canonical isomorphism

$$F_{p_0}M = j_*F_{p_0}L \cap L^{>-1} \tag{2}$$

where $L^{>a}$ is the Deligne extension of L such that the eigenvalues of the residue of the connection are contained in $(a, a+1]$.

Proof. We first consider the case $M = j_! L$, where $j_!$ is defined to be the composition $\mathbf{D} j_* \mathbf{D}$. Here, $\mathbf{D}$ denotes the functor assigning the dual, and j_* coincides with the usual direct image as $\mathcal{O}$-Modules. In this case, the filtration F on M is given by

$$F_p M = \sum_i F_i \mathcal{D}_X (F_{p-i} L^{>-1}), \tag{3}$$

(see e.g., [**26**, (3.10.8)]), where F on $\mathcal{D}_X$ is the filtration by the order of operator, and $F_p L^{>-1}$ is given by $j_* F_p L \cap L^{>-1}$ as usual. So the isomorphism (2) is clear.

In general, we use the canonical morphism $u : j_! L \to M$, see [**26**, (4.2.11)]. By the above result, it is enough to show the vanishing of F_{p_0} for $\operatorname{Ker} u$ and $\operatorname{Coker} u$, because the functor assigning F_p is an exact functor for mixed Hodge Modules. Furthermore, the functor assigning $F_p \mathrm{Gr}_k^W$ is also exact. So, we may replace u with $\mathrm{Gr}_k^W u : \mathrm{Gr}_k^W j_! L \to \mathrm{Gr}_k^W M$. This morphism is compatible with the decomposition by strict support, and condition (1) is also satisfied for $j_! L$ (using (3)). So, the assertion follows from the fact that $\mathrm{Gr}_k^W u$ induces an isomorphism between the direct factors with strict support X (this follows from the definition of the Hodge filtration on pure Hodge Modules, see e.g., [**26**, (3.10.12)]. q.e.d.

We apply this to the direct image of $\mathcal{D}$-Modules. Here, it is easier to use right $\mathcal{D}$-Modules (because it simplifies many definitions) and we use the transformation between right and left $\mathcal{D}$-Modules, which is defined by assigning $\Omega_X^{\dim X} \otimes_{\mathcal{O}_X} M$ to a left $\mathcal{D}$-Module M, where $\Omega_X^{\dim X}$ has the filtration F such that $\mathrm{Gr}_p^F = 0$ for $p \neq -\dim X$. We define the Hodge filtration F on the right $\mathcal{D}$-Module ω_X by $F_p \omega_X = \omega_X$ for $p \geq 0$ and 0 otherwise. Then, (ω_X, F) is pure of weight $-\dim X$ (and $(\Omega_X^{\dim X}, F)$ has weight $\dim X$). We can verify that $\mathrm{Gr}_{-k}^W (j_* \omega_U, F)$ is the direct sum of $(\iota_I)_* (\omega_{D_I}, F)$ with $\dim D_I = k$, where $D_I = \cap_{i \in I} D_i$ with the inclusion $\iota_I : D_I \to X$, see [**26**, (3.10.8) and (3.16.12)]. (Here the direct image $(\iota_I)_*$ is defined by tensoring the polynomial ring $\mathbf{C}[\partial_1, \ldots, \partial_r]$ over $\mathbf{C}$ if $I = \{1, \ldots, r\}$, where $\partial_i = \partial / \partial x_i$ with $(x_1, \ldots, x_n)$ a local coordinate system such that $D_i = x_i^{-1}(0)$.) We also see that $F_0 j_* \omega_U = \omega_X(D)$, and

$$F_0 H^i f_* (j_* \omega_U) = R^i f_* \omega_X(D)$$

by the definition of the direct image of filtered right $\mathcal{D}$-Modules, using the strictness of the Hodge filtration F on the direct image.

Proposition 2. *Let X, U, D, j be as above. Let $f : X \to Y$ be a proper morphism of smooth complex algebraic varieties, and D' be a divisor with normal crossings on Y. Assume that every irreducible component of any intersections of D_i is dominant to Y and smooth over $Y \setminus D'$. Then, condition* (1) *with $p_0 = 0$ is satisfied for the direct image of a filtered* (*right*) *$\mathcal{D}$-Module $H^i f_*(j_*\omega_U, F)$.*

Proof. Consider the weight spectral sequence of filtered (right) $\mathcal{D}$-Modules

$$E_1^{-k,i+k} = H^i f_* \mathrm{Gr}_k^W (j_*\omega_U, F) \Rightarrow H^i f_*(j_*\omega_U, F),$$

which underlies a spectral sequence of mixed Hodge Modules and degenerates at E_2. Since $\mathrm{Gr}_k^W (j_*\omega_U, F)$ is calculated as above and the direct image of a pure Hodge Module by a proper morphism is pure, the assertion is reduced to the proper case, where it is well known. (Indeed, it is reduced to the torsion-freeness using the decomposition by strict support as above.) q.e.d.

4.2. Finally, we add one remark for the readers' convenience.

Remark 4.1 (Deligne's extension)**.** In the above Proposition 1 and [**27**, p.513], $L^{>a}$ (resp. $L^{\geq a}$) is *Deligne's extension* of L such that the eigenvalues of the residue of the connection are contained in $(a, a+1]$ (resp. $[a, a+1)$). In our notation: Kollár's notation [**15**, Definition 2.3], $L^{>-1}$ (resp. $L^{\geq 0}$) is called the *upper* (resp. *lower*) *canonical extension* of L. In [**9**, Lemma 1.9.1], $L^{>-1}$ (resp. $L^{\geq 0}$) is called the *right* (resp. *left*) *canonical extension* of L.

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