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Some thoughts on geometries and on the nature of the gravitational field

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Abstract

This paper shows how a gravitational field generated by a given energy-momentum distribution (for all realistic cases) can be represented by distinct geometrical structures (Lorentzian, teleparallel, and nonnull nonmetricity spacetimes) or that we even can dispense all those geometrical structures and simply represent the gravitational field as a field in Faraday's sense living in Minkowski spacetime. The explicit Lagrangian density for this theory is given, and the field equations (which are Maxwell's like equations) are shown to be equivalent to Einstein's equations. Some examples are worked in detail in order to convince the reader that the geometrical structure of a manifold (modulus some topological constraints) is conventional as already emphasized by Poincaré long ago, and thus the realization that there are distinct geometrical representations (and a physical model related to a deformation of the continuum supporting Minkowski spacetime) for any realistic gravitational field strongly suggests that we must investigate the origin of its physical nature. We hope that this paper will convince readers that this is indeed the case.

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1 Introduction

Physics students learn general relativity (GR) as the modern theory of gravitation. In that theory, each gravitational field generated by a given energy-momentum tensor is represented by a Lorentzian spacetime, that is, a structure $(M, D, \mathbf{g}, \tau_{\mathbf{g}}, \uparrow)$, where M is a noncompact (locally compact) 4-dimensional Hausdorff manifold, $\mathbf{g}$ is a Lorentzian metric on M, and D is its Levi-Civita connection. Moreover, M is supposed oriented by the volume form $\tau_{\mathbf{g}}$ and the symbol $\uparrow$ means that the spacetime is time orientable[1]. From the geometrical objects in the structure $(M, D, \mathbf{g}, \tau_{\mathbf{g}}, \uparrow)$, we can calculate the *Riemann curvature tensor* $\mathbf{R}$ of D and a nontrivial GR model is one in which $\mathbf{R} \neq 0$. In that way, textbooks often say that in GR *spacetime is curved.* Unfortunately, many people mislead the curvature of a connection D on M with the fact that M can eventually be a bent surface in an (pseudo) Euclidean space

[1]For details, please consult, for example, [36, 39].

with a sufficient number of dimensions[2]. This confusion leads to all sorts of wishful thinking because many forget that GR does not fix the topology[3] of M that often must be put in "by hand" when solving a problem and thus think that they can bend spacetime if they have an appropriate kind of some exotic matter. Worse, the insistence in supposing that the gravitational field is *geometry* leads the majority of physicists to relegate the search for the real physical nature of the gravitational field as unimportant at all (see a discussion of this issue in [24]). Instead, students are advertised that GR is considered by may physicists as the most beautiful physical theory [23]. However, textbooks with a few exceptions (see, e.g., the excellent book by Sachs and Wu [39]) forget to say to their readers that in GR, there are no genuine conservation laws of energy-momentum and angular momentum unless spacetime has some *additional* structure which is not present in a general Lorentzian spacetime [27]. Only a few people tried to develop consistently theories, where the gravitational field (at least from the classical point of view) is a field in Faraday's sense living in Minkowski spacetime (see below).

In this paper, we want to recall two important results that hopefully will lead people to realize that eventually it is time to disclose the real nature of the gravitational field[4]. The first result is that the representation of gravitational fields by Lorentzian spacetimes is eventually no more than a consequence of the differential geometry knowledge of Einstein and Grossmann when they where struggling to find a consistent way to describe the gravitational field[5]. Indeed, there are some geometrical structures different from $(M, D, \boldsymbol{g}, \tau_g, \uparrow)$ that can equivalently represent such a field. The second result is that the gravitational field (in all known situations) can also be nicely represented as a field in Faraday's sense [26] living in a fixed background spacetime[6]. Concerning the alternative geometrical models, the particular cases where the connection is teleparallel (i.e., it is metrical compatible, has *null* Riemann curvature tensor and *nonnull* torsion tensor) and the one where the connection is *not* metrical compatible (i.e., its nonmetricity tensor $\mathbf{A}^\eta \neq 0$) will be addressed below. However, to understand how those alternative geometrical models (and the physical model) can be constructed, and why the Lorentzian spacetime model was Einstein's first choice, it is eventually worth to recall some historical facts concerning attempts by Einstein (and others) to build a geometrical unified theory of the gravitational and electromagnetic fields.

We start with the word torsion. Although such a word seems to have been introduced by Cartan [2] in 1922 the fact is that the concept behind the name already appeared in Ricci's paper [31] from 1895 and was also used in [32]. In those papers, Ricci introduced what is now called Cartan's *moving frames* and the *teleparallel geometry*[7].

[2] Any manifold $M, \dim M = n$ according to the Whitney theorem can be realized as a submanifold of $\mathbb{R}^m$ with $m = 2n$. However, if M carries additional structure the number m in general must be greater than $2n$. Indeed, it has been shown by Eddington [9] that if $\dim M = 4$ and if M carries a Lorentzian metric $\boldsymbol{g}$, which moreover satisfies Einstein's equations, then M can be locally embedded in a (pseudo)Euclidean space $\mathbb{R}^{1,9}$. Also, isometric embeddings of general Lorentzian spacetimes would require a lot of extra dimensions [4]. Indeed, a compact Lorentzian manifold can be embedded isometrically in $\mathbb{R}^{2,46}$ and a noncompact one can be embedded isometrically in $\mathbb{R}^{2,87}$!

[3] In particular, the topology of the universe that we live in is unknown, as yet [53].

[4] Of course, the authors know that one of the claims of string theory is that it nicely describes gravitation. In that theory, GR is only an approximated theory valid for distances much greater than the Planck length.

[5] See some details below.

[6] The preferred one is, of course, Minkowski spacetime, the simple choice, but, the true background spacetime may be eventually a more complicated one, since that manifold must represent the global topological structure of the universe, something that is not known at the time of this writing [53].

[7] Also known as Weintzböck geometry [55].

Moreover, in 1901, Ricci and Levi-Civita[8] published a joy [33], which has become the *bible* of tensor calculus and which has been extensively studied by Einstein and Grossmann in their search for the theory of the gravitational field[9]. However, Einstein and Grossmann seem to have studied only the first part of reference 4 and so missed "Cartan's moving method" and the concept of torsion. It seems also that only after 1922, Einstein became interested in the second chapter of the joy, titled *La Géométrie Intrinseque Comme Instrument de Calcul* and discovered torsion and the teleparallel geometry[10]. As it is well known, he tried to identify a certain contraction of the torsion tensor of a teleparallel geometry with the electromagnetic potential, but after sometime, he discovered that the idea did not work. Einstein's first papers on the subject[11] are [11, 12, 13]. Also, in a paper which Einstein wrote in 1925 [10], the torsion tensor concept already appeared, since he considered it as one of his field variables, the antisymmetric part of a *nonsymmetric* connection. All those papers by Einstein have been translated into English by Unzicker and Case [44] and can be downloaded from the arXiv. We also can learn in [7, 18] that Cartan tried to explain the teleparallel geometry to Einstein when he visited Paris in 1922 using the example of what we call the *Nunes connection* (*or navigator connection*) on the punctured sphere. Since this example illustrates in a crystal clear way the fact that one must not confound the Riemann curvature of a given connection defined on a manifold M with the fact that M may be viewed as a *bent* hypersurface embedded in an Euclidean space (with appropriate number of dimensions), it will be presented in Appendix A. A comparison of the parallel transport according to the Nunes connection and according to the usual Levi-Civita connection is done[12], and it is shown that the Nunes connection the Riemann curvature of the punctured sphere is *null.* In this sense, the geometry of the punctured sphere is conventional as emphasized by Poincaré [30] long ago.

As we already said that the main objective of the present paper is to clarify the fact that there are different ways of geometrically representing a gravitational field, such that the field equations in each representation result equivalent to Einstein's field equations. Explicitly, we mean by this statement the following: any model of a gravitational field in GR represented by a *Lorentzian spacetime* (*with nonnull Riemann curvature tensor and null torsion tensor* which is also parallelizable[13]) is equivalent to a teleparallel spacetime (i.e., a spacetime structure equipped with a metrical compatible teleparallel connection, which has *null* Riemann curvature tensor and *nonnull* torsion tensor)[14] or equivalent to a special spacetime structure, where the manifold M is equipped with a Minkowski metric, and where there is also defined a connection such that its *nonmetricity tensor* is not null. The teleparallel possibility is described in detail in Section 2 using the modern theory of differential forms and we claim that our presentation leaves also clear that we can even dispense with the concept of a *connection* in the description of a gravitational field[15], it is only necessary for such a representation to exist that the manifold M representing the set of all possible events

[8] A Ricci student at that time.

[9] An English translation of the joy with very useful comments has been done by the mathematical physicist Robert Hermann [21] in 1975 and that text (and many others books by Hermann) can be downloaded from http://books.google.com.br/books?q=robert+hermann.

[10] Some interesting historical details may be found in [18, 40].

[11] For a complete list of Einstein's papers on the subject see [18].

[12] The material of Appendix A follows the presentation in [36, Section 4.7.7].

[13] A manifold M is said to be parallelizable if it admits four *global* linearly independent vector fields.

[14] There are hundreds of papers (as e.g., [6]) on the subject.

[15] Explicitly, we mean that the gravitational field may be interpreted as a field in the sense of Faraday, as it is the case of the electromagnetic field.

be parallelizable, admitting four global (not all exact) 1-form fields coupled in a specific way (see below). The second possibility is illustrated with an example in Section 3. In Section 4, we present the conclusions.

2 Torsion as a description of gravity

2.1 Some notation

Suppose that a 4-dimensional M manifold is parallelizable, thus admitting a set of four global linearly independent vector $\mathbf{e_a} \in \sec TM$, $\mathbf{a} = 0,1,2,3$ fields[16] such that $\{\mathbf{e_a}\}$ is a basis for TM and let $\{\theta^{\mathbf{a}}\}$, $\theta^{\mathbf{a}} \in \sec T^*M$ be the corresponding dual basis ($\theta^{\mathbf{a}}(\mathbf{e_b}) = \delta^{\mathbf{a}}_{\mathbf{b}}$). Suppose also that not all the $\theta^{\mathbf{a}}$ are closed:

$$d\theta^{\mathbf{a}} \neq 0 \tag{2.1}$$

for a least some $\mathbf{a} = 0,1,2,3$. The 4-form field $\theta^{\mathbf{0}} \wedge \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}} \wedge \theta^{\mathbf{3}}$ defines a (positive) orientation for M.

Now, the $\{\theta^{\mathbf{a}}\}$ can be used to define a Lorentzian metric field in M by defining $\boldsymbol{g} \in \sec T^0_2 M$ by

$$\boldsymbol{g} := \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}} \tag{2.2}$$

with the matrix with entries $\eta_{\mathbf{ab}}$ being the diagonal matrix $(1,-1,-1,-1)$. Then, according to $\boldsymbol{g}$, the $\{\mathbf{e_a}\}$ are orthonormal:

$$\mathbf{e_a} \underset{\boldsymbol{g}}{\cdot} \mathbf{e_b} := \boldsymbol{g}(\mathbf{e_a}, \mathbf{e_b}) = \eta_{\mathbf{ab}}. \tag{2.3}$$

Remark 2.1. Since according to (2.3), $\mathbf{e_0}$ is a global time like vector field it follows that it defines a time orientation in M which we denote by $\uparrow$. It follows that the 4-tuple $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ is part of a structure defining a *Lorentzian spacetime* and can serve as a *substructure* to model a gravitational field in GR.

For future, use we also introduce $g \in \sec T^2_0 M$ by

$$g = \eta^{\mathbf{ab}} \mathbf{e_a} \otimes \mathbf{e_b}, \tag{2.4}$$

and we write

$$\theta^{\mathbf{a}} \underset{g}{\cdot} \theta^{\mathbf{b}} = g(\theta^{\mathbf{a}}, \theta^{\mathbf{b}}) = \eta^{\mathbf{ab}}. \tag{2.5}$$

Due to the hypothesis given by (2.1), the vector fields $\mathbf{e_a}$, $\mathbf{a} = 0,1,2,3$ will in general satisfy

$$[\mathbf{e_a}, \mathbf{e_b}] = c^{\mathbf{k}}_{\mathbf{ab}} \mathbf{e_k}, \tag{2.6}$$

where the $c^{\mathbf{k}}_{\mathbf{ab}}$ are the structure coefficients of the basis $\{\mathbf{e_a}\}$. It can be easily shown that[17]

$$d\theta^{\mathbf{a}} = -\frac{1}{2} c^{\mathbf{a}}_{\mathbf{kl}} \theta^{\mathbf{k}} \wedge \theta^{\mathbf{l}}. \tag{2.7}$$

[16]We recall that $\sec TM$ means section of the tangent bundle and $\sec T^*M$ means section of the cotangent bundle. Also $T^r_s M$ means the bundle of tensors of type (r,s) and $\sec \bigwedge^r T^*M$ means a section of the bundle of r-forms fields.

[17]See, for example, [36].

Now, we introduce two different metric compatible connections on M, namely, D (the Levi-Civita connection of $\boldsymbol{g}$) and a *teleparallel* connection ∇, such that

$$\begin{aligned} &D_{\mathbf{e_a}}\mathbf{e_b} = \omega^{\mathbf{c}}_{\mathbf{ab}}\mathbf{e_c}, \quad D_{\mathbf{e_a}}\theta^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mathbf{ac}}\theta^{\mathbf{c}}, \\ &\nabla_{\mathbf{e_a}}\mathbf{e_b} = 0, \qquad\quad \nabla_{\mathbf{e_a}}\theta^{\mathbf{b}} = 0. \end{aligned} \tag{2.8}$$

The objects $\omega^{\mathbf{c}}_{\mathbf{ab}}$ are called the connection coefficients of the connection D in the $\{\mathbf{e_a}\}$ basis and the objects $\omega^{\mathbf{a}}_{\mathbf{b}} \in \sec T^*M$ defined by

$$\omega^{\mathbf{a}}_{\mathbf{b}} := \omega^{\mathbf{a}}_{\mathbf{kb}}\theta^{\mathbf{k}} \tag{2.9}$$

are called the connection 1-forms in the $\{\mathbf{e_a}\}$ basis.

Remark 2.2. The connection coefficients $\varpi^{\mathbf{b}}_{\mathbf{ac}}$ of ∇ and the connection 1-forms of ∇ in the basis $\{\mathbf{e_a}\}$ are null according to the second line of *(2.8) and thus the basis* $\{\mathbf{e_a}\}$ is called teleparallel. So, the connection ∇ defines an absolute parallelism on M. We recall that as said in the introduction that idea has been introduced by Ricci.

Remark 2.3. Of course, as it is well known, the Riemann curvature tensor of D is in general nonnull in all points of M, but the torsion tensor of D is zero in all points of M. Moreover, the Riemann curvature tensor of ∇ is null in all points of M, whereas the torsion tensor of ∇ is nonnull in all points of M.

We recall also in order to fix notation that for a general connection, say $\mathbf{D}$ on M (not necessarily metric compatible) the *torsion and curvature operations* and the torsion and *curvature* tensors of a given general connection, say $\mathbf{D}$, are, respectively, the mappings:

$$\tau(\mathbf{u},\mathbf{v}) = \mathbf{D_u v} - \mathbf{D_v u} - [\mathbf{u},\mathbf{v}], \tag{2.10}$$

$$\rho(\mathbf{u},\mathbf{v}) = \mathbf{D_u D_v} - \mathbf{D_v D_u} - \mathbf{D}_{[\mathbf{u},\mathbf{v}]}, \tag{2.11}$$

$$\mathcal{T}(\alpha,\mathbf{u},\mathbf{v}) = \alpha\big(\tau(u,v)\big), \tag{2.12}$$

$$\mathbf{R}(\mathbf{w},\alpha,\mathbf{u},\mathbf{v}) = \alpha\big(\rho(\mathbf{u},\mathbf{v})\mathbf{w}\big) \tag{2.13}$$

for every $\mathbf{u},\mathbf{v},\mathbf{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$. In particular, we write

$$T^{\mathbf{a}}_{\mathbf{bc}} := \mathcal{T}\big(\theta^{\mathbf{a}},\mathbf{e_b},\mathbf{e_c}\big), \quad R_{\mathbf{a}\ \mathbf{cd}}^{\ \mathbf{b}} := \mathbf{R}\big(\mathbf{e_a},\theta^{\mathbf{b}},\mathbf{e_c},\mathbf{e_d}\big), \tag{2.14}$$

and define the Ricci tensor by

$$\mathrm{Ricci} := R_{\mathbf{ac}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{c}}, \quad R_{\mathbf{ac}} := R_{\mathbf{a}\ \mathbf{cb}}^{\ \mathbf{b}}. \tag{2.15}$$

Remark 2.4. Keep in mind that the Ricci tensor of a general connection is in general not symmetric[18].

We shall need also in order to fix our conventions to briefly recall the definitions of the scalar product and left and right contractions on the so-called Hodge bundle $(\bigwedge T^*M, \boldsymbol{g})$, where $\bigwedge T^*M = \bigoplus_{r=0}^{4} \bigwedge^r T^*M$ is the bundle of *nonhomogenous* multiforms.

[18]See, for example, [36, Section 4.5.8].

So, if $\mathcal{A},\mathcal{B}\in\sec\bigwedge^r T^*M$, $\mathcal{A}=\frac{1}{r!}A_{\mathbf{i}_1\ldots\mathbf{i}_\mathbf{r}}\theta^{\mathbf{i}_1}\wedge\cdots\wedge\theta^{\mathbf{i}_\mathbf{r}}$, $\mathcal{B}=\frac{1}{r!}B_{\mathbf{j}_1\ldots\mathbf{j}_\mathbf{r}}\theta^{\mathbf{j}_1}\wedge\cdots\wedge\theta^{\mathbf{j}_\mathbf{r}}$, their scalar product $\underset{g}{\cdot}$ (induced by g) is the *linear* mapping:

$$\mathcal{A}\underset{g}{\cdot}\mathcal{B}=\left(\frac{1}{r!}\right)A_{\mathbf{i}_1\ldots\mathbf{i}_\mathbf{r}}B_{\mathbf{j}_1\ldots\mathbf{j}_\mathbf{r}}\begin{pmatrix}\theta^{\mathbf{i}_1}\underset{g}{\cdot}\theta^{\mathbf{j}_1} & \ldots & \ldots & \theta^{\mathbf{i}_1}\underset{g}{\cdot}\theta^{\mathbf{j}_\mathbf{r}}\\ \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots\\ \theta^{\mathbf{i}_\mathbf{r}}\underset{g}{\cdot}\theta^{\mathbf{j}_1} & \ldots & \ldots & \theta^{\mathbf{i}_\mathbf{r}}\underset{g}{\cdot}\theta^{\mathbf{j}_\mathbf{r}}\end{pmatrix}. \tag{2.16}$$

Also, for $\mathcal{A}\in\sec\bigwedge^r T^*M$, $\mathcal{C}\in\sec\bigwedge^s T^*M$, it is $\mathcal{A}\underset{g}{\cdot}\mathcal{C}=0$. The left and right contractions of $\mathcal{X},\mathcal{Y}\in\sec\bigwedge T^*M$ are defined for arbitrary (nonhomogeneous) multiforms as the mappings $\underset{g}{\lrcorner}:\bigwedge T^*M\times\bigwedge T^*M\to\bigwedge T^*M$ and $\underset{g}{\llcorner}:\bigwedge T^*M\times\bigwedge T^*M\to\bigwedge T^*M$ such that for all $\mathcal{Z}$ $\in\sec\bigwedge T^*M$:

$$\left(\mathcal{X}\underset{g}{\lrcorner}\mathcal{Y}\right)\underset{g}{\cdot}\mathcal{Z}=\mathcal{Y}\underset{g}{\cdot}(\tilde{\mathcal{X}}\wedge\mathcal{Z}),\quad \left(\mathcal{X}\underset{g}{\llcorner}\mathcal{Y}\right)\underset{g}{\cdot}\mathcal{Z}=\mathcal{X}\underset{g}{\cdot}(Z\wedge\tilde{\mathcal{Y}}), \tag{2.17}$$

where the *tilde* means the operation of reversion, for example, if $\mathcal{B}=\frac{1}{r!}B_{\mathbf{i}_1\ldots\mathbf{i}_\mathbf{r}}\theta^{\mathbf{i}_1}\wedge\cdots\wedge\theta^{\mathbf{i}_\mathbf{r}}$, then $\tilde{\mathcal{B}}=\frac{1}{r!}B_{\mathbf{i}_1\ldots\mathbf{i}_\mathbf{r}}\theta^{\mathbf{i}_\mathbf{r}}\wedge\cdots\wedge\theta^{\mathbf{i}_1}$.

The *Hodge star operator* (or Hodge dual) is the linear mapping $\underset{g}{\star}:\bigwedge^r T^*M\to\bigwedge^{n-r}T^*M$,

$$\mathcal{A}\wedge\underset{g}{\star}\mathcal{B}=\left(\mathcal{A}\underset{g}{\cdot}\mathcal{B}\right)\tau_g \tag{2.18}$$

for every $\mathcal{A},\mathcal{B}\in\bigwedge^r T^*M$. The inverse $\underset{g}{\star}^{-1}:\bigwedge^{n-r}T^*M\to\bigwedge^r T^*M$ of the Hodge star operator is given by

$$\underset{g}{\star}^{-1}=(-1)^{r(n-r)}\operatorname{sgn}\dot{\mathbf{g}}\underset{g}{\star}, \tag{2.19}$$

where $\operatorname{sgn}\mathbf{g}=\det\mathbf{g}/|\det\mathbf{g}|$ denotes the sign of the determinant of the matrix with entries $g_{ij}=\mathbf{g}(\mathbf{e}_i,\mathbf{e}_j)$. The *Hodge coderivative operator* $\underset{g}{\delta}$ (associated to $\mathbf{g}$) is defined for $\mathcal{A}\in\sec\bigwedge^r T^*M$ by

$$\underset{g}{\delta}\mathcal{A}=(-1)^r\underset{g}{\star}^{-1}d\underset{g}{\star}\mathcal{A}. \tag{2.20}$$

2.2 Cartan's structure equations

Given that we introduced two different connections D and ∇ defined in the manifold M, we can write *two different pairs* of Cartan's structure equations, each one of the pairs describing, respectively, the geometry of the structures $(M,D,\mathbf{g},\tau_g,\uparrow)$ and $(M,\nabla,\mathbf{g},\tau_g,\uparrow)$ which will be called, respectively, a Lorentzian spacetime and a teleparallel spacetime.

2.2.1 Cartan's structure equations for D

In this case, we write

$$\Theta^{\mathbf{a}}:=d\theta^a+\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}}=0,\quad \mathcal{R}_{\mathbf{b}}^{\mathbf{a}}:=d\omega_{\mathbf{b}}^{\mathbf{a}}+\omega_{\mathbf{c}}^{\mathbf{a}}\wedge\omega_{\mathbf{b}}^{\mathbf{c}}, \tag{2.21}$$

where the $\Theta^{\mathbf{a}}\in\sec\bigwedge^2 T^*M$, $\mathbf{a}=0,1,2,3$ and the $\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}\in\sec\bigwedge^2 T^*M$, $\mathbf{a}$, $\mathbf{b}=0,1,2,3$ are, respectively, the torsion and the curvature 2-forms of D with

$$\Theta^{\mathbf{a}}=\frac{1}{2}T_{\mathbf{bc}}^{\mathbf{a}}\theta^{\mathbf{b}}\wedge\theta^{\mathbf{c}},\quad \mathcal{R}_{\mathbf{b}}^{\mathbf{a}}=\frac{1}{2}R_{\mathbf{b}\ \mathbf{cd}}^{\mathbf{a}}\theta^{\mathbf{c}}\wedge\theta^{\mathbf{d}}. \tag{2.22}$$

2.2.2 Cartan's structure equations for ∇

In this case, since $\varpi^{\mathbf{a}}_{\mathbf{b}} = 0$, we have

$$\bar{\Theta}^{\mathbf{a}} := d\theta^{a} + \varpi^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = d\theta^{a}, \quad \bar{\mathcal{R}}^{\mathbf{a}}_{\mathbf{b}} := d\varpi^{\mathbf{a}}_{\mathbf{b}} + \varpi^{\mathbf{a}}_{\mathbf{c}} \wedge \varpi^{\mathbf{c}}_{\mathbf{b}} = 0, \tag{2.23}$$

where the $\bar{\Theta}^{\mathbf{a}} \in \sec \bigwedge^2 T^*M$, $\mathbf{a} = 0,1,2,3$ and the $\bar{\mathcal{R}}^{\mathbf{a}}_{\mathbf{b}} \in \sec \bigwedge^2 T^*M$, $\mathbf{a},\mathbf{b} = 0,1,2,3$ are, respectively, the torsion and the curvature 2-forms of ∇ given by formulas analogous to the ones in (2.22).

We next suppose that the $\{\theta^{\mathbf{a}}\}$ are the basic variables representing a gravitation field. We postulate that the $\{\theta^{a}\}$ interacts with the matter fields through the following Lagrangian density[19]:

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m, \tag{2.24}$$

where $\mathcal{L}_m$ is the matter Lagrangian density and

$$\mathcal{L}_g = -\frac{1}{2} d\theta^{\mathbf{a}} \wedge \underset{g}{\star} d\theta_{\mathbf{a}} + \frac{1}{2} \underset{g}{\delta}\theta^{\mathbf{a}} \wedge \underset{g}{\star}\underset{g}{\delta}\theta_{\mathbf{a}} + \frac{1}{4}\big(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}\big) \wedge \underset{g}{\star}\big(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\big). \tag{2.25}$$

Remark 2.5. This Lagrangian is not invariant under arbitrary point dependent Lorentz rotations of the basic cotetrad fields. In fact, if $\theta^{\mathbf{a}} \mapsto \theta'^{\mathbf{a}} = \Lambda^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$, where for each $x \in M$, $\Lambda^{\mathbf{a}}_{\mathbf{b}}(x) \in L^{\uparrow}_{+}$ (the homogenous and orthochronous Lorentz group), we get that

$$\mathcal{L}'_g = -\frac{1}{2} d\theta'^{\mathbf{a}} \wedge \underset{g}{\star} d\theta'_{\mathbf{a}} + \frac{1}{2} \underset{g}{\delta}\theta'^{\mathbf{a}} \wedge \underset{g}{\star}\underset{g}{\delta}\theta'_{\mathbf{a}} + \frac{1}{4}\big(d\theta'^{\mathbf{a}} \wedge \theta'_{\mathbf{a}}\big) \wedge \underset{g}{\star}\big(d\theta'^{\mathbf{b}} \wedge \theta'_{\mathbf{b}}\big), \tag{2.26}$$

which differs from $\mathcal{L}_g$ by an exact differential. So, the field equations derived by the variational principle results invariant under a change of gauge[20].

Now, to derive the field equations directly from (2.25) is a nontrivial and laborious exercise, whose details the interested reader may find in [14, 36]. The result is

$$d \underset{g}{\star} \mathcal{S}_{\mathbf{d}} + \underset{g}{\star} t_{\mathbf{d}} = - \underset{g}{\star} \mathcal{T}_{\mathbf{d}}, \tag{2.27}$$

where

$$\begin{aligned}
\underset{g}{\star} t_{\mathbf{d}} := \frac{\partial \mathcal{L}_g}{\partial \theta^{\mathbf{d}}} &= \frac{1}{2}\Big[\Big(\theta_{\mathbf{d}} \underset{g}{\lrcorner} d\theta^{\mathbf{a}}\Big) \wedge \underset{g}{\star} d\theta_{\mathbf{a}} - d\theta^{\mathbf{a}} \wedge \Big(\theta_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star}\, d\theta_{\mathbf{a}}\Big)\Big] \\
&+ \frac{1}{2} d\Big(\theta_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star}\, \theta^{\mathbf{a}}\Big) \wedge \underset{g}{\star} d \underset{g}{\star}\, \theta_{\mathbf{a}} + \frac{1}{2} d\Big(\theta_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star}\, \theta^{\mathbf{a}}\Big) \wedge \underset{g}{\star} d \underset{g}{\star}\, \theta_{\mathbf{a}} + \frac{1}{2} d\theta_{\mathbf{d}} \wedge \underset{g}{\star}\big(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}\big) \\
&- \frac{1}{4} d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}} \wedge \Big[\theta_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star}\, \big(d\theta^{\mathbf{c}} \wedge \theta_{\mathbf{c}}\big)\Big] - \frac{1}{4}\Big[\theta_{\mathbf{d}} \underset{g}{\lrcorner} \big(d\theta^{\mathbf{c}} \wedge \theta_{\mathbf{c}}\big)\Big] \wedge \underset{g}{\star}\big(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}\big), \\
\underset{g}{\star} \mathcal{S}_{\mathbf{d}} := \frac{\partial \mathcal{L}_g}{\partial d\theta^{\mathbf{d}}} &= - \underset{g}{\star}\, d\theta_{\mathbf{d}} - \Big(\theta_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star}\, \theta^{\mathbf{a}}\Big) \wedge \underset{g}{\star} d \underset{g}{\star}\, \theta_{\mathbf{a}} + \frac{1}{2}\theta_{\mathbf{d}} \wedge \underset{g}{\star}\big(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}\big),
\end{aligned} \tag{2.28}$$

and the[21]

$$\underset{g}{\star} \mathcal{T}_{\mathbf{d}} := \frac{\partial \mathcal{L}_m}{\partial \theta^{\mathbf{d}}} = - \underset{g}{\star}\, T_{\mathbf{d}} \tag{2.29}$$

are the energy-momentum 3-forms of the matter fields[22].

[19] We observe that the first term in (2.25) is just the Lagrangian density used by Einstein in [13].

[20] See details in [36].

[21] We suppose that $\mathcal{L}_m$ does not depend explicitly on the $d\theta^{\mathbf{a}}$.

[22] In reality, due the conventions used in this paper, the true energy-momentum 3-forms are $\underset{g}{\star} T_{\mathbf{d}} = - \underset{g}{\star}\, \mathcal{T}_{\mathbf{d}}$.

Recalling that from (2.23), it is $\bar{\Theta}^{\mathbf{a}} := d\theta^{a}$, the field equations (2.27) can be written as follows:

$$d \underset{g}{\star} \mathcal{F}_{\mathbf{d}} = -\underset{g}{\star} \mathcal{T}_{\mathbf{d}} - \underset{g}{\star} t_{\mathbf{d}} - \underset{g}{\star} \mathfrak{h}_{d}, \tag{2.30}$$

where

$$\mathfrak{h}_{\mathbf{d}} = d\Big[\Big(\theta_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} \theta^{\mathbf{a}}\Big) \wedge \underset{g}{\star} d \underset{g}{\star} \theta_{\mathbf{a}} - \frac{1}{2}\theta_{\mathbf{d}} \wedge \underset{g}{\star}\big(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}\big)\Big]. \tag{2.31}$$

Finally, recalling the definition of the Hodge coderivative operator (Equation (2.20)), we can write (2.30) as follows:

$$\underset{g}{\delta} \mathcal{F}^{\mathbf{d}} = -\big(\mathcal{T}^{\mathbf{d}} + \mathbf{t}^{\mathbf{d}}\big) \tag{2.32}$$

with the $\mathbf{t}^{\mathbf{d}} \in \sec \bigwedge^{1} T^{*}M$ given by

$$\mathbf{t}^{\mathbf{d}} := t^{\mathbf{d}} + \mathfrak{h}^{\mathbf{d}}, \tag{2.33}$$

legitimate energy-momenta[23] 1-form fields for the gravitational field. Note that the total energy-momentum tensor of matter plus the gravitational field is trivially conserved in our theory:

$$\underset{g}{\delta}\big(\mathcal{T}^{\mathbf{d}} + \mathbf{t}^{\mathbf{d}}\big) = 0, \tag{2.34}$$

since $\underset{g}{\delta^{2}} \mathcal{F}^{\mathbf{d}} = 0$.

Remark 2.6. In [26], a theory of the gravitational field in Minkowski spacetime $(M \simeq \mathbb{R}^{4}, \mathring{\boldsymbol{g}}, D, \tau_{\mathring{g}}, \uparrow)$ has been presented, where a nontrivial gravitational field configuration was interpreted as generating an effective Lorentzian spacetime $(M \simeq \mathbb{R}^{4}, \boldsymbol{g}, D, \tau_{g}, \uparrow)$, where $\boldsymbol{g}$ satisfies Einstein equations and where probe particles and/or fields move. It was assumed there that the gravitational field $\mathbf{g} = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}$ is a field in Faraday sense[24], that is, the fields $\theta^{\mathbf{a}}$ have their dynamics described by a (postulated) Lagrangian density like the one in (2.25). Moreover, it was postulated that the $\theta^{\mathbf{a}}$ couple universally with the matter fields and that the presence of energy-momentum due to matter fields in some region of Minkowski spacetime distort the Lorentz vacuum in much the same way that stresses in an elastic body distorts it. Now, distortions (or deformations) in the theory of dislocations according to [56] can be of the *elastic* or *plastic* type. An elastic distortion is described by a diffeomorphism $\mathtt{h} : M \to M$. In this case, the induced metric is $\mathbf{g} = \mathtt{h}^{*}\boldsymbol{\eta}$ (analogous to the Cauchy-Green tensor [16] of elasticity theory) and according to [36, Remark 250], its Levi-Civita connection is $\mathtt{h}^{*}\mathring{D}$. This implies that the structure $(\mathtt{h}M \simeq \mathbb{R}^{4}, \mathbf{g}, \mathtt{h}^{*}\mathring{D}, \tau_{g}, \uparrow)$ is again Minkowski spacetime. In the original versions of [26, 36], this was the type of deformation considered, but this has been corrected in improved versions of those manuscripts, respectively, at the arXiv and at http://ime.unicamp.br/walrod/recentes, where an errata to [36, Chapter 10] may be found. In the quoted errata, the deformation is taken to be of the plastic type and represented by a distortion extensor field M, that is, a linear mapping $h : \bigwedge^{1} T^{*}M \to \bigwedge^{1} T^{*}M$ such that

[23]Indeed, for each index $\mathbf{d}$, the first member of (2.32) is a 1-from field and also $\mathcal{T}^{\mathbf{d}}$ is an 1-form field, so $\delta\mathcal{F}^{\mathbf{d}} + \mathcal{T}^{\mathbf{d}} = -\mathbf{t}^{\mathbf{d}}$ is a 1-form field.

[24]This means that it is interpreted as a field with an ontology analogous to the electromagnetic field.

$g = h^\dagger h$, where g is the extensor field associated with $\boldsymbol{g}$ (see below)[25]. A book describing a detailed theory of gravitation as a plastic deformation of the Lorentz vacuum will appear soon [14].

Remark 2.7. In electromagnetic theory on a Lorentzian spacetime, we have only one potential $A \in \sec \bigwedge^1 T^*M$ and the field equations are

$$dF = 0, \quad \underset{g}{\delta} F = -J, \tag{2.35}$$

where $F \in \sec \bigwedge^2 T^*M$ is the electromagnetic field and $J \in \sec \bigwedge^1 T^*M$ is the electric current. The two Maxwell equations in (2.35) can be written as a single equation using the Clifford bundle formalism [36]. In this formalism, $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, g)$. Then it can be shown that in this case $\partial = d - \underset{g}{\delta} = \theta^{\mathbf{a}} D_{\mathbf{e_a}}$ is the Dirac operator (acting on sections of $\mathcal{C}\ell(M, g)$) and we can write Maxwell equation as follows:

$$\partial F = J. \tag{2.36}$$

Now, if you fell uncomfortable in needing four distinct potentials $\theta^{\mathbf{a}}$ for describing the gravitational field, you can put then together defining a vector valued differential form:

$$\theta = \theta^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}, \tag{2.37}$$

and in this case, the gravitational field equations are

$$\begin{aligned} d\bar{\Theta} &= 0, \\ \underset{g}{\delta}\bar{\Theta} &= -(\mathcal{T} + \mathbf{t}), \end{aligned} \tag{2.38}$$

where $\bar{\Theta} = \bar{\Theta}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}$, $\mathcal{T} = \mathcal{T}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}$, $\mathbf{t} = \mathbf{t}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}$. By considering the bundle $\mathcal{C}\ell(M, g) \otimes TM$, we can even write the two equations in (2.38) as a single equation:

$$\partial\bar{\Theta} = \mathcal{T} + \mathbf{t}. \tag{2.39}$$

2.3 Relation with Einstein's theory

At this point, the reader may be asking which is the relation of the theory just presented with Einstein's GR theory? The answer is that recalling that the connection 1-forms $\omega^{\mathbf{cd}}$ of D are given by

$$\omega^{\mathbf{cd}} = \frac{1}{2}\Big[\theta^{\mathbf{d}} \underset{g}{\lrcorner} d\theta^{\mathbf{c}} - \theta^{\mathbf{c}} \underset{g}{\lrcorner} d\theta^{\mathbf{d}} + \theta^{\mathbf{c}} \underset{g}{\lrcorner}\Big(\theta^{\mathbf{d}} \underset{g}{\lrcorner} d\theta_{\mathbf{a}}\Big)\theta^{\mathbf{a}}\Big]. \tag{2.40}$$

We can show through a laborious (but standard) exercise (see [36] for details) that the first member of (2.27) is exactly $-\underset{g}{\star}\mathcal{G}_{\mathbf{d}}$ (the Einstein 3-forms). So, we have

$$\underset{g}{\star}\mathcal{G}^{\mathbf{d}} := \underset{g}{\star}\Big(\mathcal{R}^{\mathbf{d}} - \frac{1}{2}R\theta^{\mathbf{d}}\Big) \tag{2.41}$$

[25] Recently, we take knowledge that Coll [5] found that any Lorentzian metric can be written as a deformation of the Minkowski metric involving a 2-form field. We will investigate in another publication the relationship of ours and Coll's ideas.

with $\mathcal{R}^{\mathbf{d}} = R^{\mathbf{d}}_{\mathbf{a}}\theta^{\mathbf{a}}$ the Ricci 1-forms and R the scalar curvature. Then, (2.27) results are equivalent to

$$\mathcal{R}^{\mathbf{d}} - \frac{1}{2}R\theta^{\mathbf{d}} = -T^{\mathbf{d}}, \tag{2.42}$$

and taking the dot product of both members with $\theta_{\mathbf{a}}$, we get

$$R^{\mathbf{d}}_{\mathbf{a}} - \frac{1}{2}R\delta^{\mathbf{d}}_{\mathbf{a}} = -T^{\mathbf{d}}_{\mathbf{a}}, \tag{2.43}$$

which is the usual tensorial form of Einstein's equations.

Remark 2.8. When the $\theta^{\mathbf{a}}$ and the $d\theta_{\mathbf{a}}$ are packed in the form of the connection 1-forms, the Lagrangian density $\mathfrak{L}_g$ becomes

$$\mathfrak{L}_g = \mathcal{L}_{EH} + d\Big(\theta^{\mathbf{a}} \wedge \underset{g}{\star} d\theta_{\mathbf{a}}\Big), \tag{2.44}$$

where

$$\mathcal{L}_{EH} = \frac{1}{2}\mathcal{R}_{\mathbf{cd}} \wedge \underset{g}{\star}\Big(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}\Big) \tag{2.45}$$

(with $\mathcal{R}_{\mathbf{cd}}$ given by (2.21)) is the Einstein-Hilbert Lagrangian density.

3 A comment on Einstein most happy though

The exercises presented above indicate that a geometrical interpretation for the gravitational field is no more than an option among many ones. Indeed, it is not necessary to introduce any connection D or ∇ on M to have a perfectly well-defined theory of the gravitational field whose field equations are (in the sense described above) equivalent to the Einstein field equations. Note that we have not give until now any details on the *global topology* of the world manifold M. However, since we admitted that M carries four global (not all closed) 1-form fields $\theta^{\mathbf{a}}$ which defines the object $\boldsymbol{g}$, it follows that $(M, \boldsymbol{g}, \tau_g, \uparrow)$ is a spin manifold [17, 36], that is, admits spinor fields. This, of course, is necessary if the theory is to be useful in the real world since fundamental matter fields are spinor fields. The most simple spin manifold is clearly Minkowski spacetime which is represented by a structure $(M, \mathring{D}, \eta, \tau_\eta, \uparrow)$, where $M \simeq \mathbb{R}^4$, and $\mathring{D}$ is the Levi-Civita connection of the Minkowski metric η. In that case, it is possible to interpret $\boldsymbol{g}$ as a field in the Faraday sense living in $(M, \mathring{D}, \eta, \tau_\eta, \uparrow)$ or to work directly with the $\theta^{\mathbf{a}}$ which has a well-defined dynamics and coupling to the matter fields.

At last, we want to comment that as well known in Einstein's GR, one can easily distinguish in any *real physical laboratory* [28] (despite some claims on the contrary) a true gravitational field from an acceleration field of a given reference frame in Minkowski spacetime. This is because in GR, the *mark* of a real gravitational field is the nonnull Riemann curvature tensor of D, and the Riemann curvature tensor of the Levi-Civita connection of $\mathring{D}$ (present in the definition of Minkowski spacetime) is null. However, if we interpret a gravitational field as the torsion 2-forms on the structure $(M, \nabla, \boldsymbol{g}, \tau_g, \uparrow)$ viewed as a deformation of Minkowski spacetime, then one can also interpret an acceleration field of an accelerated reference frame in Minkowski spacetime as generating an effective teleparallel spacetime $(M, \overset{e}{\nabla}, \eta, \tau_\eta, \uparrow)$. This can be done as follows. Let $Z \in \sec TU$, $U \subset M$ with $\eta(Z, Z) = 1$ an accelerated reference frame on Minkowski spacetime. This means (see, e.g., [36] for details) that

$$\mathbf{a} = \mathring{D}_Z Z = 0. \tag{3.1}$$

Put $\boldsymbol{e}_0 = Z$ and define an accelerated frame as non trivial if $\vartheta^{\mathbf{0}} = \eta(\boldsymbol{e}_0,)$ is not an exact differential. Next, recall that in $U \subset M$ there always exist three other η-orthonormal vector fields $e_{\mathbf{i}}$, $\mathbf{i} = 1, 2, 3$ such that $\{e_a\}$ is an η-orthonormal basis for TU:

$$\eta = \eta_{\mathbf{ab}} \vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}},$$

where $\{\vartheta^{\mathbf{a}}\}$ is the dual basis[26] of $\{e_a\}$. We then have

$$\mathring{D}_{e_{\mathbf{a}}} e_{\mathbf{b}} = \mathring{\omega}^{\mathbf{c}}_{\mathbf{ab}} e_{\mathbf{c}}, \quad \mathring{D}_{e_{\mathbf{a}}} \vartheta^{\mathbf{b}} = -\mathring{\omega}^{\mathbf{b}}_{\mathbf{ac}} \vartheta^{\mathbf{c}}. \tag{3.2}$$

What remains in order to be possible to interpret an acceleration field as a kind of "gravitational field" is to introduce on M a η-metrical compatible connection $\overset{e}{\nabla}$ such that the $\{e_a\}$ is teleparallel according to it. We have

$$\overset{e}{\nabla}_{e_{\mathbf{a}}} e_{\mathbf{b}} = 0, \quad \overset{e}{\nabla}_{e_{\mathbf{a}}} \vartheta^{\mathbf{b}} = 0. \tag{3.3}$$

With this connection the structure $(M \simeq \mathbb{R}^4, \overset{e}{\nabla}, \eta, \tau_\eta, \uparrow)$ has null Riemann curvature tensor but a nonnull torsion tensor, whose components are related with the components of the acceleration $\boldsymbol{a}$ and with the other coefficients $\mathring{\omega}^{\mathbf{c}}_{\mathbf{ab}}$ of the connection $\mathring{D}$, which describe the motion on Minkowski spacetime of a *grid* represented by the orthonormal frame $\{e_a\}$. Schücking [40] thinks that such a description of the gravitational field makes Einstein has most happy though, that is, the equivalence principle (understood as equivalence between acceleration and gravitational field) a legitimate mathematical idea. However, a *true* gravitational field must satisfy (at least with good approximation) (2.30), whereas there is no single reason for an acceleration field to satisfy that equation.

4 A model for the gravitational field represented by the non-metricity of a connection

In this section, we suppose that the world manifold M is a 4-dimensional manifold diffeomorphic to $\mathbb{R}^4$. Let, moreover, $(t, x, y, z) = (x^0, x^1, x^2, x^3)$ be global Cartesian coordinates for M.

Next, introduce on M two metric fields:

$$\eta = dt \otimes dt - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3, \tag{4.1}$$

$$\begin{aligned}\mathbf{g} = &\left(1 - \frac{2m}{r}\right) dt \otimes dt \\ &- \left\{\left(1 - \frac{2m}{r}\right)^{-1} - 1\right\} r^{-2} \left[(x^1)^2 dx^1 \otimes dx^1 + (x^2)^2 dx^2 \otimes dx^2 + (x^3)^2 dx^3 \otimes dx^3\right] \\ &- dx^1 \otimes dx^1 + \left(1 - \frac{2m}{r}\right)^{-1} \left(dx^2 \otimes dx^2 - dx^3 \otimes dx^3\right).\end{aligned} \tag{4.2}$$

[26] In general, we will also have that $d\vartheta^{\mathbf{i}} \neq 0$, $\mathbf{i} = 1, 2, 3$.

In (4.2),

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{4.3}$$

Now, introduce $(t, r, \vartheta, \varphi) = (x^0, x'^1, x'^2, x'^3)$ as the usual spherical coordinates for M. Recall that

$$x^1 = r \sin\vartheta \cos\varphi, \quad x^2 = r \sin\vartheta \sin\varphi, \quad x^3 = r\cos\vartheta, \tag{4.4}$$

and the range of these coordinates in η are $r > 0$, $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$. For g, the range of the r variable must be $(0, 2m) \cup (2m, \infty)$.

As can be easily verified, the metric $\boldsymbol{g}$ in spherical coordinates is

$$\boldsymbol{g} = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2 \big(d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi\big), \tag{4.5}$$

which we immediately recognize as the *Schwarzschild metric* of GR. Of course, η is a Minkowski metric on M.

As next step, we introduce two *distinct* connections, $\mathring{D}$ and D on M. We assume that $\mathring{D}$ is the Levi-Civita connection of η in M, and D is the Levi-Civita connection of $\boldsymbol{g}$ in M. Then, by definition (see, e.g., [36] for more details), the nonmetricities tensors of $\mathring{D}$ relative to η and of D relative to $\boldsymbol{g}$ are null:

$$\mathring{D}\eta = 0, \quad D\boldsymbol{g} = 0. \tag{4.6}$$

However, the nonmetricity tensor $\mathbf{A}^\eta \in \sec T_3^0 M$ of $\mathring{D}$ relative to $\boldsymbol{g}$ is nonnull:

$$\mathring{D}\boldsymbol{g} = \mathbf{A}^\eta \neq 0, \tag{4.7}$$

and also the nonmetricity tensor $\mathbf{A}^g \in \sec T_3^0 M$ of D relative to η is nonnull:

$$D\eta = \mathbf{A}^g \neq 0. \tag{4.8}$$

We now calculate the components of $\mathbf{A}^\eta$ in the coordinated bases $\{\boldsymbol{\partial}_\mu\}$ for TM and $\{dx^\nu\}$ for T^*M associated with the coordinates (x^0, x^1, x^2, x^3) of M. Since $\mathring{D}$ is the Levi-Civita connection of the Minkowski metric η, we have

$$\mathring{D}_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = L^\rho_{\mu\nu} \boldsymbol{\partial}_\rho = 0, \quad \mathring{D}_{\boldsymbol{\partial}_\mu} dx^\alpha = -L^\alpha_{\mu\nu} dx^\nu = 0, \tag{4.9}$$

that is, the connection coefficients $L^\rho_{\mu\nu}$ of $\mathring{D}$ in this basis are *null.* Then, $\mathbf{A}^\eta = Q_{\mu\alpha\beta} dx^\alpha \otimes dx^\beta \otimes dx^\mu$ is given by

$$\mathbf{A}^\eta = \mathring{D}\boldsymbol{g} = \mathring{D}_{\boldsymbol{\partial}_\mu} \big(g_{\alpha\beta} dx^\alpha \otimes dx^\beta\big) \otimes dx^\mu = \left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) dx^\alpha \otimes dx^\beta \otimes dx^\mu. \tag{4.10}$$

To fix ideas, recall that for Q_{100} it is,

$$\begin{aligned} Q_{100} &= \frac{\partial}{\partial x^1}\left(1 - \frac{2m}{\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}}\right) \\ &= -2m \frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}}\right) \\ &= \frac{2mx^1}{\left[\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right]^{\frac{3}{2}}} = \frac{2mx^1}{r^3}, \end{aligned} \tag{4.11}$$

which is *nonnull* for $x^1 \neq 0$. Note also that $Q_{010} = Q_{001} = 0$.

4.1 $(M, \eta, \mathring{D})$, (M, η, D), $(M, \boldsymbol{g}, D)$, and $(M, \boldsymbol{g}, \mathring{D})$

From what has been said, it is obvious that since (M, η) and $(M, \boldsymbol{g})$ are both orientable and time orientable, then $(M, \eta, \mathring{D})$, $(M, \boldsymbol{g}, D)$ are part of the structures representing, respectively, Minkowski spacetime and Schwarzschild spacetime. More precisely, $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ represents in GR the gravitational field of a point mass with world line given by $(t, 0, 0, 0)$. As usual in GR, this world line is left out of the effective manifold[27].

We claim that $(M, \boldsymbol{g}, \mathring{D})$ or (M, η, D) are legitimate equivalent representations for the gravitational field described in GR by the substructure $(M, \boldsymbol{g}, D)$. To find, for example, the relation between the models $(M, \boldsymbol{g}, \mathring{D})$ and $(M, \boldsymbol{g}, D)$, it is necessary to recall that if in the bases $\{\partial_\mu\}$ for TM and $\{dx^\nu\}$ for T^*M, we have

$$D_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = \Gamma^\rho_{\mu\nu} \boldsymbol{\partial}_\rho, \quad D_{\boldsymbol{\partial}_\mu} dx^\alpha = -\Gamma^\alpha_{\mu\nu} dx^\nu, \tag{4.12}$$

and the Christoffel symbols are *not* all null. Moreover, in the spherical coordinates introduced above,

$$\begin{aligned} \mathring{D}_{\boldsymbol{\partial}'_\mu} \boldsymbol{\partial}'_\nu &= L'^\rho_{\mu\nu} \boldsymbol{\partial}'_\rho, \quad \mathring{D}_{\partial'_\mu} dx'^\alpha = -L'^\alpha_{\mu\nu} dx'^\nu, \\ D_{\boldsymbol{\partial}'_\mu} \boldsymbol{\partial}'_\nu &= \Gamma'^\rho_{\mu\nu} \partial'_\rho, \quad D_{\partial'_\mu} dx'^\alpha = -\Gamma'^\alpha_{\mu\nu} dx'^\nu, \end{aligned} \tag{4.13}$$

and the $L'^\rho_{\mu\nu}$ and $\Gamma'^\rho_{\mu\nu}$ are *not* all null. Now, $L'^\rho_{\mu\nu}$ and $\Gamma'^\rho_{\mu\nu}$ are related by[28]

$$L'^\rho_{\alpha\beta} = \Gamma'^\rho_{\alpha\beta} + \frac{1}{2} S'^\rho_{\alpha\beta}, \tag{4.14}$$

where $S'^\rho_{\alpha\beta}$ are the components of the so-called *strain tensor of the connection* $\mathring{D}$ relative to the connection D. For the present case, it is

$$S'^\rho_{\alpha\beta} = g'^{\rho\sigma} \left(Q'_{\alpha\beta\sigma} + Q'_{\beta\sigma\alpha} - Q'_{\sigma\alpha\beta} \right). \tag{4.15}$$

Now, since in the Cartesian coordinates $L^\rho_{\alpha\beta} = 0$, but not all $\Gamma^\rho_{\alpha\beta}$ are null, we get

$$\Gamma^\rho_{\alpha\beta} = -\frac{1}{2} S^\rho_{\alpha\beta}, \tag{4.16}$$

and thus, for example,

$$g_{1\rho} \Gamma^\rho_{00} = -\frac{1}{2} S_{100} = \frac{1}{2} Q_{100} = \frac{m x^1}{r^3}. \tag{4.17}$$

4.2 $\mathbf{A}^\eta$ as the gravitational field

Note that using coordinates (Riemann normal coordinates $\{\xi^\mu\}$ covering $V \subset U \subset M$) naturally adapted to a reference frame $Z \in \sec TV$[29] in free fall according to GR ($D_Z Z = 0$, $d\alpha \wedge \alpha = 0$, $\alpha = g(Z,))$, it is possible to put the connection coefficients of the Levi-Civita

[27]The manifold where Schwarzschild solution is obtained is one with *boundary*, that is, it is $\mathbb{R} \times [0, \infty) \times S^2$. The reason for that is that almost all mathematical physicists use manifolds with *boundary* in order to avoid the use of distributions (generalized functions). Indeed, for a rigorous point of view, taking into account that Einstein's equations are nonlinear we cannot solve it using Schwartz distributions. To solve problems involving singular distributions in GR in a rigorous way, it is necessary to use Colombeau theory of generalized functions as described, for example, in [20].

[28]See, for example, [36, Section 4.5.8].

[29]For the mathematical definitions of reference frames, naturally adapted coordinates to a reference frame and observers, see, for example, [36, Chapter 5].

connection D of $\boldsymbol{g}$ equal to zero in all points of the world line of a free fall observer (an *observer* is here *modeled* as an integral line σ of a reference frame Z, where Z is a time like vector field pointing to the future such that $Z|_{\sigma} = \sigma_*$).

In the Riemann normal coordinates system covering $U \subset M$, it is obvious that not all the connection coefficients of the connection $\mathring{D}$ (that relative to $\boldsymbol{g}$ is a nonmetrical one) are null. Moreover, the nonmetricity tensor $\mathbf{A}^{\eta}$ is not null and it represents in our model the true gravitational field. Indeed, an observer following σ does not fell any force along its world line because the gravitational force represented by the nonmetricity field $\mathbf{A}^{\eta}$ is compensated by an inertial force represented by the nonnull connection coefficients[30] $L''^{\rho}_{\mu\nu}$ of $\mathring{D}$ in the basis $\{\frac{\partial}{\partial \xi^{\nu}}\}$.

The situation is somewhat analogous to what happens in any noninertial reference frame which, of course, may be conveniently used in *any* special relativity problem (as, e.g., in a rotating disc [37]), where the connection coefficients of the Levi-Civita connection of η are not all null.

Remark 4.1. The theoretical definition of standard clocks of GR is reasonably well realized by atomic clocks, that is, under certain limits atomic clocks behave as theoretically predicted (see however [35]). Note however that atomic clocks are *not* the standard clocks of the model proposed here. We would say that the gravitational field *distorts* the period of the atomic clocks relative to the standard clocks of the proposed model, where gravity is represented by a nonmetricity tensor[31], but, who are the devices that now materialize those concepts? Well, they are *paper concepts*, like the notion of time in some Newtonian theories. They are defined and calculated in order to make *correct* predictions. However, given the status of present technology, we can easily imagine how to build devices for directly realizing the standard clocks (and rulers) of the proposed model.

5 Conclusions

In this paper, we recalled two important results. The first is that a gravitational field generated by a given energy-momentum distribution can be represented by distinct geometrical structures (Lorentzian, teleparallel, and nonnull nonmetricity spacetimes). The second important result is that we can even dispense all those geometrical structures and simply represent the gravitational field as a field in Faraday's sense living in Minkowski spacetime. The explicit Lagrangian density for this theory has been discussed, and the field equations have been shown to be equivalent to Einstein's equations. We hope that our study clarifies the real difference between mathematical models and physical reality and leads people to think about the real physical nature of the gravitational field (and also of the electromagnetic field[32])

As a final remark, we want to leave clear that after studying Einstein's papers (and also papers by many others authors) on the use of Riemann-Cartan[33] to describe a classical unified

[30]The explicit form of the coefficients $L''^{\rho}_{\mu\nu}$ may be found in [36, Chapter 5].

[31]Schwinger [41] showed with very simple arguments how the gravitational field *distorts* the period of atomic clocks making then to register the proper time predicted by GR. His arguments can be easily adapted for the alternative models studied in this paper because once $\boldsymbol{g}$ is known experimentally, we can determine η with the mathematical techniques described in [36].

[32]As suggested, for example, by the works of Laughlin [24] and Volikov [52]. Of course, it may be necessary to explore also other ideas, like, for example, existence of branes in string theory, but this is a subject for another publication.

[33]The teleparallel spaces are particular cases of the Riemann-Cartan ones. More on the classification of spacetime geometries may be found in [36].

theory of gravitation and electromagnetism, we became convinced that it seems impossible to represent the electromagnetic field using a contraction of the torsion tensor (or the torsion tensor) without introducing ad hoc hypothesis. Having said that, we recall that from time to time some authors return to the embryo of Einstein's original idea claiming to have obtained a unified theory of gravitation and electromagnetism using that tool. Among those theories that appeared in the last few years, some are completely worthless, since based in a very bad use of mathematical concepts, but some look at least at a first sight interesting enough (at least from the mathematical point of view) to deserve some comments, which will be discussed elsewhere[34].

A The Levi-Civita and the Nunes connections on $\mathring{S}^2$

Consider S^2, an sphere of radius $\mathfrak{R} = 1$ embedded in $\mathbb{R}^3$. Let $(x^1, x^2) = (\vartheta, \varphi)$ $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, be the standard spherical coordinates of S^2, which covers all the open set U which is S^2 with the *exclusion* of a semi-circle uniting the *north* and south *poles*.

Introduce the *coordinate bases*:

$$\{\boldsymbol{\partial}_\mu\}, \quad \{\theta^\mu = dx^\mu\} \tag{A.1}$$

for TU and T^*U. Next, introduce the *orthonormal bases* $\{\mathbf{e_a}\}$, $\{\theta^{\mathbf{a}}\}$ for TU and T^*U with

$$\mathbf{e_1} = \boldsymbol{\partial}_1, \quad \mathbf{e_2} = \frac{1}{\sin x^1}\boldsymbol{\partial}_2, \tag{A.2a}$$

$$\theta^{\mathbf{1}} = dx^1, \quad \theta^{\mathbf{2}} = \sin x^1 dx^2. \tag{A.2b}$$

Then,

$$[\mathbf{e_i}, \mathbf{e_j}] = c_{\mathbf{ij}}^{\mathbf{k}}\mathbf{e_k}, \quad c_{\mathbf{12}}^{\mathbf{2}} = -c_{\mathbf{21}}^{\mathbf{2}} = -\cot x^{\mathbf{1}}. \tag{A.3}$$

Moreover, the metric $\boldsymbol{g} \in \sec T_2^0 S^2$ inherited form the ambient Euclidean metric is

$$\boldsymbol{g} = dx^1 \otimes dx^1 + \sin^2 x^1 dx^2 \otimes dx^2 = \theta^{\mathbf{1}} \otimes \theta^{\mathbf{1}} + \theta^{\mathbf{2}} \otimes \theta^{\mathbf{2}}. \tag{A.4}$$

The Levi-Civita connection D of $\boldsymbol{g}$ has the following nonnull connections coefficients $\Gamma^\rho_{\mu\nu}$ in the coordinate basis (just introduced):

$$\begin{aligned} &D_{\boldsymbol{\partial}_\mu}\boldsymbol{\partial}_\nu = \Gamma^\rho_{\mu\nu}\boldsymbol{\partial}_\rho, \quad \Gamma^2_{21} = \Gamma^\varphi_{\theta\varphi} = \Gamma^2_{12} = \Gamma^\varphi_{\varphi\theta} = \cot\vartheta, \\ &\Gamma^1_{22} = \Gamma^\vartheta_{\varphi\varphi} = -\cos\vartheta\sin\vartheta. \end{aligned} \tag{A.5}$$

Also, in the basis $\{\mathbf{e_a}\}$, $D_{\mathbf{e_i}}\mathbf{e_j} = \omega_{\mathbf{ij}}^{\mathbf{k}}\mathbf{e_k}$ and the nonnull coefficients are

$$\omega_{\mathbf{21}}^{\mathbf{2}} = \cot\vartheta, \quad \omega_{\mathbf{22}}^{\mathbf{1}} = -\cot\vartheta. \tag{A.6}$$

[34] We have in mind here: (a) some papers by Vargas and Vargas and Torr, [47, 48, 49, 50], where they claim that by using the torsion tensor of some special Finsler connections it is possible to obtain a unified theory of gravitation and electromagnetism (for related papers on the subject by those authors, please consult http://cartan-einstein-unification.com/published-papers.html); (b) a paper by Unzicker, where he claims to have found a description of electromagnetism including the existence of quantized charges using teleparallel spacetimes with defects [45, 46].

The torsion and the (Riemann) curvature tensors of D (recall (2.12) and (2.13)) are

$$\begin{aligned}
&\mathcal{T}(\theta^{\mathbf{k}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{k}}(\tau(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}})) = \theta^{\mathbf{k}}(D_{\boldsymbol{e}_{\mathbf{j}}}\boldsymbol{e}_{\mathbf{i}} - D_{\boldsymbol{e}_{\mathbf{i}}}\boldsymbol{e}_{\mathbf{j}} - [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}]),\\
&\mathbf{R}(\boldsymbol{e}_{\mathbf{k}}, \theta^{\mathbf{a}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{a}}([D_{\boldsymbol{e}_{\mathbf{i}}}D_{\boldsymbol{e}_{\mathbf{j}}} - D_{\boldsymbol{e}_{\mathbf{j}}}D_{\boldsymbol{e}_{\mathbf{i}}} - D_{[\boldsymbol{e}_{\mathbf{i}},\boldsymbol{e}_{\mathbf{j}}]}]\boldsymbol{e}_{\mathbf{k}}),
\end{aligned} \tag{A.7}$$

which result in $\mathcal{T} = 0$ and that the nonnull components of $\mathbf{R}$ are $R_{\mathbf{1}\ \mathbf{21}}^{\mathbf{1}} = -R_{\mathbf{1}\ \mathbf{12}}^{\mathbf{1}} = R_{\mathbf{1}\ \mathbf{12}}^{\mathbf{2}} = -R_{\mathbf{1}\ \mathbf{12}}^{\mathbf{2}} = -1$.

Since the Riemann curvature tensor is nonnull, the parallel transport of a given vector depends on the path to be followed. We say that a vector (say $\mathbf{v}_0$) is parallel transported along a generic path $\mathbb{R} \supset I \mapsto \gamma(s) \in \mathbb{R}^3$ (say, from $A = \gamma(0)$ to $B = \gamma(1)$) with tangent vector $\gamma_*(s)$ (at $\gamma(s)$) if it determines a vector field $\mathbf{V}$ along γ satisfying

$$D_{\gamma_*}\mathbf{V} = 0, \tag{A.8}$$

and such that $\mathbf{V}(\gamma(0)) = \mathbf{v}_0$. When the path is a geodesic[35] of the connection D, that is, a curve $\mathbb{R} \supset I \mapsto c(s) \in \mathbb{R}^3$ with tangent vector $c_*(s)$ (at $c(s)$) satisfying

$$D_{c_*}c_* = 0, \tag{A.9}$$

the parallel transported vector along a c forms a constant angle with c_*. Indeed, from (A.8), it is $\gamma_* \underset{\boldsymbol{g}}{\cdot} D_{\gamma_*}\mathbf{V} = 0$. Then taking into account (A.9), it follows that

$$D_{\gamma_*}(\gamma_* \underset{\boldsymbol{g}}{\cdot} \mathbf{V}) = 0,$$

that is, $\gamma_* \underset{\boldsymbol{g}}{\cdot} \mathbf{V} =$ *constant*. This is clearly illustrated in Figure 1 (from [1]).

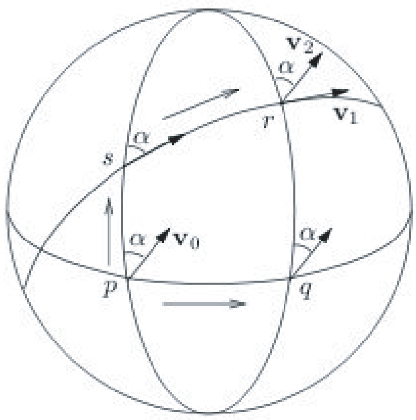

Figure 1. Levi-Civita and Nunes transport of a vector $\mathbf{v}_0$ starting at p through the paths psr and pqr. Levi-Civita transport through psr leads to $\mathbf{v}_1$, whereas Nunes transport leads to $\mathbf{v}_2$. Along pqr, both Levi-Civita and Nunes transport agree and lead to $\mathbf{v}_2$.

Consider next the manifold $\mathring{S}^2 = \{S^2 \backslash \text{north pole} + \text{south pole}\} \subset \mathbb{R}^3$, which is our sphere of radius $\mathfrak{R} = 1$ but this time *excluding* the north and south poles. Let again $\boldsymbol{g} \in \sec T_2^0 \mathring{S}^2$ be

[35] We recall that a geodesic of D also determines the minimal distance (as given by the metric $\boldsymbol{g}$) between any two points on S^2.

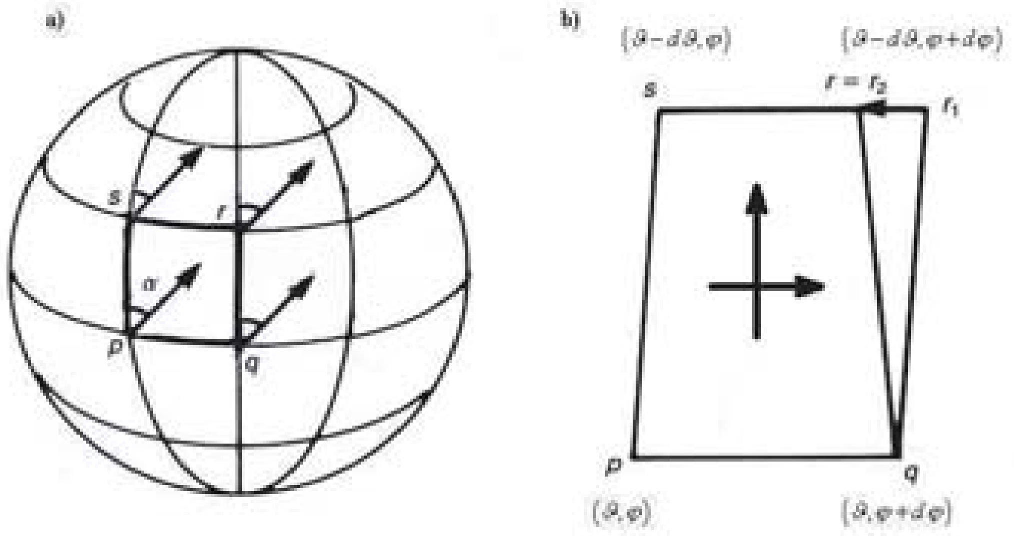

Figure 2. Characterization of the Nunes connection.

the metric field on $\mathring{S}^2$ inherited from the ambient space $\mathbb{R}^3$ and introduce on $\mathring{S}^2$ the Nunes (or navigator) connection[36] ∇ defined by the following parallel transport rule: a vector at an arbitrary point of $\mathring{S}^2$ is parallel transported along a curve γ, if it determines a vector field on γ such that at any point of γ the angle between the transported vector and the vector tangent to the latitude line passing through that point is constant during the transport. This is clearly illustrated in Figure 2. and to distinguish the Nunes transport from the Levi-Civita transport, we ask also for the reader to study with attention the caption of Figure 1.

We recall that from the calculation of the Riemann tensor $\mathbf{R}$, it follows that the structures $(\mathring{S}^2, \boldsymbol{g}, D, \tau_g)$ and also $(S^2, \boldsymbol{g}, D, \tau_g)$ are Riemann spaces of *constant curvature.* We now show that the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla, \tau_g)$ is a *teleparallel space*[37] with zero Riemann curvature tensor, but nonzero torsion tensor.

Indeed, from Figure 2, it is clear that (a) if a vector is transported along the *infinitesimal* quadrilateral $pqrs$ composed of latitudes and longitudes, first starting from p along pqr and then starting from p along psr, the parallel transported vectors that result in both cases will coincide (study also the caption of Figure 1).

[36]Pedro Salacience Nunes (1502–1578) was one of the leading mathematicians and cosmographers of Portugal during the Age of Discoveries. He is well known for his studies in cosmography, spherical geometry, astronomic navigation, and algebra, and particularly known for his discovery of loxodromic curves and the nonius. Loxodromic curves, also called rhumb lines, are spirals that converge to the poles. They are lines that maintain a fixed angle with the meridians. In other words, loxodromic curves directly related to the construction of the Nunes connection. A ship following a fixed compass direction travels along a loxodromic, this being the reason why Nunes connection is also known as navigator connection. Nunes discovered the loxodromic lines and advocated the drawing of maps in which loxodromic spirals would appear as straight lines. This led to the celebrated Mercator projection, constructed along these recommendations. Nunes invented also the Nonius scales which allow a more precise reading of the height of stars on a quadrant. The device was used and perfected at the time by several people, including Tycho Brahe, Jacob Kurtz, Christopher Clavius, and further by Pierre Vernier who in 1630 constructed a practical device for navigation. For some centuries, this device was called nonius. During the 19th century, many countries, most notably France, started to call it vernier. More details in http://www.mlahanas.de/Stamps/Data/Mathematician/N.htm.

[37]As recalled in Section 1, a teleparallel manifold M is characterized by the existence of global vector fields which is a basis for T_xM for any $x \in M$. The reason for considering $\mathring{S}^2$ for introducing the Nunes connection is that as well known (see, e.g., [8]) S^2 does not admit a continuous vector field that is nonnull at on points of it.

Now, the vector fields $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ in (A.2a) define a basis for each point p of $T_p\mathring{S}^2$, and ∇ is clearly characterized by

$$\nabla_{\boldsymbol{e_j}}\boldsymbol{e_i} = 0. \tag{A.10}$$

The components of curvature operator are

$$\bar{\mathbf{R}}(\mathbf{e_k}, \theta^{\mathbf{a}}, \mathbf{e_i}, \mathbf{e_j}) = \theta^{\mathbf{a}}([\nabla_{\mathbf{e_i}}\nabla_{\mathbf{e_j}} - \nabla_{\mathbf{e_j}}\nabla_{\mathbf{e_i}} - \nabla_{[\mathbf{e_i},\mathbf{e_j}]}]\mathbf{e_k}) = 0, \tag{A.11}$$

and the torsion operation (recall (2.10)) $\bar{\tau}$ is

$$\bar{\tau}(\mathbf{e_i}, \mathbf{e_j}) = \nabla_{\mathbf{e_j}}\mathbf{e_i} - \nabla_{\mathbf{e_i}}\mathbf{e_j} - [\mathbf{e_i}, \mathbf{e_j}] = [\mathbf{e_i}, \mathbf{e_j}], \tag{A.12}$$

which gives for the components of the torsion tensor, $\bar{T}^{\mathbf{2}}_{\mathbf{12}} = -\bar{T}^{\mathbf{2}}_{\mathbf{12}} = \cot\vartheta$. It follows that $\mathring{S}^2$, considered as part of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}})$, is *flat* (but has torsion).

If you still need more details to grasp this last result, consider Figure 2(b) which shows the standard parametrization of the points p, q, r, s in terms of the spherical coordinates introduced above. According to the geometrical meaning of torsion, its value at a given point is determined by calculating the difference between the (infinitesimal)[38] vectors $pr_1$and pr_2. If the vector pq is transported along ps, one gets (recalling that $\mathfrak{R} = 1$) the vector $\mathbf{v} = sr_1$ such that $|\mathbf{g}(\mathbf{v}, \mathbf{v})|^{\frac{1}{2}} = \sin\vartheta\triangle\varphi$. Moreover, if the vector ps is transported along pq, one gets the vector $qr_2 = qr$. Let $\mathbf{w} = sr$. Then,

$$\left|\mathbf{g}(\mathbf{w}, \mathbf{w})\right| = \sin(\vartheta - \triangle\vartheta)\triangle\varphi \simeq \sin\vartheta\triangle\varphi - \cos\vartheta\triangle\vartheta\triangle\varphi, \tag{A.13}$$

also,

$$\mathbf{u} = r_1r_2 = -u\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right), \quad u = \left|\mathbf{g}(\mathbf{u}, \mathbf{u})\right| = \cos\vartheta\triangle\vartheta\triangle\varphi. \tag{A.14}$$

Then, the connection ∇ of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}})$ has a nonnull torsion tensor $\bar{\mathcal{T}}$. Indeed, the component of $\mathbf{u} = r_1r_2$ in the direction $\partial/\partial\varphi$ is precisely $\bar{T}^{\varphi}_{\vartheta\varphi}\triangle\vartheta\triangle\varphi$. So, one gets (recalling that $\nabla_{\partial j}\partial_i = \Gamma^k_{ji}\partial_k$)

$$\bar{T}^{\varphi}_{\vartheta\varphi} = \left(\Gamma^{\varphi}_{\vartheta\varphi} - \Gamma^{\varphi}_{\varphi\vartheta}\right) = -\cot\theta. \tag{A.15}$$

To end this appendix, it is worth to show that ∇ is metrical compatible, that is $\nabla\boldsymbol{g} = 0$. Indeed, we have

$$0 = \nabla_{\mathbf{e_c}}\boldsymbol{g}(\mathbf{e_i}, \mathbf{e_j}) = (\nabla_{\mathbf{e_c}}\boldsymbol{g})(\mathbf{e_i}, \mathbf{e_j}) + \boldsymbol{g}(\nabla_{\mathbf{e_c}}\mathbf{e_i}, \mathbf{e_j}) + \boldsymbol{g}(\mathbf{e_i}, \nabla_{\mathbf{e_c}}\mathbf{e_j}) = (\nabla_{\mathbf{e_c}}\boldsymbol{g})(\mathbf{e_i}, \mathbf{e_j}). \tag{A.16}$$

Remark A.1. This appendix shows clearly that we cannot *mislead* the Riemann curvature tensor of a connection with the fact that the manifold where that connection is defined may be bend[39] as a surface in an Euclidean manifold where it is embedded. Neglecting this fact may generate a lot of *wishful* thinking when one comes to the interpretation of curvature (and torsion) in gravitational theories.

[38] This wording, of course, means that those vectors are identified as elements of the appropriate tangent spaces.

[39] Bending of surfaces embedded in $\mathbb{R}^3$ is adequately characterized by the so called shape operator discussed, e.g., in[29]. For the case of hypersurfaces (vector manifolds) embedded in $\mathbb{R}^n$ see [22].

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