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Research Article

Shape Invariant Potentials in Second-Order Supersymmetric Quantum Mechanics

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Abstract Shape invariance condition in the framework of second-order supersymmetric quantum mechanics is studied. Two classes of solvable shape invariant potentials are consequently constructed, in which the parameters a_0 and a_1 of partner potentials are related to each other by translation $a_1 = a_0 + \alpha$. In each class, general properties of the obtained shape invariant potentials are systematically investigated. The energy eigenvalues are algebraically determined and the corresponding eigenfunctions are expressed in terms of generalized associated Laguerre polynomials. It is found that these shape invariant potentials are inherently singular, characterized by the $1/x^2$ singularity at the origin.

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1 Introduction

The concept of supersymmetric quantum mechanics (SUSY QM) was initially proposed as a toy model to illustrate the problem of dynamical supersymmetry breaking in quantum field theory [19,73,74]. Soon after the formulation, it was realized that the mathematical skeleton of SUSY QM is very attractive, since it is closely related to the method of factorization [49,67,66] and to the intertwining Darboux transformations [29,31]. For a review of SUSY QM, please refer to [12,22,51] and references therein.

The basic property of one-dimensional SUSY QM is described by two supercharges Q^+ and Q^-, accompanied with a supersymmetric Hamiltonian H. The supercharges are first-order differential operators and generate the following linear superalgebra:

$$\left(Q^{\pm}\right)^2 = 0, \quad \left[H, Q^{\pm}\right] = 0, \quad \left\{Q^+, Q^-\right\} = H, \tag{1.1}$$

where $[\cdot,\cdot]$ is the commutator and $\{\cdot,\cdot\}$ is the anticommutator. In (1.1), the super-Hamiltonian H consists of two isospectral partner Hamiltonians, differing at most in the ground-state energy level. The eigenfunctions of the two partner Hamiltonians are related to each other by means of supercharges Q^+ and Q^-.

It is known that in SUSY QM supersymmetry may or may not be broken. The situation is characterized by the Witten index Δ_F, defined by the difference between the number of zero-energy states of the partner Hamiltonians. Alternatively, it can be defined by the asymptotic behavior of the superpotential [51]. In a nonperiodic quantum system, the energy spectrum, that contains a nondegenerate zero-energy ground state and two-fold degenerate positive energy states, signifies unbroken supersymmetry ($\Delta_F \neq 0$). If there is no such zero-energy ground state so that all energy states are two-fold degenerate, supersymmetry is then spontaneously broken ($\Delta_F = 0$). However, certain supersymmetric periodic quantum systems may produce a zero-energy doublet of the ground states, resulting in a completely isospectral pair of partner Hamiltonians [16,33]. If this happens, we will have $\Delta_F = 0$ even in the case of unbroken supersymmetry. Such a peculiar property is named as *self-isospectral* in the literature. Additionally, the statement that all positive energy levels are two-fold degenerate is valid only in the case of discrete spectrum. If the spectra of the partner Hamiltonians allow continuous eigenstates, the corresponding energy levels will become four-fold degenerate.

The linear superalgebraic structure described in (1.1) admits the nonlinear generalization. In a nonlinear generalization of SUSY QM, two supercharges are higher-order ($n > 1$) differential operators, which satisfy the nonlinear superalgebra[1] [3,5,6,7,9,10]

$$\left(Q^{\pm}\right)^2 = 0, \quad \left[H, Q^{\pm}\right] = 0, \quad \left\{Q^+, Q^-\right\} = \mathcal{P}_n(H), \tag{1.2}$$

[1] In the literature of SUSY, there are several synonyms for the nonlinear SUSY: the higher-order SUSY, the polynomial SUSY or the N-fold SUSY. In what follows, we will use the nonlinear SUSY and higher-order SUSY interchangeably.

where $\mathcal{P}_n(H)$ is a polynomial of the super-Hamiltonian H of order n. The number of singlet states in nonlinear SUSY QM can take values from 0 to n. The Witten index Δ_F thus cannot be used to characterize spontaneous supersymmetry breaking. Nevertheless, the formalism of nonlinear SUSY QM is very instructive. In some periodic quantum systems, the property of self-isospectrality can be realized based on the nonlinear SUSY algebra [23,24]. In particular, it can be shown that quantum periodic systems with a parity-even finite-gap potential exhibit an unusual tri-supersymmetric structure, which originates from higher-order differential operators [27,59]. Various aspects of nonlinear SUSY QM have been elaborated, including, for instance, the works on the classification of differential realizations of the nonlinear SUSY QM algebra [2] and on the intrinsic links between supersymmetric isospectrality and hidden symmetries in certain quantum systems [25,26,28,55]. In fact, the nth order nonlinear SUSY QM is closely related to the parasupersymmetric QM of order n [6,48,54,60]. It is known that the former quantum theory can be deduced from the latter one if the redundant information, namely the intermediate Hamiltonians, of the latter theory is completely truncated.

Among various super-extensions of nonlinear SUSY QM, the second-order supersymmetric quantum mechanics (2-SUSY QM) is the simplest, the best known and well-studied one [3,5,6], [8,32,37,38,61,62,64]. The original motivation of introducing 2-SUSY QM is to overcome the limitations that occur in the standard SUSY QM. It is because that the standard SUSY QM only allows us to modify the ground-state energy level of the initial Hamiltonian if no new singularity in the transformed partner potential is permitted. For the purpose of spectral design, the standard SUSY QM definitely is not a satisfactory theory. The easiest way to surpass this difficulty is to generalize the SUSY QM to the 2-SUSY QM, which involves the second-order differential operators, instead of the first-order ones. The formalism of 2-SUSY QM has been shown to be a very powerful technique to build a new family of isospectral quantum systems with desired spectral information, that is, the pre-planned energy spectra and scattering data or the potential profiles. It can be shown that either one or two more energy levels can be created above the ground-state energy level of the initial Hamiltonian [39,40,41,42,43,58,63,65]. Moreover, the 2-SUSY QM also leads to the possibility of generating the standard SUSY complex potentials which render real energy eigenvalues [4,11,15, 17]. We note that the use of a pair of SUSY transformations to modify the bound spectrum arbitrarily, effectively generating the 2-SUSY transformation, was discussed in [14,68].

The purpose of the present article is to study shape invariance condition in the formulation of 2-SUSY QM, which has not been fully addressed until today.[2] The general solution to this problem remains unsolved. Yet, we will show, upon imposing an extra relation to the 2-SUSY shape invariance condition, that two translation classes of solvable potentials can be constructed, in which the parameters of partner potentials are related to each other by translation $a_1 = a_0 + \alpha$. In each class, we find that the eigenfunctions of the obtained shape invariant potentials are expressible in terms of generalized associated Laguerre polynomials, and that the energy eigenvalues are determined algebraically. One important aspect regarding these 2-SUSY shape invariant potentials is that discontinuity of the form $1/x$ at the origin is characteristic of the superpotentials, and thus leads to $1/x^2$ singularity to the corresponding partner potentials.

The article is organized as follows. In Section 2, we briefly review the relevant formulations of 2-SUSY QM and then present the shape invariance condition in 2-SUSY QM. In Section 3, we explicitly work out two translation classes of solvable shape invariant potentials in 2-SUSY QM. The general properties of the obtained potentials are investigated. Finally, we discuss the concept of hidden shape invariance and then conclude the article in Section 4.

2 Shape invariance condition in 2-SUSY QM

This section contains two parts. In the first part, we make a quick review on the 2-SUSY intertwining relations for the partner Hamiltonians. The purpose is to reproduce the relevant properties of 2-SUSY QM. Some of the results are quoted without proof for brevity. In the second part, the shape invariance condition is introduced within the framework of 2-SUSY QM. A special form of trial solution will be used to solve the equation derived from the 2-SUSY shape invariance condition.

The 2-SUSY QM is a particular realization of the standard SUSY algebra. It is generated by the supercharge operators Q^+ and $Q^- = (Q^+)^\dagger$, together with the quasi-Hamiltonian $\mathcal{H}_{2\text{-susy}}$ of the quantum system, fulfilling the following nonlinear superalgebra [3,5,6,37,38,61,62,64]:

$$\left(Q^{\pm}\right)^2 = 0, \quad \left[\mathcal{H}_{2\text{-susy}}, Q^{\pm}\right] = 0, \quad \left\{Q^+, Q^-\right\} = \mathcal{H}_{2\text{-susy}}. \tag{2.1}$$

The 2-SUSY algebra defined above can be readily realized by the 2×2 matrices

$$Q^- = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \tag{2.2}$$

[2] A simplified version of 2-SUSY shape invariance was introduced in [1].

$$\mathcal{H}_{2\text{-susy}} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} \big(\tilde{H}-\epsilon_1\big)\big(\tilde{H}-\epsilon_2\big) & 0 \\ 0 & \big(H-\epsilon_1\big)\big(H-\epsilon_2\big) \end{pmatrix}, \tag{2.3}$$

where $\tilde{H}$ and H are two partner Hamiltonians of Schrödinger type, denoted respectively by

$$\tilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V}(x), \quad H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{2.4}$$

and A is a second-order differential shift operator defined formally by

$$A \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \eta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x). \tag{2.5}$$

It then follows, from (2.1), that the partner Hamiltonians $\tilde{H}$ and H are linked to each other by the intertwining relations of Darboux transformation with the operators A and $A^\dagger$ as

$$\tilde{H}A = AH, \quad A^\dagger\tilde{H} = HA^\dagger. \tag{2.6}$$

Furthermore, the function $\gamma(x)$ and the partner potentials $\tilde{V}(x)$ and $V(x)$ can be expressed solely in terms of $\eta(x)$ and its derivatives as

$$\gamma = d - V + \frac{\eta^2}{2} - \frac{\eta'}{2}, \tag{2.7}$$

$$\tilde{V} = V + 2\eta', \tag{2.8}$$

$$V = \frac{\eta''}{2\eta} - \frac{\eta'^2}{4\eta^2} + \frac{c}{\eta^2} + \frac{\eta^2}{4} - \eta' + d, \tag{2.9}$$

where $\eta' = \frac{\mathrm{d}\eta}{\mathrm{d}x}$, $\eta'' = \frac{\mathrm{d}^2\eta}{\mathrm{d}x^2}$ and c, d are two real constants of integration. In terms of c and d, the factorization energies of the quasi-Hamiltonian (2.3) are given by $\epsilon_1 \equiv d - \sqrt{c}$ and $\epsilon_2 \equiv d + \sqrt{c}$, respectively.

The formalism of 2-SUSY QM is a very useful tool for generating new solvable potentials which admit pre-planned spectral properties. To explain this point, let us suppose that $V(x)$ is an initially solvable potential and c and d are two fixed but arbitrary constants. The transformed partner potential $\tilde{V}(x)$ defined in (2.8) can then be completely determined if the solution $\eta(x)$ of the nonlinear second-order differential equation (2.9) is established. The spectrum of the partner potential $\tilde{V}(x)$ will depend on the chosen constant c. We consequently classify the general solution $\eta(x)$ into three cases as follows. (i) The real case with $c > 0$. In this case, we have two real non-degenerate factorization energies $\epsilon_1 \neq \epsilon_2$. The spectrum of $\tilde{V}(x)$ is obtained from that of $V(x)$ by deleting two adjacent energy levels, shifting the position of an energy level, adding two more energy levels between two adjacent ones, or adding two more levels below the ground-state energy level [39,40,65]. (ii) The confluent case with $c = 0$. We have degenerate factorization energies $\epsilon_1 = \epsilon_2$. To obtain the spectrum of $\tilde{V}(x)$, we may either add or delete an energy level to that of the initial potential $V(x)$ [42,43,58]. (iii) The complex case with $c < 0$. We have a pair of complex conjugate factorization energies $\epsilon_2 = \bar{\epsilon}_1$. In such a case, the transformed potential $\tilde{V}(x)$ is a real potential isospectral to $V(x)$. Nevertheless, the intermediate SUSY potentials are complex, but having real energy spectra [41, 63].

The concept of shape invariance can be incorporated with the structure of 2-SUSY QM. Before doing so, let us review some important properties of shape invariance in the standard SUSY QM. In the standard SUSY QM, the shape invariance condition is known to provide a key ingredient for exploring exactly solvable potentials for Schrödinger equation, since it leads immediately to an integrability condition [47]. With the help of shape invariance condition, the entire energy spectra of the partner Hamiltonians can be obtained algebraically, when SUSY is unbroken. In SUSY QM, many interesting classes of solvable shape invariant potentials that retain SUSY have been constructed and discussed [13,18,20,21,30,35,36,52,53,57,72], including all the analytically solvable potentials known in the context of nonrelativistic quantum mechanics. We mention here that the shape invariance condition is not the most general integrability condition, because certain exactly solvable potentials are shown not to be shape invariant [20].

To start with the discussion on the shape invariance condition within the framework of 2-SUSY QM, we first introduce an x-independent parameter, denoted by a_0. Then, we assume that all the functions appearing in the present section not only depend on x but also on the parameter a_0. Similarly, the constants c and d are assumed to be functions of a_0. In what follows, we will consider the case where $c(a_0) \geq 0$, so that two real factorization energies of the initial potential $V(x, a_0)$ are given by $d(a_0) - \xi(a_0)$ and $d(a_0) + \xi(a_0)$, respectively. Here, the notation $\xi(a_0) = \sqrt{c(a_0)}$ is used.

To be more explicitly, we take (2.8) and, according to the rule, write it as

$$\tilde{V}\big(x,a_0\big) = V\big(x,a_0\big) + 2\eta'\big(x,a_0\big), \tag{2.10}$$

where $\eta'(x,a_0) \equiv \frac{\mathrm{d}\eta}{\mathrm{d}x}$ is understood. What we mean by shape invariance is that the pair of partner potentials $\tilde{V}(x,a_0)$ and $V(x,a_0)$ in (2.10) are similar in shape but differ only up to a change of the parameter a_0 and an additive constant. Mathematically, it is of the form

$$\tilde{V}\big(x,a_0\big) = V\big(x,a_1\big) + e\big(a_0\big), \tag{2.11}$$

where $a_1 = f(a_0)$ is a function of a_0 and the remainder $e(a_0)$ is independent of x.

Now if we combine (2.10) and (2.11), we immediately obtain an equation that is essential for the construction of shape invariant potentials within the formalism of 2-SUSY QM. The equation reads

$$V\big(x,a_1\big) = V\big(x,a_0\big) + 2\eta'\big(x,a_0\big) - e\big(a_0\big). \tag{2.12}$$

Note that the potential $V(x,a_i)$ (for $i=0,1$) is expressible in terms of $\eta(x,a_i)$ using (2.9).

Based on (2.9) and (2.12), we can readily determine energy eigenvalues for the initial potential $V(x,a_0)$ algebraically. They are found to be ($n=0,1,2,\ldots$)

$$E_{2n} = d\big(a_n\big) - \xi\big(a_n\big) + \sum_{i=0}^{n-1} e\big(a_i\big), \quad E_{2n+1} = d\big(a_n\big) + \xi\big(a_n\big) + \sum_{i=0}^{n-1} e\big(a_i\big), \tag{2.13}$$

where the convention for the summation $\sum_{i=0}^{-1} = 0$ is used. A point is worth mentioning. For eigenenergies (2.13) to be a consistent energy spectrum of the initial potential $V(x,a_0)$, we must require that the energy gap between two adjacent eigenstates be greater than zero in order to prevent these energy levels from crossing. That is to say, the following relations must hold

$$R_n = E_{2n+1} - E_{2n} = 2\xi\big(a_n\big) > 0, \tag{2.14}$$

$$\tilde{R}_n = E_{2n+2} - E_{2n+1} = \big(d\big(a_{n+1}\big) - d\big(a_n\big) + e\big(a_n\big)\big) - \big(\xi\big(a_n\big) + \xi\big(a_{n+1}\big)\big) > 0. \tag{2.15}$$

If either one of R_n and $\tilde{R}_n$ becomes negative for a given quantum number, say $n = n_0$, it then means that the number of bound states will be finite, resulting in the corresponding 2-SUSY shape invariant potential of finite depth.

Further, the unnormalized eigenfunctions corresponding to the eigenvalue spectrum in (2.13) are given by

$$\psi_{2n}\big(x,a_0\big) \propto A^{\dagger}\big(a_0\big)A^{\dagger}\big(a_1\big)\cdots A^{\dagger}\big(a_{n-1}\big)\psi_{+}\big(x,a_n\big), \tag{2.16}$$

$$\psi_{2n+1}\big(x,a_0\big) \propto A^{\dagger}\big(a_0\big)A^{\dagger}\big(a_1\big)\cdots A^{\dagger}\big(a_{n-1}\big)\eta\big(x,a_n\big)\psi_{-}\big(x,a_n\big), \tag{2.17}$$

where the shift operator $A^{\dagger}(a_0)$ can be deduced from (2.5) and $\psi_{+}(x,a_0)$ and $\eta(x,a_0)\psi_{-}(x,a_0)$ denote the eigenstates of energy E_0 and E_1 for the initial potential $V(x,a_0)$, respectively. More explicitly, both $\psi_{\pm}$ functions are written by

$$\psi_{\pm}\big(x,a_0\big) \propto \exp\left[-\int^{x} W_{\pm}\big(x',a_0\big)dx'\right], \tag{2.18}$$

with the corresponding superpotentials [6]

$$W_{\pm}\big(x,a_0\big) = \frac{1}{2}\left[\eta\big(x,a_0\big) \pm \frac{2\xi\big(a_0\big) - \eta\big(x,a_0\big)'}{\eta\big(x,a_0\big)}\right]. \tag{2.19}$$

Normalizability of $\psi_{+}(x,a_0)$ and $\psi_{-}(x,a_0)$ will therefore be determined by the asymptotic behavior of the corresponding superpotential near its singularities and at infinity. Moreover, in terms of the superpotential $W_{+}(x,a_0)$, the initial potential $V(x,a_0)$ in (2.9) is reformulated into

$$V\big(x,a_0\big) = W_{+}\big(x,a_0\big)^2 - W_{+}'\big(x,a_0\big) + \big(d\big(a_0\big) - \xi\big(a_0\big)\big), \tag{2.20}$$

where $W_{+}'(x,a_0) \equiv \frac{\mathrm{d}W_{+}}{\mathrm{d}x}$.

At this stage, it is still too difficult to construct the general solution for (2.12), since the function $\eta(x, a_i)$ depends arbitrarily on both x and a_0. To solve the problem, we search for a particular form of $\eta(x, a_i)$ such that (2.12) can be simplified. So, let us examine the trial solution for η-function of the form $\eta(x, a_0) \equiv a_0\bar{\eta}(x)$, in which $\bar{\eta}(x)$ is any function of x.[3] Then, by substituting this trial solution into (2.12) that is established from 2-SUSY shape invariance condition, we arrive at

$$\big(a_1 + a_0\big)\bar{\eta}' = \left(\frac{c\big(a_1\big)}{a_1^2} - \frac{c\big(a_0\big)}{a_0^2}\right)\frac{1}{\bar{\eta}^2} + \big(a_1^2 - a_0^2\big)\frac{\bar{\eta}^2}{4} + \big(d\big(a_1\big) - d\big(a_0\big) + e\big(a_0\big)\big). \tag{2.21}$$

Equation (2.21) represents a nonlinear first-order differential equation satisfied by $\bar{\eta}(x)$ that is required to be independent of the parameter a_0. Hence, the detailed form of $\bar{\eta}(x)$ will depend on the specially chosen relations among the coefficients: $e(a_0)$, $c(a_i)$ and $d(a_i)$ $(i = 0, 1)$.

3 Shape invariant potentials in 2-SUSY QM

In this section, we look for solvable shape invariant potentials in 2-SUSY QM by solving the nonlinear first-order differential equation (2.21). We mention here that we have restricted our analysis of (2.21) on the condition $c(a_i) > 0$, which is known to render the associated 2-SUSY transformation real. There are two important classes to be discussed as follows.

Class 1. We choose the set of relations (for $i = 0, 1, 2, \ldots$)

$$a_1 = a_0 + \alpha, \quad c\big(a_i\big) = \big(\tilde{m}a_i\big)^2, \quad d\big(a_i\big) = \beta + \tilde{n}a_i, \quad e\big(a_i\big) = 2\tilde{n}a_i, \tag{3.1}$$

where α, β and $\tilde{n}$ are arbitrary constants, and the product $\xi(a_i) = \tilde{m}a_i > 0$ is taken to be positive. Without loss of generality, we further choose the case that is $\tilde{n} > 0$, since the result of the case $\tilde{n} < 0$ can be easily deduced by the simple transformations $\bar{\eta} \to -\bar{\eta}$ and $\alpha \to -\alpha$.

With such a choice, (2.21) is simplified to

$$\bar{\eta}' = \frac{\alpha}{4}\bar{\eta}^2 + \tilde{n}. \tag{3.2}$$

The analytical solutions of (3.2) will depend on the sign of the constant α. Before going to details, let us analyze the general structure of the shape invariant potentials in this class. From (3.2), we can readily build $\eta(x, a_0) = a_0\bar{\eta}(x)$ and consequently construct the superpotentials $W_\pm(x, a_0)$ and the initial potential $V(x, a_0)$ by using (2.19) and (2.20). They are, respectively,

$$W_\pm\big(x, a_0\big) = \frac{1}{2}\left[a_{\mp\frac{1}{4}}\bar{\eta}(x) \pm \frac{2\tilde{m} - \tilde{n}}{\bar{\eta}(x)}\right], \tag{3.3}$$

$$V\big(x, a_0\big) = \frac{1}{4}\left[a_{-\frac{3}{4}}a_{-\frac{1}{4}}\bar{\eta}(x)^2 + \frac{4\tilde{m}^2 - \tilde{n}^2}{\bar{\eta}(x)^2} + 4\beta + \frac{\alpha\tilde{n}}{2}\right], \tag{3.4}$$

where $a_n = a_0 + n\alpha$ is understood. Eigenvalues for the initial potential $V(x, a_0)$ above can be obtained algebraically from (2.13) as

$$E_{2n} = \beta + (\tilde{n} - \tilde{m})a_n + 2\tilde{n}\sum_{i=0}^{n-1} a_i, \quad E_{2n+1} = \beta + (\tilde{n} + \tilde{m})a_n + 2\tilde{n}\sum_{i=0}^{n-1} a_i. \tag{3.5}$$

In addition, the positivity condition of the energy gap between two adjacent eigenstates presented in (2.14) and (2.15) becomes

$$R_n = 2\tilde{m}a_n > 0, \quad \tilde{R}_n = 2(\tilde{n} - \tilde{m})a_{n+\frac{1}{2}}0 > 0. \tag{3.6}$$

It is clear that, to avoid level crossing, we must have $\tilde{n} > \tilde{m} > 0$ in addition to $\tilde{m}a_n > 0$.

[3] We could have written the trial solution as $\eta(x, a_0) = A(a_0)\bar{\eta}(x)$, in which $A(a_0)$ is an arbitrary function of a_0. The choice leads to the result: $A(a_0) = a_0 + k\alpha$, where α is the translation parameter (3.1). For simplicity, we take $k = 0$ in the present study.

The eigenfunctions can be explicitly constructed using (2.16) and (2.17). After some computational algebra, the first few even-number eigenstates are found,

$$\psi_2\big(x,a_0\big) \propto \left[a_0\frac{\bar{\eta}^2}{2\tilde{n}} - \left(1-\frac{\tilde{m}}{\tilde{n}}\right)\right]\psi_+\big(x,a_1\big), \tag{3.7}$$

$$\psi_4\big(x,a_0\big) \propto \left[a_0 a_{\frac{1}{2}}\frac{\bar{\eta}^4}{(2\tilde{n})^2} - 2a_{\frac{1}{2}}\left(2-\frac{\tilde{m}}{\tilde{n}}\right)\frac{\bar{\eta}^2}{2\tilde{n}} + \left(1-\frac{\tilde{m}}{\tilde{n}}\right)\left(2-\frac{\tilde{m}}{\tilde{n}}\right)\right]\psi_+\big(x,a_2\big), \tag{3.8}$$

$$\begin{aligned}\psi_6\big(x,a_0\big) \propto \Bigg[&a_0 a_{\frac{1}{2}} a_1\frac{\bar{\eta}^6}{(2\tilde{n})^3} - 3a_{\frac{1}{2}}a_1\left(3-\frac{\tilde{m}}{\tilde{n}}\right)\frac{\bar{\eta}^4}{(2\tilde{n})^2} + 3a_1\left(2-\frac{\tilde{m}}{\tilde{n}}\right)\left(3-\frac{\tilde{m}}{\tilde{n}}\right)\frac{\bar{\eta}^2}{2\tilde{n}} \\ &- \left(1-\frac{\tilde{m}}{\tilde{n}}\right)\left(2-\frac{\tilde{m}}{\tilde{n}}\right)\left(3-\frac{\tilde{m}}{\tilde{n}}\right)\Bigg]\psi_+\big(x,a_3\big),\end{aligned} \tag{3.9}$$

where $\psi_+(x,a_n)$ is the ground-state eigenfunction, expressible in terms of $W_+(x,a_n)$ via (2.18). As for the odd-number eigenstates, that is, $\psi_{2n+1}(x,a_0)$, they can be constructed directly from the corresponding even-number states $\psi_{2n}(x,a_0)$ by the replacement: $-\frac{\tilde{m}}{\tilde{n}} \to +\frac{\tilde{m}}{\tilde{n}}$ and $\psi_+(x,a_n) \to \bar{\eta}(x)\psi_-(x,a_n)$.

General expression for these eigenfunctions can be established in the forms

$$\psi_{2n}\big(x,a_0\big) \propto \mathcal{L}_n^{-\frac{\tilde{m}}{\tilde{n}}}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_{\frac{1}{2}(n-1)}\right)\psi_+\big(x,a_n\big), \tag{3.10}$$

$$\psi_{2n+1}\big(x,a_0\big) \propto \mathcal{L}_n^{\frac{\tilde{m}}{\tilde{n}}}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_{\frac{1}{2}(n-1)}\right)\bar{\eta}(x)\psi_-\big(x,a_n\big), \tag{3.11}$$

where $\mathcal{L}_n^k(x^2, a_{\frac{1}{2}m})$ is the generalized associated Laguerre polynomials defined by

$$\mathcal{L}_n^k\big(x^2, a_{\frac{1}{2}m}\big) = \sum_{s=0}^{n}(-1)^s\frac{1}{s!(n-s)!}\frac{(n+k)!}{(s+k)!}\frac{\big[a_{\frac{1}{2}m}\big]!}{\big[a_{\frac{1}{2}(m-s)}\big]!}x^{2s}. \tag{3.12}$$

Here, the notations $[a_{\frac{1}{2}m}]! = \prod_{k=0}^{m} a_{\frac{1}{2}k}$ and $[a_{-\frac{1}{2}}]! = 1$ are used. To be more explicitly, we have in (3.10) that $[a_{\frac{1}{2}(n-1)}]! = a_{\frac{1}{2}(n-1)}a_{\frac{1}{2}(n-2)}\cdots a_{\frac{1}{2}}a_0$. It is interesting to note that when $\alpha = 0$, $a_n = a_0$ and $[a_{\frac{1}{2}n}]! = (a_0)^{n+1}$, for $n \geq 0$. As a result, for $\alpha = 0$, the generalized associated Laguerre polynomials reduce to

$$\mathcal{L}_n^{-\frac{\tilde{m}}{\tilde{n}}}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_0(\alpha=0)\right) = L_n^{-\frac{\tilde{m}}{\tilde{n}}}\left(\frac{a_0\bar{\eta}^2}{2\tilde{n}}\right), \tag{3.13}$$

where $L_n^k(x^2)$ is the standard associated Laguerre polynomials.

Having established the general properties for the shape invariant potentials in this class, now let us go back to (3.2). By solving the equation, we are able to construct three different kinds of solution [34,46] as follows.

(1) If we take $\alpha > 0$, (3.2) yields this solution: $\bar{\eta}(x) = 2\sqrt{\frac{\tilde{n}}{\alpha}}\tan\frac{\sqrt{\tilde{n}\alpha}}{2}x$. A direct computation on both superpotentials $W_\pm(x,a_0)$ shows that they are singular, characterized by the $1/x$ singularity near the origin. Explicitly, we have

$$W_\pm\big(x,a_0\big) = \pm\frac{1}{2x}\left(\frac{2\tilde{m}}{\tilde{n}} - 1\right) + \frac{x}{12}\big(6\tilde{n}a_{\mp\frac{1}{6}} \mp \tilde{m}\alpha\big) + \mathcal{O}\big(x^3\big). \tag{3.14}$$

In the same vein, the singularity of the initial potential $V(x,a_0)$ around the origin behaves like $\frac{1}{x^2}(\frac{\tilde{m}^2}{\tilde{n}^2} - \frac{1}{4})$. Since $\alpha > 0$, neither R_n nor $\tilde{R}_n$ in (3.6) will become negative for the choice $\tilde{n} > \tilde{m}$. It implies that the potential accommodates infinite many numbers of bound states. This potential is known as the singular Pöschl-Teller I potential.

(2) If $\alpha < 0$ is taken, then we have from (3.2) the following: $\bar{\eta}(x) = 2\sqrt{-\frac{\tilde{n}}{\alpha}}\tanh\frac{\sqrt{-\tilde{n}\alpha}}{2}x$. A similar computation shows that both superpotentials $W_\pm(x,a_0)$ are singular at the origin, which behave as $\pm\frac{1}{2x}(\frac{2\tilde{m}}{\tilde{n}}-1)$, respectively. It results in the singularity of $V(x,a_0)$ like $\frac{1}{x^2}(\frac{\tilde{m}^2}{\tilde{n}^2} - \frac{1}{4})$ at $x = 0$. Because of $\alpha < 0$, either R_n or $\tilde{R}_n$ will become negative for large enough values of n. This singular potential therefore consists of a finite number of bound states, which is named as the singular Pöschl-Teller II potential.

(3) If we take $\alpha = 0$, (3.2) gives $\bar{\eta}(x) = \tilde{n}x$. The singularity of the superpotentials is found to be $\pm\frac{1}{2x}(\frac{2\tilde{m}}{\tilde{n}} - 1)$. As a result, the initial potentials $V(x, a_0)$ in all three kinds of solution have the same singular behavior at the origin. The present potential is called the singular harmonic potential that allows an infinite number of two shifted sets of equally energy-spaced eigenstates.

A remark is in order. As mentioned, the superpotentials in all three different solutions have the same $1/x$ singular property at the origin. In order for the wave functions in both $x > 0$ and $x < 0$ halves can have a chance to communicate to each other, we have to restrict the strength of the associated singularity to be in the domain $-1 < (\frac{2\tilde{m}}{\tilde{n}} - 1) < 1$ [50,56]. Interesting, we have the same constraint that was imposed before to avoid energy levels from crossing. Besides the singularity, all three superpotentials have an infinite discontinuity at $x = 0$. A regularization that preserves SUSY and shape invariance needs to be introduced. In effect, the corresponding regularized potentials will exhibit an extra Dirac delta-function like singularity at the origin over the unregularized ones [34,46].

Class 2. In this class, we choose another set of relations: $(i = 0, 1, 2, \dots)$

$$a_1 = a_0 + \alpha, \quad c\big(a_i\big) = \big(\tilde{m}a_i^2\big)^2, \quad d\big(a_i\big) = \beta + \tilde{n}a_i, \quad e\big(a_i\big) = 2\tilde{n}a_i, \tag{3.15}$$

where α, β and $\tilde{m} > 0$ are arbitrary constants. In addition, we set $\tilde{n} > 0$, since the situation of $\tilde{n} < 0$ can be easily obtained by the defining transformations $\bar{\eta} \to -\bar{\eta}$ and $\alpha \to -\alpha$. At this time, (2.21) becomes

$$\bar{\eta}' = \alpha\left(\frac{\tilde{m}^2}{\bar{\eta}^2} + \frac{\bar{\eta}^2}{4}\right) + \tilde{n}. \tag{3.16}$$

Before presenting the detailed results for this equation, let us investigate some general properties of the shape invariant potentials in the second class. Equation (3.16) enables us to construct the superpotentials $W_\pm(x, a_0)$ (2.19) and the initial potential $V(x, a_0)$ (2.20) as follows:

$$W_\pm\big(x, a_0\big) = \frac{1}{2}\left[a_{\mp\frac{1}{4}}\bar{\eta}(x) \pm \frac{2\tilde{m}a_0 - \tilde{n}}{\bar{\eta}(x)} \mp \frac{\alpha\tilde{m}^2}{\bar{\eta}(x)^3}\right], \tag{3.17}$$

$$V\big(x, a_0\big) = \frac{1}{4}\left[a_{-\frac{3}{4}}a_{-\frac{1}{4}}\bar{\eta}(x)^2 - \frac{5\alpha^2\tilde{m}^4}{\bar{\eta}(x)^6} - \frac{6\alpha\tilde{n}\tilde{m}^2}{\bar{\eta}(x)^4} + \frac{4\tilde{m}^2a_{-1}a_0 - \tilde{n}^2 - \frac{1}{2}\alpha^2\tilde{m}^2}{\bar{\eta}(x)^2} + 4\beta + \frac{\alpha\tilde{n}}{2}\right]. \tag{3.18}$$

We will show later that both superpotentials given in (3.17) exhibit a $1/x$ singularity at the origin. It implies that the series expansion of the $\bar{\eta}(x)$ function near the origin will be of the form $\bar{\eta}(x) \sim x^{1/3} + \cdots$.

Eigenvalues for the initial potential $V(x, a_0)$ are constructed algebraically as

$$E_{2n} = \beta + \big(\tilde{n} - \tilde{m}a_n\big)a_n + 2\tilde{n}\sum_{i=0}^{n-1} a_i, \quad E_{2n+1} = \beta + \big(\tilde{n} + \tilde{m}a_n\big)a_n + 2\tilde{n}\sum_{i=0}^{n-1} a_i. \tag{3.19}$$

The energy gap between two adjacent eigenstates in (2.14) and (2.15) becomes

$$R_n = 2\tilde{m}a_n^2, \quad \tilde{R}_n = 2\Big(\tilde{n} - \tilde{m}a_{n+\frac{1}{2}}\Big)a_{n+\frac{1}{2}} - \frac{\alpha^2\tilde{m}}{2}. \tag{3.20}$$

From the expression of $\tilde{R}_n$, we notice that no matter how large the constant $\tilde{n}$ is, the energy gap $\tilde{R}_n$ (for $\alpha \neq 0$) will eventually become negative for a large enough quantum number $n = n_0$. That is, the 2-SUSY shape invariant potentials in this second class allow only a finite number of bound states, and thus they must be of finite depth.

The associated eigenfunctions can be explicitly constructed using (2.16) and (2.17). The first few even-number eigenstates are obtained below, expressed in linear combinations of the generalized associated Laguerre polynomials (3.12), as

$$\psi_2\big(x, a_0\big) \propto \mathcal{L}_1^{-\frac{\tilde{m}}{\tilde{n}}a_1}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_0\right)\psi_+\big(x, a_1\big), \tag{3.21}$$

$$\psi_4\big(x, a_0\big) \propto \left[\mathcal{L}_2^{-\frac{\tilde{m}}{\tilde{n}}a_2}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_{\frac{1}{2}}\right) - 2\alpha\tilde{m}^2a_0\right]\psi_+\big(x, a_2\big), \tag{3.22}$$

$$\psi_6\big(x, a_0\big) \propto \left[\mathcal{L}_3^{-\frac{\tilde{m}}{\tilde{n}}a_3}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_1\right) - 2\alpha\tilde{m}^2\left[3a_0\mathcal{L}_1^{-\frac{\tilde{m}}{\tilde{n}}a_2}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_1\right) - 8(\tilde{n} - \alpha\tilde{m})a_{\frac{3}{4}}\right]\right]\psi_+\big(x, a_3\big), \tag{3.23}$$

where $\psi_+(x, a_n)$ is the ground-state eigenfunction of $V(x, a_n)$ given in (2.18). As before, the odd-number eigenstates $\psi_{2n+1}(x, a_0)$ can be deduced from the corresponding even-number states $\psi_{2n}(x, a_0)$ by the simple replacement: $-\frac{\tilde{m}}{\tilde{n}} \to +\frac{\tilde{m}}{\tilde{n}}$ and $\psi_+(x, a_n) \to \bar{\eta}(x)\psi_-(x, a_n)$.

In a similar fashion, the general expression for these eigenfunctions can be written in the series expansion of the generalized associated Laguerre polynomials as

$$\psi_{2n}\big(x, a_0\big) \propto \left[\sum_{s=0}^{n} C(n,s)\mathcal{L}_s^{-\frac{\tilde{m}}{\tilde{n}}a_{(n+s)/2}}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_{\frac{1}{2}(n-1)}\right)\right]\psi_+\big(x, a_n\big), \tag{3.24}$$

$$\psi_{2n+1}\big(x, a_0\big) \propto \left[\sum_{s=0}^{n} C(n,s)\mathcal{L}_s^{\frac{\tilde{m}}{\tilde{n}}a_{(n+s)/2}}\left(\frac{\bar{\eta}^2}{2\tilde{n}}, a_{\frac{1}{2}(n-1)}\right)\right]\bar{\eta}(x)\psi_-\big(x, a_n\big), \tag{3.25}$$

where the convention $\mathcal{L}_0^k(x^2, a_{\frac{1}{2}(n-1)}) = 1$ is used. Some of the calculated expansion coefficients $C(n, s)$ defined in (3.24) and (3.25) are listed below:

$$C(n,n) = 1, \quad C(n,n-1) = 0, \quad C(n,n-2) = -2\binom{n}{2}\alpha\tilde{m}^2,$$

$$C(n,n-3) = 16\binom{n}{3}a_{\frac{1}{4}n}(\tilde{n}-\alpha\tilde{m})\alpha\tilde{m}^2, \quad C(n,n-4) = 12\binom{n}{4}\Big(\alpha\tilde{m}^2a_0a_1 - 12(\tilde{n}-\alpha\tilde{m})^2a_{\frac{1}{3}n}\Big)\alpha\tilde{m}^2,$$

$$C(n,n-5) = 64\binom{n}{5}\Big(\big(24\tilde{n}^2 - 50\alpha\tilde{m}\tilde{n} + 28\alpha^2\tilde{m}^2\big)a_{\frac{3}{8}n} - 5\alpha\tilde{m}^2a_{\frac{3}{2}}a_{\frac{1}{4}n}\Big)(\tilde{n}-\alpha\tilde{m})\alpha\tilde{m}^2,$$

$$\begin{aligned} C(n,n-6) = 40\binom{n}{6}\alpha\tilde{m}^2\Big[&-3\alpha^2\tilde{m}^4a_0a_1a_2 + (\tilde{n}-\alpha\tilde{m})^2\Big(2\alpha\tilde{m}^2\big(86a_1 + (n+2)(2n-1)\alpha\big)a_2 \\ &- 10\Big(48(\tilde{n}-\alpha\tilde{m})^2 - 12\alpha\tilde{m}(\tilde{n}-\alpha\tilde{m}) + (14-n)\alpha^2\tilde{m}^2\Big)a_{\frac{2}{5}n}\Big)\Big], \end{aligned}$$

where $C(n, s) = 0$, for $n < s$, and the symbol $\binom{n}{m}$ denotes the binomial coefficient. In principle, the expansion coefficients $C(n, s)$ can be calculated term by term, for all s. However, a general expression for all the expansion coefficients $C(n, s)$ is not obtainable.

Now, let us turn back to solve (3.16). To obtain the solvable shape invariant potentials in this class, it is convenient to introduce the function

$$g(x) \equiv \frac{\tilde{m}}{\bar{\eta}(x)} + \frac{\bar{\eta}(x)}{2}. \tag{3.26}$$

Therefore, $\bar{\eta}(x)$ has two possibilities, depending on the asymptotic property of the corresponding superpotential near $x = 0$ and at large x,

$$\bar{\eta}(x) = g(x) \pm \sqrt{g(x)^2 - 2\tilde{m}}. \tag{3.27}$$

It is then not difficult to show that, when expressed in terms of $g(x)$, the first-order differential equation (3.16) becomes

$$\left[1 \pm \frac{g}{\sqrt{g^2 - 2\tilde{m}}}\right]g' = (\tilde{n} - \alpha\tilde{m}) + \alpha g^2, \tag{3.28}$$

and the superpotential $W_+(x, a_0)$ in (2.19) is reformulated into

$$W_+\big(x, a_0\big) = a_0g(x) - \frac{\tilde{n} - \alpha\tilde{m} + \alpha g^2(x)}{2\Big(g(x) \pm \sqrt{g(x)^2 - 2\tilde{m}}\Big)}. \tag{3.29}$$

In fact, the particular form of superpotential shown in (3.29), expressed in terms of the function $g(x)$ that satisfies the first-order differential equation (3.28), is not strange to us. Recently, both equations (3.28) and (3.29) have been developed in the study of shape invariance condition in two steps in the context of standard SUSY QM [70]. There, based on these two equations, new solvable shape invariant potentials in two steps are able to construct. Having identified the origin of both equations, we will briefly report the main results of 2-SUSY shape invariant potentials in this class, and leave the detailed discussion to the cited article.

Without loss of generality, we further choose $\tilde{n} > \alpha\tilde{m}$. The possible 2-SUSY solvable shape invariant potentials are as follows.

(1) If we take $\alpha = 0$, then we can directly solve (3.16), which yields $\bar{\eta}(x) = \tilde{n}x$. When substituting this result into the initial potential $V(x, a_0)$ (2.20), we find that it is the singular harmonic potential, presenting $\frac{1}{x^2}(\frac{\tilde{m}^2 a_0^2}{\tilde{n}^2} - \frac{1}{4})$ singularity at the origin.

(2) If $\alpha > 0$ is taken, on solving equation (3.28), we find that the $g(x)$ function can be given by patching two portions of solution. The solution in the first portion, defined in the region $0 < x < x_1 = \frac{k}{\alpha}\tan^{-1}[k\sqrt{2\tilde{m}}]$, is given by

$$\frac{\alpha}{k}x = \frac{1}{\sqrt{1+2\tilde{m}k^2}}\cot^{-1}\left[k\sqrt{\frac{g(x)^2 - 2\tilde{m}}{1+2\tilde{m}k^2}}\right] - \cot^{-1}\big[kg(x)\big], \tag{3.30}$$

where the constant $k = \sqrt{\frac{|\alpha|}{\tilde{n}-\tilde{m}\alpha}}$. The solution in the second portion, defined in the region $x_1 < x < x_2 = \frac{\pi}{2}\frac{k}{\alpha}[1 + (1+2\tilde{m}k^2)^{-1/2}]$, is

$$\frac{\alpha}{k}x = \frac{1}{\sqrt{1+2\tilde{m}k^2}}\tan^{-1}\left[k\sqrt{\frac{g(x)^2 - 2\tilde{m}}{1+2\tilde{m}k^2}}\right] + \tan^{-1}\big[kg(x)\big]. \tag{3.31}$$

The $g(x)$ function constructed above only covers the $x > 0$ region. To further extend the function $g(x)$ to cover the $x < 0$ region, we use antisymmetric property of the superpotentials.

From (3.30), we can invert this equation to obtain a series expansion in x for the function $g(x)$ as

$$g(x) = +\left(\frac{\tilde{m}}{3\alpha x}\right)^{1/3}\left[1 - \frac{2-3\tilde{m}k^2}{10k^2}\left(\frac{3\alpha x}{\tilde{m}}\right)^{2/3} + \mathcal{O}\big(x^{4/3}\big)\right]. \tag{3.32}$$

Hence, the $g(x)$ function exhibits the $x^{-1/3}$ singularity. The singular behavior near $x = 0$ for other relevant quantities can be similarly derived. For instance, the function $\bar{\eta}(x) \equiv g(x) - \sqrt{g(x)^2 - 2\tilde{m}}$ behaves like $(3\alpha\tilde{m}^2 x)^{1/3}$ at the origin, as expected. Additionally, the superpotentials $W_\pm(x, a_0)$ have a $\mp\frac{1}{6x}$ singularity and the initial potential $V(x, a_0)$ has a $-\frac{5}{36x^2}$ singularity. Unfortunately, the potential $V(x, a_0)$ constructed from the large x region, that is, from (3.31), can be shown to be of infinite depth. According to the general structure of energy spectrum (3.20), the shape invariant potentials must be of finite depth, allowing only a finite number of bound states. In short, the 2-SUSY shape invariant potentials in the second class cannot exist, for $\alpha > 0$.

(3) The case of $\alpha < 0$ results in a new solvable potential of 2-SUSY shape invariance. To see this, let us solve (3.28) for the $g(x)$ function, which can again be given by two portions of solution. The solution in the first portion, defined in the region $0 < x < x_c = \frac{k}{|\alpha|}\tanh^{-1}[k\sqrt{2\tilde{m}}]$, is of the form

$$\frac{|\alpha|}{k}x = \frac{1}{\sqrt{1-2\tilde{m}k^2}}\coth^{-1}\left[k\sqrt{\frac{g(x)^2 - 2\tilde{m}}{1-2\tilde{m}k^2}}\right] + \coth^{-1}\big[kg(x)\big], \tag{3.33}$$

where k is defined after (3.31). The solution in second portion, defined in the region $x_c < x < \infty$, is

$$\frac{|\alpha|}{k}x = \frac{1}{\sqrt{1-2\tilde{m}k^2}}\tanh^{-1}\left[k\sqrt{\frac{g(x)^2 - 2\tilde{m}}{1-2\tilde{m}k^2}}\right] + \tanh^{-1}\big[kg(x)\big]. \tag{3.34}$$

The $g(x)$ function above is defined for the $x > 0$ region, and can be extended to the $x < 0$ region by antisymmetrization.

Using (3.33), we obtain a series expansion in x for the function $g(x)$. The first few terms in the expansion looks like

$$g(x) = -\left(\frac{\tilde{m}}{3|\alpha|x}\right)^{1/3}\left[1 + \frac{2+3\tilde{m}k^2}{10k^2}\left(\frac{6|\alpha|x}{2\tilde{m}}\right)^{2/3} + \mathcal{O}\big(x^{4/3}\big)\right]. \tag{3.35}$$

In the same vein, the singularity for the function $\bar{\eta}(x) \equiv g(x) + \sqrt{g(x)^2 - 2\tilde{m}}$ behaves like $-(3|\alpha|\tilde{m}^2 x)^{1/3}$ at the origin. In addition, the superpotentials $W_\pm(x, a_0)$ have a $\mp\frac{1}{6x}$ singularity and the corresponding initial potential $V(x, a_0)$ has a $-\frac{5}{36x^2}$ singularity. Having established both portions of $g(x)$ in (3.33) and (3.34), we can readily build the complete initial potential $V(x, a_0)$. At the end, we arrive at the new solvable 2-SUSY shape invariant potential in the second class, which is not expressible in terms of elementary functions, but only in implicit forms via the function $g(x)$. As can be seen, the construction of such shape invariant potential is nontrivial, since it involves patching and adjusting different portions of the $g(x)$ function.

We mention here that the 2-SUSY solvable potentials that we have constructed in both classes possess explicit shape invariance symmetry, in which the parameters a_0 and a_1 are related to each other by $a_1 = a_0 + \alpha$. In the modern SUSY terminology [2], the corresponding 2-SUSY algebra for these shape invariant potentials is called reducible, because the corresponding algebra admits a chain of the standard SUSY realization. It means that we can gradually add or remove the two lowest energy levels of the initial Hamiltonian without breaking the positivity of the intermediate SUSY algebra. However, there is the so-called irreducible 2-SUSY algebra, in that case, the standard SUSY chain decomposition may be impossible.

4 Discussion and conclusion

In this article, the concept of shape invariance is incorporated with the formulation of 2-SUSY QM. Using a trial solution for $\eta(x, a_0)$ function, we derive the first-order differential equation (2.21) that is essential for the construction of 2-SUSY shape invariant potentials. Then, based on this equation, we establish two classes of solvable shape invariant potentials. In each class, the general properties of the obtained potentials are analyzed in detail. For instance, the eigenfunctions of these 2-SUSY shape invariant potentials are found expressible in terms of the generalized associated Laguerre polynomials and the corresponding eigenvalues are determined algebraically. Additionally, the shape invariant potentials constructed in the framework of 2-SUSY QM are similar to those shape invariant potentials in two steps discussed in the context of standard SUSY QM. That is, there must be an analogous potential algebra underlying the 2-SUSY shape invariant potentials, thus providing an alternative way of getting the energy eigenvalues by algebraic methods [34, 46, 69, 71].

As in the case of the standard SUSY QM, some solvable potentials in 2-SUSY QM may as well possess the property of hidden shape invariance, which is seen after a special choice of parameter that produces the required transformation is introduced [44]. To this purpose, let us consider the example of a free particle confined in a box ($0 < x < \pi$) of infinite walls. The initial Hamiltonian is given by $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 1$. Clearly, this Hamiltonian does not present the property of shape invariance, since we have no adjustable parameter. Therefore, we consider the related potential instead: $V_{\mathrm{IH}}(x, a_0) = a_0(a_0 - 1)\csc^2 x - a_0^2$, where the energy eigenvalues are given by $E_n = (a_0 + n)^2 - a_0^2$. When setting $a_0 = 1$, we as required recover the original potential of infinite walls. Indeed, the related potential $V_{\mathrm{IH}}(x, a_0)$ is one of the Infeld-Hull type E potentials [49], that are known to be shape invariant in the standard SUSY QM [45].

Now, we examine the property of 2-SUSY shape invariance for the initial potential $V_{\mathrm{IH}}(x, a_0)$. We first identify the factorization energies of the quasi-Hamiltonian (2.3) to be $\epsilon_1 = E_0$ and $\epsilon_2 = E_1$. This is equivalent to set the coefficients $d = \sqrt{c} = a_0 + \frac{1}{2}$. Given the expressions for c and d, we readily solve the nonlinear second-order differential equation (2.9) for the η-function. It is not hard to show that the solution is $\eta(x, a_0) = -(1 + 2a_0)\cot x$ [61, 62]. Then, via equation (2.8), the 2-SUSY partner potential $\tilde{V}_{\mathrm{IH}}(x, a_0)$ is found to be $\tilde{V}_{\mathrm{IH}}(x, a_0) = a_1(a_1 - 1)\csc^2 x - a_0^2$, where $a_1 = a_0 + \alpha$ and $\alpha = 2$ in this particular example. The eigenenergies of the partner potential $\tilde{V}_{\mathrm{IH}}(x, a_0)$ is $\tilde{E}_n = (a_1 + n)^2 - a_0^2$, which is isospectral to $V_{\mathrm{IH}}(x, a_0)$, except for the two lowest energy levels. Furthermore, when substituting both potentials $V_{\mathrm{IH}}(x, a_0)$ and $\tilde{V}_{\mathrm{IH}}(x, a_0)$ into 2-SUSY shape invariance condition (2.11), we obtain the remainder as $e(a_0) = 4(a_0 + 1)$. In sum, the partner potentials $\tilde{V}_{\mathrm{IH}}(x, a_0)$ and $V_{\mathrm{IH}}(x, a_0)$ are shown to relate to each other by 2-SUSY shape invariance symmetry.

We have discussed in the present article the solvable potentials in 2-SUSY QM that possess either explicit or hidden shape invariance symmetry. Actually, these solvable potentials belong to the translation class of shape invariance, in which the parameter of partner potentials is related to each other by translation $a_1 = a_0 + \alpha$. However, it is known that there are other classes of shape invariance. For instance, there is the so-called scaling class in the standard SUSY QM, where the parameters a_1 and a_0 are related to each other by scaling $a_1 = qa_0$, for $0 < q < 1$. The problem of scaling shape invariant potentials in 2-SUSY QM is currently under investigation.

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References

[1] A. Andrianov, F. Cannata, M. Ioffe, and D. Nishnianidze, *Systems with higher-order shape invariance: spectral and algebraic properties*, Phys. Lett. A, 266 (2000), 341–349.

[2] A. A. Andrianov and F. Cannata, *Nonlinear supersymmetry for spectral design in quantum mechanics*, J. Phys. A, 37 (2004), 10297–10321.

[3] A. A. Andrianov, M. V. Ioffe, F. Cannata, and J.-P. Dedonder, *Second order derivative supersymmetry, q deformations and the scattering problem*, Internat. J. Modern Phys. A, 10 (1995), 2683–2702.

[4] A. A. Andrianov, M. V. Ioffe, F. Cannata, and J.-P. Dedonder, *SUSY quantum mechanics with complex superpotentials and real energy spectra*, Internat. J. Modern Phys. A, 14 (1999), 2675–2688.

[5] A. A. Andrianov, M. V. Ioffe, and D. N. Nishnianidze, *Polynomial SUSY in quantum mechanics and second derivative Darboux transformations*, Phys. Lett. A, 201 (1995), 103–110.

[6] A. A. Andrianov, M. V. Ioffe, and V. P. Spiridonov, *Higher-derivative supersymmetry and the Witten index*, Phys. Lett. A, 174 (1993), 273–279.
[7] A. A. Andrianov and A. V. Sokolov, *Nonlinear supersymmetry in quantum mechanics: algebraic properties and differential representation*, Nuclear Phys. B, 660 (2003), 25–50.
[8] H. Aoyama, N. Nakayama, M. Sato, and T. Tanaka, *Classification of type A $\mathcal{N}$-fold supersymmetry*, Phys. Lett. B, 521 (2001), 400–408.
[9] H. Aoyama, M. Sato, and T. Tanaka, *General forms of a $\mathcal{N}$-fold supersymmetric family*, Phys. Lett. B, 503 (2001), 423–429.
[10] H. Aoyama, M. Sato, and T. Tanaka, *$\mathcal{N}$-fold supersymmetry in quantum mechanics: general formalism*, Nuclear Phys. B, 619 (2001), 105–127.
[11] B. Bagchi, S. Mallik, and C. Quesne, *Generating complex potentials with real eigenvalues in supersymmetric quantum mechanics*, Internat. J. Modern Phys. A, 16 (2001), 2859–2872.
[12] B. K. Bagchi, *Supersymmetry in Quantum and Classical Mechanics*, vol. 116 of Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2001.
[13] D. T. Barclay, R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta, and U. Sukhatme, *New exactly solvable Hamiltonians: shape invariance and self-similarity*, Phys. Rev. A (3), 48 (1993), 2786–2797.
[14] D. Baye, *Supersymmetry between deep and shallow nucleus-nucleus potentials*, Phys. Rev. Lett., 58 (1987), 2738–2741.
[15] C. M. Bender and S. Boettcher, *Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry*, Phys. Rev. Lett., 80 (1998), 5243–5246.
[16] H. W. Braden and A. J. Macfarlane, *Supersymmetric quantum mechanical models with continuous spectrum and the Witten index*, J. Phys. A, 18 (1985), 3151–3156.
[17] F. Cannata, G. Junker, and J. Trost, *Schrödinger operators with complex potential but real spectrum*, Phys. Lett. A, 246 (1998), 219–226.
[18] C. Chuan, *Exactly solvable potentials and the concept of shape invariance*, J. Phys. A, 24 (1991), L1165–L1174.
[19] F. Cooper and B. Freedman, *Aspects of supersymmetric quantum mechanics*, Ann. Physics, 146 (1983), 262–288.
[20] F. Cooper, J. N. Ginocchio, and A. Khare, *Relationship between supersymmetry and solvable potentials*, Phys. Rev. D (3), 36 (1987), 2458–2473.
[21] F. Cooper, J. N. Ginocchio, and A. Wipf, *Derivation of the S-matrix using supersymmetry*, Phys. Lett. A, 129 (1988), 145–147.
[22] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry and quantum mechanics*, Phys. Rep., 251 (1995), 267–385.
[23] F. Correa, V. Jakubský, L.-M. Nieto, and M. S. Plyushchay, *Self-isospectrality, special supersymmetry, and their effect on the band structure*, Phys. Rev. Lett., 101 (2008), 030403, 4.
[24] F. Correa, V. Jakubský, and M. S. Plyushchay, *Finite-gap systems, tri-supersymmetry and self-isospectrality*, J. Phys. A, 41 (2008), 485303, 35.
[25] F. Correa, V. Jakubský, and M. S. Plyushchay, *Aharonov-Bohm effect on AdS_2 and nonlinear supersymmetry of reflectionless Pöschl-Teller system*, Ann. Physics, 324 (2009), 1078–1094.
[26] F. Correa, L.-M. Nieto, and M. S. Plyushchay, *Hidden nonlinear supersymmetry of finite-gap Lamé equation*, Phys. Lett. B, 644 (2007), 94–98.
[27] F. Correa and M. S. Plyushchay, *Hidden supersymmetry in quantum bosonic systems*, Ann. Physics, 322 (2007), 2493–2500.
[28] F. Correa and M. S. Plyushchay, *Peculiarities of the hidden nonlinear supersymmetry of the Pöschl-Teller system in the light of the Lamé equation*, J. Phys. A, 40 (2007), 14403–14412.
[29] M. M. Crum, *Associated Sturm-Liouville systems*, Quart. J. Math. Oxford Ser. (2), 6 (1955), 121–127.
[30] J. W. Dabrowska, A. Khare, and U. P. Sukhatme, *Explicit wavefunctions for shape-invariant potentials by operator techniques*, J. Phys. A, 21 (1988), L195–L200.
[31] G. Darboux, *Sur une proposition relative aux équations linéaires*, C. R. Acad. Sci. Pairs, 94 (1882), 1456–1459.
[32] J. I. Díaz, J. Negro, L. M. Nieto, and O. Rosas-Ortiz, *The supersymmetric modified Pöschl-Teller and delta well potentials*, J. Phys. A, 32 (1999), 8447–8460.
[33] G. Dunne and J. Feinberg, *Self-isospectral periodic potentials and supersymmetric quantum mechanics*, Phys. Rev. D (3), 57 (1998), 1271–1276.
[34] R. Dutt, A. Gangopadhyaya, C. Rasinariu, and U. Sukhatme, *New solvable singular potentials*, J. Phys. A, 34 (2001), 4129–4142.
[35] R. Dutt, A. Khare, and U. Sukhatme, *Exactness of supersymmetric WKB spectra for shape-invariant potentials*, Phys. Lett. B, 181 (1986), 295–298.
[36] R. Dutt, A. Khare, and U. Sukhatme, *Supersymmetry, shape invariance, and exactly solvable potentials*, Am. J. Phys., 56 (1988), 163–168.
[37] D. J. Fernández, *SUSUSY quantum mechanics*, Int. J. Mod. Phys. A, 12 (1997), 171–176.
[38] D. J. Fernández, M. L. Glasser, and L. M. Nieto, *New isospectral oscillator potentials*, Phys. Lett. A, 240 (1998), 15–20.
[39] D. J. Fernández, B. Mielnik, O. Rosas-Ortiz, and B. F. Samsonov, *Nonlocal supersymmetric deformations of periodic potentials*, J. Phys. A, 35 (2002), 4279–4291.
[40] D. J. Fernández, B. Mielnik, O. Rosas-Ortiz, and B. F. Samsonov, *The phenomenon of Darboux displacements*, Phys. Lett. A, 294 (2002), 168–174.
[41] D. J. Fernández, R. Muñoz, and A. Ramos, *Second order SUSY transformations with "complex energies"*, Phys. Lett. A, 308 (2003), 11–16.
[42] D. J. Fernández and E. Salinas-Hernández, *The confluent algorithm in second order supersymmetric quantum mechanics*, J. Phys. A, 36 (2003), 2537–2543.
[43] D. J. Fernández and E. Salinas-Hernández, *Wronskian formula for confluent second-order supersymmetric quantum mechanics*, Phys. Lett. A, 338 (2005), 13–18.
[44] E. D. Filho and R. M. Ricotta, *Ladder operators for subtle hidden shape-invariant potentials*, J. Phys. A, 37 (2004), 10057–10064.
[45] T. Fukui and N. Aizawa, *Shape-invariant potentials and an associated coherent state*, Phys. Lett. A, 180 (1993), 308–313.
[46] A. Gangopadhyaya and U. P. Sukhatme, *Potentials with two shifted sets of equally spaced eigenvalues and their Calogero spectrum*, Phys. Lett. A, 224 (1996), 5–14.
[47] L. Gendenshtein, *Derivation of exact spectra of the Schrödinger equation by means of supersymmetry*, JETP Lett., 38 (1983), 356–359.

[48] Y. R. Huang and W.-C. Su, *A note on parasupersymmetric quantum mechanics of arbitrary order*, J. Phys. A, 43 (2010), 115302, 15.
[49] L. Infeld and T. E. Hull, *The factorization method*, Rev. Modern Physics, 23 (1951), 21–68.
[50] A. Jevicki and J. P. Rodrigues, *Singular potentials and supersymmetry breaking*, Phys. Lett. B, 146 (1984), 55–58.
[51] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1996.
[52] A. Khare and U. P. Sukhatme, *Scattering amplitudes for supersymmetric shape-invariant potentials by operator methods*, J. Phys. A, 21 (1988), L501–L508.
[53] A. Khare and U. P. Sukhatme, *New shape-invariant potentials in supersymmetric quantum mechanics*, J. Phys. A, 26 (1993), L901–L904.
[54] S. Klishevich and M. S. Plyushchay, *Supersymmetry of parafermions*, Modern Phys. Lett. A, 14 (1999), 2739–2752.
[55] S. M. Klishevich and M. S. Plyushchay, *Nonlinear supersymmetry, quantum anomaly and quasi-exactly solvable systems*, Nuclear Phys. B, 606 (2001), 583–612.
[56] L. Lathouwers, *The Hamiltonian* $H = (-1/2)d^2/dx^2 + x^2/2 + \lambda/x^2$ *reobserved*, J. Mathematical Phys., 16 (1975), 1393–1395.
[57] G. Lévai, *A search for shape-invariant solvable potentials*, J. Phys. A, 22 (1989), 689–702.
[58] B. Mielnik, L. M. Nieto, and O. Rosas-Ortiz, *The finite difference algorithm for higher order supersymmetry*, Phys. Lett. A, 269 (2000), 70–78.
[59] M. S. Plyushchay, *Deformed Heisenberg algebra, fractional spin fields, and supersymmetry without fermions*, Ann. Physics, 245 (1996), 339–360.
[60] M. S. Plyushchay, *Hidden nonlinear supersymmetries in pure parabosonic systems*, Internat. J. Modern Phys. A, 15 (2000), 3679–3698.
[61] J. O. Rosas-Ortiz, *Exactly solvable hydrogen-like potentials and the factorization method*, J. Phys. A, 31 (1998), 10163–10179.
[62] J. O. Rosas-Ortiz, *New families of isospectral hydrogen-like potentials*, J. Phys. A, 31 (1998), L507–L513.
[63] O. Rosas-Ortiz and R. Muñoz, *Non-Hermitian SUSY hydrogen-like Hamiltonians with real spectra*, J. Phys. A, 36 (2003), 8497–8506.
[64] B. F. Samsonov, *New features in supersymmetry breakdown in quantum mechanics*, Modern Phys. Lett. A, 11 (1996), 1563–1567.
[65] B. F. Samsonov, *New possibilities for supersymmetry breakdown in quantum mechanics and second-order irreducible Darboux transformations*, Phys. Lett. A, 263 (1999), 274–280.
[66] E. Schrödinger, *The factorization of the hypergeometric equation*, Proc. Roy. Irish Acad. Sect. A., 47 (1941), 53–54.
[67] E. Schrödinger, *Further studies on solving eigenvalue problems by factorization*, Proc. R. Irish Acad. A, 46 (1941), 183–206.
[68] J.-M. Sparenberg and D. Baye, *Supersymmetric transformations of real potentials on the line*, J. Phys. A, 28 (1995), 5079–5095.
[69] W.-C. Su, *Algebraic shape invariant potentials in two steps*, J. Phys. A, 41 (2008), 435301, 10.
[70] W.-C. Su, *Solvable potentials of shape invariance in two steps*, J. Phys. A, 41 (2008), 255307, 17.
[71] W.-C. Su, *Algebraic shape invariant potentials as the generalized deformed oscillator*, J. Phys. A, 42 (2009), 385202, 17.
[72] U. P. Sukhatme, C. Rasinariu, and A. Khare, *Cyclic shape invariant potentials*, Phys. Lett. A, 234 (1997), 401–409.
[73] E. Witten, *Dynamical breaking of supersymmetry*, Nucl. Phys. B, 188 (1981), 513–554.
[74] E. Witten, *Constraints on supersymmetry breaking*, Nuclear Phys. B, 202 (1982), 253–316.