

*Research Article*

## The Low Lying Energy-Momentum Spectrum for the Lattice Four-Fermi Model

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**Abstract** We obtain the low-lying energy-momentum spectrum for the imaginary-time lattice four-Fermi or Gross-Neveu model in  $d + 1$  space-time dimensions ( $d = 1, 2, 3$ ) and with  $N$ -component fermions. Let  $0 < \kappa \ll 0$  be the hopping parameter,  $\lambda > 0$  the four-fermion coupling,  $m > 0$  the bare fermion mass and take  $s \times s$  spin matrices ( $s = 2, 4$ ). Our analysis of the one and the two-particle spectrum is based on spectral representation for suitable two- and four-fermion correlations. The one-particle energy-momentum spectrum is obtained rigorously and is manifested by  $\frac{sN}{2}$  isolated and identical dispersion curves, and the mass of particles has asymptotic value order  $-\ln \kappa$ . The existence of two-particle bound states above or below the two-particle band depends on whether Gaussian domination does hold or does not, respectively. Two-particle bound states emerge from solutions to a lattice Bethe-Salpeter equation, in a ladder approximation. Within this approximation, the  $(\frac{sN}{2} - 1)\frac{sN}{4}$  identical bound states have  $\mathcal{O}(\kappa^0)$  binding energies at zero system momentum and their masses are all equal, with value  $\approx -2 \ln \kappa$ . Our results can be validated to the complete model as the Bethe-Salpeter kernel exhibits good decay properties.

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### 1 Introduction

The determination of elementary excitations in physical models is of great importance, as they determine the time evolution of quantum systems, and are related to the falloff rate of correlation functions (CFs). Furthermore, in models of classical statistical mechanics and quantum field theory, the spectra of the associated Hamiltonian and momentum operators can be simultaneously determined and are associated with a particle interpretation.

In this work, we determine the low-lying particle spectrum of an imaginary-time (Euclidean) lattice functional integral formulation of a four-Fermi or Gross-Neveu (GN) model, with  $N$  flavor-component fermion fields in  $d + 1$  dimensions ( $d = 1, 2, 3$  is the spatial dimension) and a global  $U(N)$  flavor symmetry. This is an interesting model, both from theoretical and experimental point of view, because it presents a set of nice features and has a non-trivial behavior. For example, in  $d = 1$ , the GN model is perturbatively renormalizable, asymptotically free in the ultraviolet scale (if  $N > 1$ ) and also exhibits the phenomenon of dynamical symmetry breaking with mass generation and dimensional transmutation (see [16]). On the other hand, the GN model is simple enough to be, in principle, accessible via purely analytical approaches (see [6, 1]). Moreover, being a model with pure fermion fields, and due to the fact that the Pauli principle implies that fermion fields are associated with bounded operators, the GN model is placed in a special position to apply both perturbative and non-perturbative techniques to obtain the rigorous control of the ultraviolet and infrared limits, as well as some important correlation properties, going far beyond the level of the most common analysis employing simply a few perturbation orders. In this way, the ultraviolet limit of the complete model was constructed in [13, 14] for  $d = 1$  and  $N > 1$ , and asymptotic completeness has been verified up to the two-particle threshold in [19]. In [23], the existence of the infrared limit and mass generation was proved for the complete model, for  $d = 1$  and  $N \gg 1$ . For  $d = 2$ , this was done in [30] and, though the model is perturbatively non-renormalizable, its ultraviolet limit was constructed for large but finite  $N$  and small values of the coupling constant in [7], using non-perturbative analytical techniques. This corresponds to one of the few examples of a theory which is not trivial but is asymptotically safe, with a stable renormalization group fixed point, and has become a source of inspiration to the search of a good quantum model for Einstein's gravity (see e.g. [3, 5, 17, 24, 25] and references therein). The existence of the ultraviolet limit of the model for  $d = 3$  is an open question. However, the question of the existence of the ultraviolet limit or how to adapt the previous constructions to our cutoff model does not bother us here since we keep the unitary lattice step fixed throughout this work. Furthermore,

it is worth mentioning that the GN model plays an important role in testing properties of more sophisticated field theoretical models [41] and in the description of interesting physical phenomena such as superconductivity and strong interactions (see e.g. [21, 26]).

Let  $0 < \kappa \ll 1$  denote the lattice hopping parameter,  $M > 0$  the fermion bare mass,  $\lambda > 0$  the four-fermion coupling and let us consider  $s \times s$  Euclidean spin matrices. Here, we consider the lattice GN model defined by the action

$$S(\bar{\psi}, \psi) = \frac{\kappa}{2} \sum_x \bar{\psi}_{\alpha,a}(x) \Upsilon_{\alpha\beta}^{\epsilon e^\mu} \psi_{\beta,a}(x + \epsilon e^\mu) - M \sum_x \bar{\psi}_{\alpha,a}(x) \psi_{\alpha,a}(x) - \frac{\lambda}{N} \sum_x (\bar{\psi}_{\alpha,a}(x) \psi_{\alpha,a}(x))^2, \quad (1.1)$$

with summation over repeated indices and where  $\bar{\psi}_{\alpha,a}(x)$  and  $\psi_{\beta,b}(y)$  are Grassmann variables,  $x = (x^0, \vec{x}) \equiv (x^0, \dots, x^d)$  is a lattice site,  $x^0 \in \mathbb{Z}_{\frac{1}{2}} \equiv \{\pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots\}$ ,  $\vec{x} \in \mathbb{Z}^d$ ,  $\alpha, \beta \in \{1, 2, \dots, s\}$  are spin indices,  $a, b \in \{1, 2, \dots, N\}$  are flavor indices,  $\epsilon = \pm 1$  and, for  $\mu = 0, \dots, d$ , the  $e^\mu$  are unitary vectors of  $\mathbb{Z}^{d+1}$ , where  $\mathbb{Z}_0^{d+1} \equiv \mathbb{Z}_{\frac{1}{2}} \times \mathbb{Z}^d$ . Originally, instead of  $M$ , the lattice formulation gives us a bare mass  $\mathcal{M}(\bar{m}, \kappa) = (\bar{m} + 2\kappa)$ . However, without loss of generality, given  $\kappa$ , we choose the bare fermion mass  $\bar{m} = \mathcal{O}(1) \gg \kappa$  such that  $\mathcal{M}(\bar{m}, \kappa) = M > 0$ . Also,  $\Upsilon^{\pm e^\mu} = -\mathbb{I} \pm \gamma^\mu$ , where  $\mathbb{I}_s$  is the  $s$ -dimensional identity matrix. For  $d = 2, 3$ , we take  $s = 4$  and  $\gamma^\mu$ ,  $\mu = 0, 1, \dots, d$  are the  $4 \times 4$  Euclidean Dirac matrices

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \text{and} \quad \gamma^j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix},$$

where  $\sigma^j$ ,  $j = 1, 2, 3$ , are hermitian, traceless and anti-commuting Pauli spin matrices. For  $d = 1$  and alternatively for  $d = 2$ , we can consider another choice of spin matrices by taking  $s = 2$ ,  $\gamma^0 = \sigma^3$ ,  $\gamma^{j=1, \dots, d} = \sigma^j$ ,  $j \neq 3$ . From now on, we will concentrate our text only on the  $d = 3$  case, with Dirac matrices. The analysis for the other dimensions is very similar and can be easily reproduced from the one given here.

To be precise, we start in a compact box of finite-volume  $\Lambda \subset \mathbb{Z}_0^{d+1}$ , of size  $|\Lambda|$ . Using fermionic (Berezin) integration (for details see [2]), the normalized statistical mechanical expectation or averages of a function  $F(\bar{\psi}, \psi)$  of the fields is formally given by

$$\langle F(\bar{\psi}, \psi) \rangle_\Lambda = \frac{1}{Z} \int_\Lambda F(\bar{\psi}, \psi) e^{-S(\bar{\psi}, \psi)} d\bar{\psi} d\psi, \quad (1.2)$$

where  $d\bar{\psi} d\psi = \prod_{x, \alpha, a} d\bar{\psi}_{\alpha,a}(x) d\psi_{\alpha,a}(x)$  and  $Z_\Lambda = \int_\Lambda e^{-S(\bar{\psi}, \psi)} d\bar{\psi} d\psi$ . The  $2n$ -point CFs are given by choosing  $F(\bar{\psi}, \psi) = \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \psi(z_1) \cdots \psi(z_n)$ , for external points  $y$ 's and  $z$ 's. The CFs vanish if the number of  $\bar{\psi}$  and  $\psi$  are different, and are all real, for  $\kappa = 0$ .

The choice of the shifted lattice for the time direction, avoiding the zero-time coordinate, follows what happens for bosonic models. In this case, in the continuum limit, two-sided equal time limits of field correlations can be accommodated. This is also the case here, at least for  $d = 1$ . Besides, the above choice of parameters guarantees, with the extra condition  $|\frac{\kappa}{M}| \leq \frac{e^{-5}}{4(d+1)sN} |1 + \frac{2\lambda}{M^2}|^{1-sN}$ , that the thermodynamic limit  $\Lambda \rightarrow \mathbb{Z}_0^{d+1}$  of the model exists, and the physical positivity holds [29]. For  $M = 0$ , the lattice thermodynamic limit (if any!) cannot be easily reached, at least in the same way as before, and taking  $M > 0$  guarantees that the one-particle free fermion dispersion curve is monotonically increasing in each momentum component, which is physically desirable.

To show the existence of the thermodynamic limit, we use cluster expansion techniques, such as a polymer expansion (see [38, 39]). In the infinite-volume limit  $|\Lambda| \rightarrow \infty$ , denoted by  $\langle \cdot \rangle$ , are lattice translation invariant and joint analytic in  $\kappa$  and  $\lambda$  for  $|\frac{\kappa}{M}| < \frac{e^{-5}}{4sN(d+1)}$ . As a consequence, the truncated CFs have exponential tree decay with rate of at least  $(1 - \varepsilon) |\ln(\mathcal{O}(1)\kappa)|$ , where  $\varepsilon \rightarrow 0$  as  $\kappa \rightarrow 0$ . From now on, we work only in the infinite-volume limit.

For completeness, we give a brief and intuitive idea on how the polymer expansion is generated. Like a standard virial expansion in statistical mechanics, in the region of parameters where the coupling between different lattice sites is weak, we write  $e^{\kappa \bar{\psi}(x) \Gamma \psi(y)} = 1 + \vartheta_{\langle xy \rangle}$  and expand the expectation value of an observable  $O$  in the small parameter  $\vartheta_{\langle xy \rangle}$ . By this expansion, we obtain decoupled expectations of the form  $\langle O \prod_{\langle xy \rangle \in \varsigma} \vartheta_{\langle xy \rangle} \rangle_{\Lambda} |_{\kappa=0}$ , where  $\varsigma$  is a set of lattice links. Note that the expectation  $\langle AB \rangle_{\Lambda} |_{\kappa=0}$  factorizes into  $\langle A \rangle_{\Lambda} |_{\kappa=0} \langle B \rangle_{\Lambda} |_{\kappa=0}$  whenever  $A$  and  $B$  depend on disjoint sets of field variables. A polymer  $\varsigma_p$  is a set of field variables that does not break up into disjoint subsets. So, we obtain an expansion for  $\langle O e^{-S} \rangle_{\Lambda} |_{\kappa=0}$  in terms of the quantities  $z(\gamma_p) = \langle O \prod_{\langle xy \rangle \in \varsigma_p} \vartheta_{\langle xy \rangle} \rangle_{\Lambda} |_{\kappa=0}$ , called the activity of the polymer  $\varsigma_p$ . Writing this expansion for both the numerator and the denominator in (1.2), we can take the quotient and show that the resulting expansion has a finite domain of convergence (see the convergence condition above) uniformly in the volume  $|\Lambda|$  of the finite box  $\Lambda$ , so that infinite-volume limit can be taken.

For  $\kappa \geq 0$ , the action of (1.1) preserves positivity. Thus, by the fundamental work of [27,28], the CFs can be analytically extended to a quantum field theory in the Minkowski space. The idea consists of using the CFs to define (through a Feynman-Kac formula [15]) an inner product in the underlying physical quantum mechanical Hilbert space of physical states  $\mathcal{H}$  (see [29,38]). This construction provides an imaginary-time lattice quantum field theory. Besides, associated with this model, we can show there is a positive metric Hilbert space, an inner product  $(F, G)_{\mathcal{H}}$ , a vacuum vector  $\Omega$  and field operators  $\bar{\psi}_{\alpha}^a(x), \psi_{\alpha}^a(x)$ . Thus, we can construct in  $\mathcal{H}$  commuting self-adjoint energy-momentum (E-M) operators  $H \geq 0, \vec{P} = (P^1, \dots, P^d)$  (with spectral points  $\vec{p} \in \mathbf{T}^d \equiv (-\pi, \pi]^d$ ). These operators are the generators of lattice semi-group of time translations  $T_0$ , with  $T_0^2 > 0$ , and the unitary group of space translations  $\vec{T} = e^{i\vec{P}\cdot\vec{x}}$ , respectively. Therefore, by the spectral theorem, the operators  $H$  and  $\vec{P}$  can be diagonalized in a common basis, and so points in the spectrum of  $H$  can be written as functions of the points in the spectrum of  $\vec{P}$ .

A straightforward consequence of the above construction is that the squared transfer matrix  $T_0^2 = e^{-2H}$ , in the infinite volume limit, is a well-defined self-adjoint contraction operator in the Hilbert space  $\mathcal{H}$  if we assume that  $T_0$  does not have a 0 eigenvalue.

The methods we apply here allow us to access the low-lying energy-momentum spectrum and were introduced and established in [8,34,35,36,37], for lattice spin chains, and in [9,11,12,33], for treating other field theoretical models. We analyze the one- and two-particle sectors of the energy-momentum spectrum. Up to near the three-particle threshold, the energy-momentum spectrum exhibits isolated dispersion curves that are identified as particles and bound states. In the one-particle sector, using our method and the  $U(N)$  flavor symmetry, we show that there are  $\frac{sN}{2}$  particles corresponding to identical and isolated dispersion curves  $w(\vec{p}) \equiv w(\vec{p}, \kappa, \lambda)$ . We thus provide a rigorous proof of both the lower and upper mass gap properties of the spectrum. Also, the particle masses  $m(\kappa) \equiv w(\vec{p} = \vec{0})$  are all equal, and we show that the asymptotic value of the one-particle masses are of order  $(-\ln \kappa)$ .

Above the one-particle spectrum, there are finite width free (noninteracting) multiparticle spectral bands. Using a lattice version of the Bethe-Salpeter (B-S) equation, we show that two-particle bound states may appear near the two-particle band. At least in a leading approximation, that we call the ladder approximation, the existence of these bound states, either above or below the free two-particle band, depends on whether or not a Gaussian domination condition holds, respectively. The *Gaussian domination* condition is characterized by the negative sign of the quantity

$$\aleph \equiv \frac{1}{2} \langle \bar{\psi}_{\alpha a}(x) \psi_{\alpha a}(x) \rangle_{\kappa=0}^{-2} - \frac{1}{2} \langle \bar{\psi}_{\alpha a}(x) \bar{\psi}_{\beta a}(x) \psi_{\alpha a}(x) \psi_{\beta a}(x) \rangle_{\kappa=0}^{-1}. \quad (1.3)$$

Otherwise, if  $\aleph > 0$ , we say that we are in the *Gaussian subjugation* regime.

Within our ladder approximation, the  $(\frac{sN}{2} - 1) \frac{sN}{2}$  bound states we detect in any case have masses which are all equal and of order  $(-2 \ln \kappa)$ , with an  $\mathcal{O}(1)$  ( $\kappa$ -independent) binding energy. This spectral pattern is similar to the one obtained in our previous work on the  $O(N)$  spin model (see [8]).

We note that a discussion on the asymptotic completeness up to the two-particle threshold, for the Gross-Neveu model in the continuum space-time, can be found in [20]. Of course, to establish this result we need to know about the existence of two-particle bound states below this threshold. This paper does not solve the problem, however, so that our bound state analysis for the model on the lattice is justified. Besides this point, our spectral bound states results are relative to the two-particle band. This is a feature of the lattice model, which is absent in the continuum spacetime case. An interesting point is to see how much of the results obtained for the lattice subsist in the continuum limit.

It is also worth emphasizing that our spectral results are obtained from spectral representations for correlations which follow from the spectral theorem applied to the energy-momentum operators and a Feynman-Kac formula. This is a fundamental point in our work that guarantees we are indeed talking about the particle spectrum. Note that the commonly found analysis in the theoretical physics literature, involving only the decay rate of correlations, is not sufficient to this end. The analysis is still more delicate when various spectral points are too close and a continuous spectrum is present.

Last, we note that our method incorporates the main ingredients to validate our bound state results beyond our ladder approximation to the complete model, as it was done in the context of the stochastic Ginzburg-Landau models and for the ferromagnetic spin models (see [9,35,36]). The basic ingredient is the fast enough decay properties satisfied by the B-S kernel  $K$  (see below). This result will be established elsewhere.

The remainder of the paper is organized as follows. In Section 2, we introduce the two- and four-point CFs and obtain the corresponding spectral representations. We also discuss some symmetry properties that are used to simplify these CFs. In Section 3, we obtain the one-particle spectrum. In Section 4, we introduce the lattice B-S equation and derive our ladder approximation to the B-S kernel. In Section 5, we solve the B-S equation in this approximation and obtain the two-particle bound states and the corresponding binding energies. Finally, we draw some conclusions in Section 6.

## 2 Spectral representations of correlations

For  $x \equiv (x^0, \vec{x}) \in \mathbb{Z}_0^{d+1}$ ,  $F$  and  $G$  functions of the fields with finite support on  $x^0 > \frac{1}{2}$ , and for the uppercase check  $\checkmark$  denoting an operator on  $\mathcal{H}$ , taking  $x^0 > 0$ , we have the Feynman-Kac formula [15] (recall that  $\Omega$  denotes the vacuum)

$$(\checkmark G \Omega, \checkmark T_0^{x^0} \checkmark T^{x^1} \checkmark T^{x^2} \checkmark T^{x^3} \checkmark F \Omega)_{\mathcal{H}} = \left\langle [T_0^{x^0} T^{\vec{x}} F] \Theta G \right\rangle, \quad (2.1)$$

where  $T_0^{x^0}$  denotes time translation by  $x^0$ ,  $T^{\vec{x}} = T^{x^1} \dots T^{x^d}$  denotes a space translation by  $\vec{x} = (x^1, \dots, x^d)$  and  $\Theta$  is an anti-linear operator which involves time reflection. More precisely, the action of  $\Theta$  on the fermion fields is given by

$$\Theta \psi_{\alpha\alpha}(x^0, \vec{x}) = \gamma_{\alpha\alpha}^0 \bar{\psi}_{\alpha\alpha}(-x^0, \vec{x}), \quad \Theta \bar{\psi}_{\alpha\alpha}(x^0, \vec{x}) = \psi_{\alpha\alpha}(-x^0, \vec{x}) \gamma_{\alpha\alpha}^0.$$

The virtue of the Feynman-Kac formula (2.1) is that it relates inner products involving functions of the field operators in the Hilbert space (left-hand side) with statistical mechanical random process (field variables) correlations (right-hand side).

In our method, the key idea to detect particles and bound states in the E-M spectrum consists of using the Feynman-Kac formula (2.1) to obtain spectral representations for convenient truncated two- and four-point CFs. It is these spectral representations that allow us to mathematically justify the relation between the complex momentum singularities of the Fourier transform of these CFs to points in the E-M spectrum (see [8]). This basic ingredient is missing in many works in the literature, where only the exponential decay rates of correlations are obtained. These decay rates may, of course, have no relation with the spectrum, specially when narrow spectral splittings and spectral bands are involved.

In order to obtain the one-particle spectrum, that is the spectrum associated with the subspace generated by vectors of the form  $\bar{\psi}_{\alpha\alpha}(x)\Omega$ , we analyze the following two-point CFs:

$$G_{\alpha\beta}^{ab}(x, y) = \chi_{y^0 \geq x^0} \langle \bar{\psi}_{\alpha\alpha}(x) \psi_{\beta\beta}(y) \rangle - \chi_{x^0 > y^0} \langle \psi_{\alpha\alpha}(x) \bar{\psi}_{\beta\beta}(y) \rangle^*, \quad (2.2)$$

where  $\chi_{\alpha}$  is the characteristic function of the set  $\alpha$  and  $*$  denotes complex conjugation. Using the spectral theorem, we can write spectral representations for the  $T_0^{x^0}$  and  $T^{\vec{x}}$  operators given as

$$\checkmark T_0 = \int_{-1}^1 \lambda^0 d\mathcal{E}_0(\lambda^0), \quad \checkmark T_{j=1, \dots, d} = \int_{-\pi}^{\pi} e^{i\lambda^j} d\mathcal{E}_j(\lambda^j),$$

where  $\mathcal{E}_0(\lambda^0)$  and  $\mathcal{E}_j(\lambda^j)$  are the respective spectral families of  $\checkmark T_0$ ,  $\checkmark T^j$ , and we define  $\mathcal{E}(\lambda) = \mathcal{E}_0(\lambda^0) \prod_1^d \mathcal{E}_j(\lambda^j)$ . Using these representations in the Feynman-Kac formula (2.1), we obtain the spectral representation for the two-point CFs:

$$G_{\alpha\beta}^{ab}(x, y) = \int_{-1}^1 \int_{\mathbf{T}^d} \lambda_0^{|x^0 - y^0| - 1} e^{i\vec{\lambda} \cdot (\vec{x} - \vec{y})} d(\bar{\psi}_{\beta\alpha}, \mathcal{E}(\lambda) \bar{\psi}_{\alpha\beta})_{\mathcal{H}}. \quad (2.3)$$

To obtain the two-particle spectrum, that is the spectrum associated with the subspace generated by vectors of the form  $\bar{\psi}_{\alpha_1 a_1}(x_1 = (x_0, \vec{x}_1)) \bar{\psi}_{\alpha_2 a_2}(x_2 = (x_0, \vec{x}_2)) \Omega$ , we analyze the partially truncated four-point CFs:

$$D_{\alpha\beta}^{ab}(x_1, x_2, y_1, y_2) = \chi_{y_1^0 \leq x_1^0} \langle \bar{\psi}_{\alpha_1 a_1}(x_1) \bar{\psi}_{\alpha_2 a_2}(x_2) \psi_{\beta b_1}(y_1) \psi_{\beta b_2}(y_2) \rangle - \chi_{x_1^0 < y_1^0} \langle \psi_{\alpha_1 a_1}(x_1) \psi_{\alpha_2 a_2}(x_2) \bar{\psi}_{\beta b_1}(y_1) \bar{\psi}_{\beta b_2}(y_2) \rangle^*, \quad (2.4)$$

where  $ab$  (resp.  $\alpha\beta$ ) means  $a_1 a_2 b_1 b_2$  (resp.  $\alpha_1 \alpha_2 \beta_1 \beta_2$ ), and we work in the equal time representation (i.e.  $x_1^0 = x_2^0$  and  $y_1^0 = y_2^0$ ). Using the lattice relative coordinates (see [33])

$$\vec{\xi} = \vec{x}_2 - \vec{x}_1, \quad \vec{\eta} = \vec{y}_2 - \vec{y}_1, \quad \tau = y_1 - x_2,$$

(2.4) can be written as, setting  $D_{\alpha\beta}^{ab}(x_1, x_2, y_1, y_2) \equiv D_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, \tau)$ ,

$$D_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, \tau) = -\chi_{\tau_0 > 0} \left\langle \psi_{\alpha_1 a_1} \left( \frac{1}{2}, \vec{0} \right) \psi_{\alpha_2 a_2} \left( \frac{1}{2}, \vec{\xi} \right) \bar{\psi}_{\beta_1 b_1}(\tau_0, \vec{\tau} + \vec{\xi}) \bar{\psi}_{\beta_2 b_2}(\tau_0, \vec{\tau} + \vec{\xi} + \vec{\eta}) \right\rangle^* + \chi_{\tau_0 \leq 0} \left\langle \bar{\psi}_{\alpha_1 a_1} \left( \frac{1}{2}, \vec{0} \right) \bar{\psi}_{\alpha_2 a_2} \left( \frac{1}{2}, \vec{\xi} \right) \psi_{\beta_1 b_1} \left( \tau_0 + \frac{1}{2}, \vec{\tau} + \vec{\xi} \right) \psi_{\beta_2 b_2} \left( \tau_0 + \frac{1}{2}, \vec{\tau} + \vec{\xi} + \vec{\eta} \right) \right\rangle.$$

Note that vacuum contributions are automatically removed in (2.4) since they vanish by the imbalance on the number of fermion fields. Thus,  $D$  has an exponential temporal decay in  $\tau^0$ .

Next, in the same way as before for the two-point CFs, we can write the spectral representation ( $\tau^0 \neq 0$ )

$$D_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, \tau) = \int_{-1}^1 \int_{\mathbf{T}^d} \lambda_0^{|\tau^0|-1} e^{i\vec{\lambda}(\vec{\xi}+\vec{\eta}+\vec{\tau})} d\left(\phi_{\beta_2\beta_1}^{b_2b_1}(-\eta), \mathcal{E}(\lambda)\phi_{\alpha_1\alpha_2}^{a_1a_2}(\xi)\right)_{\mathcal{H}}, \quad (2.5)$$

where  $\phi_{\alpha\beta}^{ab}(x) = \bar{\psi}_{\alpha}^a(0)\bar{\psi}_{\beta}^b(x)\Omega$ .

Let  $\tilde{f}(p) = \sum_{u \in \mathbb{Z}^{d+1}} f(u)e^{-ip \cdot u}$  denote the Fourier transform of  $f(u)$ ,  $u \in \mathbb{Z}^{d+1}$ . Noting that  $(x^0 - y^0) \in \mathbb{Z}$  for  $x^0, y^0 \in \mathbb{Z}_{\frac{1}{2}}$ , using (2.3) and lattice translational invariance, the Fourier transform  $\tilde{G}_{\alpha\beta}^{ab}(p)$  of  $G_{\alpha\beta}^{ab}(x - y) \equiv G_{\alpha\beta}^{ab}(x, y)$  is given by

$$\tilde{G}_{\alpha\beta}^{ab}(p) = \tilde{G}(0, \vec{p}) + \lambda_0^{-1} \int_{-1}^1 \int_{T_d} \delta(\vec{p} - \vec{\lambda}) \left[ \frac{1}{e^{ip_0} - \lambda_0} + \frac{1}{e^{-ip_0} - \lambda_0} \right] d(\bar{\psi}_{\alpha a}, \mathcal{E}(\lambda)\bar{\psi}_{\beta b})_{\mathcal{H}}. \quad (2.6)$$

Therefore, for  $\vec{p}$  fixed, the singularities in the imaginary  $p^0$  axis are associated with the one-particle spectrum.

Similarly,  $\tilde{D}_{\alpha\beta}^{ab}(p, q, \theta_0, \vec{\tau})$ , the Fourier transform of  $D_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, \tau)$  is given by

$$\int [\tilde{f}(\vec{p})]^* \tilde{f}(\vec{q}) \tilde{D}_{\alpha\beta}^{ab}(\vec{p}, \vec{q}, \theta) d^3p d^3q = \tilde{D}_{\alpha\beta}^{ab}(\vec{p}, \vec{q}, (0, \vec{\tau})) + \int \left[ \frac{1}{e^{i\theta_0} - \lambda_0} + \frac{1}{e^{-i\theta_0} - \lambda_0} \right] \times \delta(\vec{\lambda}_1 - \vec{p}) \delta(\vec{\lambda}_2 - \vec{q}) \delta(\vec{\lambda}_3 - \vec{\theta}) d\left(\phi_{\beta_2\beta_1}^{b_2b_1}(f), \mathcal{E}(\lambda)\phi_{\alpha_1\alpha_2}^{a_1a_2}(f)\right)_{\mathcal{H}}, \quad (2.7)$$

where  $\phi_{\alpha\beta}^{ab}(f) = \sum_{\vec{x}} f(\vec{x})\phi_{\alpha\beta}^{ab}(\vec{x})$ . Thus, for  $\vec{p}$  fixed, the singularities in the imaginary  $p^0$  axis determine the two-particle spectrum and so bound states.

As we stressed above, without these spectral representations, the analysis of the decay rate of suitable CFs may not lead to the identification of particles and bound states. Particles and bound states are *isolated dispersion curves* in the E-M spectrum. If there is by chance a relation with the spectrum, for example, decay rates may be related to points within a spectral band.

The CFs given in (2.2) and (2.4) are associated with matrix-valued operators on convenient spaces, and we can use symmetries at the level of CFs to simplify them. We follow [11] where the most usual symmetries were treated in detail. A symmetry operation  $\mathbb{Y}$  defined on the Grassmann algebra  $\mathcal{G}$  of the fields  $\psi$  and  $\bar{\psi}$  is an operation of  $\mathcal{G}$  onto itself such that

$$\mathbb{Y}\psi_{\alpha a}(x) = \mathcal{A}_{\alpha\beta}^{ab}\bar{\psi}_{\beta b}(\mathcal{Y}x), \quad \mathbb{Y}\bar{\psi}_{\alpha a}(x) = \tilde{\psi}_{\beta b}(\mathcal{Y}x)\mathcal{B}_{\beta\alpha}^{ba},$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are  $4 \times 4$  complex matrices and  $\mathcal{Y}$  is a bijective linear map of  $\mathbb{Z}_0^{d+1}$  into itself. The symbol  $\sim$  stands for the introduction or the removal of a bar. A symmetry of the model is defined to be a symmetry operation which leaves invariant the action (1.1) and the CFs. Usual symmetries such as discrete spatial rotations, time *reversal*  $\mathcal{T}$ , charge conjugation  $\mathcal{C}$  and parity  $\mathcal{P}$ , which can be implemented as unitary (or anti-unitary for time reversal) operators in the Hilbert space  $\mathcal{H}$ , are symmetries of the model (see [12]). We also found a new *time reflection* symmetry  $\mathbb{T}$ , which can be implemented only at the CFs level, and which is described by

$$\mathcal{Y}(x^0, \vec{x}) = (-x^0, \vec{x}), \quad \mathcal{A} = i \begin{pmatrix} 0 & -\mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \otimes \mathbb{I}_N, \quad \mathcal{B} = \mathcal{A}^{-1}.$$

By applying  $\mathcal{C}$ , the one-particle and the one anti-particle CFs are the same. So, we only need to consider the lower spin indexes (i.e.  $\alpha, \beta = 3, 4$ ) in the CFs given by (2.2) and (2.4). Next, using  $\mathcal{CPT}$ , we see that the two-point CFs  $G$  defined by (2.2) is a multiple of the identity. So, we can drop the spin and flavor (due to  $U(N)$  flavor symmetry) indices for the two-point function.

Using these properties, next we discuss the determination of the one- and the two-particle spectrum.

### 3 One-particle spectrum

Concerning the one-particle spectrum, we adapt the analysis of [34,39] for spin models. By (2.6), we just need to determine the imaginary  $p^0$  singularities of the Fourier transform of  $G$ . To identify these singularities, we use the correspondence between the spacetime decay of the two-point CFs and the width of the analyticity strip of the

two-point CFs in momentum coordinates, in accordance with the Paley-Wiener theorem (see [31,32]). Thus, we need a detailed control of the spacetime decay rates of the two-point CFs. Also, it is important to note that the polymer expansion employed here can be controlled even with the introduction of distinct hopping parameters for each lattice link. To explore these features, we use the hyperplane decoupling method (see [40]). This is done by introducing a family of new lattice link variables  $z_{a_\mu}$ ,  $\mu = 0, 1, \dots, d$ , where  $a_0 \in \mathbb{Z}_{\frac{1}{2}}$ , and  $a_i \in \mathbb{Z}$ ,  $i = 1, \dots, d$ , by changing the hopping parameter from  $\kappa$  to  $z_{a_\mu}$ , in all action terms that connect lattice point pairs such that  $(x, x + e^\mu)$  and  $(x + e^\mu, x)$ , with  $x^\mu = a$ . Thus, taking  $z_{a_\mu} = 0$  separates the model in two disconnected regions, whose frontier is given by the  $a_\mu$  hyperplane. The application of the hyperplane decoupling method corresponds to Taylor expanding the interpolated CFs up to first order in each interpolating variable  $z_{a_\mu}$ . By construction, we have  $G(x, y, z_{a_i}, \kappa) = 0$ , for  $z_{a_i} = 0$  and all  $x^i \leq a_i < y^i$ . On the other hand, the polymer expansion allows us to extend the interpolated CFs to complex  $z_{a_\mu}$ , so from the Taylor expansion, we can write

$$G(x, y, z_{a_\mu}, \kappa) = \frac{1}{2\pi} \int_0^{z_{a_\mu}} \int_C \frac{G(x, y, z_{a_\mu}, \kappa)}{z_{a_i}^2} dz_{a_i},$$

where  $C$  is a circle with center in  $z_{a_i} = 0$  and radius given by the convergence radius of the polymer expansion. This integral representation allows us to apply a Cauchy estimate and shows that  $G(x, y)$  satisfies the global bound

$$|G(x, y)| < \mathcal{O}(1) |\kappa|^{\|x-y\|},$$

where  $\|x\| = \sum_{\mu=0}^{d+1} \|x^\mu\|$ . Next, by a generalized Paley-Wiener theorem, this global bound implies that  $\tilde{G}(p)$  is analytic in a strip of width  $[-(1-\epsilon) \ln \kappa]$ ,  $0 < \epsilon \ll 1$ . Moreover, we note that the  $z_{a_\mu}$  derivative of  $G$  has the *product* formula

$$\left. \frac{\partial G}{\partial z_{a_i}} \right|_{z_{a_i}=0} = \sum_{g: g_i = a_i} G(x, g) G(g \pm e^i, y), \quad (3.1)$$

which is an instrumental property in what follows.

With the introduction of the hyperplane variables, CFs are no longer invariant under translations but, by Holmgren's lemma (see [39]) and (3.1), we can show that the modified  $G$  defines a bounded operator in the space  $\ell^2(\mathbb{Z}_0^{d+1})$ . Thus, taking the decomposition

$$G = G_d + G_n = G_d [1 + G_d^{-1} G_n]^{-1},$$

where  $G_d$  collects only local contributions to  $G$ , as a matrix operator on  $\ell^2(\mathbb{Z}_0^{d+1})$ , with  $G(x, x) \neq 0$ , and thus we can define the convolution inverse correlation  $\Gamma$  through a convergent Neumann series applied to the factor  $[1 + G_d^{-1} G_n]^{-1}$ . Using the hyperplane decoupling method again and the product formula of the  $z_{a_i}$  derivative of  $G$ , we see that

$$\left. \frac{\partial \Gamma}{\partial z_{a_i}} \right|_{z_{a_i}=0} = 0, \quad \text{for } \|x - y\| > 1.$$

Also, a straightforward application of the hyperplane decoupling method allows us to show that  $\Gamma(x, y) = 0$  and  $\frac{\partial^2 \Gamma}{\partial z_{a_i}^2} = 0$  for  $z_{a_i} = 0$  and all  $x^i \leq a_i < y^i$ . So, by a Cauchy estimate, we also get that the improved global bound

$$|\Gamma(x, y)| < \mathcal{O}(1) |\kappa|^{3\|x-y\|}, \quad (3.2)$$

which holds for  $\|x - y\| > 1$ . This bound shows that  $\Gamma(x, y)$  has a faster decay than  $G(x, y)$ . Using again the Paley-Wiener theorem, this faster decay implies that the Fourier transform  $\tilde{\Gamma}(p) = \sum_x \Gamma(x) e^{-ip \cdot x}$  is analytic up to close to the three-particle threshold, which is approximately  $-(3 - \epsilon) \ln \kappa$ . This strip of analyticity is larger than the one for  $\tilde{G}(p)$ . Hence,  $\tilde{\Gamma}^{-1}$  provides an analytic continuation for  $\tilde{G}$  on this enlarged strip. This property is important either to prove the upper mass gap property or for the analysis of the two-particle spectrum given below.

From (2.6), the one-particle E-M spectrum is determined by the zeroes of  $\det \tilde{\Gamma}$ . Using (3.2) and a Taylor expansion of  $G$  in  $\kappa = 0$  allows us to write

$$\Gamma(x) = g_2^{-1} \delta(x) - \kappa \left[ \delta(|x_0 - e^0|) \delta(\vec{x}) + \frac{\kappa}{2} \delta(x^0) \delta(|\vec{x}| - 1) \right] + \Omega(x), \quad (3.3)$$

with  $|\Omega(x)| \leq \mathcal{O}(1) |\kappa|^{3|x|}$ .

Next, let  $\tilde{\Gamma}_0(p_0, \vec{p})$  be the Fourier transform of  $\Gamma(x) - \Omega(x)$ . We see that, for  $\vec{p} \in T_d$ ,  $\det \tilde{\Gamma}_0(p_0, \vec{p})$  has exactly one root  $E(\vec{p}) = \ln \zeta_0$ , with

$$\zeta_0 = g_2^{-1} \kappa^{-1} - \sum_{\mu=1}^d \cos p^\mu + \mathcal{O}(\kappa).$$

So, by applying the analytic implicit function theorem (see [18]) and Rouché's theorem (i.e. the principle of the argument, see [4]), we can show that, since  $\kappa$  is small, the zeroes of  $\det \tilde{\Gamma}(\omega(p_0, \vec{p}))$  are close enough to  $E(\vec{p})$ . Therefore, we see that there is a lower mass gap of multiplicity  $\frac{sN}{2}$  (due to spin and flavor symmetries), corresponding to the isolated one-particle dispersion curves

$$\omega(\vec{p}) = -\ln g_2 \kappa - g_2 \kappa \sum_{j=1, \dots, d} \cos p^j + \mathcal{O}(\kappa^2). \quad (3.4)$$

The one-particle mass  $\omega(\vec{p} = \vec{0})$  is given by

$$m(\kappa) = -\ln g_2 \kappa - dg_2 \kappa + \mathcal{O}(\kappa^2), \quad (3.5)$$

where  $g_2 = \langle \bar{\psi}_{\alpha a}(x) \psi_{\alpha a}(x) \rangle|_{\kappa=0}$ . Above the one-particle spectrum there are two-, three-, ... free particle bands, with finite width, and eventually overlapping for a large particle number.

In the next section, we inspect for the existence of two-particle bound states near the free two-particle band.

#### 4 Bethe-Salpeter equation and ladder approximation

To perform our bound state analysis, we use a lattice version of the B-S equation for this model which, in operator form, reads

$$D = D_0 + DKD_0. \quad (4.1)$$

In terms of its operator kernel,  $D_0$  is given by

$$D_{0, \alpha\beta}^{ab}(x, y) = G_{\alpha_1 \beta_2}^{a_1 b_2}(x_1, y_2) G_{\alpha_2 \beta_1}^{a_2 b_1}(x_2, y_1) - G_{\alpha_1 \beta_1}^{a_1 b_1}(x_1, y_1) G_{\alpha_2 \beta_2}^{a_2 b_2}(x_2, y_2). \quad (4.2)$$

and corresponds to the Gaussianly evaluated (i.e. applying Wick's theorem) four-point function with the corrected propagator, having the above determined mass. With these definitions, we write the B-S operator

$$K = D_0^{-1} - D^{-1}. \quad (4.3)$$

Using the exponential decay of the two-point CFs, we see that  $D_0$  decays and has a finite norm. On the other hand, the polymer expansion implies that  $D^T = D - D_0$ , the connected (truncated) four-point function, has an exponential tree decay, so also  $D$  has a finite norm. Therefore, both operators exist as bounded operators in  $\ell^2(A)$ ,  $A = \{(x, y) \in \mathbb{Z}_0^{d+1} | x^0 = y^0\}^{4N^2}$ . Decomposing  $D$  and  $D_0$  into diagonal and non-diagonal contributions, we can use again Holmgren's lemma and a Neumann series argument to show that  $D_0^{-1}$ ,  $D^{-1}$  and B-S operator  $K$  also exist as bounded operators in  $\ell^2(A)$ . With this, (4.3) is validated starting from the B-S equation (4.1).

In this paper, we restrict our analysis of the two-particle bound state spectrum to the leading approximation of the B-S operator, which we call a *ladder approximation*. By computing  $K$  to the leading nonvanishing  $\kappa$  contribution  $K \equiv L$ , and using  $L$  to solve (4.1) for  $D$ , we then look for the singularities of  $\tilde{D}(\vec{\xi}, \vec{\eta}, k)$ , the  $\tau$  Fourier transform of  $D(\vec{\xi}, \vec{\eta}, \tau)$ , to determine the two-particle bound states.

We now show that  $L$  is local and  $\mathcal{O}(\kappa^0)$ . To do this, we write

$$D = D^0 + D^T,$$

where  $D^T$  is the connected (truncated) four-point function. At  $\kappa = 0$ ,  $D^T$  vanishes for non-coincident points (see [8]), as seen by writing it in terms of source derivatives of the logarithm of the generating function. Using a simple Taylor expansion in  $\kappa = 0$ , we find

$$D_{\alpha\beta}^{ab}(x, y) = -2g_4 P_{xy} (1 - S_{\alpha\beta}^{ab}) \mathbb{1}_a - 2g_2^2 (1 - P_{xy}) \mathbb{1}_a + \mathcal{O}(\kappa), \quad D_{0, \alpha\beta}^{ab}(x, y) = -2g_2^2 \mathbb{1}_a + \mathcal{O}(\kappa),$$

where

$$P_{xy} = \delta_{x_1 x_2} \delta_{y_1 y_2} \delta_{y_1 x_2}, \quad S_{\alpha\beta}^{ab} = \delta_{\alpha_1 \alpha_2} \delta_{\beta_1 \beta_2} \delta_{\beta_1 \alpha_2} \delta_{a_1 a_2} \delta_{b_1 b_2} \delta_{b_1 a_2},$$

$\mathbb{I}_a$  is the identity on the anti-symmetric subspace of  $\ell^2(A)$ , and  $g_4$  is a function of  $m, \lambda, N$  defined by

$$g_4 = \langle \bar{\psi}_{\alpha a}(x) \bar{\psi}_{\beta a}(x) \psi_{\alpha a}(x) \psi_{\beta a}(x) \rangle \Big|_{\kappa=0}.$$

With all this, we have  $K = L + \mathcal{O}(\kappa)$ , where

$$L_{\alpha\beta}^{ab}(x, y) = -\varkappa P_{xy} [\mathbb{I}_{\alpha\beta}^{ab} - S_{\alpha\beta}^{ab}],$$

where  $\varkappa$  is defined in (1.3) and with  $\mathbb{I}$  being the  $4N^2 \times 4N^2$  identity matrix.

In order to proceed, it is worth mentioning that the B-S equation (4.1) has the same structure as an operator Schrödinger equation in the resolvent form, with  $D_0$  being similar to the resolvent  $(H_0 - z)^{-1}$  for the free Schrödinger Hamiltonian operator (kinetic energy)  $H_0$ ,  $D$  the interacting resolvent  $(H - z)^{-1}$  for the Hamiltonian  $H = H_0 + V$  and, modulo a multiplicative constant,  $K$  is the potential  $V$ . We refer to [10] for more details. In view of this similarity, consider a Schrödinger Hamiltonian for two interacting particles with mass  $m > 0$  in  $(\ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d))$ , given by

$$H_2 = -\frac{1}{2m}(\Delta_1 + \Delta_2) + V_2(\vec{x}_1 - \vec{x}_2) \equiv H_0 + V_2, \quad (4.4)$$

where  $\Delta_1 = \Delta \otimes I$ ,  $\Delta_2 = I \otimes \Delta$  and  $-\Delta$  is minus the lattice Laplacian. The action of  $-\Delta$  on  $f \in \ell^2(\mathbb{Z}^d)$  is given by

$$-\Delta f(\vec{x}) = 2df(\vec{x}) - \sum_{i=1}^d [f(\vec{x} + e^i) + f(\vec{x} - e^i)].$$

The interaction potential  $V_2(x - y)$  has the form

$$V_2(x - y) = \lambda\delta(x - y) + W_{12}(x - y),$$

where, by hypothesis, the  $W$  term collects all non-local contributions to the potential  $V_2$ , and has exponential decay of order  $m^{-2}$ . Thus, for  $\lambda > 0$ , the first term is a repulsive hardcore potential.

The corresponding resolvent Schrödinger equation is given by

$$(H_2 - z)^{-1} = (H_0)^{-1} - (H_0 - z)^{-1} V_2 (H_2 - z)^{-1}. \quad (4.5)$$

By identifying  $L$  to  $-V_2$ , the resolvent equation (4.5) is analogous to our B-S equation in the ladder approximation. Using the above decomposition of  $V_2$ , we find

$$(H_2 - z)^{-1} = (H_l - z)^{-1} - (H_l - z)^{-1} W (H_2 - z)^{-1}, \quad (4.6)$$

where

$$(H_l - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1} L (H_l - z)^{-1}, \quad (4.7)$$

with

$$L(\vec{x}, \vec{y}) = \lambda\delta(\vec{x} - \vec{y}).$$

Hence, restricting our attention to the dominant contribution to the potential, we obtain the spectral properties of  $H_l = H_0 + \lambda\delta$ . Let  $\sigma(O)$  denote the spectrum of the operator  $O$ . If  $z \notin \sigma(H_l) \cup \sigma(H_0)$ , then (4.7) yields to

$$(H_l - z)^{-1}(\vec{x}, \vec{y}) = (H_0 - z)^{-1}(\vec{x}, \vec{y}) - \frac{\lambda(H_0 - z)^{-1}(\vec{x}, \vec{0})(H_l - z)^{-1}(\vec{x}, \vec{0})}{1 - \lambda(H_0 - z)^{-1}(\vec{0}, \vec{0})}. \quad (4.8)$$

For  $z \notin [0, 4d]$ ,  $\sigma(H_l)$  coincides with the  $z$  singularities of (4.7), that is the zeroes of the denominator of (4.8). Hence, taking  $\vec{k} = \vec{0}$ , we have the bound state equation

$$\lambda(H_0 + E_b)^{-1}(\vec{0}, \vec{0}) = \frac{\lambda}{(2\pi)^d} \int_{T_d} \frac{d\vec{p}}{-\tilde{\Delta}(\vec{p}) + E_b} = -1. \quad (4.9)$$

Consider the attractive case ( $\lambda < 0$ ). For small momenta,  $-\tilde{\Delta}(\vec{p}) \approx |\vec{p}|^2$ . Thus, for  $d = 1, 2$ , (4.9) diverges as  $E_b \rightarrow 0$ , and since the integrand is strictly monotone in the binding energy  $E_b > 0$ , we can find a unique bound state solution for any  $\lambda < 0$ . For  $d \geq 3$ , the integral in (4.9) converges absolutely and remains finite as  $E_b \rightarrow 0$ , therefore there is a critical value  $\lambda_c < 0$  that allows bound states to exist. This is a consequence of the Birman-Schwinger bounds given for example in [31]. Also, (4.6) and a perturbation argument can be used to show that the singularities of  $(H_2 - z)^{-1}$  are close to the singularities of  $(H_l - z)^{-1}$ .

In the next section, we rigorously determine bound states in the ladder approximation.



## 5 Bound states

In the ladder approximation, the B-S equation reads

$$D_{\alpha\beta}^{ab}(x, y) = D_{0,\alpha\beta}^{ab}(x, y) + D_{\alpha\zeta}^{ac}(x, z)L_{\zeta\epsilon}^{cd}(z, w)D_{0,\epsilon\beta}^{db}(w, y),$$

where we sum over repeated indices. Changing to relative coordinates and letting  $\tilde{f}(\vec{\xi}, \vec{\eta}, k) = \sum_{\tau} f(\vec{\xi}, \vec{\eta}, \tau)e^{-ik^0\tau^0}$  denote the Fourier transform in  $\tau$  only. At zero spatial momentum ( $\vec{k} = \vec{0}$ ), we find

$$\tilde{D}_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, k) = \tilde{D}_{0,\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, k^0) - \tilde{D}_{0,\alpha\zeta}^{ac}(\vec{\xi}, \vec{0}, k^0)\aleph[\mathbb{I}_{\epsilon\zeta}^{dc} - S_{\epsilon\zeta}^{dc}]\tilde{D}_{\epsilon\beta}^{db}(\vec{0}, \vec{\eta}, k^0). \quad (5.1)$$

Therefore, we obtain from (5.1)

$$\tilde{D}_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, k^0) = \tilde{D}_{0,\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, k^0) + \tilde{D}_{0,\alpha\zeta}^{ac}(\vec{\xi}, \vec{0}, k^0) \left\{ [\mathbb{I} + \aleph[\mathbb{I} - S]\tilde{D}_0(\vec{0}, \vec{0}, k^0)]^{-1} \right\}_{\zeta,\epsilon}^{cd} \tilde{D}_{0,\epsilon\beta}^{db}(\vec{0}, \vec{\eta}, k^0). \quad (5.2)$$

From the analysis of the two-point CFs  $G$ ,  $\tilde{D}_{0,\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, k)$  is analytic in  $[0, 2m(\kappa)) \cup (2m(\kappa) + W, 3m)$ , where

$$W = 2[w(\vec{\pi}) - w(\vec{0})] = 8\kappa g_2 d + O(\beta^2)$$

is the free two-particle band width.

Hence, from (5.2), we see that there is no bound state for  $\aleph = 0$  since this condition implies  $\tilde{D} = \tilde{D}_0$ . On the other hand, bound states do occur wherever

$$\det(\mathbb{I} + \aleph[\mathbb{I} - S]\tilde{D}_0(\vec{0}, \vec{0}, k)) = 0.$$

The operator  $(\mathbb{I} + \aleph[\mathbb{I} - S]\tilde{D}_0(\vec{0}, \vec{0}, k))$  has  $4N^2$  eigenvalues, which we denote by  $\mu_i(k)$ ,  $i = 1, \dots, 4N^2$ . With this definition, we see that a bound state is identified with zeroes of at least one of the eigenvalues  $\mu_i(k)$  of

$$\mathbb{I} + \aleph[\mathbb{I} - S]\tilde{D}_0(\vec{0}, \vec{0}, k).$$

From (4.2), by an expansion up to the leading order in  $\kappa$ , we see that

$$D_0(\vec{0}, \vec{0}, \tau) = -2G^2(\vec{0}, \tau)(\mathbb{I} - S),$$

where  $G$  is the two-point CFs given by (2.2). Thus, we just need to obtain the zeroes of the eigenvalues of

$$\mathbb{I} - 2\aleph R(k^0)(\mathbb{I} - S),$$

where

$$R(k^0) = \int \langle \bar{\psi}_{1a}(0)\psi_{1a}(\tau) \rangle^2 e^{-k^0\tau^0} d\vec{\tau} d\tau_0.$$

By inspection, there are only two distinct eigenvalues. First, for  $i = 1, \dots, 2N^2 + N$ , we have  $\mu_i(k^0) = 1$ . Thus, this eigenvalue cannot satisfy the bound state condition. Also, note that the corresponding eigenvectors lie in the symmetric subspace of  $\mathbb{R}^{4N^2}$ . Second, for  $i = 2N^2 + N, \dots, 4N^2$ , we have the eigenvalue  $\mu_i(k^0) = 1 - 2\aleph R(k^0)$ , and the corresponding eigenvectors lie in the antisymmetric subspace of  $\mathbb{R}^{4N^2}$ . These states, in principle, can satisfy our bound state condition. For them, the bound state condition is described by  $R(k^0) = (2\aleph)^{-1}$ . We now analyze this equation, using the spectral representation established in (2.3) and the conclusions of the analysis of the two-point CFs performed in Section 3, to obtain the spectral representation (see [34, 39])

$$G(0, x) = \int_0^\infty \int_{\mathbf{T}^d} e^{i\vec{p}\cdot\vec{x} - E|x_0|} d\sigma_{\vec{p}}(E) d\vec{p}, \quad (5.3)$$

where, for  $\omega(\vec{p})$  given in (3.4), we use the measure decomposition

$$d\sigma_{\vec{p}}(E) = Z(\vec{p}, \kappa)\delta(E - \omega(\vec{p}))dE + d\hat{\sigma}_{\vec{p}}(E), \quad (5.4)$$

with  $Z(\vec{p}, \kappa) = \frac{\partial \tilde{F}}{\partial \chi}(p^0 = i\chi, \vec{p})|_{\chi=\omega(\vec{p})}$ , and where  $\tilde{F}(p) = [\tilde{G}(p)]^{-1}$ . Both  $d\sigma_{\vec{p}}(E)$  and  $d\hat{\sigma}_{\vec{p}}(E)$  are positive measures. The measure  $d\hat{\sigma}_{\vec{p}}(E)$  has support in  $(\tilde{m}, \infty)$ , where  $\tilde{m} \approx -3 \ln \kappa$  approaches from below the onset of the free three-particle spectrum. The decomposition in (5.4) of  $d\sigma_{\vec{p}}(E)d\vec{p}$  separates the one-particle contribution from

the contributions involving three or more particles which are left in  $d\hat{\sigma}_{\vec{p}}(E)$ . Also, from the analysis in Section 3, it follows that  $Z(\vec{p}, \kappa)$  is positive for any  $\vec{p}$  and  $\kappa > 0$ .

Using the representation (5.3), and holding in (5.4) only the product of one-particle contributions associated with  $Z(\vec{p}, \kappa)\delta(E - \omega(\vec{p}))dE$ , we obtain (see [36] for more details)

$$R(k^0) = 2(2\pi)^{d+1} \int_{\mathbf{T}^d} \frac{\sinh(2\omega(\vec{p})) [Z(\vec{p}, \kappa)]^2}{\cosh(2\omega(\vec{p})) - \cos k^0} d\vec{p} + \mathcal{O}(\kappa),$$

where

$$Z(\vec{p}, \kappa) = (2\pi)^{-d} g_2 + \mathcal{O}(\kappa).$$

From the last expression, we can directly check that, for  $0 < \Im(k^0) < 2m$  (i.e. below the two-particle threshold), we have  $R(k^0) > 0$ . Hence, there is no bound state below the two-particle band for  $\aleph < 0$  (Gaussian domination). Also, for  $2m + W < \Im(k^0) < 3m$ , we have  $R(k^0) < 0$  and no bound state appears above the two-particle band for  $\aleph > 0$  (Gaussian subjugation). The only  $k^0$  singularities of  $\bar{D}(\xi, \vec{\eta}, k^0)$  in (5.1), for  $\Im(k^0) \in (0, 2m) \cup (2m + W, 3m)$ , are solutions of

$$2\aleph \sum_{\tau} G^2(0, \tau) e^{2m\tau_0} e^{-\delta\varepsilon\tau_0} = 1, \quad (5.5)$$

where  $m$  is the one-particle mass and  $\delta\varepsilon$  is the bound state energy measured from the two-particle threshold (i.e. we set  $-ik^0 = 2m(\kappa) - \delta\varepsilon$ ).

Next, we give an intuitive argument for the bound state formation based on the behavior of  $\langle \bar{\psi}_{1a}(0)\psi_{1a}(x) \rangle$ . A rigorous argument using the convolution form in momentum space of the above condition and the spectral representation of two-point CFs can be established. Expanding  $\langle \bar{\psi}_{1a}(0)\psi_{1a}(x) \rangle$  to the leading order in  $\kappa$ , we obtain

$$\langle \bar{\psi}_{1a}(0)\psi_{1a}(x) \rangle \approx \kappa^{|\tau^0|+|\vec{\tau}|} g_2^{|\tau^0|+|\vec{\tau}|+1}.$$

This leading behavior follows from expanding the numerator of the two-point function. In order to get nonvanishing fermionic integration, there must be a chain of overlapping bonds (propagators) connecting the points 0 to  $x$ . Taking the chains of minimal length, the right side of the above approximation to  $\langle \bar{\psi}_{1a}(0)\psi_{1a}(x) \rangle$  follows. A polymer expansion can be used to rigorously control all contributions. Using in (5.5)  $m(\kappa)$  from (3.5), we obtain the bound state condition to the leading order in  $\kappa$ ,  $2g_2\aleph(1 - e^{-\delta\varepsilon}) = 1$ .

Let  $\varepsilon^A = \delta\varepsilon$  ( $\varepsilon^R = -\delta\varepsilon - W$ ) denote the bound state binding energy for the attractive (repulsive) case. For the attractive case, we find

$$\varepsilon^A = -\ln \left[ 1 - (2\aleph g_2)^{-1} \right],$$

where  $\aleph > 0$  (Gaussian subjugation), meaning that this spectral point appears below the free two-particle threshold. Similarly, for the repulsive case, the binding energy is given by

$$\varepsilon^R = \ln \left[ 1 - (2\aleph g_2)^{-1} \right] + W.$$

Since now  $\aleph < 0$  (Gaussian domination), this spectral point appears above the free two-particle band.

## 6 Conclusions

In closing, we point out that the low-lying E-M spectra of other GN models in  $d = 1$  have been obtained (at least approximately) by other methods in the continuum massless case. For instance, the Bethe-Yang ansatz was employed in the chiral invariant version of the GN Hamiltonian (see [1]), as well as Hartree approximations (see [22]) and semiclassical approaches (as in [6]). Our results agree qualitatively with those. Contrarily to what happens in pure bosonic models, where one may claim that the dependence of the appearance of bound states on the signs of the interaction is expected, here the fact that we are dealing with a purely fermionic model makes the situation different (and we cannot make a direct prediction!) since we cannot read the sign of the interacting potentials in the expression of the model action. Besides, the results presented here are obtained in a precise mathematical setting which includes the derivation of the upper gap property that is absent in the Hamiltonian formulations. Furthermore, our method can be easily exported to other models, and our results can be extended beyond the ladder approximation using the methods, for example, of [35, 36]. To do this, a fast decay of the B-S kernel is crucial. This decay can be derived using

the hyperplane decoupling method. In terms of the lattice relative coordinates of Section 2, we show the global bound

$$|K_{\alpha\beta}^{ab}(\vec{\xi}, \vec{\eta}, \tau)| \leq \mathcal{O}(1)|\kappa|^{4|\tau^0| + \frac{1}{2}(\|\vec{\xi} + \vec{\eta} + 2\vec{\tau}\| + \|\vec{\xi}\| + \|\vec{\eta}\|)},$$

revealing a decay with a rate greater than the two-particle one. Whether or not the degeneracy of the bound states is preserved in the complete model is an open question. Finally, we remark that the spectral pattern obtained here for the  $U(N)$  GN model is similar to the one found for the  $O(2N)$  spin model in [8], except that the latter has also a bound state of a different multiplicity,  $2N$ .

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