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The Mirković–Vilonen basis and Duistermaat–Heckman measures

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1. Introduction

1.1. Biperfect bases and MV polytopes

Let G denote a simple simply-connected complex algebraic group. Going back to the work of Gel′fand–Zelevinsky [GZ], there has been great interest in finding special bases for irreducible representations $L(\lambda)$ of G. Good bases restrict to bases of weight spaces and induce bases of tensor product multiplicity spaces.

Rather than work with individual representations, it is convenient to pass to the coordinate ring $\mathbb{C}[N]$ (where $N\subset B\subset G$ is the unipotent radical of a Borel), which contains all the irreducible representations. Berenstein–Kazhdan [BeK] introduced the notion of perfect bases for $L(\lambda)$ and $\mathbb{C}[N]$; in this paper, we slightly modify their definition and work with *biperfect bases* for $\mathbb{C}[N]$. Biperfect bases have good behaviour with respect to the left and right actions of the Chevalley generators $\{e_i\}\subset\mathfrak{n}$ on $\mathbb{C}[N]$ (see §2.3).

For $G=\mathrm{SL}_2,\mathrm{SL}_3,\mathrm{SL}_4$, biperfect bases exist, are unique, and are given by explicit formulas. However, for general G, uniqueness does not hold, nor are explicit formulas available. Some constructions of biperfect bases are known in general. The first general construction was Lusztig's dual canonical basis [Lu2], aka Kashiwara's upper global basis [Kas1]. In this paper, we will focus on the *Mirković–Vilonen basis*, which is constructed for any G using the geometric Satake correspondence, and Lusztig's *dual semicanonical basis*, which is constructed for simply-laced G using preprojective algebra modules.

Though uniqueness does not hold in general, a beautiful result of Berenstein–Kazhdan [BeK] shows that every biperfect basis has the same underlying combinatorics. More precisely, any biperfect basis B comes with maps $\tilde{e}_i,\tilde{e}_i^*\colon B\to B\cup\{0\}$ (for each $i\in I$) which approximate the left and right actions of e_i on B. These maps $\tilde{e}_i$ and $\tilde{e}_i^*$ endow B with a bicrystal structure. If B and B' are two biperfect bases, then there is a unique isomorphism of bicrystals $B\to B'$ (Theorem 2.4). We write $B(\infty)$ for the abstract bicrystal common to all biperfect bases.

1.2. Bases and their polytopes

Let $G^\vee$ be the Langlands dual group and let $\mathcal{G}\mathrm{r}=G^\vee((t))/G^\vee[[t]]$ denote the affine Grassmannian of this group. Mirković–Vilonen [MV] defined a family of cycles in $\mathcal{G}\mathrm{r}$ which, under the geometric Satake correspondence, give bases for irreducible representations of G. We will give a slight modification of their construction in order to get a basis for $\mathbb{C}[N]$. Let $S_\pm^\mu:=N_\pm^\vee((t))L_\mu$ denote semi-infinite orbits in $\mathcal{G}\mathrm{r}$ (where $L_\mu\in\mathcal{G}\mathrm{r}$ is the point defined by the $G^\vee$-coweight μ). We will be concerned with the intersection of opposite semi-infinite orbits. For any $\nu\in Q_+$, the positive root cone, the irreducible components

of $\overline{S_+^0 \cap S_-^{-\nu}}$ are called *stable MV cycles*. These cycles index the MV basis $\{b_Z\}$ for $\mathbb{C}[N]$.

In addition to the MV basis, in this paper, we also study the dual semicanonical basis for $\mathbb{C}[N]$, which was introduced by Lusztig [Lu5] and further studied by Geiss–Leclerc–Schröer [GLS]. This basis $\{c_Y\}$ is indexed by irreducible components Y of Lusztig's nilpotent varieties, which parameterize representations of the preprojective algebra Λ. With each Λ-module M, we can associate a vector $\xi_M \in \mathbb{C}[N]$ and we define $c_Y := \xi_M$ for a general point $M \in Y$.

As the MV basis and dual semicanonical basis are both biperfect bases (Theorems 6.2 and 11.2), we get canonical bijections

$$\{b_Z\} \longrightarrow B(\infty) \longleftarrow \{c_Y\}.$$

Suppose that b_Z and c_Y correspond under these bijections. Outside of small rank, this does not imply that $b_Z = c_Y$ as elements of $\mathbb{C}[N]$, just that they represent the same element of the crystal. However, this combinatorial relationship is very appealing, as can be seen by the following result which combines the work of the second author [Kam2] and the first and second authors (together with Tingley) [BKT].

THEOREM 1.1. *Let Z and Y be as above and let M be a general point of Y. Then, we have an equality of polytopes*:

$$\mathrm{Conv}(\{\mu : L_\mu \in Z\}) = -\,\mathrm{Conv}(\{\underrightarrow{\dim}\, N : N \subseteq M \text{ is a } \Lambda\text{-submodule}\})$$

MV polytopes were first defined by Anderson [A] as the left-hand side of the above equality. However, in retrospect, it is more natural to view them as purely combinatorial objects: they can either be defined using a condition on their 2-faces (as in [Kam2]) or from the crystal $B(\infty)$ using Saito reflections (as in §3.3).

The above equality of polytopes motivates the following question.

Question 1.2. Let Z and Y be as above. Is there a relationship between the equivariant invariants of Z and the structure of a general point of Y? How is this connected to the relationship between the basis vectors b_Z and c_Y?

1.3. The MV basis

The first part of the paper is devoted to understanding the MV basis. We prove the following results in Theorems 6.2, 5.4 and 7.11.

THEOREM 1.3. (i) *The MV basis $\{b_Z\}$ is a biperfect basis for $\mathbb{C}[N]$.*

(ii) *For each i, the action of e_i on b_Z is given by the intersection multiplicities appearing in the intersection of Z with a hyperplane section.*

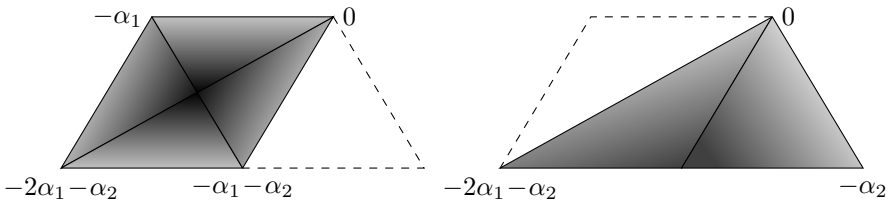

Figure 1. The SL_3 examples of $D_{\mathbf{i}}$ for $\mathbf{i}=(1,2,1)$ and $(2,1,1)$, vertices labeled by their positions, with the shading to suggest the (piecewise-linear function times Lebesgue) measure.

(iii) *Given two MV cycles Z_1 and Z_2, the product $b_{Z_1}b_{Z_2}$ in $\mathbb{C}[N]$ is given by the intersection multiplicities appearing in the intersection of the Beilinson–Drinfeld degeneration of $Z_1\times Z_2$ with the central fibre.*

In particular, the structure constants for the action of e_i and for the multiplication are non-negative integers.

Part (i) of this theorem is a 2-sided extension to $\mathbb{C}[N]$ of a like result from Braverman and Gaitsgory [BrG, Proposition 4.1]. Part (iii) was conjectured by Anderson in [A]. We prove (i) and (ii) using a result of Ginzburg [Gi] and Vasserot [V] concerning the action of the principal nilpotent under the geometric Satake correspondence. We prove (iii) by carefully considering the fusion product in the Satake category, as defined by Mirković–Vilonen [MV].

Part (iii) of this theorem is closely related to an old result of Feigin–Finkelberg–Kuznetzov–Mirković [FFKM] and a more recent result of Finkelberg–Krylov–Mirković [FKM] describing the algebra $U(\mathfrak{n})$ using Zastava spaces.

1.4. Equivariant invariants of MV cycles

In order to further our understanding of the Mirković–Vilonen basis, we relate the basis vector $b_Z\in\mathbb{C}[N]$ to equivariant invariants of the MV cycle Z. We begin by introducing a remarkable map from $\mathbb{C}[N]$ to the space $\mathcal{PP}$ of piecewise polynomial measures on $\mathfrak{t}^*_{\mathbb{R}}$. Given any sequence $\mathbf{i}=(i_1,...,i_p)\in I^p$ of simple roots, we define a measure $D_{\mathbf{i}}$ on $\mathfrak{t}^*_{\mathbb{R}}$ as follows. First, we consider a linear map $\pi_{\mathbf{i}}\colon\mathbb{R}^{p+1}\to\mathfrak{t}^*_{\mathbb{R}}$ taking the standard basis vectors to the negative partial sums $\alpha_{i_1}+...+\alpha_{i_k}$. Then, we define $D_{\mathbf{i}}$ to be the push-forward of Lesbesgue measure on the p-simplex under $\pi_{\mathbf{i}}$ (see Figure 1).

These measures $D_{\mathbf{i}}$ satisfy the shuffle relations under convolution (Lemma 8.5) and so we obtain an algebra morphism $D\colon\mathbb{C}[N]\to\mathcal{PP}$ by

$$D(f)=\sum_{\mathbf{i}}\langle e_{\mathbf{i}},f\rangle D_{\mathbf{i}},$$

where $e_{\mathbf{i}}=e_{i_1}\dots e_{i_p}\in U(\mathfrak{n})$ and where $\langle\cdot,\cdot\rangle\colon U(\mathfrak{n})\otimes\mathbb{C}[N]\to\mathbb{C}$ is the usual pairing.

Let $\overline{D}\colon\mathbb{C}[N]\to\mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$ be the map that sends a function $f\in\mathbb{C}[N]_{-\nu}$, to the coefficient of $e^{-\nu}$ in the Fourier transform of $D(f)$. Then, $\overline{D}$ is an algebra morphism, which can be described using the universal centralizer of the Lie algebra $\mathfrak{g}$. Namely, consider the map $\mathfrak{t}^{\mathrm{reg}}\to N$ that associates with $x\in\mathfrak{t}^{\mathrm{reg}}$ the unique $n_x\in N$ such that $\mathrm{Ad}_{n_x}(x+e)=x$, where e is a principal nilpotent. We prove that the map $\overline{D}\colon\mathbb{C}[N]\to\mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$ is the algebra map dual to $x\mapsto n_x$ (Proposition 8.4).

On the other hand, associated with Z, we have the *Duistermaat–Heckman measure* $\mathrm{DH}(Z)$, a measure (defined by Brion–Procesi [BP], following ideas from symplectic geometry) on $\mathfrak{t}_{\mathbb{R}}$ which captures the asymptotics of sections of equivariant line bundles on Z (see §9). This measure lives on the MV polytope of Z and its Fourier transform encodes the class of Z in the equivariant homology $H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$ (see Theorem 9.6). We also have the *equivariant multiplicity* $\varepsilon_{L_0}(Z)$ of Z; this is a rational function which represents the equivariant homology class of Z in a neighbourhood of L_0.

Our second main result (Theorem 10.2) relates the equivariant invariants of MV cycles to the above morphisms.

THEOREM 1.4. *Let Z be an MV cycle. After identifying $\mathfrak{t}$ and $\mathfrak{t}^*$, we have*

$$\mathrm{DH}(Z)=D(b_Z)\quad\textit{and}\quad\varepsilon_{L_{-\nu}}(Z)=\overline{D}(b_Z)$$

We give two proofs of this theorem. Our first proof uses Theorem 1.3 (ii) and a general result of the third author [KnA2] for computing Duistermaat–Heckman measures using hyperplane sections. Our second proof uses the work of Yun–Zhu [YZ], which relates the equivariant homology of the affine Grassmannian to the universal centralizer.

As an application of this theorem, we prove a conjecture of Muthiah [Mu]. Let λ be a dominant weight and identify the zero-weight space $L(\lambda)_0$ with $H_{2\rho^\vee(\lambda)}(\overline{\mathcal{G}\mathrm{r}^\lambda\cap S_-^0})$ by the geometric Satake correspondence. Given an irreducible component $Z\subset\overline{\mathcal{G}\mathrm{r}^\lambda\cap S_-^0}$, we can consider its equivariant multiplicity $\varepsilon_{L_0}(Z)\in\mathbb{C}(\mathfrak{t}^*)$ at the point $L_0\in\mathcal{G}\mathrm{r}$.

Muthiah [Mu] conjectured the following result and proved it when $G=\mathrm{SL}_n$ and $\lambda\leqslant n\omega_1$. We prove it for all G and λ (Theorem 10.7).

THEOREM 1.5. *The linear map $L(\lambda)_0\to\mathbb{C}(\mathfrak{t}^*)$ defined by $[Z]\mapsto\varepsilon_{L_0}(Z)$ is equivariant with respect to the actions of the Weyl group on both sides.*

1.5. Comparison of bases

As we have ring homomorphisms $D\colon\mathbb{C}[N]\to\mathcal{PP}$ and $\overline{D}\colon\mathbb{C}[N]\to\mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$, we immediately see the following result.

COROLLARY 1.6. *Suppose that* $b_Z = c_Y$. *Then*, $\mathrm{DH}(Z) = D(c_Y)$ *and* $\varepsilon_{L_0}(Z) = \overline{D}(c_Y)$.

This result is useful, since $\mathrm{DH}(Z)$ and $\varepsilon_{L_0}(Z)$ can be computed using the methods of computational commutative algebra. On the other hand, $D(c_Y)$ and $\overline{D}(c_Y)$ are relatively easy to compute using the following formula from Geiss–Leclerc–Schröer [GLS]:

$$\langle e_{\mathbf{i}}, \xi_M \rangle = \chi(F_{\mathbf{i}}(M)),$$

where $\chi(F_{\mathbf{i}}(M))$ denotes the Euler characteristic of the variety of composition series of M of type $\mathbf{i}$:

$$F_{\mathbf{i}}(M) = \{0 \subset M^1 \subset \ldots \subset M^m = M : M^k/M^{k-1} \cong S_{i_k}\}$$

In the appendix (written with Anne Dranowski and Calder Morton-Ferguson) we prove the following result (Theorem A.13).

THEOREM 1.7. *Let* $G = \mathrm{SL}_6$ *and let* $\nu = 2\alpha_1 + 4\alpha_2 + 4\alpha_3 + 4\alpha_4 + 2\alpha_5$. *There exists a specific MV cycle* Z *of weight* ν *with corresponding component* Y *of* $\Lambda(\nu)$, *such that* $\overline{D}(c_Y) = \overline{D}(b_Z) + 2\overline{D}(b)$, *where* b *is a vector which lies in both the MV and dual semi-canonical bases. In particular, we have* $b_Z \neq c_Y$.

This theorem suggests that we have $c_Y = b_Z + 2b$ in $\mathbb{C}[N]$. Remarkably, Geiss–Leclerc–Schröer found the disagreement of the dual canonical and dual semicanonical bases at the same location. Let v be the dual canonical basis element which corresponds to both b_Z and c_Y under the crystal isomorphisms. Then, from [GLS, p. 196], we have $c_Y = v + b$. Turning these equations around, we thus expect that

$$c_Y = b_Z + 2b \quad \text{and} \quad v = b_Z + b.$$

In §2.7, we explain that similar equations indeed hold in a D_4 example.

In rank 2 cluster algebras, we have a similar trichotomy of bases (see for example [La, Theorem 2.2]). In this trichotomy, the MV basis seems to match Lee–Li–Zelevinsky's [LLZ] greedy basis, which in turn coincides with Gross–Hacking–Keel–Kontsevich's theta basis [GHKK], by the work of Cheung–Gross–Muller–Musiker–Rupel–Stella–Williams [CGM+]. The theta basis exists for any cluster algebra, in particular for $\mathbb{C}[N]$. Thus, the above calculation suggests that the MV basis for $\mathbb{C}[N]$ coincides with the theta basis for this cluster algebra.

1.6. Extra-compatibility

In the Brion–Procesi definition that we use, the Duistermaat–Heckman measure is an $n \to \infty$ limit of sums of point measures. Thus it is natural to look for this extra structure, i.e. a finitely supported measure for each n, on the Λ-module side as well.

For any preprojective algebra module M and $n\in\mathbb{N}$, we consider the space of (possibly degenerate) flags of submodules

$$F_{n,\mu}(M) := \left\{ 0 \subseteq N_1 \subseteq \ldots \subseteq N_n \subseteq M : \sum_k \underrightarrow{\dim} N_k = \mu \right\}.$$

We prove the following result (Theorem 11.4) by a direct calculation; it is an analogue of Theorem 1.4, but more elementary.

THEOREM 1.8. *For any Λ-module M, $D(\xi_M)$ describes the asymptotics (as $n\to\infty$) of the function $\mu\mapsto\chi(F_{n,\mu}(M))$.*

Suppose that we have an MV cycle Z and a component Y of Lusztig's nilpotent variety, such that $b_Z=c_Y$. This implies that $D(b_Z)=D(c_Y)$, which, by Theorems 1.4 and 1.8, means that

$$\lim_{n\to\infty}(\tau_n)_*[\Gamma(Z,\mathscr{L}^{\otimes n})] = \lim_{n\to\infty}(\tau_n)_*[H^\bullet(F_n(M))],$$

where M is a general point of Y. Here, $[\cdot]$ denotes a class in the representation ring of T, regarded as a linear combination of point masses. The map τ_n represents scaling by $1/n$ and the limits are taken in the space of distributions on $\mathfrak{t}^*_{\mathbb{R}}$. Thus, it is reasonable to expect equality before taking the limits. We say that Z and Y are *extra-compatible* if, for all $n\in\mathbb{N}$ and all weights μ, we have

$$\dim\Gamma(Z,\mathscr{L}^{\otimes n})_\mu = \chi(F_{n,-\mu}(M)).$$

For example, taking $n=1$ gives

$$\dim\Gamma(Z,\mathscr{L})_\mu = \chi(\{N\subseteq M : \underrightarrow{\dim} N = -\mu\}),$$

which can be viewed as an upgrade of the equality of polytopes from Theorem 1.1.

We prove the following result (Theorem 12.11) establishing extra-compatibility in a simple class of examples.

THEOREM 1.9. *If Z is a Schubert variety in a cominuscule flag variety and M is the corresponding Λ-module, then Z and M are extra-compatible.*

In the appendix, we give also some examples for $G=\mathrm{SL}_5,\mathrm{SL}_6$ of modules which satisfy the extra-compatibility condition for $n=1,2$.

1.7. A general conjecture

Most Λ-modules M are not extra-compatibly paired with any MV cycle; for example, if $G{=}\mathrm{SL}_3$ and M is the sum of the two simple Λ-modules, then the rhombus $\mathrm{Pol}(M)$ is the union of two MV polytopes, each a triangle. However, for any Λ-module M, we expect that there will be a corresponding coherent sheaf on the affine Grassmannian, supported on a union of MV cycles. To state our precise expectation, we introduce the following space whose Euler characteristic coincides with $F_{n,\mu}(M)$.

$$\mathbb{G}_\mu(M[t]/t^n) := \{N \subseteq M \otimes \mathbb{C}[t]/t^n : N \text{ is a } \Lambda\otimes\mathbb{C}[t]\text{-submodule, with } \underrightarrow{\dim} N = \mu\}.$$

We conjecture the following result (see §12.2 for more precise motivation).

Conjecture 1.10. For any preprojective algebra module M of dimension vector ν, there exists a coherent sheaf $\mathcal{F}_M$ supported on $\overline{S_+^0 \cap S_-^{-\nu}}$ such that

$$\Gamma(\mathcal{G}\mathrm{r}, \mathcal{F}_M \otimes \mathcal{O}(n)) \cong H^\bullet(\mathbb{G}(M[t]/t^n))$$

as $T^\vee$-representations.

For example, if Z and M are extra-compatible, then we can take $\mathcal{F}_M = \mathcal{O}_Z$.

This conjecture has two important relations with recent developments. First, an earlier version of this conjecture motivated a number of recent works by Mirković and his coauthors [M], [MYZ] on the subject of local spaces.

Second, quiver varieties and affine Grassmannian slices are related using the theory of symplectic duality as introduced by Braden–Licata–Proudfoot–Webster [BLPW]. Recently, Braverman–Finkelberg–Nakajima [BFN] proved that affine Grassmannian slices are Coulomb branches associated with quiver gauge theories. Using this result, in a forthcoming paper, Hilburn, Weekes and the second author [HKW] will prove Conjecture 1.10 for those M which come from a quiver path algebra.

More generally, the relationship between MV cycles and preprojective algebra modules studied in this paper admits a generalization to arbitrary symplectic dual pairs. It would be very interesting to understand how the results presented here generalize to that setting.

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Part I. Biperfect bases

2. General background

2.1. Notation

Let G be a simple simply-connected complex algebraic group. Let B be a Borel subgroup with unipotent radical N and let T be a maximal torus of B. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{t}$ denote their Lie algebras.

Let P denote the character lattice of T, let P_+ denote the set of dominant weights, and let $Q\subset P$ denote the root lattice. Let $Q_+\subset Q$ denote the $\mathbb{N}$-span of the positive roots. We define the *dominance order* on P by declaring that $\lambda\geqslant\mu$ if $\lambda-\mu\in Q_+$.

Let $\{\alpha_i\}_{i\in I}$ denote the set of simple roots and let $\{\alpha_i^\vee\}_{i\in I}$ denote the set of simple coroots. The Cartan matrix of G is $A=(a_{i,j})_{i,j\in I}$, where $a_{i,j}=\langle\alpha_i^\vee,\alpha_j\rangle$. We define $\rho\in P$ to be the half-sum of the positive roots, and we define $\rho^\vee\in\mathrm{Hom}_{\mathbb{Z}}(P,\frac{1}{2}\mathbb{Z})$ to be the half-sum of the positive coroots.

Let $W=N_G(T)/T$ be the Weyl group, generated by the simple reflections s_i for $i\in I$. For $i\in I$, we set $h_i=\alpha_i^\vee$ and we choose root vectors e_i and f_i in $\mathfrak{g}$ of weights α_i and $-\alpha_i$, respectively, normalized so that $[e_i,f_i]=h_i$. Then, the element

$$\bar{s}_i=\exp(-e_i)\exp(f_i)\exp(-e_i)$$

is a lift of s_i in $N_G(T)$. These elements $\bar{s}_i$ satisfy the braid relations [Ti, Proposition 3], which allows us to lift any $w\in W$ to an element $\bar{w}\in N_G(T)$.

The enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ is generated by the elements e_i; it is graded by Q_+, with $\deg e_i=\alpha_i$. As is customary, for any natural number n, we denote the nth divided power of e_i by $e_i^{(n)}$.

G	a simple simply-connected complex algebraic group	§2.1
N, B, T, W	usual data associated with G	§2.1
$e_i \in \mathfrak{n}$	Chevalley generators	§2.1
$\{\alpha_i\}, \{\alpha_i^\vee\}, I$	the simple roots and coroots, with common index set I	§2.1
$P \supset P_+$	the weight lattice and dominant weights of G	§2.1
$Q \supset Q_+$	the root lattice and $\mathbb{N}$-span of the positive roots	§2.1
$\mathrm{wt}, \varepsilon_i, \tilde{e}_i, \phi_i, \tilde{f}_i$	crystal data	§2.3
$B(\infty)$	the bicrystal of $\mathbb{C}[N]$	§2.4
$L(\lambda)$	the irreducible G-representation of highest weight $\lambda \in P_+$	§2.5
$v_\lambda,\ v_\lambda^*$	a h.w. vector and a linear form on $L(\lambda)$ s.t. $v_\lambda^*(v_\lambda)=1$	§2.5
Ψ_λ	an embedding $L(\lambda) \to \mathbb{C}[N]$ taking $v \mapsto v_\lambda^*(\mathbf{?}v)$	§2.5
$G^\vee, \mathcal{G}\mathrm{r}$	the Langlands dual group and its affine Grassmannian $\mathcal{G}\mathrm{r}$	§4.1
t^μ, L_μ	the points in $G^\vee(\mathcal{K})$, $\mathcal{G}\mathrm{r}$ defined by the $G^\vee$-coweight $\mu \in P$	§4.1
$\mathcal{G}\mathrm{r}^\lambda,\ S_\pm^\mu$	the spherical and semi-infinite orbits in $\mathcal{G}\mathrm{r}$	§4.1
$F = \oplus_\mu F_\mu$	the fiber functor $H^\bullet(\mathcal{G}\mathrm{r}, \mathbf{?})$ and the weight functors	§4.1
Υ	the embedding $\mathcal{G}\mathrm{r} \hookrightarrow \mathbb{P}(V(\Lambda_0))$	§4.3
$i(Z, X \cdot V)$	the multiplicity of Z in the intersection $X \cdot V$	§5.1
Z	a typical MV cycle	§5.2
$\mathcal{I}_\lambda$	the intersection cohomology sheaf of $\overline{\mathcal{G}\mathrm{r}^\lambda}$	§5.2
$\mathcal{Z}(\lambda)$	the set of MV cycles of type λ	§5.2
$\mathcal{Z}(\infty)$	the set of stable MV cycles	§6.1
$\{b_Z\}$	the MV basis of $\mathbb{C}[N]$, indexed by stable MV cycles	§6.2
n_x	for $x \in \mathfrak{t}^{\mathrm{reg}}$, the element of N such that $\mathrm{Ad}_{n_x}(x) = e + x$	§8.1
$\overline{D}$	the corresponding algebra morphism $\mathbb{C}[N] \to \mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$	§8.1
$\mathrm{Seq}(\nu)$	the set of sequences of simple roots with sum ν	§8.2
$\overline{D}_{\mathbf{i}}$	a rational function associated with a sequence $\mathbf{i}$	§8.2
$\mathcal{PP}$	an algebra of distributions on $\mathfrak{t}_{\mathbb{R}}^*$	§8.3
$\pi_{\mathbf{i}} : \mathbb{R}^{p+1} \to \mathfrak{t}^{\mathrm{reg}}$	a linear map defined using the partial sums of a sequence $\mathbf{i}$	§8.3
$D_{\mathbf{i}}$	a measure associated with a sequence $\mathbf{i}$	§8.3
D	an algebra map $\mathbb{C}[N] \to \mathcal{PP}$	§8.3
C	the universal centralizer space $\{(b, x) : \mathrm{Ad}_b(\dot{e} + x) = \dot{e} + x\}$	§8.5
$\psi : C \to N$	a morphism given by $(tn, x) \mapsto n$	§8.5
τ_n	scaling of $\mathfrak{t}_{\mathbb{R}}^*$ by $1/n$	§9.1
$S \subset \mathbb{C}[\mathfrak{t}]$	the multiplicative set generated by $\mu \in P \setminus \{0\}$	§9.2
Λ	the preprojective algebra	§11.1
$\underrightarrow{\dim}$	the dimension vector of a Λ-module	§11.1
$\Lambda(\nu)$	Lusztig's nilpotent variety	§11.1
$F_{\mathbf{i}}(M)$	the variety of composition series of type $\mathbf{i}$	§11.2
ξ_M	an element of $\mathbb{C}[N]_{-\underrightarrow{\dim} M}$ associated with a Λ-module M	§11.2
Y	a typical component of $\Lambda(\nu)$	§11.2
c_Y	dual semicanonical basis element associated with Y	§11.2
$F_{n,\mu}(M)$	the space of length $n+1$ flags in M of total dimension μ	§11.3
$\mathbb{G}_\mu(M[t]/t^n)$	the space of dimension μ submodules in $M \otimes \mathbb{C}[t]/t^n$	§12.2

Table 1. Dramatis personæ.

The torus T acts by conjugation on N, which endows the algebra $\mathbb{C}[N]$ of regular functions on N with a weight grading

$$\mathbb{C}[N] = \bigoplus_{\nu \in Q_+} \mathbb{C}[N]_{-\nu}.$$

The group N and its Lie algebra $\mathfrak{n}$ act on both sides of $\mathbb{C}[N]$; our convention is that $f \cdot n = f(n\mathbf{?})$ and $n \cdot f = f(\mathbf{?}n)$ for each $(n, f) \in N \times \mathbb{C}[N]$. Denoting by 1_N the unit element in N, we have

$$(a \cdot f)(1_N) = (f \cdot a)(1_N)$$

for each $(a, f) \in U(\mathfrak{n}) \times \mathbb{C}[N]$, and the map $\langle a, f \rangle := (a \cdot f)(1_N)$ defines a perfect pairing $U(\mathfrak{n}) \times \mathbb{C}[N] \to \mathbb{C}$. In particular the vector space $\mathbb{C}[N]_{-\nu}$ is linearly isomorphic to the dual of $U(\mathfrak{n})_\nu$ for each $\nu \in Q_+$.

2.2. Crystals

Following for instance [KSa, §3], we recall that a G-crystal is a set $B \not\ni 0$ endowed with maps

$$\mathrm{wt}: B \longrightarrow P, \quad \varepsilon_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}, \quad \tilde{e}_i : B \longrightarrow B \cup \{0\},$$
$$\varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}, \quad \tilde{f}_i : B \longrightarrow B \cup \{0\}$$

for $i \in I$, satisfying the following axioms:

- For each $b \in B$ and $i \in I$, $\varphi_i(b) = \langle h_i, \mathrm{wt}(b) \rangle + \varepsilon_i(b)$.
- For each $b, b' \in B$ and $i \in I$, we have
$$b = \tilde{e}_i b' \quad \Longleftrightarrow \quad \tilde{f}_i b = b'.$$
- For each $b \in B$ and $i \in I$ such that $\tilde{e}_i b \neq 0$, we have
$$\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i, \quad \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \quad \text{and} \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.$$
- For each $b \in B$ and $i \in I$, if $\varphi_i(b) = \varepsilon_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

A crystal B is said to be *upper semi-normal* if, for each $b \in B$ and $i \in I$, there exists $n \in \mathbb{N}$ such that $\tilde{e}_i^n b = 0$ and

$$\varepsilon_i(b) = \max\{n \in \mathbb{N} : \tilde{e}_i^n b \neq 0\}.$$

A crystal B is said to be *semi-normal* if additionally, for each $b \in B$ and each $i \in I$, there exists $n \in \mathbb{N}$ such that $\tilde{f}_i^n b = 0$ and

$$\varphi_i(b) = \max\{n \in \mathbb{N} : \tilde{f}_i^n b \neq 0\}.$$

All the crystals that we consider in this paper are upper semi-normal. The maps ε_i and φ_i are then determined by the rest of the structure.

2.3. Perfect bases

We look for bases of $\mathbb{C}[N]$ that enjoy a form of compatibility with the left and right actions of $\{e_i\}\subset\mathfrak{n}$. The following definition matches Berenstein and Kazhdan's one ([BeK, Definition 5.30]), with the addition of a specific normalization.

Definition 2.1. A linear basis B of $\mathbb{C}[N]$ is *perfect* if it is endowed with an upper semi-normal crystal structure such that:

- The constant function equal to 1 belongs to B.
- Each b in B is homogeneous of degree $\mathrm{wt}(b)$ with respect to the weight grading
$$\mathbb{C}[N]=\bigoplus_{\nu\in Q_+}\mathbb{C}[N]_{-\nu}.$$
- For each $i\in I$ and $b\in B$, the expansion of $e_i\cdot b$ in the basis B has the form
$$e_i\cdot b=\varepsilon_i(b)\tilde{e}_i b+\sum_{\substack{b'\in B\\ \varepsilon_i(b')<\varepsilon_i(b)-1}} a_{b'}b'$$

with $a_{b'}\in\mathbb{C}$.

It follows from the definition that if a linear basis B of $\mathbb{C}[N]$ is perfect, then, for each $i\in I$ and each $b\in B$, we have
$$n=\varepsilon_i(b)\quad\Longrightarrow\quad e_i^{(n)}\cdot b=\tilde{e}_i^n b\ \text{ and }\ e_i^{n+1}\cdot b=0.$$

For $i\in I$ and $n\in\mathbb{N}$, let us define
$$K_{i,n}:=\{f\in\mathbb{C}[N]:e_i^{n+1}\cdot f=0\}.$$
Using the fact just above, we easily check that, for any perfect basis B of $\mathbb{C}[N]$, we have
$$B\cap K_{i,n}=\{b\in B:\varepsilon_i(b)\leqslant n\},$$
and moreover this set is a basis of $K_{i,n}$.

To take into account the right action of $\mathfrak{n}$ on $\mathbb{C}[N]$, we now introduce biperfect bases.

Definition 2.2. A linear basis B of $\mathbb{C}[N]$ is *biperfect*(1) if it is perfect and if it is endowed with a second upper semi-normal crystal structure $(\mathrm{wt},\varepsilon_i^*,\varphi_i^*,\tilde{e}_i^*,\tilde{f}_i^*)$ which shares the same weight map as the first crystal structure, and such that, for each $i\in I$ and $b\in B$,
$$b\cdot e_i=\varepsilon_i^*(b)\tilde{e}_i^* b+\sum_{\substack{b'\in B\\ \varepsilon_i^*(b')<\varepsilon_i^*(b)-1}} a_{b'}^*b',$$
with $a_{b'}\in\mathbb{C}$.

(1) This is the same as the notion of "basis of dual canonical type" from [McN], except that we have specialized $q=1$, and we do not require that B be invariant under the involution $*$ discussed in Remark 6.7.

We will refer to the data consisting of these two crystal structures on B as the *bicrystal structure* of B. If B is a biperfect basis of $\mathbb{C}[N]$, then for each $i\in I$ and each $n\in\mathbb{N}$, the set $\{b\in B:\varepsilon_i^*(b)\leqslant n\}$ is a basis of

$$K_{i,n}^* = \{f\in\mathbb{C}[N] : f\cdot e_i^{n+1}=0\}.$$

The algebra $\mathbb{C}[N]$ does have biperfect bases, but the explicit constructions of such bases rely on geometric constructions or on categorification methods. The first example of a biperfect basis is Lusztig's canonical basis, after specialization at $q=1$ and then dualization; in other words, Kashiwara's upper global basis, specialized at $q=1$ (see for instance [Lu3, Theorems 1.6 and 7.5], and [Kas2, §5.3]). Another example, in the simply-laced case, is the dual of Lusztig's semicanonical basis (the compatibility of the dual semicanonical basis with the subspaces $K_{i,n}$ and $K_{i,n}^*$ is established in [Lu5, §3]). The basis arising using the categorification by representations of KLR algebras is also biperfect; this fact is transparent from Khovanov and Lauda's first paper [KL] and is given a detailed proof in [McN]. Lastly, the geometric Satake correspondence also gives rise to a biperfect basis of $\mathbb{C}[N]$, as we shall see in §5.

In types A_1, A_2 and A_3, we not only have existence, but also uniqueness and explicit formulas.

Example 2.3. Suppose $G=\mathrm{SL}_3(\mathbb{C})$, with the standard choice for B, T and N. Then, $\mathbb{C}[N]=\mathbb{C}[x,y,z]$, where x, y and z are the three matrix entries of an upper unitriangular matrix

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

The unique biperfect basis of $\mathbb{C}[N]$ is

$$B = \{x^a z^b (xy-z)^c : (a,b,c)\in\mathbb{N}^3\} \cup \{y^a z^b (xy-z)^c : (a,b,c)\in\mathbb{N}^3\}.$$

The action from the left of the Chevalley generators is given by

$$e_1 = \frac{\partial}{\partial x} \quad \text{and} \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

One can check that in B, the operators e_1 and e_2 act with coefficients in $\mathbb{N}$ and that the structure constants of the multiplication belong to $\mathbb{N}$.

For the explicit formulas in type A_3, we refer to the paper [BZ2] by Berenstein and Zelevinsky, which was the starting point of the theory of cluster algebras.

2.4. Uniqueness of crystal

Berenstein and Kazhdan proved that, up to isomorphism, the crystal of a perfect basis of $\mathbb{C}[N]$ is independent of the choice of the basis ([BeK], Theorem 5.37). The same is true for biperfect bases.

THEOREM 2.4. *Let B and C be two biperfect bases of $\mathbb{C}[N]$. Then, there is a unique bijection $B \cong C$ that respects the bicrystal structure.*

Proof. We study the properties of the transition matrix $M=(m_{b,c})$ between the two bases, defined by the equation

$$c = \sum_{b \in B} m_{b,c} b \tag{$*$}$$

for each $c \in C$.

Fix $i \in I$, take $c \in C$, and set

$$n = \max\{\varepsilon_i(b) : b \in B \text{ such that } m_{b,c} \neq 0\}.$$

Then, each b occurring in the right-hand side of $(*)$ belongs to $K_{i,n}$, and therefore $c \in K_{i,n}$, that is, $\varepsilon_i(c) \leqslant n$. Applying $e_i^{(n)}$ to $(*)$, we get

$$e_i^{(n)} \cdot c = \sum_{\substack{b \in B \\ m_{b,c} \neq 0 \\ \varepsilon_i(b) = n}} m_{b,c} e_i^{(n)} \cdot b.$$

The terms $e_i^{(n)} \cdot b = \tilde{e}_i^n b$ that survive in the right-hand side belong to B, hence are linearly independent, which implies that the left-hand side is not zero, and therefore that $\varepsilon_i(c) = n$.

Let us define

$$B_{i,n} = \{b \in B : \varepsilon_i(b) = n\} \quad \text{and} \quad C_{i,n} = \{c \in C : \varepsilon_i(c) = n\}.$$

The above analysis shows that the matrix M is block upper triangular with respect to the decompositions

$$B = B_{i,0} \cup B_{i,1} \cup B_{i,2} \cup \ldots \quad \text{and} \quad C = C_{i,0} \cup C_{i,1} \cup C_{i,2} \cup \ldots$$

of the rows and the columns, and that the diagonal blocks of M are equal under the bijections

$$\tilde{e}_i^n : B_{i,n} \longrightarrow B_{i,0} \quad \text{and} \quad \tilde{e}_i^n : C_{i,n} \longrightarrow C_{i,0}.$$

We now replace the index $i \in I$ by a sequence $\mathbf{i} = (i_1, i_2, i_3, \ldots)$, in which each element of I appears infinitely many times. With an element $b \in B$, we associate its string parameters in direction $\mathbf{i}$, namely the sequence $\mathbf{n_i}(b) = (n_1, n_2, n_3, \ldots)$, where

$$n_1 = \varepsilon_{i_1}(b), \quad n_2 = \varepsilon_{i_2}(\tilde{e}_{i_1}^{n_1} b), \quad n_3 = \varepsilon_{i_3}(\tilde{e}_{i_2}^{n_2} \tilde{e}_{i_1}^{n_1} b), \quad \ldots .$$

The weights of the sequence of elements

$$b, \quad \tilde{e}_{i_1}^{n_1} b, \quad \tilde{e}_{i_2}^{n_2} \tilde{e}_{i_1}^{n_1} b, \quad \tilde{e}_{i_3}^{n_3} \tilde{e}_{i_2}^{n_2} \tilde{e}_{i_1}^{n_1} b, \quad \ldots$$

increase in $(-Q_+)$ with respect to the dominance order, so this sequence becomes eventually constant, and its final value is in

$$\bigcap_{i \in I} B_{i,0} = B \cap \left(\bigcap_{i \in I} K_{i,0} \right) = \{1\}.$$

It follows that the map $b \mapsto \mathbf{n_i}(b)$ is injective, and we can therefore transfer the lexicographic order on string parameters to a total order on B. Similarly, we define the string parameters in direction $\mathbf{i}$ of an element of C, and totally order C accordingly.

Iterating our first argument, we see that the matrix M is now upper triangular, with all diagonal elements equal—and in fact equal to 1 because of the normalization condition that 1 belongs to both B and C. In particular, we obtain a bijection between B and C that preserves string parameters in direction $\mathbf{i}$. This certainly means that this bijection preserves the weight map and the crystal operations ε_{i_1}, $\tilde{e}_{i_1}$ and $\tilde{f}_{i_1}$.

Now we can change $\mathbf{i}$, and thus replace i_1 by any element of I. The bijection does not change, because given two sets B and C and a matrix whose elements are indexed by $B \times C$, there is at most one bijection $B \cong C$ such that there exists a total order on B (and hence C) making M upper unitriangular. In other words, if M is an upper unitriangular matrix, and P and Q are permutation matrices, and PMQ^{-1} is upper triangular, then $P = Q$. (For this last assertion, it is necessary to assume that the matrices have finitely many rows and columns. This condition does not hold in our situation, but we can reduce to it by restricting to weight subspaces.)

We now have proved the existence of a bijection $B \cong C$ that respects the crystal structure $(\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$. But we can similarly construct a bijection $B \cong C$ that respects the starred crystal structure $(\mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \tilde{e}_i^*, \tilde{f}_i^*)$, and the argument in the previous paragraph shows that the two bijections necessarily coincide. □

The abstract bicrystal structure shared by all biperfect bases of $\mathbb{C}[N]$ is denoted by $B(\infty)$.

2.5. Bases in representations

Given a dominant weight λ, we denote the simple G-module of highest weight λ by $L(\lambda)$. We fix a preferred choice of a highest weight vector v_λ in $L(\lambda)$.

Definition 2.5. A linear basis B_λ of $L(\lambda)$ is *perfect* if it is endowed with an upper semi-normal crystal structure such that the following conditions hold:

- the highest weight vector v_λ belongs to B_λ;
- each b in B_λ is homogeneous of degree $\mathrm{wt}(b)$ with respect to the weight grading of $L(\lambda)$;
- for each $i\in I$ and $b\in B_\lambda$, the expansion of $e_i\cdot b$ in the basis B_λ has the form

$$e_i\cdot b=\varepsilon_i(b)\tilde{e}_i b+\sum_{\substack{b'\in B_\lambda\\ \varepsilon_i(b')<\varepsilon_i(b)-1}} a_{b'}b',$$

with $a_{b'}\in\mathbb{C}$.

Remark 2.6. (i) Any perfect basis is a good basis in the sense of Berenstein and Zelevinsky [BZ1]. It follows that any perfect basis B_λ of $L(\lambda)$ restricts to bases for tensor product multiplicity spaces. Specifically, given dominant weights μ and ν, the set

$$\{b\in B_\lambda : \mathrm{wt}(b)=\nu-\mu \text{ and } \varepsilon_i(b)\leqslant\langle h_i,\mu\rangle \text{ for all } i\in I\}$$

forms a basis for $\mathrm{Hom}(L(\nu),L(\lambda)\otimes L(\mu))$, where we use the inclusion

$$\mathrm{Hom}(L(\nu),L(\lambda)\otimes L(\mu))\hookrightarrow L(\lambda),$$
$$\phi\longmapsto(1\otimes v_\mu^*)(\phi(v_\nu))$$

(see below for the definition of v_μ^*).

(ii) Let B_λ be a perfect basis of $L(\lambda)$ and let $B_\lambda^\star$ be its dual basis with respect to a contravariant form on $L(\lambda)$. Then, for any Demazure module $W\subset L(\lambda)$, the set $B_\lambda^\star\cap W$ is a basis of W. In fact, Kashiwara's proof of the same result for the global crystal basis ([Kas3, §3.2]) only uses the axioms of a perfect basis (up to duality). In the case of the semicanonical basis, this property was observed by Savage ([Sav, Theorem 7.1]).

Let $v_\lambda^*\colon L(\lambda)\to\mathbb{C}$ be the linear form such that $v_\lambda^*(v_\lambda)=1$ and $v_\lambda^*(v)=0$ for any weight vector $v\in L(\lambda)$ of weight other than λ. We define an N-equivariant map

$$\Psi_\lambda\colon L(\lambda)\longrightarrow\mathbb{C}[N]$$

by $\Psi_\lambda(v)=v_\lambda^*(?v)$. As a matter of fact, Ψ_λ is injective and that its image is

$$\mathrm{im}\,\Psi_\lambda=\bigcap_{i\in I}K^*_{i,\langle h_i,\lambda\rangle}$$

(one can deduce this from [Hu1, Theorem 21.4] with the help of a contravariant form on $L(\lambda)$; see also [BZ3, Proposition 5.1]). It follows that if B is a biperfect basis of $\mathbb{C}[N]$, then

$$B\cap(\mathrm{im}\,\Psi_\lambda)=\{b\in B : \varepsilon_i^*(b)\leqslant\langle h_i,\lambda\rangle \text{ for all } i\in I\}$$

is a basis of $\mathrm{im}\,\Psi_\lambda$.

PROPOSITION 2.7. *Let B be a biperfect basis of $\mathbb{C}[N]$, $\lambda\in P_+$, and $B_\lambda=\Psi_\lambda^{-1}(B)$. Then, B_λ inherits from B the structure of an upper semi-normal crystal, the weight map being shifted by λ, and B_λ is a perfect basis of $L(\lambda)$.*

Proof. For each $b\in B_\lambda$, we set

$$\mathrm{wt}(b)=\mathrm{wt}(\Psi_\lambda(b))+\lambda,\quad \varepsilon_i(b)=\varepsilon_i(\Psi_\lambda(b))\quad\text{and}\quad \varphi_i(b)=\langle h_i,\mathrm{wt}(b)\rangle+\varepsilon_i(b).$$

If $\varepsilon_i(b)=0$, then we set $\tilde{e}_ib=0$. Otherwise, $\tilde{e}_i(\Psi_\lambda(b))$ appears with a non-zero coefficient in the expansion of $e_i\cdot\Psi_\lambda(b)$ in the basis B. Now, $e_i\cdot\Psi_\lambda(b)$ belongs to $(\mathrm{im}\,\Psi_\lambda)$, and this subspace is spanned by the elements of B that it contains. We conclude that $\tilde{e}_i(\Psi_\lambda(b))\in B\cap(\mathrm{im}\,\Psi_\lambda)$, and therefore we can define $\tilde{e}_ib\in B_\lambda$ by

$$\Psi_\lambda(\tilde{e}_ib)=\tilde{e}_i(\Psi_\lambda(b)).$$

The fact that B is upper semi-normal then implies that B_λ is upper semi-normal. Lastly, we define $\tilde{f}_ib\in B_\lambda\cup\{0\}$ so that

$$\Psi_\lambda(\tilde{f}_ib)=\begin{cases}\tilde{f}_i\Psi_\lambda(b), & \text{if } \tilde{f}_i(\Psi_\lambda(b))\in(\mathrm{im}\,\Psi_\lambda),\\ 0, & \text{otherwise.}\end{cases}$$ □

From Remark 2.6 (i), we immediately deduce the following corollary, which can be regarded as a generalization (from the canonical basis to arbitrary biperfect bases) of [BZ4, Corollary 3.4].

COROLLARY 2.8. *Let B be a biperfect basis of $\mathbb{C}[N]$. For any $\lambda,\mu,\nu\in P_+$, the set*

$$\{b\in B(\infty):\mathrm{wt}(b)=\nu-\mu-\lambda \text{ and } \varepsilon_i(b)\leqslant\langle h_i,\mu\rangle \text{ and } \varepsilon_i^*(b)\leqslant\langle h_i,\lambda\rangle \text{ for all } i\in I\}$$

restricts (under Ψ_λ and the inclusion from Remark 2.6 (i)) to a basis for

$$\mathrm{Hom}(L(\nu),L(\lambda)\otimes L(\mu)).$$

The proof of the following lemma relies on elementary $\mathfrak{sl}_2$-theory and is left to the reader.

LEMMA 2.9. *Let B_λ be a perfect basis of $L(\lambda)$. Then, the crystal B_λ is semi-normal, and for each $i\in I$ and $b\in B_\lambda$, the expansion of $f_i\cdot b$ in the basis B_λ has the form*

$$f_i\cdot b=\varphi_i(b)\tilde{f}_ib+\sum_{\substack{b'\in B_\lambda\\ \varphi_i(b')<\varphi_i(b)-1}}a_{b'}b',$$

with $a_{b'}\in\mathbb{C}$.

Remark 2.10. Let $\lambda\in P_+$. For each $w\in W$, the weight space $L(\lambda)_{w\lambda}$ is 1-dimensional. We choose a basis vector for this weight space by defining $v_{w\lambda}=\overline{w}v_\lambda$. These elements can also be defined by induction on the length of w: if $s_iw>w$, then

$$v_{s_iw\lambda}=f_i^{(n)}\cdot v_{w\lambda}$$

where $n=\langle h_i,w\lambda\rangle$. Using Lemma 2.9, we see that these elements $v_{w\lambda}$ belong to each perfect basis of $L(\lambda)$. Abusing slightly the standard terminology, we will call *flag minors* the functions $\Psi_\lambda(v_{w\lambda})$; they belong to each biperfect basis of $\mathbb{C}[N]$. (When λ is minuscule, they are the restrictions to N of the *flag minors* from [BFZ], [BZ3].)

We observe that, for any dominant weights λ and μ, we have $\operatorname{im}\Psi_\lambda\subset\operatorname{im}\Psi_{\lambda+\mu}$. This motivates the following definition.

Definition 2.11. A *coherent family of bases* is the datum of a basis B_λ of $L(\lambda)$ for each dominant weight $\lambda\in P_+$ such that

$$\Psi_\lambda(B_\lambda)\subset\Psi_{\lambda+\mu}(B_{\lambda+\mu})$$

for all $\lambda,\mu\in P_+$.

A biperfect basis B gives rise to a coherent family of perfect bases, namely the datum of all the bases $\Psi_\lambda^{-1}(B)$. Conversely, given a coherent family of perfect bases (B_λ), the union

$$\bigcup_{\lambda\in P_+}\Psi_\lambda(B_\lambda)$$

is a perfect basis of $\mathbb{C}[N]$. We note that the crystal structures automatically match, in the sense of Proposition 2.7.

2.6. Multiplication

We can easily describe multiplication in $\mathbb{C}[N]$ using the maps Ψ_λ. First, recall that there is a unique G-equivariant map $m_{\lambda\mu}\colon L(\lambda)\otimes L(\mu)\to L(\lambda+\mu)$ which takes $v_\lambda\otimes v_\mu$ to $v_{\lambda+\mu}$.

The following result follows immediately from the definition of Ψ_λ.

PROPOSITION 2.12. *We have the commutativity* $m\circ(\Psi_\lambda\otimes\Psi_\mu)=\Psi_{\lambda+\mu}\circ m_{\lambda\mu}$, *where* $m\colon\mathbb{C}[N]\otimes\mathbb{C}[N]\to\mathbb{C}[N]$ *is the multiplication map.*

Thus, if we have a coherent family of bases $(B_\lambda)_{\lambda\in P_+}$ and the matrix of the maps $m_{\lambda\mu}$ in these bases, then we will have a basis for $\mathbb{C}[N]$ and the structure constants for multiplication in this basis.

b	$\vec{\varepsilon}(b)$	b	$\vec{\varepsilon}(b)$
$b_1=\tilde{f}_{2134221342}b_0$	$(0,1,0,0)$	$b_7=\tilde{f}_{3221134422}b_0$	$(0,1,1,0)$
$b_2=\tilde{f}_{1342221342}b_0$	$(1,0,1,1)$	$b_8=\tilde{f}_{1221334422}b_0$	$(1,1,0,0)$
$b_3=\tilde{f}_{3422113422}b_0$	$(0,0,1,1)$	$b_9=\tilde{f}_{4422113322}b_0$	$(0,0,0,2)$
$b_4=\tilde{f}_{1422133422}b_0$	$(1,0,0,1)$	$b_{10}=\tilde{f}_{3322114422}b_0$	$(0,0,2,0)$
$b_5=\tilde{f}_{1322134422}b_0$	$(1,0,1,0)$	$b_{11}=\tilde{f}_{1122334422}b_0$	$(2,0,0,0)$
$b_6=\tilde{f}_{4221133422}b_0$	$(0,1,0,1)$	$b_{12}=\tilde{f}_{2211334422}b_0$	$(0,2,0,0)$

Table 2. The twelve elements in $\mathcal{S}$.

2.7. Non-uniqueness

In [GLS], Geiss, Leclerc and Schröer present examples where the canonical and the semicanonical bases are different. Thus, biperfect bases are not unique in general. Let us have a closer look at the simplest example (see [GLS, §19.1]). Here G is of type D_4. We enumerate the vertices in the Dynkin diagram as customary (the trivalent vertex has label 2), and we set $\lambda=\alpha_1+2\alpha_2+\alpha_3+\alpha_4$, the highest root.

We consider two biperfect bases C and C' of $\mathbb{C}[N]$. Abusing the notation, for each $b\in B(\infty)$, we denote by $C(b)$ and $C'(b)$ the elements of C and C' indexed by b. We focus on the subspace $\mathbb{C}[N]_{-2\lambda}\cap(\operatorname{im}\Psi_{2\lambda})$. It is compatible with both C and C'; specifically, both $\{C(b):b\in\mathcal{S}\}$ and $\{C'(b):b\in\mathcal{S}\}$ are bases of this subspace, where

$$\mathcal{S}=\{b\in B(\infty):\mathrm{wt}(b)=-2\lambda,\ \varepsilon_1^*(b)=\varepsilon_3^*(b)=\varepsilon_4^*(b)=0 \text{ and } \varepsilon_2^*(b)\leqslant 2\}.$$

Table 2 presents the twelve elements in $\mathcal{S}$. Here, b_0 is the element of $B(\infty)$ of weight zero, and given a word $abc\ldots$ in the alphabet $\{1,2,3,4\}$, the notation $\tilde{f}_{abc\ldots}b_0$ is a shorthand for the element $\tilde{f}_a\tilde{f}_b\tilde{f}_c\ldots b_0$ in $B(\infty)$. Lastly, $\vec{\varepsilon}(b)$ is the tuple

$$(\varepsilon_1(b),\varepsilon_2(b),\varepsilon_3(b),\varepsilon_4(b)).$$

The proof of Theorem 2.4 shows that in the expansion

$$C'(b')=\sum_{b\in B(\infty)} m_{b,b'}C(b)$$

of an element of C' in the basis C, the coordinate $m_{b,b'}$ necessarily vanishes except when $\varepsilon_i(b)\leqslant\varepsilon_i(b')$ for each i. We then deduce from the table above that $C(b)=C'(b)$ for all $b\in\{b_1,b_3,b_4,b_5,b_9,b_{10},b_{11}\}$, that $C(b_2)-C'(b_2)$ is a linear combination of $C(b_3)$, $C(b_4)$

and $C(b_5)$, and that, for $b\in\{b_6,b_7,b_8,b_{12}\}$, the difference $C(b)-C'(b)$ is a scalar multiple of $C(b_1)$. In fact, $C(b)=C'(b)$ also holds for $b\in\{b_2,b_6,b_7,b_8\}$; to prove this for $b=b_8$ for instance, one can note that

$$\varepsilon_2(b_1)=\varepsilon_2(b_8), \qquad \varepsilon_4(\tilde{e}_2b_1)=\varepsilon_4(\tilde{e}_2b_8),$$
$$\varepsilon_3(\tilde{e}_4\tilde{e}_2b_1)=\varepsilon_3(\tilde{e}_4\tilde{e}_2b_8), \qquad \varepsilon_2(\tilde{e}_3\tilde{e}_4\tilde{e}_2b_1)>\varepsilon_2(\tilde{e}_3\tilde{e}_4\tilde{e}_2b_8),$$

and refine the previous argument (see [Ba2], §2.5). To sum up, C and C' only differ at the element indexed by b_{12}, and $C(b_{12})-C'(b_{12})$ is a scalar multiple of $C(b_1)$.

Let us set $\eta=e_2(e_1e_3e_4)e_2^{(2)}(e_1e_3e_4)e_2$, an element in $U(\mathfrak{n})$. Then, $\langle\eta,C(b_1)\rangle=1$ (see [Ba2, Theorem 5.2, case III]; note that the elements b_1 and b_{12} are denoted by $b_{0,1}$ and $b_{2,0}$ in that paper), so

$$C(b_{12})-C'(b_{12})=\langle\eta,C(b_{12})-C'(b_{12})\rangle C(b_1).$$

If C is the dual semicanonical basis, then $\langle\eta,C(b_{12})\rangle=2$. If C' is the dual canonical basis/upper global basis (specialized at $q=1$), then, $\langle\eta,C'(b_{12})\rangle=1$. And if C'' is the MV basis of $\mathbb{C}[N]$ (see §6 below for the definition of this basis), then $\langle\eta,C''(b_{12})\rangle=0$. We thus see that these three bases are pairwise different. Specifically, we have the relations advertised in §1.5:

$$C(b_{12})=C''(b_{12})+2C(b_1) \quad\text{and}\quad C'(b_{12})=C''(b_{12})+C(b_1). \tag{1}$$

We will not develop enough material in the present paper to be able to provide a complete justification of the equation $\langle\eta,C''(b_{12})\rangle=0$, but we can nonetheless sketch the proof. Let $b_{13}=\tilde{f}_{1342}b_0$ and $b_{14}=\tilde{f}_{221342}b_0$; then $C''(b_{13})$ and $C''(b_{14})$ are two elements in $(\operatorname{im}\Psi_\lambda)$, that is, two matrix coefficients of the adjoint representation $L(\lambda)$. A rather straightforward calculation then gives $\langle\eta,C''(b_{13})C''(b_{14})\rangle=2$. On the other hand, using Theorem 7.11, one can expand the product

$$C''(b_{13})C''(b_{14})=2C''(b_1)+\sum_{i=2}^{8}C''(b_i)+C''(b_{12}).$$

(The forthcoming paper [BGL] will explain how to calculate the required intersection multiplicities. The actual computations are rather tedious—for the coefficient 2 one must deal with a variety of codimension 10 defined by 18 equations—and are carried out with the help of the computer algebra system Singular [DGPS].) Since the matrix coefficients of the action of the Chevalley generators in the MV basis of a representation are non-negative (a consequence of Theorem 5.4 below), we have $\langle\eta,C''(b)\rangle\geqslant 0$ for any $b\in\mathcal{S}$. This is enough to ensure that $\langle\eta,C''(b_{12})\rangle=0$.

In the appendix (Theorem A.13), we prove the analog of the first equation in (1), but only after applying the non-injective map $\overline{D}$ (see §1.4). There we also use computer algebra systems along with techniques specific to type A.

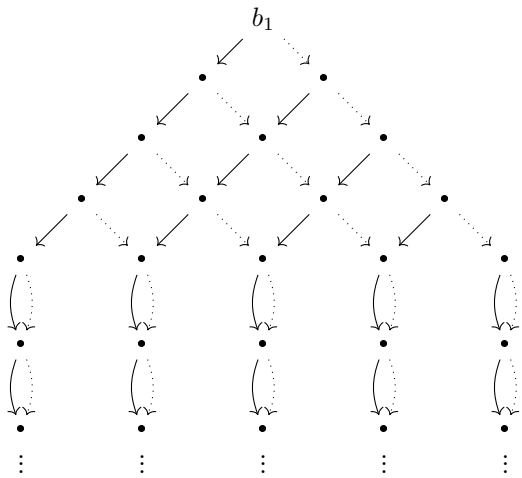

Figure 2. Local structure of $B(\infty)$.

3. More on biperfect bases

3.1. Crystal reflections

A result of Kashiwara and Saito ([KSa, Proposition 3.2.3]), extended by Tingley and Webster ([TW, Proposition 1.4]) says that the bicrystal $B(\infty)$ is characterized by the following conditions:

(i) Both crystals $(B(\infty), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ and $(B(\infty), \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \tilde{e}_i^*, \tilde{f}_i^*)$ are upper semi-normal.

(ii) $\mathrm{wt}(b) \in -Q_+$ for each element $b \in B(\infty)$.

(iii) There is a unique element $b_0 \in B(\infty)$ such that $\mathrm{wt}(b_0)=0$.

(iv) For each $i \in I$ and $b \in B$, we have $\tilde{f}_i b \neq 0$ and $\tilde{f}_i^* b \neq 0$.

(v) For each $i \neq j$ and each $b \in B(\infty)$, we have $\varepsilon_j(\tilde{f}_i^* b) = \varepsilon_j(b)$, $\varepsilon_j^*(\tilde{f}_i b) = \varepsilon_j^*(b)$ and $\tilde{f}_i \tilde{f}_j^* b = \tilde{f}_j^* \tilde{f}_i b$.

(vi) If $i \in I$ and $b_1 \in B(\infty)$ satisfy $\varepsilon_i(b_1) = \varepsilon_i^*(b_1) = 0$, then $\langle h_i, \mathrm{wt}(b_1) \rangle \geqslant 0$.

(vii) Let $i \in I$ and $b \in B$. The subset of $B(\infty)$ generated by b under the action of the operators $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{e}_i^*$, $\tilde{f}_i^*$ contains a unique element b_1 such that $\varepsilon_i(b_1) = \varepsilon_i^*(b_1) = 0$ and has the form drawn in Figure 2, where the action of $\tilde{f}_i$ is indicated by the plain arrows, the action of $\tilde{f}_i^*$ is indicated by the dotted arrows, and $\langle h_i, \mathrm{wt}(b_1) \rangle$ is the width of the shape. (The picture is drawn for $\langle h_i, \mathrm{wt}(b_1) \rangle = 4$.)

Note that (iv) is implied by (vii), and therefore not really needed.

Remark 3.1. These conditions imply that, for any $b \neq b_0$, there exists $i \in I$ such that $\varepsilon_i(b) > 0$. To show this, suppose that there exists an element $b \neq b_0$ such that $\varepsilon_i(b) = 0$ for

all $i\in I$. We may assume that, among all possible elements, b has been chosen to be of maximal weight with respect to the dominance order. Since $b\neq b_0$, the weight $\mathrm{wt}(b)$ is not dominant, and (vi) implies the existence of $i\in I$ such that $\varepsilon_i^*(b)>0$. So b is on the upper right edge of the shape drawn in Figure 2, but is not the top vertex. Let $b_1=(\tilde{e}_i^*)^{\varepsilon_i^*(b)}b$ be the top vertex of the shape. Certainly, $\langle h_i, \mathrm{wt}(b_1)\rangle\neq 0$, for the shape has a positive width, and therefore $b_1\neq b_0$. By our maximality condition, b_1 cannot satisfy the property imposed on b, so there exists $j\in I$ such that $\varepsilon_j(b_1)>0$. Necessarily $j\neq i$, and (v) implies that $\varepsilon_j(b)=\varepsilon_j(b_1)>0$, a contradiction. (This argument comes from [KSa, p. 16].)

Using this, one easily recovers the following result due to Saito ([Sai, Corollary 3.4.8]).

THEOREM 3.2. *Let $i\in I$. The map*

$$\sigma_i\colon \{b\in B(\infty)\mid \varepsilon_i^*(b)=0\}\longrightarrow \{b\in B(\infty)\mid \varepsilon_i(b)=0\}$$

given by $\sigma_i(b)=(\tilde{f}_i^*)^{\varphi_i(b)}(\tilde{e}_i)^{\varepsilon_i(b)}(b)$ is bijective and $\sigma_i^{-1}(b)=(\tilde{f}_i)^{\varphi_i^*(b)}(\tilde{e}_i^*)^{\varepsilon_i^*(b)}(b)$.

Specifically, σ_i maps the upper-left edge of the shape in Figure 2 to the upper-right edge, so that $\mathrm{wt}(\sigma_i(b))=s_i\,\mathrm{wt}(b)$. Note that $\sigma_i(b)=(\tilde{e}_i)^{\varepsilon_i(b)}(\tilde{f}_i^*)^{\varphi_i(b)}(b)$.

3.2. Biperfect bases and Weyl group action

The automorphism $\mathrm{Ad}_{\bar{s}_i}$ of the Lie algebra $\mathfrak{g}$ extends to an automorphism T_i of the enveloping algebra $U(\mathfrak{g})$. We set $U^+=U(\mathfrak{n})$, a subalgebra of $U(\mathfrak{g})$. Certainly T_i restricts to a linear isomorphism

$$U^+\cap T_i^{-1}(U^+)\xrightarrow{\simeq} T_i(U^+)\cap U^+.$$

The following theorem generalizes to any biperfect basis a property known for the dual canonical and the dual semicanonical bases (see [Lu4, Theorem 1.2] and [Ba1, §1.2]).

THEOREM 3.3. *Let B be a biperfect basis of $\mathbb{C}[N]$, let $i\in I$ and $b\in B$ be such that $\varepsilon_i^*(b)=0$, and let $u\in U^+\cap T_i^{-1}(U^+)$. Then,*

$$\langle \sigma_i(b), u\rangle = \langle b, T_i(u)\rangle.$$

The equation $\varepsilon_i(\sigma_i(b))=0$ implies that $\sigma_i(b)$ annihilates U^+e_i. Taking into account the decomposition $U^+=(U^+\cap T_i^{-1}(U^+))\oplus U^+e_i$, we see that the theorem provides a completely algebraic characterization of $\sigma_i(b)$ (previously defined only combinatorially).

Proof. Let $i\in I$ and $b\in B$ as in the statement. Set $b'=\sigma_i(b)$, $n=\varepsilon_i(b)$ and $p=\varphi_i(b)$. Then, $p=\varepsilon_i^*(b')=\varepsilon_i^*((\tilde{f}_i)^n(b'))$, which implies

$$b=(\tilde{e}_i^*)^p(\tilde{f}_i)^n(b')=((\tilde{f}_i)^n(b'))\cdot e_i^{(p)}.$$

We choose a dominant weight $\lambda\in P_+$ such that $\langle h_i,\lambda\rangle=p$ and $\langle h_j,\lambda\rangle\geqslant\varepsilon_j^*(b')$ for any $j\neq i$. We adopt the notation of §2.5; in particular, the simple $\mathfrak{g}$-module $L(\lambda)$ comes with the $\mathfrak{n}$-equivariant embedding $\Psi_\lambda\colon L(\lambda)\to\mathbb{C}[N]$. Then, $b'\in\operatorname{im}\Psi_\lambda$, the set $B_\lambda=\Psi_\lambda^{-1}(B)$ is a perfect basis of $L(\lambda)$, and we can write $b'=\Psi_\lambda(\overline{b'})$ for a certain element $\overline{b'}\in B_\lambda$.

Now $\varepsilon_i(\overline{b'})=\varepsilon_i(b')=0$, so $e_i\cdot\overline{b'}=0$ in the module $L(\lambda)$. Further,

$$\langle h_i,\operatorname{wt}(\overline{b'})\rangle=\langle h_i,\lambda+s_i(\operatorname{wt}(b))\rangle=p-\langle h_i,\operatorname{wt}(b)\rangle=p-(p-n)=n,$$

so $\bar{s}_i\cdot\overline{b'}=f_i^{(n)}\cdot\overline{b'}$. From Lemma 2.9, it follows that $\bar{s}_i\cdot\overline{b'}=(\tilde{f}_i)^n(\overline{b'})$, and therefore

$$(\tilde{f}_i)^n(b')=\Psi_\lambda((\tilde{f}_i)^n(\overline{b'}))=\Psi_\lambda(\bar{s}_i\cdot\overline{b'}).$$

Let ω be the involutive antiautomorphism of $\mathfrak{g}$ such that $\omega(e_i)=f_i$ and $\omega(h_i)=h_i$ for each $i\in I$. With respect to ω, there is a unique contravariant form $(\,\cdot\,,\,\cdot\,)$ on $L(\lambda)$ such that $(v_\lambda,v_\lambda)=1$ (see for instance [J, §1.6]). The embedding Ψ_λ is given by $\Psi_\lambda(v)=(v_\lambda,\mathbf{?}v)$; in particular, $(\tilde{f}_i)^n(b')=(v_\lambda,\mathbf{?}\bar{s}_i\cdot\overline{b'})$. Then,

$$b=(v_\lambda,e_i^{(p)}\mathbf{?}\bar{s}_i\cdot\overline{b'})=(f_i^{(p)}\cdot v_\lambda,\mathbf{?}\bar{s}_i\cdot\overline{b'})=(\bar{s}_i\cdot v_\lambda,\mathbf{?}\bar{s}_i\cdot\overline{b'})=(v_\lambda,\bar{s}_i^{-1}\mathbf{?}\bar{s}_i\cdot\overline{b'}).$$

Evaluating this equation on $T_i(u)$, where $u\in U^+\cap T_i^{-1}(U^+)$, we get $\langle b,T_i(u)\rangle=\langle b',u\rangle$, as desired. □

3.3. MV polytopes

Recall that $B(\infty)$ denotes the abstract bicrystal common to all biperfect bases. The theory of MV polytopes provides a convenient combinatorial model for $B(\infty)$. These polytopes will also serve as the support for the measures to be introduced later in Part III.

It is simplest to introduce MV polytopes using the crystal reflections σ_i. Specifically, we extend the definition of the crystal reflections to all of $B(\infty)$ by defining

$$\hat{\sigma}_i(b)=\sigma_i((\tilde{e}_i^*)^{\max}b),$$

where as usual $\tilde{e}_i^{\max}b$ means $\tilde{e}_i^{\varepsilon_i(b)}b$. These operators satisfy the braid relations

$$\underbrace{\hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_i\ldots}_{m_{i,j}\text{ factors}}=\underbrace{\hat{\sigma}_j\hat{\sigma}_i\hat{\sigma}_j\ldots}_{m_{i,j}\text{ factors}},$$

where $m_{i,j}$ is 2, 3, 4, 6 depending on $a_{ij}a_{ji}$ being 0, 1, 2, 3, respectively. To any $w\in W$ one can then attach an operator $\hat{\sigma}_w$ on $B(\infty)$ so that $\hat{\sigma}_w=\hat{\sigma}_{i_1}\dots\hat{\sigma}_{i_\ell}$ for any reduced word $w=s_{i_1}\dots s_{i_\ell}$. Further, one can show that the map $\hat{\sigma}_{w_0}$ takes every $b\in B(\infty)$ to the unique element b_0 of weight zero.

(These facts follow from the work of Saito [Sai]. Loosely speaking, $\hat{\sigma}_i$ rotates Lusztig data, dropping the first coordinate and inserting a zero at the last position. We refer to [BaK, p. 184] for a complete description.)

We define the *MV polytope* $\mathrm{Pol}(b)$ of an element $b\in B(\infty)$ as the convex hull of the weights

$$\mu_w(b):=w\,\mathrm{wt}(\hat{\sigma}_{w^{-1}}(b))$$

for $w\in W$; this polytope lies in $P\otimes_{\mathbb{Z}}\mathbb{R}$. The vertices of $\mathrm{Pol}(b)$ are the points $\mu_w(b)$. The edges of $\mathrm{Pol}(b)$ are of the form $(\mu_w(b),\mu_{ws_i}(b))$ and point in root directions; indeed, if $ws_i>w$, then

$$\mu_{ws_i}-\mu_w=\varepsilon_i(\hat{\sigma}_{w^{-1}}(b))w\alpha_i.$$

Remark 3.4. In [Kam1], the second author gave an explicit combinatorial definition of MV polytopes, defined a bicrystal structure on this set of polytopes, and proved that this bicrystal is isomorphic to $B(\infty)$. Examining the definition of this crystal structure and its relationship with Lusztig data, it follows that we defined here the same set of polytopes with the same bijection with $B(\infty)$.

Part II. Mirković–Vilonen cycles

4. Background on the geometric Satake equivalence

4.1. The geometric Satake equivalence

Let $G^\vee$ denote the Langlands dual group of G. This reductive group scheme comes with a maximal torus $T^\vee$ whose cocharacter lattice is P. Our choice of positive and negative roots provide a pair of opposite Borel subgroups $B_+^\vee$ and $B_-^\vee$ in $G^\vee$; we denote their unipotent radicals by $N_+^\vee$ and $N_-^\vee$.

Let $\mathcal{O}=\mathbb{C}[[t]]$ be the ring of formal series and let $\mathcal{K}=\mathbb{C}((t))$ be its fraction field of Laurent series. As a set, the affine Grassmannian $\mathcal{G}\mathrm{r}$ of $G^\vee$ is the homogeneous space $G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$. It is the set of $\mathbb{C}$-points of a reduced projective ind-scheme over $\mathbb{C}$; see [Ku, §§13.2.12–19] and [Z] for a thorough introduction to this object.

A weight $\mu\in P$ is a cocharacter of $T^\vee$; therefore it gives a homomorphism of groups $\mathcal{K}^\times\to T^\vee(\mathcal{K})$. We denote by $t^\mu\in T^\vee(\mathcal{K})\subset G^\vee(\mathcal{K})$ the image of $t\in\mathcal{K}^\times$ under this homomorphism, and by L_μ the image of t^μ in $\mathcal{G}\mathrm{r}$. These points L_μ are the fixed points for the action of $T^\vee(\mathbb{C})$ on $\mathcal{G}\mathrm{r}$.

Given $\lambda \in P_+$, we denote by $\mathcal{G}\mathrm{r}^\lambda$ the $G^\vee(\mathcal{O})$-orbit of L_λ in $\mathcal{G}\mathrm{r}$. This is a smooth variety of dimension $2\rho^\vee(\lambda)$. The Cartan decomposition in $G^\vee(\mathcal{K})$ implies that

$$\mathcal{G}\mathrm{r} = \bigsqcup_{\lambda \in P_+} \mathcal{G}\mathrm{r}^\lambda.$$

Each $\mathcal{G}\mathrm{r}^\lambda$ can be viewed as a (parabolic) Schubert cell; its closure is obtained by adding the orbits $\mathcal{G}\mathrm{r}^\mu$ with $\mu \in P_+$ such that $\mu \leqslant \lambda$.

Lusztig observed in [Lu1] that a great deal of information about the representation $L(\lambda)$ of G is encoded in the geometry of $\overline{\mathcal{G}\mathrm{r}^\lambda}$; for instance, the dimension of $L(\lambda)$ is equal to the dimension of the intersection homology of $\overline{\mathcal{G}\mathrm{r}^\lambda}$.

Lusztig's insight can be regarded as a categorification of the classical Satake isomorphism, where $G^\vee(\mathcal{O})$-biinvariant compactly supported functions on $G^\vee(\mathcal{K})$ are replaced by $G^\vee(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}\mathrm{r}$ with coefficients in $\mathbb{C}$. Specifically, consider the category $\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ of such sheaves with finite-dimensional support. It is possible to endow $\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ with a convolution product along with suitable associativity and commutativity constraints. The total cohomology provides a fiber functor F from $\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ to the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional $\mathbb{C}$-vector spaces. By Tannakian reconstruction, F induces an equivalence of categories from $\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ to the category $\mathrm{Rep}(\overline{G})$ of finite-dimensional representations of a group scheme $\overline{G}$ over $\mathbb{C}$ such that the diagram

$$\begin{tikzcd} \mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r}) \arrow[rr, "\simeq"] \arrow[dr, "F"'] & & \mathrm{Rep}(\overline{G}) \arrow[dl] \\ & \mathrm{Vect}_{\mathbb{C}} & \end{tikzcd}$$

commutes, where the right downward arrow is the forgetful functor. The group scheme $\overline{G}$ is algebraic, connected, reductive, and its root datum is inverse to the root datum of $G^\vee$; in other words $\overline{G}$ is isomorphic to G.

This program was carried out by Ginzburg [Gi], Beilinson–Drinfeld [BD], and finally Mirković–Vilonen [MV]. We refer the reader to the latter paper for the proof.

4.2. Weight functors

As a subgroup of $G^\vee(\mathcal{K})$, the torus $T^\vee(\mathbb{C})$ acts on $\mathcal{G}\mathrm{r}$. The regular dominant weight ρ defines a homomorphism $\mathbb{C}^\times \to T^\vee(\mathbb{C})$, so provides a $\mathbb{C}^\times$-action on $\mathcal{G}\mathrm{r}$. In their proof, Mirković and Vilonen define weight functors using the hyperbolic localization functors defined by this action [Bra]. We recall part of their construction.

Given $\mu \in P$, we denote by S^μ the $N_+^\vee(\mathcal{K})$-orbit through L_μ and by S_-^μ the $N_-^\vee(\mathcal{K})$-orbit through the same point. Then, for each L in S_+^μ, resp. S_-^μ, we have

$$\lim_{a \to 0} \rho(a) \cdot L = L_\mu, \quad \text{resp.} \quad \lim_{a \to \infty} \rho(a) \cdot L = L_\mu.$$

The Iwasawa decomposition in $G^\vee(\mathcal{K})$ implies that

$$\mathcal{G}\mathrm{r} = \bigsqcup_{\mu\in P} S^\mu = \bigsqcup_{\mu\in P} S^\mu_-.$$

It follows that the points L_μ are the fixed points for our $\mathbb{C}^\times$-action and that S^μ and S^μ_- are the attractive and repulsive varieties around L_μ.

Given $\mu\in P$, we define subsets $\overline{S^\mu}$ and $\overline{S^\mu_-}$ by

$$\overline{S^\mu} = \bigsqcup_{\nu\leqslant\mu} S^\nu \quad \text{and} \quad \overline{S^\mu_-} = \bigsqcup_{\nu\geqslant\mu} S^\nu_-.$$

These are closed subsets ([MV, Proposition 3.1]). Therefore, we can factorize the inclusion map $s_\mu\colon S^\mu_- \hookrightarrow \mathcal{G}\mathrm{r}$ as the composition of an open immersion $\mathring{s}_\mu$ and a closed immersion $\bar{s}_\mu$ as follows

$$\begin{tikzcd} S^\mu_- \arrow[r, "\mathring{s}_\mu"] \arrow[rr, bend right, "s_\mu"'] & \overline{S^\mu_-} \arrow[r, "\bar{s}_\mu"] & \mathcal{G}\mathrm{r}. \end{tikzcd}$$

For any weight $\mu\in P$ and any sheaf $\mathcal{A}\in \mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$, the cohomology $H^\bullet(S^\mu_-, s^!_\mu\mathcal{A})$ is concentrated in degree $k=2\rho^\vee(\mu)$ and we have a diagram

$$\begin{tikzcd} H^k(\overline{S^\mu_-}, \bar{s}^!_\mu\mathcal{A}) \arrow[r, "(\bar{s}_\mu)_!"] \arrow[d, "(\mathring{s}_\mu)_*"', "\simeq"] & H^k(\mathcal{G}\mathrm{r}, \mathcal{A}) \\ H^k(S^\mu_-, s^!_\mu\mathcal{A}) & \end{tikzcd}$$

where the vertical arrow is an isomorphism. Further, for each $k\in\mathbb{Z}$, the maps $(\bar{s}_\mu)_!$ provide a decomposition

$$H^k(\mathcal{G}\mathrm{r}, \mathcal{A}) = \bigoplus_{\substack{\mu\in P\\ 2\rho^\vee(\mu)=k}} H^k(\overline{S^\mu_-}, \bar{s}^!_\mu\mathcal{A})$$

by [MV, Theorem 3.6]. As a consequence, the fiber functor $F=H^\bullet(\mathcal{G}\mathrm{r}, ?)$ from §4.1 decomposes as a direct sum of weight functors

$$F = \bigoplus_{\mu\in P} F_\mu$$

defined by

$$F_\mu = H^{2\rho^\vee(\mu)}(\overline{S^\mu_-}, \bar{s}^!_\mu ?) = H^\bullet(S^\mu_-, s^!_\mu ?).$$

Since this decomposition is compatible with the convolution product, it defines a homomorphism $T\to\overline{G}$ that identifies T with a maximal torus of $\overline{G}$ ([MV, p. 122]).

4.3. Action of the principal nilpotent

To understand how G acts on the spaces $F(\mathcal{A})$, we need to fix the isomorphism $\overline{G}\cong G$. For this, we use an idea of Ginzburg [Gi].

Let $\mathfrak{g}^\vee$ be the Lie algebra of $G^\vee$ and let $q\colon P\to\mathbb{Q}$ be the W-invariant quadratic form such that $q(\alpha)=1$, if α is a short root of G. Let $B\colon P\times P\to\mathbb{Q}$ be the polar form of q and let $\iota\colon P\to\mathfrak{t}$ be the map $\mu\mapsto B(\mu,\mathbf{?})$. The invariance of q under the Weyl group implies that $\iota(\alpha_i)=q(\alpha_i)\alpha_i^\vee$ for each $i\in I$.

From this data, we can construct an affine Kac–Moody Lie algebra $\widehat{\mathfrak{g}^\vee}$, as explained in [Kac, Chapter 6]. With the standard notation setup in this reference, the dual of the Cartan subalgebra of $\widehat{\mathfrak{g}^\vee}$ can be written as

$$\mathfrak{t}\oplus\mathbb{C}\delta^\vee\oplus\mathbb{C}\Lambda_0.$$

For $\mu\in P$, we set $\pi(\mu)=\Lambda_0-\iota(\mu)-q(\mu)\delta^\vee$, an element in $\tilde{\mathfrak{t}}$.

Let V denote the integrable representation of $\widehat{\mathfrak{g}^\vee}$ of highest weight Λ_0. This representation determines a homomorphism $G^\vee(\mathcal{K})\to\mathrm{PGL}(V)$ ([Ku, Proposition 13.2.4]). This can be lifted to a representation of a central extension $E(G^\vee(\mathcal{K}))$ of $G^\vee(\mathcal{K})$ by $\mathbb{C}^\times$ ([Ku, Proposition 13.2.8]). Moreover, the cocycle that defines this extension involves the tame symbol ([Ga, Theorem 12.24]); this cocycle is trivial on $\mathcal{O}$, so this extension splits over $G^\vee(\mathcal{O})$, giving a diagram

$$\begin{tikzcd}
 & & & G^\vee(\mathcal{O}) \arrow[dl, "s"'] \arrow[d, hook] & \\
1 \arrow[r] & \mathbb{C}^\times \arrow[r, "i"] & E(G^\vee(\mathcal{K})) \arrow[r, "p"] & G^\vee(\mathcal{K}) \arrow[r] & 1.
\end{tikzcd}$$

Let v denote the highest weight vector of V. It is invariant under the group $s(G^\vee(\mathcal{O}))$, so the map $g\mapsto gv$ defines an embedding $\Upsilon\colon\mathcal{G}\mathrm{r}\hookrightarrow\mathbb{P}(V)$. This embedding is a morphism of ind-varieties ([Sl, §2]). We thereby obtain a (very ample) $G^\vee(\mathcal{K})$-equivariant line bundle $\mathscr{L}=\Upsilon^*\mathscr{O}(1)$, which incidentally is known to generate the Picard group of the identity component of $\mathcal{G}\mathrm{r}$.

Formula (6.5.4) in [Kac] implies the following statement (compare with [MV, (3.2)]).

Proposition 4.1. *For each $\mu\in P$, the line $\Upsilon(L_\mu)$ is contained in the $\pi(\mu)$ weight space of V.*

The cohomology algebra $H^\bullet(\mathcal{G}\mathrm{r},\mathbb{C})$ acts by the cup-product on $F(\mathcal{A})=H^\bullet(\mathcal{G}\mathrm{r},\mathcal{A})$ for any object $\mathcal{A}\in\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$, and the action is natural in $\mathcal{A}$. In particular, the cup-product with the first Chern class $c_1(\mathscr{L})$ is an endomorphism $\bar{e}$ of the functor F. By Lemma 5.1

in [YZ], this element $c_1(\mathscr{L})$ is primitive in $H^\bullet(\mathcal{G}\mathrm{r},\mathbb{C})$, which implies that for any sheaves $\mathcal{A}$ and $\mathcal{B}$ in $\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ we have

$$\bar{e}_{\mathcal{A}*\mathcal{B}}=\bar{e}_{\mathcal{A}}\otimes\mathrm{id}_{F(\mathcal{B})}+\mathrm{id}_{F(\mathcal{A})}\otimes\bar{e}_{\mathcal{B}}$$

under the isomorphism

$$F(\mathcal{A}*\mathcal{B})\cong F(\mathcal{A})\otimes F(\mathcal{B}).$$

It follows that $\bar{e}$ belongs to the Lie algebra $\bar{\mathfrak{g}}$ of $\overline{G}$.

Now, let $\bar{h}\in\bar{\mathfrak{g}}$ be the element that acts as the multiplication by the cohomological degree on each vector space $F(\mathcal{A})$. Clearly, we have $[\bar{h},\bar{e}]=2\bar{e}$. For any $\mathcal{A}\in\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$, the hard Lefschetz theorem guarantees the existence of an endomorphism $\bar{f}_{\mathcal{A}}$ of the vector space $F(\mathcal{A})$ such that $(\bar{e}_{\mathcal{A}},\bar{h}_{\mathcal{A}},\bar{f}_{\mathcal{A}})$ is a $\mathfrak{sl}_2$ triple. Certainly $\bar{f}_{\mathcal{A}}$ is unique, hence natural in $\mathcal{A}$, and we conclude that there is a unique element $\bar{f}\in\bar{\mathfrak{g}}$ such that $(\bar{e},\bar{h},\bar{f})$ is an $\mathfrak{sl}_2$ triple (see [Z, Theorem 5.3.23]).

By the end of §4.2, T is a maximal torus of the group $\overline{G}$, so we may decompose $\bar{\mathfrak{g}}$ into root subspaces with respect to the adjoint action of T. The root system is then the root system of G. Further, $\bar{h}$ identifies with the element $2\rho^\vee\in\mathfrak{t}$, so $\langle\bar{h},\alpha_i\rangle=2$ for each simple root α_i of G. By [Bo, Chapitre 8, §11, Proposition 8], we can then write $\bar{e}=\sum_{i\in I}\bar{e}_i$, where each $\bar{e}_i$ is a non-zero root vector of weight α_i.

With all these ingredients in hand, we can fix the isomorphism $\overline{G}\cong G$ by identifying each simple root vector $\bar{e}_i\in\bar{\mathfrak{g}}$ with its counterpart $q(\alpha_i)e_i\in\mathfrak{g}$.

5. The Mirković–Vilonen basis in representations

5.1. Some more notation

In this section we recall standard facts and notation about sheaves and cycles. Throughout this paper, we will consider sheaves of $\mathbb{C}$-vector spaces. Similarly, singular cohomology, homology, and K-theory will always be considered with $\mathbb{C}$-coefficients.

Suppose that X is a complex irreducible algebraic variety of dimension d. We denote the constant sheaf on X with stalk $\mathbb{C}$ by $\mathbb{C}_X$. The Verdier dual of $\mathbb{C}_X$ is the dualizing sheaf $\mathbb{D}_X$ on X; it can be defined as either $f^!\mathbb{C}_{\mathrm{pt}}$, where $f\colon X\to\{\mathrm{pt}\}$ is the constant map, or as the sheafification of the complex of presheaves $U\mapsto C_{-\bullet}(X,X\setminus U)$ of relative singular chains. The singular cohomology of X is identified with $H^\bullet(X,\mathbb{C}_X)$; the Borel–Moore homology of X (constructed from possibly infinite singular chains with locally finite support) is identified with $H_\bullet(X):=H^{-\bullet}(X,\mathbb{D}_X)$.

Let $D_c(X)$ denote the bounded derived category of constructible sheaves of $\mathbb{C}$-vector spaces on X. We have the (contravariant) Verdier duality functor $\mathbb{D}\colon D_c(X)\to D_c(X)$ defined by $R\mathcal{H}\mathrm{om}(-,\mathbb{D}_X)$.

5.1.1. Intersection cohomology sheaf

The open subset of regular points X^{reg} is a real connected oriented manifold of dimension $2d$, so $H_{2d}(X^{\mathrm{reg}})$ is a 1-dimensional vector space spanned by the fundamental class of X^{reg}. Since $X\setminus X^{\mathrm{reg}}$ is a pseudomanifold of dimension $\leqslant 2d-2$, the restriction map $H_{2d}(X)\to H_{2d}(X^{\mathrm{reg}})$ is an isomorphism; we denote by $[X]$ the class in $H_{2d}(X)$ that restricts to the fundamental class of X^{reg} and refer to $[X]$ as the fundamental class of X. The same notation $[X]$ will also be used to denote the image (proper push-forward) in $H_{2d}(Y)$ of this class under a closed immersion $X\to Y$.

As a topological pseudomanifold, X admits a filtration with even real-dimensional strata, which allows to define unambiguously the sheaf of intersection chains $\mathrm{IC}(X)$ with respect to the middle perversity; it restricts to the shifted local system $\mathbb{C}_S[d]$ on the open stratum S. Local sections of $\mathrm{IC}(X)$ are (possibly infinite) singular chains that satisfy specific conditions relative to how they meet the lower-dimensional strata of X; forgetting these conditions gives a map

$$\mathrm{IC}(X)\longrightarrow \mathbb{D}_X[-d] \tag{2}$$

which restricts on S to the isomorphism

$$\mathrm{IC}(X)|_S\equiv \mathbb{C}_S[d]\xrightarrow{\simeq}\mathbb{D}_S[-d]\equiv \mathbb{D}_X[-d]|_S$$

given by the orientation (see [GM, §5.1]).

5.1.2. Cup and cap products

Let Z be a locally closed subset of X and let $i\colon Z\to X$ denote the inclusion. For any object $\mathcal{A}\in D_c(X)$ on X, we write $H^\bullet_Z(X,\mathcal{A}):=H^\bullet(Z,i^!\mathcal{A})$. In particular, when $\mathcal{A}=\mathbb{C}_X$, then we write $H^\bullet_Z(X):=H^\bullet(Z,i^!\mathbb{C}_X)$. This is isomorphic to the singular cohomology $H^\bullet(X,X\setminus Z)$.

Now, let $u\in H^p_Z(X)$. So $u\colon \mathbb{C}_Z\to i^!\mathbb{C}_X[p]$ and by adjunction, we can regard u as a map

$$u\colon i_!\mathbb{C}_Z\longrightarrow \mathbb{C}_X[p].$$

For any $\mathcal{A}\in D_c(X)$, we define its *cup product* with u to be the resulting map $i_!i^*\mathcal{A}\to \mathcal{A}[p]$ given by applying $\overset{L}{\otimes}\mathcal{A}$ to the map u. Taking compactly supported global sections gives us

$$u\cup\colon H^k_c(Z,i^*\mathcal{A})\longrightarrow H^{k+p}_c(X,\mathcal{A}).$$

Similarly for any $\mathcal{A}\in D_c(X)$, we define its *cap product* with u to be the resulting map $\mathcal{A}\to i_*i^!\mathcal{A}[p]$ (given by applying $R\mathcal{H}\mathrm{om}(-,\mathcal{A})[p]$ to the map u). Taking global sections gives us

$$u\cap\colon H^{-k}(X,\mathcal{A})\longrightarrow H_Z^{-k+p}(X,\mathcal{A}).$$

If we take $\mathcal{A}=\mathbb{D}_X$, we obtain the usual cap product map in Borel–Moore homology,

$$u\cap\colon H_k(X)\longrightarrow H_{k-p}(Z).$$

For any $\mathcal{A}\in D_c(X)$, if we take the cup product map

$$i_!i^*\mathcal{A}\xrightarrow{u\cup}\mathcal{A}[p]$$

and apply Verdier duality, we obtain a map

$$\mathbb{D}(\mathcal{A})[-p]\xrightarrow{\mathbb{D}(u\cup)}i_*i^!\mathbb{D}(\mathcal{A}),$$

which coincides with the map $u\cap$ shifted by $-p$, by [KSc, (2.6.7)].

These cup and cap product maps are compatible with pull-back. Let us explain this compatibility in the case of cap product. Let $f\colon Y\to X$ be a morphism and let $W=f^{-1}(Z)$. Then, we can form $f^*u\in H_W^p(Y)$ and given any $\mathcal{A}\in D_c(Y)$, we have the following commutative diagram

$$\begin{tikzcd} H^{-k}(Y,\mathcal{A}) \arrow[r, "(f^*u)\cap"] \arrow[d, "\cong"] & H_W^{-k+p}(Y,\mathcal{A}) \arrow[d, "\cong"] \\ H^{-k}(X,f_*\mathcal{A}) \arrow[r, "u\cap"] & H_Z^{-k+p}(X,f_*\mathcal{A}), \end{tikzcd} \tag{3}$$

where the vertical isomorphisms come from the composition of push-forwards (and base change for the right-hand vertical arrow).

5.1.3. Cap products and cycles

An effective way to compute cap products with Chern classes is to reduce to calculations of intersection multiplicities. We quickly recall the definitions and the basic results from Fulton's book [F] in the very specific setup that we will need. We consider a fibre square of $\mathbb{C}$-schemes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ D \arrow[r, "i"'] & Y \end{tikzcd}$$

with i the inclusion of an effective Cartier divisor and V an irreducible variety of dimension k. We assume that V is not contained in the support of D. Let Z be an irreducible component of W; it is a subvariety of V of codimension 1. The *multiplicity* of Z in the product $D\cdot V$ is defined to be the length of the module $\mathcal{O}_{W,V}/(f)$ over the local ring $\mathcal{O}_{W,V}$ of V along W, where f is a local equation of $D|_V$ on an affine open subset of V which meets Z. Following [F, Chapter 7], this multiplicity is denoted by $i(Z, D\cdot V)$.

The inclusion i is a regular embedding of codimension 1, hence has an orientation class $u\in H^2(Y, Y\setminus D)$. Concretely (see [F, §19.2]), D is the zero-locus of the canonical section $s\colon Y\to\mathcal{O}_Y(D)$ and u is the pull-back by s of the Thom class of the line bundle $\mathcal{O}_Y(D)$. Now, consider a fibre square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ D \arrow[r, "i"'] & Y. \end{tikzcd}$$

Following the above discussion we have the cap product

$$H_k(Y') \xrightarrow{(g^*u)\cap} H_{k-2}(X').$$

The following result follows from [F, Theorem 19.2].

Proposition 5.1. *Retain the above notation and assume that V is an irreducible subvariety of Y' of dimension k, not contained in $g^{-1}(D)$. Then,*

$$(g^*u)\cap[V] = \sum_Z i(Z, D\cdot V)[Z]$$

the sum being taken over all irreducible components of $V\cap g^{-1}(D)$.

5.2. Mirković–Vilonen cycles

Let $\lambda\in P_+$, fixed for the whole section. As shown by Mirković and Vilonen ([MV, Theorem 3.2]), given $\mu\in P$, the intersection $\mathcal{G}\mathrm{r}^\lambda\cap S_-^\mu$, when non-empty, has pure dimension $\rho^\vee(\lambda-\mu)$. We define an *MV cycle* Z of type λ and weight μ to be an irreducible component of $\overline{\mathcal{G}\mathrm{r}^\lambda\cap S_-^\mu}$. Equivalently, an MV cycle of type λ and weight μ is an irreducible component of $\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S_-^\mu}$ of dimension $\rho^\vee(\lambda-\mu)$. We denote the set of these cycles by $\mathcal{Z}(\lambda)_\mu$, and define

$$\mathcal{Z}(\lambda) := \bigsqcup_{\mu\in P} \mathcal{Z}(\lambda)_\mu,$$

the set of all MV cycles of type λ.

Braverman and Gaitsgory endow $\mathcal{Z}(\lambda)$ with the structure of an upper semi-normal G-crystal [BrG]. Their definition involves a geometric construction, but one can provide the following purely combinatorial short characterization ([BaG, proof of Proposition 4.3]):

• Let $\mu\in P$ and $Z\in\mathcal{Z}(\lambda)_\mu$. We set $\mathrm{wt}(Z)=\mu$. The closed subset Z is $\mathbb{C}^\times$-invariant with respect to the action defined in §4.2 and meets the repulsive cell S^μ_-, so $L_\mu\in Z$. For each $i\in I$, we can then define

$$\varepsilon_i(Z)=\max\{n\in\mathbb{N}: L_{\mu+n\alpha_i}\in Z\} \quad \text{and} \quad \varphi_i(Z)=\varepsilon_i(Z)+\langle\alpha_i^\vee,\mu\rangle.$$

• Let $\mu\in P$, $i\in I$ and $(Z,Z')\in\mathcal{Z}(\lambda)^2$. Then,

$$Z'=\tilde{e}_iZ \quad\Longleftrightarrow\quad (Z'\subseteq Z,\ \mathrm{wt}(Z')=\mathrm{wt}(Z)+\alpha_i \text{ and } \varepsilon_i(Z')=\varepsilon_i(Z)-1).$$

We denote the intersection cohomology sheaf of the Schubert variety $\overline{\mathcal{G}\mathrm{r}^\lambda}$ by $\mathcal{I}_\lambda$. The geometric Satake equivalence maps this perverse sheaf to the simple G-module $L(\lambda)$. In other words, under the identification $\overline{G}\cong G$ specified at the end of §4, there is an isomorphism $F(\mathcal{I}_\lambda)\cong L(\lambda)$, unique up to a scalar. For each $\mu\in P$, the subspace of $L(\lambda)$ of weight μ identifies with

$$F_\mu(\mathcal{I}_\lambda)=H^k(\overline{S^\mu_-},\bar{s}^!_\mu\mathcal{I}_\lambda)=H^k_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda),$$

where $k=2\rho^\vee(\mu)$.

By (2), we have a map of sheaves $\mathcal{I}_\lambda\to\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-d]$, where $d=\dim\mathcal{G}\mathrm{r}^\lambda=2\rho^\vee(\lambda)$. Denoting the inclusion of the open stratum by $j:\mathcal{G}\mathrm{r}^\lambda\to\overline{\mathcal{G}\mathrm{r}^\lambda}$, we then get a commutative diagram

$$\begin{tikzcd}
\mathcal{I}_\lambda \arrow[r] \arrow[d] & \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-d] \arrow[d] \\
j_*j^*\mathcal{I}_\lambda \arrow[r, "\simeq"] & j_*j^*\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-d]
\end{tikzcd}$$

where the vertical arrows are adjunction maps. The bottom arrow is an isomorphism because both $\mathcal{I}_\lambda[-d]$ and $\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-2d]$ restrict to the trivial local system over $\mathcal{G}\mathrm{r}^\lambda$.

Base change in the Cartesian square

$$\begin{tikzcd}
\mathcal{G}\mathrm{r}^\lambda\cap\overline{S^\mu_-} \arrow[r, "j'"] \arrow[d, "\bar{s}'_\mu"'] & \overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-} \arrow[r] \arrow[d] & \overline{S^\mu_-} \arrow[d, "\bar{s}_\mu"] \\
\mathcal{G}\mathrm{r}^\lambda \arrow[r, "j"'] & \overline{\mathcal{G}\mathrm{r}^\lambda} \arrow[r] & \mathcal{G}\mathrm{r}
\end{tikzcd}$$

gives the isomorphism $\bar{s}_\mu^! j_* j^* \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}} \cong j'_* \mathbb{D}_{\mathcal{G}\mathrm{r}^\lambda \cap \overline{S_-^\mu}}$. Applying the functor $H^k(\overline{S_-^\mu}, \bar{s}_\mu^! \mathbf{?})$ to the commutative diagram above then yields

$$\begin{tikzcd}
F_\mu(\mathcal{I}_\lambda) = H^k(\overline{S_-^\mu}, \bar{s}_\mu^! \mathcal{I}_\lambda) \arrow[r] \arrow[d, "\simeq"'] & H^{k-d}(\overline{S_-^\mu}, \bar{s}_\mu^! \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}) \arrow[r, equal] \arrow[d] & H_{d-k}(\overline{\mathcal{G}\mathrm{r}^\lambda} \cap \overline{S_-^\mu}, \mathbb{C}) \arrow[d, "\simeq"] \\
H^k(\overline{S_-^\mu}, \bar{s}_\mu^! j_* j^* \mathcal{I}_\lambda) \arrow[r, "\simeq"] & H^{k-d}(\overline{S_-^\mu}, j'_* \mathbb{D}_{\mathcal{G}\mathrm{r}^\lambda \cap \overline{S_-^\mu}}) \arrow[r, equal] & H_{d-k}(\mathcal{G}\mathrm{r}^\lambda \cap \overline{S_-^\mu}, \mathbb{C}).
\end{tikzcd}$$

Here, the left vertical arrow is an isomorphism, as shown by Mirković and Vilonen ([MV, proof of Proposition 3.10]). The right vertical arrow is also an isomorphism, because each irreducible component of $\overline{\mathcal{G}\mathrm{r}^\lambda} \cap \overline{S_-^\mu}$ of dimension $\frac{1}{2}(d-k) = \rho^\vee(\lambda - \mu)$ meets the open stratum $\mathcal{G}\mathrm{r}^\lambda$. In fact, these irreducible components are precisely the MV cycles of type λ and of weight μ.

We denote by $[Z] \in H_{d-k}(\overline{\mathcal{G}\mathrm{r}^\lambda} \cap \overline{S_-^\mu}, \mathbb{C})$ the fundamental class of such an MV cycle Z. The set $\{[Z] : Z \in \mathcal{Z}(\lambda)_\mu\}$ is then a basis of the weight space $F_\mu(\mathcal{I}_\lambda)$. Gathering these bases for all possible weights, we obtain a basis of $F(\mathcal{I}_\lambda)$ indexed by $\mathcal{Z}(\lambda)$, which we can transport to $L(\lambda)$ by normalizing the isomorphism $F(\mathcal{I}_\lambda) \cong L(\lambda)$ in such a way that the highest weight vectors $[\{L_\lambda\}]$ and v_λ match.

The basis of $L(\lambda)$ obtained in this manner is called the *MV basis*. (By analogy with the case of the global basis and following Kashiwara's terminology, we should more accurately call it the upper MV basis.) The following result is (up to duality) Proposition 4.1 in [BrG].

THEOREM 5.2. *The MV basis is perfect.*

We will give our own proof of this result, which provides some more refined information needed in the sequel. The first step in the proof is carried out in §5.3, where we establish a formula that expresses in geometrical terms the action of a Chevalley generator e_i on a basis element $[Z]$. This formula has the form

$$e_i \cdot [Z] = \sum_{Z' \in \mathcal{Z}(\lambda)} a_{Z'} [Z'],$$

where the coefficient $a_{Z'}$ is non-zero only if

$$\mathrm{wt}(Z') = \mathrm{wt}(Z) + \alpha_i \quad \text{and} \quad Z' \subseteq Z.$$

Thus, for $[Z']$ to actually appear in $e_i \cdot [Z]$, it is necessary that $L_{\mathrm{wt}(Z') + \varepsilon_i(Z')\alpha_i} \in Z$, which in turn implies that $\varepsilon_i(Z) \geqslant \varepsilon_i(Z') + 1$. If moreover the latter relation is an equality, then necessarily $Z' = \tilde{e}_i Z$, by the characterization of the crystal structure on $\mathcal{Z}(\lambda)$ given above. At this point, it remains to show that if $\tilde{e}_i Z \neq 0$, then the coefficient $a_{\tilde{e}_i Z}$ is equal to $\varepsilon_i(Z)$. We perform this computation in §5.4.

Remark 5.3. As shown by Berenstein and Kazhdan [BeK], the crystal of a perfect basis of the highest weight module $L(\lambda)$ is independent of the choice of the basis. Therefore, the crystals of the MV basis and of the upper global basis of $L(\lambda)$ (specialized at $q{=}1$) are isomorphic. This observation provides another proof of Braverman and Gaitsgory's theorem [BrG] that states that the crystals $\mathcal{Z}(\lambda)$ are isomorphic to Kashiwara's normal crystals $B(\lambda)$.

5.3. Action of e_i on an MV cycle

Recall the notation setup in §4.3 and the statement of Proposition 4.1.

Fix $\mu{\in}P$ and pick a linear form f on V which is non-zero on the line $\Upsilon(L_\mu)$ and which vanishes on all weight subspaces of V of weight other than $\pi(\mu)$. Let $D{\subseteq}\mathcal{G}\mathrm{r}$ be the Cartier divisor defined as the intersection of $\mathcal{G}\mathrm{r}$ with the hyperplane in $\mathbb{P}(V)$ defined by f. Proposition 3.1 in [MV] tells us that

$$D\cap\overline{S^{\mu}_{-}}=\bigcup_{i\in I}\overline{S^{\mu+\alpha_i}_{-}}.$$

THEOREM 5.4. *Let $\lambda{\in}P_+$, let $i{\in}I$, and let $Z{\in}\mathcal{Z}(\lambda)_\mu$. Let*

$$e_i{\cdot}[Z]=\sum_{Z'\in\mathcal{Z}(\lambda)}a_{Z'}[Z']$$

be the expansion of the left-hand side in the MV basis of $L(\lambda)$. Then,

$$q(\alpha_i)a_{Z'}=\begin{cases} i(Z',D{\cdot}Z), & \text{if } \mathrm{wt}(Z')=\mathrm{wt}(Z)+\alpha_i \text{ and } Z'\subseteq Z,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Regarding $\mathcal{G}\mathrm{r}$ as the zero section of the total space L of the line bundle $\mathscr{L}{=}\Upsilon^*\mathscr{O}(1)$, we can consider the Thom class $\tau{\in}H^2_{\mathcal{G}}\mathrm{r}(L)$. Regarding f as a continuous map from $\mathcal{G}\mathrm{r}$ to L such that $f(\mathcal{G}\mathrm{r}\backslash D){\subseteq}L\backslash\mathcal{G}\mathrm{r}$, we can form $f^*\tau{\in}H^2_D(\mathcal{G}\mathrm{r})$. With these notations, each perverse sheaf $\mathcal{A}{\in}\mathrm{P}_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ gives rise to a diagram

$$\begin{tikzcd}
H^\bullet_{\overline{S^{\mu}_{-}}}(\mathcal{G}\mathrm{r},\mathcal{A}) \arrow[r] \arrow[d, "\cup f^*\tau"'] & H^\bullet(\mathcal{G}\mathrm{r},\mathcal{A}) \arrow[d, "\cup f^*\tau"'] \arrow[dr, "\cup c_1(\mathscr{L})"] & \\
H^\bullet_{D\cap\overline{S^{\mu}_{-}}}(\mathcal{G}\mathrm{r},\mathcal{A}) \arrow[r] & H^\bullet_D(\mathcal{G}\mathrm{r},\mathcal{A}) \arrow[r] & H^\bullet(\mathcal{G}\mathrm{r},\mathcal{A}),
\end{tikzcd}$$

which commutes following [I, II.10.2 and II.10.4].

Now let $\lambda\in P_+$ and set $\mathcal{A}=\mathcal{I}_\lambda$, $k=2\rho^\vee(\mu)$, and $d=2\rho^\vee(\lambda)$. Similarly to the isomorphism

$$H^k_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\xrightarrow{\simeq}H^{k-d}_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}})=H_{d-k}(\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-},\mathbb{C})$$

obtained in §5.2, we have an isomorphism

$$H^{k+2}_{D\cap\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\xrightarrow{\simeq}H^{k+2-d}_{D\cap\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}})=H_{d-k-2}(D\cap\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-},\mathbb{C}).$$

We then get a commutative diagram

$$\begin{tikzcd}
H^k_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda) \arrow[r, "\simeq"] \arrow[d, "\cup f^*\tau"'] & H^{k-d}_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}) \arrow[r, equal] \arrow[d, "\cup f^*\tau"'] & H_{d-k}(\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-},\mathbb{C}) \arrow[d, "\cap f^*\tau"] \\
H^{k+2}_{D\cap\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda) \arrow[r, "\simeq"] & H^{k+2-d}_{D\cap\overline{S^\mu_-}}(\mathcal{G}\mathrm{r},\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}) \arrow[r, equal] & H_{d-k-2}(D\cap\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-},\mathbb{C}).
\end{tikzcd}$$

Let $Z\in\mathcal{Z}(\lambda)_\mu$ and let $[Z]$ be its fundamental class in $H_{d-k}(\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-},\mathbb{C})$. The two commutative diagrams above and the explanations in §4.3 show that $[Z]\cap f^*\tau$ is the result of the action on $[Z]$ of the principal nilpotent $\tilde{e}$. On the other hand, $f^*\tau$ is the orientation class u of the regular embedding $D\to\mathcal{G}\mathrm{r}$, so $[Z]\cap f^*\tau$ is the homology class of the cycle $D\cdot Z$, by Proposition 5.1 applied to the fibre square

$$\begin{tikzcd}
D\cap\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-} \arrow[r] \arrow[d] & \overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^\mu_-} \arrow[d] \\
D \arrow[r] & \mathcal{G}\mathrm{r}.
\end{tikzcd}$$

Now, any irreducible component of $D\cap Z$ must be contained in $\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S^{\mu+\alpha_i}_-}$ for some $i\in I$; being of dimension

$$\dim Z-1=\rho^\vee(\lambda-\mu-\alpha_i),$$

it is then of the form Z' with $Z'\in\mathcal{Z}(\lambda)_{\mu+\alpha_i}$. We eventually obtain

$$\bar{e}\cdot[Z]=\sum_{i\in I}\sum_{Z'\in\mathcal{Z}(\lambda)_{\mu+\alpha_i}}a_{Z'}[Z'],$$

where

$$a_{Z'}=\begin{cases} i(Z',D\cdot Z), & \text{if } Z'\subseteq Z,\\ 0, & \text{otherwise.}\end{cases}$$

The claimed formula follows by isolating the contributions of the different summands in

$$\bar{e}=\sum_{i\in I}q(\alpha_i)e_i. \qquad \square$$

5.4. Computation of the leading coefficient

PROPOSITION 5.5. *Adopt the notation of Theorem 5.4 and assume that* $\tilde{e}_i Z \neq 0$. *Then,*

$$i(\tilde{e}_i Z, D\cdot Z) = q(\alpha_i)\varepsilon_i(Z).$$

Proof. Let $s_i^\vee$ be a lift in $G^\vee$ of the simple reflection s_i. The weight $s_i\rho\colon \mathbb{C}^\times \to T^\vee(\mathbb{C})$ defines an action of $\mathbb{C}^\times$ on $\mathcal{G}\mathrm{r}$. With respect to this action, the repulsive cell around the fixed point L_ν is the subset $s_i^\vee(S_-^{s_i\nu})$.

Let $x_i^\vee$ be the additive 1-parameter subgroup corresponding to the simple root $\alpha_i^\vee$ of $(G^\vee, T^\vee)$; it defines a homomorphism $\mathcal{K} \to G^\vee(\mathcal{K})$. Let $N_{-,i}^\vee$ be the unipotent radical of the parabolic subgroup of $G^\vee$ generated by $N_-^\vee$ and by the image of $x_i^\vee$. The subgroup generated by $N_{-,i}^\vee$ and the image of $x_i^\vee$ is then the maximal unipotent subgroup $s_i^\vee N_-^\vee(\mathcal{K})(s_i^\vee)^{-1}$ of $G^\vee(\mathcal{K})$. We can lift it to the central extension $E(G^\vee(\mathcal{K}))$ so as to make it act on V, and the embedding $\Upsilon\colon \mathcal{G}\mathrm{r} \to \mathbb{P}(V)$ is equivariant for this action.

After these preliminaries, let us genuinely start the proof. Let

$$m = \varepsilon_i(Z), \quad \nu = \mu + m\alpha_i, \quad r = \langle \alpha_i^\vee, \nu\rangle, \quad Z' = \tilde{e}_i^m Z,$$
$$\dot{Z} = Z \cap \mathcal{G}\mathrm{r}^\lambda \cap s_i^\vee(S_-^{s_i\nu}), \quad \dot{Z}' = Z' \cap \mathcal{G}\mathrm{r}^\lambda \cap s_i^\vee(S_-^{s_i\nu}),$$
$$\mathbb{C}[t^{-1}]_m^+ = \{a_{-m}t^{-m} + a_{1-m}t^{1-m} + \ldots + a_{-1}t^{-1} : (a_{-m}, a_{1-m}, \ldots, a_{-1}) \in \mathbb{C}^m\}.$$

From [BaG, Proposition 4.5 (ii)], we see that the assignment $(p, y) \mapsto x_i^\vee(\mathrm{pt}^r)y$ defines a homeomorphism $F\colon \mathbb{C}[t^{-1}]_m^+ \times \dot{Z}' \to \dot{Z}$; it is even an isomorphism of algebraic varieties. The cycle $\tilde{e}_i Z$ is a divisor in Z; its local equation in the open subset $\dot{Z}$ is $a_{-m} = 0$.

Our goal now is to evaluate the 'equation' $f \circ \Upsilon$ of the divisor D at a point $F(p, y)$. (We put quotation marks around the word 'equation' because $f \circ \Upsilon$ is a section of $\mathscr{L}$, not a function.)

We write $y \in \dot{Z}'$ in the form $y = gL_\nu$ with $g \in N_{-,i}^\vee(\mathcal{K})$. We write $\Upsilon(L_\nu) = \mathbb{C}v_0$ with $v_0 \in V$ of weight $\pi(\nu)$ and we decompose $v = gv_0$ as a sum of weight vectors $v = v_0 + \ldots + v_N$, with v_j of weight $\pi(\nu) - \beta_j - c_j\delta^\vee$, where $\beta_0 = c_0 = 0$ and $\beta_j \notin \mathbb{Z}\alpha_i^\vee$ for $j \neq 0$. We put

$$p = a_{-m}t^{-m} + a_{1-m}t^{1-m} + \ldots + a_{-1}t^{-1},$$

and we expand the product

$$x_i^\vee(\mathrm{pt}^r)v = \prod_{\ell=1}^{m}\Bigg(\sum_{n_\ell \geqslant 0} \frac{a_{-\ell}^{n_\ell}}{n_\ell!} X_{\alpha_i^\vee + (r-\ell)\delta^\vee}^{n_\ell}\Bigg)v,$$

where $X_{\alpha_i^\vee+q\delta^\vee}\in\widehat{\mathfrak{g}^\vee}$ is the derivative at zero of the additive subgroup $a\mapsto x_i^\vee(at^q)$ of $G^\vee(\mathcal{K})$. The linear form f vanishes on

$$\left(\prod_{\ell=1}^{m} X_{\alpha_i^\vee+(r-\ell)\delta^\vee}^{n_\ell}\right)v_j,$$

except if

$$\sum_{\ell=1}^{m} n_\ell(\alpha_i^\vee+(r-\ell)\delta^\vee)+\pi(\nu)-\beta_j-c_j\delta^\vee=\pi(\mu),$$

which can be rewritten as a system of two equations

$$\left(\sum_{\ell=1}^{m} n_\ell\right)\alpha_i^\vee-mq(\alpha_i)\alpha_i^\vee-\beta_j=0 \quad \text{and} \quad \sum_{\ell=1}^{m}(r-\ell)n_\ell-c_j=m(r-m)q(\alpha_i).$$

The first one requires that $\beta_j\in\mathbb{Z}\alpha_i^\vee$, and hence that $j=0$; the condition then becomes

$$\sum_{\ell=1}^{m} n_\ell=mq(\alpha_i) \quad \text{and} \quad \sum_{\ell=1}^{m}\ell n_\ell=m^2q(\alpha_i),$$

which, in turn, is equivalent to $n_1=...=n_{m-1}=0$ and $n_m=mq(\alpha_i)$. To sum up, we have

$$f(x_i^\vee(\mathrm{pt}^r)v)=\frac{a_{-m}^n}{n!}f(X_{\alpha_i^\vee+(r-m)\delta^\vee}^n v_0),$$

with $n=mq(\alpha_i)$.

Now, let g be another linear form on V, which is non-zero on the line $\Upsilon(L_\nu)$ and which vanishes on all weight subspaces of V of weight other than $\pi(\nu)$. Then, $g\circ\Upsilon$ does not identically vanish on $\dot{Z}$, because

$$g(x_i^\vee(\mathrm{pt}^r)v)=g(v_0)\neq 0.$$

Therefore, the rational function $(f/g)\circ\Upsilon$ has no poles on $\dot{Z}$, and has value

$$a_{-m}^n\frac{f(X_{\alpha_i^\vee+(r-m)\delta^\vee}^n v_0)}{n!g(v_0)}$$

at the point $F(p,y)$. The second factor does not depend on (p,y), and hence is a non-zero constant. We thus see that the local equation $(f/g)\circ\Upsilon$ of the divisor D vanishes along $\tilde{e}_iZ$ with multiplicity $n=q(\alpha_i)\varepsilon_i(Z)$, as asserted. □

6. The Mirković–Vilonen basis of $\mathbb{C}[N]$

6.1. Stabilization

Let $\nu\in Q_+$. Then, the subset $S_+^0\cap S_-^{-\nu}$ of $\mathcal{G}\mathrm{r}$ is non-empty and has pure dimension $\rho^\vee(\nu)$. We define a *stable MV cycle* of weight $-\nu$ to be an irreducible component of $\overline{S_+^0\cap S_-^{-\nu}}$, and we denote the set of these cycles by $\mathcal{Z}(\infty)_{-\nu}$. We further define the set of all stable MV cycles

$$\mathcal{Z}(\infty)=\bigsqcup_{\nu\in Q_+}\mathcal{Z}(\infty)_{-\nu}.$$

Let $\lambda\in P_+$. By [A, Proposition 3], for any weight $\mu\in P$, the irreducible components of $\overline{\mathcal{G}\mathrm{r}^\lambda\cap S_-^\mu}$ are the irreducible components of $\overline{S_+^\lambda\cap S_-^\mu}$ that are contained in $\overline{\mathcal{G}\mathrm{r}^\lambda}$. Additionally, the action of t^λ on $\mathcal{G}\mathrm{r}$ induces an isomorphism $\overline{S_+^0\cap S_-^{-\nu}}\xrightarrow{\simeq}\overline{S_+^\lambda\cap S_-^{\lambda-\nu}}$. It follows that the assignment $Z\mapsto t^\lambda Z$ provides a bijection

$$\{Z\in\mathcal{Z}(\infty):t^\lambda Z\subseteq\overline{\mathcal{G}\mathrm{r}^\lambda}\}\xrightarrow{\simeq}\mathcal{Z}(\lambda).$$

Recall that, for each $\lambda\in P_+$, we defined in §5.2 the MV basis $\{[Z]:Z\in\mathcal{Z}(\lambda)\}$ of the representation $L(\lambda)$.

Proposition 6.1. *The MV bases of the simple representations $L(\lambda)$ form a coherent family of perfect bases in the sense of Definition 2.11. More precisely, for each $Z\in\mathcal{Z}(\infty)$, there exists a unique element $b_Z\in\mathbb{C}[N]$ such that, for any $\lambda\in P_+$, we have*

$$t^\lambda Z\subseteq\overline{\mathcal{G}r^\lambda}\quad\Longrightarrow\quad b_Z=\Psi_\lambda([t^\lambda Z]).$$

Proof. For each $Z\in\mathcal{Z}(\infty)$, there exists $\lambda\in P_+$ such that $t^\lambda Z\subseteq\overline{\mathcal{G}\mathrm{r}^\lambda}$ ([A, Propositions 4 and 7]). The crux of the proof is to show that $\Psi_\lambda([t^\lambda Z])$ does not depend on the choice of λ.

Let $\lambda,\mu\in P_+$. The orbit $N_+^\vee(\mathcal{O})\cdot L_\lambda$ is dense in $\mathcal{G}\mathrm{r}^\lambda$ ([MV, proof of Theorem 3.2]), so

$$t^\mu\cdot\overline{\mathcal{G}\mathrm{r}^\lambda}=t^\mu\,\overline{N_+^\vee(\mathcal{O})\cdot L_\lambda}\subseteq\overline{N_+^\vee(\mathcal{O})t^\mu\cdot L_\lambda}=\overline{N_+^\vee(\mathcal{O})\cdot L_{\lambda+\mu}}=\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}.$$

As a consequence,

$$\{Z\in\mathcal{Z}(\infty):t^\lambda Z\subseteq\overline{\mathcal{G}\mathrm{r}^\lambda}\}\subseteq\{Z\in\mathcal{Z}(\infty):t^{\lambda+\mu}Z\subseteq\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}\},$$

which shows that the assignment $Z\mapsto t^\mu Z$ defines an injection $\mathcal{Z}(\lambda)\hookrightarrow\mathcal{Z}(\lambda+\mu)$.

Let $\psi\colon L(\lambda)\to L(\lambda+\mu)$ be the linear extension of this injection; in other words, ψ is the linear map that sends an element $[Z]$ of the MV basis of $L(\lambda)$ to the element $[t^\mu Z]$ of the MV basis of $L(\lambda+\mu)$. By construction, ψ raises the weight by μ and maps v_λ to $v_{\lambda+\mu}$. We claim that it intertwines the actions of N on $L(\lambda)$ and $L(\lambda+\mu)$.

To see this, let $\nu\in P$, let f be a linear form on V which is non-zero on the line $\Upsilon(L_\nu)$ and which vanishes on all weight subspaces of V of weight other than $\pi(\nu)$, and let $D\subseteq\mathcal{G}\mathrm{r}$ be the Cartier divisor defined as the intersection of $\mathcal{G}\mathrm{r}$ with the hyperplane in $\mathbb{P}(V)$ of equation f. Then, the linear form g on V defined by $g(v)=f(t^{-\mu}v)$ is non-zero on the line $\Upsilon(L_{\nu+\mu})$ and vanishes on all weight subspaces of V of weight other than $\pi(\nu+\mu)$, and $t^\mu D$ is the Cartier divisor of equation g. Thus, for any $Z\in\mathcal{Z}(\lambda)_\nu$ and $Z'\in\mathcal{Z}(\lambda)_{\nu+\alpha_i}$, we have

$$i(t^\mu Z',(t^\mu D)\cdot(t^\mu Z))=i(Z',D\cdot Z),$$

which implies (Theorem 5.4) that ψ intertwines the actions of each e_i.

Thus, the map

$$(\Psi_\lambda-\Psi_{\lambda+\mu}\circ\psi)\colon L(\lambda)\longrightarrow\mathbb{C}[N]$$

is an homomorphism of N-modules, which lowers the weight by λ and annihilates v_λ. Its image is therefore an N-invariant subspace of the augmentation ideal

$$\bigoplus_{\nu\in Q_+\setminus\{0\}}\mathbb{C}[N]_{-\nu}=\{f\in\mathbb{C}[N]:f(1_N)=0\}$$

of $\mathbb{C}[N]$. Consequently, this image is zero, and therefore $\Psi_\lambda=\Psi_{\lambda+\mu}\circ\psi$. We conclude that for any $Z\in\mathcal{Z}(\infty)$ satisfying $t^\lambda Z\subseteq\overline{\mathcal{G}\mathrm{r}^\lambda}$, we have

$$\Psi_\lambda([t^\lambda Z])=\Psi_{\lambda+\mu}([t^{\lambda+\mu}Z]).\qquad\square$$

Thus, the elements b_Z constructed in Proposition 6.1 form a perfect basis of $\mathbb{C}[N]$, which we call the MV basis of $\mathbb{C}[N]$, for it is obtained by gluing the MV bases of the representations $L(\lambda)$. By Proposition 2.7, the crystal structure on the indexing set $\mathcal{Z}(\infty)$ can be characterized by its restrictions to the sets $\mathcal{Z}(\lambda)$, where it must coincide (up to a shift in the weight map) with the structure that we used in §5. Comparing the constructions in [BrG] and [BFG], we see that this crystal structure is the one defined in this latter reference.

6.2. Biperfectness

Our aim in this section is to prove the following result.

THEOREM 6.2. *The MV basis of* $\mathbb{C}[N]$ *is biperfect.*

As a first step in the proof, we need to endow $\mathcal{Z}(\infty)$ with operators $(\varepsilon_i^*,\varphi_i^*,\tilde{e}_i^*,\tilde{f}_i^*)$ that provide the structure of a bicrystal. We do this with the help of a weight-preserving involution, which we construct as follows.

We choose an involutive antiautomorphism $g\mapsto g^\tau$ of $G^\vee$ that fixes pointwise the torus $T^\vee$ and exchanges any root subgroup with its opposite root subgroup. The automorphism $g\mapsto(g^\tau)^{-1}$ of the group $G^\vee(\mathcal{K})$ leaves $G^\vee(\mathcal{O})$ stable, hence induces an automorphism of $\mathcal{G}\mathrm{r}$ which we denote by $x\mapsto x^\dagger$. It is easy to see that $L_\mu^\dagger=L_{-\mu}$ and that $(S^\mu)^\dagger=S_-^{-\mu}$ for each weight $\mu\in P$. Given $\nu\in Q_+$ and $Z\in\mathcal{Z}(\infty)_{-\nu}$, we set $Z^*=(t^\nu Z)^\dagger$, another element in $\mathcal{Z}(\infty)_{-\nu}$. The map $Z\mapsto Z^*$ is involutive. We can now set

$$\varepsilon_i^*(Z)=\varepsilon_i(Z^*),\quad \varphi_i^*(Z)=\varphi_i(Z^*),\quad \tilde{e}_i^*Z=(\tilde{e}_i(Z^*))^*\quad\text{and}\quad \tilde{f}_i^*Z=(\tilde{f}_i(Z^*))^*$$

for each $i\in I$ and $Z\in\mathcal{Z}(\infty)$.

Remark 6.3. The involution $Z\mapsto Z^*$ on the set $\mathcal{Z}(\infty)$ was first considered by Braverman, Finkelberg and Gaitsgory [BFG], who show that $\mathcal{Z}(\infty)\cong B(\infty)$ as bicrystals. Theorem 6.2 provides an independent proof of the existence of such an isomorphism.

We extend the assignment $b_Z\mapsto b_{Z^*}$ to a linear bijection of the vector space $\mathbb{C}[N]$, which we denote by $f\mapsto f^*$.

LEMMA 6.4. *Let $i\in I$ and let u be the endomorphism of the vector space $\mathbb{C}[N]$ such that $u(f)=(e_i\cdot(f^*))^*$ for each $f\in\mathbb{C}[N]$. Then, u commutes with the left action of $\mathfrak{n}$ on $\mathbb{C}[N]$.*

Proof. We recall the formula that gives the left action of the Chevalley generators e_j on the basis elements of $\mathbb{C}[N]$. Fix $\nu\in Q_+$, pick a linear form f on the vector space V which is non-zero on the line $\Upsilon(L_{-\nu})$ and which vanishes on all weight subspaces of V of weight other than $\pi(-\nu)$, and define D to be the Cartier divisor defined as the intersection of $\mathcal{G}\mathrm{r}$ with the hyperplane in $\mathbb{P}(V)$ of equation f. Then, for each $j\in I$ and each $Z\in\mathcal{Z}(\infty)_{-\nu}$, we have

$$q(\alpha_j)e_j\cdot b_Z=\sum_{Z'\in\mathcal{Z}(\infty)_{-\nu+\alpha_j}} i(Z',D\cdot Z)b_{Z'}. \tag{4}$$

The definition of the involution $*$ leads to a similar formula for the endomorphism u. As a matter of fact, for each $Z\in\mathcal{Z}(\infty)_{-\nu}$ and $Z'\in\mathcal{Z}(\infty)_{-\nu+\alpha_i}$, we have

$$i((Z')^*,D\cdot Z^*)=i((t^{\nu-\alpha_i}Z')^\dagger,D\cdot(t^\nu Z)^\dagger;\mathcal{G}\mathrm{r})=i((t^{-\alpha_i}Z')^\dagger,(t^\nu D)\cdot Z^\dagger;\mathcal{G}\mathrm{r}).$$

Defining a Cartier divisor D' on $\mathcal{G}\mathrm{r}$ by the formula $D'=(t^\nu D)^\dagger$, we then get

$$q(\alpha_i)u(b_Z)=\sum_{Z'\in\mathcal{Z}(\infty)_{-\nu+\alpha_i}} i(t^{-\alpha_i}Z',D'\cdot Z)b_{Z'}.$$

Using [F, Corollary 2.4.2], we obtain

$$e_j\cdot u(b_Z)=\frac{1}{q(\alpha_i)q(\alpha_j)}\sum_{Z'\in\mathcal{Z}(\infty)_{-\nu+\alpha_i+\alpha_j}} i(t^{-\alpha_i}Z',D\cdot D'\cdot Z)b_{Z'}=u(e_j\cdot b_Z)$$

for each $j\in I$ and each $Z\in\mathcal{Z}(\infty)_{-\nu}$. $\square$

Remark 6.5. The left action of the Chevalley generator e_j on a basis element b_Z is obtained by intersecting Z with the divisor D so as to jettison the "bottom" part of Z. Similarly, the action of u on b_Z amounts to intersecting Z with the divisor D' so as to jettison the "top" part of Z. Thus, the lemma merely reformulates in a representation theoretic language the general fact that $D\cdot(D'\cdot Z)=D'\cdot(D\cdot Z)$.

PROPOSITION 6.6. *The involution $f\mapsto f^*$ of $\mathbb{C}[N]$ exchanges the left and the right actions of the Chevalley generators e_i.*

Proof. Let $i\in I$ and let u be the endomorphism of $\mathbb{C}[N]$ defined in Lemma 6.4. By construction, $u(\mathbb{C}[N]_{-\nu})\subseteq\mathbb{C}[N]_{-\nu+\alpha_i}$ for each $\nu\in Q_+$. Therefore, the dual of u can be restricted to an endomorphism v of the graded dual of $\mathbb{C}[N]$, namely $U(\mathfrak{n})$, and v is of degree α_i. Lemma 6.4 implies that v commutes with the right action of $\mathfrak{n}$ on $U(\mathfrak{n})$, so v is the left multiplication by an element of $U(\mathfrak{n})$. For degree reasons, this element is of the form λe_i with $\lambda\in\mathbb{C}$. Thus, u is the right action of λe_i on $\mathbb{C}[N]$.

The set $\mathcal{Z}(\infty)_{-\alpha_i}$ contains just one element—indeed $\overline{S_+^0\cap S_-^{-\alpha_i}}$ is a Riemann sphere, and hence is irreducible. Denote this element by Z_i; it is fixed by the involution $*$. Then, the basis element b_{Z_i} spans the weight subspace $\mathbb{C}[N]_{-\alpha_i}$, and by construction of the pairing between $U(\mathfrak{n})$ and $\mathbb{C}[N]$, we have $e_i\cdot b_{Z_i}=b_{Z_i}\cdot e_i$ (see §2.1; in fact, this is the constant function on N equal to 1). Now, on the one hand we have $u(b_{Z_i})=b_{Z_i}\cdot(\lambda e_i)$, and on the other hand, since $b_{Z_i}=(b_{Z_i})^*$, we have

$$u(b_{Z_i})=(e_i\cdot b_{Z_i})^*=e_i\cdot b_{Z_i}.$$

Therefore, $\lambda=1$, and we conclude that u is the right action of e_i on $\mathbb{C}[N]$.

Thus, the involution $f\mapsto f^*$ exchanges the left and the right actions of each Chevalley generator e_i. □

Proof of Theorem 6.2. Note that our constructions ensure that the involution $f\mapsto f^*$ exchanges the crystal structures $(\mathrm{wt},\varepsilon_i,\varphi_i,\tilde{e}_i,\tilde{f}_i)$ and $(\mathrm{wt},\varepsilon_i^*,\varphi_i^*,\tilde{e}_i^*,\tilde{f}_i^*)$, as well as the left and right actions of the Chevalley generators. Since the MV basis of $\mathbb{C}[N]$ is perfect, it is biperfect. □

Remark 6.7. The involution $f\mapsto f^*$ of $\mathbb{C}[N]$ is dual to the involutive algebra anti-automorphism of $U(\mathfrak{n})$ that fixes the Chevalley generators e_i. One can also easily show that $f\mapsto f^*$ is the automorphism of the algebra $\mathbb{C}[N]$ induced by the automorphism $n\mapsto(-1)^{\rho^\vee}n^{-1}(-1)^{-\rho^\vee}$ of the variety N, where $(-1)^{\rho^\vee}$ is the evaluation at -1 of the cocharacter $\rho^\vee$ of T.

6.3. MV polytopes from MV cycles

For each T-invariant closed subvariety $Z\subset\mathcal{G}\mathrm{r}$ (for example an MV cycle or stable MV cycle), we define

$$\mathrm{Pol}(Z):=\mathrm{Conv}\{\mu:L_\mu\in Z\}.$$

For any such Z and any $\gamma\in P$, note that $\mathrm{Pol}(t^\gamma Z)=\gamma+\mathrm{Pol}(Z)$.

The MV basis is biperfect (Theorem 6.2) and is indexed by the set $\mathcal{Z}(\infty)$ of stable MV cycles, so by §2.4 we get a canonical bijection $\mathcal{Z}(\infty)\cong B(\infty)$. The construction in §3.3 allows us to represent elements in $B(\infty)$ by MV polytopes. Therefore, we get a resulting bijection from $\mathcal{Z}(\infty)$ onto the set of MV polytopes. By Remark 3.4 and [Kam1, Theorem 4.7], this bijection is given by $Z\mapsto\mathrm{Pol}(Z)$.

7. Multiplication

7.1. Generalities on cosheaves

In this section, we will work with cosheaves, which we define as Verdier duals of sheaves.(2) Let X be an irreducible complex algebraic variety of dimension d.

Definition 7.1. • A (*constructible*) *cosheaf* on X is an object $\mathcal{A}$ of $D_c(X)$ which is isomorphic to $\mathbb{D}(\mathcal{F})$ for some sheaf $\mathcal{F}$.

• The *costalk* of a cosheaf $\mathcal{A}$ at a point x is the vector space $i_x^!\mathcal{A}$.

• A cosheaf $\mathcal{A}$ is *coconstant* if it is isomorphic to the Verdier dual of a constant sheaf. It is *coconstant along* V if $i_V^!(\mathcal{A})$ is coconstant as a cosheaf on V. (Here $V\subset X$ is a constructible subset and $i_V:V\to X$ is the inclusion.)

The cosheaves on X form an abelian category $\mathrm{CoSh}_c(X)$ which is the heart of a t-structure on $D_c(X)$; this t-structure is obtained by applying Verdier duality to the standard t-structure. We have cohomology functors $\widetilde{\mathcal{H}}^k:D_c(X)\to\mathrm{CoSh}_c(X)$ which are defined by $\widetilde{\mathcal{H}}^k(\mathcal{A})=\mathbb{D}(\mathcal{H}^{-k}(\mathbb{D}(\mathcal{A})))$, where $\mathcal{H}^k$ denotes the usual cohomology functor.

For any morphism $f:X\to Y$, $f^!$ is exact with respect to this t-structure and thus $f^!$ commutes with the cohomology functors $\widetilde{\mathcal{H}}^k$. This also implies that, for any cosheaf $\mathcal{A}$, the costalk $i_x^!\mathcal{A}$ is just a vector space (rather than a complex).

Cosheaves are very useful when studying the homology of fibres of a morphism of varieties.

(2) The reader may find this terminology a bit strange since this differs from the more common use of "cosheaf". We justify our use of this word in two ways. First, $i_x^!\mathcal{A}$ is often called "costalk" in the literature. Second, our category of cosheaves is equivalent to the usual category of cosheaves by [C, Proposition 7.13].

LEMMA 7.2. *Let $\pi: Y \to X$ be a morphism. For any $k \in \mathbb{Z}$, define a cosheaf $\mathcal{G}$ on X by*

$$\mathcal{G} := \widetilde{\mathcal{H}}^{-k}(\pi_* \mathbb{D}_Y).$$

Then, for any $x \in X$, we have

$$i_x^! \mathcal{G} \cong H_k(\pi^{-1}(x)).$$

Proof. We use base change and exactness of $i_x^!$ (with respect to the cosheaf t-structure) to see that

$$i_x^!(\mathcal{G}) = \widetilde{\mathcal{H}}^{-k}(i_x^! \pi_* \mathbb{D}_X) = H^{-k}(\pi^{-1}(x), \mathbb{D}_{\pi^{-1}(x)}) = H_k(\pi^{-1}(x)). \qquad \square$$

7.1.1. Cosheaves on the line

We will particularly be working with cosheaves on $\mathbb{A} = \mathbb{A}^1$ which are coconstant along $U = \mathbb{A} \setminus \{0\}$. Let $i = i_0: \{0\} \to \mathbb{A}$ and $j: U \to \mathbb{A}$ be the inclusions of the origin and its complement. We will also fix a point $x \in U$ and write $i_x: \{x\} \to \mathbb{A}$ for the inclusion.

Let $u \in H^2_{\{0\}}(\mathbb{A})$ be the usual relative orientation class. As above, we will think of u as a map $i_* \mathbb{C}_{\{0\}} \to \mathbb{C}_{\mathbb{A}}[2]$. For any vector space V, we have a constant sheaf $V_{\mathbb{A}}$ with stalks V, and the cup product gives an isomorphism

$$V = H^0_c(\{0\}, V_{\{0\}}) \xrightarrow{u \cup} H^2_c(\mathbb{A}, V_{\mathbb{A}}).$$

PROPOSITION 7.3. *Let $\mathcal{F}$ be a sheaf on $\mathbb{A}$ which is constant along U.*

(i) *We have isomorphisms*

$$i_x^* \mathcal{F} \cong \mathcal{F}(U) \quad \text{and} \quad i_0^* \mathcal{F} \cong \mathcal{F}(\mathbb{A})$$

and hence we have a restriction map $r: i_0^* \mathcal{F} \to i_x^* \mathcal{F}$.

(ii) *There is an isomorphism $i_x^* \mathcal{F} \cong H^2_c(\mathbb{A}, \mathcal{F})$ making the diagram*

$$\begin{tikzcd} i_0^* \mathcal{F} \arrow[r, "r"] \arrow[d, "\cong"] & i_x^* \mathcal{F} \arrow[d, "\cong"] \\ H^0_c(\{0\}, i_0^* \mathcal{F}) \arrow[r, "u\cup"] & H^2_c(\mathbb{A}, \mathcal{F}) \end{tikzcd} \tag{5}$$

commute.

Proof. For the purposes of this proof, let $V = i_0^* \mathcal{F}$ and $W = i_x^* \mathcal{F}$.

Part (i) is immediate; in fact, for any connected open set $U \subset \mathbb{A}$, we have

$$\mathcal{F}(U) = \begin{cases} V, & \text{if } 0 \in U, \\ W, & \text{if } 0 \notin U. \end{cases}$$

In particular, $\mathcal{F}(\mathbb{A})=V$ and $\mathcal{F}(U)=W$ and so, by restriction, we have a linear map $r\colon V\to W$.

For part (ii), consider the standard short exact sequence of sheaves

$$0\longrightarrow j_!j^!\mathcal{F}\longrightarrow\mathcal{F}\longrightarrow i_*i^*\mathcal{F}\longrightarrow 0,$$

which in our case becomes

$$0\longrightarrow j_!W_U\longrightarrow\mathcal{F}\longrightarrow i_*V_{\{0\}}\longrightarrow 0.$$

Since i is closed embedding, $i_*=i_!$ and so

$$H_c^k(\mathbb{A},i_*V_{\{0\}})=H_c^k(\{0\},V_{\{0\}})=0$$

for $k>0$. Thus, we get an isomorphism

$$W=W\otimes H_c^2(U)=H_c^2(U,W_U)=H_c^2(\mathbb{A},j_!W_U)\cong H_c^2(\mathbb{A},\mathcal{F})$$

Now that we have an isomorphism $i_x^*\mathcal{F}\cong H_c^2(\mathbb{A},\mathcal{F})$, it remains to verify the commutativity of the square (5). To that end, consider the map of sheaves $V_{\mathbb{A}}\to\mathcal{F}$, which is the identity on connected open sets containing zero and the map r on connected open sets not containing 0. By functoriality, we obtain a commutative diagram

$$\begin{array}{ccc} j_!V_U=j_!j^!V_{\mathbb{A}} & \longrightarrow & V_{\mathbb{A}} \\ \downarrow{\scriptstyle r} & & \downarrow \\ j_!W_U=j_!j^!\mathcal{F} & \longrightarrow & \mathcal{F}. \end{array}$$

Applying $H_c^2(\mathbb{A},-)$, we obtain the commutative square

$$\begin{array}{ccc} V=H_c^2(U,V_U) & \xrightarrow{=} & V=H_c^2(\mathbb{A},V_{\mathbb{A}}) \\ \downarrow{\scriptstyle r} & & \downarrow \\ W=H_c^2(U,W_U) & \xrightarrow{\cong} & H_c^2(\mathbb{A},\mathcal{F}). \end{array} \tag{6}$$

Now, by the naturality of cup product, we get a commutative square

$$\begin{array}{ccc} i_*i^*V_{\mathbb{A}} & \xrightarrow{u\cup} & V_{\mathbb{A}}[2] \\ \downarrow{\scriptstyle \cong} & & \downarrow \\ i_*i^*\mathcal{F} & \xrightarrow{u\cup} & \mathcal{F}[2]. \end{array}$$

Applying $H_c^0(\mathbb{A},-)$, we obtain a factoring of $u\cup$ as

$$V=H_c^0(\{0\},i_0^*\mathcal{F})\xrightarrow{\cong}H_c^2(\mathbb{A},V_{\mathbb{A}})\longrightarrow H_c^2(\mathbb{A},\mathcal{F}).$$

Combined with the square (6), the result follows. $\square$

Applying Verdier duality implies a similar result for cosheaves on $\mathbb{A}$.

PROPOSITION 7.4. *Let $\mathcal{G}$ be a cosheaf on $\mathbb{A}$ which is coconstant along U.*

(i) *We have isomorphisms*

$$i_x^!\mathcal{G} \cong H_c^0(U, i_U^!\mathcal{G}) \quad \textit{and} \quad i_0^!\mathcal{G} \cong H_c^0(\mathbb{A}, \mathcal{G})$$

and a corestriction map $r^\vee\colon i_x^!\mathcal{G} \to i_0^!\mathcal{G}$.

(ii) *There is an isomorphism $i_x^!\mathcal{G} \cong H^{-2}(\mathbb{A}, \mathcal{G})$ making the diagram*

$$\begin{tikzcd} i_x^!\mathcal{G} \arrow[r, "r^\vee"] \arrow[d, "\cong"] & i_0^!\mathcal{G} \arrow[d, "\cong"] \\ H^{-2}(\mathbb{A}, \mathcal{G}) \arrow[r, "u\cap"] & H^0_{\{0\}}(\mathbb{A}, \mathcal{G}) \end{tikzcd}$$

commute.

Now, we will see how we can use these results to describe degeneration of cycles in Borel–Moore homology.

Let $f\colon Y \to \mathbb{A}$ be a morphism of varieties. Fix an integer n such that all fibres of f have dimension $\leqslant n-1$ (usually $n = \dim Y$, but not necessarily). We write $Y_0 = f^{-1}(0)$, $Y_x = f^{-1}(x)$ and $Y_U = f^{-1}(U)$. Assume also that we are given an isomorphism $Y_U \cong Y_x \times U$ compatible with the projection to U. Let $\mathcal{G} = \widetilde{\mathcal{H}}^{-2n+2}(f_*\mathbb{D}_Y)$.

PROPOSITION 7.5. *We have the following results concerning $\mathcal{G}$.*

(i) *$\mathcal{G}$ is coconstant along U.*

(ii) *We have isomorphisms $i_x^!\mathcal{G} \cong H_{2n-2}(Y_x)$ and $i_0^!\mathcal{G} \cong H_{2n-2}(Y_0)$.*

(iii) *For $p < -2n+2$, we have $\widetilde{\mathcal{H}}^p(f_*\mathbb{D}_Y) = 0$, and thus we have a morphism*

$$\mathcal{G} \longrightarrow f_*\mathbb{D}_Y[-2n+2].$$

This morphism induces an isomorphism

$$H^{-2}(\mathbb{A}, \mathcal{G}) \xrightarrow{\cong} H^{-2}(\mathbb{A}, f_*\mathbb{D}_Y[-2n+2]) = H^{-2n}(Y, \mathbb{D}_Y) = H_{2n}(Y).$$

(iv) *With respect to the isomorphism $H^{-2}(\mathbb{A}, \mathcal{G}) \cong H_{2n}(Y)$ and the isomorphism $i_x^!\mathcal{G} \cong H_{2n-2}(Y_x)$, the isomorphism $i_x^!\mathcal{G} \cong H^{-2}(\mathbb{A}, \mathcal{G})$ from Proposition 7.4 (ii) is given on cycles as $[Z] \mapsto [\overline{Z \times U}]$.*

(v) *The corestriction map is compatible with the cap product in homology as follows:*

$$\begin{tikzcd} & i_x^!\mathcal{G} \arrow[r, "r^\vee"] \arrow[dl, "\cong"'] & i_0^!\mathcal{G} \arrow[d, "\cong"] \\ H^{-2}(\mathbb{A}, \mathcal{G}) \arrow[r, "\cong"] & H_{2n}(Y) \arrow[r, "(f^*u)\cap"] & H_{2n-2}(Y_0). \end{tikzcd}$$

Proof. (i) Since $j^!$ is exact on the cosheaf t-structure, $j^!\mathcal{G}=\widetilde{\mathcal{H}}^{-2n+2}(f_*\mathbb{D}_{Y_U})$ and because $Y_U\cong Y_x\times U$, we see that $j^!\mathcal{G}=\mathbb{D}_U\otimes H_{2n-2}(Y_x)$ is coconstant.

(ii) These isomorphisms follow from Lemma 7.2.

(iii) For any p and any $y\in\mathbb{A}$, we have

$$i_y^!\widetilde{\mathcal{H}}^p(f_*\mathbb{D}_Y)=H_{-p}(f^{-1}(y)).$$

Since the fibre dimension is $n-1$, we see that this homology vanishes if $p<-2n+2$ and thus $\widetilde{\mathcal{H}}^p(f_*\mathbb{D}_Y)=0$ for such p. Thus, we deduce the existence of a morphism $\mathcal{G}\to f_*\mathbb{D}_Y[-2n+2]$. We have a spectral sequence starting with $H^q(\mathbb{A},\widetilde{\mathcal{H}}^p(f_*\mathbb{D}_Y))$ and converging to $H^{p+q}(Y,\mathbb{D}_Y)$. Since $H^q(\mathbb{A},\widetilde{\mathcal{H}}^p(f_*\mathbb{D}_Y))$ vanishes for $q<-2$ and for $p<-2n+2$, we see that the only contribution to $H^{-2n}(Y,\mathbb{D}_Y)$ can come from $H^{-2}(\mathbb{A},\mathcal{G})$, and thus

$$H^{-2}(\mathbb{A},\mathcal{G})\cong H_{2n}(Y).$$

(iv) We begin by repeating part of the proof of Proposition 7.3 in the Verdier dual setting. Consider the morphism $\mathcal{G}\to j_*j^*\mathcal{G}$. As $j^*=j^!$, we obtain the morphism

$$\mathcal{G}\longrightarrow j_*\mathbb{D}_U\otimes H_{2n-2}(Y_x).$$

Applying $H^{-2}(\mathbb{A},-)$, we obtain the isomorphism

$$H^{-2}(\mathbb{A},\mathcal{G})\cong H^{-2}(\mathbb{A},j_*\mathbb{D}_U\otimes H_{2n-2}(Y_x))=H_2(U)\otimes H_{2n-2}(Y_x)=H_{2n-2}(Y_x) \tag{7}$$

Now, we have the commutative square

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] & j_*j^*\mathcal{G}=j_*\mathbb{D}_U\otimes H_{2n-2}(Y_x) \arrow[d] \\
f_*\mathbb{D}_Y[-2n+2] \arrow[r] & j_*j^*f_*\mathbb{D}_Y[-2n+2]=j_*f_*\mathbb{D}_{Y_U}[-2n+2].
\end{tikzcd}$$

Applying $H^{-2}(\mathbb{A},-)$, we obtain the commutative rectangle of isomorphisms

$$\begin{tikzcd}
H^{-2}(\mathbb{A},\mathcal{G}) \arrow[r] \arrow[d] & H^{-2}(\mathbb{A},j_*\mathbb{D}_U\otimes H_{2n-2}(Y_x)) \arrow[r, "\cong"] & H_{2n-2}(Y_x) \arrow[d] \\
H^{-2n}(Y,\mathbb{D}_Y)=H_{2n}(Y) \arrow[r] & H^{-2n}(Y_U,\mathbb{D}_{Y_U})=H_{2n}(Y_U) \arrow[r, "\cong"] & H_{2n}(Y_x\times U).
\end{tikzcd}$$

In this diagram, the bottom-left horizontal arrow is given by pull-back in Borel–Moore homology, so on cycles it is given by $[X]\mapsto[X\cap Y_U]$. The right vertical arrow is given on cycles by $[Z]\mapsto[Z\times U]$. Thus, if we trace from the top-right to the bottom-left (following the inverses of the arrows along the bottom), we see the map $[Z]\mapsto[\overline{Z\times U}\,]$.

(v) We apply the projection formula (3) to the sheaf $\mathbb{D}_Y$. We obtain the commutative square.

$$\begin{array}{ccc} H_{2n}(Y)=H^{-2n}(Y,\mathbb{D}_Y) & \xrightarrow{(f^*u)\cap} & H^{-2n+2}_{Y_0}(Y,\mathbb{D}_Y)=H_{2n-2}(Y_0) \\ \downarrow\cong & & \downarrow\cong \\ H^{-2}(\mathbb{A},\mathcal{G})\cong H^{-2n}(\mathbb{A},f_*\mathbb{D}_Y) & \xrightarrow{u\cap} & H^{-2n+2}_{\{0\}}(\mathbb{A},f_*\mathbb{D}_Y)\cong H^0_{\{0\}}(\mathbb{A},\mathcal{G}). \end{array}$$

The result now follows from part (ii) of Proposition 7.4. □

7.2. The Beilinson–Drinfeld Grassmannian

We will need to recall the definition and properties of the Beilinson–Drinfeld Grassmannian. All the definitions and results here are taken from [MV]. The original definitions are due to Beilinson–Drinfeld [BD].

For any $x\in\mathbb{C}$, let $\mathcal{O}_x=\mathbb{C}[[t-x]]$ denote the completion of the local ring of $\mathbb{A}=\mathbb{A}^1$ at x and $\mathcal{K}_x=\mathbb{C}((t-x))$ its fraction field. By the Beauville–Laszlo theorem, the $\mathbb{C}$-points of the affine Grassmannian $\mathcal{G}\mathrm{r}_x:=G(\mathcal{K}_x)/G(\mathcal{O}_x)$ can be identified with the set of pairs (P,σ), where P is a principal G-bundle on $\mathbb{A}^1$ and σ is a trivialization of P over $\mathbb{A}\setminus\{x\}$.

Since we have chosen the local coordinate $t-x$, we get an isomorphism $\mathcal{O}_x\to\mathcal{O}$, and thus $\mathcal{G}\mathrm{r}_x\to\mathcal{G}\mathrm{r}$.

Now, we consider a 2-point version of the Beilinson–Drinfeld Grassmannian. For simplicity, we will fix one point to be zero and allow the other point to vary. The set of $\mathbb{C}$-points of $\mathcal{G}\mathrm{r}_{\mathbb{A}}$ is given by

$$\begin{aligned}\mathcal{G}\mathrm{r}_{\mathbb{A}}=\{(P,\sigma,x):\ & P \text{ is a principal } G \text{ bundle on } \mathbb{A},\ x\in\mathbb{A} \\ & \text{and } \sigma \text{ is a trivialization of } P \text{ over } \mathbb{A}\setminus\{0,x\}\}.\end{aligned}$$

(See [MV, (5.1)] for a precise description of the R-points of $\mathcal{G}\mathrm{r}_{\mathbb{A}}$ for each $\mathbb{C}$-algebra R.)

We have an obvious map $\pi:\mathcal{G}\mathrm{r}_{\mathbb{A}}\to\mathbb{A}$. Let $U=\mathbb{A}^1\setminus\{0\}$. The following is diagram (5.9) in [MV].

Lemma 7.6. *There are isomorphisms*

$$\pi^{-1}(U)=\mathcal{G}r_U\cong\mathcal{G}r\times\mathcal{G}r\times U \quad \textit{and} \quad \pi^{-1}(0)\cong\mathcal{G}r.$$

The action of T by left multiplication on $\mathcal{G}\mathrm{r}$ extends to an action on $\mathcal{G}\mathrm{r}_{\mathbb{A}}$ preserving all fibres. From the fibre perspective, this is simply the diagonal action on $\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}$.

Following [MV], we introduce a global version of the semi-infinite cells. For $\mu\in P$, define $S^{\mu}_{-,\mathbb{A}}$ to be the subvariety of $\mathcal{G}\mathrm{r}_{\mathbb{A}}$ with fibres S^{μ}_{-} over zero and fibres

$$\bigcup_{\mu_1+\mu_2=\mu} S^{\mu_1}_{-}\times S^{\mu_2}_{-}$$

over $x\in U$.

We write s_μ for the inclusion of $\overline{S^{\mu}_{-,\mathbb{A}}}$ into $\mathcal{G}\mathrm{r}_{\mathbb{A}}$.

We will also need the global version of the $G^\vee(\mathcal{O})$ orbits. From [Z], for any pair $\lambda_1,\lambda_2\in P_+$, there exists a variety $\overline{\mathcal{G}\mathrm{r}^{\lambda_1,\lambda_2}_{\mathbb{A}}}\subset\mathcal{G}\mathrm{r}_{\mathbb{A}}$, whose fibres are $\overline{\mathcal{G}\mathrm{r}^{\lambda_1}}\times\overline{\mathcal{G}\mathrm{r}^{\lambda_2}}$ away from zero and $\overline{\mathcal{G}\mathrm{r}^{\lambda_1+\lambda_2}}$ over zero.

7.3. The fusion product of perverse sheaves

We will now define the fusion tensor product on $P_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ following [MV].

Consider the diagram

$$\begin{tikzcd}
\mathcal{G}\mathrm{r} \arrow[r, "i"] \arrow[d] & \mathcal{G}\mathrm{r}_{\mathbb{A}} \arrow[d, "\pi"] & \mathcal{G}\mathrm{r}_U \arrow[l, "j"'] \cong \mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}\times U \arrow[d] \\
\{0\} \arrow[r] & \mathbb{A} & U. \arrow[l]
\end{tikzcd}$$

Definition 7.7. Let $\mathcal{A}_1,\mathcal{A}_2\in P_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$. The *fusion product* (see [MV, (5.8)]) is defined by

$$\mathcal{A}_1*\mathcal{A}_2 := i^!(j_{!*}(p_{12}^!(\mathcal{A}_1\boxtimes\mathcal{A}_2)[-1]))[1],$$

where $p_{12}\colon\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}\times U\to\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}$ is the projection onto the first two factors.

Following [MV] and [BR, §8.3], we will explain the compatibility of the fusion product with the weight functors.

Fix $\mathcal{A}_1,\mathcal{A}_2\in P_{G^\vee(\mathcal{O})}(\mathcal{G}\mathrm{r})$ and let

$$\mathcal{B} := j_{!*}(p_{12}^!(\mathcal{A}_1\boxtimes\mathcal{A}_2)[-1]),$$

and, for each $\mu\in P$, let

$$\mathcal{F}_\mu := \widetilde{\mathcal{H}}^{2\rho^\vee(\mu)+1}\pi_* s_\mu^!\mathcal{B}$$

be a cosheaf on $\mathbb{A}$. The following result follows by direct computation.

PROPOSITION 7.8. (i) *The cosheaf $\mathcal{F}_\mu$ is coconstant along U with costalk at $x\in U$ given by*

$$i_x^!\mathcal{F}_\mu = \bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{A}_1)\otimes F_{\mu_2}(\mathcal{A}_2).$$

(ii) *The costalk at zero is given by*

$$i_0^!\mathcal{F}_\mu = F_\mu(\mathcal{A}_1 * \mathcal{A}_2).$$

(iii) *The cosheaf $\mathcal{F}_\mu$ is actually coconstant along all of $\mathbb{A}$ and so the corestriction map $r^\vee\colon i_x^!\mathcal{F}_\mu \to i_0^!\mathcal{F}_\mu$ provides an isomorphism*

$$\bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{A}_1)\otimes F_{\mu_2}(\mathcal{A}_2) \longrightarrow F_\mu(\mathcal{A}_1 * \mathcal{A}_2).$$

In fact, this cosheaf $\mathcal{F}_\mu$ is $\mathcal{L}[2]$, where $\mathcal{L}$ is the local system $\mathcal{L}_\mu^{2\rho^\vee(\mu)}(\mathcal{A}_1,\mathcal{A}_2)$ defined in [MV, (6.22)], and pulled back to $\mathbb{A}\subset\mathbb{A}^2$.

7.4. The multiplication map

Let λ_1 and λ_2 be two dominant weights, and let $\lambda=\lambda_1+\lambda_2$. We have a morphism

$$m_{\lambda_1\lambda_2}\colon \mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2} \longrightarrow \mathcal{I}_\lambda$$

which becomes $m_{\lambda_1\lambda_2}\colon L(\lambda_1)\otimes L(\lambda_2)\to L(\lambda)$ under the geometric Satake isomorphism.

Take $(\mathcal{A}_1,\mathcal{A}_2)=(\mathcal{I}_{\lambda_1},\mathcal{I}_{\lambda_2})$ in the setup above. Note that

$$\mathcal{B} := j_{!*}(p_{12}^!(\mathcal{I}_{\lambda_1}\boxtimes\mathcal{I}_{\lambda_2})[-1])$$

is actually the IC sheaf of $\overline{\mathcal{G}\mathrm{r}_{\mathbb{A}}^{\lambda_1,\lambda_2}}$.

Let $\mu\in P$. We will compare $\mathcal{F}_\mu := \widetilde{\mathcal{H}}^{2\rho^\vee(\mu)+1}\pi_* s_\mu^!\mathcal{B}$ to the cosheaf

$$\mathcal{G} := \widetilde{\mathcal{H}}^{2\rho^\vee(\mu)+1}\pi_* s_\mu^!\Big(\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}_{\mathbb{A}}^{\lambda_1,\lambda_2}}}[-2\rho^\vee(\lambda)-1]\Big) = \widetilde{\mathcal{H}}^{-2\rho^\vee(\lambda-\mu)}\pi_*\Big(\mathbb{D}_{\overline{\mathcal{G}\mathrm{r}_{\mathbb{A}}^{\lambda_1,\lambda_2}}\cap\overline{S_{-,\mathbb{A}}^\mu}}\Big).$$

We may apply Proposition 7.5 to the map

$$\pi\colon \overline{\mathcal{G}\mathrm{r}_{\mathbb{A}}^{\lambda_1,\lambda_2}}\cap\overline{S_{-,\mathbb{A}}^\mu} \longrightarrow \mathbb{A},$$

and thus to the sheaf $\mathcal{G}$.

So, from Proposition 7.5 (i) and (ii), it follows that $\mathcal{G}$ is coconstant along U and its costalks are as follows

$$i_0^!\mathcal{G} \cong H_{2\rho^\vee(\lambda-\mu)}(\overline{\mathcal{G}\mathrm{r}^\lambda}\cap\overline{S_-^\mu}) \cong F_\mu(\mathcal{I}_\lambda)$$

and, for $x \in U$,

$$\begin{aligned} i_x^! \mathcal{G} &= \bigoplus_{\mu_1+\mu_2=\mu} H_{2\rho^\vee(\lambda_1-\mu_1)}(\overline{\mathcal{G}\mathrm{r}^{\lambda_1}} \cap \overline{S_-^{\mu_1}}) \otimes H_{2\rho^\vee(\lambda_2-\mu_2)}(\overline{\mathcal{G}\mathrm{r}^{\lambda_2}} \cap \overline{S_-^{\mu_2}}) \\ &\cong \bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{I}_{\lambda_1}) \otimes F_{\mu_2}(\mathcal{I}_{\lambda_2}). \end{aligned}$$

LEMMA 7.9. *The following diagram commutes*:

$$\begin{tikzcd} \bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{I}_{\lambda_1}) \otimes F_{\mu_2}(\mathcal{I}_{\lambda_2}) \arrow[r, "m_{\lambda_1\lambda_2}"] \arrow[d, "\cong"] & F_\mu(\mathcal{I}_\lambda) \arrow[d, "\cong"] \\ i_x^! \mathcal{G} \arrow[r, "r^\vee"] & i_0^! \mathcal{G}. \end{tikzcd}$$

Proof. From (2), we have a map

$$\mathcal{B} \longrightarrow \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}_{\mathbb{A}}^{\lambda_1,\lambda_2}}}[-2\rho^\vee(\lambda)-1]. \tag{8}$$

Let $i: \mathcal{G}\mathrm{r} \to \mathcal{G}\mathrm{r}_{\mathbb{A}}$ denote the inclusion of the central fibre as before. Then,

$$i^! \mathcal{B}[1] = \mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2}$$

by definition. On the other hand, $\mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2} \cong \mathcal{I}_\lambda \oplus \mathcal{A}$, where $\mathcal{A}$ is supported on smaller orbits. Thus, we have a projection $\mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2} \to \mathcal{I}_\lambda$. If we apply $i^![1]$ to the map (8), we obtain $\mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2} \to \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-2\rho^\vee(\lambda)]$, which we can factor as

$$\mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2} \longrightarrow \mathcal{I}_\lambda \longrightarrow \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-2\rho^\vee(\lambda)], \tag{9}$$

because

$$\mathrm{Hom}\big(\mathcal{A}, \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-2\rho^\vee(\lambda)]\big) = \mathrm{Hom}\big(\mathbb{C}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[2\rho^\vee(\lambda)], \mathbb{D}(\mathcal{A})\big) = H^{-2\rho^\vee(\lambda)}(\overline{\mathcal{G}\mathrm{r}^\lambda}, \mathbb{D}(\mathcal{A})) = 0.$$

For the last equality, note that $\mathbb{D}(\mathcal{A})$ is a direct sum of perverse sheaves, each supported on some $\overline{\mathcal{G}\mathrm{r}^\mu} \subsetneq \overline{\mathcal{G}\mathrm{r}^\lambda}$, and $\dim \overline{\mathcal{G}\mathrm{r}^\mu} < 2\rho^\vee(\lambda)$.

On the other hand, if we apply $\widetilde{\mathcal{H}}^{2\rho^\vee(\mu)+1}\pi_* \circ s_\mu^!$ to (8), we obtain a map of cosheaves $\mathcal{F}_\mu \to \mathcal{G}$, whence a commutative diagram

$$\begin{tikzcd} i_x^! \mathcal{F}_\mu \arrow[r, "r^\vee"] \arrow[d] & i_0^! \mathcal{F}_\mu \arrow[d] \\ i_x^! \mathcal{G} \arrow[r, "r^\vee"] & i_0^! \mathcal{G}. \end{tikzcd} \tag{10}$$

The factoring (9) means that the map $i_0^! \mathcal{F}_\mu \to i_0^! \mathcal{G}$ factors as

$$H^{2\rho^\vee(\mu)}_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r}, \mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2}) \longrightarrow H^{2\rho^\vee(\mu)}_{\overline{S^\mu_-}}(\mathcal{G}\mathrm{r}, \mathcal{I}_\lambda) \longrightarrow H^{2\rho^\vee(\mu)}_{\overline{S^\mu_-}}\big(\mathcal{G}\mathrm{r}, \mathbb{D}_{\overline{\mathcal{G}\mathrm{r}^\lambda}}[-2\rho^\vee(\lambda)]\big).$$

Thus, (10) can be rewritten as

$$\begin{tikzcd}
\bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{I}_{\lambda_1}) \otimes F_{\mu_2}(\mathcal{I}_{\lambda_2}) \arrow[r] \arrow[dr, "m_{\lambda_1\lambda_2}"] \arrow[dd, "\cong"] & F_\mu(\mathcal{I}_{\lambda_1} * \mathcal{I}_{\lambda_2}) \arrow[d] \\
& F_\mu(\mathcal{I}_\lambda) \arrow[d, "\cong"] \\
i_x^! \mathcal{G} \arrow[r, "r^\vee"] & i_0^! \mathcal{G},
\end{tikzcd}$$

and the result follows. □

7.5. Multiplication on the level of cycles

Now, we will translate Lemma 7.9 to the cycle level. As before, let $\lambda_1, \lambda_2 \in P_+$, let $\lambda := \lambda_1 + \lambda_2$, let $\mu_1, \mu_2 \in P$, and let $\mu := \mu_1 + \mu_2$.

LEMMA 7.10. *Let $Z_1 \in \mathcal{Z}(\lambda_1)_{\mu_1}$ and $Z_2 \in \mathcal{Z}(\lambda_2)_{\mu_2}$. Consider*

$$Z_1 \times Z_2 \times U \subset \overline{\mathcal{G}r^{\lambda_1,\lambda_2}_{\mathbb{A}}} \cap \overline{S^\mu_{-,\mathbb{A}}}$$

Then, in $F_\mu(\mathcal{I}_\lambda) = H_{2\rho^\vee(\lambda-\mu)}(\overline{\mathcal{G}r^\lambda} \cap \overline{S^\mu_-})$, we have an equality

$$m_{\lambda_1,\lambda_2}([Z_1] \otimes [Z_2]) = \sum_Z i(Z, \pi^{-1}(0) \cdot \overline{Z_1 \times Z_2 \times U})[Z],$$

where the sum ranges over $Z \in \mathcal{Z}(\lambda)_\mu$.

Proof. By Proposition 7.5 (v) and Lemma 7.9, we obtain the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{I}_{\lambda_1}) \otimes F_{\mu_2}(\mathcal{I}_{\lambda_2}) \arrow[r, "m_{\lambda_1\lambda_2}"] \arrow[d, "\cong"] & F_\mu(\mathcal{I}_\lambda) \arrow[d, "\cong"] \\
H_{2\rho^\vee(\lambda-\mu)+2}\big(\overline{\mathcal{G}\mathrm{r}^{\lambda_1,\lambda_2}_{\mathbb{A}}} \cap \overline{S^\mu_{-,\mathbb{A}}}\big) \arrow[r, "\pi^*(u)\cap"] & H_{2\rho^\vee(\lambda-\mu)}(\overline{\mathcal{G}\mathrm{r}^{\lambda_1+\lambda_2}} \cap \overline{S^\mu_-}),
\end{tikzcd}$$

where, as before, $u\in H^2_{\{0\}}(\mathbb{A})$ denotes the usual orientation class. Now, by Proposition 7.5 (iv), the element $[Z_1]\otimes[Z_2]$ in the top-left is sent to $[\overline{Z_1\times Z_2\times U}\,]$ in the bottom-left.

Now, we apply the setup from Proposition 5.1 to $Y=\mathbb{A}^1$, $D=\{0\}$, $X'=\overline{\mathcal{G}\mathrm{r}^{\lambda}}\cap\overline{S^{\mu}_{-}}$ and $Y'=\overline{\mathcal{G}\mathrm{r}^{\lambda_1,\lambda_2}_{\mathbb{A}}}\cap\overline{S^{\mu}_{-,\mathbb{A}}}$. This gives the desired result. □

By Proposition 2.12, this immediately implies the following result concerning stable MV cycles.

THEOREM 7.11. *Let $Z_1\in\mathcal{Z}(\infty)_{-\nu_1}$ and $Z_2\in\mathcal{Z}(\infty)_{-\nu_2}$. In the algebra $\mathbb{C}[N]$, we have*

$$b_{Z_1}b_{Z_2}=\sum_{Z} i(Z,\pi^{-1}(0)\cdot\overline{Z_1\times Z_2\times U}\,)b_Z,$$

where the sum ranges over $Z\in\mathcal{Z}(\infty)_{-\nu_1-\nu_2}$.

Part III. Measures

8. Measures

All the objects defined in this section depend on the choice of a principal nilpotent element $\dot{e}\in\mathfrak{n}$ and we write $\dot{e}=\sum_i \dot{e}_i$, where each $\dot{e}_i$ is a non-zero root vector of weight α_i. These $\dot{e}_i$ are a priori unrelated to the choice of simple root vectors e_i made in §2.1.

8.1. The elements n_x

We denote the set of regular elements in $\mathfrak{t}$ by $\mathfrak{t}^{\mathrm{reg}}$.

For each $x\in\mathfrak{t}^{\mathrm{reg}}$, the subset $x+\mathfrak{n}$ of $\mathfrak{g}$ is a single orbit under the adjoint action of the group N, by [Bo, chap. 8, §11, no. 1, lemme 2]. Further, the centralizer of x in G, namely T, meets N trivially, so the action of N on $x+\mathfrak{n}$ is simply transitive. Therefore, there is a unique element $n_x\in N$ such that $\mathrm{Ad}_{n_x}(x)=x+\dot{e}$. Examining the proof in [Bo], one further notes that $x\mapsto n_x$ is a regular map $\mathfrak{t}^{\mathrm{reg}}\to N$.

We thus get an algebra map $\overline{D}\colon\mathbb{C}[N]\to\mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$ defined by $\overline{D}(f)(x)=f(n_x)$, where $f\in\mathbb{C}[N]$ and $x\in\mathfrak{t}^{\mathrm{reg}}$.

A major goal of this section is to understand the map $\overline{D}$ and to put it in a wider setting. We first study how n_x varies when the Weyl group acts on x. We denote by N_- the unipotent radical of the Borel subgroup opposite to B with respect to T. Recall that $\bar{w}$ denotes a lift to the normalizer $N_G(T)$ of an element w in the Weyl group.

PROPOSITION 8.1. *Let $x\in\mathfrak{t}^{\mathrm{reg}}$ and $w\in W$. Then, there exists $(y,t)\in N_-\times T$ such that*

$$n_{wx}=yn_x\bar{w}^{-1}t.$$

Proof. By induction on the length of w, we can reduce to the case where w is a simple reflection s_i. Choose $\dot{f}_i$ such that $(\dot{e}_i, h_i, \dot{f}_i)$ is an $\mathfrak{sl}_2$ triple. Let x_i and y_i be the additive 1-parameter subgroups of G given by $x_i(b)=\exp(b\dot{e}_i)$ and $y_i(b)=\exp(b\dot{f}_i)$ for $b\in\mathbb{C}$. Set $a=\langle\alpha_i, x\rangle$; there exists an element $t\in T$ such that

$$x_i\left(\frac{1}{a}\right)y_i(-a)x_i\left(\frac{1}{a}\right)=\bar{s}_i t.$$

Direct calculations give

$$\begin{aligned}\mathrm{Ad}_{y_i(a)}(x)&=\exp(a\,\mathrm{ad}_{\dot{f}_i})(x)=x+a[\dot{f}_i,x]=x+a^2\dot{f}_i,\\ \mathrm{Ad}_{y_i(a)}(\dot{e})&=\exp(a\,\mathrm{ad}_{\dot{f}_i})(\dot{e})=\dot{e}+a[\dot{f}_i,\dot{e}_i]+\tfrac{1}{2}a^2[\dot{f}_i,[\dot{f}_i,\dot{e}_i]]=\dot{e}-ah_i-a^2\dot{f}_i.\end{aligned}$$

Noting that $s_i x=x-\langle\alpha_i,x\rangle h_i$, we then get

$$\mathrm{Ad}_{y_i(a)}(x+\dot{e})=s_i x+\dot{e}.$$

Since t acts trivially on $\mathfrak{t}$ and $\bar{s}_i$ acts by the simple reflection s_i, we deduce that

$$\mathrm{Ad}_{(y_i(a)n_x\bar{s}_i t)}(s_i x)=\mathrm{Ad}_{(y_i(a)n_x)}(x)=\mathrm{Ad}_{y_i(a)}(x+\dot{e})=s_i x+\dot{e}. \tag{11}$$

On the other hand, let P' be the (minimal parabolic) subgroup of G generated by the Borel B and the image of the 1-parameter subgroup y_i. We denote the unipotent radical of P' by N' and the Lie algebra of N' by $\mathfrak{n}'$. Noting that $[x+\dot{e}_i,\mathfrak{n}']=\mathfrak{n}'$ and that $\dot{e}-\dot{e}_i\in\mathfrak{n}'$, we can apply Bourbaki's lemme 2 quoted above and find $n'\in N'$ such that $\mathrm{Ad}_{n'}(x+\dot{e}_i)=x+\dot{e}$. Since

$$\mathrm{Ad}_{x_i(-1/a)}(x)=x+\dot{e}_i,$$

we see that the adjoint action of $n'x_i(-1/a)$ brings x to $x+\dot{e}$, and thus $n_x=n'x_i(-1/a)$.

Since $y_i(-a)x_i(1/a)$ belongs to P' and hence normalizes N', we can find $n''\in N'$ such that

$$y_i(a)n_x\bar{s}_i t=y_i(a)n'y_i(-a)x_i\left(\frac{1}{a}\right)=y_i(a)y_i(-a)x_i\left(\frac{1}{a}\right)n''=x_i\left(\frac{1}{a}\right)n''.$$

Thus, the product $y_i(a)n_x\bar{s}_i t$ belongs to N, and by (11) it acts on $s_i x$ in the same way as $n_{s_i x}$. We conclude that $n_{s_i x}=y_i(a)n_x\bar{s}_i t$, which is of the desired form. □

8.2. Sequences and shuffles

Our next task is to find an expansion of n_x and n_x^{-1} as an infinite linear combination of Chevalley monomials.

We need some notation concerning finite sequences $\mathbf{i}=(i_1,\ldots,i_p)$ drawn from the set I.

Definition 8.2. (i) We denote by Seq the set of all such sequences $\mathbf{i}$, and for $\nu\in Q_+$, we put

$$\mathrm{Seq}(\nu):=\{\mathbf{i}=(i_1,...,i_p):\alpha_{i_1}+...+\alpha_{i_p}=\nu\}.$$

(ii) A *shuffle* of two sequences $\mathbf{j}$ and $\mathbf{k}$ is a sequence $\mathbf{i}$ produced by shuffling together the sequences $\mathbf{j}$ and $\mathbf{k}$, maintaining the same relative order among the elements of $\mathbf{j}$ and $\mathbf{k}$. We write $\mathbf{j}\sqcup\mathbf{k}$ to denote this set of shuffles. Thus, if $\mathbf{j}$ has length p and $\mathbf{k}$ has length q, then $\mathbf{j}\sqcup\mathbf{k}$ has $\binom{p+q}{p}$ elements.

(iii) With a sequence $\mathbf{i}=(i_1,...,i_p)$ in Seq, we associate the weights

$$\beta_0^{\mathbf{i}}=0,\quad \beta_1^{\mathbf{i}}=\alpha_{i_1},\quad \beta_2^{\mathbf{i}}=\alpha_{i_1}+\alpha_{i_2},\quad ...,\quad \beta_p^{\mathbf{i}}=\alpha_{i_1}+...+\alpha_{i_p}.$$

Consider the free Lie algebra $\mathfrak{f}$ on the set $\{\hat{e}_i:i\in I\}$ and its universal enveloping algebra $U(\mathfrak{f})$ (identified with the free associative algebra on this set). This algebra $U(\mathfrak{f})$ is graded by Q_+. For each $\nu\in Q_+$, we have a basis $\{\hat{e}_{\mathbf{i}}:=\hat{e}_{i_1}...\hat{e}_{i_p}\}_{\mathbf{i}\in\mathrm{Seq}(\nu)}$ for $U(\mathfrak{f})_\nu$.

The algebra $U(\mathfrak{f})$ is in fact a graded Hopf algebra with finite-dimensional components, so its graded dual $(U(\mathfrak{f}))^*$ is also a Hopf algebra. For each $\nu\in Q_+$, we consider the basis $\{\hat{e}^*_{\mathbf{i}}\}_{\mathbf{i}\in\mathrm{Seq}(\nu)}$ for $(U(\mathfrak{f}))^*_{-\nu}$ dual to the above basis $\{\hat{e}_{\mathbf{i}}\}_{\mathbf{i}\in\mathrm{Seq}(\nu)}$ for $U(\mathfrak{f})_\nu$. From the definition of the coproduct on $U(\mathfrak{f})$, we get the following shuffle identity in $(U(\mathfrak{f}))^*$:

$$\hat{e}^*_{\mathbf{j}}\hat{e}^*_{\mathbf{k}}=\sum_{\mathbf{i}\in\mathbf{j}\sqcup\mathbf{k}}\hat{e}^*_{\mathbf{i}} \tag{12}$$

There is a unique Hopf algebra map $U(\mathfrak{f})\to U(\mathfrak{n})$ that sends $\hat{e}_i$ to $\dot{e}_i$ for each $i\in I$. The dual map is an inclusion of algebras $\mathbb{C}[N]\hookrightarrow(U(\mathfrak{f}))^*$. (This inclusion was previously studied by various authors, including in [GLS, §8].) Each sequence $\mathbf{i}=(i_1,...,i_p)$ defines a monomial $\dot{e}_{\mathbf{i}}=\dot{e}_{i_1}...\dot{e}_{i_p}$ in $U(\mathfrak{n})$.

The following well-known functional identity seems to be related to cat chasing and moulds [MO] and to an identity of Littlewood [KnD, Lemma p. 149].

LEMMA 8.3. *Given p complex numbers a_1, ..., a_p, define*

$$f_p(a_1,...,a_p)=\frac{1}{a_1(a_1+a_2)...(a_1+a_2+...+a_p)}$$

whenever it makes sense. Let p and q be positive integers and let $\mathrm{Sh}(p,q)$ *denote the set of all permutations $\sigma\in\mathfrak{S}_{p+q}$ such that*

$$\sigma(1)<\sigma(2)<...<\sigma(p)\quad \textit{and}\quad \sigma(p+1)<\sigma(p+2)<...<\sigma(p+q).$$

Then, for any complex numbers $a_1,...,a_{p+q}$, we have

$$f_p(a_1,...,a_p)f_q(a_{p+1},...,a_{p+q})=\sum_{\sigma\in\mathrm{Sh}(p,q)}f_{p+q}(a_{\sigma^{-1}(1)},...,a_{\sigma^{-1}(p+q)}),$$

whenever both members make sense.

Proof. By analytic continuation, we can deduce the general result from the case where all a_i have positive real part. In this particular case,

$$f_p(a_1,...,a_p)=\int_{C_p} e^{-(a_1x_1+...+a_px_p)}\,dx_1\,...\,dx_p,$$

where $C_p=\{(x_1,...,x_p)\in\mathbb{R}^p:x_1>...>x_p>0\}$. The proposition follows by writing $C_p\times C_q$ as the disjoint union of cones

$$\sigma^{-1}\cdot C_{p+q}=\{(x_{\sigma(1)},...,x_{\sigma(p+q)}):x_1>...>x_{p+q}>0\}$$

for $\sigma\in\mathrm{Sh}(p,q)$, up to a nullset. □

For a sequence $\mathbf{i}=(i_1,...,i_p)$, we define

$$\overline{D}_{\mathbf{i}}=\prod_{k=0}^{p-1}\frac{1}{\beta_k^{\mathbf{i}}-\beta_p^{\mathbf{i}}}.$$

These rational functions $\overline{D}_{\mathbf{i}}$ can be evaluated on any $x\in\mathfrak{t}^{\mathrm{reg}}$ that satisfies $\langle\beta,x\rangle\neq 0$ for all $\beta\in Q_+\setminus\{0\}$.

Proposition 8.4. *Let $x\in\mathfrak{t}^{\mathrm{reg}}$ such that $\langle\beta,x\rangle\neq 0$ for all $\beta\in Q_+\setminus\{0\}$. Then,*

$$f(n_x)=\sum_{\mathbf{i}\in\mathrm{Seq}}\langle\dot{e}_{\mathbf{i}},f\rangle\overline{D}_{\mathbf{i}}(x)$$

for all $f\in\mathbb{C}[N]$.

These linear combinations appearing in this statement are infinite only in appearance, for $U(\mathfrak{n})$ acts locally nilpotently on $\mathbb{C}[N]$. The proposition says that the morphism $\overline{D}$ can be expanded as $\mathbb{C}(\mathfrak{t})$-linear combinations of Chevalley monomials:

$$\overline{D}=\sum_{\mathbf{i}\in\mathrm{Seq}}\overline{D}_{\mathbf{i}}\dot{e}_{\mathbf{i}}.$$

Proof. To prove this formula, we need to show that as linear forms on $\mathbb{C}[N]$

$$n_x=\sum_{\mathbf{i}\in\mathrm{Seq}}\overline{D}_{\mathbf{i}}(x)\dot{e}_{\mathbf{i}}. \tag{13}$$

We first note that Lemma 8.3 implies that

$$\overline{D}_{\mathbf{j}}(x)\overline{D}_{\mathbf{k}}(x)=\sum_{\mathbf{i}\in\mathbf{j}\sqcup\mathbf{k}}\overline{D}_{\mathbf{i}}(x)$$

for all sequences $\mathbf{j}$ and $\mathbf{k}$. Comparing with (12), it follows that

$$\sum_{\mathbf{i}\in\mathrm{Seq}} \overline{D}_{\mathbf{i}}(x)\dot{e}_{\mathbf{i}}$$

is an algebra map $(U(\mathfrak{f}))^*\to\mathbb{C}$. Thus, the right-hand side of (13) is an algebra map $\mathbb{C}[N]\to\mathbb{C}$, so is the evaluation at an element $n\in N$.

Let us compute how this n acts on x in the adjoint representation of N on $\mathfrak{g}$. Since $\overline{D}_{(k)}(x)=-1/\alpha_k(x)$, we have

$$\sum_{k\in I} \overline{D}_{(k)}(x)\,\mathrm{ad}_{\dot{e}_k}(x)=\sum_{k\in I}\left(\frac{-[\dot{e}_k,x]}{\alpha_k(x)}\right)=\dot{e}.$$

Each sequence of length greater than 2 can be written as a concatenation $(\mathbf{i},j,k)$ with $\mathbf{i}\in\mathrm{Seq}$ (possibly empty) and $(j,k)\in I^2$. Denoting by p the length of $\mathbf{i}$, we compute

$$\begin{aligned}\overline{D}_{(\mathbf{i},j,k)}(x)\,\mathrm{ad}\,\dot{e}_{(\mathbf{i},j,k)}(x)&=\left(\prod_{\ell=0}^{p}\frac{1}{\beta_\ell^{\mathbf{i}}-(\beta_p^{\mathbf{i}}+\alpha_j+\alpha_k)}\times\frac{-1}{\alpha_k}\right)(x)\,\mathrm{ad}_{\dot{e}_{\mathbf{i}}}([\dot{e}_j,[\dot{e}_k,x]])\\&=\left(\prod_{\ell=0}^{p}\frac{1}{(\beta_\ell^{\mathbf{i}}-\beta_p^{\mathbf{i}})-(\alpha_j+\alpha_k)}\right)(x)\,\mathrm{ad}_{\dot{e}_{\mathbf{i}}}([\dot{e}_j,\dot{e}_k]).\end{aligned}$$

Summing these elements with (j,k) running over I^2 gives zero, since terms pairwise cancel by antisymmetry of the Lie bracket $[\dot{e}_j,\dot{e}_k]$. Taking the sum over $\mathbf{i}$ then yields the equality

$$\sum_{\mathbf{i}\neq\varnothing}\overline{D}_{\mathbf{i}}(x)\,\mathrm{ad}_{\dot{e}_{\mathbf{i}}}(x)=\dot{e}$$

and we conclude that $\mathrm{Ad}_n(x)=x+\dot{e}$. (Note that the above sum makes sense since it is in fact finite.) As this is the definition of n_x, this completes the proof of (13). □

8.3. Measures from simplices

In the rest of §8, we explain that $\overline{D}$ is the shadow of a measure-valued morphism that carries more information. We start with its construction.

Consider the vector space of $\mathbb{C}$-valued compactly supported distributions on $\mathfrak{t}^*_{\mathbb{R}}$. It forms an algebra under convolution, the push-forward along the addition map

$$\mathfrak{t}^*_{\mathbb{R}}\times\mathfrak{t}^*_{\mathbb{R}}\xrightarrow{+}\mathfrak{t}^*_{\mathbb{R}}.$$

Define $\mathcal{PP}$ to be the subspace spanned by those distributions equal to linear combinations of piecewise-polynomial functions times Lebesgue measures on (not necessarily

full-dimensional) polytopes whose vertices lie in the weight lattice P; it is a subalgebra. All the distributions we will consider live in $\mathcal{PP}$.

Let

$$\Delta^p := \{(c_0, ..., c_p) \in \mathbb{R}^{p+1} : \text{each } c_i \geqslant 0 \text{ and } c_0 + ... + c_p = 1\}$$

be the standard p-simplex. For $\mathbf{i} \in \mathrm{Seq}$ of length p, we define the linear map $\pi_{\mathbf{i}} : \mathbb{R}^{p+1} \to \mathfrak{t}^*_{\mathbb{R}}$ by

$$\pi_{\mathbf{i}}(c_0, ..., c_p) = -\sum_{k=0}^{p} c_k \beta^{\mathbf{i}}_k$$

We define the measure $D_{\mathbf{i}}$ on $\mathfrak{t}^*_{\mathbb{R}}$ by $D_{\mathbf{i}} := (\pi_{\mathbf{i}})_*(\delta_{\Delta^p})$, the push-forward of Lebesgue measure on the p-simplex. Note that the total mass of $D_{\mathbf{i}}$ is $1/p!$.

Lemma 8.5. *The measures $D_{\mathbf{i}}$ satisfy the shuffle identity*

$$D_{\mathbf{j}} * D_{\mathbf{k}} = \sum_{\mathbf{i} \in \mathbf{j} \sqcup \mathbf{k}} D_{\mathbf{i}}.$$

Proof. Let p and q be the lengths of $\mathbf{j}$ and $\mathbf{k}$, respectively, and consider the composite map

$$\pi_{\mathbf{j}} + \pi_{\mathbf{k}} : \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \xrightarrow{\pi_{\mathbf{j}} \times \pi_{\mathbf{k}}} \mathfrak{t}^*_{\mathbb{R}} \oplus \mathfrak{t}^*_{\mathbb{R}} \xrightarrow{+} \mathfrak{t}^*_{\mathbb{R}}.$$

Then, the left side of the desired equality is exactly $(\pi_{\mathbf{j}} + \pi_{\mathbf{k}})_*(\delta_{\Delta^p \times \Delta^q})$. To get the right side, we triangulate the product $\Delta^p \times \Delta^q$ in one of the standard ways (see e.g. [Ha, pp. 277–278]) with one simplex for each shuffle. □

Comparing Lemma 8.5 with (12), we deduce that there is an algebra morphism $(U(\mathfrak{f}))^* \to \mathcal{PP}$ taking $\hat{e}^*_{\mathbf{i}}$ to $D_{\mathbf{i}}$. Composing with the inclusion of algebras $\mathbb{C}[N] \hookrightarrow (U(\mathfrak{f}))^*$ from §8.2, we get an algebra map $D : \mathbb{C}[N] \to \mathcal{PP}$. Unpacking the above definitions, we see that, for any $f \in \mathbb{C}[N]$,

$$D(f) = \sum_{\mathbf{i} \in \mathrm{Seq}} \langle \dot{e}_{\mathbf{i}}, f \rangle D_{\mathbf{i}}. \tag{14}$$

8.4. The Fourier Transform

For each weight $\beta \in P$, we define e^β to be the function $x \mapsto e^{\langle \beta, x \rangle}$ on $\mathfrak{t}_{\mathbb{C}}$. Let $\mathcal{PP}'$ be the space of meromorphic functions on $\mathfrak{t}_{\mathbb{C}}$ spanned by these exponentials over the field $\mathbb{C}(\mathfrak{t})$ of rational functions. The *Fourier Transform* is defined to be the map

$$\begin{aligned} \mathrm{FT} : \mathcal{PP} &\longrightarrow \mathcal{PP}', \\ \mu &\longmapsto \left(x \mapsto \int_{\beta \in \mathfrak{t}^*_{\mathbb{R}}} e^{\langle \beta, x \rangle} \, d\mu \right). \end{aligned}$$

LEMMA 8.6. *The Fourier transform is one-to-one and satisfies*
(i) $\mathrm{FT}(a*b)=\mathrm{FT}(a)\,\mathrm{FT}(b)$ *for all* $a,b\in\mathcal{PP}$.
(ii) *Let* $\beta\in P$. *Denoting by* δ_β *the point measure at* β, *we have* $\mathrm{FT}(\delta_\beta)=e^\beta$.

LEMMA 8.7. *For a sequence* $\mathbf{i}=(i_1,...,i_p)$, *the Fourier transform* $\mathrm{FT}(D_\mathbf{i})$ *is given by*

$$\mathrm{FT}(D_\mathbf{i})=\sum_{j=0}^{p}\frac{e^{-\beta_j^\mathbf{i}}}{\prod_{k\neq j}(\beta_k^\mathbf{i}-\beta_j^\mathbf{i})}.$$

Proof. The Fourier Transform for the Lesbesgue measure on a polytope is well known (see for example [Bri2, Proposition 5.3]). The current result then follows from the compatibility between pull-back of functions and push-forward of measures. □

The exponentials $e^{-\beta_k^\mathbf{i}}$ can be regarded as regular functions on the torus T. On the other hand, the denominators $\prod_{k\neq j}(\beta_k^\mathbf{i}-\beta_j^\mathbf{i})$ belong to the multiplicative subset $S\subset\mathbb{C}[\mathfrak{t}]$ generated by the set $Q\setminus\{0\}$. From the lemmas, we immediately obtain the following.

COROLLARY 8.8. *The composition* $\mathrm{FT}\circ D$ *defines an algebra morphism*

$$\mathbb{C}[N]\longrightarrow S^{-1}\mathbb{C}[\mathfrak{t}]\otimes\mathbb{C}[T]$$

Thus, the map $\mathrm{FT}\circ D$ can geometrically be viewed as a rational map $\mathfrak{t}\times T\to N$. (Note here that S can be replaced by a finitely generated semigroup, because $\mathbb{C}[N]$ is finitely generated.)

Remark 8.9. No open subset of $\mathfrak{t}\times T$ maps dominantly to N if $\dim N>\dim(\mathfrak{t}\times T)$, so $\mathrm{FT}\circ D$ cannot be injective if the number of positive roots exceeds twice the rank of G. Since FT is one-to-one, this means that D is not injective in general.

Remark 8.10. For any $f\in\mathbb{C}[N]$, we can write

$$\mathrm{FT}\circ D(f)=\sum_{\mathbf{i}\in\mathrm{Seq}}\langle\dot{e}_\mathbf{i},f\rangle\,\mathrm{FT}(D_\mathbf{i}).$$

If $f\in\mathbb{C}[N]_{-\nu}$, then the sum can be restricted to sequences in $\mathrm{Seq}(\nu)$, and we see from Lemma 8.7 that the exponentials $e^{-\beta}$ that appear in $\mathrm{FT}\circ D(f)$ satisfy $0\leqslant\beta\leqslant\nu$.

Further, comparing with Proposition 8.4, we see that $\overline{D}(f)$ is the coefficient of $e^{-\nu}$. On the other hand, it is not difficult to show (using Remark 6.7) that the map $x\mapsto n_x^{-1}$, corresponds to the coefficient of e^0 in $\mathrm{FT}\circ D(f)$.

THEOREM 8.11. *Let* $x\in\mathfrak{t}^{\mathrm{reg}}$, *let* $t\in T$, *and let* $f\in\mathbb{C}[N]$. *Then* $\mathrm{FT}\circ D(f)$, *viewed as a rational function on* $\mathfrak{t}\times T$, *can be evaluated at* (x,t), *and we have*

$$\mathrm{FT}\circ D(f)(x,t)=f(t^{-1}n_xtn_x^{-1}).$$

Proof. We first consider the particular case where $\langle\beta,x\rangle\neq 0$ for all $\beta\in Q\setminus\{0\}$. Given $\mathbf{i}\in\mathrm{Seq}$ of length p and $\ell\in\{0,...,p\}$, we set

$$A_\ell^{\mathbf{i}}(x,t)=\frac{t^{-\beta_\ell^{\mathbf{i}}}}{\prod\limits_{\substack{m=0\\ m\neq\ell}}^{p}(\beta_m^{\mathbf{i}}-\beta_\ell^{\mathbf{i}})(x)},$$

where $t^{-\beta_\ell^{\mathbf{i}}}$ means the evaluation at t^{-1} of the weight $\beta_\ell^{\mathbf{i}}$. In view of Lemma 8.7, we want to prove that the linear form

$$\sum_{\mathbf{i}\in\mathrm{Seq}}\left(\sum_{\ell=0}^{p}A_\ell^{\mathbf{i}}(x,t)\right)\dot{e}_{\mathbf{i}} \tag{15}$$

on $\mathbb{C}[N]$ is the evaluation at the point $t^{-1}n_xtn_x^{-1}$.

We first note that the linear form (15) is an algebra map $\mathbb{C}[N]\to\mathbb{C}$, because it is the composition of the algebra map $\mathrm{FT}\circ D$ with the evaluation at (x,t). Therefore, it is the evaluation at a point $n\in N$.

By construction, the element $t^{-1}n_xtn_x^{-1}$ is the unique element of N that brings $x+\dot{e}$ to $\mathrm{Ad}_{t^{-1}}(x+\dot{e})=x+\mathrm{Ad}_{t^{-1}}(\dot{e})$. Let us show that n fulfills this task.

Since $A_0^{(k)}(x,t)+A_1^{(k)}(x,t)=(1-t^{-\alpha_k})/\alpha_k(x)$, we have

$$\sum_{k\in I}(A_0^{(k)}(x,t)+A_1^{(k)}(x,t))\,\mathrm{ad}_{\dot{e}_k}(x)=\sum_{k\in I}(t^{-\alpha_k}-1)\dot{e}_k=\mathrm{Ad}_{t^{-1}}(\dot{e})-\dot{e}.$$

Moreover, for each sequence $\mathbf{i}$ (possibly empty) of length p and each pair (j,k) of elements from I, we have

$$A_{p+1}^{(\mathbf{i},j)}(x,t)-A_{p+1}^{(\mathbf{i},j,k)}(x,t)\times\alpha_k(x)=0,$$

and therefore

$$\begin{aligned}&\left(\sum_{\ell=0}^{p+1}A_\ell^{(\mathbf{i},j)}(x,t)\right)\mathrm{ad}_{\dot{e}_{(\mathbf{i},j)}}(\dot{e}_k)+\left(\sum_{\ell=0}^{p+2}A_\ell^{(\mathbf{i},j,k)}(x,t)\right)\mathrm{ad}_{\dot{e}_{(\mathbf{i},j,k)}}(x)\\&\quad=\left(\sum_{\ell=0}^{p}(A_\ell^{(\mathbf{i},j)}(x,t)-A_\ell^{(\mathbf{i},j,k)}(x,t)\times\alpha_k(x))-A_{p+2}^{(\mathbf{i},j,k)}(x,t)\times\alpha_k(x)\right)\mathrm{ad}_{\dot{e}_{\mathbf{i}}}([\dot{e}_j,\dot{e}_k]).\end{aligned}$$

Summing these elements with (j,k) running over I^2 therefore gives zero since terms pairwise cancel; indeed

$$A_\ell^{(\mathbf{i},j)}(x,t)-A_\ell^{(\mathbf{i},j,k)}(x,t)\times\alpha_k(x)=\frac{t^{-\beta_\ell^{\mathbf{i}}}}{\left(\prod\limits_{\substack{m=0\\ m\neq\ell}}^{p}(\beta_m^{\mathbf{i}}-\beta_\ell^{\mathbf{i}})\times(\beta_p^{\mathbf{i}}-\beta_\ell^{\mathbf{i}}+\alpha_j+\alpha_k)\right)(x)}$$

and

$$-A_{p+2}^{(\mathbf{i},j,k)}(x,t)\times\alpha_k(x)=\frac{t^{-\beta_\ell^{\mathbf{i}}-\alpha_j-\alpha_k}}{\left(\prod_{m=0}^{p}(\beta_m^{\mathbf{i}}-\beta_p^{\mathbf{i}}-\alpha_j-\alpha_k)\right)(x)}$$

are symmetric in (j,k), while the Lie bracket $[\dot{e}_j,\dot{e}_k]$ is antisymmetric. Taking the sum over all $\mathbf{i}\in\mathrm{Seq}$ then yields the equality

$$\mathrm{Ad}_n(x+\dot{e})=x+\mathrm{Ad}_{t^{-1}}(\dot{e}),$$

which completes the proof of the equality $n=t^{-1}n_xtn_x^{-1}$.

The theorem is thus established when $\langle\beta,x\rangle\neq0$ for all $\beta\in Q\setminus\{0\}$. The general case then follows from the regularity of the map $(x,t)\mapsto f(t^{-1}n_xtn_x^{-1})$ on $\mathfrak{t}^{\mathrm{reg}}\times T$. □

8.5. Universal centralizer interpretation

We now give a reinterpretation of Theorem 8.11 using a version of the universal centralizer.

For any $y\in\mathfrak{g}$, we write $C_G(y)=\{g\in G\colon \mathrm{Ad}_g(y)=y\}$ for its centralizer in G. This is an algebraic group of dimension at least r, the rank of G. An element y is said to be *regular* if $\dim C_G(y)=r$. It is well known that regular elements form a non-empty open subset of $\mathfrak{g}$ in the Zariski topology.

We will need a lemma. For $k\geqslant1$, denote by $\mathscr{C}^k$ the kth term in the lower central series of the nilpotent Lie algebra $\mathfrak{n}$; hence, $\mathscr{C}^1=\mathfrak{n}$ and $\mathscr{C}^k=0$ for k large enough. For $x\in\mathfrak{t}$, denote by $\mathfrak{n}^x=\{f\in\mathfrak{n}\colon[x,f]=0\}$ the set of elements in $\mathfrak{n}$ that centralize x.

Lemma 8.12. *Let $k\geqslant1$ be an integer and let $(x,a,b)\in\mathfrak{t}\times\mathfrak{n}^x\times\mathscr{C}^k$. Then, there is $(m,a')\in N\times\mathfrak{n}^x$ such that $\mathrm{Ad}_m(x+a+b)\in x+a'+\mathscr{C}^{k+1}$.*

Proof. The linear map $\mathrm{ad}_x\colon\mathfrak{g}\to\mathfrak{g}$ is semisimple and leaves the subspace $\mathscr{C}^k$ stable. We can thus decompose into direct sums

$$\mathfrak{g}=\mathrm{im}\,\mathrm{ad}_x\oplus\ker\mathrm{ad}_x\quad\text{and}\quad\mathscr{C}^k=\mathrm{ad}_x(\mathscr{C}^k)\oplus(\mathscr{C}^k\cap\ker\mathrm{ad}_x).$$

Let us write $b=[x,u]+b'$, with u and b' in $\mathscr{C}^k$ and $[x,b']=0$. Then, $m=\exp u$ satisfies

$$\mathrm{Ad}_m(x+a+b)\in x+a+b+[u,x]+\mathscr{C}^{k+1},$$

and one can simply take $a'=a+b'$. □

Proposition 8.13. *For any $x\in\mathfrak{t}$, the element $x+\dot{e}$ is regular and $C_G(x+\dot{e})\subset B$.*

Proof. First, we prove that $x+\dot{e}$ is regular. Consider the action of $G\times\mathbb{C}^\times$ on $\mathfrak{g}$, where G acts by the adjoint action and $\mathbb{C}^\times$ acts by scaling. Define $\mathbb{C}^\times\to G\times\mathbb{C}^\times$ by $s\mapsto(s^{-\rho^\vee},s)$. Then, $s\cdot(x+\dot{e})=sx+\dot{e}$. So $\lim_{s\to 0}s\cdot(x+\dot{e})=\dot{e}$. The set of regular elements is open in $\mathfrak{g}$ and is invariant under the action of $G\times\mathbb{C}^\times$. Since the limit point $\dot{e}$ is regular, we conclude that $x+\dot{e}$ is regular.

Now, we prove that $C_G(x+\dot{e})\subset B$. Starting with $a=0$ and $b=\dot{e}$, we apply Lemma 8.12 several times, taking successively $k=1,2,\dots$. Composing all the maps Ad_m obtained in the process, we eventually find elements $m\in N$ and $f\in\mathfrak{n}^x$ such that $\mathrm{Ad}_m(x+\dot{e})=x+f$. Note that x and f are the components of the Jordan–Chevalley decomposition of $x+f$.

On the other hand, Theorem 2.2 in [Hu2] states that the centralizer L of x is a reductive group with maximal torus T and root system

$$\Phi_L=\{\alpha \text{ root of } G:\langle\alpha,x\rangle=0\}.$$

(As a matter of fact, the statement in [Hu2] deals with the centralizer of a semisimple element in G and not of an element $x\in\mathfrak{g}$, but the proof can be adapted to our situation.) Further L is connected [Hu2, Theorem 2.11]. The intersection $B_L:=B\cap L$ is a Borel subgroup of L. We also note that f belongs to the centralizer of x in $\mathfrak{g}$, that is, the Lie algebra of L.

By the uniqueness of the Jordan–Chevalley decomposition, the centralizer $C_G(x+f)$ is the joint centralizer of x and f in G, so is the centralizer $C_L(f)$. From the fact that $x+f$ is regular in the Lie algebra $\mathfrak{g}$, we then deduce that f is a regular nilpotent element in the Lie algebra of L. We write $g=b\bar{w}b'$ in the Bruhat decomposition of L, where b and b' are in B_L and w is in the Weyl group of L. Both $\mathrm{Ad}_{b'}(f)$ and $\mathrm{Ad}_{b^{-1}}(f)$ are regular nilpotent elements in the Lie algebra of B_L, so are linear combinations of positive roots vectors, each simple root in Φ_L occurring with a non-zero coefficient ([Bo, chap. 8, §11, no. 4, proposition 10]). Since $\mathrm{Ad}_{\bar{w}}$ maps $\mathrm{Ad}_{b'}(f)$ to $\mathrm{Ad}_{b^{-1}}(f)$, this implies that w maps each simple root in Φ_L to a positive root, and it follows that $w=1$ and $g\in B_L$. We conclude that $C_L(f)\subset B_L$. Thus,

$$C_G(x+\dot{e})=\mathrm{Ad}_{m^{-1}}(C_G(x+f))=\mathrm{Ad}_{m^{-1}}(C_L(f))\subset\mathrm{Ad}_{m^{-1}}(B_L)\subset B. \qquad \square$$

We define the *universal centralizer space* to be

$$C:=\{(x,b)\in\mathfrak{t}\times B:b\in C_G(x+\dot{e})\}$$

Remark 8.14. Our space C is the base change over $\mathfrak{t}\to\mathfrak{t}/W$ of the usual universal centralizer, as defined for example in [BFM, §2.2].

From the definition, we have maps $C\to\mathfrak{t}$, $C\to T$ and $C\to N$, that send a pair $(x,tn)\in C$ to x, t and n, respectively (where $x\in\mathfrak{t}$, $t\in T$ and $n\in N$). We will be particularly interested in the map $\psi\colon C\to N$.

PROPOSITION 8.15. (i) *The above maps $C\to\mathfrak{t}$ and $C\to T$ restrict to an isomorphism $\mathfrak{t}^{\mathrm{reg}}\times_{\mathfrak{t}}C=\mathfrak{t}^{\mathrm{reg}}\times T$.*

(ii) *With respect to the isomorphism in* (i), *the map ψ restricts to*

$$\begin{aligned}\psi_{\mathrm{reg}}\colon \mathfrak{t}^{\mathrm{reg}}\times T &\longrightarrow N,\\ (x,t) &\longmapsto t^{-1}n_x t n_x^{-1}.\end{aligned}$$

(iii) *The resulting algebra morphism $\psi^*_{\mathrm{reg}}\colon\mathbb{C}[N]\to\mathbb{C}[\mathfrak{t}^{\mathrm{reg}}\times T]$ agrees with* $\mathrm{FT}\circ D$.

Proof. (i) Since $\mathrm{Ad}(n_x)(x)=x+\dot{e}$, we have $C_G(x+\dot{e})=n_xC_G(x)n_x^{-1}$. Since $x\in\mathfrak{t}^{\mathrm{reg}}$, we have $C_G(x)=T$, and so $C_G(x+\dot{e})=\{n_xtn_x^{-1}:t\in T\}$. The map $(x,t)\mapsto(x,n_xtn_x^{-1})$ from $\mathfrak{t}^{\mathrm{reg}}\times T$ to C is the converse of the desired isomorphism.

(ii) Since $n_xtn_x^{-1}=tt^{-1}n_xtn_x^{-1}$ and $t^{-1}n_xtn_x^{-1}\in N$, the result follows.

(iii) Given the previous result, this is just a restatement of Theorem 8.11. □

9. Generalities on Duistermaat–Heckman measures

9.1. Duistermaat–Heckman measures

We will now define Duistermaat–Heckman measures algebraically, following the work of Brion–Procesi [BP]. In this section, we work in a general context of a projective variety with the action of a torus. Later, we will apply these ideas to the case of an MV cycle with the action of $T^\vee$.

Let V be a (possibly infinite-dimensional) vector space with a linear action of a torus T. Let P be the weight lattice of T and let $P^\vee$ be its coweight lattice. Let $X\subset\mathbb{P}V$ be a finite-dimensional T-invariant closed subscheme of the projectivization of V. Let $\mathcal{O}(n)$ denote the usual line bundle on $\mathbb{P}(V)$. Since T acts linearly on V, $\mathcal{O}(n)$ carries a natural T-equivariant structure.

We do not assume that the torus T acts effectively on X (or even on V). We write T' for the quotient of T acting effectively on X.

On the other hand, we do assume that X^T is finite. For each $p\in X^T$, let $\Phi_T(p)$ be the weight of the action of T on the fibre of $\mathcal{O}(1)$ at the point p. Equivalently, $p=[v]$ for some weight vector $v\in V$, and $\Phi_T(p)$ is negative the weight of v.

Define the *moment polytope* $\mathrm{Pol}(X)$ to be

$$\mathrm{Pol}(X):=\mathrm{Conv}(\Phi_T(p):p\in X^T).$$

If X is connected (e.g. irreducible), then $\mathrm{Pol}(X)$ is contained in a translate of $(\mathfrak{t}')^*_{\mathbb{R}} \subset \mathfrak{t}^*_{\mathbb{R}}$.

In fact, it is easy to see that $\mathrm{Pol}(X)$ is the convex hull of all negatives of weights of the smallest linear subspace of V containing X.

The torus T acts on the space of sections $\Gamma(X, \mathcal{O}(n))$. We consider $[\Gamma(X, \mathcal{O}(n))]$, the class of $\Gamma(X, \mathcal{O}(n))$ in $R(T)$, the complexified representation ring of T. We can embed $R(T)$ into the space of distributions on $\mathfrak{t}^*_{\mathbb{R}}$ by setting

$$[U] \longmapsto \sum_{\mu \in P} \dim U_\mu \delta_\mu$$

Let $\tau_n \colon \mathfrak{t}^*_{\mathbb{R}} \to \mathfrak{t}^*_{\mathbb{R}}$ be the automorphism given by scaling by $1/n$.

Definition 9.1. The *Duistermaat–Heckman measure of the triple* (X, T, V) is defined to be the weak limit

$$\mathrm{DH}(X) = \lim_{n \to \infty} \frac{1}{n^{\dim X}} (\tau_n)_* [\Gamma(X, \mathcal{O}(n))]$$

within the space of distributions on $\mathfrak{t}^*_{\mathbb{R}}$.

Note that each $(\tau_n)_*[\Gamma(X, \mathcal{O}(n))]$ is supported on $\mathrm{Pol}(X)$, and hence so is $\mathrm{DH}(X)$. In fact, we have the following result of Brion–Procesi [BP].

PROPOSITION 9.2. *The measure* $\mathrm{DH}(X)$ *is well-defined, has support* $\mathrm{Pol}(X)$, *and is piecewise polynomial of degree* $\dim X - \dim T'$.

9.2. Fourier transform of DH measures and equivariant multiplicities

For this section, assume that each fixed point $p \in X^T$ is non-degenerate and attracting. This means that, for each p, there exists $\gamma \in P^\vee$ such that if μ is a weight of T acting on $T_p X$, then $\langle \gamma, \mu \rangle > 0$. We write σ_p^0 for the set of such γ (this is the intersection of $P^\vee$ with an open cone in $\mathfrak{t}^*_{\mathbb{R}}$ and thus is Zariski dense in $\mathfrak{t}^*$).

We can compute the Fourier transform (as defined in §8.4) of DH measures with the help of localization in equivariant K-theory and equivariant homology. Let $\widehat{S}$ be the multiplicative set in $R(T)$ generated by $1 - \delta_\mu$ for $\mu \in P \setminus \{0\}$, and let S be the multiplicative set in $H_T^\bullet = \mathrm{Sym}\, \mathfrak{t}^* = \mathbb{C}[\mathfrak{t}]$ generated by $\mu \in P \setminus \{0\}$.

Let $K^T(X)$ denote the Grothendieck group of T-equivariant coherent sheaves on X. This is a module over $R(T)$. The following result is due to Thomason [Th, Théorème 2.1], for K-theory, and many authors independently (such as Brion [Bri3, Lemma 1] or Evens and Mirković [EM, Theorem B.2]), for homology.

THEOREM 9.3. *The inclusion* $X^T \to X$ *induces isomorphisms*

$$\widehat{S}^{-1} K^T(X^T) \xrightarrow{\sim} \widehat{S}^{-1} K^T(X) \quad \textit{and} \quad S^{-1} H_\bullet^T(X^T) \xrightarrow{\sim} S^{-1} H_\bullet^T(X).$$

Because of this theorem, we can write

$$[\mathcal{O}_X] = \sum_{p\in X^T} \hat{\varepsilon}_p^T(X)[\mathcal{O}_p] \quad \text{and} \quad [X] = \sum_{p\in X^T} \varepsilon_p^T(X)[\{p\}]$$

for unique $\hat{\varepsilon}_p^T(X)\in\widehat{S}^{-1}R(T)$ and $\varepsilon_p^T(X)\in S^{-1}\mathbb{C}[\mathfrak{t}]$. Following Brion [Bri1], we call $\varepsilon_p^T(X)$ the *equivariant multiplicity* of X at p.

One advantage of these equivariant multiplicities is that they can be computed locally. Let X_p° denote an affine open T-invariant neighbourhood of p in X. (In [Bri1, Proposition 4.4], Brion observes that the only such X_p° is the attracting set of p.)

We will need the following preliminary definition. Let $A=\bigoplus_{m=0}^\infty A_m$ be an $\mathbb{N}$-graded finitely-generated commutative algebra of Krull dimension d. Then, the multiplicity of A is defined to be

$$(d-1)!\lim_{m\to\infty}\frac{\dim A_m}{m^{d-1}}$$

(this limit exists and is always non-zero).

Proposition 9.4. (i) $[\mathbb{C}[X_p^\circ]]$ *is a well-defined element of* $\widehat{S}^{-1}R(T)$, *and we have* $\hat{\varepsilon}_p^T(X)=[\mathbb{C}[X_p^\circ]]$ *in* $\widehat{S}^{-1}R(T)$.

(ii) *We have* $\varepsilon_p^T(X)[\{p\}]=[X_p^\circ]$ *in* $S^{-1}H_\bullet^T(X_p^\circ)$.

(iii) *For any* $\gamma\in\sigma_p^0$, *the rational function* $\varepsilon_p^T(X)$ *is well defined at* γ, *and* $\varepsilon_p^T(X)(\gamma)$ *is the multiplicity of the algebra* $\mathbb{C}[X_p^\circ]$ *graded with respect to* $-\gamma$.

Proof. $[\mathbb{C}[X_p^\circ]]$ is well defined because of the assumption of attractiveness (see [CG, Proposition 6.6.6]). The rest of part (i) and (ii) then follow immediately from pull-back to the open set X_p°.

Part (iii) is due to Brion [Bri1, Proposition 4.4]. □

To facilitate further computation of $\varepsilon_p^T(X)$, suppose that we have a representation W of T, all of whose weights are non-zero, and assume we are given a T-equivariant closed embedding $X_p^\circ\to W$. The *multidegree* $\operatorname{mdeg}[W](X_p^\circ)\in H_T^\bullet$ is defined by the equation $\operatorname{mdeg}[W](X_p^\circ)[W]=[X_p^\circ]$ in $H_\bullet^T(W)$. This notion of multidegree is useful, since it can be computed using the methods of commutative algebra (see for example [KZJ, §1.5]). On the other hand, the multidegree of X_p° determines the equivariant multiplicity of X at p as follows.

Proposition 9.5. *With the above setup, we have*

$$\varepsilon_p^T(X) = \frac{\operatorname{mdeg}[W](X_p^\circ)}{\prod_{\mu \text{ wt of } W}\mu}$$

Proof. We know that $[X_p^\circ]=\varepsilon_p^T(X)[\{p\}]$ in $H_\bullet^T(W)$. As $\operatorname{mdeg}[W](X_p^\circ)[W]=[X_p^\circ]$ and $(\prod_{\mu \text{ wt of } W}\mu)[W]=[\{p\}]$ in $H_\bullet^T(W)$, the result follows. □

We are ready to relate the Duistermaat–Heckman measure to equivariant multiplicity.

THEOREM 9.6. *We have*

$$\mathrm{FT}(\mathrm{DH}(X)) = \sum_{p\in X^T} \varepsilon_p^T(X) e^{\Phi(p)}$$

Proof. For sufficiently large n, we have $H^i(X,\mathcal{O}(n))=0$ for $i>0$. Thus, for sufficiently large n, $[\Gamma(X,\mathcal{O}(n))]$ equals the integral of $[\mathcal{O}_X\otimes\mathcal{O}(n)]$ in equivariant K-theory. Hence, from

$$[\mathcal{O}_X] = \sum_{p\in X^T} \hat{\varepsilon}_p^T(X)[\mathcal{O}_p],$$

we deduce that, for sufficiently large n, we have

$$[\Gamma(X,\mathcal{O}(n))] = \sum_{p\in X^T} \hat{\varepsilon}_p^T(X)\delta_{n\Phi(p)}.$$

Let $d=\dim X$. We see that

$$\begin{aligned}\mathrm{FT}(\mathrm{DH}(X)) &= \mathrm{FT}\left(\lim_{n\to\infty}\frac{1}{n^d}(\tau_n)_*\left(\sum_p \hat{\varepsilon}_p^T(X)\delta_{n\Phi(p)}\right)\right)\\ &= \sum_p \lim_{n\to\infty}\frac{1}{n^d}\,\mathrm{FT}((\tau_n)_*(\hat{\varepsilon}_p^T(X)))e^{\Phi(p)}.\end{aligned}$$

So it suffices to show that, for each p, we have

$$\lim_{n\to\infty}\frac{1}{n^d}\,\mathrm{FT}((\tau_n)_*(\hat{\varepsilon}_p^T(X))) = \varepsilon_p^T(X)$$

Now, pick $\gamma\in\sigma_p^0$ and let c be the multiplicity of the algebra $\mathbb{C}[X_p^\circ]$ graded with respect to $-\gamma$. By Proposition 9.4 (iii), it suffices to show that

$$\lim_{n\to\infty}\frac{1}{n^d}\,\mathrm{FT}((\tau_n)_*(\hat{\varepsilon}_p^T(X)))(\gamma) = c.$$

Using Proposition 9.4 (i), we see that

$$\lim_{n\to\infty}\frac{1}{n^{\dim X}}\,\mathrm{FT}((\tau_n)_*(\hat{\varepsilon}_p^T(X)))(\gamma) = \lim_{n\to\infty}\frac{1}{n^d}\sum_{m=0}^{\infty} a_m e^{-m/n},$$

where

$$a_m = \sum_{\langle\mu,-\gamma\rangle=m} \dim \mathbb{C}[X_p^\circ]_\mu.$$

Using the fact that

$$a_m = \frac{c}{(d-1)!}m^{d-1}+\ldots,$$

by elementary calculus, we compute that this limit equals c, as desired. □

Remark 9.7. The above theorem holds without the assumption that the fixed points are attractive (though this assumption suffices for our purposes). In fact, the only place that attractiveness assumption is used in this section is in Proposition 9.4 (i). We can avoid using attractiveness by using degeneration to normal cones of the fixed points (thereby making them attractive with respect to the attendant new circle action).

9.3. An extension to coherent sheaves

We continue in the above setup, but we consider a T-equivariant coherent sheaf $\mathcal{F}$ on X. Following [CG, Definition 5.9.4], we define the *support cycle* of $\mathcal{F}$ as

$$[\operatorname{supp}\mathcal{F}] := \sum_S \operatorname{mult}_S(\mathcal{F})[S] \in H_\bullet(X)$$

where the sum ranges over all maximal dimensional irreducible components of the support of $\mathcal{F}$. These components are necessarily T-invariant.

We define the Duistermaat–Heckman measure of $\mathcal{F}$ by

$$\mathrm{DH}(\mathcal{F}) := \lim_{n\to\infty} \frac{1}{n^{\dim\operatorname{supp}\mathcal{F}}}(\tau_n)_*[\Gamma(X, \mathcal{F}\otimes\mathcal{O}(n))]$$

The main result of this section is that $\mathrm{DH}(\mathcal{F})$ only depends on the support cycle.

THEOREM 9.8. *We have the following expansion*:

$$\mathrm{DH}(\mathcal{F}) = \sum_S \operatorname{mult}_S(\mathcal{F})\,\mathrm{DH}(S),$$

where the sum ranges over all maximal-dimensional irreducible components of the support of $\mathcal{F}$.

Proof. As in the previous section, we can consider the expansion of $[\mathcal{F}]\in\widehat{S}^{-1}K^T(X)$ in terms of the fixed points, and we define $\hat{\varepsilon}_p^T(\mathcal{F})$ by

$$[\mathcal{F}] = \sum_p \hat{\varepsilon}_p^T(\mathcal{F})[\mathcal{O}_p].$$

Then, proceeding as in the proof of Theorem 9.6, we obtain that

$$\mathrm{FT}(\mathrm{DH}(\mathcal{F})) = \sum_p \lim_{n\to\infty} \frac{1}{n^d}\,\mathrm{FT}((\tau_n)_*(\hat{\varepsilon}_p^T(\mathcal{F})))e^{\Phi(p)},$$

where $d=\dim\operatorname{supp}\mathcal{F}$.

Now, as above $\hat{\varepsilon}_p^T(\mathcal{F})=[\Gamma(X_p^\circ,\mathcal{F})]$ in $\widehat{S}^{-1}R(T)$. Thus, we are reduced to a local computation on each open set X_p°. The desired equation then follows from a result in Chriss–Ginzburg [CG, Theorem 6.6.12]. □

9.4. A formula using BB strata

The third author has given a formula in [KnA2] for computing this DH measure using components of Białynicki–Birula strata. We now recall this formula, and for simplicity assume X is irreducible.

Choose a coweight ν of T and thus an embedding $\mathbb{C}^\times \hookrightarrow T$. Assume that $X^{\mathbb{C}^\times} = X^T$ and moreover that the composite map $X^{\mathbb{C}^\times} \xrightarrow{\Phi_T} P \xrightarrow{\nu} \mathbb{Z}$ is injective. (If Φ_T is injective, then it is easy to construct such ν, and otherwise it is of course impossible.) By this assumption, $X^{\mathbb{C}^\times}$ acquires the structure of a totally-ordered set (very much depending on the choice of ν). We write p_{top} and p_{bot} for the maximal and minimal points in this set, respectively.

Definition 9.9. For each $p \in X^{\mathbb{C}^\times}$, let X_p denote the Białynicki–Birula stratum of p, defined by

$$X_p := \left\{ x \in X : \lim_{s \to 0} sx = p \right\}$$

An irreducible component Y of $\overline{X}_p$ will be a called a *BB cycle* in X based at p.

Note that the only BB cycle of weight $\Phi(p_{\mathrm{top}})$ is X and the only BB cycle of weight $\Phi(p_{\mathrm{bot}})$ is $\{p_{\mathrm{bot}}\}$.

For each $p \in X^{\mathbb{C}^\times}$, fix $f_p \in V^*$ such that f_p is a T-weight vector and $f_p(p) \neq 0$. Note that this implies that f_p vanishes at every other fixed point (since they live in different weight spaces in $\mathbb{P}(V)$). The section f_p defines a T-invariant Cartier divisor $D_p \subset X$.

For any two BB cycles Y_1 and Y_2 based at $p_1 < p_2$, we define

$$c(Y_1, Y_2) := i(Y_1, D_{p_2} \cdot Y_2).$$

Definition 9.10. A *chain* of BB cycles is a sequence $Y_\bullet = (Y_0, Y_1, ..., Y_m)$ such that $Y_0 = \{p_{\mathrm{bot}}\}$, $Y_m = X$, and $c(Y_{k-1}, Y_k) \neq 0$ for all k. A *weight chain* is a sequence

$$\mu_\bullet = (\mu_0, ..., \mu_m)$$

such that there exists a chain of BB cycles $Y_\bullet$ such that the weight of Y_k is μ_k for all k.

Note that, if Y is a BB cycle based at p, then $\dim D_p \cap Y = \dim Y - 1$, so the length of any chain is the dimension of X.

For any weight chain $\mu_\bullet$, define

$$c(\mu_\bullet) = \sum_{Y_\bullet} c(Y_0, Y_1) \dots c(Y_{m-1}, Y_m),$$

where the sum is over all chains of BB cycles $Y_\bullet$ of weight $\mu_\bullet$.

As before, let Δ^m be the standard m-simplex.

For any weight chain $\mu_\bullet$, define $\pi_{\mu_\bullet}\colon \mathbb{R}^{m+1}\to\mathfrak{t}^*_{\mathbb{R}}$ by

$$\pi_{\mu_\bullet}(a_0,\dots,a_m)=\sum_{k=0}^{m} a_k\mu_k.$$

The following result (due to the third author [KnA2]) explains how these weight chains can be used to compute the DH measure.

THEOREM 9.11.

$$\mathrm{DH}(X)=\sum_{\mu_\bullet} c(\mu_\bullet)(\pi_{\mu_\bullet})_*(\delta_{\Delta^m}),$$

where the sum is over all weight chains.

10. Measures from MV cycles

Recall that at the beginning of §8, we fixed a choice of regular nilpotent, $\dot{e}$. From this point on, we specialize $\dot{e}=\bar{e}$, where $\bar{e}$ is the principal nilpotent coming from the geometric Satake correspondence §4.3. In particular, we have $\dot{e}_i=q(\alpha_i)e_i$.

10.1. DH measures of MV cycles

Let $\nu\in Q_+$ and let Z be a stable MV cycle of weight $-\nu$. Note that Z is a $T^\vee$-invariant subvariety of the affine Grassmannian and we have a projective embedding $\Upsilon\colon \mathcal{G}\mathrm{r}\to\mathbb{P}(V)$, from §4.3.

We will apply the constructions of the last two sections to the triple $(Z,T^\vee,V)$. In particular, in §9.1 we defined a polytope $\mathrm{Pol}(Z)$; note that this polytope lives in $\mathfrak{t}_{\mathbb{R}}=(\mathfrak{t}^\vee_{\mathbb{R}})^*$. On the other hand, in §6.3 we defined the MV polytope (also denoted $\mathrm{Pol}(Z)$) which lives in $\mathfrak{t}^*_{\mathbb{R}}$.

Recall that we have the map $\iota\colon P\to\mathfrak{t}_{\mathbb{R}}$ which we can extend to a linear bijection $\iota\colon \mathfrak{t}^*_{\mathbb{R}}\to\mathfrak{t}_{\mathbb{R}}$.

LEMMA 10.1. *For any stable MV cycle Z, we have an equality $\iota(\mathrm{Pol}(Z))=\mathrm{Pol}(Z)$.*

Proof. The action of $T^\vee$ on $\mathcal{G}\mathrm{r}$ has fixed points $\{L_\mu:\mu\in P\}$. Moreover, the weight of $T^\vee$ acting on the fibre of $\mathcal{O}(1)$ at the point L_μ is $\iota(\mu)$, by Proposition 4.1. □

Similarly, by definition, $\mathrm{DH}(Z)$ is a distribution on $\mathfrak{t}_{\mathbb{R}}=(\mathfrak{t}^\vee_{\mathbb{R}})^*$ and we also have the distribution $D(b_Z)$ on $\mathfrak{t}^*_{\mathbb{R}}$.

THEOREM 10.2. *For any stable MV cycle Z, we have an equality $\iota_*(D(b_Z))=\mathrm{DH}(Z)$.*

Proof. We choose a generic antidominant embedding $\mathbb{C}^\times \hookrightarrow T^\vee$. Then, the BB cycles inside Z are simply the stable MV cycles contained in Z. Moreover, for each weight chain $\mu_\bullet$, there exists a unique $\mathbf{i}\in\mathrm{Seq}(\nu)$ such that $\mu_k=\alpha_{i_1}+\ldots+\alpha_{i_k}$ for all k.

Thus, by Theorem 9.11, we conclude that upon identifying $\mathfrak{t}_\mathbb{R}$ with $\mathfrak{t}^*_\mathbb{R}$ using ι, we have

$$\mathrm{DH}(Z)=\sum_{\mathbf{i}\in\mathrm{Seq}(\nu)} c(\mathbf{i})D_\mathbf{i},\quad c(\mathbf{i})=\sum_{Y_\bullet} c(Y_0,Y_1)\ldots c(Y_{m-1},Y_m), \tag{16}$$

where the second sum ranges over the set of sequences $Y_\bullet$, where Y_k is a stable MV cycle of weight $-(\alpha_{i_1}+\ldots+\alpha_{i_k})$ and where

$$c(Y_k,Y_{k+1})=i(Y_k,D_{\mu_{k+1}}\cdot Y_{k+1}).$$

Here $D_{\mu_{k+1}}$ is the divisor in $\mathcal{G}\mathrm{r}$ coming from a vector $f\in V^*$ such that $f(L_{\mu_{k+1}})\neq 0$ and f vanishes on all other weight spaces.

On the other hand, we begin with (14)

$$D(b_Z)=\sum_{\mathbf{i}\in\mathrm{Seq}(\nu)}\langle \dot{e}_{i_1}\ldots\dot{e}_{i_m},b_Z\rangle D_\mathbf{i}=\sum_{\mathbf{i}\in\mathrm{Seq}(\nu)}(\dot{e}_{i_1}\ldots\dot{e}_{i_m}b_Z)(1_N)D_\mathbf{i}.$$

Then, we expand out the right-hand side using (4) from the proof of Lemma 6.4, and we reach exactly the same formula (16). □

10.2. Reformulation and alternate proof of Theorem 10.2

We apply the Fourier Transform and then $(\iota^{-1})^*$ to the equality in Theorem 10.2 and we obtain

$$\mathrm{FT}(D(b_Z))=(\iota^{-1})^*\,\mathrm{FT}(\mathrm{DH}(Z)).$$

This is an equality of analytic functions on $\mathfrak{t}_\mathbb{C}$; more precisely, by the analysis in §8.4, this is an equality in the space $S^{-1}\mathbb{C}[\mathfrak{t}]\otimes\mathbb{C}[T]$.

Now, by Proposition 8.15 (iii), we have that $\mathrm{FT}(D(b_Z))=\psi^*_{\mathrm{reg}}(b_Z)$.

On the other hand, by Theorem 9.6, we know that $\mathrm{FT}(\mathrm{DH}(Z))$ is closely related to the expansion of the homology class $[Z]$ in the fixed point basis under the isomorphism

$$\iota(S)^{-1}H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})\cong\iota(S)^{-1}H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r}^{T^\vee}).$$

In order to avoid confusion, in this section we will write $[Z]_{T^\vee}$ for the image of $[Z]$ in $\iota(S)^{-1}H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r}^{T^\vee})$.

The fixed point set $\mathcal{G}\mathrm{r}^{T^\vee}$ is in correspondence with the weight lattice P, and thus has a group structure. Hence, $H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r}^{T^\vee})$ is an algebra, and we have an obvious isomorphism

$$\iota(S)^{-1}\mathbb{C}[\mathfrak{t}^*]\otimes\mathbb{C}[T]\cong H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r}^{T^\vee}).$$

From these observations (and the injectivity of the Fourier Transform), we see that Theorem 10.2 is equivalent to the following statement.

THEOREM 10.3. *Let Z be a stable MV cycle. In the algebra $S^{-1}\mathbb{C}[\mathfrak{t}]\otimes\mathbb{C}[T]$, we have an equality*

$$\psi^*_{\mathrm{reg}}(b_Z)=(\iota^{-1})^*([Z]_{T^\vee}).$$

We will now give an alternate proof of Theorem 10.3 using results of Yun–Zhu [YZ]. These authors define a commutative convolution algebra structure on $H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$, and describe this algebra using the geometric Satake correspondence.

To formulate their result, recall the universal centralizer space C and its morphisms $C\to\mathfrak{t}\times T$ and $\psi\colon C\to N$ from §8.5. The following theorem follows from combining Propositions 3.3, 5.7 and Remark 3.4 from [YZ].

THEOREM 10.4. *There is an isomorphism $\theta\colon\mathbb{C}[C]\to H_\bullet^{T^\vee}(\mathcal{G}r)$ making the following diagram commute*:

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{t}]\otimes\mathbb{C}[T] \arrow[r, "\iota\otimes 1"] \arrow[d] & \mathbb{C}[\mathfrak{t}^*]\otimes\mathbb{C}[T] \arrow[r] & H_\bullet^{T^\vee}(\mathcal{G}r^{T^\vee}) \arrow[d] \\
\mathbb{C}[C] \arrow[rr, "\theta"] & & H_\bullet^{T^\vee}(\mathcal{G}r).
\end{tikzcd}$$

In order to apply this result, we will need to setup a bit more notation.

For any $\lambda\in P_+$, we have a map

$$\tau_\lambda\colon H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\longrightarrow H^{T^\vee}_{d-\bullet}(\overline{\mathcal{G}\mathrm{r}^\lambda})\longrightarrow H^{T^\vee}_{d-\bullet}(\mathcal{G}\mathrm{r}) \tag{17}$$

where the first arrow comes from (2) (with $d=\rho^\vee(\lambda)$). We will also make use of the inclusion

$$H^\bullet(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\hookrightarrow H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)$$

coming from the isomorphism

$$H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\cong H^\bullet(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\otimes\mathbb{C}[\mathfrak{t}^*]$$

given in [YZ, Lemma 2.2].

Also, recall the map

$$\Psi_\lambda\colon H^\bullet(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)=L(\lambda)\hookrightarrow\mathbb{C}[N]$$

from §2.5.

In this section, we will also need to consider the automorphism $t^{-\lambda}$ of $\mathcal{G}\mathrm{r}$, and we write $t^{-\lambda}_*\colon H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})\to H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$ for the resulting map on equivariant homology.

LEMMA 10.5. *For any* $v\in H^\bullet(\mathcal{G}r,\mathcal{I}_\lambda)$, *we have*

$$\theta(\psi^*(\Psi_\lambda(v)))=(t^{-\lambda})_*\tau_\lambda(v).$$

Proof. To begin, fix the isomorphism of varieties $T\times N\cong B$ given by $(t,n)\mapsto tn$. Thus, we get an isomorphism on coordinate rings $\mathbb{C}[B]\cong\mathbb{C}[T]\otimes\mathbb{C}[N]$.

Recall the linear map $v_\lambda^*\colon L(\lambda)\to\mathbb{C}$ defined in §2.5. We extend v_λ^* to a $\mathbb{C}[\mathfrak{t}^*]$-linear map $v_\lambda^*\colon H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\to\mathbb{C}[\mathfrak{t}^*]$ using the isomorphism

$$H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\cong H^\bullet(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)\otimes\mathbb{C}[\mathfrak{t}^*].$$

Note that $v_\lambda^*=p_*\circ\tau_\lambda$, where τ_λ is defined in (17) and where $p_*\colon H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})\to H_\bullet^{T^\vee}(\mathrm{pt})$ is the proper push-forward in Borel–Moore homology.

Given $v\in L(\lambda)$, we define $f_v\in\mathbb{C}[B]$ by $f_v(b)=v_\lambda^*(bv)$. Under the isomorphism

$$\mathbb{C}[B]\cong\mathbb{C}[T]\otimes\mathbb{C}[N],$$

we find that $f_v=\lambda\otimes\Psi_\lambda(v)$.

We can factor $\psi\colon C\to N$ into $C\to B\to N$, which gives rise to a factorization of ψ^* as

$$\mathbb{C}[N]\hookrightarrow\mathbb{C}[B]\xrightarrow{\psi_B^*}\mathbb{C}[C].$$

Following [YZ, §3], we will now describe $\theta\circ\psi_B^*$.

Let $\{h^i\}$ and $\{h_i\}$ be dual bases of the free $\mathbb{C}[\mathfrak{t}^*]$ modules $H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r})$ and $H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$. By the definition of σ_{can} from [YZ, §3], for any $v\in L(\lambda)=H^\bullet(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)$, we have

$$\theta(\psi_B^*(f_v))=\sum_i v_\lambda^*(h^iv)h_i\in H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r}),$$

where we use the action of $H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r})$ on $H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda)$.

Thus,

$$\theta(\psi_B^*(f_v))=\sum_i p_*(\tau_\lambda(h^iv))h_i=\sum_i p_*(h^i\tau_\lambda(v))h_i,$$

since τ_λ is a $H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r})$-module morphism.

Now, since the pairing between $H^\bullet_{T^\vee}(\mathcal{G}\mathrm{r})$ and $H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$ is given by $a\otimes h\mapsto p_*(ah)$, we conclude that

$$\theta(\psi_B^*(f_v))=\tau_\lambda(v). \tag{18}$$

The commutative diagram in Theorem 10.4 implies that the isomorphism

$$\theta\colon\mathbb{C}[C]\longrightarrow H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$$

is a $\mathbb{C}[T]$-module isomorphism, where $\mathbb{C}[T]$ acts on $\mathbb{C}[C]$ using $C\to T$ and acts on $H_\bullet^{T^\vee}(\mathcal{G}\mathrm{r})$ using the maps $(t^\mu)_*$. Thus, we see that (18) along with $f_v=\lambda\otimes\Psi_\lambda(v)$ implies the desired result. $\square$

Finally, here is our promised alternate proof.

Alternate proof of Theorem 10.3. Let Z be a stable MV cycle. Choose $\lambda\in P_+$ such that $t^\lambda Z\subset\overline{\mathcal{G}\mathrm{r}^\lambda}$. Then, $t^\lambda Z$ is an MV cycle of type λ. We consider its class

$$[t^\lambda Z]_{IH}\in H^k(\mathcal{G}\mathrm{r},\mathcal{I}_\lambda).$$

By the definition of τ_λ, we have that $\tau_\lambda([t^\lambda Z]_{IH})=[t^\lambda Z]$ in $H^{T^\vee}_{d-k}(\mathcal{G}\mathrm{r})$ where $d=2\rho^\vee(\lambda)$.

Thus, by Lemma 10.5, we have $\theta(\psi^*(b_Z))=[Z]$ in $H^{T^\vee}_{d-k}(\mathcal{G}\mathrm{r})$.

By Theorem 10.4, the map $H^{T^\vee}_\bullet(\mathcal{G}\mathrm{r}^{T^\vee})\to H^{T^\vee}_\bullet(\mathcal{G}\mathrm{r})$ is dual to $C\to\mathfrak{t}^*\times T$. Thus passing to $\mathfrak{t}^{\mathrm{reg}}$ and inverting this map, we obtain the desired equality. □

10.3. Proof of Muthiah's conjecture

Let Z be a stable MV cycle. As a corollary of Theorem 10.3, it is easy to see that $\overline{D}(b_Z)$ is given by equivariant multiplicities at the bottom of Z.

More precisely, applying Remark 8.10 and Theorem 9.6, we immediately deduce the following.

Corollary 10.6. *Let Z be a stable MV cycle of weight $-\nu$. We have the following equality in $\mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$:*

$$\overline{D}(b_Z)=(\iota^{-1})^*\varepsilon^{T^\vee}_{L_{-\nu}}(Z).$$

Now, we are in a position to prove Mutiah's conjecture (Theorem 1.5). We begin by recalling the setup. Let $\lambda\in P_+\cap Q$, and let Z be an MV cycle of type λ and weight zero, so $t^{-\lambda}Z$ is a stable MV cycle of weight $-\lambda$.

Note that we have an equality

$$\varepsilon^{T^\vee}_{L_{-\lambda}}(t^{-\lambda}Z)=\varepsilon^{T^\vee}_{L_0}(Z).$$

Thus, in light of Corollary 10.6, in order to establish Theorem 1.5 we are left to prove the following result.

Theorem 10.7. *The map*

$$\begin{aligned} L(\lambda)_0 &\longrightarrow \mathbb{C}[\mathfrak{t}^{\mathrm{reg}}],\\ v &\longmapsto \overline{D}(\Psi_\lambda(v)),\end{aligned}$$

is W-equivariant.

Proof. Let $w\in W$ and $x\in\mathfrak{t}^{\mathrm{reg}}$. Then, by Proposition 8.1, there exist $(y,t)\in N_-\times T$ such that

$$n_{w^{-1}x}=yn_x\overline{w}t.$$

Hence, we have

$$\overline{D}(\Psi_\lambda(v))(w^{-1}x)=v_\lambda^*(n_{w^{-1}x}v)=v_\lambda^*(yn_x\overline{w}tv)=v_\lambda^*(n_x\overline{w}v)=\overline{D}(\Psi_\lambda(\overline{w}v))(x),$$

where we used that $tv=v$, as v is of weight zero, and that $v_\lambda^*(yu)=v_\lambda^*(u)$, as $y\in N_-$.

Since this holds for all $w\in W$ and $x\in\mathfrak{t}^{\mathrm{reg}}$, we conclude that

$$v\longmapsto\overline{D}(\Psi_\lambda(v))$$

is W-equivariant. □

11. Preprojective algebra modules

From this point on, we assume that G is simply-laced. In particular, this means that $q(\alpha_i)=1$ for all i, and thus $\iota(\alpha_i)=\alpha_i^\vee$, so ι corresponds to the usual identification of the root and coroot lattices. Thus, we can (and will) drop ι from our notation without possibility of confusion.

11.1. Preprojective algebras and their modules

Let H denote the set of oriented edges of the Dynkin diagram (so $(i,j),(j,i)\in H$ whenever i and j are connected in I). If $h=(i,j)$, write $\bar{h}=(j,i)$. Fix a map $\tau\colon H\to\{1,-1\}$ such that for each h, $\tau(h)+\tau(\bar{h})=0$ (such a τ corresponds to an orientation of each edge of the Dynkin diagram).

Define the *preprojective algebra* Λ to be the quotient of the path algebra of (I,H) by the relation

$$\sum_{h\in H}\tau(h)h\bar{h}=0.$$

So a Λ-module M consists of vector spaces M_i for $i\in I$ and linear maps $M_h\colon M_i\to M_j$ for each $h=(i,j)\in H$, such that

$$\sum_{h\in H}\tau(h)M_hM_{\bar{h}}=0. \tag{19}$$

Given a Λ-module M, we define its dimension vector by

$$\underrightarrow{\dim}\,M=\sum_{i\in I}(\dim M_i)\alpha_i.$$

We write S_i for the simple module at vertex i, i.e. the module with $\underrightarrow{\dim} S_i = \alpha_i$. The map $M \mapsto \underrightarrow{\dim} M$ gives rise to an isomorphism $K(\Lambda\text{-mod}) \cong Q$.

For each $\nu = \sum_{i \in I} \nu_i \alpha_i \in Q_+$, we consider the affine variety of Λ-module structures on $\bigoplus_{i \in I} \mathbb{C}^{\nu_i}$. More precisely, we define

$$\Lambda(\nu) \subset \bigoplus_{(i,j) \in H} \operatorname{Hom}(\mathbb{C}^{\nu_i}, \mathbb{C}^{\nu_j})$$

to be the subvariety defined by the equation (19).

11.2. The dual semicanonical basis

Let M be a Λ-module with dimension vector ν. Following Lusztig [Lu5] and Geiss–Leclerc–Schröer [GLS, §5], we define an element $\xi_M \in \mathbb{C}[N]_{-\nu}$ as follows.

First, for each $\mathbf{i} \in \operatorname{Seq}(\nu)$, we define the projective variety of composition series of type $\mathbf{i}$,

$$F_{\mathbf{i}}(M) = \{0 = M^0 \subseteq M^1 \subseteq \ldots \subseteq M^m = M : M^k / M^{k-1} \cong S_{i_k} \text{ for all } k\},$$

and then we define $\xi_M \in \mathbb{C}[N]_{-\nu}$ by requiring that

$$\langle e_{i_1} \ldots e_{i_m}, \xi_M \rangle = \chi(F_{\mathbf{i}}(M))$$

for any $\mathbf{i} \in \operatorname{Seq}(\nu)$, where χ denotes topological Euler characteristic. (Note: Lusztig and Geiss–Leclerc–Schröer consider decreasing composition series, whereas we chose to use increasing ones. Our choice accounts for the use of the dual setup and ensures that the crystal structure on the dual semicanonical basis coincides with the crystal structure defined in [KSa, §5].)

With this definition, the following result is immediate (see [GLS, Lemma 7.3]).

Lemma 11.1. *For any Λ-modules M and N, we have $\xi_{M \oplus N} = \xi_M \xi_N$.*

This map $M \mapsto \xi_M$ is constructible and so for any component $Y \subset \Lambda(\nu)$, we can define $c_Y \in \mathbb{C}[N]_{-\nu}$ by setting $c_Y = \xi_M$, for M a general point in Y.

The following result is due to Lusztig [Lu5].

Theorem 11.2. (i) *For each $\nu \in Q_+$, $\{c_Y : Y \in \operatorname{Irr} \Lambda(\nu)\}$ is a basis for $\mathbb{C}[N]_{-\nu}$.*

(ii) *The union of these bases forms a biperfect basis of $\mathbb{C}[N]$.*

Proof. Statement (i) is a direct consequence of [Lu5, Theorem 2.7]. From [Lu5, §2.9] and the definition of the bicrystal structure on the set of irreducible components of the nilpotent varieties (see [KSa, §5]), we deduce that for any irreducible component

$Y \subset \Lambda(\nu)$ and any $i \in I$, if we set $p = \varepsilon_i^*(Y)$ and $Y' = (\tilde{e}_i^*)^p Y$, then $c_Y \cdot e_i^{(p)} = c_{Y'}$. With these notation, write

$$c_Y \cdot e_i = \sum_{Y''} \alpha_{Y''} c_{Y''}.$$

Routine arguments show then that

$$\alpha_{Y''} = \begin{cases} p, & \text{if } Y'' = \tilde{e}_i^* Y, \\ 0, & \text{if } \varepsilon_i^*(Y'') \geqslant p-1 \text{ and } Y'' \neq \tilde{e}_i^* Y, \end{cases}$$

which proves the half of the statement (ii) related to the right action of N on $\mathbb{C}[N]$. The other half can be proved analogously or deduced from [Lu5, Theorem 3.8]. □

This basis for $\mathbb{C}[N]$ is called the dual semicanonical basis. By §2.4, it carries a bicrystal structure isomorphic to $B(\infty)$. Thus, a MV polytope is uniquely associated with each element in the dual semicanonical basis.

On the other hand, if M is a Λ-module, then we define its *Harder–Narasimhan polytope* to be

$$\mathrm{Pol}(M) := \mathrm{Conv}\{-\underrightarrow{\dim} N : N \subseteq M \text{ is a submodule}\}.$$

This map $M \mapsto \mathrm{Pol}(M)$ is constructible and so for any component $Y \subset \Lambda(\nu)$, we can define $\mathrm{Pol}(Y) := \mathrm{Pol}(M)$ for M a general point in Y. (We added the minus sign into the definition since ξ_M has weight $-\underrightarrow{\dim} M$.)

The following result was obtained by the first two authors with Tingley (see [BKT, §1.3]).

THEOREM 11.3. *Let Y be a component of $\Lambda(\nu)$. Then,* $\mathrm{Pol}(Y)$ *is the MV polytope of the basis vector c_Y.*

11.3. Measures from Λ-modules

Let M be a Λ-module of dimension vector ν. By the definition of ξ_M and the map $D: \mathbb{C}[N] \to \mathcal{PP}$, we have that

$$D(\xi_M) = \sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} \chi(F_{\mathbf{i}}(M)) D_{\mathbf{i}}.$$

Note that the measure $D(\xi_M)$ is supported on the polytope $\mathrm{Pol}(M)$.

In the previous section (Theorem 10.2), we showed that the measure $D(b_Z)$ of an MV basis vector equals the push-forward of the Duistermaat–Heckman measure $\mathrm{DH}(Z)$ of the corresponding MV cycle. The Duistermaat–Heckman measure is defined as the

asymptotics of sections of line bundles on Z. In a similar fashion, we will now explain that $D(\xi_M)$ can also be regarded as an asympototic.

We define $F_n(M)$ to be the space of $(n+1)$-step chains of submodules of M, so

$$F_n(M)=\{0=M^0\subseteq M^1\subseteq\ldots\subseteq M^{n+1}=M\},$$

and let $F_{n,\mu}(M)$ denote the locus in $F_n(M)$ where $\varinjlim M^1+\ldots+\varinjlim M^n=\mu$.

We will record the information of the Euler characteristics of $F_{n,\mu}(M)$ as an element of $R(T)$ by

$$[H^\bullet(F_n(M))]=\sum_{\mu\in Q_+}\chi(F_{n,\mu}(M))\delta_{-\mu}.$$

Note that $[H^\bullet(F_n(M))]$ is supported on the polytope $n\mathrm{Pol}(M)$.

THEOREM 11.4. *For any Λ-module M, with $\nu=\varinjlim M$, we have*

$$D(\xi_M)=\lim_{n\to\infty}\frac{1}{n^{\rho^\vee(\nu)}}(\tau_n)_*[H^\bullet(F_n(M))].$$

Proof. Let $m=\dim M=\rho^\vee(\nu)$.

The proof is largely parallel to that of [KnA1, §3]. We begin by defining the locally constant function

$$\begin{aligned}\mu_\bullet\colon F_n(M)&\longrightarrow Q_+^{\,n},\\ M^\bullet&\longmapsto(\varinjlim(M^k/M^{k-1}))_{k=1,\ldots,n}.\end{aligned}$$

For any $M^\bullet\in F_n(M)$, the number of non-zero terms in $\mu_\bullet(M^\bullet)$ is at most m. Given a sequence $\gamma\in Q_+^{\,n}$, let $\gamma^{\neq 0}$ be the sequence with its zeros removed. Thus, we can decompose

$$[H^\bullet(F_n(M))]=\sum_{\ell=0}^{m}\sum_{\theta\in(Q_+\setminus\{0\})^\ell}\sum_{\substack{\gamma\in Q_+^n\\ \gamma^{\neq 0}=\theta}}\chi(\{M^\bullet\in F_n(M):\mu_\bullet(M^\bullet)=\gamma\})\delta_{-\sum_{k=1}^n(n+1-k)\gamma_k}.$$

Now observe that, if $\theta=\gamma^{\neq 0}$, we have an obvious isomorphism

$$\{M^\bullet\in F_n(M):\mu_\bullet(M^\bullet)=\gamma\}\cong\{M^\bullet\in F_\ell(M):\mu_\bullet(M^\bullet)=\theta\}.$$

We let χ_θ denote the Euler characteristic of this space. Thus,

$$[H^\bullet(F_n(M))]=\sum_{\ell=0}^{m}\sum_{\theta\in(Q_+\setminus\{0\})^\ell}\chi_\theta\sum_{\substack{\gamma\in Q_+^n\\ \gamma^{\neq 0}=\theta}}\delta_{-\sum_{k=1}^n(n+1-k)\gamma_k}.$$

The number of γ with $\gamma^{\neq 0}=\theta$ is plainly $\binom{n}{\ell}=O(n^\ell)$. We now rescale

$$\frac{[H^\bullet(F_n(M))]}{n^{\rho^\vee(\nu)}}=\sum_{\ell=0}^{m}\sum_{\theta\in(Q_+\setminus\{0\})^\ell}\chi_\theta\frac{1}{n^m}\sum_{\substack{\gamma\in Q_+^n\\ \gamma^{\neq 0}=\theta}}\delta_{-\sum_{k=1}^n(n+1-k)\gamma_k},\quad \text{using } \rho^\vee(\nu)=m,$$

and let

$$D_{\theta,n}:=\frac{1}{n^m}\sum_{\gamma:\gamma^{\neq 0}=\theta}\delta_{-\sum_{k=1}^n(n+1-k)\gamma_k}.$$

This term has total mass $O(n^{\ell-m})$. In the limit $n\to\infty$, we can therefore neglect all $\ell<m$, and independently we apply $(\tau_n)_*$:

$$\lim_{n\to\infty}(\tau_n)_*\frac{[H^\bullet(F_n(M))]}{n^{\rho^\vee(\nu)}}=\sum_{\theta\in(Q_+\setminus\{0\})^m}\chi_\theta\lim_{n\to\infty}(\tau_n)_*D_{\theta,n}.$$

Since $\theta\in(Q_+\setminus\{0\})^m$ (i.e. θ has the same length as $\dim M$), for the locus

$$\{M^\bullet\in F_\ell(M):\mu_\bullet(M^\bullet)=\theta\}$$

to be non-empty, we need each θ_k to be a simple root and that $\sum_k\theta_k=\nu$. Such a θ uniquely determines (and is determined by) a sequence $\mathbf{i}\in\mathrm{Seq}(\nu)$ with $\theta_k=\alpha_{i_k}$ for all k. Moreover, we have $\chi_\theta=\chi(F_{\mathbf{i}}(M))$.

Now that $\ell=m$, the term $D_{\theta,n}$ has mass $1/m!$ as $n\to\infty$, the volume of the m-simplex. We proceed to determine how that mass is distributed.

To index the $\binom{n}{m}$ terms in the $D_{\theta,n}$ sum, we count how long the individual strings of zeros are between the m non-zero terms in γ: there are $1+m$ strings of zeros, of total length $n-m$. Thus, the set of all γ such that $\gamma^{\neq 0}=\theta$ are naturally in bijection with lattice points in the dilated simplex $(n-m)\Delta^m$. (We will soon apply $(\tau_n)_*$, resulting in the nearly-standard simplex $(1-m/n)\Delta^m$.) The jth vertex of this simplex corresponds to the case that γ has j non-zero terms up front, its $n-m$ zeros all in the middle, and $m-j$ non-zero terms at the end (its non-zero terms θ determined by γ).

Recall the linear map $\pi_{\mathbf{i}}:\mathbb{R}^{m+1}\to\mathfrak{t}_{\mathbb{R}}^*$ which takes the $(k+1)$-st standard basis vector to the negative partial sum $-\beta_k^{\mathbf{i}}=-(\theta_1+\ldots+\theta_k)$. The map $\pi_{\mathbf{i}}$ intertwines the above bijection with the map

$$\gamma\longmapsto-\sum_{k=1}^n(n+1-k)\gamma_k+\zeta,$$

where $\zeta:=\sum_{k=1}^m(m+1-k)\theta_k$ is a shift which is independent of n. Thus, we obtain that

$$D_{\theta,n}=\sum_{x\in(n-m)\Delta^m\cap\mathbb{Z}^{m+1}}\delta_{\pi_{\mathbf{i}}(x)-\zeta},$$

and thus

$$\lim_{n\to\infty}(\tau_n)_*\frac{1}{n^m}D_{\theta,n}=D_{\mathbf{i}},$$

as desired. $\square$

12. Comparing measures from MV cycles and from Λ-modules

12.1. From measures to sections

Let Z be a stable MV cycle of weight ν. Let Y be an irreducible component of $\Lambda(\nu)$. We say that Y and Z *correspond* if $\mathrm{Pol}(Y)=\mathrm{Pol}(Z)$; in other words, if the corresponding basis elements c_Y and b_Z give the same element in the bicrystal $B(\infty)$.

For the remainder of this section, fix a pair (Y,Z) which correspond. Recall that the measures $D(c_Y)$ and $D(b_Z)$ are both supported on $\mathrm{Pol}(Y)=\mathrm{Pol}(Z)$. Thus, an enhancement of the equality of polytopes would be the equality of measures. Note that the equality of basis vectors $c_Y=b_Z$ would imply the equality of measures $D(c_Y)=D(b_Z)$, but not vice versa (because of Remark 8.9).

By Theorems 10.2 and 11.4, we see that each of the measures $D(c_Y)$ and $D(b_Z)$ are the limits of (scalings of) measures $[H^\bullet(F_n(M))]$ (where M is a general point of Y) and $[\Gamma(Z,\mathscr{L}^{\otimes n})]$. This motivates the following definition.

Definition 12.1. We say that Y and Z are *extra-compatible* if, for all $n\in\mathbb{N}$ and $\mu\in Q_+$, we have

$$\dim\Gamma(Z,\mathscr{L}^{\otimes n})_{-\mu}=\chi(F_{n,\mu}(M)),$$

where M is a general point of Y.

The following is clear from the above results.

Proposition 12.2. *Consider the following four statements*:
(i) *Y and Z are extra-compatible.*
(ii) *$c_Y=b_Z$.*
(iii) *$D(c_Y)=D(b_Z)$.*
(iv) *$\overline{D}(c_Y)=\overline{D}(b_Z)$.*
We have the implications

$$\text{(i)} \implies \text{(iii)} \quad \text{and} \quad \text{(ii)} \implies \text{(iii)} \implies \text{(iv)}.$$

12.2. General conjecture

We can translate the Euler characteristic of $F_{n,\mu}(M)$ into the Euler characteristic of another variety. Consider the algebra $\Lambda[t]:=\Lambda\otimes_{\mathbb{C}}\mathbb{C}[t]$. We define

$$\mathbb{G}_\mu(M[t]/t^n)=\{N\subset M\otimes\mathbb{C}[t]/t^n : N \text{ is a } \Lambda[t]\text{-submodule, with } \underrightarrow{\dim}\, N=\mu\},$$

Lemma 12.3. *For any M, n and μ, we have $\chi(\mathbb{G}_\mu(M[t]/t^n))=\chi(F_{n,\mu}(M))$.*

Proof. First, define an inclusion $F_{n,\mu}(M)\to\mathbb{G}_\mu(M[t]/t^n)$ by

$$(M^1,...,M^n)\longmapsto M^1\otimes\mathbb{C}t^0\oplus M^2\otimes\mathbb{C}t^1\oplus...\oplus M^n\otimes\mathbb{C}t^{n-1}\subset M\otimes\mathbb{C}[t]/t^n.$$

It is easy to see that the right-hand side really is a $\Lambda[t]$-submodule.

On the other hand, we can define an action of $\mathbb{C}^\times$ on $\mathcal{G}\mathrm{r}_{n,\mu}(M)$ using the action of $\mathbb{C}^\times$ on $\mathbb{C}[t]/t^n$ given by $s\cdot p(t)=p(st)$. It is easy to see that the above map gives an isomorphism

$$F_{n,\mu}(M)\longrightarrow\mathbb{G}_\mu(M[t]/t^n)^{\mathbb{C}^\times},$$

and so the result follows. □

Thus, we deduce that Y and Z are extra-compatible if, for all $n\in\mathbb{N},\mu\in Q_+$ we have

$$\dim\Gamma(Z,\mathcal{L}^{\otimes n})_{-\mu}=\chi(\mathbb{G}_\mu(M[t]/t^n)),$$

where M is a general point of Y.

If we assume that the odd cohomology of $\mathbb{G}_\mu(M[t]/t^n)$ vanishes, this implies that there is an equality of $T^\vee$-representations,

$$\Gamma(Z,\mathscr{L}^{\otimes n})=H^\bullet(\mathbb{G}(M[t]/t^n))\quad\text{for all } n\in\mathbb{N},\tag{20}$$

where $T^\vee$ acts on the right-hand side through the decomposition

$$\mathbb{G}(M[t]/t^n)=\bigsqcup_\mu\mathbb{G}_\mu(M[t]/t^n).$$

Remark 12.4. The right-hand side of (20) carries a cohomological grading. We expect that (up to appropriate shift) this matches the $\mathbb{Z}$-grading on the left-hand side which comes from the loop rotation $\mathbb{C}^\times$ action on Z.

The left-hand sides of (20) form the components of a graded algebra, so it is natural to search for a similar structure on the right-hand side. After studying this question for some time, we are pessimistic about finding this algebra structure. On the other hand, $\bigoplus_n\Gamma(Z,\mathscr{L}^{\otimes n})$ is also a graded module over the ring

$$\bigoplus_n\Gamma\big(\overline{S_+^0\cap S_-^{-\nu}},\mathscr{L}^{\otimes n}\big).$$

We believe that such a module structure naturally exists for the direct sums of the right-hand side of (20). In fact, we believe that this structure is present for any module M, not just general modules corresponding to extra-compatible components.

Conjecture 12.5. For any preprojective algebra module M of dimension vector ν, the $\mathbb{N}\times P$-graded vector space

$$\bigoplus_{n\in\mathbb{N}} H^\bullet(\mathbb{G}(M[t]/t^n))$$

carries the structure of a $T^\vee$-equivariant graded $\bigoplus_n \Gamma\big(\overline{S_+^0\cap S_-^{-\nu}},\mathcal{L}^{\otimes n}\big)$-module.

If we assume this conjecture, then we get a coherent sheaf $\mathcal{F}_M$ on $\overline{S_+^0\cap S_-^{-\nu}}$ such that, for large enough n,

$$\Gamma\big(\overline{S_+^0\cap S_-^{-\nu}},\mathcal{F}_M\otimes\mathscr{L}^{\otimes n}\big)\cong H^\bullet(\mathbb{G}(M[t]/t^n)),$$

as $T^\vee$-representations. Assuming the vanishing of odd cohomology, Theorem 11.4 implies that $\mathrm{DH}(\mathcal{F}_M)=D(\xi_M)$.

On the other hand, we have the support cycle $[\operatorname{supp}\mathcal{F}_M]=\sum a_k[Z_k]$, where Z_k ranges over the stable MV cycles of weight $-\nu$. By Theorem 9.8, we know that

$$\mathrm{DH}(\mathcal{F}_M)=\sum a_k\,\mathrm{DH}(Z_k).$$

Thus, by Theorem 10.2, we reach $\sum_k a_k D(b_{Z_k})=D(\xi_M)$.

This suggests that $\xi_M=\sum_k a_k b_{Z_k}$ in $\mathbb{C}[N]$. In conclusion, the expansion of ξ_M in the MV basis should be given by the support cycle of $\mathcal{F}_M$.

The conjecture extends to direct sums $M=\bigoplus_{k=1}^d M_k$ of Λ-modules. Such an M carries a $(\mathbb{C}^\times)^d$-action with

$$\mathbb{G}(M[t]/t^n)^{(\mathbb{C}^\times)^d}\cong\prod_{k=1}^d\mathbb{G}(M_k[t]/t^n).$$

We will explain a conjectural relation between the sheaf $\mathcal{F}_M$ and the sheaves $\mathcal{F}_{M_k}$. For this we will use the Beilinson–Drinfeld Grassmannian. In §7.2, we recalled the definition of $\mathcal{G}\mathrm{r}_{\mathbb{A}}$, a family over $\mathbb{A}$. In a similar fashion, there is the BD Grassmannian $\mathcal{G}\mathrm{r}_{\mathbb{A}^d}$ defined by G-bundles trivialized away from a collection of d points. (There is a small difference: in §7.2, we fixed one of the points to be zero; here we let all the points vary.) As in §7.2, the fibre of $\mathcal{G}\mathrm{r}_{\mathbb{A}^d}\to\mathbb{A}^d$ over a point $(x_1,\dots,x_d)$ is isomorphic to a product of copies of $\mathcal{G}\mathrm{r}$, indexed by the set $\{x_1,\dots,x_d\}$.

Conjecture 12.6. For any tuple $(M_k)_{k=1}^d$ of Λ-modules, there is a $T^\vee$-equivariant sheaf $\mathcal{F}_{(M_k)}$ on $\mathcal{G}\mathrm{r}_{\mathbb{A}^d}$, flat over $\mathbb{A}^d$, that

(i) has global sections $\Gamma(\mathcal{G}\mathrm{r}_{\mathbb{A}},\mathcal{F}_{(M_k)}\otimes\mathscr{L}^{\otimes n})\cong H^\bullet_{(\mathbb{C}^\times)^d}(\mathbb{G}(\oplus_k M_k[t]/t^n))$ as representations of $T^\vee$ and as modules over $\mathbb{C}[\mathbb{A}^d]=H^\bullet_{(\mathbb{C}^\times)^d}(\mathrm{pt})$;

(ii) over points on the diagonal, restricts to the sheaf $\mathcal{F}_{\bigoplus_k M_k}$ from Conjecture 12.5 associated with $\bigoplus_k M_k$;

(iii) over general points of $\mathbb{A}^d$, restricts to the $\boxtimes$ product $\boxtimes_k \mathcal{F}_{M_k}$ of the sheaves from Conjecture 12.5 associated with the individual M_k.

The simplest case is that each M_k is 1-dimensional, and so there exists $\mathbf{i}\in\mathrm{Seq}(\nu)$ such that $M_k = S_{i_k}$. In this case, we expect that $\mathcal{F}_{M_k} = \mathcal{O}_{\overline{S_+^0 \cap S_-^{-\alpha_{i_k}}}}$ for each k and that $\mathcal{F}_{(M_k)} = \mathcal{O}_{\overline{Z^\nu}}$, the structure sheaf of the compatified Zastava space. In fact, we will explain in [HKW], that in this case, the conjecture follows from Remark 3.7 in Braverman–Finkelberg–Nakajima [BFN].

12.3. Shifting MV cycles

In order to compute the sections of line bundles over stable MV cycles, it will sometimes be useful to shift them, since they often appear more naturally as MV cycles in some $\overline{\mathcal{G}\mathrm{r}^\lambda}$. So, the following result will be helpful for us.

Proposition 12.7. *Let Z be any MV cycle. Let $\nu\in P$ and $n\in\mathbb{N}$. We have an isomorphism of $T^\vee$-representations*

$$\Gamma(Z, \mathscr{L}^{\otimes n}) = \Gamma(t^\nu Z, \mathscr{L}^{\otimes n}) \otimes \mathbb{C}_{n\iota(\nu)}.$$

Proof. We can lift t^ν to an element of $E(G^\vee(\mathcal{K}))$. Using this lift, we extend the obvious isomorphism $Z \to t^\mu Z$ to an isomorphism of line bundles, and thus to an isomorphism of sections. However, this isomorphism is not $T^\vee$-equivariant, because $T^\vee$ and t^ν do not commute inside $E(G^\vee(\mathcal{K}))$.

Let $\mu\in P$ and let $a\in\mathbb{C}^\times$. In $E(G^\vee(\mathcal{K}))$, the commutator $t^\nu a^\mu t^{-\nu} a^{-\mu}$ lies in the central $\mathbb{C}^\times$ and equals $\iota(\nu)(\mu)$. Since this central $\mathbb{C}^\times$ acts by weight n on $\mathscr{L}^{\otimes n}$, the result follows. $\square$

12.4. Spherical Schubert varieties

Let $\lambda\in P_+$ and consider the spherical Schubert variety $\overline{\mathcal{G}\mathrm{r}^\lambda}$. We shift it to form the stable MV cycle $t^{-\lambda}\overline{\mathcal{G}\mathrm{r}^\lambda}$, a component of $\overline{S_+^0 \cap S_-^{-\lambda + w_0\lambda}}$. The MV polytope of $t^{-\lambda}\overline{\mathcal{G}\mathrm{r}^\lambda}$ is the shifted Weyl polytope

$$\mathrm{Conv}(w\lambda : w\in W) - \lambda.$$

The corresponding Λ-module is injective. More precisely, for each $i\in I$, let $I(\omega_i)$ be the injective hull of the simple module S_i. Let

$$I(\lambda) = \bigoplus_i I(\omega_i)^{\oplus\langle\alpha_i^\vee,\lambda\rangle}.$$

This is a rigid module, and so the closure of the corresponding locus (consisting of those module structures isomorphic to M) in $\Lambda(\lambda - w_0\lambda)$ is an irreducible component $Y(\lambda)$.

We have equalities of basis vectors

$$b_{t^{-\lambda}\overline{\mathcal{G}\mathrm{r}^\lambda}} = \Psi_\lambda(v_{w_0\lambda}) = c_{Y(\lambda)},$$

as both are flag minors (see Remark 2.10).

We conjecture that $t^{-\lambda}\overline{\mathcal{G}\mathrm{r}^\lambda}$ and $Y(\lambda)$ are extra-compatible in the above sense, in other words, for all $n \in N$ and $\mu \in Q_+$, we have

$$\dim \Gamma(t^{-\lambda}\overline{\mathcal{G}\mathrm{r}^\lambda}, \mathscr{L}^{\otimes n})_{-\mu} = \chi(F_{n,\mu}(I(\lambda))).$$

We will now prove a stronger version of this statement when $n=1$.

Consider the Nakajima quiver variety $\mathcal{M}(\mathbf{w})$, defined using the framing vector $\mathbf{w}$, with $w_i = \langle \alpha_i^\vee, \lambda \rangle$. This quiver variety has a "core" $\mathcal{L}(\mathbf{w})$, and there is a homotopy retraction of $\mathcal{M}(\mathbf{w})$ onto $\mathcal{L}(\mathbf{w})$. We have the following result of Shipman [Sh, Corollary 3.2].

THEOREM 12.8. *There is an isomorphism* $\mathcal{L}(\mathbf{w}) \cong F_1(I(\lambda))$.

Thus, we get a chain of isomorphisms

$$H^\bullet(\mathcal{M}(\mathbf{w})) \cong H^\bullet(\mathcal{L}(\mathbf{w})) \cong H^\bullet(F_1(I(\lambda)).$$

By work of Varagnolo, there is an action of $\mathfrak{g}^\vee[[t]]$ on $H^\bullet(\mathcal{M}(\mathbf{w}))$. Since $\mathcal{G}\mathrm{r}^\lambda$ is an orbit of $G^\vee(\mathcal{O})$, we also have a $\mathfrak{g}^\vee[[t]]$ action on $\Gamma(\overline{\mathcal{G}\mathrm{r}^\lambda}, \mathscr{L})$. The following result is essentially due to Kodera–Naoi [KN] and Fourier–Littelmann [FL] (see [KTW+, Theorem 8.5]).

THEOREM 12.9. $\Gamma(\overline{\mathcal{G}r^\lambda}, \mathscr{L}) \cong H^\bullet(F_1(I(\lambda))$ *as representations of* $\mathfrak{g}^\vee[[t]]$.

Unpacking the weight spaces on both sides (and using the odd cohomology vanishing established by Nakajima [N, Proposition 7.3.4]), the above theorem implies the $n=1$ condition appearing in the definition of extra-compatibility. (Note that on both sides the weight spaces get shifted by λ. On the left-hand side, this is because of Proposition 12.7 and, on the right-hand side, this is because of the definition of the action of the Cartan in the Varagnolo action.)

COROLLARY 12.10. *We have*

$$\dim \Gamma(t^{-\lambda}\overline{\mathcal{G}r^\lambda}, \mathscr{L})_{-\mu} = \chi(F_{1,\mu}(I(\lambda))).$$

More generally, it seems reasonable that $H^\bullet(F_n(I(\lambda)))$ should carry an action of $\mathfrak{g}^\vee[[t]]$, extending the torus action which comes from the decomposition

$$F_n(I(\lambda)) = \bigsqcup_\mu F_{n,\mu}(I(\lambda)).$$

More generally, we expect that the extra compatibility extends to those MV cycles, and the quiver variety components which represent the flag minors from Remark 2.10.

12.5. Schubert varieties inside cominuscule Grassmannians

We now examine a class of MV cycles which we can prove are extra-compatible: Schubert varieties inside cominuscule Grassmannians. These represent flag minors from minuscule representations.

Let $\omega_i \in P_+$ be *cominuscule*, meaning that ω_i is a minimal element of $P_+ \setminus \{0\}$ with respect to the dominance order. In this case, it is easy to see that $\mathcal{G}\mathrm{r}^{\omega_i}$ is closed and that the action of $G^\vee$ on $\mathcal{G}\mathrm{r}^{\omega_i}$ gives rise to an isomorphism $G^\vee/P_i \cong \mathcal{G}\mathrm{r}^{\omega_i}$, where P_i is a maximal parabolic subgroup of $G^\vee$.

Moreover, the MV cycles of type ω_i are the Schubert varieties in $G^\vee/P_i$. For each $\gamma \in W\omega_i$, we have a Schubert variety $\mathcal{G}\mathrm{r}^{\omega_i} \cap \overline{S_-^\gamma}$. We can then translate to obtain a stable MV cycle $Z(\gamma) := t^{-\omega_i}(\mathcal{G}\mathrm{r}^{\omega_i} \cap \overline{S_-^\gamma})$ of weight $\gamma - \omega_i$.

We introduce a partial order (the Bruhat order) on $W\omega_i$ by $\tau \leqslant \gamma$ if $\tau - \gamma \in Q_+$ (caution: this is opposite to our convention for dominance order). This partial order corresponds to the order on MV cycles of type ω_i by containment. The minimal element of this partial order is ω_i and the maximal element is $w_0\omega_i$. We will be particularly interested in intervals in this poset of the form

$$[\omega_i, \gamma] = \{\tau \in W\omega_i : \omega_i \leqslant \tau \leqslant \gamma\}.$$

The MV polytope $P(\gamma)$ of $Z(\gamma)$ is easily described using this order (see for instance [KNS, Proposition 2.5.1]) as

$$P(\gamma) = \mathrm{Conv}\{-\omega_i + \tau : \tau \in [\omega_i, \gamma]\}.$$

On the other hand, for each $\gamma \in W\omega_i$, there is a unique (up to isomorphism) Λ-module $N(\gamma)$ with dimension-vector $\omega_i - \gamma$ and socle S_i (except if $\gamma = \omega_i$, in which case $N(\gamma) = 0$). We have a corresponding component $Y(\gamma)$ of $\Lambda(\omega_i - \gamma)$. Note that $Y(\gamma)$ and $Z(\gamma)$ correspond, because they both represent the unique elements of $B(\infty)$ of weight $\gamma - \omega_i$ which lies in the image of Ψ_{ω_i}.

As a special case, we have $N(w_0\omega_i)=I(\omega_i)$, the injective hull of S_i. In fact, for each $\gamma\in W\omega_i$, $N(\gamma)$ occurs once as a submodule of $I(\omega_i)$, and these are all the submodules of $I(\omega_i)$. The map $\gamma\mapsto N(\gamma)$ is an isomorphism of posets between $W\omega_i$ and the poset of submodules of $I(\omega_i)$ (under inclusion).

This implies that $\mathrm{Pol}(N(\gamma))=P(\gamma)$.

THEOREM 12.11. *Let $\gamma\in W\omega_i$. The pair $(Z(\gamma),Y(\gamma))$ is extra-compatible.*

Proof. For each $n\in\mathbb{N}$, let

$$F_n(\gamma):=\{(\tau_1,...,\tau_n):\omega_i\leqslant\tau_1\leqslant...\leqslant\tau_n\leqslant\gamma\}$$

be the set of chains in the poset $[\omega_i,\gamma]$.

From the above discussion, we see that $\tau\mapsto N(\tau)$ gives a bijection $F_n(\gamma)=F_n(N(\gamma))$.

On the other hand, Seshadri [Se] has given a bijection between $F_n(\gamma)$ and the standard monomial basis for $\Gamma(Z(\gamma),\mathscr{L}^{\otimes n})$ (see also [LR, Theorem 8.1.0.2]). □

Remark 12.12. More generally, we can consider a pair $\tau,\gamma\in W\omega_i$ such that $\tau\leqslant\gamma$. Then, we can consider the translated Richardson variety

$$Z(\tau,\gamma):=t^{-\tau}(\mathcal{G}\mathrm{r}^{\omega_i}\cap\overline{S^\tau}\cap\overline{S^\gamma_-})$$

This will also be an MV cycle. (In the case of type A, these MV cycles were studied by Anderson–Kogan [AK, §2.6].)

The corresponding Λ-module is $N(\gamma)/N(\tau)$ which underlies a component $Y(\tau,\gamma)$ of $\Lambda(\tau-\gamma)$. The above analysis carries over to this case by considering chains in the interval $[\tau,\gamma]$. Thus we see that the pair $(Z(\tau,\gamma),Y(\tau,\gamma))$ is also extra-compatible.

Appendix A. Extra-compatibility of MV cycles and preprojective algebra modules and non-equality of bases

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A.1. Tableaux, MV cycles, and preprojective algebra modules

In this appendix, we will work with $G=\mathrm{SL}_m$ and $G^\vee=\mathrm{PGL}_m$ (and in fact with $m=5,6$). Our goal is to compare the MV and dual semicanonical basis elements b_Z and c_Y, where Z and Y correspond in the sense of §12.1. In order to construct compatible pairs (Z,Y), we will work with the combinatorics of semistandard Young tableaux.

A.1.1. Tableaux and Lusztig data

We identify $P=\mathbb{Z}^m/\mathbb{Z}(1,...,1)$ and $P_+=\{\lambda=(\lambda_1,...,\lambda_m)\in\mathbb{Z}^m:\lambda_1\geqslant...\geqslant\lambda_m=0\}$, and we identify P_+ with the set of Young diagrams having at most $m-1$ rows. We also identify $Q=\{(\nu_1,...,\nu_m)\in\mathbb{Z}^m:\nu_1+...+\nu_m=0\}$, and we have the simple roots $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$.

We will write the positive roots for SL_m as $\Delta_+=\{\varepsilon_i-\varepsilon_j:1\leqslant i<j\leqslant m\}$. For any choice of convex order for Δ_+, there is a bijection $\psi: B(\infty)\to\mathbb{N}^{\Delta_+}$ constructed by Lusztig (see [Lu2, §2]). In this appendix, we will work with one choice of convex order

$$\varepsilon_1-\varepsilon_2\leqslant...\leqslant\varepsilon_1-\varepsilon_m\leqslant\varepsilon_2-\varepsilon_3\leqslant...\leqslant\varepsilon_2-\varepsilon_m\leqslant...\leqslant\varepsilon_{m-1}-\varepsilon_m \tag{21}$$

(this convex order corresponds to the reduced word $s_1...s_{m-1}...s_1s_2s_1$ for w_0). A *Lusztig datum* is an element $n_\bullet\in\mathbb{N}^{\Delta_+}$. The *weight* of $n_\bullet$ is defined to be $\sum_{\beta\in\Delta_+}n_\beta\beta$.

Fix $\lambda\in P_+$ and let $N=\sum_i\lambda_i$. Let $\mathrm{YT}(\lambda)$ be the set of semi-standard Young tableaux of shape λ. We say that a tableau $\tau\in\mathrm{YT}(\lambda)$ has weight $\mu\in\mathbb{Z}^m$ if τ has μ_i boxes numbered i for $1\leqslant i\leqslant m$. Let $\mathrm{YT}(\lambda)_\mu\subset\mathrm{YT}(\lambda)$ denote the tableaux having weight μ. For $1\leqslant i\leqslant m$, let $\tau^{(i)}$ denote the restricted tableau obtained from τ by deleting all boxes numbered j, for $j>i$. Let $\mathrm{sh}(\tau^{(i)})\in\mathbb{N}^m$ denote the shape of $\tau^{(i)}$.

Given a tableau $\tau\in\mathrm{YT}(\lambda)_\mu$, we define its Lusztig datum $n(\tau)_\bullet$ by setting

$$n(\tau)_{\varepsilon_i-\varepsilon_j}:=\mathrm{sh}(\tau^{(j)})_i-\mathrm{sh}(\tau^{(j-1)})_i=\text{number of boxes on the } i\text{-row filled with } j.$$

Note that $n(\tau)_\bullet$ has weight $\lambda-\mu$.

A.1.2. Generalized orbital varieties

We will study MV cycles by identifying open affine subsets of them with generalized orbital varieties. This identification comes from the Mirković–Vybornov isomorphism, which we now recall. Fix $\lambda\in P_+$ and let $N=\sum_i\lambda_i$. Fix $\mu\in P_+$ such that $\lambda-\mu\in Q_+$ (in particular, $N=\sum_i\mu_i$). We will work inside the space of $N\times N$ matrices, and we consider such matrices in block form, where we have $m\times m$ blocks, with the (i,j) block of size $\mu_i\times\mu_j$.

We will need the following spaces of matrices:

$$\begin{aligned}\mathfrak{n}&=\text{upper triangular matrices},\\ \mathbb{O}_\lambda&=\text{matrices of Jordan type }\lambda,\\ \mathbb{T}_\mu&=\{J_\mu+x:\text{all entries of } x \text{ are zero, except the first } \min(\mu_i,\mu_j)\\ &\qquad\qquad\text{columns of the last row of each } (i,j) \text{ block}\},\end{aligned}$$

where J_μ is the Jordan form matrix of type μ.

Example A.1. If $\mu=(3,2)$ then an element of $\mathbb{T}_\mu$ takes the form

$$\left[\begin{array}{ccc|cc} & 1 & & & \\ & & 1 & & \\ * & * & * & * & * \\ \hline & & & & 1 \\ * & * & & * & * \end{array}\right],$$

where all blank entries are zero.

We will study $\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu$ and $\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu\cap\mathfrak{n}$. The irreducible components of this latter variety are called *generalized orbital varieties*. Note that, if $\mu=(1,...,1)$, then $\mathbb{T}_\mu$ is the space of all matrices, and so we recover the usual orbital varieties.

Given $\tau\in\mathrm{YT}(\lambda)_\mu$, define

$$\mathring{\mathring{Z}}_\tau=\{A\in\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu\cap\mathfrak{n}: A|_{\mathbb{C}^{\mu_1+\ldots+\mu_i}}\in\mathbb{O}_{\mathrm{sh}(\tau^{(i)})} \text{ for } 1\leqslant i\leqslant m\}. \tag{22}$$

Here and elsewhere, $\mathbb{C}^k$ denotes the subspace of $\mathbb{C}^N$ spanned by the first k standard basis elements.

The following result is due to the first appendix author.

THEOREM A.2. ([D]) (i) *For each τ, $\mathring{\mathring{Z}}_\tau$ has a unique irreducible component of dimension $\rho^\vee(\lambda-\mu)$, and all other components have smaller dimension.*

(ii) *Letting $\mathring{Z}_\tau$ denote the closure of this component of maximal dimension, the map $\tau\mapsto\mathring{Z}_\tau$ gives a bijection between $\mathrm{YT}(\lambda)_\mu$ and the set of irreducible components of $\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu\cap\mathfrak{n}$.*

The affine space $\mathbb{T}_\mu$ (and the subvariety $\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu\cap\mathfrak{n}$) carries an action of the maximal torus $T^\vee\subset\mathrm{PGL}_m$, where $[t_1,...,t_m]\in T^\vee$ acts by conjugating by the $N\times N$ diagonal matrix whose entries $t_1,...,t_1,...,t_m,...,t_m$ are constant in each block.

In this appendix, we will work with polynomials in the weights of $T^\vee$. In particular, we introduce the notation

$$p(\mu):=\prod_{1\leqslant i<j\leqslant m}(\varepsilon_i-\varepsilon_j)^{\mu_j}=\prod_{1\leqslant i<j\leqslant m}(\alpha_i+...+\alpha_{j-1})^{\mu_j}$$

for the product of the weights of $T^\vee$ acting on the affine space $\mathbb{T}_\mu\cap\mathfrak{n}$.

A.1.3. Generalized orbital varieties and MV cycles

We will now recall the Mirković–Vybornov isomorphism [MVy] in the form of [CK].

Given a nilpotent $N\times N$ matrix $A\in\mathbb{T}_\mu$, we define

$$\varphi(A)\in G_1^\vee[t^{-1}]t^\mu$$

(where $G^\vee=\mathrm{PGL}_m$ and $G_1^\vee[t^{-1}]$ denotes the kernel of $G^\vee[t^{-1}]\to G^\vee$) by the formulas

$$\begin{aligned}\varphi(A)&=t^\mu I_m+(a_{ij}(t)),\\ a_{ij}(t)&=-\sum_{k=1}^{\mu_i}A_{ij}^k t^{k-1},\\ A_{ij}^k&=k\text{th entry on the last row of the }(j,i)\text{ block of }A.\end{aligned}$$

The following result follows from [CK, Theorem 3.1] (except we have applied transpose to the domain).

THEOREM A.3. *The map $A\mapsto[\varphi(A)]$ gives an isomorphism*

$$\varphi\colon\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu\longrightarrow\overline{\mathcal{G}r^\lambda}\cap G_1^\vee[t^{-1}]L_\mu.$$

Note that $A\mapsto[\varphi(A)]$ is $T^\vee$-equivariant with respect to the above defined action of $T^\vee$ on $\mathbb{T}_\mu$ and the usual $T^\vee$ action on $\mathcal{G}\mathrm{r}$.

In [D], the first appendix author restricted this isomorphism to the generalized orbital varieties and obtained the following result.

THEOREM A.4. (i) *φ restricts to an isomorphism*

$$\varphi\colon\overline{\mathbb{O}}_\lambda\cap\mathbb{T}_\mu\cap\mathfrak{n}\longrightarrow\overline{\mathcal{G}r^\lambda}\cap S_-^\mu$$

(ii) *For any tableau $\tau\in\mathrm{YT}(\lambda)_\mu$, the closure of the image of the generalized orbital variety, $Z_\tau:=\overline{\varphi(\mathring{Z}_\tau)}$, is the MV cycle whose Lusztig datum equals $n(\tau)_\bullet$. In other words, the following diagram commutes*

$$\begin{tikzcd}\mathrm{YT}(\lambda)_\mu \arrow[r] \arrow[dd] & \mathcal{Z}(\lambda)_\mu \arrow[d] \\ & \mathcal{Z}(\infty) \arrow[d] \\ \mathbb{N}^{\Delta_+} & B(\infty), \arrow[l]\end{tikzcd}$$

where the top horizontal arrow is $\tau\mapsto Z_\tau$, the left vertical arrow is $\tau\mapsto n(\tau)_\bullet$, the right vertical arrows are constructed in §6 and the bottom arrow is Lusztig's bijection.

We will use this theorem to compute the value of $\overline{D}$ on an MV basis vector, with the aid of the following proposition.

PROPOSITION A.5. *We have*

$$\overline{D}(b_{t^{-\lambda}Z_\tau}) = \varepsilon_{L_{\mu-\lambda}}(t^{-\lambda}Z_\tau) = \varepsilon_{L_\mu}(Z_\tau) = \frac{\mathrm{mdeg}[\mathfrak{n}\cap\mathbb{T}_\mu]\mathring{Z}_\tau}{p(\mu)}.$$

Proof. The first equality is Corollary 10.6 and the third one is a consequence of Proposition 9.5. □

A.1.4. A Plücker embedding

We will now explain how the knowledge of $\mathring{Z}_\tau$ allows us to compute the sections of line bundles over Z_τ.

Recall that, in the lattice model for $\mathcal{G}\mathrm{r}$, the orbit $\mathcal{G}\mathrm{r}^\lambda$ can be described as

$$\mathcal{G}\mathrm{r}^\lambda = \{L \in \mathcal{G}\mathrm{r} : L \subset L_0 = \mathbb{C}[[t]]^m \text{ and } t|_{L_0/L} \text{ has Jordan type } \lambda\}.$$

Let $p \geqslant \lambda_1$. Thus, $L \in \overline{\mathcal{G}\mathrm{r}^\lambda}$ contains t^pL_0, and the quotient L/t^pL_0 has dimension $mp-N$.

PROPOSITION A.6. *The map*

$$\begin{aligned} \overline{\mathcal{G}r^\lambda} &\longrightarrow \mathbb{G}\left(mp-N, \frac{L_0}{t^pL_0}\right), \\ L &\longmapsto \frac{L}{t^pL_0}, \end{aligned}$$

is a closed embedding. Moreover, the standard determinant line bundle on

$$\mathbb{G}\left(mp-N, \frac{L_0}{t^pL_0}\right)$$

restricts to the line bundle $\mathscr{L}$ on $\overline{\mathcal{G}r^\lambda}$.

Now, assume that $N=m$, $\mu=(1,...,1)$ and $p=2$, so that $mp-N=m$. We consider the Plücker embedding

$$\mathbb{G}\left(m, \frac{L_0}{t^2L_0}\right) \longrightarrow \mathbb{P}\left(\bigwedge^m \frac{L_0}{t^2L_0}\right).$$

Our aim is to study the chain of maps connecting $\overline{\mathbb{O}}_\lambda \cap \mathfrak{n}$ with this projective space.

Let $A \in \overline{\mathbb{O}}_\lambda \cap \mathfrak{n}$. In this case, the Mirković–Vybornov isomorphism is simply given by $\varphi(A) = [t\mathrm{Id} - A^{tr}]$ and the corresponding lattice is

$$L_A = (t\mathrm{Id} - A^{tr})L_0 = \mathrm{Span}_{\mathbb{C}[[t]]}\{e_it - A^{tr}e_i : 1 \leqslant i \leqslant m\}.$$

Fix the basis

$$(v_1, ..., v_m, v_{\bar{1}}, ..., v_{\bar{m}}) \equiv ([e_1 t], ..., [e_m t], [-e_1], ..., [-e_m]) \quad \text{for } \frac{L_0}{t^2 L_0}.$$

We write $S=(1, ..., m, \bar{1}, ..., \bar{m})$ for the index set of this basis. We have a resulting basis for $\bigwedge^m L_0/t^2 L_0$ indexed by subsets $C \subset S$ of size m.

In this basis, $L_A/t^2 L_0$ is the row space of the $m \times 2m$ matrix $\tilde{A}=[I\ A]_\beta$. For any subset $C \subset S$ of size m, we can consider the minor $\Delta_C(\tilde{A})$ of this matrix using the columns C. Thus, we have established the following result.

Proposition A.7. *Under the chain of maps*

$$\overline{\mathbb{O}}_\lambda \cap \mathfrak{n} \longrightarrow \overline{\mathcal{G}r^\lambda} \cap S^0_- \longrightarrow \mathbb{G}(m, L_0/t^2 L_0) \longrightarrow \mathbb{P}\left(\bigwedge^m \frac{L_0}{t^2 L_0}\right),$$

A is sent to $[\Delta_C(\tilde{A})]_{C \in \binom{S}{m}}$.

We also note that these maps are $T^\vee$-equivariant (in fact PGL_m-equivariant), where $T^\vee$ acts on $\mathbb{P}\big(\bigwedge^m L_0/t^2 L_0\big)$ using the natural GL_m action on $L_0/t^2 L_0$. In particular, the basis vectors v_i and $v_{\bar{i}}$ both have weight ε_i.

From this proposition, it is immediate that $\overline{\mathbb{O}}_\lambda \cap \mathfrak{n}$ is mapped into the affine space

$$\mathbb{A}^{\binom{2m}{m}-1} \subset \mathbb{P}\left(\bigwedge^m \frac{L_0}{t^2 L_0}\right),$$

defined by the condition that $\Delta_{1,...,m} \neq 0$, and moreover that, for any τ, $\mathring{Z}_\tau$ is the intersection of Z_τ with this open affine space. Thus, we deduce the following corollary.

Corollary A.8. *The ideal of $Z_\tau \subset \mathbb{P}\big(\bigwedge^m L_0/t^2 L_0\big)$ equals the homogenization of the kernel of the map*

$$\mathbb{C}\left[\left\{\Delta_C : C \in \binom{S}{m}\right\}\right] \longrightarrow \mathbb{C}[\mathring{Z}_\tau].$$

A.1.5. Preprojective algebra modules

Fix $\nu = Q_+$. We have a bijection $\mathrm{Irr}\,\Lambda(\nu) \to B(\infty)_{-\nu}$. We will now recall the composition

$$\bigcup_\nu \mathrm{Irr}\,\Lambda(\nu) \longrightarrow B(\infty) \longrightarrow \mathbb{N}^{\Delta_+},$$

which was studied in [BKT].

The convex order (21) on Δ_+ determines a sequence of indecomposable bricks $(B_\beta)_{\beta \in \Delta_+}$ labelled by the positive roots. For this convex order,

$$B_{\tilde{\varepsilon}_i - \varepsilon_j} = i \longleftarrow i+1 \longleftarrow ... \longleftarrow j-1.$$

Let M be a Λ-module. The *Harder–Narasimhan filtration* of M is the unique decreasing filtration $(M_\beta)_{\beta\in\Delta_+}$ such that $M_\beta/M_{\beta'}\cong B_\beta^{\bigoplus n(M)_\beta}$, where $\beta<\beta'$ is a consecutive pair in the convex order on Δ_+. We define the Lusztig datum $n(M)_\bullet$ of M to be these multiplicities.

We define $n(Y)_\bullet$ for any component $Y\subset\Lambda(\nu)$ by setting $n(Y)_\bullet:=n(M)_\bullet$, where M is a general point of Y. The following result is contained in [BKT] (see Remark 5.25 (ii)).

THEOREM A.9. *The map* $\bigcup_\nu \operatorname{Irr}\Lambda(\nu)\to\mathbb{N}^{\Delta_+}$ *defined by* $Y\mapsto n(Y)_\bullet$ *is a bijection. Moreover, this bijection agrees with the composition* $\bigcup_\nu \operatorname{Irr}\Lambda(\nu)\to B(\infty)\to\mathbb{N}^{\Delta_+}$ *(where the second map is Lusztig's bijection).*

From Theorems A.4 and A.9, we deduce the following.

COROLLARY A.10. *Let* $\tau\in\mathrm{YT}(\lambda)_\mu$ *be a Young tableau and let* $\nu=\lambda-\mu$. *Let* Z_τ *be the MV cycle constructed in* §A.1.3. *Let* $Y_\tau\subset\Lambda(\nu)$ *be component such that* $n(Y)_\bullet=n(\tau)_\bullet$.

Then, the stable MV cycle $t^{-\lambda}Z_\tau$ *and the component* Y_τ *correspond in the sense of* §12.1.

A.1.6. Evidence for extra-compatibility in an A_4 example

We take $m=N=5$, $\lambda=(2,2,1)$, $\mu=(1,1,1,1,1)$ and consider

$$\tau=\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}.$$

Let $Z=Z_\tau$ be the MV cycle defined from τ in §A.1.3. Let $Y=Y_\tau\subset\Lambda(\nu)$ and let M be a general point of Y_τ, i.e. a general module with $n(M)=n(\tau)$. This gives the simplest indecomposable M not covered by the analysis in §12.5.

The pair $(t^{-\lambda}Z,Y)$ are compatible, by Corollary A.10. Moreover, $b_{t^{-\lambda}Z}=c_Y$, as can be seen by a variant of the analysis from §2.7. In fact, we expect that $t^{-\lambda}Z$ and Y are extra-compatible. We will now prove the following result which gives evidence in this direction.

THEOREM A.11. (i) *For all* $n\in\mathbb{N}$, *we have* $\dim\Gamma(t^{-\lambda}Z,\mathscr{L}^{\otimes n})=\dim H^\bullet(F_n(M))$.
(ii) *For* $n=1,2$ *and all* μ, *we have* $\dim\Gamma(t^{-\lambda}Z,\mathscr{L}^{\otimes n})_\mu=\dim H^\bullet(F_{n,\mu}(M))$.
(iii) *We have* $\overline{D}(b_{t^{-\lambda}Z})=\overline{D}(\xi_M)$.

A.1.7. Generalized orbital variety and multidegree

Let $A \in \mathring{\mathring{Z}}_\tau$ and write

$$A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We apply (22) to find the following conditions along with the equations they impose on the matrix entries of A:

$$\begin{aligned} \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} &\in \mathbb{O}_{(2)}, & a_1 &\neq 0, \\ \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_5 \\ 0 & 0 & 0 \end{bmatrix} &\in \mathbb{O}_{(2,1)}, & a_5 &= 0, \\ \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 \end{bmatrix} &\in \mathbb{O}_{(2,2)}, & a_1a_6 + a_2a_8 &= 0, \\ & & a_8 &\neq 0, \\ & & a_{10} &= 0, \\ A \in \mathbb{O}_\lambda, & & \det \begin{bmatrix} a_6 & a_7 \\ a_8 & a_9 \end{bmatrix} &= 0, \\ & & a_1a_7 + a_2a_9 &= 0. \end{aligned}$$

Altogether, we find that $\mathring{Z} = \mathring{Z}_\tau$ is the vanishing locus of

$$I = (a_5, a_{10}, a_1a_6 + a_2a_8, a_7a_8 - a_6a_9, a_1a_7 + a_2a_9),$$

and we verified using a computer that this ideal is prime.

Applying the algorithm from [KZJ, §1.5] (or by computer), we obtain the following:

$$\operatorname{mdeg}[\mathfrak{n}]\mathring{Z} = (\alpha_1\alpha_2 + \alpha_2^2 + \alpha_2\alpha_3)\alpha_4^2 + (\alpha_1\alpha_2^2 + \alpha_2^3 + \alpha_2\alpha_3^2 + 2(\alpha_1\alpha_2 + \alpha_2^2)\alpha_3)\alpha_4. \tag{23}$$

A.1.8. The MV cycle and its sections

In this example, the description of the MV cycle Z from Corollary A.8 can be simplified, since $\mathring{Z}$ is contained in the subspace defined by the vanishing of a_5 and a_{10}. By ignoring

minors which are forced to be zero among the set of $\binom{10}{5}$ total possibilities, we can exhibit Z as a subvariety of $\mathbb{P}^{16}$ using the following set of minors:

$$u=\Delta_{12345},\quad b_1=\Delta_{1345\bar{1}},\quad b_2=\Delta_{1245\bar{1}},\quad b_3=\Delta_{1235\bar{1}},\quad b_4=\Delta_{1234\bar{1}},\quad b_5=\Delta_{1235\bar{2}},$$
$$b_6=\Delta_{1234\bar{2}},\quad b_7=\Delta_{1235\bar{3}},\quad b_8=\Delta_{1234\bar{3}},\quad b_9=\Delta_{123\overline{12}},\quad b_{10}=\Delta_{124\overline{12}},\quad b_{11}=\Delta_{124\overline{13}},$$
$$b_{12}=\Delta_{125\overline{12}},\quad b_{13}=\Delta_{125\overline{13}},\quad b_{14}=\Delta_{123\overline{13}},\quad b_{15}=\Delta_{134\overline{13}},\quad b_{16}=\Delta_{135\overline{13}}.$$

Let $P=\mathbb{C}[b_1,...,b_{16},u]$ and

$$\begin{aligned}J_1=(&b_9-b_3b_6+b_4b_5,b_{10}-b_2b_6,b_{11}-b_2b_8,b_{12}-b_2b_5,\\&b_{13}-b_2b_7,b_{14}-b_3b_8+b_4b_7,b_{15}-b_1b_8,b_{16}-b_1b_7),\\J_2=(&b_1b_5+b_2b_7,b_6b_7-b_5b_8,b_1b_6+b_2b_8),\end{aligned}$$

where J_1 just comes from $\mathbb{C}[\mathbf{n}]$ and J_2 represents the additional relations coming from I. Thus, $Z=\mathrm{Proj}(P/J^h)$ for $J=J_1+J_2$, where J^h is the homogenization of J with respect to u. Using Macauley2, we obtain the following expression for the space of sections:

$$\dim\Gamma(Z,\mathcal{O}(n))=\frac{(n+1)^2(n+2)^2(n+3)(5n+12)}{144}. \tag{24}$$

A.1.9. The preprojective algebra module

A general module M with Lusztig data $n(\tau)=(1,0,0,0,1,1,0,0,1,0)$ is

$$\begin{tikzcd}
1 \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & \\
& 22 \arrow[dr] & & 4 \arrow[dl] \\
& & 3, &
\end{tikzcd}$$

with the maps chosen such that $\ker(M_2\to M_3)$, $\mathrm{im}(M_3\to M_2)$ and $\mathrm{im}(M_1\to M_2)$ are all distinct.

In fact, M has HN filtration with subquotients

$$\begin{array}{c}1\\\hline 2\\\hline 3\to 2\\\hline 4\to 3\end{array}$$

ν	$F_1(M)^\nu$	$\dim H^\bullet(F_1(M)^\nu)$	$\dim H^\bullet(F_n(M)^\nu)$
$(0,0,0,0)$	point	1	1
$(0,1,0,0)$	point	1	n
$(0,0,1,0)$	point	1	n
$(0,0,1,1)$	point	1	$\frac{1}{2}n(n+1)$
$(0,1,1,0)$	$\mathbb{P}^1$	2	$\frac{1}{2}n(3n+1)$
$(0,1,1,1)$	$\mathbb{P}^1$	2	$\frac{1}{6}n(n+1)(5n+1)$
$(0,2,1,0)$	point	1	$\frac{1}{2}n^2(n+1)$
$(1,1,1,0)$	point	1	$\frac{1}{6}n(n+1)(n+2)$
$(0,1,2,1)$	point	1	$\frac{1}{12}n(n+1)^2(n+2)$
$(0,2,1,1)$	point	1	$\frac{1}{6}n^2(n+1)(2n+1)$
$(1,1,1,1)$	point	1	$\frac{1}{24}n(n+1)(n+2)(3n+1)$
$(1,2,1,0)$	point	1	$\frac{1}{6}n^2(n+1)(n+2)$
$(0,2,2,1)$	point	1	$\frac{1}{12}n^2(n+1)^2(n+2)$
$(1,2,1,1)$	point	1	$\frac{1}{24}n^2(n+1)(n+2)(3n+1)$
$(1,2,2,1)$	point	1	$\frac{1}{144}n^2(n+1)^2(n+2)(5n+7)$

Table 2. Spaces of (chains of) submodules of M.

A.1.10. Flags of submodules

We will now outline a recursive method for computing $\dim H^\bullet(F_n(M))$. Given a dimension vector $\nu\in Q_+$, let $F_n(M)^\nu$ be the component of $F_n(M)$ consisting of $(n+1)$-step flags for which the nth submodule in the chain has dimension vector ν. Note that, for each ν, all submodules of M of dimension ν are isomorphic (this is a special property of M). This means that for any ν, there exists a submodule N such that

$$F_n(M)^\nu \cong F_{n-1}(N)\times F_{1,\nu}(M).$$

Using this recursive definition, we compute $\dim H^\bullet(F_n(M)^\nu)$ in Table 2.

Since $F_n(M)$ is the disjoint union of each of these varieties, we have

$$\dim H^\bullet(F_n(M)) = \sum_\nu \dim H^\bullet(F_n(M)^\nu),$$

and so, summing the above polynomials, we get

$$\dim H^\bullet(F_n(M)) = \frac{(n+1)^2(n+2)^2(n+3)(5n+12)}{144}. \tag{25}$$

Together with (24), this establishes Theorem A.11 (i).

When $n=1,2$, we can take this computation further and prove Theorem A.11 (ii). (When $n=1$, this can be easily seen by comparing the weights of the variables $u, b_1, ..., b_{16}$, with the dimension vectors of submodules of M from Table 2, taking into account the shifting of weights given by Proposition 12.7.)

A.1.11. Computation of the "flag function"

By Proposition 8.4 and the definition of ξ_M from §11.1, we have that

$$\overline{D}(\xi_M) = \sum_{\mathbf{i}\in\mathrm{Seq}(\nu)} \chi(F_{\mathbf{i}}(M))\overline{D}_{\mathbf{i}}.$$

We call $\overline{D}(\xi_M)$ the flag function of M.

For $\mathbf{i}\in\mathrm{Seq}(\nu)$ among

$$\begin{array}{llll} (3,4,2,3,2,1) & (3,2,4,3,2,1) & (2,3,4,2,3,1) & (2,3,2,1,4,3) \\ (2,3,2,4,3,1) & (2,3,4,2,1,3) & (2,3,2,4,1,3) & (3,2,1,2,4,3) \\ (3,4,2,1,2,3) & (3,2,4,1,2,3) & (3,2,1,4,2,3), & \end{array}$$

the variety $F_{\mathbf{i}}(M)$ is a point, so $\chi(F_{\mathbf{i}}(M))=1$. For $\mathbf{i}$ among

$$\begin{array}{llll} (3,4,2,2,3,1) & (3,2,4,2,3,1) & (3,2,2,4,3,1) & (3,4,2,2,1,3) \\ (3,2,4,2,1,3) & (3,2,2,4,1,3) & (3,2,2,1,4,3), & \end{array}$$

we see that $F_{\mathbf{i}}(M)\cong\mathbb{P}^1$, so $\chi(F_{\mathbf{i}}(M))=2$. For all other values of $\mathbf{i}$, we have $F_{\mathbf{i}}(M)=\varnothing$.

The flag function is a rational function, but we can use $p(\mu)$ to clear the denominator. By direct computation, we obtain that the flag function of M is given by

$$\overline{D}(\xi_M)p(\mu) = (\alpha_1\alpha_2+\alpha_2^2+\alpha_2\alpha_3)\alpha_4^2+(\alpha_1\alpha_2^2+\alpha_2^3+\alpha_2\alpha_3^2+2(\alpha_1\alpha_2+\alpha_2^2)\alpha_3)\alpha_4.$$

To complete the proof of Theorem A.11 (iii), we compare with (23) and apply Proposition A.5.

A.2. Weak evidence for extra-compatibility in an A_5 example

Let $\lambda=(2,2,1,1)$, let $\mu=(1,1,1,1,1,1)$ and consider

$$\tau = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \cline{1-1} 6 & \\ \cline{1-1}\end{array}.$$

As before, let $Z=Z_\tau$ be the MV cycle defined from τ in §A.1.3. Let $Y=Y_\tau\subset\Lambda(\nu)$. A general point of Y is of the form M_a, where $a\in\mathbb{C}$ is a parameter. This is the simplest example of a component whose general point is not rigid.

The pair $(t^{-\lambda}Z,Y)$ are compatible and $b_Z=c_Y$, as can be seen by a variant of the analysis from §2.7. We have the following weak evidence for extra-compatibility and for the equality of basis vectors.

THEOREM A.12. (i) *For all $\nu\in Q_+$, we have*

$$\dim\Gamma(t^{-\lambda}Z,\mathscr{L})_{-\nu}=\dim H^\bullet(F_{1,\nu}(M_a)).$$

(ii) *We have*

$$\overline{D}(b_{t^{-\lambda}Z})=\overline{D}(\xi_{M_a}).$$

A.2.1. Generalized orbital variety and multidegree

We apply (22) with the aid of a computer to find that the generalized orbital variety $\mathring{Z}_\tau$ is cut out by the prime ideal

$$\begin{aligned}I=(&a_{15},a_{10},a_1,a_3a_6-a_2a_7,a_2a_{12}+a_3a_{14},a_6a_{12}+a_7a_{14},a_2a_{11}+a_3a_{13},\\&a_6a_{11}+a_7a_{13},a_{12}a_{13}-a_{11}a_{14},a_5a_6a_{13}-a_2a_9a_{13}-a_4a_6a_{14}+a_2a_8a_{14},\\&a_5a_7a_{13}-a_3a_9a_{13}-a_4a_7a_{14}+a_3a_8a_{14},a_5a_7a_{11}-a_3a_9a_{11}-a_4a_7),\end{aligned}$$

where $a_1,\dots,a_{15}$ are the matrix entries of a 6×6 upper triangular matrix

$$A=\begin{bmatrix}0&a_1&a_2&a_3&a_4&a_5\\0&0&a_6&a_7&a_8&a_9\\0&0&0&a_{10}&a_{11}&a_{12}\\0&0&0&0&a_{13}&a_{14}\\0&0&0&0&0&a_{15}\\0&0&0&0&0&0\end{bmatrix}$$

in $\mathfrak{n}$.

From here, it is easy to compute $\operatorname{mdeg}[\mathfrak{n}]\mathring{Z}$ using a computer.

As in the previous section, we can use the ideal of the orbital variety $\mathring{Z}$ to find the homogeneous ideal of the MV cycle Z, and thus to determine $\Gamma(Z,\mathscr{L})$. We omit the details.

A.2.2. The preprojective algebra module and its flag function

In this example

$$n(\tau)=(0,1,0,0,0,0,0,1,0,1,0,0,0,1,0).$$

The general module $M_a\in Y$ is of the form

$$\begin{tikzcd}
 & 2 \arrow[dl, "1"'] \arrow[dr, "\binom{1}{0}"] & & 4 \arrow[dl, "\binom{0}{1}"'] \arrow[dr, "1"] & \\
1 \arrow[dr, "-a"'] & & 33 \arrow[dl, "(a\ 1)"'] \arrow[dr, "(1\ 1)"] & & 5. \arrow[dl, "-1"] \\
 & 2 & & 4 &
\end{tikzcd}$$

It is easy to determine all submodules of M_a, and thus the space $F_1(M_a)$. Comparing with the computation of $\Gamma(Z,\mathscr{L})$ yields the proof of Theorem A.12 (i). We omit the details.

We can compute $\overline{D}(\xi_{M_a})$ by enumerating composition series in the same manner as in the A_4 example; there are 148 sequences $\mathbf{i}$ with $F_{\mathbf{i}}(M_a)=\mathbb{P}^1$ and 104 sequences $\mathbf{i}$ with $F_{\mathbf{i}}(M_a)=\mathrm{pt}$. Computing in this way, we get that $\overline{D}(\xi_{M_a})p(\mu)$ is equal to the multidegree given in the previous section, yielding the proof of Theorem A.12 (ii).

A.3. Non-equality of basis vectors

Let $\lambda=(4,4,2,2)$, let $\mu=(2,2,2,2,2,2)$ and consider

$$\tau=\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2&5&5\\\hline 4&4\\\cline{1-2} 6&6\\\cline{1-2}\end{array},$$

so that $\nu=2\alpha_1+4\alpha_2+4\alpha_3+4\alpha_4+2\alpha_5$ and

$$n(\tau)=(0,2,0,0,0,0,0,2,0,2,0,0,0,2,0).$$

As before, we let $Z=Z_\tau$ and $Y=Y_\tau$, giving a corresponding pair $(t^{-\lambda}Z,Y)$. However, we will now prove that $b_Z\neq c_Y$. Using Proposition 12.2, it suffices to prove that

$$\overline{D}(b_Z)\neq\overline{D}(c_Y).$$

A general point of Y is $M_a\oplus M_{a'}$, with $a\neq a'$. Let $I(\omega_2+\omega_4):=I(\omega_2)\oplus I(\omega_4)$ be the injective Λ module (using the notation of §12.4). We prove the following.

THEOREM A.13. *We have*

$$\overline{D}(b_Z) = \overline{D}(\xi_{M_a \oplus M_{a'}}) - 2\overline{D}(\xi_{I(\omega_2+\omega_4)})$$

and, in particular,

$$\overline{D}(b_Z) \neq \overline{D}(\xi_{M_a \oplus M_{a'}}),$$

and thus $b_Z \neq c_Y$.

Proof. By Lemma 11.1, the computation of the right-hand side is reduced to the previous section (and the easy computation of $\overline{D}(\xi_{I(\omega_i)})$).

On the other hand, for the left-hand side, we use (22) to give a description of the generalized orbital variety $\mathring{Z}_\tau$. Using the aid of a computer, we find that it is cut out by a prime ideal I inside a polynomial ring with 24 generators.

From there, it is easy to compute the multidegree of $\mathring{Z}_\tau$, and thus $\overline{D}(b_Z)$. □

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