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Modules Over Color Hom-Poisson Algebras

Ibrahima Bakayoko*

Department de Mathematiques, UJNK/Centre Universitaire de N'Zerekore, BP: 50, N'Zerekore, Guinea

Abstract

In this paper we introduce color Hom-Poisson algebras and show that every color Hom-associative algebra has a non-commutative Hom-Poisson algebra structure in which the Hom-Poisson bracket is the commutator bracket. Then we show that color Poisson algebras (respectively morphism of color Poisson algebras) turn to color Hom-Poisson algebras (respectively morphism of Color Hom-Poisson algebras) by twisting the color Poisson structure. Next we prove that modules over color Hom–associative algebras A extend to modules over the color Hom-Lie algebras L(A), where L(A) is the color Hom-Lie algebra associated to the color Hom-associative algebra A. Moreover, by twisting a color Hom-Poisson module structure map by a color Hom-Poisson algebra endomorphism, we get another one.

Keywords: Color hom-associative algebras; Color hom-Lie algebras; Homomorphism; Formal deformation;Hom-modules; Modules over color Hom-Lie algebras; Modules over color Hom-Poisson algebras

Introduction

Color Hom-Poisson algebras are generalizations of Hom-Poisson algebras introducedin [1], where they emerged naturally in the study of 1-parameter formaldeformations of commutative Hom-associative algebras. Color Hom-Poisson algebrasgeneralize, on the one hand, color Hom-associative [2,3] and color Hom-Lie algebras[2,3] which have been recently investigated by various authors. Onthe other hand, they generalize Hom-Lie superalgebras [4]. These structures arewell-known to physicists and to mathematicians studying differential geometry and homotopy theory. The cohomology theory of Lie superalgebras [5] has been generalizedto the cohomology of Hom-Lie superalgebras in [6]. A cohomology of colorLie algebras was introduced and investigated in [7], and the representations ofcolor Lie algebras were explicitly described in [8]. Modules over Poisson algebras receive various definitions [9,10] we will use theone introduced in [9]. The aim of this paper is to study color Hom-Poisson algebras and modules over color Hom-Poisson algebras. The paper is organized as follows. In section 4, we recall some basic notions related to color Hom-associative algebras and color Hom-Lie algebras. In section 5, we define color Hom-Poisson algebras and point out that to any color Hom-associative algebra ones can associate a color Hom-Poisson algebra. Next, starting from a color Poisson algebra and Poisson algebra morphism we get another one by twisting the associative product and Lie bracket. In section 6, we introduce modules over color Hom-Poisson algebras and prove that starting from a color Hom-Poisson module we get another one by twisting the module structure map by a Hom-Poisson algebra endomorphism. All vector spaces considered are supposed to be over fields of characteristics different from 2.

Preliminaries

Let G be an abelian group. A vector space V is said to be a G-graded if, there exist a family $(V_a)_{a \in G}$ of vector subspaces of V such that

$$V = \bigoplus_{a\in G} V_a$$

An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$ We denote H (V) the set of all homogeneous elements in V.

Let $V = \bigoplus_{a\in G} V_a$ and $V' = \bigoplus_{a\in G} V'_a$ be two G-graded vector spaces. A linear mapping $f: V \to V'$ is said to be homogeneous of degree b if

$$f(V_a) \subseteq V_{a+b}, \forall_a \in G$$

If,f is homogeneous of degree zero i.e. $f(V_a) \subseteq V'_a$ holds for any $a \in G$ then f is said to be even.

An algebra (*A*, μ) is said to be G-graded if its underlying vector space is G-graded i.e. $A = \bigoplus_{a\in G} A_a$ and if furthermore $\mu(A_a, A_b) \subseteq A_{a+b}$ for all $a, b \in$ G.

Let *A*' be another G-graded algebra. A morphism $f: A \to A'$ of G-graded algebras is by definition an algebra morphism from *A* to *A*' which is, in addition an even mapping.

Definition

Let G be an abelian group. A map $\varepsilon: GXG \to K^*$ is called a skew-symmetric bicharacter on G if the following identities hold,

1. ε(a,b) ε(b,a)=1
2. ε (a,b+c)= ε(a,b) ε(a,c)
3. ε (a+b,c)= ε(a,c) ε(b,c)

a,b,c ∈ G

Remark that

$\varepsilon(a,0)=\varepsilon(0,a)=1$, $\varepsilon(a,a)=\pm 1$ For all $a \in G$

Where, 0 is the identity of G. If x and y are two homogeneous elements of degree a and b respectively and ε is a skew-symmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$

Definition

A color Hom-associative algebra is a quadruple $(A, \mu, \varepsilon, \alpha)$ consisting of a G-graded vector space A, an even bilinear map $\mu: AXA \to A$ and an even linear map such $\alpha: A \to A$ that

***Corresponding author:** Ibrahima Bakayoko, Department de Mathematiques, UJNK/Centre Universitaire de N'Zerekore, BP: 50, N'Zerekore, Guinea, E-mail: ibrahimabakayoko27@gmail.com

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$$\mu(\alpha(x), \alpha(y)) = \alpha(\mu(x, y)) \quad (1)$$

$$\mu(\alpha(x),\mu(y, z))=\mu(\mu(x, y),\alpha(z)) \quad (2)$$

If in addition $\mu(x, y) = \varepsilon(x, y)\, \mu(y, x)$ the color Hom-associative algebra $(A, \mu, \varepsilon, \alpha)$ is said to be a ε -commutative color Hom-associative algebra.

Remark

When α=*Id* we recover the classical associative color algebra.

Recall that the Hom-associator, aS_A of a Hom-algebra A is defined as : $asA : A\otimes A\otimes A \to A$, 1

Observe that $as_A \equiv 0$ when A is a color-Hom-associative algebra.

Definition

Let $(A, \mu, \varepsilon, \alpha)$ and $(A', \mu', \varepsilon', \alpha')$ be two color Hom-associative algebras. An even linear map $f : A \to A'$ is said to be a morphism of color Hom-associative algebras if $f\circ\alpha = \alpha'\circ f$ and

$f(\mu(x, y))=\mu'(f(x),f(y))$

For all $x, y \in A$.

Lemma

([17]) Let $(A, \mu, \varepsilon,)$ be a color associative algebra and α be an even algebra endomorphism. Then $(A, \mu\alpha, \varepsilon, \alpha)$ where $\mu_\alpha = \alpha\circ\mu$ is a color Hom-associative algebra. Moreover, suppose that (A', μ', ε) be another color associative algebra and $\alpha' : A' \to A'$ be an even algebra endomorphism such that $f\circ\alpha = \alpha'\circ f$ then $f : (A,\mu',\varepsilon,\alpha) \to (A',\mu',\varepsilon,\alpha')$ is also a morphism of color Hom-associative algebras.

Definition

([17]) A color Hom-Lie algebra is a quadruple $(A, \{.,.\},\varepsilon, \alpha)$ consisting of a G-graded vector space A, an even bilinear map

$\{.,.\} : AXA \to A$ (i.e $\{A_a, A_b\} \subseteq A_{a+b}$ for all $a,b \in G$) a bicharacter, and an even linear map $\alpha : A \to A$ such that for any $x, y \in H(A)$ we have

$$\{x, y\} = -\varepsilon(x, y)\{y, x\} \quad (\varepsilon\text{ -Skew-symmetry}) \quad (3)$$

$$\alpha\circ\{.,.\} = \{.,.\}\circ\alpha^{\otimes 2} \quad (\text{Multiplicativity}) \quad (4)$$

$$\oint \varepsilon(z, x)\{\alpha(x), \{y, z\}\} = 0 \quad (\varepsilon\text{ -Hom-Jacobi identity}) \quad (5)$$

Where $\oint$ means cyclic summation.

By the ε skew symmetry 3 of the color Hom-Lie bracket {. , .}, the color Hom-Jacobi identity 5 is equivalent to

$$\{\{x, y\}, \alpha(z)\} = \{\alpha(x), \{y, z\}\} - \varepsilon(x, y)\{\alpha(y), \{x, z\}\} \quad (6)$$

Remark that a color Lie algebra $(A, \{.,.\}, \varepsilon)$ is a color Hom-Lie algebra with α=Id

Morphism of color Hom-Lie algebras are defined similarly to the Definition 4.3, where the color Hom-associative product is replaced by the color Hom-Lie bracket. Examples of color Hom-Lie algebras are provided in [2,3].

The following lemma connects color Hom-associative algebras to color Hom-Lie algebras.

Lemma

([17]) Let (A, μ, ε, α) be a color Hom-associative algebra.

Then $(A, \{.,.\} = \mu - \varepsilon(.,.)\mu^{op}, \varepsilon, \alpha)$ is a color Hom-Lie algebra, denoted by L (A).

Color Hom-Poisson algebras

Definition

A color Hom-Poisson algebra consists of a G-graded vector space A, a multiplication $\mu : AXA \to A$, an even bilinear bracket $\{.,.\} : AXA \to A$ and an even linear map $\alpha : A \to A$ such that

1. $(A, \mu, \varepsilon, \alpha)$ is a color Hom-associative algebra,
2. $(A, \{.,.\},\varepsilon, \alpha)$ is a color Hom-Lie algebra,
3. the color Hom-Leibniz identity is satisfied i.e.

$$\{\alpha(x), \mu(y, z)\} = \mu(\{x, y\}, \alpha(z)) + \varepsilon(x, y)\mu(\alpha(y), \{x, z\}), \quad (7)$$

For $x, y, z \in H(A)$ any

If in addition μ is ε commutative, the color Hom-Poisson algebra $(A, \{.,.\},\varepsilon, \alpha)$ is said to be a ε commutative color Hom-Poisson algebra.

The condition 7 expresses the compatibility between the color Hom-associative product μ and the color Hom-Lie bracket {.,.} it can be written equivalently

$$\{\mu(x, y), \alpha(z)\} = \mu(\alpha(x), \{y, z\}) + \varepsilon(x, y)\mu(\alpha(y), \{x, z\}). \quad (8)$$

Remark

We recover Poisson algebras ([6, 5]) when α = Id and $\varepsilon \equiv 1$

We need the following lemma in Proposition 6.1.

Lemma

If $(A, \mu \{.,.\}, \varepsilon, \alpha)$ is a ε commutative color Hom-Poisson algebra, then$(A, -\mu - \{.,.\},\varepsilon, \alpha)$ is also a ε commutative color Hom-Poisson algebra.

The following theorem is the color version of ([11], Proposition 4.6).

Theorem

Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra.

Then $(A, \mu, \{.,.\} = \mu - \varepsilon(.,.)\mu^{op}, \varepsilon, \alpha)\sqrt{b^2 - 4ac}$ is a color Hom-Poisson algebra.

Proof:

According to Lemma 4.2, it remains to prove the color Hom-Leibniz identity 7. For any $x, y, z \in H(A)$

$$\begin{aligned}
&\{\alpha(x), \mu(y,z)\} - \mu(\{x,y\},\alpha(z)) - \varepsilon(x,y)\mu(\alpha(y),\{x,z\}) = \\
&= \mu(\alpha(x),\mu(y,z)) - \varepsilon(x,y=z)\mu(\mu(x,y),\alpha(z)) \\
&= \varepsilon(x,)\mu(\mu(y,x),\alpha(z)) - \varepsilon(x,y)\mu(\alpha(y),\mu(x,z)) + \varepsilon(x,y)\varepsilon(x,z)\mu(\alpha(y),\mu(z,x)) \\
&= \mu(\alpha(x),\mu(y,z)) - \mu(\mu(x,y),\alpha(z)) - \varepsilon(x,y+z)\mu(\mu(y,z),\alpha(x)) \\
&+\varepsilon(x,y)\mu(\mu(y,x),\alpha(z)) + \varepsilon(x,y)\mu(\mu(y,x),\alpha(z)) - \varepsilon(x,y)\mu(\alpha(y),\mu(,z)) \\
&= -as_{\mu,\alpha}(x,y,z) - \varepsilon(x,)\varepsilon(x,z)as_{\mu,\alpha}(y,z,x) + \varepsilon(x,y)as_{\mu,\alpha}(y,x,z) = 0
\end{aligned}$$

This finishes the proof.

Corollary

Let $(A, \mu \{.,.\}, \varepsilon, \alpha)$ be a color associative algebra and α an even color algebra endomorphism. Then $(A, \mu_\alpha, \{.,.\}_\alpha, \varepsilon, \alpha)$ where

$\mu_\alpha = \alpha\circ\mu, \{.,.\}_\alpha = \mu_\alpha - \varepsilon(.,.)\mu_\alpha^{op}$ is a color Hom-Poisson algebra.

Proof

The proof follows from Lemma 4.1 and Theorem 3.1.

Lemma

Let *(A, μ {.,.} ,ε)* be a color Poisson algebra and α be an even color Poisson algebra endomorphism. Then $(A,\mu_\alpha,\{.,.\}_\alpha,\varepsilon,\alpha)$ is a color Hom Poisson algebra.

Proof

By Lemma 4.1 and ([3] Example1.2), we only need to prove the color Hom-Leibniz identity. For any $x,y,z\in H(A)$

$$\{\alpha(x),\mu_\alpha(y,z)\}_\alpha - \mu_\alpha(\{x,y\}_\alpha,\alpha(z)) - \varepsilon(x,y)\mu_\alpha(\alpha(y),\{x,z\}_\alpha) =$$

$$= \alpha^2(x),\alpha^2(\mu(y,z))\} - \mu(\alpha^2(\{x,y\}),\alpha^2(z)) - \varepsilon(x,y)\mu(\alpha^2(y),\alpha^2(\{x,z\}))$$

$$= \alpha^2(\{x,\mu(y,z)\} - \mu(\{x,y\},z) - \varepsilon(x,y)\mu(y,\{x,z\}))$$

$$= 0$$

This completes the proof.

Theorem

Let *(A, μ {.,.} ,ε, α)* be a color Hom-Poisson algebra and $\beta: A\to A$ be an even color Poisson algebra endomorphism. Then, $A_\beta = (A,\mu_\beta = \beta\circ\mu,\{.,.\}_\beta = \beta\circ\{.,.\},\varepsilon,\beta\circ\alpha)$ is a color Hom-Poisson algebra.

Moreover, suppose that *(A′, μ′, {.,.}′ ε)* is a color Poisson algebra and $(A,\{.,.\}_\beta,\varepsilon,\beta\circ\alpha)$ is an even color Poisson algebra endomorphism. If $f: A\to A'$ is a color Poisson algebra morphism that satisfies $f\circ\beta = \alpha'\circ f$, then

$$f:(A,\mu_\beta,\{.,.\}_\beta,\varepsilon,\beta\circ\alpha)\to(A',\mu'_\alpha,\{.,.\}'_\alpha,\varepsilon,\alpha)$$

is a color Hom-Poisson algebra homomorphism.

Proof

It is straightforward to show that $(A,\mu_\beta,\varepsilon,\beta\circ\alpha)$ is a color Hom associative algebra and $(A,\{.,.\}_\beta,\varepsilon,\beta\circ\alpha)$ is a color Hom-Lie algebra ([3] Theorem1.1). The proof of the color Hom-Leibniz identity is similar to that of Lemma 5.2.For the second assertion, we have

$$f(\mu_\beta(x,y)) = f(\mu(\beta(x),\beta(y))) = \mu'(f(\beta(x)),f(\beta(y)))$$

$$= \mu'(\alpha'(f(x)),\alpha'(f(y)) = \mu'_{\alpha'}(f(x),(y)).$$

We have a similar proof for the color Hom-Poisson bracket.

Corollary 5.2

Let *(A, μ {.,.} ,ε, α)*be a color Hom-Poisson algebra. Then

$$A^n = (A,\{.,.\}^{(n)} = \alpha^n\circ\{.,.\},\mu^{(n)} = \alpha^n\circ\mu,\varepsilon,\alpha^{n+1})$$

is a color Hom-Poisson algebra for each integer $n\geq 0$. We finish this section by studying deformations by composition of color HomPoisson algebras.

Definition 5.2

Let *(A, μ {.,.} ,ε, α)*be a color Hom-Poisson algebra. A one parameter formal deformation of A is given by K[[t]]-bilinear maps $\mu_t(.,.): A[[t]]XA[[t]]\to A[[t]]$ and $\{.,.\}_t: A[[t]]XA[[t]]\to A[[t]]$ of the form $\mu_t = \sum_{i\geq 0} t^i\mu_i$ and $\{.,.\} = \sum_{i\geq 0} t^i\{.,.\}_i$ where each μ_i and $\{.,.\}_i$ are K-bilinear maps and (extended to K[[t]]-bilinear maps), and such that for all following conditions be satisfied.

$$\mu_t(\alpha_t(x),\alpha_t(y)) = \alpha_t(\mu_t(x,y)), \tag{9}$$

$$\mu_t(\alpha_t(x),\mu_t(y,z)) = \mu_t(\mu_t(x,y),\alpha_t(z)), \tag{10}$$

$$\alpha_t\circ\{.,.\}_t = \{.,.\}_t\circ\alpha_t^{\otimes 2}, \tag{11}$$

$$\{x,y\}_t = -\varepsilon(x,y)\{y,x\}_t, \tag{12}$$

$$\oint\varepsilon(z,x)\{\alpha_t(x),\{y,z\}_t\}_t = 0, \tag{13}$$

$$\{\alpha_t(x),\mu_t(y,z)\}_t = \mu_t(\{x,y\}_t,\alpha_t(z)) + \varepsilon(x,y)\mu_t(\alpha_t(y),\{x,z\}_t). \tag{14}$$

The deformation is said to be of order k if $\mu_t = \sum_{i=0}^k t^i\mu_i$ and $\{.,.\}_t = \sum_{i=0}^k t^i\{.,.\}_i$

Proposition 5.1

Let *(A, μ {.,.} ,ε)* be a color Poisson algebra and α_t an even color Poisson algebra endomorphism of the form $\alpha_t = \alpha_0 + \sum_{i=0}^k t^i\alpha_i$ where α_i are endomorphism of A (as color Poisson algebra), t is a parameter in K and k is an entiger. Let μ_t α_t μ and $\{.,.\}_t = \alpha_t\circ\{.,.\}$ then $(A,\mu_t,\{.,.\}_t,\varepsilon,\alpha_t)$ is a color Hom-Poisson algebra which is a deformation of the color Poisson algebra *(A, μ {.,.} ,ε* viewed as a color Hom-Poisson algebra *(A, μ {.,.} ,ε, Id)*.

Proof

The proof follows from Theorem.2.

As in the case of Poisson algebras ([10,12,13]), the cohomology of color Hom-Poisson algebras is described by the cohomology of the underlying color Hom-Lie algebras ([3]).

Modules Over Color Hom-Poisson Algebras

Definition 6.1

Let G be an abelian group. A Hom-module is a pair (M, α_M) in which M is a G-graded vector space and $\alpha_M: M\to M$ is an even linear map.

Definition 6.2

Let $(A, M_A, \varepsilon, \alpha_A)$ be a color Hom-associative algebra. An A-module is a Hom-module (M, α_M) together with a bilinear map $\mu_M: A\otimes M\to M$ called structure map, such that

$$\mu_M(A_a.M_b)\subseteq M_{a+b} \tag{15}$$

$$\alpha_M\circ\mu_M = \mu_M\circ(\alpha_A\otimes\alpha_M), \tag{16}$$

$$\mu_M\circ(\alpha_A\otimes\mu_M) = \mu_M\circ(\mu_A\otimes\alpha_M) \tag{17}$$

Twisting a module structure map by an algebra endomorphism, we get another one as stated in the following Lemma.

Lemma 6.1

Let $(A, M_A, \varepsilon, \alpha_A)$ be a color Hom-associative algebra and M an A-module with structure map $\mu_M: A\otimes M\to M$ Define the map

$$\tilde{\mu}_M = \mu_M\circ(\alpha_A^2\otimes Id_M): A\otimes M\to M. \tag{18}$$

Then $\tilde{\mu}_M$ is the structure map of another A-module structure on M.

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Proof

The proof is similar to that of ([14], Lemma 4.5).

Definition 6.3

([3]) Let $(L, \{.,.\}, \varepsilon, \alpha_L)$ be a color Hom-Lie algebra and (M, α_M) a Hom-module. An L-module on M consists of a K-bilinear map $\mu_M : A\otimes M \to M$ such that

$$\mu_M(A_a, M_b) \subseteq M_{a+b} \tag{19}$$

$$\alpha_M(\mu_M(\mathrm{x},\mathrm{m})) = \mu_M(\alpha_L(\mathrm{x}), \alpha_M(m)),$$

$$\mu_M(\{\mathrm{x},\mathrm{y}\}, \alpha_M(\mathrm{m})) = \mu_M(\alpha_L(\mathrm{x}), \mu_M(,m)) \tag{20}$$

$$-\varepsilon(x,y)\mu_M(\alpha_L(\mathrm{y}), \mu_M(\mathrm{x},\mathrm{m})), \tag{21}$$

for any $\mathrm{m}\in H(\mathrm{M}), \mathrm{x},\mathrm{y}\in H(L)$

Remark 6.1

When $\alpha_M = Id_M$ and $\alpha = Id_L$ we recover the definition of Lie modules ([15-17]).

The following statement is the Lie analogue of Lemma 6.1.

Lemma 6.2

Let $(L, \{.,.\}, \varepsilon, \alpha)$ be a color Hom-Lie algebra and M an L-module with structure map $\mu_M = L\otimes M \to M$ Define the map

$$\tilde{\mu}_M = \mu_M \circ (\alpha^2 \otimes Id_M) : L\otimes M \to M \tag{22}$$

Then $\tilde{\mu}_M$ is the structure map of another L-module structure on M.

Proof

Equalities 19 and 20 are proved as in Lemma 6.1. Now, we prove 21 for $\tilde{\mu}_M$ .For any $x,y\in L, m\in M$

$$\begin{aligned}
\tilde{\mu}_M(\{x,y\}, \alpha_M(\mathrm{m})) &= \mu_M(\alpha^2\otimes Id)(\{x,y\}, \alpha_M(m))\\
&= \mu_M(\{\alpha^2(x), \alpha^2(y)\}, \alpha_M(m))\\
&= \mu_M(\alpha^3(\mathrm{x}), \mu_M(\alpha^2(\mathrm{y}),\mathrm{m})) - \varepsilon(\mathrm{x},\mathrm{y})\mu_M(\alpha^3(\mathrm{y}), \mu_M(\alpha^2(x),m))\\
&= \mu_M(\alpha^3(x), \mu_M(\alpha^2\otimes Id)(\mathrm{y}\otimes \mathrm{m}))\\
&\quad - \varepsilon(\mathrm{x},\mathrm{y})\mu_M(\alpha^3(y), \mu_M(\alpha^2\otimes Id)(x\otimes m))\\
&= \mu_M(\alpha^2(\alpha(x), \tilde{\mu}_M(y\otimes m)) - \varepsilon(x,y)\mu_M(\alpha^2(\alpha(Y)), \tilde{\mu}_M(\mathrm{x}\otimes \mathrm{m}))\\
&= \mu_M(\alpha^2\otimes Id)(\alpha(x)\otimes \tilde{\mu}_M(y\otimes m))\\
&\quad - \varepsilon(x,y)\mu_M(\alpha^2\otimes Id)(\alpha(\mathrm{y})\otimes \tilde{\mu}_M(x\otimes m))\\
&= \tilde{\mu}_M(\alpha(\mathrm{x}), \tilde{\mu}_M(y,m)) - \varepsilon(x,y)\tilde{\mu}_M(\alpha(y), \tilde{\mu}_M(x,m)).
\end{aligned}$$

Hence the conclusion holds.

The following result shows that A-modules extend to L(A)-modules with samemodule structure map.

Theorem 6.1

Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra and (M, A_M) an A-module with structure map μ_M . Then, M is a L(A)-module with structure map μ_M.

Proof

In fact, it suffices to show the relation 21. For any $x,y\in H(A), m\in H(M)$,We have

$$\begin{aligned}
&\mu_M(\alpha(x), \mu_M(y,m)) - \varepsilon(x,y)\mu_M(\alpha(y), \mu_M(x,m))\\
&= \mu_M(\mu(x,y), \alpha_M(m)) - \varepsilon(x,y)\mu_M(\mu(y,x), \alpha_M(m))\\
&= \mu_M(\mu(x,y) - \varepsilon(x,y)\mu(y,x), \alpha_M(m))\\
&= \mu_M(\{x,y\}, \alpha_M(\mathrm{m})).
\end{aligned}$$

This establishes the Theorem.

The corollaries below give a large class of examples of L(A)-modules.

Corollary 6.1

Let $A_\alpha = (\mathrm{A}, \mu_\alpha, \varepsilon, \alpha)$ be a color Hom-associative algebra as in Lemma 4.1 and (M, α_M) an A_α module with structure map μ_M. Then, M is an L(A)-module with structure map μ_M.

Proof

Prove 21. Indeed, for $x,y\in H(A), m\in H(M)$ we have

$$\begin{aligned}
&\mu_M(\alpha(x), \mu_M(y,m)) - \varepsilon(x,y)\mu_M(\alpha(y), \mu_M(x,m))\\
&= \mu_M(\mu_\alpha(x,y)), \alpha_M(m)) - \varepsilon(x,y)\mu_M(\mu_\alpha(y,x), \alpha_M(m))\\
&= \mu_M(\mu_\alpha(x,y) - \varepsilon(x,y)\mu_\alpha(y,x), \alpha_M(m))\\
&= \mu_M(\{x,y\}, \alpha_M(\mathrm{m})).
\end{aligned}$$

We conclude that M is an L(A)-module with structure map μ_M

Corollary 6.2

Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra and (M, α_M) an A-module with structure map μ_M Put

$$\tilde{\mu}_M = \mu_M \circ (\alpha_A^2 \otimes Id_M)$$

Then M is an L(A)-module with structure map $\tilde{\mu}_M$.

Proof

We know from Lemma (6.1) that $\tilde{\mu}_M$ is an A-module structure map. And, for $x,y\in H(A),\ m\in H(M)$, one has

$$\begin{aligned}
&\tilde{\mu}_M(\alpha(x), \tilde{\mu}_M(y,m)) - \varepsilon(x,y)\tilde{\mu}_M(\alpha(\mathrm{y}), \tilde{\mu}_M(\mathrm{x},m))\\
&= \mu_M(\alpha^2(\alpha(x)), \mu_M(\alpha^2(y),m)) - \varepsilon(x,y)\mu_M(\alpha^2(\alpha(\mathrm{y})), \mu_M(\alpha^2(\mathrm{x}),m))\\
&= \mu_M(\alpha(\alpha^2(x)), \mu_M(\alpha^2(y),m)) - \varepsilon(x,y)\mu_M(\alpha(\alpha^2(\mathrm{y})), \mu_M(\alpha^2(\mathrm{x}),m))\\
&= \mu_M(\mu(\alpha^2(x), \alpha^2(y)), \alpha_M(m)) - \varepsilon(x,y)\mu_M(\mu(\alpha^2(\mathrm{y}), \alpha^2(\mathrm{x})), \alpha_M(m))\\
&= \mu_M(\alpha^2(\mu(x,y) - \varepsilon(x,y)\mu(y,x)), \alpha_M(m))\\
&= \mu_M(\alpha^2(\{x,y\}), \alpha_M(m))\\
&= \tilde{\mu}_M(\{x,y\}, \alpha_M(m)).
\end{aligned}$$

This is similar to the relation 21 for $\tilde{\mu}_M$

Now we define modules for color Hom-Poisson algebras.

Definition 6.4

Let $(A, \mu, \{.,.\}, \varepsilon, \alpha)$ be a ε commutative color Hom-Poisson algebra and (M, α_M) a Hom-module.

A color Hom-Poisson module structure on M consists of two K-bilinear maps $\mu_M : A\otimes M \to M$ and $\lambda_M : A\otimes M \to M$ such that

(i) M is an A-module and an L-module,

(ii) And for any $x, y \in H(A), m \in H(M)$,

$$\lambda_M(\alpha(x), \mu_M(y,m)) = \mu_M(\{x,y\}, \alpha_M(m)) + \varepsilon(x,y)\mu_M(\alpha(y), \lambda_M(x,m)), \tag{23}$$

$$\lambda_M(\mu(x,y), \alpha_M(m)) = \mu_M(\alpha(x), \lambda_M(y,\mathrm{m})) + \varepsilon(x,y)\mu_M(\alpha(y), \lambda_M(x,m)). \tag{24}$$

When $\alpha = Id_A, \alpha_M = Id_M and\ \varepsilon \equiv 1$

We recover the definition of modules over Poisson algebras ([9]).

Example 6.1

(i) Any module over a ε commutative color Hom-associative algebra (resp. color Hom-Lie algebra) can be seen as a module over a ε commutative color Hom-Poisson algebra with the trivial color Hom-Lie bracket (resp. trivial color Hom-associative product).

(ii) Any ε commutative color Hom-Poisson algebra is a module over itself.

Example 6.2

Let $(V, \mu_V, \lambda_V, \alpha_V)$ and $(W, \mu_W, \lambda_W, \alpha_W)$ be two modules over the ε Commutative color Hom-Poisson algebra $(A, \mu, \{.,.\}, \varepsilon, \alpha)$

Then the direct product $M = V \times W$ is a module over A with structure maps $\mu_M : A\otimes M \to M$, $\lambda_M : A\otimes M \to M$ and $\alpha_M : M \to M$

Defined by

$\mu_M(x(v,w)) = (\mu_V(x,v), \mu_W(x,\mathrm{w})), \lambda_M(x,(\mathrm{v},\mathrm{w})) = (\lambda_V(x,v), \lambda_W(x,\mathrm{w}))$ and $\alpha_M(v,w) = (\alpha_V(v), \alpha_W(w))$

for any $x \in H(A), v \in H(V)\ and\ w \in H(W)$.

Proposition 6.1

If $(M, \mu_M, \lambda_M, \alpha_M)$ is a module over the ε commutative color Hom-Poisson algebra $(A, \mu, \{.,.\}, \varepsilon, \alpha)$ then $(M, -\mu_M, -\lambda_M, \alpha_M)$ is also a module

Over the ε commutative color Hom-Poisson algebra $(A, -\mu, -\{.,.\}, \varepsilon, \alpha)$

Proof

The proof comes from Definition 6.4 and Lemma 5.1.

Theorem 6.2

Let $(A, \mu, \{.,.\}, \varepsilon, \alpha)$ be A ε commutative color Hom-Poisson algebra and $(M, \mu_M, \lambda_M, \alpha_M)$ color Hom-Poisson module. Then

$$\tilde{\mu}_M = \mu_M \circ (\alpha^2 \otimes Id_M) : \mathrm{A}\otimes M \to M, \tag{25}$$

$$\tilde{\lambda}_M = \lambda_M \circ (\alpha^2 \otimes Id_M) : \mathrm{A}\otimes M \to M, \tag{26}$$

Define another color Hom-Poisson module structure on M.

Proof

We know that $\tilde{\mu}_M$ is a structure of another A-module structure map on M (Lemma 6.1) and $\tilde{\lambda}_M$ is a structure of another L-module structure map on M (Lemma 6.2). Show relations 23 and 24 for $\tilde{\mu}_M$ and $\tilde{\lambda}_M$. For all $x, y \in H(A)$ and $m \in H(M)$

$$\begin{aligned}
&\tilde{\lambda}_M(\alpha \otimes \tilde{\mu}_M)(x\otimes y\otimes m) = \tilde{\lambda}_M(\alpha(x), \tilde{\mu}_M(y\otimes m))\\
&= \tilde{\lambda}_M(\alpha(x), \mu_M(\alpha^2 \otimes Id))(y\otimes m)\\
&= \lambda_M(\alpha^2 \otimes Id)(\alpha(x), \mu_M(\alpha^2(y)\otimes m))\\
&= \lambda_M(\alpha^3, \mu_M(\alpha^2 \otimes m))\\
&= \mu_M(\{\alpha_A^2(x), \alpha_A^2(y)\}, \alpha_M(m))\\
&\quad + \varepsilon(\mathrm{x},\mathrm{y})\mu_M(\alpha_A^3(y), \lambda_M(\alpha_A^2(x), m)) \quad (by\ (23))\\
&= \mu_M(\alpha^2(\{x,y\}), \alpha_M(m))\\
&\quad + \varepsilon(\mathrm{x},\mathrm{y})\mu_M(\alpha^2 \otimes Id)(\alpha(y), \lambda_M(\alpha^2 \otimes Id)(x\otimes m))\\
&= \mu_M(\alpha^2 \otimes Id)(\{x,y\}\otimes \alpha_M(m)) + \varepsilon(\mathrm{x},\mathrm{y})\tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x\otimes m))\\
&= \tilde{\mu}_M(\{x,y\}\otimes \alpha_M(m)) + \varepsilon(\mathrm{x},\mathrm{y})\tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x\otimes m)).
\end{aligned}$$

And, for $x, y \in H(A)$ and $m \in H(M)$

$$\begin{aligned}
&\tilde{\lambda}_M(\mu\otimes \alpha_M)(x\otimes y\otimes m)\\
&= \lambda_M(\alpha^2 \otimes Id)(\mu(x\otimes y)\otimes \alpha_M(m))\\
&= \lambda_M(\alpha^2(\mu(x\otimes y))\otimes \alpha_M(m))\\
&= \mu_M(\alpha^3(x), \lambda_M(\alpha^2(y), m))\\
&\quad + \varepsilon(\mathrm{x},\mathrm{y})\mu_M(\alpha^3(y), \lambda_M(\alpha^2(x), m)) \quad (by\ (24))\\
&= \mu_M(\alpha^2 \otimes Id)(\alpha(x), \lambda_M(\alpha^2 \otimes Id)(y\otimes m))\\
&\quad + \varepsilon(\mathrm{x},\mathrm{y})\mu_M(\alpha^2 \otimes Id)(\alpha(y), \lambda_M(\alpha^2 \otimes Id)(x\otimes m))\\
&= \tilde{\mu}_M(\alpha(x), \tilde{\lambda}_M(y\otimes m)) + \varepsilon(\mathrm{x},\mathrm{y})\tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x\otimes m)).
\end{aligned}$$

Hence equations 23 and 24 hold for $\tilde{\mu}_M$ and $\tilde{\lambda}_M$. This completes the proof.

Corollary 6.3

Let $(A,\mu,\{.,.\},\varepsilon)$ be A ε commutative color Poisson algebra and $(M, M_M, \lambda_M, \alpha_M)$ A module over the color Hom-Poisson algebra $(A, \mu_\alpha = \alpha\circ\mu, \{.,.\}_\alpha = \alpha\circ\{.,.\}, \varepsilon, \alpha)$. Then $\tilde{\mu}_M, \tilde{\lambda}_M$ define another color Hom- Poisson module structure on M.

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