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A CANONICAL BUNDLE FORMULA

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Abstract

A higher dimensional analogue of Kodaira's canonical bundle formula is obtained. As applications, we prove that the log-canonical ring of a klt pair with $\kappa \leq 3$ is finitely generated, and that there exists an effectively computable natural number M such that $|MK_X|$ induces the Iitaka fibering for every algebraic threefold X with Kodaira dimension $\kappa = 1$.

1. Introduction

If $f : X \to C$ is a minimal elliptic surface over $\mathbb{C}$, then the relative canonical divisor $K_{X/C}$ is expressed as

$$K_{X/C} = f^*L + \sum_P \frac{m_P - 1}{m_P} f^*(P), \tag{1}$$

where L is a nef divisor on C and P runs over the set of points such that $f^*(P)$ is a multiple fiber with multiplicity $m_P > 1$. It is the key in the estimates on the plurigenera $P_n(X)$ that the coefficients $(m_P - 1)/m_P$ are 'close' to 1 [12]. Furthermore $12L$ is expressed as

$$12K_{X/C} = f^*j^*\mathcal{O}_{\mathbb{P}}(1) + 12\sum_P \frac{m_P - 1}{m_P} f^*(P) + \sum \sigma_Q f^*(Q), \tag{2}$$

where σ_Q is an integer $\in [0, 12)$ and $j : C \to \mathbb{P}^1$ is the j-function [5, (2.9)]. The computation of these coefficients is based on the explicit classification of the singular fibers of f, which made the generalization difficult.

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We note that L in the exact analogue of the formula (1) for the case $\dim X/C = 2$ need not be a divisor (Example 2.7) and that the formula (2) is more natural to look at if L is allowed to be a $\mathbb{Q}$-divisor.

The higher dimensional analogue of the formula (2) is treated in Section 2 as a refinement of [15, §5 Part II] and the log version in Section 4. We give the full formula only in 4.5 to avoid repetition. The estimates of the coefficients are treated in 2.8, 3.1 and 4.5. (See 3.9 on the comparison of the formula (2) and our estimates.) We note that the "coefficients" in the formula (2) are of the form $1 - 1/m$ except for the finite number of exceptions $1/12, \cdots, 11/12$. In the generalized formula, the coefficients are in a more general form (cf. 4.5.(v)), which still enjoys the DCC (Descending Chain Condition) property of Shokurov.

The following are some of the applications.

1. (Corollary 5.3) If (X, Δ) is a klt pair with $\kappa(X, K_X + \Delta) \leq 3$, then its log-canonical ring is finitely generated.

2. (Corollary 6.2) There exists an effectively computable natural number M such that $|MK_X|$ induces the Iitaka fibering for every algebraic threefold X with Kodaira dimension $\kappa(X) = 1$.

To get the analogue for an $(m+1)$-dimensional X ($m \geq 3$) with $\kappa(X) = 1$, it remains to show that an arbitrary m-fold F with $\kappa(F) = 0$ and $p_g(F) = 1$ is birational to a smooth projective model with effectively bounded m-th Betti number.

Notation. Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the set of positive (resp. non-negative) integers. We work over $\mathbb{C}$ in this note. Let X be a normal variety and B, B' $\mathbb{Q}$-divisors on X.

If $B - B'$ is effective, we write $B \succ B'$ or $B' \prec B$.

We write $B \sim B'$ if $B - B'$ is a principal divisor on X (linear equivalence of $\mathbb{Q}$-divisors).

Let B_+, B_- be the effective $\mathbb{Q}$-divisors on X without common irreducible components such that $B_+ - B_- = B$. They are called the *positive* and the *negative* parts of B.

Let $f : X \to C$ be a surjective morphism. Let B^h, B^v be the $\mathbb{Q}$-divisors on X with $B^h + B^v = B$ such that an irreducible component of Supp B is contained in Supp B^h iff it is mapped onto C. They are called the *horizontal* and the *vertical* parts of B over C. B is said to be *horizontal* (resp. *vertical*) over C if $B = B^h$ (resp. $B = B^v$). The phrase "over C" might be suppressed if there is no danger of confusion.

As for other notions, we mostly follow [14]. However we introduce a slightly different terminology to distinguish the pairs with non-effective boundaries (cf. [10]).

A pair (X, D) consists of a normal variety X and a $\mathbb{Q}$-divisor D. If $K_X + D$ is $\mathbb{Q}$-Cartier, we can pull it back by an arbitrary resolution $f : Y \to X$ and obtain the formula

$$K_Y = f^*(K_X + D) + \sum_i a_i E_i,$$

where E_i are prime divisors and $a_i \in \mathbb{Q}$. The pair (X, D) is said to be sub klt (resp. sub lc) if $a_i > -1$ (resp. ≥ -1) for every resolution f and every i. Furthermore, (X, D) is said to be klt (resp. lc) if D is effective.

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2. Semistable part of $K_{X/C}$

In this section, we refine the results of [15, §5, Part II] after putting the basic results together.

2.1. Let $f : X \to C$ be a surjective morphism of a normal projective variety X of dimension $n = m + l$ to a nonsingular projective l-fold C such that

(i) X has only canonical singularities, and

(ii) the generic fiber F of f is a geometrically irreducible variety with Kodaira dimension $\kappa(F) = 0$. We fix the smallest $b \in \mathbb{Z}_{>0}$ such that the b-th plurigenus $P_b(F)$ is non-zero.

Proposition 2.2. *There exists one and only one $\mathbb{Q}$-divisor D modulo linear equivalence on C with a graded $\mathcal{O}_C$-algebra isomorphism*

$$\oplus_{i\geq 0}\mathcal{O}(\lfloor iD \rfloor) \cong \oplus_{i\geq 0}(f_*\mathcal{O}(ibK_{X/C}))^{**},$$

where M^{**} denotes the double dual of M.

Furthermore, the above isomorphism induces the equality

$$bK_X = f^*(bK_C + D) + B,$$

where B is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\lfloor iB_+ \rfloor) = \mathcal{O}_C$ $(\forall i > 0)$ and $\operatorname{codim}(f(\operatorname{Supp} B_-) \subset C) \geq 2$. We note that for an arbitrary open set U of C, $D|_U$ and $B|_{f^{-1}(U)}$ depend only on $f|_{f^{-1}(U)}$.

Proof. By [15, (2.6.i)], there exists $c > 0$ such that

$$(f_*\mathcal{O}(ibcK_{X/C}))^{**} = \{(f_*\mathcal{O}(bcK_{X/C}))^{**}\}^{\otimes i} \quad (\forall i > 0).$$

Choose an embedding $\phi : (f_*\mathcal{O}(bK_{X/C}))^{**} \subset \mathbb{Q}(C)$ into the function field of C, and we can define a Weil divisor cD by

$$\phi^{\otimes c} : (f_*\mathcal{O}(bcK_{X/C}))^{**} \cong \mathcal{O}(cD) \subset \mathbb{Q}(C).$$

D modulo linear equivalence does not depend on the choice of ϕ.

Since taking the double dual has no effect on codimension 1 points, there is a natural inclusion

$$f^*\mathcal{O}_C(cD) \subset \mathcal{O}_X(bcK_{X/C}) \text{ on } X \setminus f^{-1}(\text{some codim 2 subset of } C).$$

Extending it to X, we obtain a $\mathbb{Q}$-divisor B such that $B = bK_{X/C} - f^*D$. It is easy to see that B satisfies the required conditions. q.e.d.

Definition 2.3. Under the notation of 2.2, we denote D by $L_{X/C}$. It is obvious that $L_{X/C}$ depends only on the birational equivalence class of X over C.

If X has bad singularities, then we take a nonsingular model X' of X and use $L_{X'/C}$ as our definition of $L_{X/C}$.

Proposition 2.4 (Viehweg). *Let $\pi : C' \to C$ be a finite surjective morphism from a nonsingular l-fold C' and let $\overline{f}' : \overline{X}' \to C'$ be a nonsingular model of $X \times_C C' \to C'$. Then there is a natural relation*

$$L_{\overline{X}'/C'} \prec \pi^* L_{X/C}.$$

Furthermore if $X \times_C C'$ has a semistable resolution over a neighborhood of a codimension 1 point P' of C', or if $X_{\pi(P')}$ has only canonical singularities, then $P' \notin \operatorname{Supp}(\pi^* L_{X/C} - L_{\overline{X}'/C'})$.

Proof. Except for the last assertion, this is due to [22, §3] (cf. [15, (4.10)]). If $X_{\pi(P')}$ has only canonical singularities, then $X \times_C C'$ has only canonical singularities in a neighborhood of $f'^{-1}(P')$ by [20, Proposition 7] or [9] because so does the generic fiber F of f. Thus 2.4 follows. q.e.d.

Corollary 2.5. *There exists one and only one $\mathbb{Q}$-divisor $L^{ss}_{X/C}$ $(\prec L_{X/C})$ such that*

(i) $\pi^* L^{ss}_{X/C} \prec L_{\overline{X}'/C'}$ *for arbitrary $\pi : C' \to C$ as in* 2.4, *and*

(ii) $\pi^* L^{ss}_{X/C} = L_{\overline{X}'/C'}$ *at P' if π in* 2.4 *is such that $X \times_C C' \to C'$ has a semistable resolution $\overline{X}' \to C'$ over a neighborhood of P' or $X_{\pi(P')}$ has only canonical singularities.*

There exists an effective divisor $\Sigma \subset C$ such that every birational morphism $\pi : C' \to C$ from a nonsingular projective l-fold with $\pi^*(\Sigma)$ an snc divisor has the following property: Let X' be a projective resolution of $X \times_C C'$ and $f' : X' \to C'$ the induced morphism. Then $L^{ss}_{X'/C'}$ is nef.

Proof. When C'/C is Galois with group G, $L_{\overline{X}'/C'}$ is G-invariant and therefore descends to a $\mathbb{Q}$-subdivisor of $L_{X/C}$. The minimum $L^{ss}_{X/C}$ $(\prec L_{X/C})$ of all the descents exists by 2.4, whence the uniqueness follows.

The last assertion is proved in [15, §5, part II] though it is not explicitly stated there. In [15, (5.13)], First our Σ is a divisor containing the discriminant locus of f and h constructed in the proof of [15, (5.15.2)] (which are f' and h' in our 2.6). Then [15, (5.14.1)] shows that our $L^{ss}_{X/C}/b$ is equal to $P_{f',bc}$ for sufficiently divisible $c \in \mathbb{Z}_{>0}$. Finally [15, (5.15.3)] shows that $P_{f',bc}$ is nef. q.e.d.

Remark 2.6. Under the notation of 2.1, consider the following construction. Since $\dim |bK_F| = 0$, there exists a Weil divisor W on X such that

(i) W^h is effective and $f_* \mathcal{O}(iW^h) = \mathcal{O}_C$ for all $i > 0$, and

(ii) $bK_X - W$ is a principal divisor (ψ) for some non-zero rational function ψ on X.

Let $s : Z \to X$ be the normalization of X in $\mathbb{Q}(X)(\psi^{1/b})$. Then the above proof actually shows the following:

Fix resolutions X' and Z' of X and Z. We write $f' : X' \to C$ and $h' : Z' \to C$. Then as the divisor $\Sigma \subset C$ in 2.5, we can take an arbitrary effective divisor $\Sigma \subset C$ such that $f' : X' \to C$ and $h' : Z' \to C$ are smooth over $C \setminus \Sigma$.

Example 2.7. Let F be a K3 surface with a free involution $\iota : F \to F$ so that $E = F/\{1, \iota\}$ is an Enriques surface. Let $j : \mathbb{P}^1 \to \mathbb{P}^1$ be the involution $x \mapsto -x$, so that 0 and ∞ are the only fixed points. Let $f : X = \mathbb{P}^1 \times F/\{1, j \times \iota\} \to C = \mathbb{P}^1/\{1, j\}$ be the map induced by the first projection. Then f is smooth over $C \setminus \{0, \infty\}$, and $f^*(0) = 2E_0$ and $f^*(\infty) = 2E_\infty$, where $E_0 \simeq E_\infty \simeq E$. Using $K_{E_t} \sim K_X + E_t|_{E_t} \not\sim 0$ for $t = 0$ and ∞, one easily sees that $K_{X/C} \sim f^*\mathcal{O}(1)$ and that $L^{ss}_{X/C} = \frac{1}{2}(0) - \frac{1}{2}(\infty)$. Thus $L^{ss}_{X/C}$ is only a $\mathbb{Q}$-divisor fitting in the analogue of the formula (1):

$$K_{X/C} = f^*L^{ss}_{X/C} + \sum_{t=0,\infty} \frac{1}{2} f^*(t).$$

The main part (i) of the following is included in 4.5.(v) (see also 4.7). We still leave it here for reference.

Proposition 2.8. *Under the notation and the assumptions of* 2.5*, let $N \in \mathbb{Z}_{>0}$ be such that $Nf^*(L^{ss}_{X/C})$ is a Weil divisor. Then we have*

$$L_{X/C} = L^{ss}_{X/C} + \sum_P s_P P,$$

where $s_P \in \mathbb{Q}$ for every codimension 1 point P of C such that:

(i) *For each P, there exist $u_P, v_P \in \mathbb{Z}_{>0}$ such that $0 < v_P \leq bN$ and $s_P = (bNu_P - v_P)/(Nu_P)$.*

(ii) *$s_P = 0$ if $f^*(P)$ has only canonical singularities or if $X \to C$ has a semistable resolution in a neighborhood of P.*

In particular, s_P depends only on $f|_{f^{-1}(U)}$ where U is an open set of C containing P.

The assertion (ii) is proved in 2.5.(ii).

3. Bounding the denominator

In applications, it is important to bound the denominator of $L^{ss}_{X/C}$.

Theorem 3.1. *Let $E \to F$ be the cover associated to the b-th root of the unique element of $|bK_F|$. Let $\overline{E}$ be a nonsingular projective model of E and let B_m be its m-th Betti number. Then there is a natural number $N = N(B_m)$ depending only on B_m such that $NL^{ss}_{X/C}$ is a divisor.*

Remark 3.2. We have $N(x) = \mathrm{lcm}\{y \in \mathbb{Z}_{>0} \mid \varphi(y) \leq x\}$, where $\varphi(y)$ is Euler's function.

3.3. To prove 3.1, we can replace the base space C by a general hyperplane-section H and X by f^*H. Repeating this procedure, we may assume that C is a curve in the rest of this section.

We use the construction given in 2.6, i.e., let W, $s : Z \to X$ and $h = f \circ s : X \to C$ be as in 2.6. We note that the branch locus of s is contained in the singular locus of X and $\operatorname{Supp} W$.

Lemma 3.4 ([15, (5.15.8)]). $bL^{ss}_{Z/C} = L^{ss}_{X/C}$.

Proof. Because of the definition 2.3, we may replace C with its base change and change the model X birationally. Hence we may assume that X is smooth and $W^h_{red} + f^*\Sigma$ is a reduced snc divisor, where Σ is a reduced effective divisor of C such that $f^{-1}(\Sigma)$ contains W^v and f is smooth over $C \setminus \Sigma$. We may also assume that $h : Z \to C$ has a semistable resolution $h' : Z' \to C$ and that h is smooth over $C \setminus \Sigma$. Hence $K_X + W^h_{red} + f^*\Sigma$ is log-canonical and

$$K_Z + (s^{-1}W^h)_{red} + h^*\Sigma = s^*(K_X + W^h_{red} + f^*\Sigma)$$

is also log-canonical (cf. [13, (20.3)] for the explicit statement, which goes back to [19, §1]). Hence for all $n > 0$ we have

$$h'_*\mathcal{O}(n(K_{Z'} + (s'^{-1}W^h)_{red} + h'^*\Sigma)) = h_*\mathcal{O}(n(K_Z + (s^{-1}W^h)_{red} + h^*\Sigma)),$$

where $s' : Z' \to X$ is the induced morphism. By 2.6.(i), we see

$$\begin{aligned} h_*\mathcal{O}(bn(K_Z + (s^{-1}W^h)_{red} + h^*\Sigma)) &= h'_*\mathcal{O}(bnK_{Z'}) \otimes \mathcal{O}(bn\Sigma) \\ &= \mathcal{O}(bnL^{ss}_{Z/C} + bnK_C + bn\Sigma). \end{aligned}$$

By a similar computation, we get

$$f_*\mathcal{O}(bn(K_X + W^h_{red} + f^*\Sigma)) = \mathcal{O}(nL^{ss}_{X/C} + bnK_C + bn\Sigma).$$

Whence the natural inclusion $bL^{ss}_{Z/C} \succ L^{ss}_{X/C}$ becomes an equality. q.e.d.

3.5. Let $\overline{h} : \overline{Z} \to C$ be a nonsingular projective model of $Z \to C$ whose generic fiber is isomorphic to $\overline{E}$. Let $\pi : C' \to C$ be a finite Galois morphism with group G such that $\overline{Z} \times_C C'$ has a semistable resolution $\overline{h}' : \overline{Z}' \to C'$ whose generic fiber is isomorphic to $\overline{E}$. We give names as in the next diagram.

$$\begin{array}{ccc} \overline{Z} & \xleftarrow{\sigma} & \overline{Z}' \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle \overline{h}'} \\ C & \xleftarrow[\pi]{} & C' \end{array}$$

By 2.5, we have

$$\overline{h}'_*\mathcal{O}(K_{\overline{Z}'/C'}) = \mathcal{O}(\pi^* L^{ss}_{Z/C}).$$

We note that $\pi^* L^{ss}_{Z/C}$ is a Weil divisor [15, (2.6.ii)].

3.6. Due to its birational invariance, $\overline{h}'_*\mathcal{O}(K_{\overline{Z}'/C'})$ is G-linearized. Let $P' \in C'$ and let $G_{P'}$ be the stabilizer. Then $G_{P'}$ acts on $\overline{h}'_*\mathcal{O}(K_{\overline{Z}'/C'}) \otimes \mathbb{C}(P')$ through a character $\chi_{P'} : G_{P'} \to \mathbb{C}^*$. Then $NL^{ss}_{Z/C}$ is a divisor iff $\chi^N_{P'} = 1$ for all P' because $\mathcal{O}(NL^{ss}_{Z/C})$ should be $(\pi_*\mathcal{O}(N\pi^* L^{ss}_{Z/C}))^G$.

We can now localize everything on neighborhoods of $P = \pi(P')$ and P'. Let e be the ramification index of π at P'. Let z be a local coordinate for the germ (C, P) and $z' = z^{1/e}$ for (C', P'). We have a natural homomorphism $G_{P'} \to \mu_e = \{x \in \mathbb{C} \mid x^e = 1\}$. Let H' be the canonical extension of $H'_\circ = \mathcal{O}_{C'_\circ} \otimes (R^m\overline{h}'_\circ\mathbb{C}_{\overline{Z}'_\circ})_{prim}$ to C', where $C'_\circ = C' \setminus \{P'\}$, $\overline{Z}'_\circ = \overline{Z}' \setminus \overline{h}'^{-1}(P')$ and $\overline{h}'_\circ : \overline{Z}'_\circ \to C'_\circ$ is the restriction of $\overline{h}'$. By its construction, H' admits a μ_e-action. Let $\zeta = e^{2\pi i/e} \in \mu_e$.

We quote the following to compare $\chi_{P'}$ and H'.

Proposition 3.7 [7, Proposition 1]). *There exists a μ_e-equivariant injection*

$$\overline{h}'_*\mathcal{O}(K_{\overline{Z}'/C'}) \otimes \mathbb{C}(P') \subset H' \otimes \mathbb{C}(P').$$

3.8. Thus if ℓ is the order of $\chi_{P'}$, then $\varphi(\ell) \leq B_m$, where $\varphi(\ell)$ is Euler's function. Let $N(x) = \mathrm{lcm}\{\ell \mid \varphi(\ell) \leq x\}$. Then we see that $N(B_m)L^{ss}_{Z/C}$ is a divisor. Hence 3.1 is proved.

Remark 3.9. When $f : X \to C$ is a minimal elliptic surface, we mentioned the formula (2). By 2.5 and [5, (2.9)], we have $b = 1$,

$12L^{ss}_{X/C} \simeq j^*\mathcal{O}_{\mathbb{P}}(1)$ and

$$s_P = \begin{cases} (m_P - 1)/m_P & \text{if} \quad m_P > 1 \\ \sigma_P/12 & \text{if} \quad m_P = 1. \end{cases}$$

Our estimates are the following. Since $B_1(F) = 2$ and $N(2) = 12$, our 3.1 shows that $12L^{ss}_{X/C}$ is a Weil divisor and the estimates 2.8.(i) is compatible with the above.

4. Log-canonical bundle formula

In this section, we give the log analogue of the semistable part defined in Section 2 and give a log-canonical bundle formula.

4.1. Let $f : X \to C$ be a surjective morphism of a normal projective variety X of dimension $n = m + l$ to a nonsingular projective l-fold C such that:

(i) (X, Δ) is a sub klt pair (assumed klt from 4.4 and on).

(ii) The generic fiber F of f is a geometrically irreducible variety with $\kappa(F, (K_X + \Delta)|_F) = 0$. We fix the smallest $b \in \mathbb{Z}_{>0}$ such that the $f_*\mathcal{O}_X(b(K_X + \Delta)) \neq 0$.

The following is proved similarly to 2.2.

Proposition 4.2. *There exists one and only one $\mathbb{Q}$-divisor D modulo linear equivalence on C with a graded $\mathcal{O}_C$-algebra isomorphism*

$$\oplus_{i\geq 0}\mathcal{O}_C(\lfloor iD \rfloor) \cong \oplus_{i\geq 0}(f_*\mathcal{O}_X(\lfloor ib(K_X + \Delta)\rfloor - ibf^*K_C))^{**}.$$

Furthermore, the above isomorphism induces the equality

$$b(K_X + \Delta) = f^*(bK_C + D) + B^{\Delta},$$

where B^{Δ} is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\lfloor iB^{\Delta}_{+}\rfloor) = \mathcal{O}_C$ $(\forall i > 0)$ and $\operatorname{codim}(f(\operatorname{Supp} B^{\Delta}_{-}) \subset C) \geq 2$. For an arbitrary open set $U \subset C$, $D|_U$ and $B^{\Delta}|_{f^{-1}(U)}$ depend only on $f|_{f^{-1}(U)}$ and $\Delta|_{f^{-1}(U)}$.

Definition 4.3. We denote D given in 4.2 by $L_{(X,\Delta)/C}$ or simply by $L^{log}_{X/C}$ if there is no danger of confusion.

We set $s^{\Delta}_P := b(1 - t^{\Delta}_P)$, where t^{Δ}_P is the log-canonical threshold of f^*P with respect to $(X, \Delta - B^{\Delta}/b)$ over the generic point η_P of P:

$$t^{\Delta}_P := \max\{t \in \mathbb{R} \mid (X, \Delta - B^{\Delta}/b + tf^*P) \text{ is sub lc over } \eta_P\}.$$

Note that $s_P^{\Delta} \neq 0$ only for a finite number of codimension 1 points P because there exists a nonempty Zariski open set $U \subset C$ such that $s_P^{\Delta} = 0$ for every prime divisor P with $P \cap U \neq \emptyset$. We may simply write s_P rather than s_P^{Δ} if there is no danger of confusion. We note that s_P^{Δ} depends only on $f|_{f^{-1}(U)}$ and $\Delta|_{f^{-1}(U)}$ where U is an open set containing P.

We will see in 4.7 that this coincides with s_P introduced in 2.8.

We set $L^{ss}_{(X,\Delta)/C} := L_{(X,\Delta)/C} - \sum_P s_P^{\Delta} P$ and call it the *log-semistable part* of $K_{X/C}(\Delta)$. We may simply denote it by $L^{log,ss}_{X/C}$ if there is no danger of confusion.

We note that D, $L_{(X,\Delta)/C}$, s_P^{Δ}, t_P^{Δ} and $L^{ss}_{(X,\Delta)/C}$ are birational invariants of (X, Δ) over C in the following sense. Let (X', Δ') be a projective sub klt pair and $\sigma : X' \to X$ a birational morphism such that $K_{X'} + \Delta' - \sigma^*(K_X + \Delta)$ is an effective σ-exceptional $\mathbb{Q}$-divisor. Then the above invariants for $f \circ \sigma$ and (X', Δ') are equal to those for f and (X, Δ).

Putting the above symbols together, we have *the log-canonical bundle formula* for (X, Δ) over C:

$$b(K_X + \Delta) = f^*(bK_C + L^{log,ss}_{X/C}) + \sum_P s_P^{\Delta} f^* P + B^{\Delta}, \tag{3}$$

where B^{Δ} is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\lfloor iB^{\Delta}_+ \rfloor) = \mathcal{O}_C$ $(\forall i > 0)$ and $\operatorname{codim}(f(\operatorname{Supp} B^{\Delta}_-) \subset C) \geq 2$.

We need to pass to a certain birational model $f' : X' \to C'$ to understand the log-semistable part more clearly and to make the log-canonical bundle formula more useful.

4.4. From now on we assume that (X, Δ) is klt.

Let $g : Y \to X$ be a log resolution of (X, Δ) with Θ a $\mathbb{Q}$-divisor on Y such that $K_Y + \Theta = g^*(K_X + \Delta)$. Let $\Sigma \subset C$ be an effective divisor satisfying the following conditions:

(1) $h := f \circ g$ is smooth and Supp Θ^h is relatively normal crossing over $C \setminus \Sigma$.

(2) $h(\operatorname{Supp} \Theta^v) \subset \Sigma$.

(3) f is flat over $C \setminus \Sigma$.

Let $\pi : C' \to C$ be a birational morphism from a nonsingular projective variety such that:

(i) $\Sigma' := \pi^{-1}(\Sigma)$ is an snc divisor.

(ii) π induces an isomorphism $C' \setminus \Sigma' \cong C \setminus \Sigma$.

(iii) The irreducible component X_1 of $X \times_C C'$ dominating C' is flat over C'.

Let X' be the normalization of X_1, and $f' : X' \to C'$ the induced morphism. Let $g' : Y' \to X'$ be a log resolution such that $Y' \setminus {h'}^{-1}(\Sigma') \cong Y \times_X X_1 \setminus \alpha^{-1}(\Sigma')$, where $h' := f' \circ g'$ and $\alpha : Y \times_X X_1 \to C'$ (see [21, Resolution Lemma] or [1, Theorem 12.4]). Let Θ' be the $\mathbb{Q}$-divisor on Y' such that $K_{Y'} + \Theta' = (\tau \circ g')^*(K_X + \Delta)$, where $\tau : X' \to X$ is the induced morphism. Furthermore, we can assume that $\mathrm{Supp}({h'}^{-1}(\Sigma') \cup \Theta')$ is an snc divisor, and $h'(\mathrm{Supp}(\Theta')^v) \subset \Sigma'$.

Later we treat horizontal or vertical divisors on X, X' or Y' over C without referring to C. Note that a $\mathbb{Q}$-divisor on X' or Y' is horizontal (resp. vertical) over C iff it is horizontal (resp. vertical) over C'.

We note that the horizontal part $(\Theta')^h_-$ of the negative part Θ'_- of Θ' is g'-exceptional.

$$\begin{array}{ccccc}
Y & \longleftarrow & Y \times_X X_1 & \longleftarrow & Y' \\
{\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle g'} \\
X & \longleftarrow & X_1 & \longleftarrow & X' \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'} \\
C & \underset{\pi}{\longleftarrow} & C' & = & C'
\end{array}$$

The following formula is the main theorem of this section.

Theorem 4.5 (Log-canonical bundle formula). *Under the above notation and assumptions, let Ξ be a $\mathbb{Q}$-divisor on Y' such that (Y', Ξ) is sub klt and $\Xi - \Theta'$ is effective and exceptional over X. (Note that Ξ exists since (X, Δ) is klt.) Then the log-canonical bundle formula*

$$b(K_{Y'} + \Xi) = (h')^*(bK_{C'} + L^{ss}_{(Y',\Xi)/C'}) + \sum_P s^{\Xi}_P (h')^*(P) + B^{\Xi}$$

for (Y', Ξ) over C' has the following properties:

(i) $h'_* \mathcal{O}_{Y'}(\lfloor iB^{\Xi}_+ \rfloor) = \mathcal{O}_{C'}$ *for all* $i > 0$.

(ii) B^{Ξ}_- *is g'-exceptional and* $\mathrm{codim}(h'(\mathrm{Supp}\ B^{\Xi}_-) \subset C') \geq 2$.

(iii) *The following holds for every $i > 0$:*

$$\begin{aligned} H^0(X, \lfloor ib(K_X + \Delta)\rfloor) &= H^0(Y', \lfloor ib(K_{Y'} + \Xi)\rfloor) \\ &= H^0(C', \lfloor ibK_{C'} + iL^{log,ss}_{Y'/C'} + \sum i\, s^{\Xi}_P P\rfloor). \end{aligned}$$

(iv) *$L^{log,ss}_{Y'/C'}$ is nef.*

If furthermore (Y', Ξ) is klt, then:

(v) *Let N be a positive integer such that $Nh'^*(L^{log,ss}_{Y'/C'})$ and $bN\Xi^v$ are Weil divisors. Then for each P, there exist $u_P, v_P \in \mathbb{Z}_{>0}$ such that $0 < v_P \leq bN$ and $s_P = (bNu_P - v_P)/(Nu_P)$.*

Proof. First, (i) is obvious by the formula (3). Similarly, (ii) follows because $(g')_*(B^{\Xi}_{-}) = 0$ by the equidimensionality of f'.

By (ii) and the conditions on Ξ, the following holds for all $i > 0$:

$$\begin{aligned} H^0(X, \lfloor ib(K_X + \Delta)\rfloor) &= H^0(Y', \lfloor ib(K_{Y'} + \Xi)\rfloor) \\ &= H^0(Y', \lfloor ib(K_{Y'} + \Xi) + iB^{\Xi}_{-}\rfloor). \end{aligned}$$

By the log-canonical bundle formula and then by (i), we have

$$\begin{aligned} &H^0(Y', \lfloor ib(K_{Y'} + \Xi) + iB^{\Xi}_{-}\rfloor) \\ &\quad = H^0(Y', \lfloor i(h')^*(bK_{C'} + L^{ss}_{(Y',\Xi)/C'} + \sum s^{\Xi}_P P) + iB^{\Xi}_{+}\rfloor) \\ &\quad = H^0(C', \lfloor ibK_{C'} + iL^{log,ss}_{Y'/C'} + \sum i\, s^{\Xi}_P P\rfloor). \end{aligned}$$

Thus (iii) is settled. The property (iv) will be settled by 4.8, and (v) at the end of this section. q.e.d.

Proposition 4.6. *Under the notation and the assumptions of Theorem 4.5, $L^{ss}_{(Y',\Xi)/C'}$ does not depend on the choice of Ξ. In particular, $L^{ss}_{(Y',\Theta')/C'} = L^{ss}_{(Y',\Theta'_+)/C'} = L^{ss}_{(Y',\Xi)/C'}$.*

Proof. Let $H \subset Y'$ be an effective horizontal $\mathbb{Q}$-divisor which is exceptional over X. If H is added to Ξ, then B^{Ξ} increases by bH and s^{Ξ}_P stays the same. Hence $L^{ss}_{(Y',\Xi)/C'}$ stays the same in this case.

Assume now that $A := \Xi - \Theta'$ is exceptional over X and vertical, and let

$$\alpha_P := \max\{t \in \mathbb{R} \mid (B^{\Theta'})^v + bA^v - \alpha(h')^*P \text{ is effective over } \eta_P\},$$

where η_P is the generic point of P.

Then it is easy to see that $B^{\Xi} = B^{\Theta'} + bA - \sum \alpha_P (h')^*P$ and $\sum s^{\Xi}_P P = \sum (s^{\Theta'}_P + \alpha_P)P$. Hence $L^{ss}_{(Y',\Xi)/C'} = L^{ss}_{(Y',\Theta')/C'}$. q.e.d.

Proposition 4.7. *Assume that $f : X \to C$ is as in* 2.1. *Then $(X, 0)$ is klt, and the definitions in* 4.3 *for $f : (X, 0) \to C$ are compatible with the corresponding ones for $f : X \to C$ in Section* 2.

In other words, $B^0 = B$ (in 2.2*), $L_{(X,0)/C} \sim L_{X/C}$ (in* 2.3*) and $s^0_P = s_P$ (in* 2.8*) and hence $L^{ss}_{(X,0)/C} \sim L^{ss}_{X/C}$ (in* 2.5*).*

Proof. First, $B^0 = B$ and $L_{(X,0)/C} \sim L_{X/C}$ are obvious because the definitions for X and $(X, 0)$ coincide. During the proof, we will denote B by B_X to avoid confusion.

Since $L^{ss}_{X/C} = L_{X/C} - \sum s_P P$ and $L^{ss}_{(X,0)/C} = L_{(X,0)/C} - \sum s^0_P P$, it is enough to show $s^0_P = s_P$. Let $\Sigma \subset C$ be the reduced divisor given in 4.4. Then $s^0_P = s_P = 0$ for $P \not\subset \Sigma$ by 2.8 and 4.3. Thus it is sufficient to prove $s^0_P = s_P$ for every $P \subset \Sigma$.

Replacing C by a general hyperplane-section H, Σ by $\Sigma \cap H$ and X by f^*H, we can further assume that C is a curve. Note that we are free to enlarge Σ.

Let $t_P := 1 - s_P/b$ for all P. Then we have the following equality.

$$f^*(K_C + \frac{1}{b}L^{ss}_{X/C} + \Sigma) = K_X - \frac{1}{b}B_X + \sum_{P\in\Sigma} t_P f^*P. \tag{4}$$

By the semistable reduction theorem, there is a finite surjective morphism $\pi : C' \to C$ from a smooth curve C' such that the normalization X' of $X \times_C C'$ has a semistable resolution over C' as in 2.5. Let $\beta : X' \to X$ and $f' : X' \to C'$ be the induced morphisms as in the following commutative diagram,

$$\begin{array}{ccc} X & \xleftarrow{\beta} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ C & \xleftarrow[\pi]{} & C'. \end{array}$$

Enlarging Σ, we may assume that π is étale on $C' \setminus \Sigma'$, where $\Sigma' := \pi^{-1}(\Sigma)$. In particular, $s_{P'} = 0$ for $P' \in \Sigma'$ by 2.8.(ii), and the formula (4) for f' reduces to

$$(f')^*(K_{C'} + \frac{1}{b}L^{ss}_{X'/C'} + \Sigma') = K_{X'} - \frac{1}{b}B_{X'} + \sum_{P'\in\Sigma'} (f')^*P'. \tag{5}$$

Since $\pi^*L^{ss}_{X/C} = L^{ss}_{X'/C'}$ by 2.5 and $f^*(K_C + \Sigma) = K_{C'} + \Sigma'$, π pulls

back the formula (4) to (5) and we have the following equality.

$$\beta^*(K_X - \frac{1}{b}B_X + \sum_{P\in\Sigma} t_P f^*P) = K_{X'} - \frac{1}{b}B_{X'} + \sum_{P'\in\Sigma'} (f')^*P'. \tag{6}$$

Since $t^0_P = 1 - s^0_P/b$ by 4.3, it is enough to prove $t_P = t^0_P$, the log-canonical threshold of f^*P with respect to $(X, -B_X/b)$. Since β is finite and dominating, it is enough to prove that the right hand side of the formula (6) is sub lc and not sub klt on a neighborhood $U_{P'}$ of $(f')^{-1}(P')$ for every $P' \in \Sigma'$ (see [13, (20.3)]).

Indeed, the right hand side is sub lc by the semistable resolution of f' and not sub klt on $U_{P'}$ by $B_{X'} \not\supset (f')^{-1}(P')$. q.e.d.

Theorem 4.8. *The log-semistable part $L^{log,ss}_{Y'/C'}$ is nef.*

Proof. By 4.6, we can assume that $\Xi = \Theta'_+$. By the definition of B^{Ξ}_+, $h'_*\mathcal{O}_{Y'}(\lceil B^{\Xi}_+/b\rceil) = \mathcal{O}_{C'}$ holds. Therefore, the condition (3) in 4.10 is satisfied. The condition (1) is also satisfied because $h'(\mathrm{Supp}\ \Xi^v) \subset \mathrm{Supp}\ \Sigma' = \mathrm{Supp}\ \pi^*\Sigma$ by 4.4.(i). Applying 4.10 to $(Y', \Xi - B^{\Xi}/b)$, we see that $L^{log,ss}_{Y'/C'}$ is nef. q.e.d.

4.9. We recall Kawamata's positivity theorem [8, Theorem 2] for the reader's convenience (see also [2]).

Theorem 4.10. *Let $g : Y \to T$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$ and $Q = \sum_l Q_l$ be normal crossing divisors on Y and T, respectively, such that $g^{-1}(Q) \subset P$ and g is smooth over $T\setminus Q$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor on Y (d_j's may be negative) such that:*

(1) *$g : \mathrm{Supp}(D^h) \to T$ is relatively normal crossing over $T \setminus Q$, and $g(\mathrm{Supp}(D^v)) \subset Q$.*

(2) *$d_j < 1$ if P_j is horizontal.*

(3) *$\dim_{k(\eta)} g_*\mathcal{O}_Y(\lceil -D\rceil) \otimes_{\mathcal{O}_T} k(\eta) = 1$ for the generic point η of T.*

(4) *$K_Y + D \sim_{\mathbb{Q}} g^*(K_T + L)$ for some $\mathbb{Q}$-divisor L on T.*

Let

$$\begin{aligned} g^*Q_l &= \sum_j w_{lj}P_j, \\ \overline{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}} \text{ if } g(P_j) = Q_l, \\ \delta_l &= max\ \{\overline{d}_j \mid g(P_j) = Q_l\}, \\ \Delta_0 &= \sum_l \delta_l Q_l, \text{ and} \\ M &= L - \Delta_0. \end{aligned}$$

Then M is nef.

Remark 4.11. (i) It can be checked easily that $1 - \delta_l$ is the log-canonical threshold of g^*Q_l with respect to (Y, D) over the generic point η_{Q_l}.

(ii) In [8, Theorem 2], it is assumed that $d_j < 1$ for all P_j. However, the assumption that $d_j < 1$ for vertical P_j was not used in the proof (see the proof of [8, Theorem 2]).

We start with a lemma to prove 4.5.(v).

Lemma 4.12. *Under the notation and the assumptions of* 4.5.(v)*, assume that C' is a curve. Then the following holds.*

$$b(K_{Y'/C'} + \Xi + ((h')^{-1}\Sigma')_{red}) \succ (h')^*(L^{log,ss}_{Y'/C'} + b\Sigma').$$

Proof. Note that $B^{\Xi}_{-} = 0$ since C is a curve. Then we have

$$\begin{aligned} K_{Y'/C'} + \Xi + ((h')^{-1}\Sigma')_{red} &\succ K_{Y'/C'} + \Xi - B^{\Xi}_{+}/b + \sum_{P\in\Sigma'} t^{\Xi}_P (h')^*P \\ &= \frac{1}{b}(h')^*L^{ss}_{(Y',\Xi)/C'} + (h')^*\Sigma', \end{aligned}$$

where the first relation follows from $\Xi \succ 0$ and the definitions of Σ' and t^{Ξ}_P, and the second from the log-canonical bundle formula and the relation $s^{\Xi}_P/b + t^{\Xi}_P = 1$ for all P. q.e.d.

Proof of 4.5.(v). Replacing C' by a general hyperplane-section H and Y' by $(h')^*(H)$, we can immediately reduce to the case where C' is a curve. For simplicity, Ξ in B^{Ξ}, s^{Ξ}_P and t^{Ξ}_P will be suppressed during the proof. We note that B is effective.

By the hypothesis, the vertical part D of the Weil divisor

$$bN(K_{Y'/C'}+\Xi^v)-(h')^*NL^{log,ss}_{Y'/C'}=N\sum_P s_P(h')^*P+NB-bN\Xi^h$$

is a Weil divisor. We note that

$$D=N\sum_P s_P(h')^*P+NB^v=bN(K_{Y'/C'}+\Xi)-(h')^*NL^{log,ss}_{Y'/C'}-NB^h.$$

By 4.12, we have

$$bN(K_{Y'/C'}+\Xi)-(h')^*NL^{log,ss}_{Y'/C'}+bN\left((h')^{-1}\Sigma'\right)_{red}\succ(h')^*bN\Sigma'.$$

Whence

$$D+NB^h+bN\left((h')^{-1}\Sigma'\right)_{red}\succ(h')^*bN\Sigma'. \tag{7}$$

Let D_P and B^v_P be the parts of D and B^v lying over P. Let $(h')^*P=\sum_k a_kF_k$ be the irreducible decomposition. Then $D_P-NB^v_P=Ns_P(h')^*P$ and $\mathrm{Supp}(D_P-Ns_P(h')^*P)\not\supset F_c$ for some c by the definition of B^v_P. In particular $Ns_Pa_c\in\mathbb{Z}$. Furthermore, comparing the coefficients of F_c in the formula (7), we obtain $Ns_Pa_c+bN\geq bNa_c$, that is, $Na_cs_P\geq bN(a_C-1)$. Since (Y',Ξ) is klt, we have $t_P>0$ and hence $s_P<b$. Hence $u_P:=a_c$ works. q.e.d.

5. Log-canonical rings

In this section, we consider an application of the log-canonical bundle formula 4.5 to the log-canonical ring of a pair (X,Δ):

$$R(X,K_X+\Delta)=\oplus_{i\geq 0}H^0(X,\lfloor iK_X+i\Delta\rfloor).$$

We recall a well-known proposition on graded rings.

Proposition 5.1. *Let $R=\oplus_{i\geq 0}R_i$ be a graded integral domain with degree i part R_i such that $R_0=\mathbb{C}$. For $m\in\mathbb{Z}_{>0}$, let $R^{(m)}:=\oplus_{i\geq 0}R_{im}$ be the graded ring whose degree i part is R_{im}. Then R is finitely generated over R_0 iff so is $R^{(m)}$ over R_0.*

Sketch of the proof. If R is finitely generated, we can let $\mathbb{Z}/(m)$ act on R so that $R^{(m)}$ is the invariant part. Hence $R^{(m)}$ is finitely generated. If $R^{(m)}$ is finitely generated, then R is contained in the integral closure of $R^{(m)}$ in the quotient field of R. Hence R is a finite $R^{(m)}$-module and thus finitely generated. q.e.d.

Theorem 5.2. *Let (X, Δ) be a proper klt pair with*

$$\kappa(X, K_X + \Delta) = l \geq 0.$$

Then there exist an l-dimensional klt pair (C', Δ') with $\kappa(C', K_{C'} + \Delta') = l$, two $e, e' \in \mathbb{Z}_{>0}$ and an isomorphism

$$R(X, K_X + \Delta)^{(e)} \cong R(C', K_{C'} + \Delta')^{(e')}$$

of graded rings.

Proof. Let $f : X \dashrightarrow C$ be the Iitaka fibering with respect to $K_X + \Delta$. By replacing X and C, we assume that the following conditions hold:

1. X and C are projective smooth varieties and $f : X \to C$ is a proper surjective morphism with connected fibers.
2. Supp Δ is an snc divisor and $\lfloor \Delta \rfloor = 0$.
3. There exists an effective divisor $\Sigma \subset C$ such that f is smooth and Supp Δ^h is relatively normal crossing over $C \setminus \Sigma$ and such that $f(\mathrm{Supp}(\Delta^v)) \subset \Sigma$.

Note that adding effective exceptional $\mathbb{Q}$-divisors does not change the log-canonical ring.

We use the same notation as 4.5. By the hypothesis, $bK_{C'} + L^{log,ss}_{Y'/C'} + \sum_P s_P P$ is big on C'. Thus let A an ample $\mathbb{Q}$-divisor and G an effective $\mathbb{Q}$-divisor on C' such that

$$bK_{C'} + L^{log,ss}_{Y'/C'} + \sum_P s_P P \sim_{\mathbb{Q}} A + G.$$

Then by 4.5.(v), there exists a sufficiently small rational number $\varepsilon > 0$ such that the pair

$$(C', \sum_P (s_P/b)P + (\varepsilon/b)G)$$

is klt. Since $L^{log,ss}_{Y'/C'}$ is nef, $L^{log,ss}_{Y'/C'} + \varepsilon A$ is ample and it is $\mathbb{Q}$-linearly equivalent to δH for some rational $\delta > 0$ and a very ample divisor H. We can furthermore assume that

$$(C', \sum_P (s_P/b)P + (\varepsilon/b)G + (\delta/b)H)$$

is klt.

Now choose $a \in \mathbb{Z}_{>0}$ such that $a\varepsilon, a\delta \in \mathbb{Z}$ and $aL^{log,ss}_{Y'/C'} + a\varepsilon A \sim a\delta H$. Then we have

$$\begin{aligned}(a+a\varepsilon)(bK_{C'} + L^{log,ss}_{Y'/C'} + \sum s_P P) & \\ \sim abK_{C'} + a\delta H + a(\sum s_P P + \varepsilon G) & \\ = ab(K_{C'} + \sum \frac{s_P}{b} P + \frac{\varepsilon}{b} G + \frac{\delta}{b} H). &\end{aligned}$$

Thus we can take $e = (a + a\varepsilon)b$, $e' = ab$ and

$$\Delta' = \sum_P (s_P/b)P + (\varepsilon/b)G + (\delta/b)H.$$

q.e.d.

The following corollary contains a generalization of Moriwaki's result [16, Theorem (3.1)].

Corollary 5.3. *Let (X, Δ) be a proper klt pair such that $\kappa(X, K_X + \Delta) \leq 3$. Then the log-canonical ring of (X, Δ) is finitely generated.*

Proof. By 5.1 and 5.2, our problem is reduced to the case $\dim X = \kappa(X, K_X + \Delta) \leq 3$. The case $\dim X = 2$ is settled by [4, (1.5)] and the case $\dim X = 3$ by [11, Corollary]. q.e.d.

The following theorem is a generalization of [18, Corollary] and [17, (3.7) Proposition].

Theorem 5.4. *Let (X, Δ) be a proper klt pair. Assume that the log-canonical ring $R(X, K_X + \Delta)$ is finitely generated. Then there exists an effective $\mathbb{Q}$-divisor Ξ on $S := \operatorname{Proj} R(X, K_X + \Delta)$ such that (S, Ξ) is klt. Especially, S has only rational singularities.*

Proof. By 5.2, we can assume that $\dim X = \kappa(X, K_X + \Delta)$. By changing the model birationally, we can further assume that X is non-singular and Δ is an snc divisor. Then it is settled by [17, (3.7) Proposition]. q.e.d.

6. Pluricanonical systems

Theorem 6.1. *For arbitrary $b, k \in \mathbb{Z}_{>0}$, there exists an effectively computable natural number $M = M(b, k)$ with the following property.*

Let X be a nonsingular projective variety of dimension $m+1$ and Kodaira dimension $\kappa(X) = 1$, C a nonsingular projective curve and $f : X \to C$ a surjective morphism with connected fibers. Assume that its generic fiber F has $\kappa(F) = 0$, $P_b(F) = 1$ and the m-th Betti number $B_m(F) \leq k$. Then $|M(b,k)K_X|$ induces the rational map $f : X \dashrightarrow C$.

Since one can change the smooth model birationally so that $B_2(F) \leq 22$ and $|12K_F| \neq \emptyset$ for threefolds X, we need no extra conditions for 3-folds.

Corollary 6.2. *There exists an effectively computable natural number M such that, for every algebraic 3-fold X with Kodaira dimension $\kappa(X) = 1$, $|MK_X|$ induces the Iitaka fibering of X.*

Proof of 6.1. By 2.3 and 2.8, we have the following for all $n \geq 0$.

$$H^0(X, nbK_X) = H^0(C, \lfloor n(bK_C + L^{ss}_{X/C} + \sum_P s_P P)\rfloor).$$

We note that $\deg(bK_C + L^{ss}_{X/C}) + \sum_P s_P > 0$ by $\kappa(X) = 1$. Let g be the genus of C and let $N = N(k)$. We treat the problem in 3 cases.

Case 1 ($g \geq 2$). Setting $n = iN$ in the above, we obtain the divisor

$$iNbK_C + iNL^{ss}_{X/C} + \lfloor iNb(\sum_P s_P P)\rfloor$$

on C. We note that $NL^{ss}_{X/C}$ is nef Cartier divisor by 2.5 and 3.1 and that $\lfloor iNb(\sum_P s_P P)\rfloor \succ 0$ by 2.8. Hence the divisor is very ample if $iNb \geq 3$ because $\deg \geq 3 \deg K_C \geq 2g+1$. So $M(b,k) = 3Nb$ works.

Case 2 ($g = 1$). We have $\deg L^{ss}_{X/C} + \sum_P s_P > 0$. It is enough to find one $i \in \mathbb{Z}_{>0}$ such that the divisor $\lfloor iN(L^{ss}_{X/C} + \sum_P s_P P)\rfloor$ has $\deg \geq 3$, because such a divisor is very ample on C. Hence this case is reduced to 6.3 with $a = 0$ and $c = bN$.

As pointed out by the referee, the Case 1 above is in [6, §3] and the Case 2 can probably be deduced as well. Thus the following is the key case.

Case 3 ($g = 0$). We have $\deg(bK_C + L^{ss}_{X/C}) + \sum_P s_P > 0$ and need to one $i \in \mathbb{Z}_{>0}$ such that $\deg\lfloor iN(L^{ss}_{X/C} + \sum_P s_P P)\rfloor - 2biN > 0$. Thus this case is again reduced to 6.3 with $a = 2bN$ and $c = bN$.

q.e.d.

Thus it remains to prove the following.

Proposition 6.3. *Let $a \in \mathbb{Z}$ and $c \in \mathbb{Z}_{>0}$. Let $S(a,c)$ be the set of*

$$\xi = (r; m_1, \cdots, m_r; b_1, \cdots, b_r)$$

where $r \in \mathbb{Z}_{\geq 0}$, $m_1, \cdots, m_r \in \mathbb{Z}_{>0}$ and $b_1, \cdots, b_r \in \mathbb{Z}_{\geq 0}$ satisfy the conditions $b_i < \min\{m_i, c+1\}$ for all i and

$$-a + \sum_{i=1}^{r} \frac{m_i - b_i}{m_i} > 0.$$

For $n \in \mathbb{Z}_{>0}$, let

$$f(n,\xi) = -an + \sum_{i=1}^{r} \lfloor n \frac{m_i - b_i}{m_i} \rfloor.$$

Then there exists $n = n(a,c) \in \mathbb{Z}_{>0}$ such that $f(\nu n, \xi) \geq \nu$ for all $\xi \in S(a,c)$ and $\nu \in \mathbb{Z}_{>0}$.

Proof. We note that $f(nm, \xi) \geq mf(n,\xi)$ for $n, m \in \mathbb{Z}_{>0}$. Hence it is enough to get $f(n,\xi) \geq 1$. For each $\xi \in S(a,c)$, let

$$I_1(\xi) = \{i \in [1,r] \mid m_i \leq 2c\},\ I_2(\xi) = [1,r] \setminus I_1(\xi),$$

and $J(\xi) = (m_i)_{i \in [1,r]}$. It is easy to see that if $|I_2(\xi)| \geq 2a+1$ then $f(2,\xi) \geq 1$. Thus we may assume that $|I_2(\xi)| \leq 2a$. It is also easy to see that if $\alpha(\xi) := \sum_{i \in I_1(\xi)} (m_i - b_i)/m_i > a$, then $(2c)!\alpha(\xi) \in \mathbb{Z}$ and

$$f((2c)!, \xi) = (2c)!\,(\alpha(\xi) - a) + \sum_{i \in I_2(\xi)} \lfloor (2c)! \cdot \frac{m_i - b_i}{m_i} \rfloor > 0.$$

Thus we may also assume that $\alpha(\xi) \leq a$. Hence by $m_i \leq 2c$ $(i \in I_1(\xi))$, we have $|I_1(\xi)| \leq 2c\alpha(\xi) \leq 2ac$.

Under the conditions $r = |I_1(\xi)| + |I_2(\xi)| \leq 2ac + 2a$ and $b_i \leq c$, the set $T = \{(r, b_1, \cdots, b_r)\}$ is finite. Hence for each $\eta \in T$, we work on the subset $S(a,c)_\eta$ of $S(a,c)$ with the fixed r and b_i's.

We note that $f(n,\xi) \leq f(n,\xi')$ if $\xi, \xi' \in S(a,c)_\eta$ satisfy $J(\xi) \leq J(\xi')$ (coordinatewise). Hence we are done by the following lemma. q.e.d.

Lemma 6.4. *Let $r \in \mathbb{Z}_{>0}$ and $S \subset \mathbb{Z}^r_{>0}$. With respect to the total order, S contains only a finite number of minimal elements.*

Proof. We use the induction on r. If $r = 1$, this is obvious. Fix any $s = (s_1, \cdots, s_r) \in S$ and let $m := \max\{s_i\}$. For $i \in [1,r]$ and $n \in [1,m]$, let $T_{n,i} = \{(t_1, \cdots, t_r) \in S \mid t_i = n\}$. Since each $T_{n,i}$ has only a finite number of minimal elements by the induction hypothesis, so does the finite union $\cup_{n,i} T_{n,i}$. If $t \in S \setminus \cup_{n,i} T_{n,i}$, then $t > s$. q.e.d.

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