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Research Article

The Generalized Burnside Theorem in Noncommutative Deformation Theory*

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Abstract Let A be an associative algebra over a field k, and let $\mathcal{M}$ be a finite family of right A-modules. A study of the noncommutative deformation functor $\mathrm{Def}_{\mathcal{M}}$ of the family $\mathcal{M}$ leads to the construction of the algebra $\mathcal{O}^A(\mathcal{M})$ of observables and the generalized Burnside theorem, due to Laudal (2002). In this paper, we give an overview of aspects of noncommutative deformations closely connected to the generalized Burnside theorem.

MSC 2010: 13D10, 14D15

1 Introduction

Let k be a field and let A be an associative k-algebra. For any right A-module M, there is a commutative deformation functor $\mathrm{Def}_M : \mathsf{l} \to \mathsf{Sets}$ defined on the category l of local Artinian commutative k-algebras with residue field k. We recall that for an algebra R in l, a deformation of M to R is a pair (M_R, τ), where M_R is an R-A bimodule (on which k acts centrally) that is R-flat, and $\tau : k \otimes_R M_R \to M$ is an isomorphism of right A-modules.

Let a_r be the category of r-pointed Artinian k-algebras for $r \geq 1$, the natural noncommutative generalization of l. We recall that an algebra R in a_r is an Artinian ring, together with a pair of structural ring homomorphisms $f : k^r \to R$ and $g : R \to k^r$ with $g \circ f = \mathrm{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. Any algebra R in a_r has r simple right modules of dimension one, the natural projections $\{k_1, \dots, k_r\}$ of k^r.

In [2], a noncommutative deformation functor $\mathrm{Def}_{\mathcal{M}} : \mathsf{a}_r \to \mathsf{Sets}$ of a finite family $\mathcal{M} = \{M_1, \dots, M_r\}$ of right A-modules was introduced, as a generalization of the commutative deformation functor $\mathrm{Def}_M : \mathsf{l} \to \mathsf{Sets}$ of a right A-module M. In the case $r = 1$, this generalization is completely natural, and can be defined word for word as in the commutative case. The generalization to the case $r > 1$ is less obvious and has further-reaching consequences, but is still very natural. A deformation of $\mathcal{M}$ to R is defined to be a pair $(M_R, \{\tau_i\}_{1 \leq i \leq r})$, where M_R is an R-A bimodule (on which k acts centrally) that is R-flat, and $\tau_i : k_i \otimes_R M_R \to M_i$ is an isomorphism of right A-modules for $1 \leq i \leq r$. We remark that M_R is R-flat if and only if

$$M_R \cong \big(R_{ij} \otimes_k M_j\big) = \begin{pmatrix} R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \cdots & R_{1r} \otimes_k M_r \\ R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \cdots & R_{2r} \otimes_k M_r \\ \vdots & \vdots & \ddots & \cdots \\ R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \cdots & R_{rr} \otimes_k M_r \end{pmatrix},$$

considered as a left R-module, and that a deformation in $\mathrm{Def}_{\mathcal{M}}(R)$ may be thought of as a right multiplication $A \to \mathrm{End}_R(M_R)$ of A on the left R-module M_R that lifts the multiplication $\rho : A \to \oplus_i \mathrm{End}_k(M_i)$ of A on the family $\mathcal{M}$.

There is an obstruction theory for $\mathrm{Def}_{\mathcal{M}}$, generalizing the obstruction theory for the commutative deformation functor. Hence there exists a formal moduli (H, M_H) for $\mathrm{Def}_{\mathcal{M}}$ (assuming a mild condition on $\mathcal{M}$). We consider the *algebra of observables* $\mathcal{O}^A(\mathcal{M}) = \mathrm{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \mathrm{Hom}_k(M_i, M_j))$ and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & \mathcal{O}^A(\mathcal{M}) \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\pi} \\ & & \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i) \end{array}$$

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given by the versal family $M_H \in \mathrm{Def}_{\mathcal{M}}(H)$. The algebra $B = \mathcal{O}^A(\mathcal{M})$ has an induced right action on the family $\mathcal{M}$ extending the action of A, and we may consider $\mathcal{M}$ as a family of right B-modules. In fact, $\mathcal{M}$ is the family of simple B-modules since π can be identified with the quotient morphism $B \to B/\operatorname{rad} B$.

When A is an algebra of finite dimension over an algebraically closed field k and $\mathcal{M}$ is the family of simple right A-modules, Laudal proved the *generalized Burnside theorem* in [2], generalizing the structure theorem for semi-simple algebras and the classical Burnside theorem. Laudal's result is stated in the following form.

Theorem (The generalized Burnside theorem)**.** *Let A be a finite-dimensional algebra over a field k, and let $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ be the family of simple right A-modules. If $\mathrm{End}_A(M_i) = k$ for $1 \le i \le r$, then $\eta : A \to \mathcal{O}^A(\mathcal{M})$ is an isomorphism. In particular, η is an isomorphism when k is algebraically closed.*

Let A be an algebra of finite dimension over an algebraically closed field k and let $\mathcal{M}$ be any finite family of right A-modules of finite dimension over k. Then the algebra $B = \mathcal{O}^A(\mathcal{M})$ has the property that $\eta_B : B \to \mathcal{O}^B(\mathcal{M})$ is an isomorphism, or equivalently, that the assignment $(A, \mathcal{M}) \mapsto (B, \mathcal{M})$ is a closure operation. This means that the family $\mathcal{M}$ has exactly the same module-theoretic properties, in terms of (higher) extensions and Massey products, considered as a family of modules over B as over A.

2 Noncommutative deformations of modules

Let k be a field. For any integer $r \ge 1$, we consider the category a_r of r-pointed Artinian k-algebras. We recall that an object in a_r is an Artinian ring R, together with a pair of structural ring homomorphisms $f : k^r \to R$ and $g : R \to k^r$ with $g \circ f = \mathrm{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of a_r are the ring homomorphisms that commute with the structural morphisms. It follows from this definition that $I(R)$ is the Jacobson radical of R, and therefore that the simple right R-modules are the projections $\{k_1, \dots, k_r\}$ of k^r.

Let A be an associative k-algebra. For any family $\mathcal{M} = \{M_1, \dots, M_r\}$ of right A-modules, there is a noncommutative deformation functor $\mathrm{Def}_{\mathcal{M}} : \mathrm{a}_r \to \mathrm{Sets}$, introduced by Laudal [2]; see also Eriksen [1]. For an algebra R in a_r, we recall that a deformation of $\mathcal{M}$ over R is a pair $(M_R, \{\tau_i\}_{1 \le i \le r})$, where M_R is an R-A bimodule (on which k acts centrally) that is R-flat, and $\tau_i : k_i \otimes_R M_R \to M_i$ is an isomorphism of right A-modules for $1 \le i \le r$. Moreover, $(M_R, \{\tau_i\}) \sim (M'_R, \{\tau'_i\})$ are equivalent deformations over R if there is an isomorphism $\eta : M_R \to M'_R$ of R-A bimodules such that $\tau_i = \tau'_i \circ (1 \otimes \eta)$ for $1 \le i \le r$. We may prove that M_R is R-flat if and only if

$$M_R \cong \left(R_{ij} \otimes_k M_j\right) = \begin{pmatrix} R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \cdots & R_{1r} \otimes_k M_r \\ R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \cdots & R_{2r} \otimes_k M_r \\ \vdots & \vdots & \ddots & \cdots \\ R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \cdots & R_{rr} \otimes_k M_r \end{pmatrix},$$

considered as a left R-module, and a deformation in $\mathrm{Def}_{\mathcal{M}}(R)$ may be thought of as a right multiplication $A \to \mathrm{End}_R(M_R)$ of A on the left R-module M_R that lifts the multiplication $\rho : A \to \oplus_i \mathrm{End}_k(M_i)$ of A on the family $\mathcal{M}$.

Let us assume that $\mathcal{M}$ is a *swarm*, that is, $\mathrm{Ext}^1_A(M_i, M_j)$ has finite dimension over k for $1 \le i, j \le r$. Then $\mathrm{Def}_{\mathcal{M}}$ has a pro-representing hull or a formal moduli (H, M_H); see Laudal [2, Theorem 3.1]. This means that H is a complete r-pointed k-algebra in the pro-category $\hat{\mathrm{a}}_r$, and that $M_H \in \mathrm{Def}_{\mathcal{M}}(H)$ is a family defined over H with the following versal property: for any algebra R in a_r and any deformation $M_R \in \mathrm{Def}_{\mathcal{M}}(R)$, there is a homomorphism $\phi : H \to R$ such that $\mathrm{Def}_{\mathcal{M}}(\phi)(M_H) = M_R$. The formal moduli (H, M_H) is unique up to non-canonical isomorphism. However, the morphism ϕ is not uniquely determined by (R, M_R).

When $\mathcal{M}$ is a swarm with formal moduli (H, M_H), right multiplication on the H-A bimodule M_H by elements in A determines an algebra homomorphism

$$\eta : A \longrightarrow \mathrm{End}_H(M_H).$$

We write $\mathcal{O}^A(\mathcal{M}) = \mathrm{End}_H(M_H)$ and call it the *algebra of observables*. Since M_H is H-flat, we have that $\mathrm{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \mathrm{Hom}_k(M_i, M_j))$, and it follows that $\mathcal{O}^A(\mathcal{M})$ is explicitly given as the matrix algebra

$$\begin{pmatrix} H_{11} \widehat{\otimes}_k \mathrm{End}_k(M_1) & H_{12} \widehat{\otimes}_k \mathrm{Hom}_k(M_1, M_2) & \cdots & H_{1r} \widehat{\otimes}_k \mathrm{Hom}_k(M_1, M_r) \\ H_{21} \widehat{\otimes}_k \mathrm{Hom}_k(M_2, M_1) & H_{22} \widehat{\otimes}_k \mathrm{End}_k(M_2) & \cdots & H_{2r} \widehat{\otimes}_k \mathrm{Hom}_k(M_2, M_r) \\ \vdots & \vdots & \ddots & \cdots \\ H_{r1} \widehat{\otimes}_k \mathrm{Hom}_k(M_r, M_1) & H_{r2} \widehat{\otimes}_k \mathrm{Hom}_k(M_r, M_2) & \cdots & H_{rr} \widehat{\otimes}_k \mathrm{End}_k(M_r) \end{pmatrix}.$$

Let us write $\rho_i : A \to \mathrm{End}_k(M_i)$ for the structural algebra homomorphism defining the right A-module structure on M_i for $1 \leq i \leq r$, and

$$\rho : A \longrightarrow \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i)$$

for their direct sum. Since H is a complete r-pointed algebra in $\hat{\mathrm{a}}_r$, there is a natural morphism $H \to k^r$, inducing an algebra homomorphism

$$\pi : \mathcal{O}^A(\mathcal{M}) \longrightarrow \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i).$$

By construction, there is a right action of $\mathcal{O}^A(\mathcal{M})$ on the family $\mathcal{M}$ extending the right action of A, in the sense that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & \mathcal{O}^A(\mathcal{M}) \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\pi} \\ & & \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i) \end{array}$$

commutes. This makes it reasonable to call $\mathcal{O}^A(\mathcal{M})$ the algebra of observables.

3 The generalized Burnside theorem

Let k be a field and let A be a finite-dimensional associative k-algebra. Then the simple right modules over A are the simple right modules over the semi-simple quotient algebra $A/\operatorname{rad}(A)$, where $\operatorname{rad}(A)$ is the Jacobson radical of A. By the classification theory for semi-simple algebras, it follows that there are finitely many non-isomorphic simple right A-modules.

We consider the noncommutative deformation functor $\mathrm{Def}_{\mathcal{M}} : \mathrm{a}_r \to \mathrm{Sets}$ of the family $\mathcal{M} = \{M_1, M_2, \ldots, M_r\}$ of simple right A-modules. Clearly, $\mathcal{M}$ is a swarm, hence $\mathrm{Def}_{\mathcal{M}}$ has a formal moduli (H, M_H), and we consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & \mathcal{O}^A(\mathcal{M}) \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\pi} \\ & & \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i). \end{array}$$

By a classical result, due to Burnside, the algebra homomorphism ρ is surjective when k is algebraically closed. This result is conveniently stated in the following form.

Theorem 1 (Burnside's theorem). *If $\mathrm{End}_A(M_i) = k$ for $1 \leq i \leq r$, then ρ is surjective. In particular, ρ is surjective when k is algebraically closed.*

Proof. There is a factorization $A \to A/\operatorname{rad}(A) \to \oplus_i \mathrm{End}_k(M_i)$ of ρ. If $\mathrm{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $A/\operatorname{rad}(A) \to \oplus_i \mathrm{End}_k(M_i)$ is an isomorphism by the classification theory for semi-simple algebras. Since $\mathrm{End}_A(M_i)$ is a division ring of finite dimension over k, it is clear that $\mathrm{End}_A(M_i) = k$ whenever k is algebraically closed. □

Let us write $\overline{\rho} : A/\operatorname{rad} A \to \oplus_i \mathrm{End}_k(M_i)$ for the algebra homomorphism induced by ρ. We observe that ρ is surjective if and only if $\overline{\rho}$ is an isomorphism. Moreover, let us write $J = \operatorname{rad}(\mathcal{O}^A(\mathcal{M}))$ for the Jacobson radical of $\mathcal{O}^A(\mathcal{M})$. Then we see that

$$J = \big(\operatorname{rad}(H)_{ij} \widehat{\otimes}_k \mathrm{Hom}_k\left(M_i, M_j\right)\big) = \ker(\pi).$$

Since $\rho(\operatorname{rad} A) = 0$ by definition, it follows that $\eta(\operatorname{rad} A) \subseteq J$. Hence there are induced morphisms

$$\mathrm{gr}(\eta)_q : \operatorname{rad}(A)^q / \operatorname{rad}(A)^{q+1} \to J^q / J^{q+1}$$

for all $q \geq 0$. We may identify $\mathrm{gr}(\eta)_0$ with $\overline{\rho}$, since $\mathcal{O}^A(\mathcal{M})/J \cong \oplus_i \mathrm{End}_k(M_i)$. The conclusion in Burnside's theorem is therefore equivalent to the statement that $\mathrm{gr}(\eta)_0$ is an isomorphism.

Theorem 2 (The generalized Burnside theorem)**.** *Let A be a finite-dimensional algebra over a field k, and let $\mathcal{M} = \{M_1, M_2, \ldots, M_r\}$ be the family of simple right A-modules. If $\operatorname{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\eta : A \to \mathcal{O}^A(\mathcal{M})$ is an isomorphism. In particular, η is an isomorphism when k is algebraically closed.*

Proof. It is enough to prove that η is injective and that $\operatorname{gr}(\eta)_q$ is an isomorphism for $q = 0$ and $q = 1$, since A and $\mathcal{O}^A(\mathcal{M})$ are complete in the $\operatorname{rad}(A)$-adic and J-adic topologies. By Burnside's theorem, we know that $\operatorname{gr}(\eta)_0$ is an isomorphism. To prove that η is injective, let us consider the kernel $\ker(\eta) \subseteq A$. It is determined by the obstruction calculus of $\operatorname{Def}_{\mathcal{M}}$; see Laudal [2, Theorem 3.2] for details. When A is finite-dimensional, the right regular A-module A_A has a decomposition series

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = A_A$$

with F_p/F_{p-1} a simple right A-module for $1 \leq p \leq n$. Namely, A_A is an *iterated extension* of the modules in $\mathcal{M}$. This implies that η is injective; see Laudal [2, Corollary 3.1]. Finally, we must prove that $\operatorname{gr}(\eta)_1 : \operatorname{rad}(A)/\operatorname{rad}(A)^2 \to J/J^2$ is an isomorphism. This follows from the Wedderburn-Malcev theorem; see Laudal [2, Theorem 3.4], for details. □

4 Properties of the algebra of observables

Let A be a finite-dimensional algebra over a field k, and let $\mathcal{M} = \{M_1, \ldots, M_r\}$ be any family of right A-modules of finite dimension over k. Then $\mathcal{M}$ is a swarm, and we denote the algebra of observables by $B = \mathcal{O}^A(\mathcal{M})$. It is clear that

$$B/\operatorname{rad}(B) \cong \bigoplus_i \operatorname{End}_k(M_i)$$

is semi-simple, and it follows that $\mathcal{M}$ is the family of simple right B-modules. In fact, we may show that $\mathcal{M}$ is a swarm of B-modules, since B is complete and $B/(\operatorname{rad} B)^n$ has finite dimension over k for all positive integers n.

Proposition 3. *If k is an algebraically closed field, then $\eta_B : B \to \mathcal{O}^B(\mathcal{M})$ is an algebra isomorphism.*

Proof. Since $\mathcal{M}$ is a swarm of A-modules and of B-modules, we may consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta^A} & B = \mathcal{O}^A(\mathcal{M}) & \xrightarrow{\eta^B} & C = \mathcal{O}^A(\mathcal{M}) \\ & {\scriptstyle\rho}\searrow & \downarrow & \swarrow & \\ & & \bigoplus_{1 \leq i \leq r} \operatorname{End}_k(M_i). & & \end{array}$$

The algebra homomorphism η^B induces maps $B/\operatorname{rad}(B)^n \to C/\operatorname{rad}(C)^n$ for all $n \geq 1$. Since k is algebraically closed and $B/\operatorname{rad}(B)^n$ has finite dimension over k, it follows from the generalized Burnside theorem that $B/\operatorname{rad}(B)^n \to C/\operatorname{rad}(C)^n$ is an isomorphism for all $n \geq 1$. Hence η^B is an isomorphism. □

In particular, the proposition implies that the assignment $(A, \mathcal{M}) \mapsto (B, \mathcal{M})$ is a closure operation when k is algebraically closed. In other words, the algebra $B = \mathcal{O}^A(\mathcal{M})$ has the following properties:

(1) the family $\mathcal{M}$ is the family of the simple B-modules;
(2) the family $\mathcal{M}$ has exactly the same module-theoretic properties, in terms of (higher) extensions and Massey products, considered as a family of modules over B as over A.

Moreover, these properties characterize the algebra $B = \mathcal{O}^A(\mathcal{M})$ of observables.

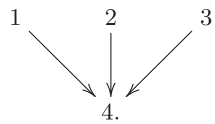
5 Examples: representations of ordered sets

Let k be an algebraically closed field, and let Λ be a finite ordered set. Then the algebra $A = k[\Lambda]$ is an associative algebra of finite dimension over k. The category of right A-modules is equivalent to the category of presheaves of vector spaces on Λ, and the simple A-modules correspond to the presheaves $\{M_\lambda : \lambda \in \Lambda\}$ defined by $M_\lambda(\lambda) = k$ and $M_\lambda(\lambda') = 0$ for $\lambda' \neq \lambda$. The following results are well known:

(1) if $\lambda > \lambda'$ in Λ and $\{\gamma \in \Lambda : \lambda > \gamma > \lambda'\} = \emptyset$, then $\operatorname{Ext}^1_A(M_\lambda, M_{\lambda'}) \cong k$;
(2) if $\{\gamma \in \Lambda : \lambda \geq \gamma \geq \lambda'\}$ is a simple loop in Λ, then $\operatorname{Ext}^2_A(M_\lambda, M_{\lambda'}) \cong k$;
(3) in all other cases, $\operatorname{Ext}^1_A(M_\lambda, M_{\lambda'}) = \operatorname{Ext}^2_A(M_\lambda, M_{\lambda'}) = 0$.

5.1 A hereditary example

Let us first consider the following ordered set. We label the elements by natural numbers, and write $i \to j$ when $i > j$:



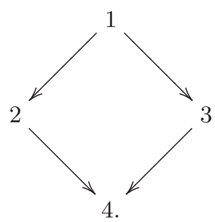
In this case, the simple modules are given by $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$, and we can easily compute the algebra $\mathcal{O}^A(\mathcal{M})$ of observables since $\mathrm{Ext}^2_A(M_i, M_j) = 0$ for all $1 \leq i, j \leq 4$. We obtain

$$\mathcal{O}^A(\mathcal{M}) = \left(H_{ij} \widehat{\otimes}_k \operatorname{Hom}_k\left(M_i, M_j\right)\right) \cong H \cong \begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.$$

It follows from the generalized Burnside theorem that $\eta : A \to \mathcal{O}^A(\mathcal{M})$ is an isomorphism. Hence we recover the algebra $A \cong \mathcal{O}^A(\mathcal{M}) \cong H$.

5.2 The diamond

Let us also consider the following ordered set, called *the diamond*. We label the elements by natural numbers, and write $i \to j$ when $i > j$:



In this case, the simple modules are given by $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$. Since $\mathrm{Ext}^2_A(M_1, M_4) \cong k$, we must compute the cup-products

$$\begin{aligned} \mathrm{Ext}^1_A(M_1, M_2) \cup \mathrm{Ext}^1_A(M_2, M_4) &\longrightarrow \mathrm{Ext}^2_A(M_1, M_4), \\ \mathrm{Ext}^1_A(M_1, M_3) \cup \mathrm{Ext}^1_A(M_3, M_4) &\longrightarrow \mathrm{Ext}^2_A(M_1, M_4) \end{aligned}$$

in order to compute H. These cup-products are non-trivial; see Laudal [2, Remark 3.2] for details. Hence we obtain

$$\mathcal{O}^A(\mathcal{M}) = \left(H_{ij} \widehat{\otimes}_k \operatorname{Hom}_k\left(M_i, M_j\right)\right) \cong H \cong \begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Note that H_{14} is two-dimensional at the tangent level and has a relation. Also in this case, it follows from the generalized Burnside theorem that $\eta : A \to \mathcal{O}^A(\mathcal{M})$ is an isomorphism. Hence we recover the algebra $A \cong \mathcal{O}^A(\mathcal{M}) \cong H$.

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