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Eggert's Conjecture for 2-Generated Nilpotent Algebras

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Abstract

Let A be a commutative nilpotent finitely-dimensional algebra over a field F of characteristic $p > 0$. A conjecture of Eggert says that $p\cdot$ dim $A^{(p)}$ dim A, where $A^{(p)}$ is the subalgebra of A generated by elements a^p, $a \in A$. We show that the conjecture holds if $A^{(p)}$ is at most 2-generated.

Keywords: Nilpotent algebra; Eggert's conjecture; Commutative nilpotent ring; Polynomial bases

Introduction

Let F be a field of characteristic $p>0$ and A a commutative (associative) nilpotent finite-dimensional algebra over F. Let $A^{(p)}$ be the subalgebra generated by the set $\{a^p |\ a \in A\}$. N. Eggert [1] conjectured that

$p\cdot$ dim $A^{(p)} \leq$ dim A.

This conjecture gives an answer to the problem, when a finite abelian group is isomorphic to the adjoint group of some finite commutative nilpotent F-algebra. Recall that the adjoint group of A is the set A with the operation $x\circ y=x + y + x\,y$ for every $x, y \in A$.

Validity of this hypothesis would also have influence on an estimation of a (Prüfer) rank of a product of two (abelian) p-groups.

N. Eggert proved his conjecture only when dim $A^{(p)} \leq 2$. Five years later, R. Bautista [2] proved it when dim $A^{(p)}= 3$. C. Stack confirmed this results in Stack et al. [3,4], but provided shorter proofs. Finally, Amberg and Kazarin [5] proved the conjecture for the case dim $A^{(p)} \leq 4$.

Another type of results presented by McLean [6,7]. He showed that this conjecture is true if the algebra A is either radical of a group algebra of a finite abelian group or A is graded and at least one of the following conditions is fulfilled:

(i) $p = 2$ and $(A^{(p)})^4 = 0$.

(ii) $A^{(p)}$ is 2-generated.

(iii) $(A^{(p)})^3 = 0$.

(iv) $n < 3p$ and $3 \leq s - 1 \leq p$, where n is the number of generators of $A^{(p)}$ and s is the index of nilpotence of $A^{(p)}$.

We also should mention the result of Gorlov [8]. He proved the conjecture for nilpotent algebras A with a metacyclic adjoint group.

One paper concerning Eggert's conjecture appeared in 2002 and the author L. Hammoudi [9] claimed he proved it. But, as Amberg [10] and McLean [7] have shown, his proof was incorrect.

In this short note we sketch out the main steps of the proof that Eggert's conjecture is true if the subalgebra $A^{(p)}$ has at most two generators. For the details, the reader is referred to Korbelar [11].

Since we will deal with nilpotency and commutativity only, we point out that the word 'algebra' will mean a commutative one and not necessary possesing a unit.

For an algebra A and a subset $X \subseteq A$ we denote $\langle X \rangle$ ($[X]$, resp.) the algebra (vector space, resp.) generated by X.

An algebra A is called nilpotent if $A^m=0$ for some $m \in \mathbb{N}$.

Through this paper let always F be a field of characteristic $p > 0$ and $R = F\,[x, y]$ be the ring of polynomials over the variables x, y and the field F.

We start with the remark, that the number of any minimal generating set of a finite generated nilpotent F-algebra A is equal to dim A/A^2. This implies the following:

Lemma 1.1. *Suppose that Eggert's conjecture holds for every nilpotent 2-generated F-algebra. Then it also holds for every nilpotent F-algebra A such that A(p) is a 2-generated F-algebra.*

In the rest we deal with 2-generated nilpotent algebras.

Bases of Nilpotent Algebras

We will use the well-known concept of monomial ordering and standard bases.

For $\alpha = (i, j) \in \mathbb{N}_0^2$ put

$x^\alpha = x^i y^j \in F[x, y]$.

Denote $[X]_0 = \{x^\alpha \mid \alpha \in \mathbb{N}_0^2\} \cup \{0\}$ the multiplicative monoid with the *lexicographical* ordering $\leq$ such that

$$x^{(i,j)} \leq x^{(i',j')} \Leftrightarrow i < i' \vee (i = i' \wedge j \leq j')$$

and

$$x^{(i,j)} \leq 0$$

for every $(i, j), (i', j') \in \mathbb{N}_0^2$

For $0 \neq f = \sum_\alpha \lambda_\alpha x^\alpha \in F[x, y]$ put

$$m(f) =| \min\{x^\alpha \mid \lambda_\alpha \neq 0\}$$

$$m(0) = 0.$$

Finally, f will be called normal iff $\lambda_{\alpha 0} = 1$, where m(f) = $\boldsymbol{x}^{\alpha 0}$, and m($f$) $< \pi\, \boldsymbol{x}^\alpha$ implies $\lambda_\alpha = 0$ for every

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$\alpha \in \mathbb{N}_0^2$

This function m: $F[x, y] \to [X]_0$ has common properties of a valuation:

(i) $\mathrm{m}(fg) = \mathrm{m}(f)\,\mathrm{m}(g)$.

(ii) $\mathrm{m}(f + g) \geq \min\{\mathrm{m}(f); \mathrm{m}(g)\, g$. Moreover, $\mathrm{m}(f + g) = \mathrm{m}(f)$ if $\mathrm{m}(f) < \mathrm{m}(g)$.

(iii) $\mathrm{m}(f(x^p, y^p)) = \mathrm{m}(f)^p$.

for every $f, g \in F[x, y]$.

Finally. a set $\mathcal{X} \subseteq \{x^\alpha \mid \alpha \in \mathbb{N}_0^2\}$ will be called *upper* (*lower*, resp.) if $x^\alpha \in \mathcal{X}$ and $x^\alpha \mid x^\beta$ ($x^\beta \mid x^\alpha$, resp.) implies $x^\beta \in \mathcal{X}$ for every $x^\alpha, x \in [X]_0$.

Definition 2.1. Let A be a nilpotent F-algebra generated by $\{a_1, a_2\}$. Put

$$C_A(a_1,a_2) = \{u \in [X]_0 (\exists f \in Rx + Ry)\, \mathrm{m}(f) = u \wedge f(a_1,a_2) = 0\}$$

and

$$\mathcal{B}_A(a_1,a_2) = [X]_0 \setminus \mathcal{C}_A(a_1,a_2).$$

Proposition 2.2. *Let A be a nilpotent F -algebra generated by* $\{a_1, a_2\}$. *Then*:

(i)$\mathcal{C}_A(a_1, a_2)$ *is an upper set and* $0 \in \mathcal{C}_A(a_1, a_2)$.

(ii)$\mathcal{B}_A(a_1, a_2)$ *is a lower set and* $1 \in \mathcal{B}_A(a_1, a_2)$.

(iii)*The set* $\{x^\alpha (a_1, a_2) \mid 1 \neq x^\alpha \in \mathcal{B}_A(a_1, a_2)\}$ *is a basis of A. In particular,* $\mathcal{B}_A(a_1, a_2)$ *is finite.*

(iv) $\mathcal{C}_A(a_1, a_2) = \{u\, [X]_0 \mid (\exists f \in Rx + Ry)\, \mathrm{m}(f) = u \wedge f(a_1, a_2) = 0 \wedge f$ is normal$\}\ \{0\}$.

Definition 2.3. Let A be a nilpotent F-algebra generated by $\{a_1, a_2\}$. Denote

$$n_0 = \#\{x^\alpha \in \mathcal{B}_A(a_1,a_2) \mid \alpha \in \mathbb{N}_0 \times \{0\} - 1,$$
$$d_i = \#\{x^\alpha \in \mathcal{B}_A(a_1,a_2) \alpha \in \{i\} \times \mathbb{N}_0\},$$
$$\overline{n_0} = \#\{x^\alpha \in \mathcal{B}_{A^{(p)}}(a_1^p,a_2^p) \alpha \in \mathbb{N}_0 \times \{0\} - 1,$$
$$\overline{d}_i = \#\mathrm{x}^\alpha \in \mathcal{B}_{A^{(p)}}(a_1^p,a_2^p) \alpha \in \{i\} \times \mathbb{N}_0$$

and

$$D_i = \sum_{k=pi}^{pi+p-1} d_k$$

for $i \in \mathbb{N}_0$

Lemma 2.4. *Let A be a nilpotent F -algebra generated by* $\{a_1, a_2\}$. *Then*:

(*i*) $\overline{d} \;+\; +\overline{d}^{\,-} =| \qquad (a_1^p,a_2^p) |= 1 + \dim A^{(\;)}$.

(*ii*) $D_0 + \;\; + D^{-} =| \qquad (a_1,a_2) |= 1 + \dim A.$

(*iii*) *The set* $\{x^\alpha (a_1^p,a_2^p) \mid 1 \neq x^\alpha \in \qquad (a_1^p,a_2^p)\}$ is *a basis of* $A^{(\mathrm{p})}$

Eggert's Conjecture for 2-generated Algebras

Let $I \subseteq Rx + Ry$ be an ideal in R such that $A = Rx + Ry/I$ is a non-zero nilpotent F-algebra.

We have $A = \langle x + I, y + I \rangle$ and $A^{(p)} = \langle x^p + I, y^p + I \rangle$.

By definition of $\mathcal{C}_A(x + I; y + I)$ there are $f_i \in Rx + Ry$, $0 \leq i \leq n_0 + 1$, such that $\mathrm{m}(f_i) = x^{(i,di)}$, $f_i \in I$ and f_i are normal.

The main idea of the proof lies in the fact that taking a normal polynomial from I, dividing it by x and then multiplying by some suitable y^k, we get again a member of I (3.3). Then, using binomial formula in a suitable way, we obtain a polynomial that will estimate the number $\overline{d_i z}$ (see 3.4 and the definition of $\mathcal{B}_{A^{(p)}}(a_1^p, a_2^p)$.)

Lemma 3.1. (i) $f_0 = y^{d_0} - xh_0, where\, h_0 \in R, and\; f_{n_0+1} = x^{n_0}.$

(ii) $xf_i \in Rf_{i+1} + \cdots + Rf_{n_0+1}\; for\, i = 0, \ldots, n_0.$

Definition 3.2. Denote

$w_A = \max \mathcal{B}_A(x + I, y + I)$.

For $0 \leq i \leq \overline{n_0}$ denote

$m_i \in \mathbb{N}_0$

the least integer such that $pi \leq m_i \leq pi + p\text{-}1$ and $d_{pi} \geq \ldots \geq d_{mi} = d_{mi+1} = \ldots = d_{pi+p-1}$. Put

$$l_i = \left(\sum_{k=pi}^{m_i-1} (d_k - 1)\right) - (p-1)d_{m_i}.$$

Following lemma is obtained using induction.

Lemma 3.3. *Let* $1 \leq i \leq n_0 + 1$ *and* $0 \neq f \in I$ *be such that* $\mathrm{m}(f)\, x^i$. Then $y^{d_{i-1}-1}(f/x) + \mathrm{I} \in [w_A + I]$.

The proof of the next proposition uses only the binomial formula. It finds the particular polynomial the we need to make an estimation of the numbers D_i and thus of the dimension of $A^{(p)}$.

Proposition 3.4.

(i) $If\; 0 \leq i < \overline{n_0}\, and\, l_i \geq 0, then\, y^{l_i} x^{pi} (f_{m_i} / x^{m_i})^p + I \in [w_A + I]$

(ii) $If\; 0 \leq i < \overline{n_0}\, and\, l_i < 0, then\, x^{pi} (f_{m_i} / x^{m_i})^p \in I$

(iii) $If\; i = \overline{n_0}, then\, y^{D_i - 1} x^{pi} + I \in [w_A + I].$

Now, only exploring carefully the previous cases for i and l_i we get the following interesting claim. It says that the inequality " $p\overline{d}_i \leq D_i$ " holds for almost every i.

Theorem 3.5. *One of the following cases takes place:*

(i)$p\overline{d}_{\overline{n_0}} \leq D_{\overline{n_0}} + p - 2\, and\; p\overline{d}_i \leq D_i + 1\, for\, every\, 0 \leq i < \overline{n_0}. Moreover,$

$p\overline{d}_{i_0} = D_{i_0} + 1\, for\, at\, most\, one\, 0 \leq i_0 < \overline{n_0}$

(ii) $p\overline{d}_{\overline{n_0}} \leq D_{\overline{n_0}} + p - 1\, and\; p\overline{d}_i \leq D_i\, for\, every\, 0 \leq i < \overline{n_0}$

And our main result is just an easy corollary of this and 1.1.

Theorem 3.6. *Let A be a nilpotent F -algebra, char F=p>0, such that* $A^{(p)}$ *is 2-generated. Then p·dim* $A^{(p)}$ *dim A.*

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