Unitary braid matrices: bridge between topological and quantum entanglements

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Abstract

Braiding operators corresponding to the third Reidemeister move in the theory of knots and links are realized in terms of parametrized unitary matrices for all dimensions. Two distinct classes are considered. Their (nonlocal) unitary actions on separable pure product states of three identical subsystems (i.e., the spin projections of three particles) are explicitly evaluated for all dimensions. This, for our classes, is shown to generate entangled superposition of four terms in the base space. The 3-body and 2-body entanglements (in three 2-body subsystems), the 3 tangles, and 2 tangles are explicitly evaluated for each class. For our matrices, these are parametrized. Varying parameters they can be made to sweep over the domain $(0, 1)$. Thus, braiding operators corresponding to over- and undercrossings of three braids and, on closing ends, to topologically entangled Borromean rings are shown, in another context, to generate quantum entanglements. For higher dimensions, starting with different initial triplets one can entangle by turns, each state with all the rest. A specific coupling of three angular momenta is briefly discussed to throw more light on three body entanglements.

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1 Introduction (two faces of unitary braid matrices)

The third Reidemeister move in the theory of knots and links imposes equivalence between two specific sequences of over- and undercrossing of three braids. Such sequences can be repeated and closing the ends of the braids one obtains topologically entangled Borromean rings whose history reaches back far into the past (see [10, Figure 7]). Braid matrices, "Baxterized" to depend on spectral (rapidity) parameters, satisfy an equation—the braid equation—which corresponds precisely to the above-mentioned Reidemeister move. They provide matricial representations of a particular type of topological entanglements. When a braid matrix is also unitary, it can also be implemented to induce unitary transformations in base spaces of corresponding dimensions representing possible quantum states of an object (i.e., the spin projections of particles).

Let us make this more precise. Let $\widehat{R}(\theta)$ be a unitary $N^2 \times N^2$ matrix, I be the $N \times N$ unit matrix and

$$\widehat{R}_{12} = \widehat{R} \otimes I, \quad \widehat{R}_{23} = I \otimes \widehat{R},$$

where $\widehat{R}_{12}$, $\widehat{R}_{23}$ act on triple tensor products $V_N \otimes V_N \otimes V_N$ of N-dimensional vector spaces V_N. To be a braid matrix, $\widehat{R}(\theta)$ must satisfy

$$\widehat{R}_{12}(\theta)\widehat{R}_{23}(\theta + \theta')\widehat{R}_{12}(\theta') = \widehat{R}_{23}(\theta')\widehat{R}_{12}(\theta + \theta')\widehat{R}_{23}(\theta).$$

The indices (12), (23) correspond to successive crossings (braid 2 overcrossing braid 1 and undercrossing braid 3). The equality sign imposes the essential Reidemeister constraint (the 3$^{\mathrm{rd}}$ move). This will be the link of our matrices to topological entanglement. The role of braid matrices satisfying unitarity:

$$\left(\widehat{R}(\theta)\right)^{+} = \widehat{R}(\theta),$$

in quantum entanglements has been noted and discussed by Kauffman and Lomonaco [10] in their paper "Braiding operators are universal quantum gates". A large number of relevant sources are cited in [10]. We have presented before two quite distinct classes of unitary $N^2 \times N^2$ braid matrices [2, 3], one real and for even N and the other complex, for all N. There is no upper limit to N. [2, 3] cite other sources.

In the following sections, we will systematically, explicitly derive the measures of 2-body and 3-body entanglements generated by the action of the braiding operator:

$$\widehat{B} = \widehat{R}_{12}(\theta)\widehat{R}_{23}(\theta + \theta')\widehat{R}_{12}(\theta') \quad \left(= \widehat{R}_{23}(\theta')\widehat{R}_{12}(\theta + \theta')\widehat{R}_{23}(\theta)\right), \tag{1.1}$$

acting on the pure separable product states:

$$|a\rangle \otimes |b\rangle \otimes |c\rangle = |abc\rangle,$$

spanning the basis $V_N \otimes V_N \otimes V_N$. One can then, if necessary, evaluate the following:

$$\widehat{B}\left(\sum_{a,b,c} f_{abc}|abc\rangle\right).$$

The measures of 2 tangles and 3 tangles derived in [7] will be used throughout. Topological and quantum entanglements, two domains of $\widehat{B}$, will thus be brought together. One essential point must be noted: the unitary matrix $\widehat{R}$ is not locally unitary. $\widehat{R}$ cannot be expressed as $\widehat{R}_1 \otimes \widehat{R}_2$ acting on $V_N \otimes V_N$, where $\widehat{R}_1$, $\widehat{R}_2$ are each a unitary $N \times N$ matrix acting on V_N. Such an $\widehat{R}$ would have been trivial in the context of braiding. Nor would they have induced quantum entanglements acting on a product state $|a\rangle \otimes |b\rangle$ in $V \otimes V$. Non-local unitarity is crucial in the action of $\widehat{B}$. Local unitary transformations can however be used to classify already entangled states, as has been done systematically by Carteret and Sulbery [4]. We aim at generating quantum entanglements.

We close the introduction with some notations we will use throughout. For $N = 2$, V_2 is usually taken to be spanned by the spin projections of spin-$\frac{1}{2}$ particles:

$$(|+\rangle, |-\rangle) \equiv (|1\rangle, |0\rangle).$$

With passage to higher dimensions in mind, we will often use the state vectors:

$$|1\rangle = \left|{1 \atop 0}\right\rangle, \quad |\bar{1}\rangle = \left|{0 \atop 1}\right\rangle.$$

Generalization is direct. Thus, for $N = 4$,

$$|1\rangle = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad |2\rangle = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \quad |\bar{2}\rangle = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, \quad |\bar{1}\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}.$$

For all N,

$$\bar{i} = N + 1 - i, \quad \bar{\bar{i}} = N + 1 - \bar{i} = i.$$

For convenience, we will continue to use the terminology of spin projections. But the indices above can also correspond to other suitably enumerated quantum states of a system.

2   Unitary braid matrices and their actions

2.1   Real, unitary, and even-dimensional braid matrices

A class of real, unitary, $(2n)^2 \times (2n)^2$ dimensional braid matrices [3] is given by the following:

$$\left(\widehat{\mathrm{R}}(z)\right)^{\pm 1} = \frac{1}{\sqrt{1 + z^2}} (I \otimes I \pm z K \otimes J),$$

where $z = \tanh\theta$ and (K, J) are $(2n) \times (2n)$ matrices given by the following:

$$J = \sum_{i=1}^{n} \left((-1)^{\bar{i}}(i\bar{i}) + (-1)^{i}(\bar{i}i) \right), \quad K = \sum_{i=1}^{n} \left((i\bar{i}) + (\bar{i}i) \right) \tag{2.1}$$

with $\bar{i} = 2n + 1 - i$ and I is the $(2n) \times (2n)$ unit matrix. We always denote by (ij) the matrix with a single nonzero element, unity on row i and column j. A detailed study of this class can be found in [3]. An equivalent construction, without explicit introduction of the tensor product structure $(K \otimes J)$, can be found in [12]. From (2.1):

$$JK = -KJ = \sum_{i=1}^{n} \left((-1)^{\bar{i}}(ii) + (-1)^{i}(\bar{i}\bar{i}) \right), \quad K^2 = -J^2 = I. \tag{2.2}$$

Denoting $(\tanh\theta, \tanh\theta', \tanh(\theta + \theta')) = (z, z', z'')$ with $z'' = \frac{z+z'}{1+zz'}$ and using (2.2), one obtains unitarity:

$$\left(\widehat{\mathrm{R}}(z)\right)^{+} = \widehat{\mathrm{R}}(z)^{-1}$$

and the explicit evaluation:

$$\widehat{\mathrm{B}} = \widehat{\mathrm{R}}_{12}(z)\widehat{\mathrm{R}}_{23}(z'')\widehat{\mathrm{R}}_{12}(z') = \widehat{\mathrm{R}}_{23}(z')\widehat{\mathrm{R}}_{12}(z'')\widehat{\mathrm{R}}_{23}(z)$$

$$= \frac{1}{\sqrt{\left(1 + z^2\right)\left(1 + z'^2\right)\left(1 + z''^2\right)}}$$

$$\times \left((1 - zz')I \otimes I \otimes I + (z + z')(I \otimes K \otimes J + K \otimes J \otimes I) + z''(z' - z)(K \otimes KJ \otimes J) \right).$$

(A misprint in the overall factor in [3, (2.16)] is corrected above. We have also set $z''(1 + zz') = (z + z')$ in the second term there.) We now consider the action of $\widehat{\mathrm{B}}$ on basis states:

$$|abc\rangle = |a\rangle \otimes |b\rangle \otimes |c\rangle, \tag{2.3}$$

where (a, b, c) each span the $(2n)$ dimensional V_{2n} in $V_{2n} \otimes V_{2n} \otimes V_{2n}$ with for

$$a = i, \quad \bar{a} = 2n + 1 - a = \bar{i}, \quad a = \bar{i}, \quad \bar{a} = 2n + 1 - a = i, \quad (i = 1, \dots, n). \tag{2.4}$$

Similarly (b, c) can be $(j, \bar{j})$, $(k, \bar{k})$, respectively, over the same domain. The notations (2.3), (2.4) make the formalism much more compact.

Using (2.1)–(2.4), one obtains the following:

$$\widehat{B}|abc\rangle = f_0|abc\rangle + f_1|a\bar{b}\bar{c}\rangle + f_2|\bar{a}b\bar{c}\rangle + f_3|\bar{a}\bar{b}c\rangle \tag{2.5}$$

with

$$(f_0, f_1, f_2, f_3) = \frac{1}{\sqrt{\left(1 + z^2\right)\left(1 + z'^2\right)\left(1 + z''^2\right)}} \\ \times \left((1 - zz'), (-1)^c(z + z'), (-1)^b(z + z'), (-1)^{b+c}z''(z' - z)\right). \tag{2.6}$$

Noting that

$$(1 - zz')^2 + (z + z')^2 = (1 + z^2)(1 + z'^2), \\ (z + z')^2 + z''^2(z' - z)^2 = z''^2(1 + z^2)(1 + z'^2), \tag{2.7}$$

one immediately verifies that, consistently with unitarity of $\widehat{B}$,

$$f_0^2 + f_1^2 + f_2^2 + f_3^2 = 1.$$

On the left of (2.5), $|abc\rangle$ is by definition a separable product of pure states, hence unentangled. One the right, the superposition can be shown to imply entanglement. On assuming it can be expressed as a product $\left(\sum x_i|x_i\rangle\right) \otimes \left(\sum y_i|y_i\rangle\right) \otimes \left(\sum z_i|z_i\rangle\right)$, one runs into contradictions. In Section 3, we will go much further. We will obtain explicitly the intrinsic 3-body entanglement (3 tangles) and the 2-body entanglements (2 tangles) of the three subsystems. They will be expressed in terms of (f_0, f_1, f_2, f_3) of (2.6).

2.2 Complex, even-dimensional, multiparameter unitary braid matrices

In a series of papers (some of which are cited in [2]), we have constructed a class multiparameter braid matrices, the number of such parameters increasing as N^2 with the dimension $N^2 \times N^2$ of the matrix for both even and odd dimensions ($N = 2, 3, 4, 5, 6, \dots$). It was then noted [2, 3] that for all these parameters pure imaginary, this class corresponds to unitary braid matrices. (One may also consider real parameters with θ imaginary.) In this subsection, we restrict our considerations to even-dimensional matrices. For N odd special features arise which are best treated separately in Section 4. The even-dimensional, unitary $(2n)^2 \times (2n)^2$ matrix is given by the following:

$$\widehat{R}(\theta) = \sum_{\epsilon} \sum_{i,j} e^{m_{ij}^{(\epsilon)} \theta} \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right), \tag{2.8}$$

the definitions of the projectors being

$$P_{ab}^{(\epsilon)} = \frac{1}{2}\left\{(aa) \otimes (bb) + (\bar{a}\bar{a}) \otimes (\bar{b}\bar{b}) + \epsilon\left[(a\bar{a}) \otimes (b\bar{b}) + (\bar{a}a) \otimes (\bar{b}b)\right]\right\}, \tag{2.9}$$

where $a = (i, \bar{i})$, $b = (j, \bar{j})$ runs over $2n$ values and $\bar{a} = 2n + 1 - a$ and $\bar{b} = 2n + 1 - b$. The explicit forms of $\hat{R}(\theta)$ for $N = (2, 4)$ are given in [2] where the transition to unitarity is formulated in Section 3 ($m_{ij}^{(\epsilon)} \longrightarrow im_{ij}^{(\epsilon)}$, with the coefficient $i = \sqrt{-1}$).

Consider again the action of

$$\widehat{B} = \widehat{R}_{12}(\theta)\widehat{R}_{23}(\theta + \theta')\widehat{R}_{12}(\theta').$$

One obtains (compare (2.5), (2.6)):

$$\widehat{B}|abc\rangle = f_0|abc\rangle + f_1|a\bar{b}\bar{c}\rangle + f_2|\bar{a}b\bar{c}\rangle + f_3|\bar{a}\bar{b}c\rangle, \tag{2.10}$$

but now with coefficients defined below. Set

$$\lambda_\pm = m_{ab}^{(\pm)}(\theta + \theta'), \quad \mu_\pm = m_{bc}^{(\pm)}(\theta + \theta') \tag{2.11}$$

with only the sum $(\theta + \theta')$ as factor above (in contrast to $\tanh\theta$, $\tanh\theta'$, $\tanh(\theta + \theta')$ all playing roles in the previous case). In terms of (λ, μ), one has

$$f_0 = \frac{1}{4}\left(e^{i\lambda_+} + e^{i\lambda_-}\right)\left(e^{i\mu_+} + e^{i\mu_-}\right), \quad f_1 = \frac{1}{4}\left(e^{i\lambda_+} + e^{i\lambda_-}\right)\left(e^{i\mu_+} - e^{i\mu_-}\right),$$
$$f_2 = \frac{1}{4}\left(e^{i\lambda_+} - e^{i\lambda_-}\right)\left(e^{i\mu_+} - e^{i\mu_-}\right), \quad f_3 = \frac{1}{4}\left(e^{i\lambda_+} - e^{i\lambda_-}\right)\left(e^{i\mu_+} + e^{i\mu_-}\right), \tag{2.12}$$

satisfying the unitarity constraints:

$$f_0 f_0^+ + f_1 f_1^+ + f_2 f_2^+ + f_3 f_3^+ = 1.$$

Again, one can easily verify that the right side of (2.10) represents an entangled state (as for (2.5)). We will obtain explicitly the 3-tangle and the 2-tangle in Section 3.

Repeated actions of $\widehat{B}$ with different parameters will modify the coefficients as follows:

$$\widehat{B}'\widehat{B}|abc\rangle = \widehat{B}'\left(f_0|abc\rangle + f_1|a\bar{b}\bar{c}\rangle + f_2|\bar{a}b\bar{c}\rangle + f_3|\bar{a}\bar{b}c\rangle\right)$$
$$= g_0|abc\rangle + g_1|a\bar{b}\bar{c}\rangle + g_2|\bar{a}b\bar{c}\rangle + g_3|\bar{a}\bar{b}c\rangle$$

with

$$g_0 = \left(f_0 f_0' + f_1 f_1' + f_2 f_2' + f_3 f_3'\right), \quad g_1 = \left(f_0 f_1' + f_1 f_0' + f_2 f_3' + f_3 f_2'\right),$$
$$g_2 = \left(f_0 f_2' + f_1 f_3' + f_2 f_0' + f_3 f_1'\right), \quad g_3 = \left(f_0 f_3' + f_1 f_2' + f_2 f_1' + f_3 f_0'\right).$$

Here, (f_0', f_1', f_2', f_3') are coefficients due to the action of $\widehat{B}'$ alone. This set may belong to the same class as (f_0, f_1, f_2, f_3) or to the other one of our two classes (see (2.6) and (2.12)). The process can be repeated remaining always in the closed subspace ($|abc\rangle, |a\bar{b}\bar{c}\rangle, |\bar{a}b\bar{c}\rangle, |\bar{a}\bar{b}c\rangle$). (Starting with the same $|a\rangle$ but with different $|b'\rangle$, $|c'\rangle$, e.g., and continuing thus one can entangle each individual state with all the others successively in the total base space. See the relevant remarks in Section 6.) The essential features of the aspects that interest us principally (analyzed in Section 3) can be shown to be conserved under iterations indicated above. For that reason, and also for simplicity, we will restrict our study (Section 3) to the two sets of coefficients (2.6) and (2.12).

3   Computation of quantum entanglements

We now extract from (2.5), (2.6), (2.10), and (2.12), respectively, the quantum entanglements generated by $\widehat{\mathrm{B}}$ acting on the pure product state $|abc\rangle$, where $|abc\rangle$ can be any triplet selected from the N^3 dimensional base space. For spin $\frac{1}{2}$ particles,

$$|a\rangle \in \big(|+\rangle, |-\rangle\big) \equiv \big(|1\rangle, |0\rangle\big).$$

For higher spins (with $|\bar{a}\rangle$ now written as $|-a\rangle$),

$$|a\rangle \in \big(|N\rangle, |N-1\rangle, \ldots, |1-N\rangle, |-N\rangle\big),$$

and similarly for $(|b\rangle, |c\rangle)$. Under the action of the braiding operator $\widehat{\mathrm{B}}$ (defined in (1.1) with $|\bar{a}\rangle = |N - a + 1\rangle \equiv |-a\rangle$),

$$\widehat{\mathrm{B}}|abc\rangle = f_0|abc\rangle + f_1|a\bar{b}\bar{c}\rangle + f_2|\bar{a}b\bar{c}\rangle + f_3|\bar{a}\bar{b}c\rangle, \tag{3.1}$$

where (f_0, f_1, f_2, f_3) are given by (2.6) and (2.12) for our two classes, respectively. For spin $\frac{1}{2}$, one has two subsets:

$$\big(|111\rangle, |100\rangle, |010\rangle, |001\rangle\big), \quad \big(|000\rangle, |011\rangle, |101\rangle, |110\rangle\big).$$

For higher spin, as noted before, one can start with the same $|a\rangle$ but $(|b\rangle, |c\rangle)$ chosen from all the other possibilities. But for each initial choice, one remains, under the action $\widehat{\mathrm{B}}$, in the subspace given by the right hand of (3.1). This is the very special, fundamental, property of our unitary matrices. This allows us, even for higher spins, to implement systematically the formalism and concepts of Coffman, Kundu, and Wootters (CKW) concerning 3-particle entanglements (and corresponding 2-particle ones for the three subsystems) in [7]. This we now proceed to do.

The density matrix corresponding (3.1) is as follows:

$$\rho_{123} = \big(f_0|abc\rangle + f_1|a\bar{b}\bar{c}\rangle + f_2|\bar{a}b\bar{c}\rangle + f_3|\bar{a}\bar{b}c\rangle\big)\big(f_0^+\langle cba| + f_1^+\langle \bar{c}\bar{b}a| + f_2^+\langle \bar{c}b\bar{a}| + f_3^+\langle c\bar{b}\bar{a}|\big).$$

Tracing out c, one obtains

$$\begin{aligned}
\rho_{12} =\ & f_0 f_0^+ |ab\rangle\langle ba| + f_0 f_3^+ |ab\rangle\langle \bar{b}\bar{a}| + f_1 f_1^+ |a\bar{b}\rangle\langle \bar{b}a| + f_1 f_2^+ |a\bar{b}\rangle\langle b\bar{a}| \\
& + f_2 f_2^+ |\bar{a}b\rangle\langle b\bar{a}| + f_2 f_1^+ |\bar{a}b\rangle\langle \bar{b}a| + f_3 f_3^+ |\bar{a}\bar{b}\rangle\langle \bar{b}\bar{a}| + f_3 f_0^+ |\bar{a}\bar{b}\rangle\langle ba|.
\end{aligned} \tag{3.2}$$

Tracing out b in ρ_{12}, one obtains a diagonal:

$$\rho_1 = \big(f_0 f_0^+ + f_1 f_1^+\big)|a\rangle\langle a| + \big(f_2 f_2^+ + f_3 f_3^+\big)|\bar{a}\rangle\langle \bar{a}|. \tag{3.3}$$

One can write down (ρ_{13}, ρ_{13}), (ρ_2, ρ_3) from symmetry. Thus, for example,

$$\rho_2 = \big(f_0 f_0^+ + f_2 f_2^+\big)|b\rangle\langle b| + \big(f_3 f_3^+ + f_1 f_1^+\big)|\bar{b}\rangle\langle \bar{b}|,$$

and so on.

The spin-flipped matrix:

$$\widetilde{\rho}_{AB} = \begin{vmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{vmatrix} \rho_{AB}^+ \begin{vmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{vmatrix}$$

can now be obtained for $(\rho_{12}, \rho_{23}, \rho_{13})$ and then the products $(\rho_{AB}\widetilde{\rho}_{AB})$. Thus,

$$
\rho_{12}\widetilde{\rho}_{12} = 2
\begin{vmatrix}
f_0 f_0^+ f_3 f_3^+ & 0 & 0 & f_0^2 f_0^+ f_3^+ \\
0 & f_1 f_1^+ f_2 f_2^+ & f_1^2 f_1^+ f_2^+ & 0 \\
0 & f_2^2 f_1^+ f_2^+ & f_1 f_1^+ f_2 f_2^+ & 0 \\
f_3^2 f_0^+ f_3^+ & 0 & 0 & f_0 f_0^+ f_3 f_3^+
\end{vmatrix}.
$$

The products $(\rho_{13}\widetilde{\rho}_{13})$, $(\rho_{23}\widetilde{\rho}_{23})$ are related to the result above through evident permutations of the indices $(1,2,3)$. The eigenstates can be read off as $\begin{vmatrix} f_0/f_3 \\ 0 \\ 0 \\ \pm 1 \end{vmatrix} \Big\rangle$, $\begin{vmatrix} 0 \\ f_1/f_2 \\ \pm 1 \\ 0 \end{vmatrix} \Big\rangle$. The eigenvalues of $(\rho_{12}\widetilde{\rho}_{12})$ are as follows:

$$
\left(\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2\right) = 4\left(f_0 f_0^+ f_3 f_3^+, f_1 f_1^+ f_2 f_2^+, 0, 0\right).
$$

(The ordering of the first two roots depends on values of the parameters in $\widehat{\mathrm{B}}$.)

Taking square roots, the "concurrence" is as follows:

$$
C_{12} = 2\left|\left(f_0 f_0^+ f_3 f_3^+\right)^{1/2} - \left(f_1 f_1^+ f_2 f_2^+\right)^{1/2}\right|. \tag{3.4}
$$

Similarly,

$$
C_{23} = 2\left|\left(f_0 f_0^+ f_1 f_1^+\right)^{1/2} - \left(f_2 f_2^+ f_3 f_3^+\right)^{1/2}\right|, \tag{3.5}
$$

$$
C_{13} = 2\left|\left(f_0 f_0^+ f_2 f_2^+\right)^{1/2} - \left(f_1 f_1^+ f_3 f_3^+\right)^{1/2}\right|. \tag{3.6}
$$

Note that the product $(\lambda_1 \lambda_2)$ is the same for the three subsystems. Thus, from equations (17) and (24) of CKW [7], the 3-tangle is as follows:

$$
\tau_{123} = 16\left(f_0 f_0^+ f_1 f_1^+ f_2 f_2^+ f_3 f_3^+\right)^{1/2}. \tag{3.7}
$$

The invariance of τ_{123} under permutations of the particles $(1,2,3)$ (i.e., (a,b,c)) is evident above. Having obtained the results in terms of (f_0, f_1, f_2, f_3), we proceed below to study them for our two classes implementing (2.6) and (2.12). We start with τ_{123} since the crucial role of $\widehat{\mathrm{B}}$ is to entangle 3 particles.

(I). From (2.6) for real (θ, θ'),

$$
\begin{aligned}
\tau_{123} &= 16\left(f_0 f_0^+ f_1 f_1^+ f_2 f_2^+ f_3 f_3^+\right)^{1/2} = 16\left(f_0 f_1 f_2 f_3\right) \\
&= 16\frac{(1 - zz')(z + z')^2 z''|z - z'|}{\left(\left(1 + z^2\right)\left(1 + z'^2\right)\left(1 + z''^2\right)\right)^2}
\end{aligned} \tag{3.8}
$$

Here, $z = \tanh\theta$, $z' = \tanh\theta'$, $z'' = \tanh(\theta + \theta')$, and $1 \geq t(z, z', z'') \geq 0$. Special points:

(1) $(z = z')$: the difference $|z - z'|$ in (3.8) arises from anticommutativity of (J, K) in (2.2). For $z = z'$, $f_3 = 0$ and hence $\tau_{123} = 0$ in (3.7). From (3.4), (3.5), (3.6), one has nonzero 2 tangles. We exclude this point, being particulary interested in 3 tangles.

(2) $(z = -z')$: now $z'' = 0$, that is, $(\theta + \theta') = 0$. This is a trivial point. Now, in (1.1),

$$
\widehat{\mathrm{B}} = \widehat{\mathrm{R}}_{12}(\theta)\widehat{\mathrm{R}}_{23}(0)\widehat{\mathrm{R}}_{12}(-\theta) = \widehat{\mathrm{R}}_{23}(-\theta)\widehat{\mathrm{R}}_{12}(0)\widehat{\mathrm{R}}_{23}(\theta), \tag{3.9}
$$

where

$$\big(\widehat{R}_{23}(0), \widehat{R}_{12}(0)\big) = I \otimes I \otimes I, \tag{3.10}$$

and due to unitarity,

$$\widehat{R}_{12}(\theta)\widehat{R}_{12}(-\theta) = \widehat{R}_{23}(\theta)\widehat{R}_{23}(-\theta) = I \otimes I \otimes I. \tag{3.11}$$

We exclude also this point.

(3) $(z = 1, z' = 0)$, $(z = 0, z' = 1)$: for both points, $z'' = 1$. These limiting cases provide the maximal value $\tau_{123} = 1$. They correspond to the following:

$$f_0 = f_1 = f_2 = f_3 = \frac{1}{2}, \quad B|abc\rangle = \frac{1}{2}\big(|abc\rangle + |a\bar{b}\bar{c}\rangle + |\bar{a}b\bar{c}\rangle + |\bar{a}\bar{b}c\rangle\big).$$

The overall factor $\frac{1}{2}$ contributes $\frac{1}{2^4}$ to cancel exactly the factor 16 in (3.8), which arose from the fact that our $\widehat{B}$ acting on a product state $|abc\rangle$ gives a superposition of 4 states leading to entanglements.

Away from such points, for the generic case, keeping in mind (2.7), we define

$$x = \frac{(1 - zz')}{(z + z')}, \quad y = \frac{z''|z - z'|}{(z + z')},$$

and write

$$\tau_{123} = \frac{16xy}{\big(x^2 + y^2 + 2\big)^2} < \frac{4xy}{(xy + 1)^2} = \left(\frac{\big((xy)^{1/2} + (xy)^{-1/2}\big)}{2}\right)^{-2} < 1. \tag{3.12}$$

From (2.6), (3.4), (3.5), (3.6), the 2-particle concurrences are

$$C_{12} = 2\big|f_0 f_3 - f_1 f_2\big| = \frac{4z''z'}{\big(1 + z'^2\big)\big(1 + z''^2\big)}, \quad \text{(for } |z - z'| = z - z'\text{)},$$

and

$$C_{12} = \frac{4z''z}{\big(1 + z^2\big)\big(1 + z''^2\big)}, \quad \text{(for } |z - z'| = z' - z\text{)},$$

$$C_{23} = C_{13} = 2\big|f_1(f_0 - f_3)\big| = \frac{2(z+z')\big(1-z^2\big)}{\big(1+z^2\big)\big(1+z''^2\big)(1+zz')}, \quad \text{(for } |z-z'|=z-z'\text{)},$$

and

$$C_{23} = C_{13} = \frac{2(z + z')\big(1 - z'^2\big)}{\big(1 + z'^2\big)\big(1 + z''^2\big)(1 + zz')}, \quad \text{(for } |z - z'| = z' - z\text{)}.$$

For $(z = 1, z' = 0, z'' = 1)$ and also for $(z = 0, z' = 1, z'' = 1)$ $C_{12} = C_{23} = C_{13} = 0$, while $\tau_{123} = 1$ attaining the maximum. It is the situation one finds in GHZ state though that is quite different otherwise. (We refer to the comments below equation (24) of CKW [7]). One can also compare a Borromean ring (e.g., [10, Section 8.3]). If any one of the three entangled braids is cut, the remaining two fall apart, they are no longer entangled.

(II). We now obtain the quantum entanglements induced by the action of $\widehat{B}$ in a product state $|abc\rangle$ for our complex, multiparameter, unitary braid operator. From (2.12),

$$f_0 f_0^+ = \left(\cos\left(\frac{\lambda_+ - \lambda_-}{2}\right) \cos\left(\frac{\mu_+ - \mu_-}{2}\right) \right)^2,$$

$$f_1 f_1^+ = \left(\cos\left(\frac{\lambda_+ - \lambda_-}{2}\right) \sin\left(\frac{\mu_+ - \mu_-}{2}\right) \right)^2,$$

$$f_2 f_2^+ = \left(\sin\left(\frac{\lambda_+ - \lambda_-}{2}\right) \sin\left(\frac{\mu_+ - \mu_-}{2}\right) \right)^2,$$

$$f_3 f_3^+ = \left(\sin\left(\frac{\lambda_+ - \lambda_-}{2}\right) \cos\left(\frac{\mu_+ - \mu_-}{2}\right) \right)^2,$$

satisfying

$$f_0 f_0^+ + f_1 f_1^+ + f_2 f_2^+ + f_3 f_3^+ = 1.$$

Here, from (2.11),

$$\lambda_+ - \lambda_- = \left(m_{ab}^{(+)} - m_{ab}^{(-)}\right)(\theta + \theta'), \quad \mu_+ - \mu_- = \left(m_{bc}^{(+)} - m_{bc}^{(-)}\right)(\theta + \theta'). \tag{3.13}$$

The 3-tangle is now from (3.7), along with (3.13),

$$\tau_{123} = 16\left(f_0 f_0^+ f_1 f_1^+ f_2 f_2^+ f_3 f_3^+\right)^{1/2} = \left(\sin\left(\lambda_+ - \lambda_-\right) \sin\left(\mu_+ - \mu_-\right)\right)^2. \tag{3.14}$$

From (3.4), (3.5), (3.6), the 2 tangles are

$$C_{12} = \left| \sin\left(\lambda_+ - \lambda_-\right) \cos\left(\mu_+ - \mu_-\right) \right|, \quad C_{23} = \left| \cos\left(\lambda_+ - \lambda_-\right) \sin\left(\mu_+ - \mu_-\right) \right|, \quad C_{13} = 0.$$

Let us take a closer look at these results.

(1) The vanishing of C_{13} is related to the fact that in the action of $\widehat{B}$ on $|abc\rangle$, the terms involving $m_{ab}^{(\pm)}$ act on $|ab\rangle$ and those involving $m_{bc}^{(\pm)}$ on $|bc\rangle$. Thus, $|b\rangle$ is acted on by both parts while $|a\rangle$ and $|c\rangle$ are decoupled in the above sense. One can alter the actions on them independently by varying the two sets of parameters. One the other hand, the presence of $|b\rangle$ generates a coupling with $|a\rangle$ on one hand and with $|c\rangle$ on the other. A parallel feature was absent for our class (I). There apart from (θ, θ') there are no free parameters (like the m's for class (II)). And $z'' = (z + z')(1 + zz')^{-1}$ combines (z, z') nonlinearly.

(2) Here, the presence of the sum $(\theta + \theta')$ in (3.13) makes zero entanglements for $(\theta + \theta') = 0$ evident. Compare the discussion related to (3.9)–(3.11).

(3) The domain $0 \leq \tau_{123} \leq 1$ is evident from (3.14). Compare the discussion leading to (3.12).

(4) If the ratio (for $+1$ or -1 below),

$$\left(\frac{m_{ab}^{(+)} - m_{ab}^{(-)}}{m_{bc}^{(+)} - m_{bc}^{(-)}} \right)^{\pm 1} = 1, 3, 5, \ldots$$

an odd integer the upper limit $(\tau_{123} = 1)$ is attained periodically in the space of rapidities as the sum $(\theta + \theta')$ is varied. If the ratio on the right is incommensurable, one can have quasiperiodicity. Varying the parameters $(m_{ab}^{(\pm)}, m_{bc}^{(\pm)})$, one can sweep through different possibilities.

(5) For spin $\frac{1}{2}$ ($\widehat{R}$ a 4×4 matrix), there is only one set $m_{11}^{(\pm)}$ ($m_{bc}^{(\pm)} = m_{ab}^{(\pm)} = m_{11}^{(\pm)}$). Hence, in (3.14):

$$\tau_{123} = \big(\sin \big((m_{11}^{(+)} - m_{11}^{(-)})(\theta + \theta') \big) \big)^4.$$

This is always periodic in $(\theta + \theta')$.

4   Odd dimensions

The real matrices ([2, equation (2.1)]), our class (I), have no odd dimensional counterparts. The complex, multiparameter matrices (2.8), (2.9) are not thus restricted. In fact, the odd dimensional sequences based on "nested sequences of projectors" were the first to be constructed. The lowest odd dimensional (9×9) case, with imaginary parameters for unitarity, is explicitly presented in [3, Section 11]. The crucial difference with $N = 2n$ is that for $N = (2n - 1)$, $(n = 2, 3, \ldots)$ $\bar{i} = (2n - 1) - i + 1 = 2n - i$ and hence

$$\bar{n} = 2n - n = n.$$

For $N = 2n$, $\bar{i} \neq i$ for each i. Correspondingly in (2.9), now (for $(a, b) \neq n$)

$$P_{an}^{(\epsilon)} = P_{a\bar{n}}^{(\epsilon)} = \frac{1}{2} \big\{ (aa) + (\bar{a}\bar{a}) + \epsilon \big[(a\bar{a}) + (\bar{a}a) \big] \big\} \otimes (nn), \tag{4.1}$$

$$P_{nb}^{(\epsilon)} = \frac{1}{2} (nn) \otimes \big\{ (bb) + (\bar{b}\bar{b}) + \epsilon \big[(b\bar{b}) + (\bar{b}b) \big] \big\}, \tag{4.2}$$

$$P_{nn} = (nn) \otimes (nn). \tag{4.3}$$

If in (2.10)

$$(a, b, c) \neq n,$$

the results of Section 3 can be formally carried over unchanged. But for

$$|abc\rangle = \big(|nbc\rangle, |anc\rangle, |abn\rangle, |nnc\rangle, |nbn\rangle, |ann\rangle, |nnn\rangle \big), \tag{4.4}$$

fairly evident modifications are necessary. Some indications are given in [3]. Odd dimensional $\widehat{R}$ is necessary in dealing with particles of integer spins. If in a 3-photon state, each one is in a state of polarization $|\pm\rangle$ (i.e., $|\pm 1\rangle$), there being no $|0\rangle$ states the results of Section 2.2 can be used.

Consider now the action of $\widehat{B}$ on the states (4.4). From (4.1), (4.2), (4.3),

$$P_{nb}^{(\epsilon)} |nb\rangle = \frac{1}{2} |n\rangle \otimes \big(|b\rangle + \epsilon |\bar{b}\rangle \big) = \frac{1}{2} \big(|nb\rangle + \epsilon |n\bar{b}\rangle \big),$$

$$P_{an}^{(\epsilon)} |an\rangle = \frac{1}{2} \big(|a\rangle + \epsilon |\bar{a}\rangle \big) \otimes |n\rangle = \frac{1}{2} \big(|an\rangle + \epsilon |\bar{a}n\rangle \big),$$

$$P_{nn}^{(\epsilon)} |nn\rangle = |nn\rangle.$$

The action of $\widehat{B}$ on the states (4.4) can now be studied. From our point of view (links with quantum entanglements), not only $|nnn\rangle$ but also $(|nnc\rangle, |nbn\rangle, |ann\rangle)$ are trivial since one obtains under action of $\widehat{B}$ superposition of states $|nn\rangle \otimes (|c\rangle, |\bar{c}\rangle)$ and so on. Only one spin is affected. Entanglement is not produced. The states $(|nbc\rangle, |abn\rangle)$ can also be set aside,

under the action of $\widehat{B}$ the state $|n\rangle$ remains a bystander to 2-particle entanglements of $|bc\rangle$ and $|ab\rangle$.

The state $|anc\rangle$ deserves a closer look. One obtains the following:

$$\widehat{B}|anc\rangle = f_0|anc\rangle + f_1|an\bar{c}\rangle + f_2|\bar{a}n\bar{c}\rangle + f_3|\bar{a}nc\rangle.$$

The coefficients (f_0, f_1, f_2, f_3) are obtained by setting, in (2.11), (2.12),

$$\lambda_\pm = m_{an}^{(\pm)}(\theta + \theta'), \quad \mu_\pm = m_{nc}^{(\pm)}(\theta + \theta').$$

But now tracing out indices (with $\bar{n} = n$) leads to differences. As compared to (3.2), now (with $b = \bar{b} = n$):

$$\rho_{12} = \left(f_0 f_0^+ + f_1 f_1^+\right)|an\rangle\langle na| + \left(f_0 f_3^+ + f_1 f_2^+\right)|an\rangle\langle n\bar{a}|$$
$$+ \left(f_3 f_0^+ + f_2 f_1^+\right)|\bar{a}n\rangle\langle na| + \left(f_2 f_2^+ + f_3 f_3^+\right)|\bar{a}n\rangle\langle n\bar{a}|.$$

One has 2×2 matrix now. However, tracing out n in ρ_{12},

$$\rho_1 = \left(f_0 f_0^+ + f_1 f_1^+\right)|a\rangle\langle a| + \left(f_0 f_3^+ + f_1 f_2^+\right)|a\rangle\langle \bar{a}|$$
$$+ \left(f_3 f_0^+ + f_2 f_1^+\right)|\bar{a}\rangle\langle a| + \left(f_2 f_2^+ + f_3 f_3^+\right)|\bar{a}\rangle\langle \bar{a}|.$$

This is no longer diagonal-like (3.3).

We will not analyze such cases further in this paper. For $N = 2n$, at the center of the matrix, $\widehat{R}$ is the square lattice with corners, which can be denoted as $(nn, n\bar{n}, \bar{n}n, \bar{n}\bar{n})$. For $N = 2n - 1$, (since $\bar{n} = n$) this reduces to the point (nn) common to the diagonal and the antidiagonal. This is the source of difference. When this common point is not involved in $|abc\rangle$, the results correspond for even and odd dimensions.

5   Entanglement via a special coupling of 3 spins

This section is a brief digression. We restrict our remarks here to 3 spin $\frac{1}{2}$ particles. For this case, in the study of entanglements, basic roles are usually attributed to the states:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big), \tag{5.1}$$

$$|W\rangle = \frac{1}{\sqrt{3}}\big(|001\rangle + |010\rangle + |100\rangle\big), \tag{5.2}$$

$$|\widetilde{W}\rangle = \frac{1}{\sqrt{3}}\big(|110\rangle + |101\rangle + |011\rangle\big). \tag{5.3}$$

(See [9] and sources cited there.) Their local unitary transformations can also be considered [4].

Our approach via the braiding operator $\widehat{B}$ led to states of the type $(f_0|000\rangle + f_1|011\rangle + f_2|101\rangle + f_3|110\rangle)$ and $(g_0|111\rangle + g_1|100\rangle + g_2|010\rangle + g_3|001\rangle)$ with normalized coefficients (with parameter dependence):

$$f_0 f_0^+ + f_1 f_1^+ + f_2 f_2^+ + f_3 f_3^+ = 1, \quad g_0 g_0^+ + g_1 g_1^+ + g_2 g_2^+ + g_3 g_3^+ = 1,$$

given in Section 2. The states $|000\rangle$ and $|111\rangle$ of $|GHZ\rangle$ are separately superposed, respectively, with those of $|\widetilde{W}\rangle$, $|W\rangle$, respectively, and these 4 terms in the superpositions have

been thoroughly studied in the preceding sections. Though we are particularly interested in such cases, it is interesting to point out the direct relations of $|GHZ\rangle$, $|\widetilde{W}\rangle$ and $|W\rangle$ to a coupling of 3 angular momenta ($\overrightarrow{\mathbf{J}}_1$, $\overrightarrow{\mathbf{J}}_2$, $\overrightarrow{\mathbf{J}}_3$) to obtain eigenstates of the following:

$$Z = (\overrightarrow{\mathbf{J}}_1 \times \overrightarrow{\mathbf{J}}_2) \cdot \overrightarrow{\mathbf{J}}_3.$$

These also, of course, eigenstates of

$$\overrightarrow{\mathbf{J}}^2 = \overrightarrow{\mathbf{J}}_1^2 + \overrightarrow{\mathbf{J}}_2^2 + \overrightarrow{\mathbf{J}}_3^2$$

and $J^0 = J_1^0 + J_2^0 + J_3^0$. J_0 being the third component in the circular ones (J_+, J_-, J_0). Such a coupling was proposed by A. Chakrabarti long ago [6]. It was also proposed by J. M. Lévy-Leblond and M. Nahas [11]. From all the results of [6], concerning states $|jm\zeta\rangle$ ($j(j+1), m, \zeta$) denoting eigenvalues of (($\overrightarrow{\mathbf{J}}^2, J^0, Z$), resp.), we mention one. For the maximal value, $j = j_1 + j_2 + j_3$ always $\zeta = 0$. For $j_1 = j_2 = j_3 = \frac{1}{2}$ and $j = \frac{3}{2}$,

$$\frac{1}{\sqrt{2}}\left(\left|\begin{matrix}3\\2\end{matrix}\,\begin{matrix}3\\2\end{matrix}\,0\right\rangle \pm \left|\begin{matrix}3\\2\end{matrix}\,-\begin{matrix}3\\2\end{matrix}\,0\right\rangle\right) = \frac{1}{\sqrt{2}}\big(|000\rangle \pm |111\rangle\big), \quad \left|\begin{matrix}3\\2\end{matrix}\,\begin{matrix}1\\2\end{matrix}\,0\right\rangle = |W\rangle, \quad \left|\begin{matrix}3\\2\end{matrix}\,-\begin{matrix}1\\2\end{matrix}\,0\right\rangle = |\widetilde{W}\rangle.$$

(The states on the right, unlike those on the left, are those of (5.1), (5.2), (5.3).) For $j = \frac{1}{2}$, one has nonzero ζ. From (A.13) of [6] (with $\zeta = \pm\frac{\sqrt{3}}{4}$),

$$\left|\begin{matrix}1\\2\end{matrix}\,\begin{matrix}1\\2\end{matrix}\,\pm\frac{\sqrt{3}}{4}\right\rangle = \frac{1}{\sqrt{3}}\Big(e^{\pm\mathrm{i}\frac{2\pi}{3}}|011\rangle + e^{\mp\mathrm{i}\frac{2\pi}{3}}|101\rangle + |110\rangle\Big),$$

$$\left|\begin{matrix}1\\2\end{matrix}\,-\begin{matrix}1\\2\end{matrix}\,\pm\frac{\sqrt{3}}{4}\right\rangle = \frac{1}{\sqrt{3}}\Big(e^{\mp\mathrm{i}\frac{2\pi}{3}}|100\rangle + e^{\pm\mathrm{i}\frac{2\pi}{3}}|010\rangle + |001\rangle\Big).$$

We conclude by noting the following.

(1) This coupling was proposed due to its symmetries under the permutations of the spins ($\overrightarrow{\mathbf{J}}_1$, $\overrightarrow{\mathbf{J}}_2$, $\overrightarrow{\mathbf{J}}_3$). Such symmetries are notoriously lacking for the standard 2-step coupling via C. G. coefficients, where permutations are related to 6-j symbols. One has to label the intermediate step additionally with (j_1, j_2), (j_1, j_3), or (j_2, j_3). Reduction under the rotation group implementing Z gives simultaneous reduction, without additional effort, under S_3 the group of permutations of the 3 particles.

(2) Here are see how this formalism leads directly to states famous in the study of quantum entanglements.

6 Discussions

We want to emphasize how in larger dimensions each object (i.e., spin states of component particles) is seen to be entangled with all the others through a full exploitation of our formalism. Since our approach is via the braiding operator $\widehat{\mathrm{B}}$ (defined in (1.1)), we start by picking out a triplet:

$$|a\rangle \otimes |b\rangle \otimes |c\rangle \equiv |abc\rangle,$$

where (a, b, c) is any element among the basis states spanning $\mathrm{V}_N \otimes \mathrm{V}_N \otimes \mathrm{V}_N$ and obtain the entangled superpositions studied (Section 3):

$$\widehat{\mathrm{B}}|abc\rangle = f_0|abc\rangle + f_1|a\bar{b}\bar{c}\rangle + f_2|\bar{a}b\bar{c}\rangle + f_3|\bar{a}\bar{b}c\rangle, \tag{6.1}$$

where $|\bar{a}\rangle = |N - a + 1\rangle$, and so on. The states $(|a\rangle, |\bar{a}\rangle)$, $(|b\rangle, |\bar{b}\rangle)$, $(|c\rangle, |\bar{c}\rangle)$ of the subsystems are involved above. But now one can start again with any triplet $|ab'c'\rangle$, $|a''bc''\rangle, \ldots$, $(b' \neq b$ or $\bar{b}$, and so on) covering thus systematically all possible choices in $V_N \otimes V_N \otimes V_N$ and then implement the action of $\widehat{B}$. Thus, at the end each, $|a\rangle$ will be entangled with each $|b\rangle$ and each $|c\rangle$. At each step a quadruplet $(|abc\rangle, |a\bar{b}\bar{c}\rangle, |\bar{a}b\bar{c}\rangle, |\bar{a}\bar{b}c\rangle)$ will be involved, this being the essential property of both classes of unitary braid matrices we propose (with nonzero terms on the diagonal and the antidiagonal only).

Consider the simplest nontrivial case. For three spin half particles, the two quadruplets will be $(|000\rangle, |011\rangle, |101\rangle, |110\rangle)$ $(|111\rangle, |100\rangle, |010\rangle, |001\rangle)$. But already it is evident that starting by turns with, say $(|000\rangle, |001\rangle)$ finally $|a\rangle = |0\rangle$ will be entangled with $|bc\rangle = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$, i.e., with all possible states of $|b\rangle$ and $|c\rangle$. This is a general feature.

In Section 5, we have contrasted our typical superposition (6.1), to the prominent roles in the study of 3-particle entanglements of the states $(|GHZ\rangle, |W\rangle, |\widetilde{W}\rangle)$ given in (5.1)–(5.3). It is implicit in our formalism that the maximum 3-tangle is obtained for the following:

$$f_0 = f_1 = f_2 = f_3 = \frac{1}{2},$$

namely, $\frac{1}{2}(|111\rangle + |100\rangle + |010\rangle + |001\rangle)$ and $\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$. The (d_1, d_2, d_3) defined in equation (21) of CKW [7] are for both the cases above:

$$d_1 = 0, \quad d_2 = 0, \quad d_3 = \frac{1}{16}, \tag{6.2}$$

and hence (see equations (22)–(24) of CKW [7]):

$$\tau_{123} = 4\big|d_1 - 2d_2 + 4d_3\big| = 1.$$

The value (6.2) are the same as for $|GHZ\rangle$. For the maximal superposition (normalized sum of the quadruplets with all coefficients equal) $\frac{1}{\sqrt{8}}(|000\rangle + |011\rangle + |101\rangle + |111\rangle + |100\rangle + |010\rangle + |001\rangle)$, there is a striking change. Now,

$$\big(d_1, d_2, d_3\big) = \frac{1}{2^6}(4, 6, 2), \quad \tau_{123} = \frac{1}{2^6}(4 - 2 \cdot 6 + 4 \cdot 2) = 0.$$

One reaches now the lower bound. CKW [7] notes below equation (25) "It would be very interesting to know which of the results of this paper generalize to larger objects or to larger collections of objects". Our formalism furnishes one possible approach to many component objects and their collections. We have not answered the question whether the entanglement in larger dimensions can be formulated in a systematically hierarchical fashion, involving simultaneously more and more objects. Our motivation has been "entangling topological and quantum entanglements" via the braiding operator $\widehat{B}$ corresponding to third Reidemeister move. Having constructed unitary classes $\widehat{B}$, we were able to implement them to generate quantum entanglements.

Three particle entanglements were emphasized before [8]. We have studied, for our cases, the permutation invariant measures of entanglement of [7]. The crucial feature of our treatment is the study of $\widehat{B}$ acting on $V \times V \times V$ rather than $\widehat{R}$ acting on $V \times V$. The treatment starting in Section 2.2 displays one aspect of the multiple possibilities inherent in our multiparameter models. This can be put side by side with their role (for real parameters) in statistical models [1]. At the end of Section 3, we briefly evoke possible periodicity in the space of parameters. Introducing a magnetic field (and a simple generalization of the formalism of [5]), one would obtain periodicity in time of our entangled 3-particle states. This will be studied elsewhere.

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