J. DIFFERENTIAL GEOMETRY
58 (2001) 421-455

AUTOMORPHISMS AND EMBEDDINGS OF SURFACES AND QUADRUPLE POINTS OF REGULAR HOMOTOPIES

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Abstract

Let F be a closed surface. If $i, i' : F \to \mathbb{R}^3$ are two regularly homotopic generic immersions, then it has been shown in [5] that all generic regular homotopies between i and i' have the same number mod 2 of quadruple points. We denote this number by $Q(i, i') \in \mathbb{Z}/2$. For F orientable we show that for any generic immersion $i : F \to \mathbb{R}^3$ and any diffeomorphism $h : F \to F$ such that i and $i \circ h$ are regularly homotopic,

$$Q(i, i \circ h) = \Big(\mathrm{rank}(h_* - \mathrm{Id}) + (n+1)\epsilon(h)\Big) \bmod 2,$$

where h_* is the map induced by h on $H_1(F, \mathbb{Z}/2)$, n is the genus of F and $\epsilon(h)$ is 0 or 1 according to whether h is orientation preserving or reversing, respectively.

We then give an explicit formula for $Q(e, e')$ for any two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$. The formula is in terms of homological data extracted from the two embeddings.

1. Introduction

For F a closed surface and $i, i' : F \to \mathbb{R}^3$ two regularly homotopic generic immersions, we are interested in the number mod 2 of quadruple points occurring in generic regular homotopies between i and i'. It has been shown in [5] that this number is the same for all such regular homotopies, and so it is a function of i and i' which we denote $Q(i, i') \in \mathbb{Z}/2$. There then arises the problem of finding explicit formulae for $Q(i, i')$. (*Generic* immersions and *generic* regular homotopies are defined in a natural way. For a brief discussion see [4].)

Received January 24, 2001.

Assuming F is orientable, we give an explicit formula for $Q(i, i\circ h)$, where $i : F \to \mathbb{R}^3$ is any generic immersion and $h : F \to F$ is any diffeomorphism such that i and $i \circ h$ are regularly homotopic (Theorem 5.6: Also fully stated in Abstract above).

Using the formula for $Q(i, i\circ h)$ we then give an explicit formula for $Q(e, e')$ for any two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$ (Theorem 7.3). This formula depends on the following data: If $e : F \to \mathbb{R}^3$ is an embedding then $e(F)$ splits $\mathbb{R}^3$ into two pieces, one compact and one noncompact, which will be denoted $M^0(e)$ and $M^1(e)$ respectively. By restriction of range e induces maps $e^k : F \to M^k(e)$ $(k = 0, 1)$ and let $A^k(e) \subseteq H_1(F, \mathbb{Z}/2)$ be the kernel of the map induced by e^k on $H_1(\cdot, \mathbb{Z}/2)$. Let $o(e)$ be the orientation on F which is induced from $M^0(e)$ to $\partial M^0(e) = e(F)$ and then via e to F. Our formula for $Q(e, e')$ will be in terms of the two triplets $A^0(e), A^1(e), o(e)$ and $A^0(e'), A^1(e'), o(e')$. We will also extend our formula to finite unions of closed orientable surfaces.

For two special cases a formula for $Q(e, e')$, for e, e' embeddings, has already been known: The case where F is a sphere has appeared in [4] and [5], and the case where F is a torus has appeared in [5].

Based on the Smale-Hirsch Theorem ([2]) Pinkall in [6] gave a useful tool for determining when two immersions $i, i' : F \to \mathbb{R}^3$ are regularly homotopic, namely, any immersion $i : F \to \mathbb{R}^3$ induces a quadratic form $g^i : H_1(F, \mathbb{Z}/2) \to \mathbb{Z}/2$, and two immersions $i, i' : F \to \mathbb{R}^3$ are regularly homotopic iff $g^i = g^{i'}$. Let $\widehat{\mathcal{M}}$ denote the group of all diffeomorphisms $h : F \to F$ up to isotopy. Given $i : F \to \mathbb{R}^3$ we will be interested in the group of all $h \in \widehat{\mathcal{M}}$ such that $Q(i, i \circ h)$ is defined, that is the group of all $h \in \widehat{\mathcal{M}}$ such that i and $i \circ h$ are regularly homotopic. It follows from the above criterion that this is precisely the group $\widehat{\mathcal{M}}_{g^i}$ of all $h \in \widehat{\mathcal{M}}$ which preserve the quadratic form g^i on $H_1(F, \mathbb{Z}/2)$. We are thus led to study the groups $\widehat{\mathcal{M}}_g$, starting with their index 2 subgroup $\mathcal{M}_g$ of orientation preserving maps.

The plan of the paper is as follows: In Section 2 we present the known results on quadratic forms which we will need. In Section 3 we show that the expression $\mathrm{rank}(T - \mathrm{Id}) \bmod 2$ appearing in our proposed formula for $Q(i, i\circ h)$ defines a homomorphism on appropriate subgroups of $GL(H_1(F, \mathbb{Z}/2))$ (Theorem 3.3). In Section 4 we show that (except for one special case) the group $\mathcal{M}_g$ is generated by a certain family of Dehn twists and squares of Dehn twists (Theorem 4.9). The formula for $Q(i, i\circ h)$ is then proved in Section 5. In Section 6 we study further properties of quadratic forms. In Section 7 we give the precise statement of our formula for $Q(e, e')$ for e, e' embeddings, which is then proved in Section 8.

2. Quadratic Forms over $\mathbb{Z}/2$

In this section we summarize the definitions and known properties of quadratic forms over $\mathbb{Z}/2$ which will be needed in our work. Proofs to all facts stated in this section may be found in [1], except for those relating to the Arf invariant, which may be found in [3].

Let V be a finite dimensional vector space over $\mathbb{Z}/2$. A function $g : V \to \mathbb{Z}/2$ is called a *quadratic form* if g satisfies: $g(x+y) = g(x) + g(y) + B(x, y)$ for all $x, y \in V$, where $B(x, y)$ is a bilinear form. The following properties follow:

(a) $g(0) = 0$.

(b) $B(x, x) = 0$ for all $x \in V$.

(c) $B(x, y) = B(y, x)$ for all $x, y \in V$.

g is called *nondegenerate* if B is nondegenerate, i.e., for any $0 \neq x \in V$ there is $y \in V$ with $B(x, y) \neq 0$.

Proposition 2.1. *If g is nondegenerate then V is necessarily of even dimension and there exists a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ for V such that $B(a_i, a_j) = B(b_i, b_j) = 0$ and $B(a_i, b_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$ and such that one of the following two possibilities holds:*

(1) *$g(a_i) = g(b_i) = 0$ for $i = 1, \ldots, n$.*

(2) *$g(a_1) = g(b_1) = 1$ and $g(a_i) = g(b_i) = 0$ for $i = 2, \ldots, n$.*

g is completely determined by the values $g(v_i)$ and $B(v_i, v_j)$ on a basis $v_1, \ldots, v_{2n}$ and so for given dimension $2n$ there are two isomorphism classes of nondegenerate quadratic forms, and they are in fact distinct. The invariant $\mathrm{Arf}(g) \in \mathbb{Z}/2$ is then defined to be 0 or 1 according to whether (1) or (2) of Proposition 2.1 holds respectively. (In the more general setting of [1], this is equivalent to g having index n or $n-1$ respectively.) The Arf invariant is additive in the following sense:

Proposition 2.2. *If $g_i : V_i \to \mathbb{Z}/2$, $i = 1, 2$, are nondegenerate quadratic forms, then $g_1 \oplus g_2 : V_1 \oplus V_2 \to \mathbb{Z}/2$ defined by $(g_1 \oplus g_2)(x_1, x_2) = g_1(x_1) + g_2(x_2)$ is a nondegenerate quadratic form with $\mathrm{Arf}(g_1 \oplus g_2) = \mathrm{Arf}(g_1) + \mathrm{Arf}(g_2)$.*

From now on we will always assume that our quadratic form g is nondegenerate.

Proposition 2.3. *If $a_1, \dots, a_k \in V$ are independent and $B(a_i, a_j) = 0$ for all $1 \le i, j \le k$ then there are $b_1, \dots, b_k \in V$ with $B(b_i, b_j) = 0$ and $B(a_i, b_j) = \delta_{ij}$ for all $1 \le i, j \le k$ ($a_1, \dots, a_k, b_1, \dots, b_k$ are then necessarily independent).*

A linear map $T : V \to V$ is called *orthogonal* with respect to g if $g(T(x)) = g(x)$ for all $x \in V$. It then follows that $B(T(x), T(y)) = B(x, y)$ for all $x, y \in V$ and that T is invertible. The group of all orthogonal maps of V with respect to g will be denoted $O(V, g)$.

Definition 2.4. Given $a \in V$, define $T_a : V \to V$ by $T_a(x) = x + B(x, a)a$.

Proposition 2.5. *$T_a \in O(V, g)$ iff $g(a) = 1$ or $a = 0$.*

Theorem 2.6 (Cartan, Dieudonne)**.** *Except for the case when $\dim V = 4$ and $\mathrm{Arf}(g) = 0$, $O(V, g)$ is generated by the elements T_a with $g(a) = 1$.*

Theorem 2.6 will also follow from Theorem 4.9 below. See Remark 4.10.

If $W \subseteq V$ is a subspace, then the conjugate space $W^\perp$ of W is defined by $W^\perp = \{x \in V : B(x, y) = 0 \text{ for all } y \in W\}$. If $a \in V$ we similarly define $a^\perp = \{x \in V : B(x, a) = 0\}$. Let Id denote the identity map on V, let $\mathrm{Im}(T)$ denote the image of T and let $\mathbf{F}(T) = \{x \in V : T(x) = x\}$.

Proposition 2.7. *If $T \in O(V, g)$ then $\mathrm{Im}(T - \mathrm{Id}) = (\mathbf{F}(T))^\perp$.*

3. A Homomorphism from $O(V, g)$ to $\mathbb{Z}/2$.

Let $a \in V$, then $\mathbf{F}(T_a) = a^\perp$ and so if $a \ne 0$ $\dim \mathbf{F}(T_a) = 2n - 1$, where $2n = \dim V$.

Lemma 3.1. *Let $T \in O(V, g)$ and $a \in V$ with $g(a) = 1$.*

(1) *If $\mathbf{F}(T) \subseteq \mathbf{F}(T_a)$ then $\dim \mathbf{F}(T \circ T_a) = \dim \mathbf{F}(T) + 1$.*

(2) *If $\mathbf{F}(T) \not\subseteq \mathbf{F}(T_a)$ then $\dim \mathbf{F}(T \circ T_a) = \dim \mathbf{F}(T) - 1$.*

Proof. We first note:

(a) If $x \notin a^\perp$ then $B(x, a) = 1$ so $T_a(x) = x + a$ and so $T \circ T_a(x) = T(x + a)$.

(b) For any $x \in V$, if $T(x+a) = x$ then $g(x) = g(T(x+a)) = g(x+a) = g(x) + g(a) + B(x,a)$ and so $B(x,a) = g(a) = 1$, that is $x \notin a^\perp$.

We get that $\mathbf{F}(T \circ T_a) \not\subseteq a^\perp$ iff $\exists x \notin a^\perp$ with $T(x+a) = x$ iff $\exists x \in V$ with $T(x+a) = x$ iff $\exists x \in V$ with $(T - \mathrm{Id})(x+a) = a$ iff $a \in \mathrm{Im}(T - \mathrm{Id}) = \mathbf{F}(T)^\perp$ (Proposition 2.7) iff $\mathbf{F}(T) \subseteq a^\perp$. Since $a^\perp = \mathbf{F}(T_a)$ we conclude that $\mathbf{F}(T \circ T_a) \not\subseteq \mathbf{F}(T_a)$ iff $\mathbf{F}(T) \subseteq \mathbf{F}(T_a)$.

Clearly $\mathbf{F}(T) \cap \mathbf{F}(T_a) = \mathbf{F}(T \circ T_a) \cap \mathbf{F}(T_a)$ and let k denote the dimension of this subspace. $\mathbf{F}(T_a)$ is of codimension 1 in V and so it follows:

(1) If $\mathbf{F}(T) \subseteq \mathbf{F}(T_a)$ then $\dim \mathbf{F}(T) = k$ and $\mathbf{F}(T \circ T_a) \not\subseteq \mathbf{F}(T_a)$ and so $\dim \mathbf{F}(T \circ T_a) = k+1$.

(2) If $\mathbf{F}(T) \not\subseteq \mathbf{F}(T_a)$ then $\dim \mathbf{F}(T) = k+1$ and $\mathbf{F}(T \circ T_a) \subseteq \mathbf{F}(T_a)$ and so $\dim \mathbf{F}(T \circ T_a) = k$.

q.e.d.

We now define $\psi : O(V,g) \to \mathbb{Z}/2$ by:

$$\psi(T) \;=\; \mathrm{rank}(T - \mathrm{Id}) \quad \mathrm{mod}\ 2.$$

Remark 3.2. Since $\mathbf{F}(T) = \ker(T - \mathrm{Id})$ (or by Proposition 2.7) we may also write: $\psi(T) = \mathrm{codim}\mathbf{F}(T) \bmod 2$, and since V is of even dimension we also have: $\psi(T) = \dim \mathbf{F}(T) \bmod 2$.

Theorem 3.3. $\psi : O(V,g) \to \mathbb{Z}/2$ *is a (nontrivial) homomorphism.*

Proof. We will be using the equivalent definition

$$\psi(T) = \dim \mathbf{F}(T) \bmod 2$$

of Remark 3.2. Assume first that (V,g) is not of the special case excluded from Theorem 2.6, and so $O(V,g)$ is generated by the elements T_a with $g(a) = 1$. If $T = T_{a_1} \circ \cdots \circ T_{a_k}$ $(g(a_i) = 1)$ then since $\psi(\mathrm{Id}) = \dim V \bmod 2 = 0$, induction on Lemma 3.1 implies $\psi(T) = k \bmod 2$ which clearly implies that ψ is a homomorphism.

We are left with the case $\dim V = 4, \mathrm{Arf}(g) = 0$. By Proposition 2.2, $(V,g) \cong (V' \oplus V', g' \oplus g')$ where $\dim V' = 2, \mathrm{Arf}(g') = 1$. We identify V with $V' \oplus V'$ via such an isomorphism. The set of all elements in V with $g = 1$ is $V_1 \cup V_2$ where $V_1 = \{(x,0) : 0 \neq x \in V'\}$ and $V_2 = \{(0,x) : 0 \neq x \in V'\}$. If $a \in V_1$ and $b \in V_2$ then $B(a,b) = 0$, whereas if $a \neq b$

are in the same V_k then $B(a,b)=1$. It follows that any $T \in O(V,g)$ must either map each V_k into itself or map V_1 into V_2 and V_2 into V_1. So T is of the form $(x,y) \mapsto (T_1(x), T_2(y))$ or $(x,y) \mapsto (T_1(y), T_2(x))$ where $T_1, T_2 \in O(V', g')$ $(= GL(V'))$. Such a map will be denoted by $(T_1,T_2)_0$ or $(T_1,T_2)_1$ respectively. If $T=(T_1,T_2)_0$ then $T(x,y)=(x,y)$ iff $T_1(x)=x$ and $T_2(y)=y$ and so $\mathbf{F}(T)=\mathbf{F}(T_1)\oplus\mathbf{F}(T_2)$ so $\dim\mathbf{F}(T) = \dim\mathbf{F}(T_1)+\dim\mathbf{F}(T_2)$ so $\psi(T)=\psi(T_1)+\psi(T_2)$. (The ψ on the left is the function on $O(V,g)$ and the ψ on the right is the function on $O(V',g')$.) If $T=(T_1,T_2)_1$ then $T(x,y)=(x,y)$ iff $T_1(y)=x$ and $T_2(x)=y$ that is (x,y) is of the form $(x, T_2(x))$ with $T_1\circ T_2(x)=x$. And so $\dim\mathbf{F}(T)=\dim\mathbf{F}(T_1\circ T_2)$ so $\psi(T)=\psi(T_1\circ T_2)$. Now, since $\dim V'=2$, V' belongs to the general case, and so we already know $\psi(T_1\circ T_2)=\psi(T_1)+\psi(T_2)$. So we have shown for both $u=0$ and $u=1$ that $\psi((T_1,T_2)_u)=\psi(T_1)+\psi(T_2)$. Now if $T=(T_1,T_2)_u$ and $S=(S_1,S_2)_{u'}$ then $T\circ S$ is of the form $(T_1\circ S_1, T_2\circ S_2)_{u''}$ or $(T_1\circ S_2, T_2\circ S_1)_{u''}$. In any case (again using the fact that ψ on V' is a homomorphism) we get that $\psi(T\circ S)=\psi(T_1)+\psi(T_2)+\psi(S_1)+\psi(S_2)=\psi(T)+\psi(S)$.

Finally, $\psi : O(V,g) \to \mathbb{Z}/2$ is not trivial since $\psi(T_a)=1$ for any $a\in V$ with $g(a)=1$. q.e.d.

Remark 3.4. 1. For $A \in O_k(\mathbb{R})$ (the group of $k\times k$ orthogonal matrices over $\mathbb{R}$) $\operatorname{codim}\mathbf{F}(A)=0 \bmod 2$ iff $\det A = 1$. And so by Remark 3.2, $\psi : O(V,g)\to\mathbb{Z}/2$ may be thought of as an analogue of the homomorphism $\det : O_k(\mathbb{R}) \to \{1,-1\}$ (det on $O(V,g)$ is of course trivial).

2. Our expression for ψ is meaningful on the whole of $GL(V)$, however ψ is in general not a homomorphism on $GL(V)$ or even on its subgroup $Sp(V)\supseteq O(V,g)$ of maps preserving $B(x,y)$.

For $\dim V=4, \mathrm{Arf}(g)=0$, we note that though the identification of V with $V'\oplus V'$ in the proof of Theorem 3.3 is not unique, the (unordered) pair of sets V_1,V_2 is uniquely defined by its mentioned properties, namely, $V_1\cup V_2=\{v\in V : g(v)=1\}$, $B(a,b)=0$ for $a\in V_1, b\in V_2$, and $B(a,b)=1$ for $a\neq b\in V_k$, $k=1,2$. It follows, as we have noticed, that any $T\in O(V,g)$ either preserves each V_k (then $T=(T_1,T_2)_0$), or interchanges the V_ks (then $T=(T_1,T_2)_1$).

Definition 3.5. Let $\dim V=4, \mathrm{Arf}(g)=0$. $T\in O(V,g)$ will be called a *U-map* if T interchanges V_1 and V_2.

Lemma 3.6. *Let* $\dim V=4, \mathrm{Arf}(g)=0$. *If T is a U-map such that* $T^2=\mathrm{Id}$ *then* $\psi(T)=0$.

Proof. $T = (T_1, {T_1}^{-1})_1$ and so by the proof of Theorem 3.3, $\psi(T) = \psi(T_1) + \psi({T_1}^{-1}) = 0$. q.e.d.

4. Generators for the Orthogonal Mapping Class Group

Let F be a closed orientable surface. H_1 from now on will always denote $H_1(F, \mathbb{Z}/2)$ (considered as a vector space over $\mathbb{Z}/2$). Let $g : H_1 \to \mathbb{Z}/2$ be a quadratic form whose associated bilinear form $B(x, y)$ is the algebraic intersection form $x \cdot y$ of H_1. (In particular, g is nondegenerate.) Let $\mathcal{M}$ denote the mapping class group of F i.e., the group of all orientation preserving diffeomorphisms $h : F \to F$ up to isotopy. For $h : F \to F$, let h_* denote the map it induces on H_1. The *orthogonal mapping class group of F with respect to g* will be the subgroup $\mathcal{M}_g$ of $\mathcal{M}$ defined by $\mathcal{M}_g = \{h \in \mathcal{M} : h_* \in O(H_1, g)\}$.

A simple closed curve will be called a *circle*. If c is a circle in F, the homology class of c in H_1 will be denoted by $[c]$. Given a circle in F, a Dehn twist along c will be denoted $\mathcal{T}_c$. (We will not establish a convention as to which of the two possible Dehn twists is $\mathcal{T}_c$ and which is ${\mathcal{T}_c}^{-1}$, rather, it will be clear in each case which of the two should be used.) The map induced on H_1 by $\mathcal{T}_c$ is $T_{[c]}$ of Definition 2.4. And so by Proposition 2.5, $\mathcal{T}_c \in \mathcal{M}_g$ iff $g([c]) = 1$ or $[c] = 0$. Also, since $(T_{[c]})^2 = \mathrm{Id}$, $(\mathcal{T}_c)^2 \in \mathcal{M}_g$ for any circle c. In view of this we make the following definition:

Definition 4.1. A map $h : F \to F$ will be called *good* if it is of one of the following forms:

(1) $h = (\mathcal{T}_c)^2$ for some circle c.

(2) $h = \mathcal{T}_c$ for a circle c with $g([c]) = 1$.

(3) $h = \mathcal{T}_c$ for a circle c with $[c] = 0$.

A good map will be called of Type 1, 2 or 3 accordingly.

The purpose of this section is to show that except for the special case when $\mathrm{genus}(F) = 2$ and $\mathrm{Arf}(g) = 0$, $\mathcal{M}_g$ is generated by the good maps. For the mentioned special case, we will show that one more generator is required.

Whenever we consider two circles in F, we will assume that they intersect transversally. $|c_1 \cap c_2|$ will then denote the number of intersection points between circles c_1 and c_2. (And so the algebraic intersection

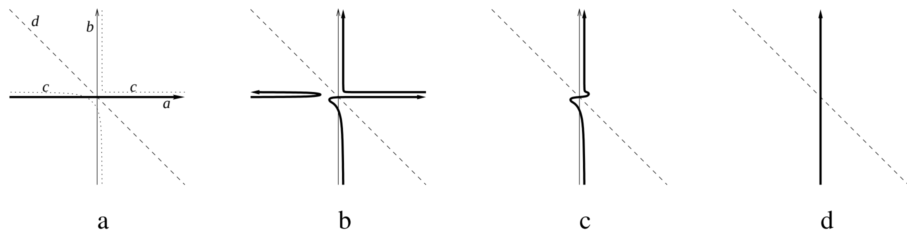

Figure 1:

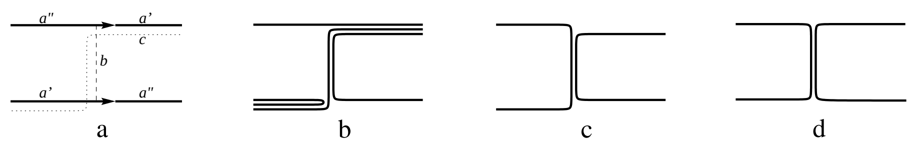

Figure 2:

$[c_1] \cdot [c_2]$ in H_1 is the reduction mod 2 of $|c_1 \cap c_2|$.) Given two circles a, b in F with $|a \cap b| = 1$, there are two ways for joining them into one circle c via surgery at their intersection point. c will be called a *merge* of a and b. If a and b are oriented, then c will be called the *positive* or *negative* merge of a and b, according to whether the surgery is performed so that the orientations of a and b match or do not match, respectively.

Lemma 4.2. *Let a, b be two oriented circles in F with $|a \cap b| = 1$, let P be their intersection point and let c be their negative merge.*

(1) *$\mathcal{T}_c$ followed by an isotopy performed in a thin neighborhood of $a \cup b$, maps a orientation preservingly onto b.*

(2) *If d is another circle passing P, and otherwise disjoint form a and b, and if a and b cross d at P in the same direction* (*as in Figure* 1a)*, then the above Dehn twist and isotopy may be performed while fixing d.*

Proof. See Figure 1. q.e.d.

Lemma 4.3. *Let a be a circle in F and b an arc connecting two points of a, and whose interior is disjoint from a. Assume that at the two endpoints of b, a passes b in the same direction* (*as in Figure* 2a)*. Let a', a'' be the two parts of a into which it is separated by ∂b and let $c = b \cup a'$. Then $(\mathcal{T}_c)^2$ followed by an isotopy performed in a thin* *neighborhood of* c*, maps* a *onto the circle obtained by surgering* a *along the arc* b *as in Figure* 2d.

Proof. See Figure 2. q.e.d.

The setting for the following lemma is that of Sections 2 and 3:

Lemma 4.4. *Assume* (V, g) *is not of the two special cases* $\dim V = 2$, $\mathrm{Arf}(g) = 0$ *and* $\dim V = 4$, $\mathrm{Arf}(g) = 0$. *Let* $w_1, \ldots, w_k \in V$, $k \geq 0$, *be independent vectors with* $g(w_i) = 1$ *and* $B(w_i, w_j) = 0$ *for all* $1 \leq i, j \leq k$. *Let* $W = \langle w_1, \ldots, w_k \rangle$ *(the subspace spanned by* $w_1, \ldots, w_k$*) and let* $a_1, a_2 \in W^\perp - W$ *be two vectors with* $g(a_1) = g(a_2) = 1$ *and* $B(a_1, a_2) = 0$. *Then there exists* $c \in W^\perp$ *with* $g(c) = 1$ *and* $B(a_1, c) = B(a_2, c) = 1$.

Proof. Assume first $k > 0$. We first find $b \in W^\perp$ such that $B(a_1, b) = B(a_2, b) = 1$. Since $a_1, a_2 \notin W = (W^\perp)^\perp$ there are $b_1, b_2 \in W^\perp$ with $B(a_1, b_1) = B(a_2, b_2) = 1$. If also $B(a_1, b_2) = 1$ or $B(a_2, b_1) = 1$ then we have a b. Otherwise $b_1 + b_2$ is our b. If $g(b) = 1$ we are done with $c = b$, otherwise take $c = b + w_1$.

Now assume $k = 0$, so $W = \{0\}$ and $W^\perp = V$. If $a_1 = a_2 = a$, take $b \in V$ with $B(a, b) = 1$. If $g(b) = 1$ we are done with $c = b$, otherwise define $U = \langle a, b \rangle$. If $a_1 \neq a_2$, take $b_1, b_2 \in V$ with $B(b_1, b_2) = 0$ and $B(a_i, b_j) = \delta_{ij}$ (Proposition 2.3). If $g(b_1 + b_2) = 1$ we are done with $c = b_1 + b_2$, otherwise define $U = \langle a_1, a_2, b_1, b_2 \rangle$. In either case $B(x, y)$ is nondegenerate on U, and so it is nondegenerate on $U^\perp$. If $\dim V > \dim U$ it follows that g cannot be identically 0 on $U^\perp$. Take any element $d \in U^\perp$ with $g(d) = 1$ then we are done with $c = b + d$ or $c = b_1 + b_2 + d$ respectively. So we are left with the case $\dim V = \dim U$. If $\dim V = 2$ then we have assumed $\mathrm{Arf}(g) = 1$ and so we must have had $g(b) = 1$. If $\dim V = 4$ then again we have assumed $\mathrm{Arf}(g) = 1$ and so $g(b_1) \neq g(b_2)$ (since $g(a_1) = g(a_2) = 1$) so again we must have had $g(b_1 + b_2) = 1$. q.e.d.

Remark 4.5. When $\dim V = 4$ then in the proof of Lemma 4.4 above, we haven't used the additional assumption that $\mathrm{Arf}(g) = 1$ in the following two cases:

(1) When $k > 0$.

(2) When $k = 0$ and $a_1 = a_2$ (since then $\dim V > \dim U$).

Lemma 4.6. *Assume* F, g *are not of the two special cases* $\mathrm{genus}(F) = 1$, $\mathrm{Arf}(g) = 0$ *and* $\mathrm{genus}(F) = 2$, $\mathrm{Arf}(g) = 0$. *Let* $w_1, \ldots, w_k$ *be disjoint circles in* F *with* $g([w_i]) = 1$ *and such that* $[w_1], \ldots, [w_k]$ *are* *independent in* H_1 *(which is equivalent to* $\bigcup_i w_i$ *not separating* F*). Let* a_1, a_2 *be oriented circles in* F *with* $g([a_1]) = g([a_2]) = 1$ *and such that* a_1, a_2 *are each disjoint from* $\bigcup_i w_i$ *and* $[a_1], [a_2] \notin \langle [w_1], \dots, [w_k] \rangle$*. Then there is a sequence* $h_1, \dots, h_m$ *of good maps of Type* 1 *and* 2 *which all fix* $\bigcup_i w_i$ *and such that* $h_1 \circ \cdots \circ h_m$ *(followed by an isotopy fixing* $\bigcup_i w_i$*) maps* a_1 *orientation preservingly onto* a_2*.*

Proof. Assume first that $[a_1] \cdot [a_2] = 1$. If actually $|a_1 \cap a_2| = 1$ then we are done by Lemma 4.2 since the merge c of a_1 and a_2 satisfies $g([c]) = g([a_1]) + g([a_2]) + [a_1] \cdot [a_2] = 1$ and so $\mathcal{T}_c$ is a good map of Type 2. So assume $|a_1 \cap a_2|$ is some odd number > 1. Then necessarily there exist two consecutive crossings along a_2, at which a_1 crosses a_2 in the same direction. Applying the map $(\mathcal{T}_c)^2$ of Lemma 4.3 (a is here a_1 and b is a portion of a_2) reduces $|a_1 \cap a_2|$ by precisely 2, and so again $[a_1] \cdot [a_2] = 1$ and so we may continue by induction.

Assume now $[a_1] \cdot [a_2] = 0$. By Lemma 4.4 there is $x \in H_1$ with $g(x) = 1$, $x \cdot [a_1] = x \cdot [a_2] = 1$ and $x \cdot [w_i] = 0$ for all i. ($x \notin \langle [w_1], \dots, [w_k] \rangle$ follows.) There exists a circle c in F with $[c] = x$ and such that c is disjoint from each w_i. (Start with any embedded representative and surger it along the w_is until it is disjoint from all of them. This is possible since the number of intersection points with each w_i is even. Then connect the various components to each other by surgery. This is possible since $F - \bigcup_i w_i$ is connected and since there are no orientations to consider.) By the previous case we may now map a_1 onto c, and from there, orientation preservingly onto a_2. q.e.d.

Remark 4.7. If F, g are of the special case $\mathrm{genus}(F) = 2, \mathrm{Arf}(g) = 0$ then if in Lemma 4.6 we further assume either that $[a_1] \cdot [a_2] = 1$ or that $[a_1] = [a_2]$ or that $k > 0$ then it follows from the proof of Lemma 4.6 and from Remark 4.5, that the conclusion of Lemma 4.6 still holds.

Definition 4.8. Let $\mathrm{genus}(F) = 2, \mathrm{Arf}(g) = 0$. A map $\mathcal{U} \in \mathcal{M}_g$ such that $\mathcal{U}_*$ is a U-map on H_1 (Definition 3.5) will again be called a U-map. (Such maps clearly exist.)

Theorem 4.9. *If* F, g *are not of the special case* $\mathrm{genus}(F) = 2$, $\mathrm{Arf}(g) = 0$ *then* $\mathcal{M}_g$ *is generated by the good maps. In the mentioned special case,* $\mathcal{M}_g$ *is generated by the good maps and any one* U*-map.*

Proof. We first assume we are not in the two special cases appearing in Lemma 4.6, in particular, we are not in the special case of this theorem. Let $n = \mathrm{genus}(F)$ and let $a_1, \dots, a_n, b_1, \dots, b_n$ be circles such that $|a_i \cap a_j| = |b_i \cap b_j| = 0$ $(i \neq j)$, $|a_i \cap b_j| = \delta_{ij}$ and $g([a_i]) = 1$.

(Start with such circles without the assumption on g. Then if for some i both circles have $g = 0$, replace one of them with their merge. Then exchange a_i with b_i if necessary.)

Now let $h \in \mathcal{M}_g$. We will compose h with good maps until (after isotopy) we arrive at the identity. We may first use Lemma 4.6 to bring the a_is one by one back to place. Indeed, after $a_1, \dots, a_k$ are in place, consider $a_1, \dots, a_k$, $h(a_{k+1})$, a_{k+1}, as the $w_1, \dots, w_k$, a_1, a_2 of Lemma 4.6, respectively, assigning orientations to a_{k+1} and $h(a_{k+1})$ which correspond via h.

So assuming h fixes all a_is, we bring each b_i back to place, and assume we have already done this for all $i < j$. Denote $a = a_j$, $b = b_j$ and let P be the intersection point of a and b. Since a is fixed by h, $h(b)$ must also pass P, and since h is orientation preserving, b and $h(b)$ must cross a at P in the same direction (where orientations on b and $h(b)$ correspond via h). Our permanent assumption that any circles we consider intersect transversally, may still be maintained for the intersection of b and $h(b)$ at P. Assume first that P is the only intersection point between b and $h(b)$. Let c be the negative merge of b and $h(b)$. $g([h(b)]) = g([b])$ (since h preserves g) and so $g([c]) = g([b]) + g([h(b)]) + [b] \cdot [h(b)] = 1$. By Lemma 4.2, $\mathcal{T}_c$ and an isotopy bring $h(b)$ orientation preservingly onto b, and the Dehn twist and isotopy may be performed while fixing a and while fixing all other a_is and all b_is with $i < j$. (Our $h(b)$, b, c and a, correspond to a, b, c and d of Lemma 4.2 respectively.)

So assume now that there are additional intersection points between $h(b)$ and b besides P. Choose a side of a in F. Let X be the first additional intersection point, when moving from P along b into the chosen side. Let a' be a circle parallel and close to a on the chosen side. $g([a']) = 1$. If a' is close enough to a, then a' intersects both b and $h(b)$ each at a single point, and so $|\mathcal{T}_{a'} \circ h(b) \cap b| = |h(b) \cap b| + 1$. We denote this new intersection point by P'. If we choose the orientation of the Dehn twist $\mathcal{T}_{a'}$ correctly, then $\mathcal{T}_{a'} \circ h(b)$ will cross b at P' in the same direction that it crosses b at X. Let c be the circle which is the union of the subarcs of b and $\mathcal{T}_{a'} \circ h(b)$ which are defined by P' and X, and which do not contain P, and so c is disjoint from a. By Lemma 4.3, $|(\mathcal{T}_c)^2 \circ \mathcal{T}_{a'} \circ h(b) \cap b| = |\mathcal{T}_{a'} \circ h(b) \cap b| - 2$. And so we have first increased the intersection by 1 and then decreased it by 2, and so we may continue by induction.

We are now in the situation that all a_is and b_is are fixed, so we are left with performing a map fixing all a_is and b_is, which is equivalent to performing a boundary fixing map on a sphere with n holes. The group of all such maps is known to be generated by Dehn twists. Now, any circle in the complement of the a_is and b_is is bounding in F, and so these Dehn twists are good maps of Type 3. This completes the proof for the general case.

The case $\mathrm{genus}(F) = 1, \mathrm{Arf}(g) = 0$ does not rely on the above, and will not be used in the sequel. We defer it to the end of Section 5.1.

We are left with the case $\mathrm{genus}(F) = 2, \mathrm{Arf}(g) = 0$. We will show that by adding any U-map $\mathcal{U}$, the above proof can be made to work. By Remark 4.7, the only problem we have is when moving the first circle a_1. Let V_1, V_2 be the pair of subsets of H_1 from the definition of U-map, and say $[a_1] \in V_1$. If $[h(a_1)] \in V_2$ then $[\mathcal{U} \circ h(a_1)] \in V_1$, and so we may assume $[a_1]$ and $[h(a_1)]$ are both in V_1. But then either $[a_1] = [h(a_1)]$ or $[a_1] \cdot [h(a_1)] = 1$. By Remark 4.7 again, Lemma 4.6 applies, and so the above process works. q.e.d.

Remark 4.10. If $p : \mathcal{M} \to GL(H_1)$ denotes the map $h \mapsto h_*$ then it is known that $p(\mathcal{M}) = Sp(H_1)$, the group of maps preserving the intersection form on H_1. It follows that $p(\mathcal{M}_g) = O(H_1, g)$ (since $O(H_1, g) \subseteq Sp(H_1)$ and $\mathcal{M}_g = p^{-1}(O(H_1, g))$.) Since good maps of Type 1 and 3 induce the identity on H_1 and since any V with non-degenerate quadratic form is isomorphic to (H_1, g) for appropriate F and g, we see that our Theorem 4.9 implies Theorem 2.6 (the Cartan-Dieudonne Theorem for the field $\mathbb{Z}/2$).

5. Quadruple Points of Regular Homotopies

Any immersion $i : F \to \mathbb{R}^3$ induces a quadratic form $g^i : H_1(F, \mathbb{Z}/2) \to \mathbb{Z}/2$ whose associated bilinear form $B(x, y)$ is the algebraic intersection form $x \cdot y$ of $H_1(F, \mathbb{Z}/2)$, as follows: For $x \in H_1(F, \mathbb{Z}/2)$ let $A \subseteq F$ be an annulus bounded by circles c, c' with $[c] = x$, let $j : A \to \mathbb{R}^3$ be an embedding which is regularly homotopic to $i|_A$ and define $g^i(x)$ to be the $\mathbb{Z}/2$ linking number between $j(c)$ and $j(c')$ in $\mathbb{R}^3$. One needs to verify that $g^i(x)$ is independent of the choices being made and that $g^i(x+y) = g^i(x) + g^i(y) + x \cdot y$. This has been done in [6] in the more general setting of surfaces which are not necessarily orientable. (It is then necessary for the quadratic form to attain values in $(\frac{1}{2}\mathbb{Z})/2 = \mathbb{Z}/4$ rather than $\mathbb{Z}/2$, to accommodate for the half twists of Mobious bands, and the Arf invariant attains values in $\mathbb{Z}/8$.) For immersions $i, i' : F \to \mathbb{R}^3$,

$i \sim i'$ will denote that i and i' are regularly homotopic in $\mathbb{R}^3$. The following has been shown in [6]:

Theorem 5.1. *Let $i, i' : F \to \mathbb{R}^3$ be two immersions.*

(1) $g^i = g^{i'}$ *iff* $i \sim i'$.

(2) $\mathrm{Arf}(g^i) = \mathrm{Arf}(g^{i'})$ *iff there exists a diffeomorphism* $h : F \to F$ *such that* $i \sim i' \circ h$.

(3) $\mathrm{Arf}(g^e) = 0$ *for any embedding* $e : F \to \mathbb{R}^3$.

Let $\widehat{\mathcal{M}}$ be the group of *all* diffeomorphisms of F (not necessarily orientation preserving) up to isotopy. Given a quadratic form g on H_1 (whose associated bilinear form is the intersection form) let $\widehat{\mathcal{M}}_g$ be the subgroup of $\widehat{\mathcal{M}}$ defined by $\widehat{\mathcal{M}}_g = \{h \in \widehat{\mathcal{M}} : h_* \in O(H_1, g)\}$. ($\mathcal{M}_g$ is then a subgroup of index 2 in $\widehat{\mathcal{M}}_g$.) Now let $i : F \to \mathbb{R}^3$ be an immersion and $h : F \to F$ a diffeomorphism. By Theorem 5.1(1), $i \sim i \circ h$ iff $g^i = g^{i \circ h}$. It is easy to see that $g^{i \circ h} = g^i \circ h_*$ and so we get:

Proposition 5.2. $i \sim i \circ h$ *iff* $h \in \widehat{\mathcal{M}}_{g^i}$.

Let $H_t : F \to \mathbb{R}^3$ be a generic regular homotopy. We denote by $q(H_t) \in \mathbb{Z}/2$ the number mod 2 of quadruple points occurring in H_t. The following has been shown in [5]:

Theorem 5.3. *Let F be any closed surface* (*not necessarily orientable or connected*). *If $H_t, G_t : F \to \mathbb{R}^3$ are two generic regular homotopies between the same two generic immersions, then $q(H_t) = q(G_t)$.*

Definition 5.4. Let $i, i' : F \to \mathbb{R}^3$ be two regularly homotopic generic immersions. We define $Q(i, i') \in \mathbb{Z}/2$ by $Q(i, i') = q(H_t)$, where H_t is any generic regular homotopy between i and i'. This is well defined by Theorem 5.3.

If $H_t, G_t : F \to \mathbb{R}^3$ are two regular homotopies such that the final immersion of H_t is the initial immersion of G_t, then $H_t * G_t$ will denote the regular homotopy that performs H_t and then G_t.

Lemma 5.5. *Let $i : F \to \mathbb{R}^3$ be a generic immersion. The map $\widehat{\mathcal{M}}_{g^i} \to \mathbb{Z}/2$ given by $h \mapsto Q(i, i \circ h)$ is a homomorphism.*

Proof. Let $h_1, h_2 \in \widehat{\mathcal{M}}_{g^i}$ and let H_t^k be a generic regular homotopy from i to $i \circ h_k$, $k = 1, 2$. Then $H_t^1 * (H_t^2 \circ h_1)$ is a regular homotopy from i to $i \circ h_2 \circ h_1$ and $q(H_t^1 * (H_t^2 \circ h_1)) = q(H_t^1) + q(H_t^2)$. q.e.d.

Recall that for $T \in O(V,g)$ we have defined

$$\psi(T) = \text{rank}(T - \text{Id}) \bmod 2,$$

and have shown that $\psi : O(V,g) \to \mathbb{Z}/2$ is a homomorphism (Theorem 3.3). For $h \in \widehat{\mathcal{M}}_g$ let $\epsilon(h) \in \mathbb{Z}/2$ be 0 or 1 according to whether h is orientation preserving or reversing, respectively, and let $n = \text{genus}(F)$. Since $\epsilon : \widehat{\mathcal{M}}_g \to \mathbb{Z}/2$ and $h \mapsto h_*$ are also homomorphisms, the following $\Psi : \widehat{\mathcal{M}}_g \to \mathbb{Z}/2$ is a homomorphism:

$$\begin{aligned}\Psi(h) &= \psi(h_*) + (n+1)\epsilon(h) \\ &= \big(\text{rank}(h_* - \text{Id}) + (n+1)\epsilon(h)\big) \bmod 2.\end{aligned}$$

Our purpose is to show:

Theorem 5.6. *Let $i : F \to \mathbb{R}^3$ be a generic immersion. Then for any $h \in \widehat{\mathcal{M}}_{g^i}$:*

$$Q(i, i \circ h) = \Psi(h).$$

Let $i : F \to \mathbb{R}^3$ be an immersion and let c be a circle in F such that c is disjoint from the multiplicity set of i. Adding a *ring* to i along c will mean to change i into a new immersion i' in the following way: If $X \subseteq Y$ then $N(X,Y)$ will denote a regular neighborhood of X in Y. Let $U = N(i(c), \mathbb{R}^3)$, thin enough so that $A = i^{-1}(U)$ is an annulus which is still disjoint from the multiplicity set. Let D and a be a disc and an arc. Let $f_1 : a \to D$ be a proper embedding and let $f_2 : a \to D$ be a proper immersion with one transverse intersection and such that $f_1|_{N(\partial a, a)} = f_2|_{N(\partial a, a)}$. Parametrize U and A as $D \times S^1$ and $a \times S^1$ so that $i|_A : A \to U$ will be given by $f_1 \times \text{Id} : a \times S^1 \to D \times S^1$. Now, the new immersion i' will be given by $f_2 \times \text{Id}$ on A, and will be the same as i outside A. There are basically two ways for adding a ring to i along c, depending on what side of A in $\mathbb{R}^3$ the ring will be facing (which in turn depends on our choice of $f_2 : a \to D$). If i is an embedding, then $i(F)$ separates $\mathbb{R}^3$ into two pieces, one compact and one noncompact. They will be denoted $M^0(i)$ and $M^1(i)$ respectively. And so if i is an embedding then we have a natural way for distinguishing the two possibilities for adding a ring along a given circle c, namely, the ring is facing either $M^0(i)$ or $M^1(i)$.

Note that $f_1 \times \text{Id}, f_2 \times \text{Id} : A \to U$ are homotopic relative ∂A (but not regularly homotopic). And so if $N = N(i(F), \mathbb{R}^3)$ with $N \supseteq U$, then i and i' are homotopic in N (but in general not regularly homotopic).

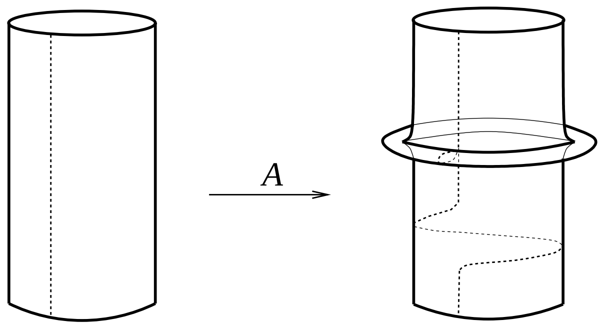

Figure 3: Move A

We now present two moves on immersions, that have been introduced in [5]. Let S_0 be an annulus and S_1 be a disc. Move A (resp. B) is a regular homotopy which is applied to a proper immersion of S_0 (resp. S_1) into a ball E and which fixes $N(\partial S_k, S_k), k = 0, 1$. Move A begins with the standard embedding of S_0 in E, and adds a ring and a Dehn twist along parallel essential circles in S_0. The ring may face either side of S_0 in E and the Dehn twist may have either orientation. Figure 3 shows one of the possibilities. The reverse move, going from right to left in Figure 3, will be called an A^{-1} move. Move B is described in Figure 4. It begins with a specific immersion of S_1, with two arcs of intersection, and replaces them with two other arcs of intersection. It is easy to see (and has been shown in [5]) that the initial and final immersions that we have presented for the A and B moves, are indeed regularly homotopic in E (while keeping $N(\partial S_k, S_k)$ fixed). Move A (resp. B) will be applied to an immersion $i : F \to \mathbb{R}^3$ inside a ball E in $\mathbb{R}^3$ for which $i^{-1}(E)$ is an annulus (resp. disc) and $i|_{i^{-1}(E)} : i^{-1}(E) \to E$ is as above. (The rest of F will be kept fixed.) In particular, an A move may be applied to a neighborhood of a circle c in F iff c is disjoint from the multiplicity set of i and there is an embedded disc D in $\mathbb{R}^3$ such that $D \cap i(F) = i(c)$. The move will then be performed in a thin $N(D, \mathbb{R}^3)$. If the circle along which we perform an A move happens to bound a disc in F, then the Dehn twist that is produced is trivial, and may be cancelled by rotating this disc. The following has been shown in [5]:

Proposition 5.7. *Let S_0, S_1, E denote an annulus, disc and ball respectively.*

(1) *For any generic regular homotopy $A_t : S_0 \to E$ that realizes an A*

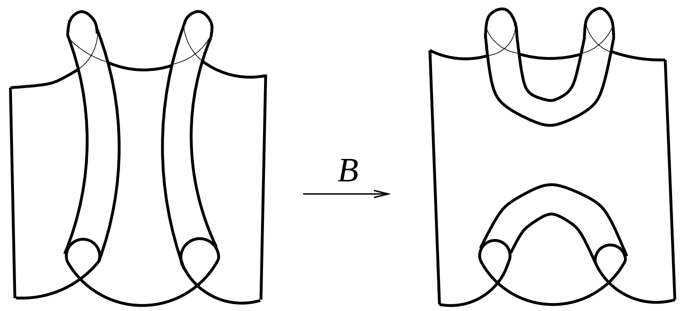

Figure 4: Move B

move, $q(A_t) = 1$.

(2) *For any generic regular homotopy* $B_t : S_1 \to E$ *that realizes the* B *move,* $q(B_t) = 1$.

If $H_t : F \to \mathbb{R}^3$ is given by $H : F \times [0,1] \to \mathbb{R}^3$ then we denote by H_{-t} the regular homotopy given by $(x,t) \mapsto H(x, 1-t)$. Clearly H_{-t} is generic iff H_t is generic, and $q(H_t) = q(H_{-t})$.

Lemma 5.8. *If* $i \sim i'$ *then* $Q(i, i \circ h) = Q(i', i' \circ h)$ *for any* $h \in \widehat{\mathcal{M}}_{g^i} = \widehat{\mathcal{M}}_{g^{i'}}$.

Proof. Let J_t be a regular homotopy from i to i' and H_t a regular homotopy from i to $i \circ h$. Then $J_{-t} * H_t * (J_t \circ h)$ is a regular homotopy from i' to $i' \circ h$ and $q(J_{-t} * H_t * (J_t \circ h)) = q(H_t)$. q.e.d.

Remark 5.9. After we prove Theorem 5.6, we will know that the assumption $i \sim i'$ in Lemma 5.8 is actually unnecessary, as long as $h \in \widehat{\mathcal{M}}_{g^i} \cap \widehat{\mathcal{M}}_{g^{i'}}$. This is so since $\Psi(h)$ does not depend on i. (It is a function of h only.)

We begin the proof of Theorem 5.6. First let genus$(F) = 0$. By Lemma 5.8 (and since all immersions of S^2 in $\mathbb{R}^3$ are regularly homotopic) we may assume i is an embedding onto the unit sphere. Let $r : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection with respect to the xy-plane, and $h \in \widehat{\mathcal{M}}$ be such that $i \circ h = r \circ i$, then h is the unique nontrivial element of $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_{g^i}$ and $\Psi(h) = \epsilon(h) = 1$. The following is a regular homotopy from i to $i \circ h$: Perform an A move on the equator of the sphere, such that the ring formed will be facing $M^1(i)$. Then exchange the northern and southern hemispheres, arriving at $i \circ h$. By Proposition 5.7 the

A move contributed 1 mod 2 quadruple points, then exchanging the hemispheres involved only double curves, and so $Q(i, i \circ h) = 1$. We proceed by induction on genus(F) beginning with genus$(F) = 1$. For the induction step we will use the generators of $\mathcal{M}_g$ introduced in Theorem 4.9, but for the starting point genus$(F) = 1$ we will introduce a different set of generators which will be more suitable. Furthermore, for genus$(F) = 1$ we will need to separate the cases $\mathrm{Arf}(g^i) = 0$ and $\mathrm{Arf}(g^i) = 1$.

5.1 The case genus$(F) = 1$, Arf$(g^i) = 0$

This has basically been done in [5]. We present it here with slight variation. Let T denote the torus. We will say an embedding $e : T \to \mathbb{R}^3$ is *standard* if its image is the torus $\widetilde{T} \subseteq \mathbb{R}^3$ obtained by rotating the circle $\{y = 0, (x-2)^2+z^2 = 1\}$ around the z-axis. Let $\widetilde{m}, \widetilde{l}$ be the circles in $\widetilde{T}$ given by $\widetilde{m} = \{y = 0, (x-2)^2+z^2 = 1\}$ and $\widetilde{l} = \{z = 0, x^2+y^2 = 1\}$ and choose some orientations for $\widetilde{m}$ and $\widetilde{l}$. For a standard embedding $e : T \to \mathbb{R}^3$, let m_e and l_e denote the oriented circles in T such that $e(m_e) = \widetilde{m}$ and $e(l_e) = \widetilde{l}$ (respecting orientations.)

Since $\mathrm{Arf}(g^i) = 0$ then by Theorem 5.1(2,3) i is regularly homotopic to a standard embedding. (Take an arbitrary standard embedding i', then $i \sim i' \circ h$ for some diffeomorphism $h : T \to T$, but $i' \circ h$ is again a standard embedding.) So by Lemma 5.8 we may assume i itself is a standard embedding. By viewing $m = m_i, l = l_i$ as the basis for $H_1(T, \mathbb{Z})$ (note $\mathbb{Z}$ coefficients) we identify $\widehat{\mathcal{M}}$ with $GL_2(\mathbb{Z})$. We will think of any $h \in \widehat{\mathcal{M}}$ both as a map from F to F and as a 2×2 matrix. If $h \in \widehat{\mathcal{M}}$ then $h_* : H_1 \to H_1$ (now $\mathbb{Z}/2$ coefficients) presented with respect to the basis $[m], [l]$ is simply the $\mathbb{Z}/2$ reduction of the matrix h. $g^i([m]) = g^i([l]) = 0$ and for $v = [m] + [l]$, $g^i(v) = 1$. A matrix $h \in GL_2(\mathbb{Z})$ is in $\widehat{\mathcal{M}}_{g^i}$ if h_* preserves g^i. This will happen iff $h_*(v) = v$ and $h_*(\{[m], [l]\}) = \{[m], [l]\}$, i.e., iff the $\mathbb{Z}/2$ reduction of h is either $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ or $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. By means of row and column operations one can show that this subgroup of $GL_2(\mathbb{Z})$ is generated by the following four elements: $A_1 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, $A_3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $A_4 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Since the $\mathbb{Z}/2$ reduction of A_1, A_2, A_3 is I and that of A_4 is J, and since $\psi(I) = 0$, $\psi(J) = 1$ and $n = 1$, we have $\Psi(A_1) = \Psi(A_2) = \Psi(A_3) = 0$ and $\Psi(A_4) = 1$. So we need to show that $Q(i, i \circ A_k) = 0$ for $k = 1, 2, 3$ and $Q(i, i \circ A_4) = 1$.

A_1 and A_2 are $(\mathcal{T}_m)^2$ and $(\mathcal{T}_l)^2$ respectively. Let D be a compressing disc for $i(m) = \widetilde{m}$ in $\mathbb{R}^3$ and $E = N(D, \mathbb{R}^3)$ thin enough so that $i(T) \cap E$

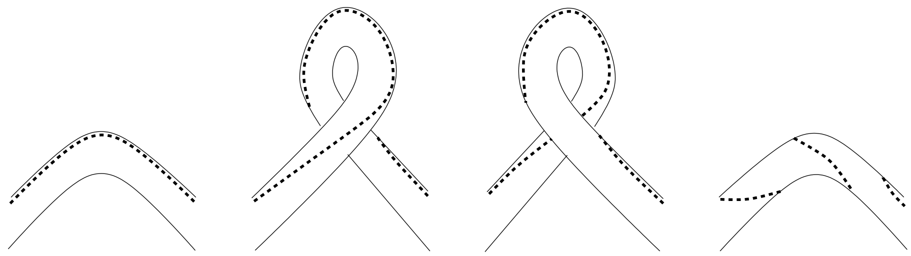

Figure 5:

is a standard annulus in E. In E we isotope $i(T)\cap E$ to be a thin tube, we then perform the "belt trick" on this tube (Figure 5) and then isotope $i(T)\cap E$ back to place. The effect of this regular homotopy is precisely $(\mathcal{T}_m)^2$ and it involves only double curves, and so $Q(i, i\circ A_1)=0$. In the same way $Q(i, i\circ A_2)=0$.

Up to isotopy in T, $i\circ A_3 = r\circ i$ where r is the reflection of $\mathbb{R}^3$ with respect to the xy-plane. This may be achieved by a regular homotopy similar to the one we have used for the case genus$(F)=0$, as follows: Perform an A move on each of the two circles $\{z=0, x^2+y^2=1\}$ and $\{z=0, x^2+y^2=3\}$, so that the ring formed by each of them is facing $M^1(i)$ and such that the two Dehn twists formed will have opposite orientations and so will cancel each other. (The A moves are performed inside thin neighborhoods of compressing discs for the two circles.) So we remain with just the two rings. We may now exchange the upper and lower halves of T until we arrive at $i\circ A_3$. The two A moves each contributed 1 mod 2 quadruple points and the final stage involved only double curves, and so all together $Q(i, i\circ A_3)=0$.

For A_4, isotope T until it has the shape of a large sphere with a tiny handle at its north pole. Now exchange the northern and southern hemispheres. This will involve only double curves, and will result in a sphere having a tiny handle at the south pole and a ring along the equator. We cancel this ring with an A^{-1} move in a thin neighborhood of the plane of the equator, resulting in an embedding again. We may think of m and l as being contained in the tiny handle, and so tiny compressing discs for m and l may be pulled along with the regular homotopy. The compressing disc of m now lies in $M^1(i)$, and the compressing disc of l now lies in $M^0(i)$. It follows that the final embedding may be isotoped to a standard embedding i.e., to a map of the form $i\circ h$, and this h

is orientation reversing and exchanges m and l. And so h must be either $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. These two maps composed with i are isotopic in $\mathbb{R}^3$ (via a half revolution about the x-axis) and so we may assume $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = A_4$. Our regular homotopy had a portion involving just double curves, then one A^{-1} move, and finally some isotopy, and so $Q(i, i \circ A_4) = 1$. This completes the proof that $Q(i, i \circ h) = \Psi(h)$ for every $h \in \widehat{\mathcal{M}}_{g^i}$ for F a torus and $\mathrm{Arf}(g^i) = 0$.

We now give the promised completion of the proof of Theorem 4.9. We need to show that $\mathcal{M}_g$ is generated by good maps when genus$(F) = 1, \mathrm{Arf}(g) = 0$. Choose two oriented circles a, b with $g([a]) = g([b]) = 0$ and $|a \cap b| = 1$ as a basis for $H_1(T, \mathbb{Z})$, thus identifying $\mathcal{M}$ with $SL_2(\mathbb{Z})$. Again we see $h \in SL_2(\mathbb{Z})$ is in $\mathcal{M}_g$ iff its $\mathbb{Z}/2$ reduction is either I or J. By row and column operations, we then see that $\mathcal{M}_g$ is generated by $A_1 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $A' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Now A_1 and A_2 are $(\mathcal{T}_a)^2$ and $(\mathcal{T}_b)^2$. If c is the positive merge of a and b then $g([c]) = 1$ and $\mathcal{T}_c$ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ we see $\mathcal{M}_g$ is generated by $(\mathcal{T}_a)^2$, $(\mathcal{T}_b)^2$ and $\mathcal{T}_c$.

5.2 The case genus$(F) = 1$, Arf$(g^i) = 1$

By Theorem 5.1(1), i is regularly homotopic to an immersion which is obtained from a standard embedding $e : T \to \mathbb{R}^3$ by adding a ring along the circle c which is the positive merge of m_e and l_e, and such that the ring is facing $M^1(e)$. (One checks directly that such an immersion i' has $\mathrm{Arf}(g^{i'}) = 1$, but then $g^i = g^{i'}$ since on V of dimension 2 there is only one g with $\mathrm{Arf}(g) = 1$.) By Lemma 5.8 we may assume i itself is this new immersion (Figure 6a). Since all nonzero elements in H_1 have $g^i = 1$, it follows that $O(H_1, g^i) = GL(H_1)$ and so $\widehat{\mathcal{M}}_{g^i} = \widehat{\mathcal{M}}$. Since $\mathbb{Z}/2$ is abelian, it is enough to verify $Q(i, i \circ h) = \Psi(h)$ only on normal generators, and we claim that $B_1 = \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $B_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are normal generators of $\widehat{\mathcal{M}}$ where the identification with $GL_2(\mathbb{Z})$ is via m_e, l_e. (There are no m_i, l_i since such are defined only for standard embeddings.) Indeed, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ which as we have noticed in the end of Section 5.1, is a Dehn twist along c (the positive merge of m_e and l_e.) Now, any Dehn twist is a normal generator of $\mathcal{M}$, and since B_1, B_2 are orientation reversing, they normally generate the whole of $\widehat{\mathcal{M}}$. As above, we see that $\Psi(B_1) = 0$ and $\Psi(B_2) = 1$, so we need to show that $Q(i, i \circ B_1) = 0$ and $Q(i, i \circ B_2) = 1$.

The regular homotopy we construct for B_1 is as follows: Let the ring become thicker, and at the same time let the "body" of the torus be-

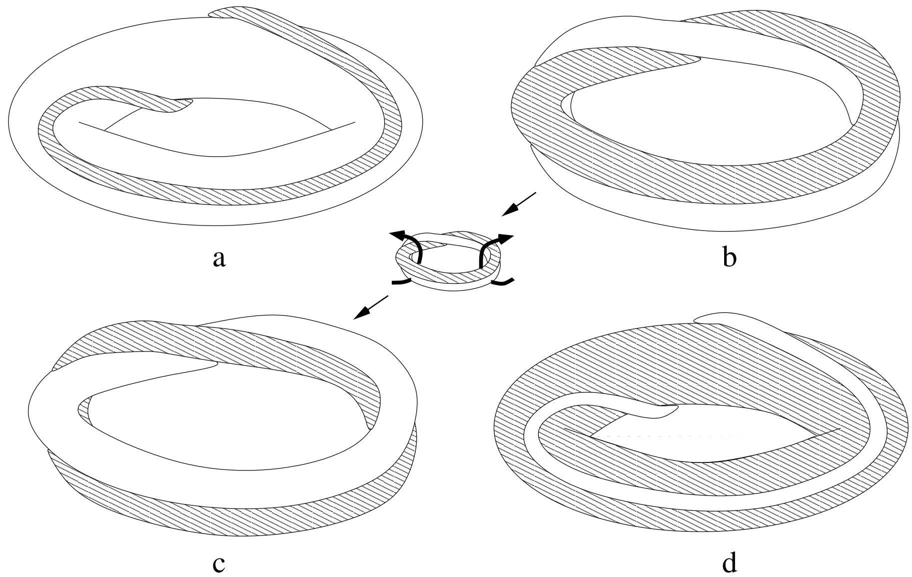

Figure 6:

come thinner, until they are equal in width, (Figure 6a→b.) Now using the intersection circle as an axis of rotation, we perform a half revolution of the torus around it, interchanging the two equal-width rings, (Figure 6b→c.) We then return to our original position by reversing our first step. (Figure 6c→d.) We thus arrive at an immersion of the form $i \circ h$. The map $h : T \to T$ may best be understood by looking at the intermediate stage of the regular homotopy, Figure 6b→c. We see here that h maps m_e to $-m_e$ and c to itself. i.e., the column $(1,0)^t$ to $(-1,0)^t$ and $(1,1)^t$ to itself. And so indeed $h = B_1$. We had no quadruple points (actually no singular occurrences at all) and so $Q(i, i \circ B_1) = 0$.

For B_2, we first imitate the regular homotopy we have had for $A_4 = B_2$ of Section 5.1 i.e., we perform that regular homotopy on e and carry the ring along. If before exchanging the upper and lower hemispheres we make sure that the ring is situated at the tiny handle, then this exchange will have at most triple points, and the ring will not interfere with the A^{-1} move, and so at the end of this process we will have $q = 1$. The immersion j we arrive at, is the immersion obtained by adding a ring R to the embedding $e \circ B_2$ along the circle $e \circ B_2(c) = e(c)$, that is the same circle along which R was originally situated, but now R is facing $M^0(e)$ instead of $M^1(e)$ and so j is not of the form $i \circ h$. We resolve

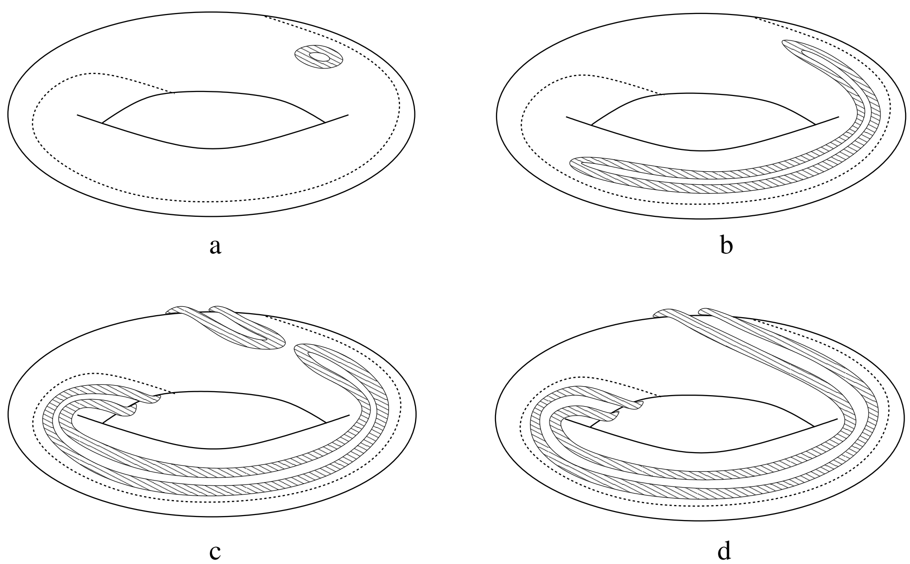

Figure 7:

this with a regular homotopy as follows: Perform an A move on a little disc bounding circle in T near R, forming a ring R' facing $M^1(e)$. See Figure 7a. (The dotted line in Figure 7 is the intersection curve of R. R itself is not seen since it is facing $M^0(e)$.) We then elongate R' along side R, until it approaches itself from all the way around. (Figure 7a→b→c.) We then perform a B move, turning R' into two rings which are parallel to R, but facing $M^1(e)$. (Figure 7c→d.) It is then easy to construct an explicit regular homotopy so that R and the new ring which is adjacent to it, will cancel each other, and with no quadruple points at all. (The idea is as follows: Let $f : a \to D$ be a proper immersion of an arc a into a disc D with two loops facing opposite sides, as in the front disc of Figure 8a. There is a regular homotopy $f_t : a \to D$ fixing $N(\partial a, a)$, from f to an embedding as in the front disc of Figure 8b. If f_t is generic then it has at most triple points. Now the regular homotopy $f_t \times \mathrm{Id} : a \times S^1 \to D \times S^1$ is a regular homotopy which begins with a pair of rings facing opposite sides, and cancels them. Figure 8 depicts a portion of $a \times S^1$ in the initial and final immersions. Indeed since f_t has at most triple points, so will $f_t \times \mathrm{Id}$, but $f_t \times \mathrm{Id}$ is not generic. Nevertheless, any specific $f_t \times \mathrm{Id}$ may serve as a guide for constructing an explicit generic regular homotopy with no quadruple points, and which

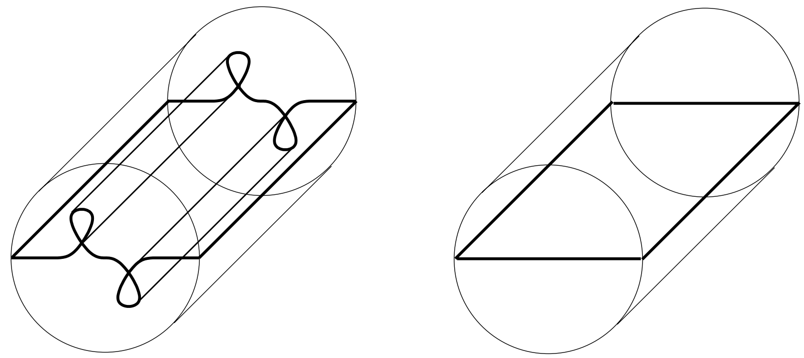

Figure 8:

begins and ends with the same immersions.)

Since R and one of the new rings have disappeared, we are left with one ring facing $M^1(e)$ and situated along a circle parallel to $e(c)$. By pushing it precisely to $e(c)$ we finally get an immersion of the form $i \circ h$. The regular homotopy which started with j and replaced the ring R with a ring facing $M^1(e)$, took place inside some $N = N(e(F), \mathbb{R}^3)$, and so the homotopy class into N is still that of $e \circ B_2$ and so (up to isotopy in F) the new immersion is indeed $i \circ B_2$. ($i : F \to N$ is a homotopy equivalence, and so two diffeomorphisms $h, h' : F \to F$ are isotopic in F iff $i \circ h, i \circ h' : F \to N$ are homotopic in N.) Finally, our regular homotopy from i to j involved 1 mod 2 quadruple points, then from j to $i \circ B_2$ we had one A move, one B move, and a regular homotopy with no quadruple points, and so all together indeed $Q(i, i \circ B_2) = 1$.

5.3 The general case

Assume genus$(F) > 1$. By Theorem 4.9 $\mathcal{M}_{g^i}$ is generated by a set of Dehn twists and squares of Dehn twists, and in the special case genus$(F) = 2, \mathrm{Arf}(g^i) = 0$ we also need a U-map. Other than the special U-map generator, which will be dealt with last, each generator h fixes all but a regular neighborhood of a circle a (and we may assume a does not bound a disc in F). If a is non-separating then there is a circle b in F with $|a \cap b| = 1$. $N(a \cup b, F)$ is a punctured torus, and so $c = \partial N$ is a circle separating F into two subsurfaces F_1, F_2 of smaller genus than F and with $h(F_k) = F_k$. If a is separating, then a nearby

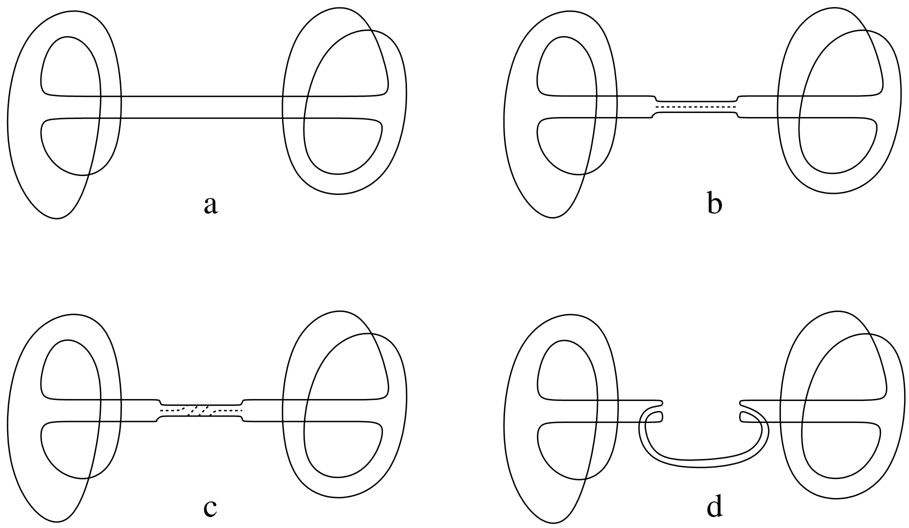

Figure 9:

parallel circle c will again separate F in this way. Now $\widehat{\mathcal{M}}_{g^i}$ needs one additional generator. Choosing any separating circle c in F there is clearly an orientation reversing $h : F \to F$ which preserves the two sides of c and which induces the identity on H_1 and so $h \in \widehat{\mathcal{M}}_{g^i}$. And so finally we have a set of generators for $\widehat{\mathcal{M}}_{g^i}$, each of which (except for the U-map) preserves such a separation of F by a circle c into F_1, F_2 of smaller genus.

Let $A = N(c, F)$. Slightly diminishing F_1, F_2 to be the components of $F - \mathrm{int}A$, we may still assume $h(F_k) = F_k$, $k = 1, 2$. Since c is separating in F, $g^i([c]) = 0$ and so $i|_A$ is regularly homotopic to a standard embedding of A, in the shape of a thin tube. By means of [7] (namely the proof of Theorem 2.1) we may extend such a regular homotopy of A to the whole of F. We now stretch this tube to be very long, at the same time pulling F_1 and F_2 rigidly away from each other until they are disjoint. See Figure 9a. By taking a smaller A if necessary, we may assume $i(A)$ is disjoint from $i(F - A)$, Figure 9b. By Lemma 5.8, we may assume that this is in fact our immersion i. Let $\overline{F_1}, \overline{F_2}$ be the closed surfaces obtained by gluing a disc D_k to F_k and let $h_k : \overline{F_k} \to \overline{F_k}$ be an extension of $h|_{F_k} : F_k \to F_k$. If the tube $i(A)$ is very thin, then there is also a naturally defined extension $i_k : \overline{F_k} \to \mathbb{R}^3$ of $i|_{F_k}$. We may further assume that the thin ball B in $\mathbb{R}^3$ which is bounded by the sphere $i_1(D_1) \cup i(A) \cup i_2(D_2)$, is disjoint from $i(F - A)$.

Since $h|_{F_k}$ preserves $g^i|_{H_1(F_k,\mathbb{Z}/2)}$ then h_k preserves g^{i_k}. It follows that there is a regular homotopy H_t^k between i_k and $i_k \circ h_k$. We perform H_t^1 and H_t^2 inside disjoint balls, and we let the thin tube A be carried along. We can make sure that no quadruple point of H_t^k occurs in D_k ($k = 0, 1$) and that the very thin tube A does not pass triple points. The regular homotopy H_t induced on F in this way will then have the sum of the numbers of quadruple points of H_t^1 and H_t^2.

Now if h is orientation preserving then so are h_k, in particular $h_k|_{D_k}$ is orientation preserving. So if we had carried the thin ball B along with the tube A, then it would now approach the D_ks from the same side it had for i. And so we may continue the regular homotopy on the tube A, still not passing through triple points, and cancelling all knotting by having the thin tube pass itself, until it is back to its original place, and this will not contribute any quadruple points. However, the new embedding of A may differ from $i \circ h|_A$ by some number of Dehn twists as in Figure 9c. We may resolve this by rigidly rotating say F_1 around the axis of the tube.

If on the other hand h is orientation reversing, then after applying H_t^1 and H_t^2 and carrying the tube along, the thin ball B will approach both D_ks from the wrong side. And so after we cancel all knotting, the tube A will be as in Figure 9d. Figure 10 presents a regular homotopy that resolves this, and has 1 mod 2 quadruple points. Figure 10a depicts the relevant part of Figure 9d, where the regular homotopy will take place. Figure 10a→b→c is a regular homotopy with no singular occurrences, or alternatively may be thought of as an ambient isotopy of $\mathbb{R}^3$. It shows that we may view the immersion of A as a sphere with two rings facing outward, each of which has a tube attached to it. We now perform a B move which joins the two rings into one ring with two tubes attached to it, Figure 10c→d. Again by ambient isotopy, the ring may be brought to the equator, Figure 10d→e. Finally we exchange the northern and southern halves of the sphere, arriving at an embedding, Figure 10e→f. This regular homotopy involved a B move and a portion involving only double curves, and so indeed it had 1 mod 2 quadruple points. We then continue to bring A back to place. As above, the new embedding of A may differ from $i \circ h|_A$ by some number of Dehn twists, and those may be cancelled by rigidly rotating F_1.

We have thus constructed a regular homotopy between i and $i \circ h$ such that the number mod 2 of quadruple points, is the sum of the numbers

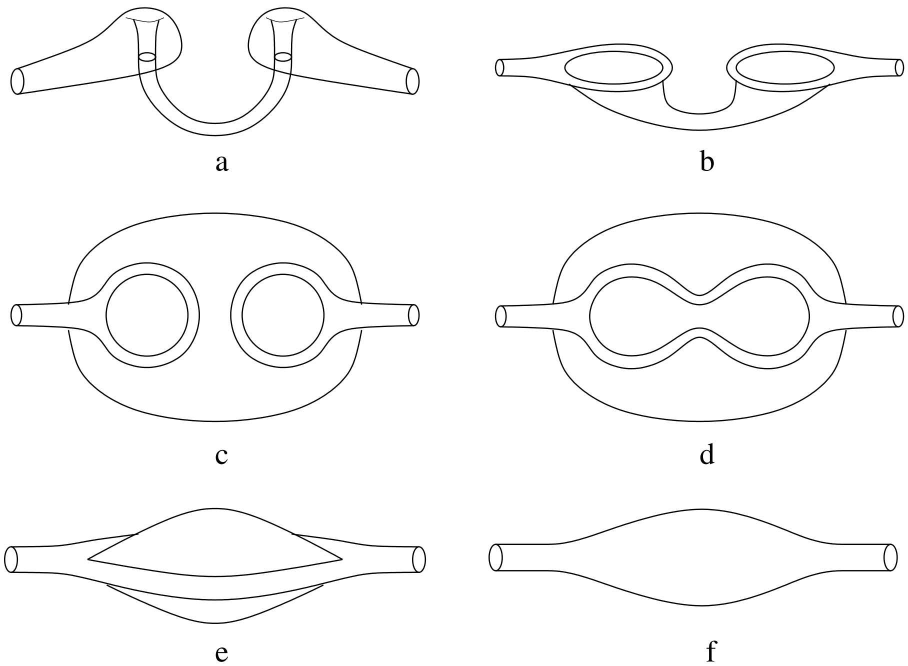

Figure 10:

occurring in the $\overline{F_k}$s in case h is orientation preserving, and the sum plus 1, in case h is orientation reversing. In other words $Q(i, i \circ h) = Q(i_1, i_1 \circ h_1) + Q(i_2, i_2 \circ h_2) + \epsilon(h)$. By the induction hypothesis, $Q(i_k, i_k \circ h_k) = \Psi(h_k)$, $k = 1, 2$. Let $n_k = \text{genus}(\overline{F_k})$ and notice $n = n_1 + n_2$, $\epsilon(h_k) = \epsilon(h)$ and $\text{rank}(h_* - \text{Id}) = \text{rank}(h_{1*} - \text{Id}) + \text{rank}(h_{2*} - \text{Id})$ so $\psi(h_*) = \psi(h_{1*}) + \psi(h_{2*})$. So finally: $Q(i, i \circ h) = \Psi(h_1) + \Psi(h_2) + \epsilon(h) = \psi(h_{1*}) + (n_1 + 1)\epsilon(h) + \psi(h_{2*}) + (n_2 + 1)\epsilon(h) + \epsilon(h) = \psi(h_*) + (n_1 + n_2 + 1)\epsilon(h) = \Psi(h)$.

We now deal with the special U-map generator which appears when $\text{genus}(F) = 2, \text{Arf}(g^i) = 0$. Since $\text{Arf}(g^i) = 0$, then by Theorem 5.1(2,3) and Lemma 5.8 as before, we may assume i is an embedding whose image is two embedded tori connected to each other with a tube, and such that a half revolution around some line in $\mathbb{R}^3$ maps $i(F)$ onto itself and interchanges the two tori. Let $h : F \to F$ be the map such that $i \circ h$ is the final embedding of this half revolution, then h is orientation preserving and so $h \in \mathcal{M}_{g^i}$. Take a circle c in one of the tori with $g^i([c]) = 1$, then $h(c)$ lies in the other torus, and so $[c] \neq h_*([c])$ and

$[c] \cdot h_*([c]) = 0$. It follows that $[c]$ and $h_*([c])$ are not in the same V_k of the definition of U-map, so h_* must be a U-map, and so h is indeed a U-map on F. Since $h^2 = \mathrm{Id}$ then by Lemma 3.6, $\psi(h_*) = 0$ and so $\Psi(h) = \psi(h_*) + \epsilon(h) = 0$. Since there is a rigid rotation between i and $i \circ h$, $Q(i, i \circ h) = 0$. This completes the proof of Theorem 5.6.

6. Totally Singular Decompositions

Back to the setting of Sections 2 and 3, let V be a finite dimensional vector space over $\mathbb{Z}/2$ with nondegenerate quadratics form $g : V \to \mathbb{Z}/2$.

A subspace $A \subseteq V$ such that $g|_A \equiv 0$ is called a *totally singular* subspace. A pair (A, B) of subspaces of V will be called a *totally singular decomposition* of V (abbreviated TSD) if $V = A \oplus B$ and both A and B are totally singular. It then follows that $\dim A = \dim B$. (From Lemma 6.1 below it follows that (V, g) admits a TSD iff $\mathrm{Arf}(g) = 0$.)

The proof of the following lemma appears in [1]:

Lemma 6.1. *Let* $\dim V = 2n$.

(1) *If $A \subseteq V$ is a totally singular subspace of dimension n then there exists a $B \subseteq V$ such that (A, B) is a TSD of V.*

(2) *If (A, B) is a TSD of V and $a_1, \ldots, a_n$ is a given basis for A then there is a basis $b_1, \ldots, b_n$ for B such that $B(a_i, b_j) = \delta_{ij}$.*

Definition 6.2. If (A, B) is a TSD of V then a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ of V will be called (A, B)-*good* if $a_i \in A$, $b_i \in B$ and $B(a_i, b_j) = \delta_{ij}$.

The following two lemmas follow directly from the definition of quadratic form:

Lemma 6.3. *Let (A, B) be a TSD of V and $a_1, \ldots, a_n, b_1, \ldots, b_n$ an (A, B)-good basis for V. If $v = \sum x_i a_i + \sum y_i b_i$ and $v' = \sum x'_i a_i + \sum y'_i b_i$ then $g(v) = \sum x_i y_i$ and $B(v, v') = \sum x_i y'_i + \sum y_i x'_i$.*

Lemma 6.4. *Let (A, B) and (A', B') be two TSDs of V. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be an (A, B)-good basis for V and $a'_1, \ldots, a'_n, b'_1, \ldots, b'_n$ an (A', B')-good basis for V. If $T : V \to V$ is the linear map defined by $a_i \mapsto a'_i$, $b_i \mapsto b'_i$ then $T \in O(V, g)$.*

We now study the relationship between TSDs and our homomorphism $\psi : O(V, g) \to \mathbb{Z}/2$.

Lemma 6.5. *If (A, B) is a TSD of V and $T \in O(V, g)$ satisfies $T(A) = A$ and $T(B) = B$ then $\psi(T) = 0$.*

Proof. By Lemma 6.1 there exists an (A, B)-good basis $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ for V. Using Lemma 6.3 it is easy to verify that the matrix of T with respect to such a basis has the form: $\begin{pmatrix} S^t & 0 \\ 0 & S^{-1} \end{pmatrix}$ where $S \in GL_n(\mathbb{Z}/2)$. It follows that $\psi(T) = 0$. q.e.d.

Given two TSDs (A, B), (A', B') of V then by Lemmas 6.1 and 6.4 there exists a $T \in O(V, g)$ such that $T(A) = A'$ and $T(B) = B'$. It follows from Lemma 6.5 that if T_1, T_2 are two such Ts then $\psi(T_1) = \psi(T_2)$. And so the following is well defined:

Definition 6.6. For a pair (A, B), (A', B') of TSDs of V let $\widehat{\psi}(A, B; A', B') \in \mathbb{Z}/2$ be defined by $\widehat{\psi}(A, B; A', B') = \psi(T)$ for some (thus all) $T \in O(V, g)$ with $T(A) = A'$ and $T(B) = B'$.

Definition 6.7. For two TSDs (A, B), (A', B') of V, we will write $(A, B) \sim (A', B')$ if $\widehat{\psi}(A, B; A', B') = 0$.

Since ψ is a homomorphism, $\widehat{\psi}(A, B; A'', B'') = \widehat{\psi}(A, B; A', B') + \widehat{\psi}(A', B'; A'', B'')$ for any three TSDs $(A, B), (A', B'), (A'', B'')$. It follows that $\sim$ is an equivalence relation with precisely two equivalence classes and that $\widehat{\psi}(A, B; A'', B'') = \widehat{\psi}(A', B'; A'', B'')$ whenever $(A, B) \sim (A', B')$.

Lemma 6.8. *Let* $\dim V = 2n$ *and let $A \subseteq V$ be a totally singular subspace of dimension n. If $T \in O(V, g)$ satisfies $T(x) = x$ for every $x \in A$ then $\psi(T) = 0$.*

Proof. By Lemma 6.1 there is a $B \subseteq V$ such that (A, B) is a TSD of V and an (A, B)-good basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ for V. Using Lemma 6.3 it is easy to verify that the matrix of T with respect to such a basis has the form: $\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$ where I is the $n \times n$ identity matrix and $S \in M_n(\mathbb{Z}/2)$ is an alternating matrix, i.e., if $S = \{s_{ij}\}$ then $s_{ii} = 0$ and $s_{ij} = s_{ji}$ $(1 \leq i, j \leq n.)$ Since alternating matrices have even rank, it follows that $\psi(T) = 0$. q.e.d.

Corollary 6.9. *Let (A, B) and (A', B') be two TSDs of V. If $A = A'$ or $B = B'$ then $(A, B) \sim (A', B')$.*

Proof. Say $A = A'$. By Lemmas 6.1 and 6.4 there exists a $T \in O(V, g)$ with $T(x) = x$ for all $x \in A = A'$ and $T(B) = B'$. The conclusion follows from Lemma 6.8. q.e.d.

Let $V_0, V_1 \subseteq V$ be two subspaces of V. We will write $V_0 \perp V_1$ if $B(x,y) = 0$ for every $x \in V_0$, $y \in V_1$. The following is clear:

Lemma 6.10. *Let $V_0, V_1 \subseteq V$ satisfy $V = V_0 \oplus V_1$ and $V_0 \perp V_1$.*

(1) *If for $l = 0, 1$, (A_l, B_l) is a TSD of V_l (with respect to $g|_{V_l}$ which is indeed nondegenerate) then $(A_0 + A_1, B_0 + B_1)$ is a TSD of V.*

(2) *If (A'_l, B'_l) is another TSD of V_l and $(A_l, B_l) \sim (A'_l, B'_l)$ $(l = 0, 1)$ then $(A_0 + A_1, B_0 + B_1) \sim (A'_0 + A'_1, B'_0 + B'_1)$.*

7. Statement of Result on Embeddings

If $e : F \to \mathbb{R}^3$ is an embedding then $e(F)$ splits $\mathbb{R}^3$ into two pieces, one compact and one noncompact, which we have denoted $M^0(e)$ and $M^1(e)$ respectively. By restriction of range, e induces maps $e^k : F \to M^k(e)$, $k = 0, 1$. Let $e^k_* : H_1(F, \mathbb{Z}/2) \to H_1(M^k(e), \mathbb{Z}/2)$ be the map induced by e^k and finally let $A^k(e) = \ker e^k_*$, $k = 0, 1$.

Lemma 7.1. *Let $e : F \to \mathbb{R}^3$ be an embedding, then $(A^0(e), A^1(e))$ is a TSD of $H_1(F, \mathbb{Z}/2)$ with respect to the quadratic form g^e.*

Proof. We first show that each $A^k(e)$ is totally singular: For $x \in A^k(e)$ let A, c, c' be as in the definition of $g^e(x)$ and simply take $j = e|_A$. Since $e^k_*(x) = 0$, $e(c)$ bounds a properly embedded (perhaps non-orientable) surface S in $M^k(e)$. Since $e(c')$ is disjoint from S, the $\mathbb{Z}/2$ linking number between $e(c)$ and $e(c')$ in $\mathbb{R}^3$ is 0, and so $g^e(x) = 0$. Now, the fact that $H_1(F, \mathbb{Z}/2) = A^0(e) \oplus A^1(e)$ is a consequence of the $\mathbb{Z}/2$ Mayer-Vietoris sequence for $\mathbb{R}^3 = M^0(e) \cup M^1(e)$ where F is identified with $M^0(e) \cap M^1(e)$ via e. q.e.d.

If $e, e' : F \to \mathbb{R}^3$ are two regularly homotopic embeddings then $g^e = g^{e'}$ so $(A^0(e), A^1(e))$ and $(A^0(e'), A^1(e'))$ are TSDs of $H_1(F, \mathbb{Z}/2)$ with respect to the same quadratic form and so $\widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e'))$ is defined. We spell out the actual computation involved in $\widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e'))$:

(1) Find a basis $a_1, \dots, a_n, b_1, \dots, b_n$ for $H_1(F, \mathbb{Z}/2)$ such that $e^0_*(a_i) = 0$, $e^1_*(b_i) = 0$ and $a_i \cdot b_j = \delta_{ij}$.

(2) Find a similar basis $a'_1, \dots, a'_n, b'_1, \dots, b'_n$ using e' in place of e.

(3) Let m be the dimension of the subspace of $H_1(F,\mathbb{Z}/2)$ spanned by:

$$a_1' - a_1 \ , \ \dots \ , \ a_n' - a_n \ , \ b_1' - b_1 \ , \ \dots \ , \ b_n' - b_n.$$

(4) $\widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) = m \bmod 2$, (an element in $\mathbb{Z}/2$.)

Definition 7.2. If $e : F \to \mathbb{R}^3$ is an embedding then we define $o(e)$ to be the orientation on F which is induced from $M^0(e)$ to $\partial M^0(e) = e(F)$ and then via e to F (and where the orientation on $M^0(e)$ is the restriction of the orientation of $\mathbb{R}^3$.) If $e, e' : F \to \mathbb{R}^3$ are two embeddings then we define $\widehat{\epsilon}(e,e') \in \mathbb{Z}/2$ to be 0 if $o(e) = o(e')$ and 1 if $o(e) \neq o(e')$.

Our purpose is to show:

Theorem 7.3. *Let n be the genus of F. If $e, e' : F \to \mathbb{R}^3$ are two regularly homotopic embeddings then:*

$$Q(e,e') \ = \ \widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) \ + \ (n+1)\widehat{\epsilon}(e,e').$$

8. Equivalent Embeddings and k-Extendible Regular Homotopies

Let $e : F \to \mathbb{R}^3$ be an embedding, let $P \subseteq \mathbb{R}^3$ be a plane and assume $e(F)$ intersects P transversally in a unique circle. Let $c = e^{-1}(P)$ then c is a separating circle in F. Let A be a regular neighborhood of c in F and let F_0, F_1 be the connected components of $F - \mathrm{int}A$. (A lower index will always be related to the splitting of $\mathbb{R}^3$ via a plane, the assignment of 0 and 1 to the two sides being arbitrary. An upper index on the other hand is related to the splitting of $\mathbb{R}^3$ via the image of a closed surface, assigning 0 to the compact side and 1 to the noncompact side.) Let $\overline{F_l}$ $(l = 0,1)$ be the closed surface obtained by gluing a disc D_l to F_l. Let $e_l : \overline{F_l} \to \mathbb{R}^3$ be the embedding such that $e_l|_{F_l} = e|_{F_l}$ and $e_l(D_l)$ is parallel to P. Let $i_{F_l F} : F_l \to F$ and $i_{F_l \overline{F_l}} : F_l \to \overline{F_l}$ denote the inclusion maps. The induced map $i_{F_l \overline{F_l}*} : H_1(F_l, \mathbb{Z}/2) \to H_1(\overline{F_l}, \mathbb{Z}/2)$ is an isomorphism and let $s_l : H_1(\overline{F_l}, \mathbb{Z}/2) \to H_1(F, \mathbb{Z}/2)$ be the map $s_l = i_{F_l F*} \circ (i_{F_l \overline{F_l}*})^{-1}$.

Lemma 8.1. *Under the above assumptions and definitions: $A^k(e) = s_0(A^k(e_0)) + s_1(A^k(e_1))$, $k = 0,1$.*

Proof. This follows from the fact that the inclusions $F_0 \cup F_1 \to \overline{F_0} \cup \overline{F_1}$, $F_0 \cup F_1 \to F$, $M^0(e_0) \cup M^0(e_1) \to M^0(e)$ and $M^1(e) \to \mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ all induce isomorphisms on $H_1(\cdot, \mathbb{Z}/2)$ and the splitting of each of the above spaces via P induces a direct sum decomposition of $H_1(\cdot, \mathbb{Z}/2)$. We only check that the inclusion $M^1(e) \to \mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ induces isomorphism on $H_1(\cdot, \mathbb{Z}/2)$. Indeed $\mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ is obtained from $M^1(e)$ by gluing a 2-handle along $e(A)$, and the inclusion of $e(A)$ in $M^1(e)$ is null-homotopic. q.e.d.

Definition 8.2. Two embeddings $e, f : F \to \mathbb{R}^3$ will be called *equivalent* if:

(1) There is a regular homotopy between e and f with no quadruple points.

(2) $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$.

(3) $o(e) = o(f)$.

Definition 8.3. An embedding $e : F \to \mathbb{R}^3$ will be called *standard* if its image $e(F)$ is a surface in $\mathbb{R}^3$ as in Figure 11.

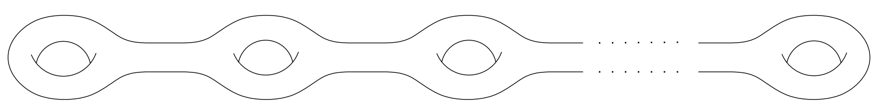

Figure 11: Image of a standard embedding.

In Proposition 8.8 below we will show that any embedding $e : F \to \mathbb{R}^3$ is equivalent to a standard embedding. The proof will be by induction on genus(F) and the following lemma will be used in the induction step:

Lemma 8.4. *Let $e : F \to \mathbb{R}^3$ be an embedding. Assume $e(F)$ intersects a plane $P \subseteq \mathbb{R}^3$ transversally in one circle and let $c, A, F_l, \overline{F_l}, D_l, e_l$ be as above. If $e_l : \overline{F_l} \to \mathbb{R}^3$ $(l = 0, 1)$ are both equivalent to standard embeddings, then e is equivalent to a standard embedding.*

Proof. Changing e by isotopy, we may assume $e(A)$ is a very thin tube. $e_l : \overline{F_l} \to \mathbb{R}^3$ is equivalent to a standard embedding f_l via a regular homotopy $(H_l)_t : \overline{F_l} \to \mathbb{R}^3$ having no quadruple points. We may further assume that each $(H_l)_t$ moves $\overline{F_l}$ only within the corresponding half-space defined by P, that each $f_l(D_l)$ is situated at the point of

$f_l(\overline{F_l})$ which is closest to P and that these two points are opposite each other with respect to P. We now perform both $(H_l)_t$, letting the thin tube A be carried along. If we make sure the thin tube A does not pass triple points occurring in F_1 and F_2 then the regular homotopy H_t induced on F in this way will also have no quadruple points. Since $e(A)$ has approached $e_l(\overline{F_l})$ from $M^1(e_l)$ and since $o_{e_l} = o_{f_l}$, we also have at the end of H_t that A approaches $f_l(\overline{F_l})$ from $M^1(f_l)$. And so we may continue moving the tube A until it is all situated in the region between $f_0(\overline{F_0})$ and $f_1(\overline{F_1})$, then canceling all knotting by having the thin tube pass itself (this involves only double lines) until A is embedded as a straight tube connecting $f_0(\overline{F_0})$ to $f_1(\overline{F_1})$ and so the final map $f : F \to \mathbb{R}^3$ thus obtained is indeed a standard embedding. By assumption $(A^0(e_l), A^1(e_l)) \sim (A^0(f_l), A^1(f_l))$, $l = 0, 1$ which implies that $(s_l(A^0(e_l)), s_l(A^1(e_l))) \sim (s_l(A^0(f_l)), s_l(A^1(f_l)))$, $l = 0, 1$ as TSDs of $V_l = s_l(H_1(\overline{F_l}, \mathbb{Z}/2)) \subseteq H_1(F, \mathbb{Z}/2)$. (Note that s_l preserves the corresponding quadratic forms.) But $H_1(F, \mathbb{Z}/2) = V_0 \oplus V_1$ and $V_0 \perp V_1$ and so by Lemma 6.10 and Lemma 8.1 $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$. Finally, from $o_{e_l} = o_{f_l}$ it follows that $o(e) = o(f)$. q.e.d.

Definition 8.5. Let $e, f : F \to \mathbb{R}^3$ be two embeddings. A regular homotopy $H_t : F \to \mathbb{R}^3$ $(a \leq t \leq b)$ with $H_a = e$, $H_b = f$ will be called *k-extendible* (where k is either 0 or 1) if there exists a regular homotopy $G_t : M^k(e) \to \mathbb{R}^3$ $(a \leq t \leq b)$ satisfying:

(1) G_a is the inclusion map of $M^k(e)$ in $\mathbb{R}^3$.

(2) $H_t = G_t \circ e^k$. (Recall that $e^k : F \to M^k(e)$ is simply e with range restricted to $M^k(e)$.)

(3) G_b is an embedding with $G_b(M^k(e)) = M^k(f)$.

Lemma 8.6. *If for a given k there is a k-extendible regular homotopy between the embeddings e and f then $A^k(e) = A^k(f)$.*

Proof. $f = H_b = G_b \circ e^k$ and so $f^k = G_b^k \circ e^k$ where $G_b^k : M^k(e) \to M^k(f)$ is the map G_b with range restricted to $M^k(f)$. Since G_b^k is a diffeomorphism it follows that $\ker f_*^k = \ker e_*^k$. q.e.d.

Corollary 8.7. *If there is a k-extendible regular homotopy between the embeddings e and f for either $k = 0$ or $k = 1$ then:*

(1) $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$.

(2) $o(e) = o(f)$.

Proof. (1) follows from Lemma 8.6 and Corollary 6.9. Since G_a is the inclusion and G_t is a regular homotopy it follows that G_b is orientation preserving. This implies (2). q.e.d.

Proposition 8.8. *Every embedding $e : F \to \mathbb{R}^3$ is equivalent to a standard embedding.*

Proof. The proof is by induction on the genus of F. If $F = S^2$ then any e is isotopic to a standard embedding and isotopic embeddings are equivalent. So assume F is of positive genus and so there is a compressing disc D for $e(F)$ in $\mathbb{R}^3$ (i.e., $D \cap e(F) = \partial D$ and ∂D does not bound a disc in $e(F)$). Let $c = e^{-1}(\partial D) \subseteq F$ and let A be a regular neighborhood of c in F. Isotoping A along D as before we may assume A is embedded as a thin tube. There are four cases to be considered according to whether D is contained in $M^0(e)$ or $M^1(e)$ and whether ∂D separates or does not separate $e(F)$.

Case 1. $D \subseteq M^0(e)$ and ∂D separates $e(F)$. It then follows that D separates $M^0(e)$. If F_0, F_1 denote the two components of $F - \mathrm{int}A$ and $e_l : \overline{F_l} \to \mathbb{R}^3$ are defined as before then it follows from the assumptions of this case that $M^0(e_0)$ and $M^0(e_1)$ are disjoint and the tube $e(A)$ approaches each $e_l(\overline{F_l})$ from its noncompact side, i.e., from $M^1(e_l)$. Move each foot of the tube $e(A)$ (see Figure 12) along the corresponding surface $e_l(\overline{F_l})$ until they are each situated at the point p_l of $e_l(\overline{F_l})$ having maximal z-coordinate. In particular it follows that now $e(A)$ approaches each $e_l(\overline{F_l})$ from above. We now uniformly shrink each $e(F_l)$ towards the point p_l until it is contained in a tiny ball B_l attached from below to the corresponding foot of $e(A)$, arriving at a new embedding $e' : F \to \mathbb{R}^3$. This regular homotopy is clearly 0-extendible, and since no self intersection may occur within each of F_0, F_1 and A, this regular homotopy has no quadruple points. And so by Corollary 8.7 e' is equivalent to e. We now continue by isotopy, deforming the thin tube $e'(A)$ until it is a straight tube, and rigidly carrying B_0 and B_1 along. We finally arrive at an embedding e'' for which there is a plane P intersecting $e''(F)$ as in Lemma 8.4 with our F_0 and F_1 on the two sides of P. Since the genus of both $\overline{F_0}$ and $\overline{F_1}$ is smaller than that of F then by our induction hypothesis and Lemma 8.4, e'' is equivalent to a standard embedding.

Case 2. $D \subseteq M^1(e)$ and ∂D separates $e(F)$. Let $F_l, \overline{F_l}, e_l$ $(l = 0, 1)$ be as above. This time either $M^0(e_0) \subseteq M^0(e_1)$ or $M^0(e_1) \subseteq M^0(e_0)$ and assume the former holds. In this case $e(A)$ approaches only $e_0(\overline{F_0})$ from its noncompact side and so we push the tube and perform the

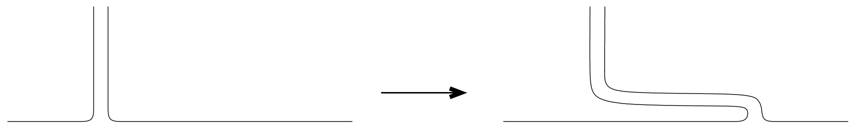

Figure 12: Moving the foot of a tube.

uniform shrinking as above only with F_0. This is a 1-extendible regular homotopy since we are shrinking $M^0(e_0)$ which is part of $M^1(e)$. Now, if B is the tiny ball into which we have shrunken $e(F_0)$ then ∂B supplies separating compressing discs on both sides of $e(F)$ and so we are done by Case 1.

Case 3. $D \subseteq M^0(e)$ and ∂D does not separate $e(F)$. If $F' = F - \mathrm{int}A$ and $e' : \overline{F'} \to \mathbb{R}^3$ is induced as above (where $\overline{F'}$ is the surface obtained from F' by gluing two discs to it) then both feet of the tube $e(A)$ approach $e'(\overline{F'})$ from its noncompact side. Push the feet of $e(A)$ until they are both situated near the same point p in $e'(\overline{F'})$ having maximal z coordinate. As before, this implies that the two feet are approaching p from above. Let P be a horizontal plane passing slightly below p (so that in a neighborhood of p it intersects F in only one circle). We may pull the tube $e(A)$ until it is all above P. We then let it pass through itself until it is unknotted. This is a 0-extendible regular homotopy with no quadruple points, at the end of which we have an embedding intersecting P as in Lemma 8.4 with an embedding of a torus above the plane P, this embedding being already standard and an embedding of a subsurface F'' of F below the plane P, F'' being of smaller genus than that of F. Again we are done by induction and Lemma 8.4.

Case 4. $D \subseteq M^1(e)$ and ∂D does not separate $e(F)$. We may proceed as in Case 3 (this time via a 1-extendible regular homotopy) to obtain a standard embedding of a torus connected with a tube to $e'(\overline{F'})$ but this time the torus is contained in $M^0(e')$ and the tube connects to $e'(\overline{F'})$ from its compact side. But once we have such an embedding then the little standardly embedded torus has non-separating compressing discs on both sides and so we are done by Case 3. q.e.d.

Lemma 8.9. *If $e : F \to \mathbb{R}^3$ is an embedding and $h : F \to F$ is a diffeomorphism such that e and $e \circ h$ are regularly homotopic, then $\widehat{\psi}(A^0(e), A^1(e); A^0(e \circ h), A^1(e \circ h)) = \psi(h_*)$ and $\widehat{\epsilon}(e, e \circ h) = \epsilon(h)$.*

(*Recall that* h_* *is the map induced by* h *on* $H_1(F, \mathbb{Z}/2)$ *and* $\epsilon(h) \in \mathbb{Z}/2$ *is* 0 *or* 1 *according to whether* h *is orientation preserving or reversing, respectively.*)

Proof. $x \in \ker(e \circ h)_*^k$ iff $h_*(x) \in \ker e_*^k$ and so $A^k(e \circ h) = h_*^{-1}(A^k(e))$, $k = 0, 1$. By definition then $\widehat{\psi}(A^0(e), A^1(e); A^0(e \circ h), A^1(e \circ h)) = \psi(h_*^{-1}) = \psi(h_*)$. (Recall that if e and $e \circ h$ are regularly homotopic then indeed $h_*^{-1} \in O(H_1(F, \mathbb{Z}/2), g^e)$.) $\widehat{\epsilon}(e, e \circ h) = \epsilon(h)$ is clear. q.e.d.

We are now ready to prove Theorem 7.3. For two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$ let $\widehat{\Psi}(e, e') = \widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) + (n+1)\widehat{\epsilon}(e, e')$. We need to show $Q(e, e') = \widehat{\Psi}(e, e')$. If $f : F \to \mathbb{R}^3$ is another embedding in the same regular homotopy class then $Q(e, e') = Q(e, f) + Q(f, e')$ and $\widehat{\Psi}(e, e') = \widehat{\Psi}(e, f) + \widehat{\Psi}(f, e')$. And so if e is equivalent to f and $Q(f, e') = \widehat{\Psi}(f, e')$ then also $Q(e, e') = \widehat{\Psi}(e, e')$. And so we may replace e with an equivalent standard embedding f (Proposition 8.8) and similarly replace e' with an equivalent standard embedding f'. Now f and f' have isotopic images and so after isotopy we may assume $f(F) = f'(F)$ and so $f' = f \circ h$ for some diffeomorphism $h : F \to F$. By Lemma 8.9 and Theorem 5.6 the proof of Theorem 7.3 is complete.

We conclude with a remark on systems of surfaces. If $S = F_1 \cup \cdots \cup F_r$ is a system of closed orientable surfaces, and $e : S \to \mathbb{R}^3$ is an embedding, then we can rigidly move $e(F_i)$ one by one, until they are all contained in large disjoint balls. When it is the turn of F_i to be rigidly moved, then the union of all other components is embedded and so only double lines occur. If $e' : S \to \mathbb{R}^3$ is another embedding then we can similarly move $e'(F_i)$ into the corresponding balls. It follows that $Q(e, e') = \sum_{i=1}^r Q(e|_{F_i}, e'|_{F_i})$ and so we obtain a formula for systems of surfaces, namely: $Q(e, e') = \sum_{i=1}^r \widehat{\Psi}(e|_{F_i}, e'|_{F_i})$.

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