

The m -Derivations of Distribution LieAlgebras

Princy Randriambolondrantomalala*

Département de Mathématiques et Informatique, Faculté des Sciences, Université d'Antananarivo, Antananarivo 101, Madagascar

Abstract

Let M be a N -dimensional smooth differentiable manifold. Here, we are going to analyze $(m>1)$ -derivations of Lie algebras relative to an involutive distribution on subrings of real smooth functions on M . First, we prove that any $(m>1)$ -derivations of a distribution Ω on the ring of real functions on M as well as those of the normalizer of Ω are Lie derivatives with respect to one and only one element of this normalizer, if Ω doesn't vanish everywhere. Next, suppose that $N = n + q$ such that $n>0$, and let S be a system of q mutually commuting vector fields. The Lie algebra of vector fields \mathfrak{A}_S on M which commutes with S , is a distribution over the ring $F_0(M)$ of constant real functions on the leaves generated by S . We find that m -derivations of \mathfrak{A}_S is local if and only if its derivative ideal coincides with \mathfrak{A}_S itself. Then, we characterize all non local m -derivation of \mathfrak{A}_S . We prove that all m -derivations of \mathfrak{A}_S and the normalizer of \mathfrak{A}_S are derivations. We will make these derivations and those of the centralizer of \mathfrak{A}_S more explicit.

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Introduction and Preliminary

Let m be a natural integer greater than or equal to 2. We recall that a m -derivation D of a Lie \mathbb{R} -algebra \mathfrak{A} is an endomorphism of \mathfrak{A} , such that for all $X_1, X_2, \dots, X_m \in \mathfrak{A}$

$$D[X_1, [X_2, \dots, [X_{m-1}, X_m] \dots]] = [D(X_1), [X_2, \dots, [X_{m-1}, X_m] \dots]] + [X_1, [D(X_2), \dots, [X_{m-1}, X_m] \dots]] + \dots + [X_1, [X_2, \dots, [D(X_{m-1}), X_m] \dots]] + [X_1, [X_2, \dots, [X_{m-1}, D(X_m)] \dots]].$$

This map is inner with respect to Lie algebra \mathfrak{B} if D equals to a Lie derivative with respect to $X \in \mathfrak{B}$; if $X \in \mathfrak{A}$, it is an inner m -derivation. A standard m -derivation D is a sum of derivations of \mathfrak{A} and \mathbb{R} -linear maps of \mathfrak{A} into the center of \mathfrak{A} such that $D[\mathfrak{A}, [\mathfrak{A}, [\dots, [\mathfrak{A}, \mathfrak{A}] \dots]] = \{0\}$.

Is it sufficient to study derivation of Lie algebras? What is the reason for studying the more general notion? " $(m>2)$ -derivation"? In other words, can we find $(m>2)$ -derivations of a vector fields Lie algebra which are not derivations? In [1], we found m -derivations all polynomial vector fields Lie algebras P on \mathbb{R}^n , where P contains Euler vector fields E and all constant vector fields. We remark that all these m -derivations are derivations when m is even. If m is an odd number, m -derivations are generally sum of derivations and m -derivations with homogeneous degree -2 . Over \mathbb{R}^3 , we can take a simple example where the Lie \mathbb{R} -algebra is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, E, z \frac{\partial}{\partial x}, (z)^2 \frac{\partial}{\partial x}$ and the \mathbb{R} linear map D is defined by $D\left((z)^2 \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}$ and vanishing otherwise. It is a 3-derivation, but not a derivation. In [2], some graded Lie algebra m -derivations are discussed. Here, we are interested in m -derivations of distribution Lie algebra on a N -smooth manifold M over an M -real functions ring. We know that all smooth vector fields can be locally approximated to polynomial vector fields, so we think that all results in [1] are naturally true in the case of distributions. But, the results which

follow are different. The differential operator theory see [3] is the main tool throughout our proofs.

We denote by $F(M)$ the ring of all real functions on M , $\chi(M)$ (resp. $\chi(TM)$) the vector fields Lie algebras over M (resp. over the tangent bundle TM).

At first, we consider an involutive distribution Ω over $F(M)$. That is to say, Ω is a $F(M)$ -sub-module of the module of all vector fields on M . Assuming that the open set $O_\Omega = \{x \in M / \Omega(x) \neq \{0\}\}$ equals M , we are looking for characteristics of m -derivations of Lie algebras relative to Ω and applications of the obtained results on some remarkable distributions. We propose to prove that each m -derivation of Ω (resp. of the normalizer in $\chi(M)$ of Ω) is simply a Lie derivative with respect to one and only one normalizer's vector fields (resp. is inner). These theorems can be extended where O_Ω is dense over M .

Secondly, let be $N=n+q$ with $n \geq 1$ and $q>0$, S a system of q non-vanishing vector fields which commute mutually. We know by results in [4] that S yields a generalized foliation on M . We assume that all leaves are regular and we notice that $F_0(M)$, the ring of real smooth functions which are constant on the leaves over M . Let U be a p -dimensional adapted chart domain relative to the foliation and (x^a, y^i) (resp. (U, x^a)), where $1 \leq a \leq n+q-p, 1 \leq i \leq p$ if $p \geq 1$ (resp. where $1 \leq a \leq n+q$ if $p=0$). Then, there are two modules over $F_0(U)$, $A_S^1(U)$ spanned by $\left(\frac{\partial}{\partial x^a}\right)_{1 \leq a \leq n+q-p}$ and $\mathfrak{A}_S^2(U)$ generated by $\left(\frac{\partial}{\partial y^i}\right)_{1 \leq i \leq p}$. These previous modules are Lie algebras such

*Corresponding author: Princy Randriambolondrantomalala, Département de Mathématiques et Informatique, Faculté des Sciences, Université d'Antananarivo, Antananarivo 101, BP 906, Madagascar, Tel: +261 20 22 326 39; E-mail: Princypcpc@yahoo.fr

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that $\mathfrak{A}_S(U)$ is equal to the semi-direct product of these two algebras: $\mathfrak{A}_S(U) = \mathfrak{A}_S^1(U) \oplus \mathfrak{A}_S^2(U)$ for all distinguished U . We can say that \mathfrak{A}_S is a smooth distribution of M over $F_0(M)$. Throughout this paper, we assume that this chart is $(p > 0)$ -dimensional in the sense of the foliation, unless expressly stated. Our aims are to characterize all m -derivations of \mathfrak{A}_S , of the normalizer of \mathfrak{A}_S and of the centralizer of \mathfrak{A}_S in $\chi(M)$. The corresponding work where $S = \{0\}$ has been done in the previous section. Because of \mathfrak{A}_S 's lower central series constancy, which coincides with module direct sum of $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$, our work on these m -derivations is non-trivial. The main results of this section are: all m -derivation is local iff the derivative ideal of \mathfrak{A}_S is \mathfrak{A}_S itself, which is equivalent to the fact that $\mathfrak{A}_S^1(F_0(M))$ has non-vanishing elements over the whole M . Moreover, all m -derivations of \mathfrak{A}_S or of the normalizer \mathfrak{N} of \mathfrak{A}_S are sums of a Lie derivative with respect to one \mathfrak{N} 's element, of one local m -derivation which takes its value in \mathfrak{A}_S^2 depending on two non-vanishing 1-differential forms over \mathfrak{A}_S , and of a non-local m -derivation of \mathfrak{A}_S . We give some recommendations for constructing all these non local m -derivations. In addition, all \mathbb{R} -linear maps of $\langle S \rangle$ the centralizer of \mathfrak{A}_S into itself are m -derivations. We characterize all local \mathbb{R} -endomorphisms of $\langle S \rangle$ in the case where all elements of S are densely supported or $\langle S \rangle$ is spanned by singleton, and those which are non local. It is well known that the open set of all foliation regular points is dense in M , then one can extend these results where the foliation is singular and if the above 1-forms prolongs smoothly on M .

Several applications of our results about Lie algebras relative to: all vector fields, all compactly supported vector fields, generalized foliations, μ -projected vector fields cf. [5], k -nullity space of connection curvature, and vector fields Lie algebras on TM commuting with Liouville vector fields cf. [6]; are given at the end of this paper.

Throughout this article, the Lie derivative with respect to $X \in \chi(M)$ is denoted L_X . We adopt the Einstein index summation and suppose that all considered objects are smooth.

The m -derivations of Lie algebras attached to Ω

According the hypothesis about Ω , we can affirm that Ω is a Lie sub-algebra of $\chi(M)$. A generalization of [7,8]'s theorems in the sense of derivation or triple derivation can be stated as follows:

Theorem 2.1. All m -derivations of Ω (resp. of the normalizer of in $\chi(M)$) are Lie derivative with respect to one and only one vector field of the normalizer of Ω (resp. is inner).

Proof. Assume that $x \in M$, $\exists X \in \Omega$ such that $X(x) \neq 0$. By Frobenius theorem, we find one chart (U_x, φ_x) which contains x and local coordinate system (x^1, \dots, x^{n-1}, y) where $X = \frac{\partial}{\partial y}$. Letting D be an m -derivation of Ω , we know that the Lie algebra spanned by brackets of all elements in Ω is the derivative ideal of Ω denoted by $[\Omega; \Omega]$. Local behavior of D can be proved by adapting one of Proposition 2.4 in [7] and using that the derivative ideal of Ω is Ω itself. Therefore D_{U_x} is an m -derivation of Ω_{U_x} . Let's give $f \in F(U_x)$, as we know, $\frac{\partial}{\partial y} \in \Omega_{U_x}$ then $D_{U_x} \left(f \frac{\partial}{\partial y} \right) = D^0(f) \frac{\partial}{\partial y} + D^{\alpha}(f) \frac{\partial}{\partial x^\alpha}$ is uniquely determined, where each D^α is differential operator over the trivial bundle $U_x \times \mathbb{R}$ cf. [3]. Thus, if necessary we can write $D^0 = \sum_{|\alpha| \geq 0, |\beta| \geq 1, \alpha \neq \beta} \chi^{\alpha,0} \frac{\partial^{|\alpha|}}{\partial x^\alpha} + \chi^{\beta,\alpha} \frac{\partial^{|\alpha|+\beta}}{\partial x^\beta \partial y^\alpha} + \chi^{\alpha,\beta} \frac{\partial^\alpha}{\partial y^\beta}$, where A, B are multi-indices corresponding to coordinates.

Let's apply D_{U_x} to $\left[x^j \frac{\partial}{\partial y}, \left[y \frac{\partial}{\partial y}, \dots, \left[y \frac{\partial}{\partial y}, f \frac{\partial}{\partial y} \right] \dots \right] \right]$, where $x^n = y$. By

definition of m -derivations and when f is replaced by monomials, we have:

- If $\text{deg}(f) \geq 2$, $D^0(f)(0, \dots, 0) = 0$ except for $f \equiv y^2$.
- If $\text{deg}(f) \geq 1$, $D^\alpha(f)(0, \dots, 0) = 0$ except for $f \equiv yP(x^i, i \neq n)$

where $P(x^i)$ are free x^n monomials.

By reasoning as in the previous, we compute $D_{U_x} \left[\frac{\partial}{\partial y}, \left[\frac{\partial}{\partial y}, \dots, \left[\frac{\partial}{\partial y}, y^m P(x^i, i \neq n) \frac{\partial}{\partial y} \right] \dots \right] \right]$. It is easy, using both the previous relation and the previous proof, to obtain the nullity of $D^\alpha \left(y^m P(x^i) \right) (0, \dots, 0)$. By coordinates translations, we can affirm that each $D^j, j \neq 0$ is a differential operator of order 0 and D^0 is a sum of one of order 1 with one other $\chi^{0,2} \frac{\partial^2}{(\partial y)^2}$ of order 2.

Computing in the same way as the previous calculus,

$D_{U_x} \left[f \frac{\partial}{\partial y}, \left[y \frac{\partial}{\partial y}, \dots, \left[y \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right] \dots \right] \right]$ gives:

- for $f \equiv y$, $D^0(1) = -\frac{\partial \chi^{0,1}}{\partial y}$.
- for $f \equiv x^j$ except $j \neq n$, $D^j(1) = -\frac{\partial \chi^{(0, \dots, 0, 1, 0, \dots, 0)}}{\partial y} (1_j \text{ means } 1 \text{ is in } j\text{-th rank})$.

By these results, $D_{U_x} \left(f \frac{\partial}{\partial y} \right) = L_{\chi^{0,1} \frac{\partial}{\partial y} + \chi^{(0, \dots, 0, 1, 0, \dots, 0)} \frac{\partial}{\partial x^j}} \left(f \frac{\partial}{\partial y} \right) + \chi^{0,2} \frac{\partial^2 f}{(\partial y)^2} \frac{\partial}{\partial y}$. Consequently,

$D_0 = \chi^{0,2} \frac{\partial^2}{(\partial y)^2} \otimes \frac{\partial}{\partial y}$ is a derivation of the $F(U_x)$ -sub-module spanned by $\frac{\partial}{\partial y}$. Applying D_0 to $\left[y^2 \frac{\partial}{\partial y}, \left[\frac{\partial}{\partial y}, \dots, \left[\frac{\partial}{\partial y}, y^{m-1} \frac{\partial}{\partial y} \right] \dots \right] \right]$, we have $\chi^{0,2}(y=0) = 0$. By coordinate's translations, we can write that $\chi^{0,2} = 0$.

We take Proposition 2.6 of [7] and we have $D_{U_x} = L_{\chi^{0,1} \frac{\partial}{\partial y} + \chi^{(0, \dots, 0, 1, 0, \dots, 0)} \frac{\partial}{\partial x^j}}$.

Following the arguments of the proof of Theorem 2.7 in [7], we end the demonstration of the first assertion of our theorem. Taking that the derivative ideal of Ω is Ω itself into account, we can adapt the proof of Theorem 2.12 in [7] to state the second assertion.

Remark 2.2. These theorems are correct if we consider O_Ω to be dense over M and if the corresponding vector of the Lie derivative relative to the m -derivation cited by Theorem 2.1 can be smoothly extended towards M .

The m -derivations of Lie algebras defined by \mathfrak{A}_S

We know that nil potency of order $m-1$ of \mathfrak{A}_S forces any endomorphism of \mathfrak{A}_S to be an m -derivation. To avoid this triviality, we prove that:

Proposition 3.1. The lower central series of \mathfrak{A}_S are constant and equal to the module $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$.

Proof. The lower central series of \mathfrak{A}_S is determined by $\mathfrak{C}(\mathfrak{A}_S) \mathfrak{A}_S$ and for all $p > 0$,

$\mathfrak{C}^p(\mathfrak{A}_S) = [\mathfrak{A}_S, \mathfrak{C}^{p-1}(\mathfrak{A}_S)]$ cf. [1]. By Proposition 3.7 of [4], the derivative ideal of \mathfrak{A}_S is $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$. From the linearity of brackets, the Jacobi identity and the fact that \mathfrak{A}_S^2 is an ideal of \mathfrak{A}_S , we deduce $[\mathfrak{A}_S, [\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]]] = \mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$. Then, we deduce the result.

We assume the following conventions about the index,

$i, j, k \in \{1, \dots, p\}, a, b, c \in \{1, \dots, n + q - p\}$, and each index indexed by 0 is fixed.

Proposition 3.2. Let D be a m -derivation of \mathfrak{A}_S and U a domain of distinguished chart such that if $X \in \mathfrak{A}_S$ over U vanishes, then $D(X)$ over U on \mathfrak{A}_S^1 is zero.

Proof. Let D be a such m -derivation and X an element of \mathfrak{A}_S satisfying the above hypothesis. We assume that $D(X)|_{U_0} \neq 0$, then it exists an open set V_z containing z , such that the a_0 -th component of $D(X)|_{U_0}$ on V_z is everywhere non zero. Let's consider $f \in F_0(M)$ such that $f|_{V_z} = (x^{a_0})^2$ where $\text{Supp}(f) \subset U$, and Y, X_3, \dots, X_m are elements of \mathfrak{A}_S with $Y|_{V_z} = \frac{\partial}{\partial y^{i_0}}, X_3|_{V_z} = \dots = X_{m-1}|_{V_z} = x^{a_0} \frac{\partial}{\partial x^{a_0}}, X_m|_{V_z} = \frac{\partial}{\partial x^{a_0}}$.

By definition, we obtain

$$D[X, [Y, [X_3, \dots, [X_{m-1}, X_m] \dots]](z) = [D(X), [Y, [X_3, \dots, [X_{m-1}, X_m] \dots]](z) + [X, Z](z) \quad (3.1)$$

With $Z \in \mathfrak{A}_S$, a contradiction.

Proposition 3.3. The centralizer \mathcal{C} of $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$ coincides with the vector \mathbb{R} -space $\langle S \rangle$ spanned by S .

Proof. Recall that

$$\mathcal{C} = \left\{ X \in \chi(M) / [X, \mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]] = \{0\} \right\}.$$

Choose $X \in \mathcal{C}$ and let be U a distinguished connected chart domain of the foliation. When $p = 0$, we have $X = 0$. For $p \geq 1$, we put $X_U = X^a \frac{\partial}{\partial x^a} + X^i \frac{\partial}{\partial y^i} \in \mathfrak{C}_U$. By the fact $[X, \mathfrak{A}_S^1] = \{0\}$, $X^a = 0$ for all a and each $X^i \in (F(U) - F_0(U)) \cup (F(U) \cap F_0(U))$. Therefore, $X_{\mathfrak{A}_S} = 0$ and $X = f^i X_i$ where all $f^i \in (F(M) - F_0(M)) \cup (F(M) \cap F_0(M))$. Assume $Y^1(g^i) X_i \in [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$ with $g^j X_j \in \mathfrak{A}_S^2, Y^1 \in \mathfrak{A}_S^1$. It's known that $[X, \mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]] = \{0\}$ so $Y^1(g^i) X_i (f^j) X_j = 0$ for all g^i and Y^1 . Then all f^j are in $F_0(M)$ and consequently they are constant, and \mathcal{C} is a subset of the \mathbb{R} -vector space spanned by S . The converse inclusion obvious.

Proposition 3.4. All non-local m -derivations of \mathfrak{A}_S vanish on $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$ and take their values in $\langle S \rangle$. Conversely, all \mathbb{R} -endomorphisms D of \mathfrak{A}_S which have these properties, is a m -derivation of \mathfrak{A}_S . All these maps are standard m -derivations.

Proof. To simplify, we pose a such m -derivation D . Then there is $X \in \mathfrak{A}_S$ and a distinguished chart domain U so that $X_U \equiv 0$ with $D(X)(z \in U) \neq 0$. Thus, we have an open set V_z in U containing z , with $D(X)$ everywhere non-vanishing. Recall that the center of \mathfrak{A}_S is the intersection of its centralizer with itself. We reason by contradiction, we suppose that $D(X)$ doesn't belong to the center of \mathfrak{A}_S . By Proposition 3.2, we claim that on V_z the i_0 -th component of $D(X)$ is everywhere non vanishing. So, this component is not a constant function. Consequently, we can assume that its partial derivative with respect to a x^{a_0} is non-zero at z . Then, we consider X_2, \dots, X_m to be elements of \mathfrak{A}_S such that $\text{Supp}(X_2) \subset U$ and $X_2|_{V_z} = \dots = X_{m-1}|_{V_z} = x^{a_0} \frac{\partial}{\partial x^{a_0}}, X_m|_{V_z} = \frac{\partial}{\partial x^{a_0}}$. By the m -derivation definition,

$$D[X, [X_2, [X_3, \dots, [X_{m-1}, X_m] \dots]](z) = [D(X), [X_2, [X_3, \dots, [X_{m-1}, X_m] \dots]](z) + [X, Z](z)$$

Where $Z \in \mathfrak{A}_S$, we have a contradiction. Moreover, Proposition

3.1 and the previous result lead to nullity of D over $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$

It is easy to prove the last assertions of our proposition.

We can note immediately that,

Lemma 3. For all $k \geq 2$, if D is a k -derivation of Lie algebra \mathfrak{A} then the center \mathcal{C} of \mathfrak{A} satisfies the following equation $[D(\mathcal{C}), \mathfrak{C}^{k-2}(\mathfrak{A})] = \{0\}$.

Proposition 3.6. Local m -derivations of \mathfrak{A}_S stabilize \mathfrak{A}_S^2 .

Proof. We set a local m -derivation D , D_U is still an m -derivation. Without trivial case $p = 0$, let a, b, i be some fixed indices, we write

$$D_U \left(x^b \frac{\partial}{\partial y^i} \right) = D_{i,b}^c \frac{\partial}{\partial x^c} + D_{i,b}^{j+m+q-p} \frac{\partial}{\partial y^j}$$

And

$$(-1)^{m-1} \delta_a^b D_U \left(\frac{\partial}{\partial y^i} \right) = D_U \left[\left[x^b \frac{\partial}{\partial y^i}, \left[x^a \frac{\partial}{\partial x^a}, \left[\dots, \left[x^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^a} \right] \dots \right] \right] \right] \right], \forall a. \quad (3.2)$$

By using the (3.2), Lemma 3.5 and Proposition 3.3, we deduce that each $D_{i,b}^c$ is constant.

Let f be an element of $F_0(U)$, We remark that

$$\left[f \frac{\partial}{\partial x^b}, \left[x^c \frac{\partial}{\partial x^c}, \left[\dots, \left[x^c \frac{\partial}{\partial x^c}, x^b \frac{\partial}{\partial y^i} \right] \dots \right] \right] \right] = f \frac{\partial}{\partial y^i}. \quad (3.3)$$

By mapping D_U to (3.3) in the case where f is a polynomial of degree greater or equal than two, the previous result and the fact that $D_{i,b}^c$ is a differential operator over $U \times \mathbb{R}$, proves that $D_U \left(f \frac{\partial}{\partial y^i} \right) \in \mathfrak{A}_S^2(U)$ for all i . Furthermore, combining the previous results and the obtained relation by

$$D_U \left[(x^b)^2 \frac{\partial}{\partial x^b}, \left[x^c \frac{\partial}{\partial x^c}, \left[\dots, \left[x^c \frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^b} \right] \dots \right] \right] \right] = -2D_U \left(x^b \frac{\partial}{\partial y^i} \right)$$

We see that $D_{i,b}^c = 0$. Then, $D_U \left(x^b \frac{\partial}{\partial y^i} \right) \in \mathfrak{A}_S^2(U)$

In addition,

$$D_U \left(f \frac{\partial}{\partial y^i} \right) = D_U \left[f \frac{\partial}{\partial x^b}, \left[x^c \frac{\partial}{\partial x^c}, \left[\dots, \left[x^c \frac{\partial}{\partial x^c}, x^b \frac{\partial}{\partial y^i} \right] \dots \right] \right] \right]$$

and the previous statement leads to $D_U \left(f \frac{\partial}{\partial y^i} \right) \in \mathfrak{A}_S^2(U)$ for all U .

Proposition 3.7. The Lie algebra \mathfrak{A}_S^2 is stabilized by m -derivations of \mathfrak{A}_S .

Proof. We deduce the result from Propositions 3.4, 3.6.

Theorem 3.8. We have equivalences between:

1. All m -derivation of \mathfrak{A}_S is local.
2. There is an $X \in \mathfrak{A}_S^1$ and $h \in F_0(M)$ such that $X(h)(x) \neq 0 \forall x \in M$.
3. The derivative ideal of \mathfrak{A}_S , $[\mathfrak{A}_S, \mathfrak{A}_S]$ coincides with \mathfrak{A}_S itself.

Proof. In $2. \Rightarrow 1.$, we use the same reasoning as the one of the proof of Theorem 3.11 in [4]. As for $1. \Rightarrow 2.$ we suppose that there is an $f \notin F_0(M) \mathfrak{A}_S^1(F_0(M))$. Since $S \neq 0$, then it exists k such that X_k is non-zero on the open set U_k and $g \in F_0(M)$ vanishing on U_k with $(fg)|_{U_k} \equiv 0$. So, it is immediate that the \mathbb{R} -linear map defined by

$$D(X) = \begin{cases} 0 & \text{if } X \in \mathfrak{A}_S - \{\text{Rfg}X_k\}, \\ D^j X_j & \text{if } X = \text{fg}X_k \text{ where } D^j \in \mathbb{R} \text{ for all } j=1, \dots, q. \end{cases}$$

Is a non-local m -derivation when $D^k \neq 0$. Thus $1. \Leftrightarrow 2.$

We reason in the same way as in [4] for $1. \Leftrightarrow 3.$

Remark 3.9. We assert that if the derivative ideal of \mathfrak{A}_S doesn't coincide with \mathfrak{A}_S , then it exists $f \notin F_0(M)\mathfrak{A}_S^1(F_0(M))$, zero on the open set where one X_k is non-vanishing. To realize a non local m -derivation D , we exploit the non-vanishing on $\mathfrak{A}_S - (\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2])$ of the following \mathbb{R} -linear map:

For $X = gX_i$, $D(X) = D^j_{g,i} X_j$ where

$$\begin{cases} D^j_{g,i} \neq 0 \text{ or} \\ \text{if there is non vanishing } X_{h_0} \text{ over the previous cited open set and } D^j_{g,i} \neq 0 \text{ or} \\ \exists i_0, j_0; h \notin F_0(M)\mathfrak{A}_S^1(F_0(M)) / \text{Supp}(X_{h_0}) = M \text{ and } \text{CSupp}(X_{h_0}) \subset \text{Supp}(X_{h_0}) \exists D^j_{h_0} \neq 0 \\ \text{or it exists } X_{h_0} \text{ and } h \notin F_0(M)\mathfrak{A}_S^1(F_0(M)) \text{ such that their multiplication is zero on an} \\ \text{open set } V \text{ without non vanishing both over } V \text{ and if there is } X_{h_0} \text{ on } V D^j_{h_0} \neq 0 \end{cases}$$

These results are immediate by using Theorem 3.8, Proposition 3.4 and the definition of non local \mathfrak{A}_S m -derivation.

Proposition 3.10. The normalizer \mathfrak{N} of \mathfrak{A}_S in $\chi(M)$ is locally isomorphic to $\mathfrak{A}_S \oplus \mathfrak{gl}(p, \mathbb{R})$ as a vector space, where p is the corresponding leaf local dimension. So $[\mathfrak{N}, \mathfrak{A}_S] = \mathfrak{A}_S$ locally $[\mathfrak{N}, \mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]] = \mathfrak{A}_S$. Moreover, all local m -derivations of \mathfrak{N} stabilize \mathfrak{A}_S

Proof. We define \mathfrak{N} by the set of all vector fields X such that $[X, \mathfrak{A}_S] \subset \mathfrak{A}_S$. So, we are in a distinguished chart U , all \mathfrak{N}_U 's elements are obtained with direct use of the definition of the normalizer of $\mathfrak{A}_S(U)$. Indeed, \mathfrak{N}_U is the sum of $\mathfrak{A}_S(U)$ and the vector \mathbb{R} -space spanned by $y^i \frac{\partial}{\partial y^j}$. It's clear that, this last space is isomorphic to $\mathfrak{gl}(p, \mathbb{R})$. The two results which follow are easily proved by the same argument as the previous. As for the last assertion, let's take $X_1 \in \mathfrak{N}$, $X_2, \dots, X_m \in \mathfrak{A}_S$ and D a local m -derivation of \mathfrak{N} . In accordance with the m -derivation definition, we have $D[X_1, [X_2, \dots, [X_{m-1}, X_m] \dots]] \in \mathfrak{A}_S$. By local equation $[\mathfrak{N}, \mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]] = \mathfrak{A}_S$, Proposition 3.1 and the previous result where each X_i runs through over the respective sets, we affirm that $D_{\mathfrak{A}_S}(\mathfrak{A}_S)$ is subset of \mathfrak{A}_S .

Theorem 3.11. Given that we have a local m -derivation D of \mathfrak{A}_S towards \mathfrak{A}_S^2 . We find 1 differential closed forms α^i and ω^i over U , where $i = 1, \dots, p$ with $D_v = (\alpha^j + \omega^j) \otimes \frac{\partial}{\partial y^j}$ denoted $D_v^{(\alpha, \omega)}$ such that $\ker(\alpha^j) \supset \mathfrak{A}_S^2(U)$ and $\ker(\omega^j) \supset \mathfrak{A}_S(U)$. Besides,

$$\alpha^i [X_1, X_2] = X_1 \cdot \alpha^i(X_2) - X_2 \cdot \alpha^i(X_1)$$

$\forall X_1, X_2 \in \mathfrak{A}_S(U)$. The converse of this result is also true. Furthermore, the condition that the maps $D_v^{(\alpha, \omega)}$, with $\alpha = (\alpha^1, \dots, \alpha^p)$ and $\omega = (\omega^1, \dots, \omega^p)$, are inner is equivalent to, for all i , $\omega^i \equiv 0$ and α^i are exact. Then we get $D_v^{(\alpha, \omega)} = -L_{f^i} \frac{\partial}{\partial y^j}$ Where, $\alpha^i = df^i$ with $f^i \in F_0(U)$. Generally if $\alpha = 0$, then $D_v^{(\alpha, \omega)} = L_{y^i c^i} \frac{\partial}{\partial y^j}$ where $C_i^j \in \mathbb{R}$

Proof. Agreeing with the above hypothesis, we pose $D_v = \gamma^i \otimes \frac{\partial}{\partial y^j}$, where the $\gamma^i = \gamma^i_a dx^a + \gamma^i_j dy^j$ with γ^i_a, γ^i_j belong to $F_0(U)$. By the relations of m -derivations which come from

$$D_U \left[\frac{\partial}{\partial y^{j_0}}, \left[x^a \frac{\partial}{\partial x^a}, \dots, \left[x^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{a_0}} \right] \dots \right] \right],$$

we state that $\frac{\partial \gamma^i_j}{\partial x^{a_0}} = 0$ for all j_0 and a_0 . We write the subsequent equality

$$D_U \left[\frac{\partial}{\partial x^{a_0}}, \left[x^a \frac{\partial}{\partial x^a}, \dots, \left[x^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{b_0}} \right] \dots \right] \right] \text{ at } (0, \dots, 0).$$

Then, we can have $\left(\frac{\partial \gamma^i_{a_0}}{\partial x^{b_0}} - \frac{\partial \gamma^i_{b_0}}{\partial x^{a_0}} \right) (0, \dots, 0) = 0$, for all a_0, b_0 . So,

with the help of coordinate's translations, we get the previous equality at other arbitrary points in U . Thus, each γ^i is closed. By exploiting all these assertions, we can adapt the demonstrations of Proposition 3.14, 3.15 et 3.16 of [4] and we achieve our proof.

Let \mathfrak{R} be the set of pair of forms (α, ω) quoted before. We will denote by $\mathfrak{Z}(U)$, the complement set of those of $(\alpha, \omega) \in \mathfrak{R}$ such that α is exact and $\omega = 0$ or $\alpha = 0$. We might assume that $\mathfrak{Z} = \mathfrak{Z}(M)$.

Theorem 3.12. The form of m -derivations of \mathfrak{A}_S is $L_X + D^{(\alpha, \omega) \in \mathfrak{Z}} + D_l$ where $X \in \mathfrak{N}$, for all distinguished chart U , $D^{(\alpha, \omega)}|_U = 0$ if the leaf dimension over U is zero; $D^{(\alpha, \omega)}|_U = D^{(\alpha, \omega) \in \mathfrak{Z}_U}$ otherwise. And D_l is a non local m -derivation analogous to the one of Remark 3.9. Particularly, these m -derivations are derivations.

Proof. Taking D an m -derivation of \mathfrak{A}_S , it is split into a sum of local m -derivation D_0 and of a non-local m -derivation D_l of \mathfrak{A}_S . So, D_l has the same form as the one of Remark 3.9. We can write D_0 as $D_0^{11} + D_0^{12} + D_0^{21} + D_0^{22}$ with D_0^l the \mathbb{R} -linear component of D_0 mapping \mathfrak{A}_S^l to \mathfrak{A}_S^l , where $l, t = 1, 2$. By Proposition 3.7, $D_0^{21} = 0$. In accordance with the same proposition, we can divide \mathfrak{A}_S by \mathfrak{A}_S^2 and the quotient m derivation of D_0 is denoted $\overline{D_0}$. The map $\overline{D_0}$ becomes an m -derivation of $\overline{\mathfrak{A}_S^1}$ by the splitting of $\overline{D_0}$ we know that $\overline{\mathfrak{A}_S^1}$ is locally isomorphic to $\chi(\mathbb{R}^{n+q-p})$. Then $\overline{D_0}$ coincides with L_X where $X \in \overline{\mathfrak{A}_S^1}$. Consequently, $D_0 = L_Y + D^{(\alpha, \omega)}$ with $Y \in \mathfrak{N}$ and $D^{(\alpha, \omega)} \in \mathfrak{Z}$ is defined by Theorem 3.11. With the help of Proposition 3.4, respectively Theorem 3.11, D_l respectively D_0 is a derivation.

Proposition 3.13. All m -derivation of the Lie \mathbb{R} -algebra of all linear fields taking value in the constant fields Lie \mathbb{R} -algebra of \mathbb{R}^t is Lie derivative with respect to one constant field.

Proof. Let D be such m -derivation and $(z^i)_{1 \leq i \leq t}$ one coordinates system of \mathbb{R}^t . We note that $D \left(z^u \frac{\partial}{\partial z^v} \right) = D_v^u \frac{\partial}{\partial z^t}$ with $u, v \in \{1, \dots, t\}$. It's easy to verify that D vanishes if and only if $D(E)$ is zero too by using the following equation

$$D \left[E, \left[E, \dots, \left[E, z^u \frac{\partial}{\partial z^v} \right] \dots \right] \right] = \left[E, \left[E, \dots, \left[E, D \left(z^u \frac{\partial}{\partial z^v} \right) \right] \dots \right] \right] \quad (3.4)$$

Where E is the Euler vector field. Then, we write $D(E) = C^l \frac{\partial}{\partial z^l}$. For different and fixed u, v , we exploit the obtained relation from

$$D \left[z^u \frac{\partial}{\partial z^u}, \left[z^u \frac{\partial}{\partial z^u}, \dots, \left[z^u \frac{\partial}{\partial z^u}, z^u \frac{\partial}{\partial z^v} \right] \dots \right] \right].$$

Therefore, we have $D_v^u = 0$ where $l \neq u, v; D_v^u = D_u^v$. In accordance with (3.4) when $u=v$, we state that $C^u = D_u^u$ and $D_u^u = 0$ for $l \neq u$. In

addition, (3.4) gives us $D_v^{uu} = 0$. Thus, we proved that $D = L_{C^j \frac{\partial}{\partial x^j}}$.

Proposition 3.14. All m -derivation of \mathfrak{N} taking its value in is a sum of m -derivations of \mathfrak{A}_S towards $\langle S \rangle$ and m -derivation of $(\mathfrak{N} - \mathfrak{A}_S) \cup \{0\}$ to $\langle S \rangle$.

Proof. Let D be a m -derivation of \mathfrak{N} towards $\langle S \rangle$. It is known that every local m -derivation of \mathfrak{N} stabilizes \mathfrak{A}_S , $(\mathfrak{N} - \mathfrak{A}_S) \cup \{0\}$ is a Lie algebra and there is a direct sum of modules $\mathfrak{N} = \mathfrak{A}_S \oplus ((\mathfrak{N} - \mathfrak{A}_S) \cup \{0\})$. Moreover, every non local m -derivation of \mathfrak{N} vanishes on $\mathfrak{A}_S \subset [\mathfrak{N}, \mathfrak{N}]$. Then $D = D^{11} + D^{21}$ with D^{11} (resp. D^{21}) the linear component map of D from \mathfrak{A}_S to $\langle S \rangle$ (resp. of $(\mathfrak{N} - \mathfrak{A}_S) \cup \{0\}$ towards $\langle S \rangle$). By the fact that D is linear and D^{11} takes value in $\langle S \rangle$; D^{11} , D^{21} are m -derivations.

Proposition 3.15. All m -derivations from $(\mathfrak{N} - \mathfrak{A}_S) \cup \{0\}$ towards $\langle S \rangle$ are Lie derivative with respect to an element of $\langle S \rangle$.

Proof. For $k \in \{1, \dots, q\}$, we consider the open set $U_k = \{x \in M / X_k(x) \neq 0\}$. We know that all element of $(\mathfrak{N} - \mathfrak{A}_S) \cup \{0\}$ is of the form $Cf^i X_j$ where $C \in \mathbb{R}$, $f^i \in F_0(M)$ such that $X_i(f^j)$ in U_i equals 0 for all $j \neq i$, equals 1 for $j = i$. First, we show that such an m -derivation D is local. We fix i, j belonging to $\{1, \dots, q\}$ with $i \neq j$. Only in $\mathbb{C}\text{Supp}(X_i)$ we can find an distinguished open set U such that $(f^j X_i)|_U \equiv 0$ or $(f^i X_j)|_U \equiv 0$. It is immediate that $[f^i X_i, [f^i X_i, \dots, [f^i X_i, f^j X_i] \dots]] = (-1)^{m-1} f^j X_i$ by applying D to this last expression, we obtain $(D(f^j X_i))|_U \equiv 0$. Now, we let the map D acts on the following bracket $[f^j X_j, [f^j X_j, \dots, [f^j X_j, f^i X_i] \dots]] = 0$. We have $[f^j X_j, [f^j X_j, \dots, [f^j X_j, D(f^i X_i)] \dots]]|_U \equiv 0$ and $(D(f^i X_i))|_U \equiv 0$. In addition, when we are unable to choose $i \neq j$, the proof is trivial. Thus, D is local.

Second, suppose that D is local. Agreeing with the result of Proposition 3.13, we achieve our proof.

Theorem 3.16. All m -derivations of the normalizer \mathfrak{N} of \mathfrak{A}_S have a form like the one of Theorem 3.12. Moreover, the normalizer of \mathfrak{N} is \mathfrak{N} itself.

Proof. Given $D = D_0 + D_2$ an m -derivation of \mathfrak{N} , where D_0 is local and D_2 non local. By Proposition 3.10, $D_{0|\mathfrak{A}_S}$ is a m -derivation of \mathfrak{A}_S . Let be $X_1 \in \mathfrak{N}$ and $X_2, \dots, X_m \in \mathfrak{A}_S$, we write the equation relative to m -derivation corresponding to $D_0[X_1, [X_2, \dots, [X_{m-1}, X_m] \dots]]$. By the definition of m -derivations, the previous result and Theorem 3.12, we prove that

$$\left[(D_0 - (L_X + D^{\alpha, \omega} + D_1))(X_1), [X_2, \dots, [X_{m-1}, X_m] \dots] \right] = 0 \quad (3.5)$$

$\forall X_2, \dots, X_m \in \mathfrak{A}_S$. Let's denote by D' the m -derivation defined by $D_0 - (L_X + D^{\alpha, \omega} + D_1)$, $D'(X_1)$ belongs to the intersection of the centralizer of $\mathfrak{A}_S^1 \oplus [\mathfrak{A}_S^1, \mathfrak{A}_S^2]$ with \mathfrak{N} . With Proposition 3.3, $D'(X_1)$ becomes an element of $\langle S \rangle$. In addition, we know that $D_{2|\mathfrak{A}_S} = 0$. By using $X_{i=1} \in \mathfrak{A}_S$ in the relation of m -derivation similar to (3.5), the Proposition 3.1 and Proposition 3.3, we have $D_2(X_1) \in \langle S \rangle$. Then, Proposition 3.14 and Proposition 3.15 split D' to a sum of a derivation of \mathfrak{A}_S and L_X with $X \in \langle S \rangle$, D_2 is zero. Moreover, we can affirm that the normalizer of \mathfrak{N} coincides with \mathfrak{N} itself.

Proposition 3.17. Every endomorphism D of the commutative

Lie algebra $\langle S \rangle$ is a m -derivation of $\langle S \rangle$. If D is local, it is a Lie derivative with respect to one element of \mathfrak{N} . In the case where D is non local, then it is determined by the existence of $i \neq j$ such that $\emptyset \neq \mathbb{C}\text{Supp}(X_i) \subset \text{Supp}(X_j)$ with $D(X_i) = \lambda_i^k X_k$, where each $\lambda_i^k \in \mathbb{R}$ and $\lambda_i^j \in \mathbb{R}^*$.

Proof. The first assertion is obvious. Moreover, it is clear that the normalizer of $\langle S \rangle$ is \mathfrak{N} , and its centralizer is \mathfrak{A}_S . Let D be a local endomorphism of $\langle S \rangle$. On U , we put $D\left(\frac{\partial}{\partial y^i}\right) = \lambda_i^k \frac{\partial}{\partial y^k}$. Then $D = L_X$ with $X = \sum \lambda_i^k y^i \frac{\partial}{\partial y^k} + Y$ and $Y \in \mathfrak{A}_{SU}$, for all U . Thus, X belongs to \mathfrak{N} . In addition, if D' is a non-local endomorphism of $\langle S \rangle$, let's write $D'(X_i) = \lambda_i^k X_k$. It is easy to see that D' is non-local iff our last assertion is true.

With the help of the previous proposition, we can confirm immediately

Corollary 3.18. If all elements of S are densely supported over M or if S is reduced to a singleton, then all endomorphisms of $\langle S \rangle$ is local.

Applications

The following is a list of some Lie algebras for which our theorems hold.

We denote by \mathfrak{C}_c the Lie algebra of all compactly supported vector fields on M which is an involutive distribution over M . We know that the normalizer of \mathfrak{C}_c in $\chi(M)$ is $\chi(M)$ and $O_{\mathfrak{C}_c} = M$ see [7].

We suppose that M is a differential manifold equipped with a nonsingular generalized foliation \mathfrak{F} see [1]. We denote $\chi(\mathfrak{F})$ (resp. $\chi_c(\mathfrak{F})$) the involutive distribution of tangent vector fields to the foliation (resp. of compactly supported vector fields in $\chi(\mathfrak{F})$). The normalizer of $\chi(\mathfrak{F})$ in $\chi(M)$ is denoted $\mathfrak{N}(\mathfrak{F})$. The foliation preserving vector fields Lie algebra is named $\mathfrak{L}(\mathfrak{F})$.

Here, V is a smooth manifold and μ is surjective smooth map from M to V see [5]. It is well known that the set of μ -projected vector fields \mathcal{N}_0 is a Lie algebra, and μ -zero-projected vector fields set η_0 is an involutive distribution of M . The normalizer of η_0 in $\chi(M)$ is denoted \mathcal{N} and we assume that $O_{\eta_0} = M$.

Now, let Γ be a connection in the Grifone sense over M cf. [9]. We can cite the curvature horizontal nullity distribution space \mathfrak{N}_R^h (R is the curvature), the distribution of horizontal projected vector fields in the curvature nullity space \mathfrak{A}_Γ^h . Their respective normalizers in $\chi(TM)$ are designated by \mathcal{N}_R , \mathcal{N}_Γ see [10] and we suppose that $O_{\mathfrak{N}_R^h} = O_{\mathfrak{A}_\Gamma^h} = TM$.

We call \mathcal{N}^k the k -nullity space distribution of vector fields in the Finsler space considered by Bidabad [11] such that the nullity index doesn't vanish everywhere. Let's note that \mathcal{N}^k is its normalizer in the vector fields Lie algebra.

Thus, replacing respectively \mathfrak{A} by $\chi(M)$, \mathfrak{C}_c , $\chi(\mathfrak{F})$, $\chi_c(\mathfrak{F})$, $\mathfrak{L}(\mathfrak{F})$, η_0 , \mathcal{N}_0 , \mathfrak{N}_R^h , \mathfrak{A}_Γ^h , \mathcal{N}^k and \mathfrak{B} by $\chi(M)$, $\chi(M)$, $\mathfrak{N}(\mathfrak{F})$, $\mathfrak{N}(\mathfrak{F})$, $\mathfrak{N}(\mathfrak{F})$, \mathcal{N} , \mathcal{N} , \mathcal{N}_R , \mathcal{N}_Γ , \mathcal{N}^k ; we state that "All m -derivation of \mathfrak{A} (resp. of \mathfrak{B}) is inner with respect to \mathfrak{B} (resp. is inner)".

In addition, let's consider the system S composed by the Liouville vector field C on TM . We work on TM without zero section, we find all m -derivation of \mathfrak{A}_S by our theorem, as well as its normalizer which is locally isomorphic to $\mathfrak{A}_S \oplus gl(1, \mathbb{R})$. By density of the foliation regular points set defined by S , we obtain analogous results on TM . All \mathbb{R} -linear maps of $\langle C \rangle$ into itself are local.

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