How to Prove the Riemann Hypothesis

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Abstract
The aim of this paper is to prove the celebrated Riemann Hypothesis. I have already discovered a simple proof of the Riemann Hypothesis. The hypothesis states that the nontrivial zeros of the Riemann zeta function have real part equal to 0.5. I assume that any such zero is $s = a + bi$. I use integral calculus in the first part of the proof. In the second part I employ variational calculus. Through equations (50) to (59) I consider (a) as a fixed exponent, and verify that $a = 0.5$. From equation (60) onward I view (a) as a parameter ($a < 0.5$) and arrive at a contradiction. At the end of the proof (from equation (73)) and through the assumption that (a) is a parameter, I verify again that $a = 0.5$.

Keywords: Definite integral; Indefinite integral; Variational calculus

Introduction
The Riemann zeta function is the function of the complex variable $s = a + bi$ (where $i = \sqrt{-1}$), defined in the half plane $a > 1$ by the absolute convergent series.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(1)

and in the whole complex plane by analytic continuation.

The function $\zeta(s)$ has zeros at the negative even integers -2, -4, … and one refers to them as the trivial zeros. The Riemann hypothesis states that the nontrivial zeros of $\zeta(s)$ have real part equal to 0.5 [1].

Proof of the Hypothesis
We begin with the equation,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (0 < a < 1)$$

(2)

and with,

$$s = a + bi$$

(3)

$$\zeta(a + bi) = 0$$

(4)

It is known that the nontrivial zeros of $\zeta(s)$ are all complex. Their real parts lie between zero and one.

If $0 < a < 1$ then,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (0 < a < 1)$$

(5)

$[x]$ is the integer function.

Hence,

$$\int_{0}^{\infty} \frac{x^{-1} - a^{-1}}{x^{a+1}} dx = 0$$

(6)

Therefore,

$$\int_{0}^{\infty} ([x] - x)x^{-1} - a^{-1} - b^{-1} dx = 0$$

(7)

$$\int_{0}^{\infty} ([x] - x)x^{-1} - a^{-1} - b^{-1} dx = 0$$

(8)

$$\int_{0}^{\infty} ([x] - x)(\cos(b \log x) - i \sin(b \log x)) dx = 0$$

(9)

Separating the real and imaginary parts we get,

$$\int_{0}^{\infty} x^{-1} - a^{-1} \cos(b \log x) dx = 0$$

(10)

$$\int_{0}^{\infty} x^{-1} - a^{-1} \sin(b \log x) dx = 0$$

(11)

According to the functional equation, if $\zeta(S) = 0$ then $\zeta(1-S) = 0$. Hence we get besides equation (11)

$$\int_{0}^{\infty} y^{-1} - a^{-1} ([y] - y) \sin(b \log y) dy = 0$$

(12)

In equation (11) replace the dummy variable $x$ by the dummy variable $y$.

$$\int_{0}^{\infty} y^{-1} - a^{-1} ([y] - y) \sin(b \log y) dy = 0$$

(13)

We form the product of the integrals (12) and (13). This is justified by the fact that both integrals (12) and (13) are absolutely convergent. As to integral (12) we notice that [2]

$$\int_{0}^{\infty} x^{-2} + a^{-1} ([x] - x) \sin(b \log x) dx \leq \int_{0}^{\infty} x^{-2} + a^{-1} ([x] - x) dx$$

(14)

We begin with the equation,

$$\zeta(S) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (0 < a < 1)$$

(2)

And with,

$$s = a + bi$$

(3)

$$\zeta(a + bi) = 0$$

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It is known that the nontrivial zeros of $\zeta(S)$ are all complex. Their real parts lie between zero and one.

If $0 < a < 1$ then,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (0 < a < 1)$$

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$[x]$ is the integer function.

Hence,

$$\int_{0}^{\infty} \frac{x^{-1} - a^{-1}}{x^{a+1}} dx = 0$$

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Therefore,

$$\int_{0}^{\infty} ([x] - x)x^{-1} - a^{-1} - b^{-1} dx = 0$$

(7)

$$\int_{0}^{\infty} ([x] - x)x^{-1} - a^{-1} - b^{-1} dx = 0$$

(8)

$$\int_{0}^{\infty} ([x] - x)(\cos(b \log x) - i \sin(b \log x)) dx = 0$$

(9)

Separating the real and imaginary parts we get,
And as to integral (13) \( \int_0^\infty y^{a-1} \sin(b \log y) dy \)
\[
\leq \int_0^\infty y^{a-1} \sin(b \log y) dy
\leq \int_0^\infty y^{a-1} dy
\]

\[
= \lim_{t \to 0} \int_0^t y^{a-1} dy + \lim_{t \to 0} \int_t^\infty y^{a-1} dy
\]

\[
= \frac{1}{1-a} + \lim_{t \to 0} \int_t^\infty y^{a-1} dy
\leq \frac{1}{1-a} + \frac{1}{1-a}.
\]

Since the limits of integration do not involve \( x \) or \( y \), the product can be expressed as the double integral,
\[
\int_0^\infty \int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) \sin(b \log y) dx dy = 0
\]
(14)

Thus
\[
\int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) \sin(b \log y) dx dy = 0
\]
(15)

That is,
\[
\int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) \sin(b \log y) dx dy = 0
\]
(16)

Consider the integral on the right-hand side of equation (17)
\[
\int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) \sin(b \log y) dx dy
\]
(18)

In this integral make the substitution \( x = \frac{z}{y} \) \( dx = \frac{-dz}{y^2} \).

The integral becomes,
\[
\int_0^\infty \frac{z^{a-1}}{y} \sin(b \log z) \frac{-dz}{y^2} dy
\]
(19)

That is,
\[
\int_0^\infty z^{a-1} \sin(b \log z) dz dy
\]
(20)

This is equivalent to
\[
\int_0^\infty \frac{z^{a-1}}{y} \sin(b \log y) dz dy
\]
(21)

If we replace the dummy variable \( z \) by the dummy variable \( x \), the integral takes the form [3]
\[
\int_0^\infty \frac{x^{a-1}}{y} \sin(b \log x) dx dy
\]
(22)

Rewrite this integral in the equivalent form,
\[
\int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) dx dy
\]
(23)

Thus equation 17 becomes,
\[
\int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) dx dy
\]
(24)

Write the last equation in the form,
\[
\int_0^\infty x^{a-1} y^{a-1} \sin(b \log x) dx dy
\]
(25)

Let \( 0 < b \) be an arbitrary small positive number. We consider the following regions in the \( x \)-plane [4].

The region of integration \( I = [0, \infty) \times (0, \infty) \)
(26)

The large region \( II = [p, \infty) \times [p, \infty) \)
(27)

The narrow strip \( I 2 = [p, \infty) \times [0, p] \)
(28)

The narrow strip \( I 3 = [0, p] \times [0, \infty) \)
(29)

Note that,
\[
I = I 1 \cup I 2 \cup I 3
\]
(30)

Denote the integer and in the left hand side of equation (25) by:
\[
F(x, y) = x^{a-1} y^{a-1} \sin(b \log x) \cos(b \log y)
\]
(31)

Let us find the limit of \( F(x, y) \) as \( x \to \infty \) and \( y \to \infty \). This limit is given by:
\[
\lim_{x \to \infty} a_{a-1} \cos(b \log y) \cos \left( \frac{1}{x} \right)
\]
(32)

\((tz)\) is the fractional part of the number \( z \), \( 0 \leq (t) < 1 \)

The above limit vanishes, since all the functions \( \cos(b \log y), \cos \left( \frac{1}{x} \right) \) and \( (t) \) remain bounded as \( x \to \infty \) and \( y \to \infty \).

Note that the function \( F(x, y) \) is defined and bounded in the region
\[
I 1
\]

We can prove that the integral,
\[
\int_0^\infty \frac{1}{x} \sin(b \log x) dx dy
\]
(33)

is bounded as follows
\[
\leq \int_0^\infty x^{-a} y^{-a} \sin(b \log x) \sin(b \log y) dx dy
\]
(34)

That is,
\[
\int_0^\infty x^{-a} y^{-a} \sin(b \log x) \sin(b \log y) dx dy
\]
(35)

This is equivalent to
\[
\int_0^\infty x^{-a} y^{-a} \sin(b \log x) \sin(b \log y) dx dy
\]
(36)

If we replace the dummy variable \( z \) by the dummy variable \( x \), the integral takes the form [3]
\[
\int_0^\infty x^{-a} y^{-a} \sin(b \log x) dx dy
\]
(37)

Rewrite this integral in the equivalent form,
\[
\int_0^\infty x^{-a} y^{-a} \sin(b \log x) dx dy
\]
(38)

\[
\leq \int_0^\infty x^{-a} \cos(b \log x) \sin \left( \frac{1}{x} \right) + \cos \left( \frac{1}{x} \right) \right) dx dy
\]
(39)

\[
\leq \int_0^\infty x^{-a} \cos(b \log x) \sin \left( \frac{1}{x} \right) + \cos \left( \frac{1}{x} \right) \right) dx dy
\]
(40)

\[
\leq \int_0^\infty x^{-a} \cos(b \log x) \sin \left( \frac{1}{x} \right) + \cos \left( \frac{1}{x} \right) \right) dx dy
\]
(41)
\[
\int \frac{x-a}{x}(1-x) \, dx \leq \int \frac{x-a}{x}(1-x) \, dx + \sum_{n=1}^{\infty} \frac{1}{a^n (1-a)} \int \frac{x-a}{x}(1-x) \, dx \, dy
\]

Since \( a \) is a very small arbitrary positive number. Since the integral:

\[
\lim_{t \to 0} \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx
\]

is bounded, it remains to show that \( \lim_{t \to 0} \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded.

Consider the integral:

\[
\int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx = \int_{x=a}^{\infty} \frac{x-a}{x} \, dx - \int_{x=a}^{\infty} \frac{x-a}{x} \, dx
\]

is bounded. Also it is evident that the integral

\[
\int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx
\]

is bounded, this was proved in the course of proving that the integral \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded. Also it is evident that the integral,

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy
\]

is bounded. Thus we deduce that the integral (40) \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded.

Hence, according to equation (39), the integral \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded [6].

Now consider the integral,

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy
\]

We write it in the form:

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy = \int_{y=0}^{\infty} y^{-1/2} (\cos(\log y)) \, dy \frac{1}{y^2} \, dx
\]

(34)

This is because in this region \((y)=y\). It is evident that the integral

\[
\int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx
\]

is bounded, was proved in the course of proving that the integral \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded. Also it is evident that the integral,

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy
\]

is bounded. Thus we deduce that the integral (40) \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded.

Hence, according to equation (39), the integral \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded [6].

Now consider the integral,

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy
\]

We write it in the form:

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy = \int_{y=0}^{\infty} y^{-1/2} (\cos(\log y)) \, dy \frac{1}{y^2} \, dx
\]

(34)

This is because in this region \((y)=y\)

\[
\int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx
\]

is bounded, this was proved in the course of proving that the integral \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded. Also it is evident that the integral,

\[
\int_{x=a}^{\infty} \frac{1}{y^2} \, dy
\]

is bounded. Thus we deduce that the integral (40) \( \int_{x=a}^{\infty} \frac{x-a}{x}(1-x) \, dx \) is bounded.

Consider the region:

\[
I_4=I_2
\]

(35)

We know that,

\[
0 = \int_{x=a}^{\infty} F(x,y) \, dx \, dy = \int_{x=a}^{\infty} F(x,y) \, dx \, dy + \int_{x=a}^{\infty} F(x,y) \, dx \, dy
\]

(36)

and that,

\[
\int_{x=a}^{\infty} F(x,y) \, dx \, dy
\]

is bounded.

From which we deduce that the integral [5]

\[
\int_{x=a}^{\infty} F(x,y) \, dx \, dy
\]

is bounded.

Remember that,

\[
\int_{x=a}^{\infty} F(x,y) \, dx \, dy = \int_{x=a}^{\infty} F(x,y) \, dx \, dy + \int_{x=a}^{\infty} F(x,y) \, dx \, dy
\]

(39)

Consider the integral,

\[
\int_{x=a}^{\infty} F(x,y) \, dx \, dy \leq \int_{x=a}^{\infty} F(x,y) \, dx \, dy
\]

(40)

\[
\int_{x=a}^{\infty} \left( (\frac{1}{x}) - (x) \right) \, dx \leq \int_{x=a}^{\infty} \left( (\frac{1}{x}) - (x) \right) \, dx \, dy
\]

\[
\leq \int_{x=a}^{\infty} \left( (\frac{1}{x}) - (x) \right) \, dx \, dy
\]

\[
\leq \int_{x=a}^{\infty} \left( (\frac{1}{x}) - (x) \right) \, dx \cdot \frac{1}{y^2} \, dy
\]

(41)

(42)

(43)

(44)
Now equation (44) gives us,
\[-K \leq \int y^{\alpha-x}((y)) \cos(b \log y) dy \leq K\]  
(45)
According to equation (42) we have,
\[\int f(x,y) dx dy = \int \int y^{\alpha-x}((y)) \cos(b \log y) \frac{(1/x) - x^{2a-1}}{x^a} dx\]  
(46)
\[\geq \int \left(-(K)\right)\frac{(1/x) - x^{2a-1}}{x^a} dx = K \int \frac{(1/x) - x^{2a-1}}{x^a} dx\]
Since
\[\int f(x,y) dx dy\] is bounded, then \[\int \frac{(1/x) - x^{2a-1}}{x^a} dx\] is also bounded. Therefore the integral,
\[G = \int \frac{(1/x) - x^{2a-1}}{x^a} dx\] is bounded
(47)
We denote the integral and of (47) by:
\[F = \int \frac{(1/x) - x^{2a-1}}{x^a}\]
(48)
Let \(G[F]\) be the variation of the integral \(G\) due to the variation of the integrand \(F\).

Since,
\[G[F] = \int F dx\] (the integral (49) is indefinite)
(49)
(here we do not consider \(a\) as a parameter, rather we consider it as a given exponent)

We deduce that \(\frac{\delta G[F]}{\delta F(x)} = 1\)
that is,
\[\delta G[F] = \delta F(x)\]
(50)
But we have,
\[\delta G[F] = \int dx \frac{\delta G[F]}{\delta F(x)} \delta F(x)\] (the integral (51) is indefinite)
(51)
Using equation (50) we deduce that,
\[\delta G[F] = \int dx \delta F(x)\] (the integral (52) is indefinite)
(52)
Since \(G[F]\) is bounded across the elementary interval \([0,p]\), we must have that,
\[\delta G[F] \text{is bounded across this interval}\]
(53)
From (52) we conclude that,
\[\delta G = \int_0^p dx \delta F(x) = \int_0^p \frac{dF\delta x}{dx} \delta x = [F\delta x](at x=p) - [F\delta x](at x = 0)\]
(54)
Since the value of \([F\delta x]\) (at \(x=p\))is bounded, we deduce from equation (54) that,
\[\lim(x \to 0) F \delta x \text{ must remain bounded}.\]
(55)
Thus we must have that,
\[\lim(x \to 0) \frac{\delta x}{x^a} \{(1/x) - x^{2a-1}\}\]
(56)
First we compute,
\[\lim(x \to 0) \frac{\delta x}{x^a}\]
(57)
Applying L’Hospital’ rule we get,
\[
\lim (x \to 0) \frac{1-x^{2a-1}}{x^2}
\]
must remain bounded. (71)

But,

\[
\lim (x \to 0) \frac{1-x^{2a-1}}{x^2} = \lim (x \to 0) \frac{1-x^{2a-1}}{x^2} = \lim (x \to 0) \frac{1-x^{2a-1}}{x^2}
\]

It is evident that this last limit is unbounded. This contradicts our conclusion (71) that:

\[
\lim (x \to 0) \frac{1-x^{2a-1}}{x^2}
\]
must remain bounded (for \(a < 0.5\))

Therefore the case \(a < 0.5\) is rejected. We verify here that, for \(a = 0.5\) (71) remains bounded as \((x \to 0)\).

We have that:

\[
(1) - x^{2a-1} < 1 - x^{2a-1}
\]

Therefore:

\[
\lim (a \to 0.5) (x \to 0) \frac{1-x^{2a-1}}{x^2} < \lim (a \to 0.5) (x \to 0) \frac{1-x^{2a-1}}{x^2}
\]

We consider the limit:

\[
\lim (a \to 0.5) (x \to 0) \frac{1-x^{2a-1}}{x^2}
\]

We write:

\[
a = (\lim x \to 0) (0.5 + x)
\]

Hence we get:

\[
\lim (a \to 0.5) (x \to 0) x^{2a-1} = \lim (x \to 0) x^{2a-1} = 1
\]

(77)

(Since \(\lim (x \to 0) x^{2a-1} = 1\))

Therefore we must apply L’Hospital’ rule with respect to \(x\) in the limiting process (75):

\[
\lim (a \to 0.5) (x \to 0) \frac{1-x^{2a-1}}{x^2} = \lim (a \to 0.5) (x \to 0) \frac{-2a}{ax^{a-1}} x^{2a-1}
\]

(78)

\[
\lim (a \to 0.5) (x \to 0) \frac{1-2}{x^2}
\]

Now we write again,

\[
a = (\lim x \to 0) (0.5 + x)
\]

Thus the limit (78) becomes:

\[
\lim (a \to 0.5) (x \to 0) \frac{1-2}{x^2} = \lim (x \to 0) \frac{(0.5 + x)^{2a-2}}{x^2} = \lim (x \to 0) \frac{2x^2}{x^2} = 0
\]

(81)

Thus we have verified here that, for \(a = 0.5\) (71) approaches zero as \((x \to 0)\) and hence remains bounded.

We consider the case \(a > 0.5\). This case is also rejected, since according to the functional equation, if \(\zeta(s) = 0\) \((s = a + bi)\) has a root with \(a > 0.5\), then it must have another root with another value of \(a < 0.5\). But we have already rejected this last case with \(a < 0.5\).

Thus we are left with the only possible value of \(a\) which is \(a = 0.5\)

Therefore \(a = 0.5\)

This proves the Riemann Hypothesis.

**Conclusion**

The Riemann Hypothesis is now proved. The hypothesis states that the nontrivial zeros of the Riemann zeta function have real part equal to 0.5. I assume that any such zero is \(s = a + bi\). I use integral calculus in the first part of the proof. In the second part I employ variational calculus. Through equations (50) to (59) I consider \(a\) as a fixed exponent, and verify that \(a = 0.5\). From equation (60) onward I view \(a\) as a parameter \((a < 0.5)\) and arrive at a contradiction. At the end of the proof (from equation (73)) and through the assumption that \(a\) is a parameter, I verify again that \(a = 0.5\).

**References**