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# THE WILLMORE FLOW WITH SMALL INITIAL ENERGY

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**Abstract**

We consider the $L^2$ gradient flow for the Willmore functional. In [5] it was proved that the curvature concentrates if a singularity develops. Here we show that a suitable blowup converges to a nonumbilic (compact or noncompact) Willmore surface. Furthermore, an $L^\infty$ estimate is derived for the tracefree part of the curvature of a Willmore surface, assuming that its $L^2$ norm (the Willmore energy) is locally small. One consequence is that a properly immersed Willmore surface with restricted growth of the curvature at infinity and small total energy must be a plane or a sphere. Combining the results we obtain long time existence and convergence to a round sphere if the total energy is initially small.

## 1. Introduction

For a closed, immersed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional (as introduced initially by Thomsen [11]) is

$$\mathcal{W}(f) = \int_\Sigma |A^\circ|^2 \, d\mu, \tag{1}$$

where $A^\circ = A - \frac{1}{2} g \otimes H$ denotes the tracefree part of the second fundamental form $A = D^2 f^\perp$ and $\mu$ is the induced area measure. The associated Euler-Lagrange operator is

$$\mathbf{W}(f) = \Delta H + Q(A^\circ) H. \tag{2}$$

Here $H$ is the mean curvature vector and $Q(A^\circ)$ acts linearly on normal vectors along $f$ by the formula (using summation with respect to a $g$-orthonormal basis $\{e_1, e_2\}$)

$$Q(A^\circ)\phi = A^\circ(e_i, e_j)\langle A^\circ(e_i, e_j), \phi\rangle. \tag{3}$$

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In (2) the Laplace operator $\Delta\phi = -\nabla^*\nabla\phi$ is understood with respect to the connection $\nabla_X\phi = (D_X\phi)^\perp$ on normal vector fields along $f$, where $\nabla^*$ denotes the formal adjoint of $\nabla$.

In this paper we continue our study from [5] of the $L^2$ gradient flow for (1), briefly called the Willmore flow, which is the fourth order, quasilinear geometric evolution equation

$$\partial_t f = -\mathbf{W}(f). \tag{4}$$

As a main result we have shown in [5] that the existence time is bounded from below in terms of the concentration of the measure $f_\#(\mu\llcorner|A|^2)$ in $\mathbb{R}^n$ at time $t = 0$. Here we study the operator (2) and the flow (4) under the assumption that $\mathcal{W}(f)$ is — either locally or globally — small. This condition is natural from the variational point of view and may be interpreted geometrically by saying that the deviation of $f$ from being round is small in an averaged sense. One of our results is:

**Theorem 5.1.** *There exists $\varepsilon_0(n) > 0$ such that if at time $t = 0$ we have $\mathcal{W}(f_0) < \varepsilon_0$, then the Willmore flow exists smoothly for all times and converges to a round sphere.*

The smallness assumption implies, if $\varepsilon_0$ is not too big, that $\Sigma$ is topologically a sphere and that $f$ is an embedding (see [13] for the case $n = 3$). Moreover, any sequence $f_k$ with $\mathcal{W}(f_k) \to 0$ subconverges, after appropriate translation and rescaling, to some round sphere in the sense of both Hausdorff distance and measure [8]. However the $f_k$ need not be graphs over the limit sphere, as can be seen by modifying Example 1 in [12]. At present we do not know an example ruling out the possibility of dropping the smallness condition in Theorem 5.1 entirely; in any case it is desirable to replace the number $\varepsilon_0$ by a more explicit constant.[1]

The statement of the theorem was recently proved in [9] under the stronger assumption that $f_0$ is close to a round sphere in the $C^{2,\alpha}$-topology, using a center manifold analysis which gives related stability results for a couple of other flows; see [2] for an overview. Our method, which is (and has to be) entirely different, involves deriving a priori estimates from the equation satisfied by the curvature, somewhat analogous to the work of Huisken [3, 4]. However, in our problem the crucial estimates are of integral type and the smallness condition is essential in

[1] *Note added in proof:* a numerical example of a singularity was recently contributed by U. Mayer & G. Simonett: `http://www.math.utah.edu/~mayer/math/numerics.html`.

ruling out possible concentrations related to the scale invariance of the functional.

## 2. Estimates for surfaces with locally small Willmore energy

Here we derive some bounds for immersed surfaces $f : \Sigma \to \mathbb{R}^n$ depending on the $L^2$ norms of their curvature $A$ and of their Willmore gradient $\mathbf{W}(f) = \Delta H + Q(A^\circ)H$, under the assumption that the $L^2$ norm of $A^\circ$, the tracefree part of the curvature, is locally small.

Recall the equations of Mainardi-Codazzi, Gauß and Ricci:

$$(\nabla_X A)(Y,Z) = (\nabla_Y A)(X,Z); \quad \nabla H = -\nabla^* A = -2\nabla^* A^\circ, \tag{5}$$

$$K = \frac{1}{4}|H|^2 - \frac{1}{2}|A^\circ|^2, \tag{6}$$

$$R^\perp(X,Y)\phi = A^\circ(e_i,X)\langle A^\circ(e_i,Y),\phi\rangle - A^\circ(e_i,Y)\langle A^\circ(e_i,X),\phi\rangle. \tag{7}$$

Note $\langle R^\perp(X,Y)\phi,\phi\rangle = 0$ and in particular $R^\perp = 0$ for $n = 3$, i.e., codimension one. The Codazzi equations imply that $\nabla A$ and $\nabla^2 A$ can be expressed by $\nabla A^\circ$ and $\nabla^2 A^\circ$, respectively. In particular one has inequalities

$$|\nabla A| \leq c\,|\nabla A^\circ|, \quad |\nabla^2 A| \leq c\,|\nabla^2 A^\circ|. \tag{8}$$

**Lemma 2.1.** *For any $p$-linear form $\phi$ along $f$ we have*

$$\begin{aligned}
&\big((\nabla\nabla^* - \nabla^*\nabla)\phi\big)(X_1,\ldots,X_p) \\
&= K\phi(X_1,\ldots,X_p) + K\sum_{k=2}^{p}\phi(X_k,X_2,\ldots,X_1,\ldots,X_p) \\
&\quad - K\sum_{k=2}^{p} g(X_1,X_k)\,\phi(e_i,X_2,\ldots,e_i,\ldots,X_p) \\
&\quad + R^\perp(e_i,X_1)\,\phi(e_i,X_2,\ldots,X_p) - (\nabla^* T)(X_1,\ldots,X_p).
\end{aligned} \tag{9}$$

*Here the tensor $T$ is given by*

$$T(X_0,X_1,\ldots,X_p) = (\nabla_{X_0}\phi)(X_1,X_2,\ldots,X_p) - (\nabla_{X_1}\phi)(X_0,X_2,\ldots,X_p).$$

*Proof.* From the proof of Lemma 2.1 in [5] we have

$$\begin{aligned}
&\big((\nabla\nabla^* - \nabla^*\nabla)\phi\big)(X_1,\ldots,X_p) \\
&\quad = \big(R^p(e_i,X_1)\,\phi\big)(e_i,X_2,\ldots,X_p) - (\nabla^* T)(X_1,\ldots,X_p).
\end{aligned}$$

Now the curvature operator $R^p$ is given by

$$
\begin{aligned}
&\big(R^p(e_i,X_1)\phi\big)(e_i,X_2,\ldots,X_p)\\
&\quad= R^\perp(e_i,X_1)\,\phi(e_i,X_2,\ldots,X_p)\\
&\qquad-\ \phi\big(R(e_i,X_1)\,e_i,X_2,\ldots,X_p\big)\\
&\qquad-\sum_{k=2}^{p}\phi\big(e_i,X_2,\ldots,R(e_i,X_1)X_k,\ldots,X_p\big)\\
&\quad= R^\perp(e_i,X_1)\,\phi(e_i,X_2,\ldots,X_p)\\
&\qquad-K\,\phi\big(g(X_1,e_i)e_i,X_2,\ldots,X_p\big)+K\,\phi\big(g(e_i,e_i)X_1,X_2,\ldots,X_p\big)\\
&\qquad-K\sum_{k=2}^{p}\ \phi\big(e_i,X_2,\ldots,g(X_1,X_k)e_i,\ldots,X_p\big)\\
&\qquad+\ K\sum_{k=2}^{p}\ \phi\big(g(e_i,X_k)e_i,X_2,\ldots,X_1,\ldots,X_p\big).
\end{aligned}
$$

Inserting yields the desired formula. q.e.d.

We will need three different choices for $\phi$ in (9). Taking first $\phi = A$ yields $T = 0$ and $\nabla^*\phi = -\nabla H$ by (5), and we get Simons' identity ([10])

$$\Delta A = \nabla^2 H + 2KA^\circ + R^\perp(e_i,\cdot\,)\,A(e_i,\cdot\,).$$

To bring this in a more useful form, let us denote by $S^\circ(B)$ the symmetric, tracefree part of any bilinear form with normal values along $f$. In particular, we have

$$S^\circ(\nabla^2 H) = \nabla^2 H - \frac{1}{2}\,g(\,\cdot\,,\cdot\,)\,\Delta H - \frac{1}{2}\,R^\perp(\,\cdot\,,\cdot\,)\,H.$$

Now $\Delta\big(\frac{1}{2}\,g(\,\cdot\,,\cdot\,)\,H\big) = \frac{1}{2}\,g(\,\cdot\,,\cdot\,)\,\Delta H$ and $R^\perp(e_i,X)\frac{1}{2}\,g(e_i,Y)\,H = -\frac{1}{2}R^\perp(X,Y)\,H$, which implies, using (6) and (7),

$$\Delta A^\circ = S^\circ(\nabla^2 H) + \frac{1}{2}|H|^2\,A^\circ + A^\circ * A^\circ * A^\circ. \tag{10}$$

Here and in the following we denote by $A * B$ any universal, linear combination of tensors obtained by tensor product and contraction from $A$ and $B$. Our second choice in (9) is $\phi = \nabla H$, where now

$$T(X,Y) = \nabla^2_{X,Y}H - \nabla^2_{Y,X}H = R^\perp(X,Y)\,H.$$

Using again (5), (6) and (7), we infer

$$\nabla^*(\nabla^2 H) = \nabla(\nabla^*\nabla H) - \frac{1}{4}|H|^2\nabla H + A * A^\circ * \nabla A^\circ. \tag{11}$$

Finally taking $\phi = \nabla A^\circ$ in (9) yields

$$\begin{aligned} T(X,Y,Z,V) &= \left(R^2(X,Y)\, A^\circ\right)(Z,V) = (A * A * A)(X,Y,Z,V), \\ \nabla^* T &= A * A * \nabla A^\circ. \end{aligned}$$

Thus we obtain from (9) and (6), (7)

$$\nabla^*(\nabla^2 A^\circ) = \nabla(\nabla^*\nabla A^\circ) + A * A * \nabla A^\circ. \tag{12}$$

We now convert (10), (11) and (12) into integral estimates.

**Lemma 2.2.** *If $f : \Sigma \to \mathbb{R}^n$ is an immersion with $\mathbf{W}(f) = W$ and $\gamma \in C^1_c(\Sigma)$ satisfies $|\nabla\gamma| \le \Lambda$, then*

(13)
$$\int |\nabla A|^2\gamma^2\, d\mu \le \frac{c}{\Lambda^2}\int |W|^2\gamma^4\, d\mu + c\int |A^\circ|^4\gamma^2\, d\mu + c\Lambda^2 \int_{[\gamma>0]} |A|^2\, d\mu.$$

*Proof.* Multiply (10) by $\gamma^2 A^\circ$ and integrate by parts to obtain, after applying (5),

$$\begin{aligned} &\int |\nabla A^\circ|^2\gamma^2\, d\mu + \frac{1}{2}\int |H|^2\, |A^\circ|^2\, \gamma^2\, d\mu \\ &\quad \le \frac{1}{2}\int |\nabla H|^2\, \gamma^2\, d\mu + c\int |A^\circ|^4\, \gamma^2\, d\mu + \int \gamma\, \nabla\gamma * A^\circ * \nabla A^\circ\, d\mu. \end{aligned}$$

Using the equation $\Delta H + Q(A^\circ)H = W$ we have

$$\begin{aligned} \frac{1}{2}\int |\nabla H|^2\, \gamma^2\, d\mu &= -\frac{1}{2}\int \langle H, \Delta H\rangle \gamma^2\, d\mu + \int \gamma\, \nabla\gamma * A * \nabla A^\circ\, d\mu \\ &= -\frac{1}{2}\int \langle H, W\rangle\gamma^2\, d\mu + \frac{1}{2}\int \langle H, Q(A^\circ)H\rangle\gamma^2\, d\mu \\ &\quad + \int \gamma\, \nabla\gamma * A * \nabla A^\circ\, d\mu \\ &\le \frac{c}{\Lambda^2}\int |W|^2\, \gamma^4\, d\mu + c\Lambda^2 \int_{[\gamma>0]} |H|^2\, d\mu \\ &\quad + \frac{1}{2}\int \langle H, Q(A^\circ)H\rangle\gamma^2\, d\mu + \int \gamma\, \nabla\gamma * A * \nabla A^\circ\, d\mu. \end{aligned}$$

It is easy to see the inequality

$$0 \leq \langle Q(A^\circ)H, H\rangle \leq |A^\circ|^2\,|H|^2. \tag{14}$$

Furthermore we have

$$\int \gamma\,\nabla\gamma * A * \nabla A^\circ\,d\mu \leq \frac{1}{2}\int |\nabla A^\circ|^2\,\gamma^2\,d\mu + c\Lambda^2 \int_{[\gamma>0]} |A|^2\,d\mu.$$

Inserting these inequalities, absorbing and recalling (8) proves the claim. q.e.d.

**Lemma 2.3.** *Under the assumptions of Lemma* 2.2 *we have for* $\eta = \gamma^4$

$$\begin{aligned}\int |\nabla^2 H|^2\,\eta + \int |A|^2\,|\nabla A|^2\eta\,d\mu + \int |A|^4\,|A^\circ|^2\,\eta\,d\mu \\ \leq c\int |W|^2\,\eta\,d\mu + c\int (|A^\circ|^2\,|\nabla A^\circ|^2 + |A^\circ|^6)\eta\,d\mu \\ + c\,\Lambda^4 \int_{[\gamma>0]} |A|^2\,d\mu.\end{aligned} \tag{15}$$

*Proof.* We start multiplying (11) by $\eta\,\nabla H$ and integrating by parts. This yields

$$\begin{aligned}&\int |\nabla^2 H|^2\,\eta\,d\mu + \frac{1}{4}\int |H|^2\,|\nabla H|^2\,\eta\,d\mu \\ &\leq \int |\Delta H|^2\,\eta\,d\mu + \int A * A^\circ * \nabla A^\circ * \nabla A^\circ \eta\,d\mu \\ &\quad + \int \gamma^3\,\nabla\gamma * \nabla H * \nabla^2 H\,d\mu \\ &\leq c\int |W|^2\eta\,d\mu + c\int |A^\circ|^4\,|H|^2\eta\,d\mu \\ &\quad + \varepsilon\int |H|^2\,|\nabla A^\circ|^2\eta\,d\mu + c(\varepsilon)\int |A^\circ|^2\,|\nabla A^\circ|^2\eta\,d\mu \\ &\quad + \frac{1}{2}\int |\nabla^2 H|^2\eta\,d\mu + c\,\Lambda^2\int |\nabla H|^2\,\gamma^2\,d\mu.\end{aligned}$$

Now by (13) we can estimate

$$\int |\nabla H|^2\,\gamma^2\,d\mu \leq \frac{c}{\Lambda^2}\int |W|^2\eta\,d\mu + \frac{c}{\Lambda^2}\int |A^\circ|^6\eta\,d\mu + c\,\Lambda^2\int_{[\gamma>0]} |A|^2\,d\mu.$$

Using the inequality $c\,|A^\circ|^4\,|H|^2 \leq \varepsilon\,|H|^4|A^\circ|^2 + c(\varepsilon)\,|A^\circ|^6$ and rearranging, we arrive at

$$\begin{aligned}
&\int |\nabla^2 H|^2 \eta\,d\mu + \int |H|^2\,|\nabla H|^2\,\eta\,d\mu \\
&\quad\leq c\int |W|^2\,\eta\,d\mu + c\,\Lambda^4 \int_{[\gamma>0]} |A|^2\,d\mu \\
&\qquad + c(\varepsilon)\int |A^\circ|^2\,|\nabla A^\circ|^2\,\eta\,d\mu + c(\varepsilon)\int |A^\circ|^6\,\eta\,d\mu \\
&\qquad + \varepsilon\int |H|^4\,|A^\circ|^2 \eta\,d\mu + \varepsilon\int |H|^2\,|\nabla A^\circ|^2\,\eta\,d\mu.
\end{aligned} \tag{16}$$

Next we use (10) to compute

$$\begin{aligned}
&\int |H|^2\,|\nabla A^\circ|^2\,\eta\,d\mu \\
&\quad= -\int |H|^2 \langle A^\circ, \Delta A^\circ\rangle \eta\,d\mu + \int H * \nabla H * A^\circ * \nabla A^\circ\,\eta\,d\mu \\
&\qquad + \int |H|^2\,A^\circ * \nabla A^\circ * \nabla\eta\,d\mu \\
&\quad= -\int |H|^2 \langle A^\circ, \nabla^2 H + \frac{1}{2}|H|^2\,A^\circ + A^\circ * A^\circ * A^\circ\rangle \eta\,d\mu \\
&\qquad + \int H * \nabla H * A^\circ * \nabla A^\circ \eta\,d\mu + \int |H|^2\,A^\circ * \nabla A^\circ * \nabla\eta\,d\mu \\
&\quad\leq \frac{1}{2}\int |H|^2\,|\nabla H|^2 \eta\,d\mu + \int H * \nabla H * A^\circ * \nabla A^\circ\,\eta\,d\mu \\
&\qquad + \int |H|^2\,A^\circ * \nabla A^\circ * \gamma^3\,\nabla\gamma\,d\mu \\
&\qquad - \frac{1}{2}\int |H|^4\,|A^\circ|^2\,\eta\,d\mu + c\int |H|^2\,|A^\circ|^4 \eta\,d\mu \\
&\quad\leq \left(\frac{1}{2} + \delta\right)\int |H|^2\,|\nabla H|^2 \eta\,d\mu + c(\delta)\int |A^\circ|^2\,|\nabla A^\circ|^2 \eta\,d\mu \\
&\qquad + \delta\int |H|^2\,|\nabla A^\circ|^2\,\eta\,d\mu + c(\delta)\,\Lambda^2\int |H|^2\,|A^\circ|^2\,\gamma^2\,d\mu \\
&\qquad - \frac{1}{2}\int |H|^4\,|A^\circ|^2\,\eta\,d\mu + c\int |H|^2\,|A^\circ|^4 \eta\,d\mu.
\end{aligned}$$

From the inequalities

$$c\int |H|^2\,|A^\circ|^4\,\eta\,d\mu \le \delta\int |H|^4\,|A^\circ|^2\,\eta\,d\mu + c(\delta)\int |A^\circ|^6\eta\,d\mu,$$

$$c(\delta)\,\Lambda^2\int |H|^2\,|A^\circ|^2\,\gamma^2\,d\mu \le \delta\int |H|^4\,|A^\circ|^2\eta\,d\mu + c(\delta)\,\Lambda^4\int_{[\gamma>0]} |A^\circ|^2\,d\mu,$$

we see that

$$\begin{aligned}(17)\quad (1-\delta)&\int |H|^2|\nabla A^\circ|^2\,\eta\,d\mu + \left(\frac{1}{2}-2\delta\right)\int |H|^4\,|A^\circ|^2\eta\,d\mu\\ &\le \left(\frac{1}{2}+\delta\right)\int |H|^2\,|\nabla H|^2\,\eta\,d\mu\\ &\quad + c(\delta)\left(\int |A^\circ|\,|\nabla A^\circ|^2\eta\,d\mu + \int |A^\circ|^6\eta\,d\mu + \Lambda^4\int_{[\gamma>0]} |A|^2\,d\mu\right).\end{aligned}$$

Adding the inequalities (16) and (17) yields

$$\begin{aligned}&\int |\nabla^2 H|^2\eta\,d\mu + \left(\frac{1}{2}-\delta\right)\int |H|^2|\nabla H|^2\eta\,d\mu\\ &\quad + (1-\delta-\varepsilon)\int |H|^2\,|\nabla A^\circ|^2\eta\,d\mu + \left(\frac{1}{2}-2\delta-\varepsilon\right)\int |H|^4\,|A^\circ|^2\eta\,d\mu\\ &\le c(\delta,\varepsilon)\left(\int |A^\circ|^2\,|\nabla A^\circ|^2\eta\,d\mu + \int |A^\circ|^6\eta\,d\mu + \Lambda^4\int_{[\gamma>0]} |A|^2\,d\mu\right)\\ &\quad + c\int |W|^2\eta\,d\mu.\end{aligned}$$

The claim of the Lemma follows by choosing $\varepsilon = \delta = \frac{1}{8}$. q.e.d.

**Proposition 2.4.** *If $f : \Sigma \to \mathbb{R}^n$ is an immersion with $\mathbf{W}(f) = W$ and $\eta = \gamma^4$, where $\gamma \in C^1_c(\Sigma)$ satisfies $|\nabla\gamma| \le \Lambda$, then*

$$\begin{aligned}&\int |\nabla^2 A|^2\eta\,d\mu + \int |A|^2\,|\nabla A|^2\eta\,d\mu + \int |A|^4\,|A^\circ|^2\eta\,d\mu\\ &\quad \le c\int |W|^2\eta\,d\mu + c\,\Lambda^4\int_{[\gamma>0]} |A|^2\,d\mu + c\int (|A^\circ|^2\,|\nabla A^\circ|^2 + |A^\circ|^6)\eta\,d\mu.\end{aligned}$$

*Proof.* Multiply (12) by $\eta \nabla A^\circ$, integrate by parts and apply (10) to get

$$\begin{aligned}\int |\nabla^2 A^\circ|^2 \eta \, d\mu &\le \int |\Delta A^\circ|^2 \eta \, d\mu + c \int |A|^2 \, |\nabla A|^2 \eta \, d\mu \\ &\quad + \int \gamma^3 \nabla \gamma * \nabla A^\circ * \nabla^2 A^\circ \, d\mu \\ &\le c \int |\nabla^2 H|^2 \eta \, d\mu \\ &\quad + c \int |A|^2 \, |\nabla A|^2 \, d\mu + c \int |A|^4 \, |A^\circ|^2 \eta \, d\mu \\ &\quad + \frac{1}{2} \int |\nabla^2 A^\circ|^2 \eta \, d\mu + c \, \Lambda^2 \int |\nabla A^\circ|^2 \gamma^2 \, d\mu.\end{aligned}$$

The claim now follows from Lemma 2.2 and Lemma 2.3, recalling (8).

q.e.d.

We next need a multiplicative Sobolev inequality.

**Lemma 2.5.** *Under the assumptions of Proposition* 2.4 *we have*

$$\begin{aligned}(18) \qquad & \int (|A^\circ|^2 \, |\nabla A^\circ|^2 + |A^\circ|^6) \eta \, d\mu \\ & \le c \int_{[\gamma > 0]} |A^\circ|^2 \, d\mu \, \cdot \int (|\nabla^2 A|^2 + |A|^2 \, |\nabla A|^2 + |A|^4 \, |A^\circ|^2) \eta \, d\mu \\ & \quad + c \, \Lambda^4 \left( \int_{[\gamma > 0]} |A^\circ|^2 \, d\mu \right)^2 .\end{aligned}$$

*Proof.* Recall the Michael-Simon Sobolev inequality ([7])

$$(19) \qquad \left( \int_\Sigma u^2 \, d\mu \right)^{\frac{1}{2}} \le c \left( \int_\Sigma |\nabla u| \, d\mu + \int_\Sigma |H| \, |u| \, d\mu \right),$$

with $c = c(n)$. Letting $u = |A^\circ|\,|\nabla A^\circ|\,\gamma^2$ we obtain

$$\begin{aligned}
&\int |A^\circ|^2\,|\nabla A^\circ|^2 \eta\, d\mu \\
&\quad \le c \Big( \int |A^\circ|\,|\nabla^2 A^\circ|\,\gamma^2\, d\mu \Big)^2 + c \Big( \int |\nabla A^\circ|^2\,\gamma^2\, d\mu \Big)^2 \\
&\qquad + c \Big( \int |A|\,|A^\circ|\,|\nabla A^\circ|\,\gamma^2\, d\mu \Big)^2 + c \Big( \int |A^\circ|\,|\nabla A^\circ|\,\gamma\,|\nabla\gamma|\, d\mu \Big)^2 \\
&\quad \le c \int_{[\gamma>0]} |A^\circ|^2\, d\mu \int (|\nabla^2 A^\circ|^2 + |A|^2\,|\nabla A^\circ|^2)\eta\, d\mu \\
&\qquad + c\,\Lambda^4 \Big( \int_{[\gamma>0]} |A^\circ|^2\, d\mu \Big)^2 + c \Big( \int |\nabla A^\circ|^2\,\gamma^2\, d\mu \Big)^2 .
\end{aligned}$$

In the last term, we integrate by parts to get

$$\begin{aligned}
(20) \quad \int |\nabla A^\circ|^2\,\gamma^2\, d\mu &\le c \int |A^\circ|\,|\nabla^2 A^\circ|\,\gamma^2\, d\mu + c\,\Lambda \int |A^\circ|\,|\nabla A^\circ|\,\gamma\, d\mu \\
&\le c \Big( \int_{[\gamma>0]} |A^\circ|^2\, d\mu \Big)^{\frac12} \cdot \Big( \int |\nabla^2 A^\circ|^2\,\eta\, d\mu \Big)^{\frac12} \\
&\quad + \frac{1}{2} \int |\nabla A^\circ|^2\,\gamma^2\, d\mu + c\Lambda^2 \int_{[\gamma>0]} |A^\circ|^2\, d\mu.
\end{aligned}$$

Absorbing and inserting proves the claimed inequality for the first term in (18). For the other term, choose $u = |A^\circ|^3\,\gamma^2$ in (19) and compute

$$\begin{aligned}
&\int |A^\circ|^6\,\eta\, d\mu \\
&\quad \le c \Big( \int |A^\circ|^2\,|\nabla A^\circ|\,\gamma^2\, d\mu + \int |A|\,|A^\circ|^3\,\gamma^2\, d\mu + c\,\Lambda \int |A^\circ|^3\,\gamma\, d\mu \Big)^2 \\
&\quad \le c \Big( \int |\nabla A^\circ|^2\,\gamma^2\, d\mu \Big)^2 + c \int_{[\gamma>0]} |A^\circ|^2\, d\mu \cdot \int |A|^2 |A^\circ|^4\,\eta\, d\mu \\
&\qquad + c\,\Lambda^4 \Big( \int_{[\gamma>0]} |A^\circ|^2\, d\mu \Big)^2.
\end{aligned}$$

Combining with (20) proves the estimate for the second term on the left of (18). q.e.d.

**Proposition 2.6.** *Let $f : \Sigma \to \mathbb{R}^n$ be an immersed surface, and let $\Lambda = \|\nabla\gamma\|_{L^\infty}$, where $\gamma$ has compact support on $\Sigma$. There exists a constant $\varepsilon_0 = \varepsilon_0(n) > 0$ such that if*

$$\int_{[\gamma>0]} |A^\circ|^2 \, d\mu < \varepsilon_0,$$

*then we have for a constant $c = c(n) < \infty$*

$$\begin{aligned}\int \left(|\nabla^2 A|^2 + |A|^2 \, |\nabla A|^2 + |A|^4 \, |A^\circ|^2\right) \gamma^4 \, d\mu \\ \leq c \int |\mathbf{W}(f)|^2 \gamma^4 \, d\mu + c\,\Lambda^4 \int_{[\gamma>0]} |A|^2 \, d\mu.\end{aligned}$$

This is an immediate consequence of Proposition 2.4 and Lemma 2.5. As a first application we deduce the following result.

**Theorem 2.7** (Gap Lemma)**.** *Let $f : \Sigma \to \mathbb{R}^n$ be a properly immersed (compact or noncompact) Willmore surface, and let $\Sigma_\varrho(0) = f^{-1}(B_\varrho(0))$. If*

$$\begin{aligned}\liminf_{\varrho\to\infty} \frac{1}{\varrho^4} \int_{\Sigma_\varrho(0)} |A|^2 \, d\mu \quad &= \quad 0, \quad \textit{and} \\ \int_{\Sigma} |A^\circ|^2 \, d\mu < \varepsilon_0 \quad &= \quad \varepsilon_0(n),\end{aligned}$$

*then $f$ is an embedded plane or sphere.*

*Proof.* We take $\gamma(p) = \varphi(\frac{1}{\varrho}\,|f(p)|)$, where $\varphi \in C^1(\mathbb{R})$ satisfies $\varphi(s) = 1$ for $s \leq \frac{1}{2}$, $\varphi(s) = 0$ for $s \geq 1$ and $\varphi \geq 0$. Then we have $\Lambda = c/\varrho$ in Proposition 2.6. Since $\mathbf{W}(f) = 0$ by assumption, we can let $\varrho \to \infty$ and conclude $A^\circ \equiv 0$. This implies, by a standard result of differential geometry [13], that $f$ maps into a fixed, round 2-sphere or plane $S \subset \mathbb{R}^n$. As $f$ is complete, it follows that $f : (\Sigma, g) \to S$ is a global isometry. q.e.d.

We shall now derive an $L^\infty$ bound for $A^\circ$ from Proposition 2.6.

**Lemma 2.8.** *For $\gamma \in C^1_c(\Sigma)$ with $|\nabla\gamma| \leq \Lambda$ and any normal $p$-form $\phi$ along $f$ we have the inequality*

$$\|\gamma^2 \, \phi\|^4_{L^\infty} \leq c \, \|\gamma^2 \, \phi\|^2_{L^2} \left[\int \left(|\nabla^2\phi|^2 + |H|^4 \, |\phi|^2\right) \gamma^4 \, d\mu + \Lambda^4 \int_{[\gamma>0]} |\phi|^2 \, d\mu\right].$$

*Proof.* This is Lemma 4.3 in [5], except that there a bound on the second derivatives of $\gamma$ was assumed. Letting $\psi = \gamma^2\phi$ we apply Theorem 5.6 in [5] to obtain

(21)

$$\begin{aligned}\|\psi\|_{L^\infty}^6 &\le c\,\|\psi\|_{L^2}^2 \left(\|\nabla\psi\|_{L^4}^4 + \|H\,\psi\|_{L^4}^4\right)\\ &\le c\,\|\psi\|_{L^2}^2 \left(\int \gamma^8|\nabla\phi|^4\,d\mu + \Lambda^4\int\gamma^4|\phi|^4\,d\mu + \int |H|^4\,|\psi|^4\,d\mu\right).\end{aligned}$$

The three integrals on the right are estimated as follows (starting with the third):

$$\int |H|^4\,|\psi|^4\,d\mu \le \|\psi\|_{L^\infty}^2 \int |H|^4\,|\phi|^2\,\gamma^4\,d\mu, \tag{22}$$

$$\Lambda^4\int\gamma^4|\phi|^4\,d\mu \le \|\psi\|_{L^\infty}^2\,\Lambda^4 \int_{[\gamma>0]} |\phi|^2\,d\mu. \tag{23}$$

By partial integration, we infer

$$\begin{aligned}\int |\nabla\phi|^2\,\gamma^2\,d\mu &\le c\int |\phi|\,|\nabla^2\phi|\,\gamma^2\,d\mu + c\Lambda\int|\phi|\,|\nabla\phi|\,\gamma\,d\mu\\ &\le \frac{c}{\Lambda^2}\int|\nabla^2\phi|^2\,\gamma^4\,d\mu + c\Lambda^2\int_{[\gamma>0]}|\phi|^2\,d\mu + \frac{1}{2}\int|\nabla\phi|^2\,\gamma^2\,d\mu.\end{aligned}$$

Using again integration by parts and Cauchy-Schwarz

$$\begin{aligned}\int |\nabla\phi|^4\,\gamma^8\,d\mu &\le c\left(\int|\phi|\,|\nabla\phi|^2\,|\nabla^2\phi|\,\gamma^8\,d\mu + \Lambda\int|\phi|\,|\nabla\phi|^3\,\gamma^7\,d\mu\right)\\ &\le c\,\|\psi\|_{L^\infty}\left(\int|\nabla\phi|^4\,\gamma^8\,d\mu\right)^{\frac12}\left(\int|\nabla^2\phi|^2\,\gamma^4\,d\mu\right)^{\frac12}\\ &\quad + c\Lambda\,\|\psi\|_{L^\infty}\left(\int|\nabla\phi|^4\,\gamma^8\,d\mu\right)^{\frac12}\left(\int|\nabla\phi|^2\,\gamma^2\,d\mu\right)^{\frac12}.\end{aligned}$$

Combining the last two inequalities, we get

$$\int|\nabla\phi|^4\,\gamma^8\,d\mu \le c\,\|\psi\|_{L^\infty}^2\left(\int|\nabla^2\phi|^2\,\gamma^4\,d\mu + c\,\Lambda^4\int_{[\gamma>0]}|\phi|^2\,d\mu\right). \tag{24}$$

Inserting (22)–(24) into (21) proves the claim. q.e.d.

Combining Proposition 2.6 and Lemma 2.8, where $\phi = A^\circ$ and $\gamma$ is a cutoff function depending on extrinsic distance as in Theorem 2.7, we obtain the following "partial" curvature estimate.

**Theorem 2.9** (Tracefree Curvature Estimate)**.** *Let $f : \Sigma \to \mathbb{R}^n$ be an immersed surface with $\Sigma_\varrho = f^{-1}\big(B_\varrho(x_0)\big) \subset\subset \Sigma$, and suppose that*

$$\int_{\Sigma_\varrho} |A^\circ|^2 \, d\mu < \varepsilon_0,$$

*where $\varepsilon_0 = \varepsilon_0(n) > 0$ is a fixed constant. Then*

$$\|A^\circ\|^2_{L^\infty(\Sigma_{\varrho/2})} \leq c \left( \|\mathbf{W}(f)\|_{L^2(\Sigma_\varrho)} + \frac{1}{\varrho^2} \|A\|_{L^2(\Sigma_\varrho)} \right) \|A^\circ\|_{L^2(\Sigma_\varrho)}. \tag{25}$$

Assuming smallness of the full second fundamental form $A$, one easily adapts the arguments above to also prove the following:

**Theorem 2.10** (Curvature Estimate)**.** *Let $f : \Sigma \to \mathbb{R}^n$ be an immersed surface, $\Sigma_\varrho = f^{-1}(B_\varrho(x_0)) \subset\subset \Sigma$ and suppose*

$$\int_{\Sigma_\varrho} |A|^2 \, d\mu < \varepsilon_0,$$

*where $\varepsilon_0 = \varepsilon_0(n)$ is a fixed constant. Then*

$$\|A\|^2_{L^\infty(\Sigma_{\varrho/2})} \leq c \left( \|\mathbf{W}(f)\|_{L^2(\Sigma_\varrho)} + \frac{1}{\varrho^2} \|A\|_{L^2(\Sigma_\varrho)} \right) \|A\|_{L^2(\Sigma_\varrho)}. \tag{26}$$

**Remark 2.11.** The statements of the Theorems 2.7, 2.9 and 2.10 clearly also hold with the extrinsic distance sets $\Sigma_\varrho(x_0)$ replaced by distance sets with respect to the *intrinsic* distance function, since only a bound on the first derivatives of the cutoff function was needed.

## 3. Local estimates for the flow

We now consider solutions $f : \Sigma \times [0, T) \to \mathbb{R}^n$ to the gradient flow for the Willmore integral,

$$\partial_t f = -\mathbf{W}(f).$$

We abbreviate $\mathbf{W}(f) =: W$ in the following and compute first a precise formula for the evolution of the energy density. Recall from [5]:

$$\partial_t(d\mu) = \langle H, W\rangle\, d\mu, \tag{27}$$

$$\partial_t^\perp H = -\left(\Delta W + Q(A^\circ)W + \frac{1}{2}H\langle H, W\rangle\right), \tag{28}$$

$$\begin{aligned} \partial_t^\perp A(X,Y) = {} & -\nabla^2_{X,Y}W + \frac{1}{2}\,g(X,Y)\left[Q(A^\circ)W + \frac{1}{2}H\langle H,\, W\rangle\right] \\ & + \frac{1}{2}H\langle A^\circ(X,Y), W\rangle + \frac{1}{2}A^\circ(X,Y)\langle H,\, W\rangle \\ & + \frac{1}{2}R^\perp(X,Y)W. \end{aligned} \tag{29}$$

Here we used (2.18), (2.6) and (2.3) from [5]. Furthermore, using (2.15) in [5] we infer

$$\begin{aligned} \partial_t^\perp\left(\frac{1}{2}\,g(X,Y)\,H\right) = {} & -\frac{1}{2}\,g(X,Y)\,\left(\Delta W + Q(A^\circ)W + \frac{1}{2}H\langle H, W\rangle\right) \\ & + \langle A^\circ(X,Y), W\rangle H + \frac{1}{2}g(X,Y)H\,\langle H,\, W\rangle, \end{aligned}$$

and subtracting this from (29) yields

$$\begin{aligned} \partial_t^\perp A^\circ(X,Y) = {} & -S^\circ(\nabla^2 W) + g(X,Y)\,Q(A^\circ)W \\ & + \frac{1}{2}\,A^\circ(X,Y)\langle H,\, W\rangle - \frac{1}{2}\,H\langle A^\circ(X,Y), W\rangle. \end{aligned} \tag{30}$$

Recall that $S^\circ(\dots)$ denotes the symmetric, tracefree component. We compute separately for $H$ and $A^\circ$. By (27) and (28)

$$\begin{aligned} & \partial_t\left(\frac{1}{2}\,|H|^2\,d\mu\right) \\ & \quad = -\left\langle \Delta W + Q(A^\circ)W + \frac{1}{2}\,H\langle H,\, W\rangle, H\right\rangle\,d\mu + \frac{1}{2}|H|^2\,\langle H,\, W\rangle\,d\mu \\ & \quad = -\langle \Delta H + Q(A^\circ)\,H, W\rangle\,d\mu + \big(\langle \Delta H,\, W\rangle - \langle H, \Delta W\rangle\big)\,d\mu \\ & \quad = -|W|^2\,d\mu + \nabla_{e_i}\big(\langle \nabla_{e_i} H,\, W\rangle - \langle H,\, \nabla_{e_i} W\rangle\big)\,d\mu, \end{aligned}$$

whence

$$\partial_t\left(\frac{1}{2}\,|H|^2\,d\mu\right) + |W|^2\,d\mu = (\nabla^*\alpha)\,d\mu, \tag{31}$$

where $\alpha$ is the 1-form given by

$$\alpha(X) = \nabla_X\,\langle H,\, W\rangle - 2\langle \nabla_X\, H,\, W\rangle. \tag{32}$$

In order to compute for $A^\circ$, we first have (using again (2.15) in [5]) for a $g$-orthonormal basis

$$
\begin{aligned}
g(\partial_t e_i,\, e_j) + g(e_i,\, \partial_t e_j) &= \partial_t\big(g(e_i, e_j)\big) - (\partial_t g)(e_i, e_j) \\
&= -2\langle A(e_i, e_j), W\rangle \\
&= -2\langle A^\circ(e_i, e_j), W\rangle - \delta_{ij}\langle H,\, W\rangle.
\end{aligned}
$$

This implies further

$$
\begin{aligned}
&\langle A^\circ(\partial_t e_i, e_k),\, A^\circ(e_i, e_k)\rangle \\
&\quad = g(\partial_t e_i, e_j)\langle A^\circ(e_j, e_k),\, A^\circ(e_i, e_k)\rangle \\
&\quad = -\left(\langle A^\circ(e_i, e_j),\, W\rangle + \frac{1}{2}\,\delta_{ij}\langle H,\, W\rangle\right)\,\langle A^\circ(e_i, e_k), A^\circ(e_j, e_k)\rangle \\
&\quad = -\big\langle A^\circ(e_i, e_k)\langle A^\circ(e_i, e_j), W\rangle, A^\circ(e_j, e_k)\big\rangle - \frac{1}{2}\,|A^\circ|^2\langle H,\, W\rangle \\
&\quad = -\left\langle \frac{1}{2}\, g(e_j, e_k) Q(A^\circ)W, A^\circ(e_j, e_k)\right\rangle - \frac{1}{2}\,|A^\circ|^2\langle H,\, W\rangle \\
&\quad = -\frac{1}{2}|A^\circ|^2\langle H,\, W\rangle,
\end{aligned}
$$

where we used (2.5) from [5]. We use this and (30) to compute

$$
\begin{aligned}
&\partial_t\left(|A^\circ|^2\, d\mu\right) \\
&\quad = 2\langle(\partial_t\, A^\circ)\,(e_i, e_k),\, A^\circ(e_i, e_k)\rangle d\mu \\
&\qquad + 2\langle A^\circ(\partial_t e_i, e_k) + A^\circ(e_i, \partial_t e_k), A^\circ(e_i, e_k)\rangle\, d\mu + |A^\circ|^2\,\langle H, W\rangle d\mu \\
&\quad = -2\langle\nabla^2 W, A^\circ\rangle d\mu + |A^\circ|^2\,\langle H,\, W\rangle d\mu \\
&\qquad - \langle A^\circ(e_i, e_k),\, W\rangle\,\langle A^\circ(e_i, e_k),\, H\rangle d\mu \\
&\qquad - 2|A^\circ|^2\langle H,\, W\rangle d\mu + |A^\circ|^2\,\langle H,\, W\rangle d\mu \\
&\quad = -2\langle\nabla^2 W, A^\circ\rangle d\mu - \langle Q(A^\circ)H, W\rangle d\mu \\
&\quad = \big(-2\,\nabla_{e_i}\langle\nabla_{e_j} W,\, A^\circ(e_i, e_j)\rangle + \langle\nabla_{e_j} W,\, \nabla_{e_j} H\rangle\big) d\mu \\
&\qquad - \langle Q(A^\circ)H, W\rangle d\mu \\
&\quad = \big(-2\,\nabla_{e_i}\langle\nabla_{e_j} W,\, A^\circ(e_i, e_j)\rangle + \nabla_{e_i}\langle W,\, \nabla_{e_i} H\rangle\big) d\mu \\
&\qquad - \langle W, \Delta H\rangle d\mu - \langle Q(A^\circ)H, W\rangle d\mu.
\end{aligned}
$$

Thus we have shown

$$
\partial_t(|A^\circ|^2\, d\mu) + |W|^2\, d\mu = (\nabla^*\beta)\, d\mu, \tag{33}
$$

where $\beta$ is the 1-form defined by

$$
\beta(X) = 2\langle\nabla_{e_j} W, A^\circ(X, e_j)\rangle - \langle\nabla_X H, W\rangle. \tag{34}
$$

**Lemma 3.1.** *If $f$ is a Willmore flow, then for any function $\eta$ and $W = \mathbf{W}(f)$ we have:*

$$\begin{aligned}(35)\quad &\partial_t \int \frac{1}{2}|H|^2 \eta\, d\mu + \int |W|^2 \eta\, d\mu \\ &\qquad = \int \Big( \frac{1}{2}|H|^2 \partial_t \eta - \langle H, W\rangle \Delta \eta - 2\langle \nabla_{\operatorname{grad}\eta} H, W\rangle \Big)\, d\mu,\end{aligned}$$

$$\begin{aligned}(36)\quad &\partial_t \int |A^\circ|^2 \eta\, d\mu + \int |W|^2 \eta\, d\mu \\ &= \int \Big( |A^\circ|^2\, \partial_t \eta - 2\langle A^\circ(e_i, e_j),\, W\rangle \nabla^2_{e_i,e_j}\eta - 2\langle \nabla_{\operatorname{grad}\eta} H,\, W\Big)\, d\mu.\end{aligned}$$

*Proof.* Formula (35) is immediate from (31) and (32). For (36) we compute for $\beta$ as in (34):

$$\begin{aligned}\int \eta\, \nabla^* \beta\, d\mu &= \int \big( 2\langle \nabla_{e_j} W, A^\circ(\operatorname{grad}\eta, e_j)\rangle - \langle \nabla_{\operatorname{grad}\eta} H,\, W\rangle \big) d\mu \\ &= -\int 2\langle (\nabla_{e_j} A^\circ)\,(e_j, \operatorname{grad}\eta), W\rangle d\mu \\ &\quad - \int 2\langle A^\circ(\nabla_{e_j} \operatorname{grad}\eta, e_j),\, W\rangle d\mu - \int \langle \nabla_{\operatorname{grad}\eta} H,\, W\rangle d\mu \\ &= -\int 2\, \nabla^2_{e_i,e_j}\eta\, \langle A^\circ(e_i, e_j), W\rangle d\mu - \int 2\langle \nabla_{\operatorname{grad}\eta} H,\, W\rangle d\mu,\end{aligned}$$

which, together with (33), proves (36). q.e.d.

In controlling the energy density in time, difficulties arise because of the dependence of $\partial_t \eta$ and $\nabla^2 \eta$ on $f$, and since $\mathbf{W}(f)$ differs from $\Delta H$ by the term $Q(A^\circ)H$. For a ball $B_\varrho = B_\varrho(x_0) \subset \mathbb{R}^n$ and $f : \Sigma \to \mathbb{R}^n$ we adopt as in Section 2 the notation

$$\Sigma_\varrho(x_0) = f^{-1}(B_\varrho(x_0))$$

and consider a cutoff function $\widetilde{\gamma} \in C^1_c(B_\varrho)$, $\widetilde{\gamma} \geq 0$, such that

$$(37)\qquad |D\,\widetilde{\gamma}| \leq \frac{c}{\varrho}, \quad |D^2\,\widetilde{\gamma}| \leq \frac{c}{\varrho^2}.$$

We put $\gamma = \widetilde{\gamma} \circ f$ and observe

$$\begin{aligned}(38)\qquad \nabla \gamma &= (D\,\widetilde{\gamma} \circ f) \cdot Df \\ \nabla^2 \gamma &= (D^2\,\widetilde{\gamma} \circ f)\,(Df, Df) + (D\,\widetilde{\gamma} \circ f) \cdot A(\cdot\,,\cdot\,).\end{aligned}$$

**Lemma 3.2.** *If $f : \Sigma \times [0,T) \to \mathbb{R}^n$ is a Willmore flow and $\eta = \gamma^4$ for $\gamma = \widetilde{\gamma} \circ f$ with $\widetilde{\gamma}$ as in (37), then we have for $W = \mathbf{W}(f)$*

$$\partial_t \int |A^\circ|^2 \eta \, d\mu + \frac{1}{2} \int |W|^2 \eta \, d\mu \leq \frac{c}{\varrho^2} \int |A|^2 \, |A^\circ|^2 \, \gamma^2 \, d\mu + \frac{c}{\varrho^4} \int_{[\gamma>0]} |A|^2 \, d\mu \tag{39}$$

$$\partial_t \int |A|^2 \eta \, d\mu + \int |W|^2 \eta \, d\mu \leq \frac{c}{\varrho^2} \int |A|^4 \, \gamma^2 \, d\mu + \frac{c}{\varrho^4} \int_{[\gamma>0]} |A|^2 \, d\mu. \tag{40}$$

*Proof.* We estimate the terms in (36) and (35). We have

$$\begin{aligned} \int \gamma^2 |\nabla H|^2 \, d\mu &= -\int \gamma^2 \langle H, \, \Delta H \rangle d\mu + \frac{c}{\varrho} \int \gamma |H| \, |\nabla H| \, d\mu \\ &\leq -\int \gamma^2 \langle H, W \rangle d\mu + c \int \gamma^2 |A^\circ|^2 \, |H|^2 d\mu \\ &\quad + \frac{1}{2} \int \gamma^2 |\nabla H|^2 d\mu + \frac{c}{\varrho^2} \int_{[\gamma>0]} |H|^2 d\mu. \end{aligned}$$

As

$$\int |\nabla \eta| \, |\nabla H| \, |W| d\mu \leq \varepsilon \int |W|^2 \eta \, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int \gamma^2 \, |\nabla H|^2 d\mu,$$

we obtain by combining

$$\begin{aligned} -\int 2 \langle \nabla_{\operatorname{grad} \eta} H, \, W \rangle d\mu &\leq \varepsilon \int |W|^2 \eta d\mu + \frac{c(\varepsilon)}{\varrho^2} \int \gamma^2 |A^\circ|^2 \, |H|^2 d\mu \\ &\quad + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |H|^2 d\mu. \end{aligned}$$

Next using (38)

$$\begin{aligned} -\int 2 \langle A^\circ(e_i, e_j), \, W \rangle \nabla^2_{e_i, e_j} \eta \, d\mu &\leq c \int |A^\circ| \, |W| \, \left( \frac{1}{\varrho^2} \gamma^2 + \frac{1}{\varrho} \gamma^3 |A^\circ| \right) d\mu \\ &\leq \varepsilon \int |W|^2 \eta \, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int |A^\circ|^4 \gamma^2 \, d\mu \\ &\quad + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A^\circ|^2 \, d\mu. \end{aligned}$$

Finally

$$\begin{aligned}\int |A^\circ|^2\, \partial_t \eta\, d\mu &\leq \frac{c}{\varrho} \int |A^\circ|^2\, |W|\, \gamma^3\, d\mu \\ &\leq \varepsilon \int |W|^2\, \gamma^4\, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int |A^\circ|^4 \gamma^2\, d\mu.\end{aligned}$$

Combining the three estimates and absorbing for $\varepsilon > 0$ small, we obtain (39). The estimate (40) follows analogously from (35). q.e.d.

**Lemma 3.3.** *Let $f : \Sigma \times [0,T) \to \mathbb{R}^n$ be a Willmore flow. If*

$$\int_{\Sigma_\varrho(x_0)} |A^\circ|^2\, d\mu < \varepsilon_0 \quad \text{at some time } t \in [0,T), \tag{41}$$

*then for a constant $c_0 > 0$ we have at time $t$*

$$\partial_t \int |A^\circ|^2 \gamma^4\, d\mu + c_0 \int (|\nabla^2 A|^2 + |A|^2\, |\nabla A|^2 + |A|^4\, |A^\circ|^2) \gamma^4\, d\mu \leq \frac{c}{\varrho^4} \int_{\Sigma_\varrho(x_0)} |A|^2\, d\mu, \tag{42}$$

*and*

$$\partial_t \int |H|^2 \gamma^4\, d\mu + c_0 \int (|\nabla^2 A|^2 + |A|^2\, |\nabla A|^2 + |A|^4\, |A^\circ|^2) \gamma^4\, d\mu \leq \frac{c}{\varrho^4} \int_{\Sigma_\varrho(x_0)} |A|^2\, d\mu. \tag{43}$$

*Proof.* (42) follows by combining (39) with Proposition 2.6, after estimating

$$\frac{c}{\varrho^2} \int |A|^2\, |A^\circ|^2\, \gamma^2\, d\mu \leq \varepsilon \int |A|^4\, |A^\circ|^2\, \gamma^4\, d\mu + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A^\circ|^2\, d\mu.$$

For the other bound we must go back to (35), estimating the three terms on the right hand side. We have

$$\int \frac{1}{2} |H|^2\, \partial_t\, \eta\, d\mu = - \int \frac{1}{2} |H|^2\, D\, \tilde{\eta} \circ f \cdot \big(\Delta H + Q(A^\circ)\, H\big)\, d\mu.$$

By Young's inequality with 4 and 4/3, we have

$$(44)\quad \frac{1}{\varrho}\int |A^\circ|^2\,|A|^3\,\gamma^3\,d\mu \le \varepsilon \int |A^\circ|^{8/3}\,|A|^{10/3}\,\gamma^4\,d\mu + \frac{c(\varepsilon)}{\varrho^4}\int_{[\gamma>0]} |A|^2\,d\mu.$$

Using integration by parts, we infer

$$\begin{aligned}\int H * H\langle D\,\widetilde{\eta}\circ f, \Delta H\rangle\,d\mu & \\ &\le \frac{c}{\varrho}\int |H|\,|\nabla H|^2\,\gamma^3\,d\mu + \frac{c}{\varrho^2}\int |H|^2\,|\nabla H|\,\gamma^2\,d\mu \\ &\le \varepsilon\int |H|^2\,|\nabla H|^2\,\gamma^4\,d\mu + \frac{c(\varepsilon)}{\varrho^2}\int |\nabla H|^2\,\gamma^2\,d\mu \\ &\quad + \frac{c(\varepsilon)}{\varrho^4}\int_{[\gamma>0]} |H|^2\,d\mu.\end{aligned}$$

In the proof of Lemma 3.2, we have already shown by partial integration that

$$\begin{aligned}\int \gamma^2|\nabla H|^2\,d\mu \quad &\le \quad \delta\varrho^2\int |W|^2\,\gamma^4\,d\mu + \frac{c(\delta)}{\varrho^2}\int_{[\gamma>0]} |A|^2\,d\mu \\ &\qquad +\delta\varrho^2\int |A|^2\,|A^\circ|^4\,\gamma^4\,d\mu,\end{aligned}$$

so that by combining we obtain

$$\begin{aligned}(45)\qquad \int H * H\langle D\,\widetilde{\eta}\circ f,\, \Delta H\rangle\,d\mu & \\ &\le \varepsilon\int (|W|^2 + |A|^2\,|\nabla A|^2 + |A|^2\,|A^\circ|^4)\,\gamma^4 d\mu \\ &\quad + \frac{c(\varepsilon)}{\varrho^4}\int_{[\gamma>0]} |A|^2\,d\mu.\end{aligned}$$

Thus (44) and (45) estimate the first of the three terms on the right hand side of (35). For the second we use

$$\begin{aligned} -\int \langle H,\, W\rangle\,\Delta\eta\,d\mu & \\ &\le -\int \langle H, \Delta H\rangle\,\langle D\,\widetilde{\eta}\circ f,\, H\rangle d\mu + \frac{c}{\varrho^2}\int |H|\,|\Delta\, H|\,\gamma^2\,d\mu \\ &\quad + \frac{c}{\varrho}\int |A^\circ|^2\,|A|^3\,\gamma^3\,d\mu + \frac{c}{\varrho^2}\int |A^\circ|^2\,|A|^2\gamma^2\,d\mu.\end{aligned}$$

The first integral is estimated by (45), the third integral by (44). Furthermore

$$\frac{c}{\varrho^2}\int |H|\,|\Delta H|\,\gamma^2\,d\mu \;\le\; \varepsilon\int |\nabla^2 A|^2\,\gamma^4\,d\mu + \frac{c(\varepsilon)}{\varrho^4}\int_{[\gamma>0]} |A|^2\,d\mu$$

$$\frac{c}{\varrho^2}\int |A^\circ|^2\,|A|^2\,\gamma^2\,d\mu \;\le\; \varepsilon\int |A^\circ|^4\,|A|^2\,\gamma^4\,d\mu + \frac{c(\varepsilon)}{\varrho^4}\int_{[\gamma>0]} |A|^2\,d\mu.$$

The third integral on the right of (35) satisfies

$$\int |\nabla\eta|\,|\nabla H|\,|W|\,d\mu \le \varepsilon\int |W|^2\gamma^4\,d\mu + \frac{c(\varepsilon)}{\varrho^2}\int |\nabla H|^2\,\gamma^2\,d\mu$$

and the right hand side is already estimated. Thus putting things together we have shown

$$\partial_t\left(\int \frac{1}{2}|H|^2\,\eta\,d\mu\right) + \frac{3}{4}\int |W|^2\eta\,d\mu$$
$$\le \varepsilon\int (|\nabla^2 A|^2 + |A|^2\,|\nabla A|^2 + |A|^4\,|A^\circ|^2)\eta\,d\mu + \frac{c(\varepsilon)}{\varrho^4}\int_{[\gamma>0]} |A|^2\,d\mu.$$

Now (43) follows from Proposition 2.6. q.e.d.

**Proposition 3.4.** *Let $f : \Sigma \times [0,T) \to \mathbb{R}^n$ be a Willmore flow with $\int_\Sigma |A|^2\,d\mu \le \varkappa$. There exist constants $\varepsilon_1 = \varepsilon_1(n) > 0$ and $c_1 = c(n)/\varkappa > 0$, such that if $\varrho > 0$ is chosen with*

(46) $$\int_{\Sigma_\varrho} |A^\circ|^2\,d\mu \le \varepsilon < \varepsilon_1 \quad \text{at time } t = 0 \quad \text{for all } \Sigma_\varrho = \Sigma_\varrho(x_0) \subset \mathbb{R}^n,$$

*then for any time $0 \le t < t_1 = \min\{c_1\,\varrho^4,\, T\}$ we have*

(47) $$\int_{\Sigma_\varrho} |A^\circ|^2\,d\mu + \int_0^t\int_{\Sigma_\varrho} (|\nabla^2 A|^2 + |A|^2\,|\nabla A|^2 + |A|^4\,|A^\circ|^2)d\mu\,d\tau \le c(\varepsilon + \varkappa\varrho^{-4}\,t),$$

(48) $$\int_0^t \|A^\circ\|^4_{L^\infty(\Sigma_\varrho)}d\tau \le c(\varepsilon + \varkappa\,\varrho^{-4}\,t).$$

*Moreover, for* $0 < \sigma \leq \varrho$ *and* $\tau < \min\{c_1 \sigma^4, T\}$ *we then also have*

$$\int\limits_{\Sigma_{\sigma/2}(x)} |A|^2 \, d\mu\Big|_{t=\tau} \leq \int\limits_{\Sigma_\sigma(x)} |A|^2 \, d\mu\Big|_{t=0} + c\,\varkappa\,\sigma^{-4}\,\tau \quad \forall x \in \mathbb{R}^n. \tag{49}$$

*Proof.* Let $N = N(n)$ be the number of balls $B_{\varrho/2} \subset \mathbb{R}^n$ needed to cover $B_\varrho \subset \mathbb{R}^n$ and choose $\varepsilon_1 \leq \frac{\varepsilon_0}{4N}$, where $\varepsilon_0 > 0$ is as in Lemma 3.3. Assume (41) is satisfied on $[0, t]$ for all $B_\varrho \subset \mathbb{R}^n$ , and integrate (42) to obtain using (46)

$$\begin{aligned} \int\limits_{\Sigma_{\varrho/2}} |A^\circ|^2 \, d\mu + c_0 \int_0^t \int\limits_{\Sigma_{\varrho/2}} (|\nabla^2 A|^2 + |A|^2 \, |\nabla A|^2 + |A|^4 \, |A^\circ|^2) \, d\mu \, ds \\ \leq \varepsilon + c\varkappa\varrho^{-4} t. \end{aligned}$$

Assuming $t \leq c_1 \varrho^4$ where $c_1$ is chosen with $0 < c_1 \leq \frac{\varepsilon_0}{4N\,c\,\varkappa}$, we conclude

$$\begin{aligned} \int\limits_{\Sigma_\varrho} |A^\circ|^2 \, d\mu \quad &+ \quad c_0 \int_0^t \int\limits_{\Sigma_\varrho} (|\nabla^2 A|^2 + |A|^2 \, |\nabla A|^2 + |A|^4 \, |A^\circ|^2) \, d\mu \, ds \\ &\leq \quad N(\varepsilon + c\,\varkappa\,\varrho^{-4}\,t) \\ &\leq \quad N(\varepsilon_1 + c\,\varkappa\,c_1) \\ &\leq \quad \frac{\varepsilon_0}{2}. \end{aligned}$$

It follows that (41) holds up to time $t = t_1$ for all $x_0 \in \mathbb{R}^n$, and (47) follows. In particular $\int_{\Sigma_\varrho} |A^\circ|^2 \, d\mu \leq c\,(\varepsilon_1 + \varkappa c_1)$, whence a covering argument with possibly smaller $\varepsilon_1, c_1$ implies the smallness hypothesis in Theorem 2.9 for any ball $B_{2\varrho} \subset \mathbb{R}^n$ and any $t \in [0, t_1]$. Inequality (48) now follows from combining (25) with (47), again using a covering. Finally (49) is obtained by integrating (43) and (42) on $[0, t]$. q.e.d.

We next state a version of the higher order estimates obtained in [5] which is localized in time.

**Theorem 3.5** (Interior Estimates)**.** *Let* $f : \Sigma \times (0, T] \to \mathbb{R}^n$ *be a Willmore flow satisfying the condition*

$$\sup_{0<t\leq T} \int\limits_{\Sigma_\varrho(0)} |A|^2 \, d\mu \leq \varepsilon < \varepsilon_0(n), \tag{50}$$

*where $T \leq c(n)\,\varrho^4$. Then for any $k \in \mathbb{N}_0$ we have at time $t \in (0,T]$ the estimates*

$$\|\nabla^k A\|_{L^\infty\left(\Sigma_{\varrho/2}(0)\right)} \leq c(k)\sqrt{\varepsilon}\, t^{-\frac{k+1}{4}} \tag{51}$$

$$\|\nabla^k A\|_{L^2\left(\Sigma_{\varrho/2}(0)\right)} \leq c(k)\sqrt{\varepsilon}\, t^{-\frac{k}{4}}. \tag{52}$$

*Proof.* By scaling $f_\varrho(p,t) = \frac{1}{\varrho} f(p, \varrho^4 t)$ we may assume $\varrho = 1$. Using (4.13) and (4.9) from [5], see also Proposition 4.6 in [5], we obtain on $B = B_{3/4}(0)$ the inequalities

$$\int_0^T \int_{\Sigma_{3/4}} \left(|\nabla^2 A|^2 + |A|^6\right) d\mu\, dt \leq c\,\varepsilon, \tag{53}$$

$$\int_0^T \|A\|^4_{L^\infty(\Sigma_{3/4})}\, dt \leq c\,\varepsilon. \tag{54}$$

Fix a cutoff function $\widetilde{\gamma} \in C_c^\infty(\mathbb{R}^n)$ with $\chi_{B_{1/2}} \leq \widetilde{\gamma} \leq \chi_B$ and $\|D\,\widetilde{\gamma}\|_{L^\infty} + \|D^2\,\widetilde{\gamma}\|_{L^\infty} \leq c$. Also define cutoff functions in time by

$$\chi_j(t) = \begin{cases} 0 & \text{for } t \leq (j-1)\,\frac{T}{m} \\ \frac{m}{T}\left(t - (j-1)\,\frac{T}{m}\right) & \text{in between} \\ 1 & \text{for } t \geq j\,\frac{T}{m}, \end{cases}$$

where $0 \leq j \leq m$ and $m \in \mathbb{N}_0$. Note $\chi_0 \equiv 1$ on $[0,T]$, $\chi_m(T) = 1$ and

$$0 \leq \dot{\chi}_j \leq \frac{m}{T}\,\chi_{j-1}. \tag{55}$$

Introducing the notation $\alpha(t) = \|A\|^4_{L^\infty(\Sigma_{3/4})}$, $E_j(t) = \int |\nabla^{2j} A|^2\,\gamma^{4j+4} d\mu$ (where $\gamma = \widetilde{\gamma} \circ f$), we have by (4.14) in [5]

$$\frac{d}{dt} E_j(t) + \frac{1}{2} E_{j+1}(t) \leq c\,\alpha(t)\,E_j(t) + c\left(1 + \alpha(t)\right)\varepsilon.$$

Letting $e_j(t) = \chi_j(t)\,E_j(t)$ this implies, using also (55),

$$\begin{aligned} \frac{d}{dt} e_j(t) \leq\ & c\,\alpha(t)\,e_j(t) - \frac{1}{2}\chi_j(t)\,E_{j+1}(t) \\ & + c\left(1 + \alpha(t)\right)\varepsilon + \frac{m}{T}\,\chi_{j-1}(t)\,E_j(t). \end{aligned} \tag{56}$$

We now prove by induction for $0 \leq j \leq m$ and all $t \in (0,T]$ the inequality

$$e_j(t) + \frac{1}{2}\int_0^t \chi_j(s)\,E_{j+1}(s)\,ds \leq \frac{c(m)\,\varepsilon}{T^j}.$$

For $j = 0$ this follows from assumption (50) and estimate (53). Integrating (56) on $[0,T]$ yields, for $j \geq 1$,

$$\begin{aligned} e_j(t) + \frac{1}{2}\int_0^t \chi_j(s)\, E_{j+1}(s)\, ds \\ \leq c\int_0^t \alpha(s)\, e_j(s)\, ds + c\,\varepsilon \int_0^t \big(1+\alpha(s)\big) ds \\ + \frac{m}{T}\int_0^t \chi_{j-1}(s)\, E_j(s)\, ds. \end{aligned}$$

Now since $\int_0^T \alpha(t)\, dt \leq c\,\varepsilon$ by (54), we may apply Gronwall's inequality to get

$$\begin{aligned} e_j(t) + \frac{1}{2}\int_0^t \chi_j(s)\, E_{j+1}(s)\, ds &\leq c\,\varepsilon + \frac{c\, m}{T}\,\frac{c(m)\,\varepsilon}{T^{j-1}} \\ &\leq \frac{c(m)\,\varepsilon}{T^j}, \end{aligned}$$

as $T \leq c(n)$ by assumption. Thus we have at time $t = T$

$$\int |\nabla^{2m} A|^2\, \gamma^{4m+4}\, d\mu \leq \frac{c(m)\varepsilon}{T^m}.$$

The estimate for $\nabla^{2m+1}A$ follows by interpolation as in Lemma 5.1 of [5], taking $r = 1$, $p = q = 2$, $\alpha = 1$, $\beta = 0$, $s = 4m + 6$ and $t = \frac{1}{s} \in [-\frac{1}{2}, \frac{1}{2}]$ there and using again $T \leq c(n)$. Renaming $T$ into $t$, the $L^2$-estimate (52) is proved. Using (4.9) and (4.7) in [5], the $L^\infty$-estimate (51) follows. q.e.d.

## 4. Construction of the blowup

In this section we rescale the Willmore flow at an assumed singularity at finite or infinite time, thereby constructing a static Willmore surface as a limit. We shall need the following local area bound due to L. Simon [8].

**Lemma 4.1.** *Let $f : \Sigma \to \mathbb{R}^n$ be a properly immersed surface. Then for $0 < \sigma \leq \varrho < \infty$ and $\Sigma_\varrho = \Sigma_\varrho(x_0)$ one has*

$$\frac{\mu(\Sigma_\sigma)}{\sigma^2} \leq c\left(\frac{\mu(\Sigma_\varrho)}{\varrho^2} + \int_{\Sigma_\varrho} |H|^2\, d\mu\right).$$

*In particular if* $\Sigma$ *is compact without boundary*

$$\frac{\mu(\Sigma_\sigma)}{\sigma^2} \leq c\big(\mathcal{W}(f) + 4\pi\chi(\Sigma)\big).$$

The following compactness theorem, whose proof is omitted, is a localized version of the result of J. Langer [6].

**Theorem 4.2.** *Let* $f_j : \Sigma_j \longrightarrow \mathbb{R}^n$ *be a sequence of proper immersions, where* $\Sigma_j$ *is a two-dimensional manifold without boundary. Let*

$$\Sigma_j(R) = \{p \in \Sigma_j : |f_j(p)| < R\}$$

*and assume the bounds*

$$\begin{aligned} \mu_j\big(\Sigma_j(R)\big) &\leq c(R) \quad \textit{for any } R > 0, \\ \|\nabla^k A_j\|_{L^\infty} &\leq c(k) \quad \textit{for any } k \in \mathbb{N}_0. \end{aligned}$$

*Then there exists a proper immersion* $\hat{f} : \hat{\Sigma} \to \mathbb{R}^n$*, where* $\hat{\Sigma}$ *is again a two-manifold without boundary, such that after passing to a subsequence we have a representation*

$$f_j \circ \varphi_j = \hat{f} + u_j \quad \textit{on } \hat{\Sigma}(j) = \{p \in \hat{\Sigma} : |\hat{f}(p)| < j\} \tag{57}$$

*with the following properties:*

$$\begin{aligned} &\varphi_j : \hat{\Sigma}(j) \to U_j \subset \Sigma_j \quad \textit{is diffeomorphic}, \\ &\Sigma_j(R) \subset U_j \quad \textit{if } j \geq j(R), \\ &u_j \in C^\infty(\hat{\Sigma}(j), \mathbb{R}^n) \quad \textit{is normal along } \hat{f}, \\ &\|\hat{\nabla}^k u_j\|_{L^\infty(\hat{\Sigma}(j))} \to 0 \quad \textit{as } j \to \infty, \textit{ for any } k \in \mathbb{N}_0. \end{aligned}$$

Roughly speaking, the theorem says that on any ball $B_R(0)$ the immersion $f_j$ can be written as a normal graph $\hat{f} + u_j$ with small norm for $j$ large over a limit immersion $\hat{f}$, after suitably reparametrizing with $\varphi_j$.

Now let $f : \Sigma \times [0, T) \to \mathbb{R}^n$ be a smooth Willmore flow defined on a closed surface $\Sigma$, where $0 < T \leq \infty$. Define

$$\varkappa(r, t) = \sup_{x \in \mathbb{R}^n} \int_{\Sigma_r(x)} |A(t)|^2 \, d\mu_t.$$

Choose an arbitrary sequence $r_j \searrow 0$ and assume concentration in the sense that for all $j$

$$t_j = \inf\{t \geq 0 : \varkappa(r_j, t) > \varepsilon_1\} < T, \tag{58}$$

where $\varepsilon_1 = \varepsilon_0/c$ and $\varepsilon_0, c$ are the constants from Theorem 1.2 in [5]. Clearly

$$\int_{\Sigma_{r_j}(x)} |A(t_j)|^2 \, d\mu_{t_j} \leq \varepsilon_1 \quad \text{for any } x \in \mathbb{R}^n.$$

On the other hand, choosing an appropriate sequence of balls at times $\tau_\nu \searrow t_j$, we find a point $x_j \in \mathbb{R}^n$ satisfying

$$\int_{f^{-1}(\overline{B_{r_j}(x_j)})} |A(t_j)|^2 \, d\mu_{t_j} \geq \varepsilon_1.$$

Now we rescale by considering

$$\begin{gathered} f_j : \Sigma \times [-r_j^{-4} t_j, r_j^{-4}(T - t_j)) \to \mathbb{R}^n, \\ f_j(p,t) = \tfrac{1}{r_j}(f(p, t_j + r_j^4 t) - x_j). \end{gathered} \tag{59}$$

By the above we have $\varkappa_j(1,t) \leq \varepsilon_1$ for $t \leq 0$ and

$$\int_{f^{-1}(\overline{B_1(0)})} |A_j(0)|^2 \, d\mu_j \geq \varepsilon_1. \tag{60}$$

Furthermore Theorem 1.2 of [5] yields $r_j^{-4}(T - t_j) \geq c_0$ and in fact

$$\varkappa_j(1,t) \leq \varepsilon_0 \quad \text{for } 0 < t \leq c_0.$$

We may now apply Theorem 3.5 on parabolic cylinders $B_1(x) \times (t-1, t]$ to obtain

$$\|\nabla^k A_j(t)\|_{L^\infty} \leq c(k) \quad \text{for } -r_j^{-4} t_j + 1 \leq t \leq c_0. \tag{61}$$

Furthermore Lemma 4.1 yields

$$\frac{\mu_j(t)(\Sigma_R(0))}{R^2} \leq c\big(\mathcal{W}(f_0) + 4\pi\, \chi(\Sigma)\big) < \infty.$$

We apply Theorem 4.2 to the sequence $f_j = f_j(\,\cdot\,, 0\,) : \Sigma \to \mathbb{R}^n$, thus obtaining a limit immersion $\hat{f}_0 : \hat{\Sigma} \to \mathbb{R}^n$. Let $\varphi_j : \hat{\Sigma}(j) \to U_j \subset \Sigma$ be as in (57). Then the reparametrization

$$f_j(\varphi_j, \,\cdot\,) : \hat{\Sigma}(j) \times [0, c_0] \to \mathbb{R}^n \tag{62}$$

is a Willmore flow with initial data

$$f_j(\varphi_j, 0) = \hat{f}_0 + u_j : \hat{\Sigma}(j) \to \mathbb{R}^n. \tag{63}$$

The flows (62) satisfy the curvature bounds (61) and have initial data converging locally in $C^k$ to the immersion $\hat{f}_0 : \Sigma \to \mathbb{R}^n$. By standard estimates for geometric evolution equations, see (4.24)–(4.28) in [5], we deduce the locally smooth convergence

$$f_j(\varphi_j, \cdot) \to \hat{f}, \tag{64}$$

where $\hat{f} : \hat{\Sigma} \times [0, c_0] \to \mathbb{R}^n$ is a Willmore flow with initial data $f_0$. But on the other hand we have

$$\begin{aligned}
&\int_0^{c_0} \int_{\hat{\Sigma}(j)} \big|\mathbf{W}\big(f_j(\varphi_j, t)\big)\big|^2 \, d\mu_{f_j(\varphi_j, \cdot)} dt \\
&\quad = \int_0^{c_0} \int_{U_j} \big|\mathbf{W}\big(f_j(\,\cdot\,, t)\big)\big|^2 \, d\mu_j dt \\
&\quad \le \int_\Sigma |A_j(c_0)|^2 \, d\mu_j - \int_\Sigma |A_j(0)|^2 \, d\mu \\
&\quad = \int_\Sigma |A(t_j + r_j^4 \, c_0)|^2 \, d\mu - \int_\Sigma |A(t_j)|^2 \, d\mu,
\end{aligned}$$

which converges to zero as $j \to \infty$. This implies that $\mathbf{W}(\hat{f}) \equiv 0$ which means that $\hat{f}(\,\cdot\,, t) \equiv \hat{f}_0$ is an immersed Willmore surface, which is independent of time. Furthermore (60) implies, because of the smooth convergence in (64),

$$\int\limits_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2 \, d\hat{\mu} \ge \varepsilon_1 > 0. \tag{65}$$

Thus $\hat{f}$ is not a union of planes.

**Lemma 4.3.** *Let $\hat{f} : \hat{\Sigma} \longrightarrow \mathbb{R}^n$ be the blowup constructed above. If $\hat{\Sigma}$ contains a compact component $C$, then in fact $\hat{\Sigma} = C$ and $\Sigma$ is diffeomorphic to $C$.*

*Proof.* For $j$ sufficiently large, $\varphi_j(C)$ is open and closed in $\Sigma$. Hence by connectedness of $\Sigma$ we have $\Sigma = \varphi_j(C)$ and thus $\hat{\Sigma} = C$. q.e.d.

**Theorem 4.4** (Nontriviality of the Blowup)**.** *Let $\hat{f} : \hat{\Sigma} \to \mathbb{R}^n$ be the blowup of a Willmore flow as constructed above. Then none of* *the components of $\hat{f}$ parametrizes a round sphere. In particular, the blowup has a component which is a nonumbilic (compact or noncompact) Willmore surface.*

*Proof.* Otherwise, Lemma 4.3 implies that the blowup surface $\hat{f} : \hat{\Sigma} \to \mathbb{R}^n$ is an embedded round sphere, i.e., has no further components. It follows that, up to the diffeomorphism $\varphi_j : \hat{\Sigma} \to \Sigma$, the map $f_j(\,\cdot\,, 0)$ is $C^k$-close to a round sphere and therefore

$$\begin{aligned} \int_\Sigma |A^\circ(t_j)|^2 \, d\mu &= \int_\Sigma |A_j^\circ(0)|^2 d\mu_j \to 0, \\ \mu(t_j)(\Sigma) &= r_j^2 \, \mu_j(0)(\Sigma) \to 0. \end{aligned}$$

This contradicts the lower area bound which will be proved in Theorem 5.2. q.e.d.

## 5. Small initial energy

In this section we finally prove our main result:

**Theorem 5.1** (Global Existence and Convergence for Small Initial Energy). *There exists an $\varepsilon_0 = \varepsilon_0(n) > 0$ such that if at time $t = 0$ there holds*

$$\mathcal{W}(f_0) = \int_\Sigma |A^\circ|^2 \, d\mu < \varepsilon_0,$$

*then the Willmore flow exists smoothly for all times and converges exponentially to a round sphere as $t \to \infty$.*

We split the proof into several steps. The first step was already used in Theorem 4.4 and is of independent interest.

**Theorem 5.2** (Area Estimate). *Let $f : \Sigma \times [0, T) \to \mathbb{R}^n$ be a Willmore flow with $\mathbf{W}(f) = \int_\Sigma |A^\circ|^2 \, d\mu \le \varepsilon < \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(n)$ is as in Proposition* 3.4. *Then*

$$(1 - c\varepsilon)\, \mu_0(\Sigma) \le \mu(\Sigma) \le (1 + c\varepsilon)\, \mu_0(\Sigma) \tag{66}$$

$$\int_0^t \int_\Sigma (|\nabla A|^2 + |A|^2 \, |A^\circ|^2) \, d\mu \, ds \le c\varepsilon \, \mu_0(\Sigma). \tag{67}$$

*Proof.* We have

$$\frac{d}{dt} \mu(\Sigma) = -\int_\Sigma |\nabla H|^2 \, d\mu + \int_\Sigma \langle Q(A^\circ) H, H \rangle d\mu.$$

Multiplying Simons' identity (10) by $A^\circ$ and integrating yields (cf. Lemma 2.2):

$$(68) \qquad \begin{aligned} 2\int_\Sigma |\nabla A^\circ|^2\,d\mu + \int_\Sigma |H|^2\,|A^\circ|^2\,d\mu \\ = \int_\Sigma |\nabla H|^2\,d\mu + \int_\Sigma A^\circ * A^\circ * A^\circ * A^\circ\,d\mu. \end{aligned}$$

As $\langle Q(A^\circ)H, H\rangle \leq |A^\circ|^2\,|H|^2$ by (14), we obtain

$$\begin{aligned} \frac{d}{dt}\,\mu(\Sigma) + 2\int_\Sigma |\nabla A^\circ|^2 d\mu &\leq c\int_\Sigma |A^\circ|^4\,d\mu \\ &\leq c\,\|A^\circ\|^4_{L^\infty}\,\mu(\Sigma). \end{aligned}$$

From (48) with $\varrho = \infty$ we have

$$\int_0^t \|A^\circ\|^4_{L^\infty}\,ds \leq c\,\varepsilon,$$

and the Gronwall inequality yields

$$\mu(\Sigma) + 2\int_0^t\int_\Sigma |\nabla A^\circ|^2\,d\mu\,ds \leq (1 + c\,\varepsilon)\,\mu_0(\Sigma).$$

On the other hand, by Michael-Simon

$$\begin{aligned} \int |\nabla H|^2\,d\mu &\leq c\left(\int \big(|\nabla^2 H| + |H||\nabla H|\big)\,d\mu\right)^2 \\ &\leq c\,\mu(\Sigma)\int_\Sigma \big(|\nabla^2 H|^2 + |H|^2|\nabla H|^2\big)\,d\mu. \end{aligned}$$

As $\langle Q(A^\circ)H, H\rangle \geq 0$ we obtain

$$\frac{d}{dt}\,\mu(\Sigma) \geq -c\,\mu(\Sigma)\int_\Sigma (|\nabla^2 H|^2 + |H|^2\,|\nabla H|^2)\,d\mu.$$

Using (47) with $\varrho = \infty$ implies the remaining inequality in (66). In particular we obtain

$$\int_0^t\int_\Sigma |\nabla A^\circ|^2\,d\mu\,ds \leq c\,\varepsilon\,\mu_0(\Sigma),$$

and

$$\begin{aligned} \int_0^t\int_\Sigma |A^\circ|^4\,d\mu\,ds &\leq (1 + c\varepsilon)\,\mu_0(\Sigma)\int_0^t \|A^\circ\|^4_{L^\infty} ds \\ &\leq c\,\varepsilon\,\mu_0(\Sigma). \end{aligned}$$

Finally from (68) and Codazzi (5)

$$\begin{aligned}\int_0^t \int_\Sigma |H|^2\,|A^\circ|^2\,d\mu\,ds &\leq c\int_0^t\int_\Sigma (|\nabla A^\circ|^2+|A^\circ|^4)d\mu\,ds\\ &\leq c\,\varepsilon\,\mu_0(\Sigma).\end{aligned}$$

This proves the theorem. q.e.d.

**Remark.** The extrinsic diameter is bounded above and below by the diameter of the initial surface, cf. [8].

**Lemma 5.3.** *Under the assumptions of Theorem* 5.1 *there exists a radius* $r_0 > 0$ *such that*

$$\int\limits_{\Sigma_{r_0}(x)} |A(t)|^2\,d\mu \leq \varepsilon_1 \quad \text{for all } x\in\mathbb{R}^n,\ t\in[0,\infty),$$

*where* $\varepsilon_1 > 0$ *is as in* (58).

*Proof.* Otherwise, the blowup construction of Section 4 yields an immersed Willmore surface $\hat{f}:\hat{\Sigma}\to\mathbb{R}^n$ with

$$\int\limits_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2\,d\hat{\mu} \geq \varepsilon_1,$$

whereas

$$\int_{\hat{\Sigma}} |\hat{A}^\circ|^2\,d\hat{\mu} < \varepsilon_0.$$

By Theorem 2.7 the surface $\hat{f}$ must be a union of embedded planes and round spheres, which however contradicts the nontriviality of the blowup, Theorem 4.4. q.e.d.

**Lemma 5.4.** *For any sequence* $t_j \to \infty$ *there exist* $x_j \in \mathbb{R}^n$ *and* $\varphi_j \in \mathrm{Diff}\,(\Sigma)$ *such that, after passing to a subsequence, the immersions* $f(\varphi_j,t_j)-x_j$ *converge smoothly to an embedded round sphere.*

*Proof.* Let $x_j = f(p,t_j)$ where $p\in\Sigma$ is arbitrary. By the previous lemma and the interior curvature estimates from Theorem 3.5, we have for $t_j \geq 1$

$$\|\nabla^k A(\,\cdot,t_j)\|_{L^\infty} \leq c(k). \tag{69}$$

Furthermore, Lemma 4.1 yields the area bound

$$\frac{\mu(t_j)\big(B_R(x_j)\big)}{R^2} \le c\big(\mathcal{W}\big(f(\,\cdot\,,t_j)\big) + 4\pi\chi(\Sigma)\big).$$

According to Theorem 4.2, there exist a properly immersed surface $\hat{f} : \hat{\Sigma} \longrightarrow \mathbb{R}^n$ and diffeomorphisms $\varphi_j : \hat{\Sigma}(j) \longrightarrow U_j \subset \Sigma$, such that (after selection of a subsequence)

$$f(\varphi_j, t_j) - x_j \longrightarrow \hat{f}$$

locally in $C^k$ on $\hat{\Sigma}$. On $\hat{\Sigma}(j)$ we consider the Willmore flows

$$g_j(p,t) = f\big(\varphi_j(p), t_j + t\big) - x_j \quad (t \ge -t_j).$$

These satisfy the curvature estimates (69), and the initial data (at $t = 0$) converge to $\hat{f}$. Arguing as in (64), we obtain that $g_j$ converges locally smoothly on $\hat{\Sigma} \times [0,\infty)$ to a Willmore flow $g : \hat{\Sigma} \times [0,\infty) \to \mathbb{R}^n$ with initial data $\hat{f}$. But now

$$\int_0^1 \int_{\hat{\Sigma}(j)} |\mathbf{W}(g_j)|^2 \, d\mu_{g_j} dt \le \int_{t_j}^{t_j+1} \int_{\Sigma} |\mathbf{W}(f)|^2 d\mu \, dt \to 0 \quad \text{as } j \to \infty.$$

Therefore we have $\mathbf{W}(g) \equiv 0$ which proves that $\hat{f}$ is a Willmore surface, and Theorem 2.7 implies that $\hat{f}$ is a union of embedded planes and round spheres. Using the upper area bound in (66) and excluding several components as in Lemma 4.3 we conclude that $\hat{f}$ must be a round sphere, and that the subconvergence is smooth. q.e.d.

As a consequence of the above, we obtain that

(70) $$\mathcal{W}(f) = \int_{\Sigma} |A^\circ|^2 \, d\mu \to 0 \quad \text{as } t \to \infty.$$

Moreover, Theorem 5.2 implies the existence of the nonzero limit

(71) $$\omega = \lim_{t\to\infty} \mu(t)(\Sigma) \in (0,\infty).$$

Finally we now prove exponential decay of the curvature; from this one obtains smooth convergence of $f$ to a round sphere $\hat{f}$ and thus Theorem 5.1 in a standard way.

**Lemma 5.5.** *As $t \nearrow \infty$, the following asymptotic statements hold, where $\lambda > 0$ is a constant:*

$$\|\nabla^k A(t)\|_{L^\infty} \leq c_k\, e^{-\lambda t} \quad \text{for } k \geq 1, \tag{72}$$

$$\|A^\circ\|_{L^\infty} \leq c_0\, e^{-\lambda t}. \tag{73}$$

*Proof.* For $\omega$ as in (71), the previous Lemma implies that the sectional curvature and the mean curvature of $f(\,\cdot\,,t)$ satisfy

$$\left\| K(\,\cdot\,,t) - \frac{4\pi}{\omega} \right\|_{L^\infty} \longrightarrow 0,$$

$$\left\| |H|^2 - \frac{16\pi}{\omega} \right\|_{L^\infty} \longrightarrow 0 \quad \text{as } t \to \infty.$$

In particular, we may assume after a fixed time translation that

$$|H|^2 \geq c > 0 \quad \text{for all } t \geq 0.$$

By Lemma 3.3, we then have for all $t$

$$\partial_t \int_\Sigma |A^\circ|^2 d\mu + c \int_\Sigma (|\nabla^2 A|^2 + |\nabla A|^2 + |A^\circ|^2) d\mu \leq 0,$$

which implies

$$\int_\Sigma |A^\circ|^2 d\mu + \int_t^\infty \int_\Sigma (|\nabla^2 A|^2 + |\nabla A|^2) d\mu\, ds \leq c\, e^{-2\lambda t}, \tag{74}$$

for a constant $\lambda = \lambda(n) > 0$. From here we easily derive exponential decay of all derivatives of $A$. Namely, letting $\varrho \to \infty$ in Corollary 3.4 of [5], we have for $\phi = \nabla^m A$ ($m \geq 1$)

$$\frac{d}{dt} \int |\phi|^2 d\mu + \frac{3}{4} \int |\nabla^2 \phi|^2 d\mu \leq \int \big(P_3^{m+2}(A) + P_5^m(A)\big) * \phi\, d\mu.$$

Using that $A$ and all its derivatives remain bounded as $t \to \infty$, we can estimate

$$\int P_2^0(A) * \nabla^{m+2} A * \phi\, d\mu \leq \varepsilon \int |\nabla^2 \phi|^2 d\mu + c(\varepsilon) \int |\nabla^m A|^2 d\mu,$$

$$\int \big(\hat{P}_3^{m+2}(A) + P_5^m(A)\big) * \phi d\mu \leq c \sum_{j=1}^{m+1} \int |\nabla^j A|^2 d\mu.$$

Here $\hat{P}_3^{m+2}(A)$ denotes all terms of type $P_3^{m+2}(A)$ that do not contain the $(m+2)$-th derivative, and of course $c$ is not a universal constant here. We obtain

$$\frac{d}{dt}\int |\phi|^2 d\mu + \frac{1}{2}\int |\nabla^2\phi|^2 d\mu \le c \sum_{j=1}^{m+1} \int |\nabla^j A|^2 d\mu,$$

and now by induction using (74)

$$\|\nabla^m A\|_{L^2}^2 + \int_t^\infty \|\nabla^{m+2} A\|_{L^2}^2 \, ds \le c\, e^{-2\lambda t}.$$

By a Sobolev inequality, e.g., the Michael-Simon inequality, we deduce

$$\|A^\circ\|_{L^\infty} \le c_0\, e^{-\lambda t}, \tag{75}$$

$$\|\nabla^k A\|_{L^\infty} \le c_k\, e^{-\lambda t}, \tag{76}$$

which ends the proof of Lemma 5.5 and of Theorem 5.1. q.e.d.

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