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Several Properties of Multiple Hypergeometric Euler Numbers

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Abstract. In this paper, we introduce the higher order hypergeometric Euler numbers and show several interesting expressions. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers. One advantage of hypergeometric numbers, including Bernoulli, Cauchy and Euler hypergeometric numbers, is the natural extension of determinant expressions of the numbers. As applications, we can get the inversion relations such that Euler numbers are elements in the determinant.

1. Introduction

Euler numbers E_n, defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}, \tag{1}$$

have been investigated by many authors in various aspects. Euler numbers also occur in combinatorics, in particular, when the number of alternating permutations of a set with an even number of elements. Recently, poly-Euler numbers [22] are introduced and studied via special values of an L-function as a generalization of the Euler numbers.

The *hypergeometric Euler numbers* $E_{N,n}$ ([20]) are defined by

$$\frac{1}{{}_1F_2(1; N+1, (2N+1)/2; x^2/4)} = \sum_{n=0}^{\infty} E_{N,n} \frac{x^n}{n!}, \tag{2}$$

where ${}_1F_2(a; b, c; z)$ is the hypergeometric function given by

$${}_1F_2(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}} \frac{z^n}{n!}$$

with the rising factorial $(x)^{(n)} = x(x+1)\dots(x+n-1)$ $(n \ge 1)$ and $(x)^{(0)} = 1$. When $N = 0$, then $E_n = E_{0,n}$ are classical Euler numbers, given in (1). A similar concept in the

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polynomial version of hypergeometric Euler numbers leads to *Truncated Euler polynomials* [17].

Several other kinds of analogues or generalizations of the Euler numbers have been considered by many authors. For example, poly-Euler number [22], multiple Euler numbers, Apostol-Euler numbers, Frobenius-Euler numbers, various types of q-Euler numbers. However, the hypergeometric Euler numbers have several interesting properties.

Similar hypergeometric numbers are hypergeometric Bernoulli numbers $B_{N,n}$ and *hypergeometric Cauchy numbers* $c_{N,n}$. For $N \ge 1$, define *hypergeometric Bernoulli numbers* $B_{N,n}$ ([8, 9, 10, 11, 14]) by

$$\frac{1}{{}_1F_1(1; N+1; x)} = \frac{x^N/N!}{e^x - \sum_{n=0}^{N-1} x^n/n!} = \sum_{n=0}^{\infty} B_{N,n}\frac{x^n}{n!}, \tag{3}$$

where

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!}$$

is the confluent hypergeometric function. When $N = 1$, $B_{1,n} = B_n$ are classical Bernoulli numbers, given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!}.$$

For $N \ge 1$, define the *hypergeometric Cauchy numbers* $c_{N,n}$ ([15]) by

$$\frac{1}{{}_2F_1(1, N; N+1; -x)} = \frac{(-1)^{N-1}x^N/N}{\log(1+x) - \sum_{n=1}^{N-1}(-1)^{n-1}x^n/n} = \sum_{n=0}^{\infty} c_{N,n}\frac{x^n}{n!}, \tag{4}$$

where

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}$$

is the Gauss hypergeometric function. When $N = 1$, $c_n = c_{1,n}$ are classical Cauchy numbers ([4]) given by

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} c_n\frac{x^n}{n!}.$$

Notice that $b_n = c_n/n!$ are sometimes called Bernoulli numbers of the second kind.

These hypergeometric numbers have several interesting properties, which have not been seen in other generalized numbers. One advantage of hypergeometric numbers is the natural extension of determinant expressions of the numbers.

In [13], the hypergeometric Bernoulli numbers $B_{N,n}$ $(N \geq 1,\, n \geq 1)$ can be expressed as

$$B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix}.$$

When $N = 1$, we have a determinant expression of Bernoulli numbers ([5, p. 53]).

In [1, 19], the hypergeometric Cauchy numbers $c_{N,n}$ $(N \geq 1,\, n \geq 1)$ can be expressed as

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix}.$$

When $N = 1$, we have a determinant expression of Cauchy numbers ([5, p. 50]).

In [13], such concepts have been generalized as multiple hypergeometric Bernoulli numbers $B_{N,n}^{(r)}$, defined by

$$\frac{1}{\bigl({}_1F_1(1; N+1; x)\bigr)^r} = \sum_{n=0}^{\infty} B_{N,n}^{(r)} \frac{x^n}{n!}.$$

When $r = 1$, $B_{N,n} = B_{N,n}^{(1)}$ are the original hypergeometric Bernoulli numbers. In [12, 13], multiple hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ are introduced and studied, so that $B_{N,n}^{(r)} = B_{N,n}^{(r)}(0)$.

In [19], multiple hypergeometric Cauchy numbers $c_{N,n}^{(r)}$, defined by

$$\frac{1}{\bigl({}_2F_1(1, N; N+1; -x)\bigr)^r} = \sum_{n=0}^{\infty} c_{N,n}^{(r)} \frac{x^n}{n!}.$$

When $r = 1$, $c_{N,n} = c_{N,n}^{(1)}$ are the original hypergeometric Cauchy numbers. In [19], multiple hypergeometric Cauchy polynomials are introduced and studied, holding the similar interesting properties.

In [16, 20], the hypergeometric Euler numbers $E_{N,2n}$ $(N \ge 0, n \ge 1)$ are expressed as

$$E_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N)!}{(2N+2)!} & 1 & & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} \end{vmatrix} . \tag{5}$$

When $N = 0$, this is reduced to a famous determinant expression of Euler numbers (*cf.* [5, p. 52]).

Such determinant expressions may be obvious or artificial for the readers with different backgrounds. However, there are motivations from combinatorics, in particular, graph theory. In 1989, Cameron [3] considered the operator A defined on the set of sequences of non-negative integers as follows. For $x = \{x_n\}_{n\ge 1}$ and $z = \{z_n\}_{n\ge 1}$, let $Ax = z$, where

$$1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{\infty} x_n t^n\right)^{-1} . \tag{6}$$

Suppose that x enumerates a class C. Then Ax enumerates the class of disjoint unions of members of C, where the order of the "component" members of C is significant. The operator A also plays an important role for free associative (non-commutative) algebras. More motivations and background together with many concrete examples (in particular, in the aspects of Graph theory) by this operator can be seen in [3].

In this paper, we give several expressions of multiple hypergeometric Euler numbers, including determinant expressions. As one of the applications, we give the inversion relations such that Euler numbers are elements in the determinant. We also show some inversion expressions related to the Cameron's operator in Trudi's formula.

2. Some basic properties of the hypergeometric Euler numbers

From definition (2), we have the following proposition. Note that $E_{N,n} = 0$ when n is odd.

$$\sum_{i=0}^{n/2} \frac{1}{(2N+n-2i)!(2i)!} E_{N,2i} = 0 \quad (n \ge 2 \text{ is even}) \tag{7}$$

and $E_{N,0} = 1$.

By identity (7) or the identity

$$E_{N,n} = -n!(2N)! \sum_{i=0}^{n/2-1} \frac{E_{N,2i}}{(2N+n-2i)!(2i)!} ,$$

we can obtain the values of $E_{N,n}$ $(n = 0, 2, 4, \dots)$. We record the first few values of $E_{N,2n}$ in Appendix.

We have an explicit expression of $E_{N,n}$ for each even n ([20]). Namely, for $N \ge 0$ and $n \ge 1$ we have

$$E_{N,2n} = (2n)! \sum_{r=1}^{n} (-1)^r \sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r \ge 1}} \frac{\big((2N)!\big)^r}{(2N+2i_1)! \cdots (2N+2i_r)!} \,. \tag{8}$$

For $N = 1$, there is a relation between hypergeometric Euler numbers and Bernoulli numbers [20, Theorem 3.1]:

$$E_{1,n} = -(n-1)B_n \quad (n \ge 1) \,.$$

3. Multiple hypergeometric Euler numbers

For positive integers N and r, define the *higher order hypergeometric Euler numbers* $E_{N,n}^{(r)}$ by the generating function

$$\begin{aligned}\frac{1}{\big({}_1F_2(1; N+1, (2N+1)/2; x^2/4)\big)^r} &= \left(\frac{x^{2N}/(2N)!}{\cosh x - \sum_{i=0}^{N-1} x^{2i}/(2i)!}\right)^r \\ &= \sum_{n=0}^{\infty} E_{N,n}^{(r)} \frac{x^n}{n!} \,. \end{aligned} \tag{9}$$

From the definition (9),

$$\begin{aligned}\left(\frac{x^{2N}}{(2N)!}\right)^r &= \left(\sum_{i=0}^{\infty} \frac{x^{2N+2i}}{(2N+2i)!}\right)^r \left(\sum_{n=0}^{\infty} E_{N,n}^{(r)} \frac{x^n}{n!}\right) \\ &= x^{2rN} \Bigg(\sum_{l=0}^{\infty} \sum_{\substack{i_1+\cdots+i_r=l\\ i_1,\dots,i_r \ge 0}} \frac{l!}{(2N+i_1)! \cdots (2N+i_r)!} \\ &\quad \times \frac{\big(1+(-1)^{i_1}\big) \cdots \big(1+(-1)^{i_1}\big)}{2^r} \frac{x^l}{l!} \Bigg) \left(\sum_{m=0}^{\infty} E_{N,m}^{(r)} \frac{x^m}{m!} \right) \\ &= x^{2rN} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\substack{i_1+\cdots+i_r=n-m\\ i_1,\dots,i_r \ge 0}} \binom{n}{m} \frac{(n-m)!}{(2N+i_1)! \cdots (2N+i_r)!} \\ &\quad \times \frac{\big(1+(-1)^{i_1}\big) \cdots \big(1+(-1)^{i_1}\big)}{2^r} E_{N,m}^{(r)} \frac{x^n}{n!} \,. \end{aligned}$$

Hence, as a generalization of (7), for $n \ge 1$, we have the following relation.

PROPOSITION 1.

$$\sum_{m=0}^{n}\sum_{\substack{i_1+\cdots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{E_{N,2m}^{(r)}}{(2m)!(2N+2i_1)!\cdots(2N+2i_r)!}=0$$

with $E_{N,0}^{(r)}=1$.

Note that by (9), $E_{N,n}^{(r)}=0$ for odd n.

By Proposition 1 or

$$E_{N,2n}^{(r)}=-(2n)!\big((2N)!\big)^r\sum_{m=0}^{n-1}\sum_{\substack{i_1+\cdots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{E_{N,2m}^{(r)}}{(2m)!(2N+2i_1)!\cdots(2N+2i_r)!} \tag{10}$$

with $E_{N,0}^{(r)}=1$ ($N\ge 1$), some initial values of $E_{N,2n}^{(r)}$ ($0\le n\le 4$) are explicitly given (see Appendix).

As a generalization of (8), we have an explicit expression of $E_{N,n}^{(r)}$.

THEOREM 1. *For* $N\ge 0$ *and* $n\ge 1$, *we have*

$$E_{N,2n}^{(r)}=(2n)!\sum_{k=1}^{n}(-1)^k\sum_{\substack{e_1+\cdots+e_k=n\\ e_1,\dots,e_k\ge 1}}R_r(2e_1)\cdots R_r(2e_k)\,,$$

where

$$R_r(2e)=\sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}}\frac{\big((2N)!\big)^r}{(2N+2i_1)!\cdots(2N+2i_r)!}\,. \tag{11}$$

The first few values of $R_r(2e)$ are given in Appendix.

We shall introduce the Hasse-Teichmüller derivative in order to prove Theorem 1 easily. Let $\mathbf{F}$ be a field of any characteristic, $\mathbf{F}[[z]]$ the ring of formal power series in one variable z, and $\mathbf{F}((z))$ the field of Laurent series in z. Let n be a nonnegative integer. We define the Hasse-Teichmüller derivative $H^{(n)}$ of order n by

$$H^{(n)}\Big(\sum_{m=R}^{\infty}c_m z^m\Big)=\sum_{m=R}^{\infty}c_m\binom{m}{n}z^{m-n}$$

for $\sum_{m=R}^{\infty}c_m z^m\in\mathbf{F}((z))$, where R is an integer and $c_m\in\mathbf{F}$ for any $m\ge R$. Note that $\binom{m}{n}=0$ if $m<n$.

The Hasse-Teichmüller derivatives satisfy the product rule [23], the quotient rule [6] and the chain rule [7]. One of the product rules can be described as follows.

LEMMA 1. *For $f_i \in \mathbf{F}[[z]]$ $(i = 1, \dots, k)$ with $k \ge 2$ and for $n \ge 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k \ge 0}} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k)\,.$$

The quotient rules can be described as follows.

LEMMA 2. *For $f \in \mathbf{F}[[z]]\backslash\{0\}$ and $n \ge 1$, we have*

$$H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k \ge 1}} H^{(i_1)}(f) \cdots H^{(i_k)}(f) \tag{12}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k \ge 0}} H^{(i_1)}(f) \cdots H^{(i_k)}(f)\,. \tag{13}$$

PROOF OF THEOREM 1. Put $h(x) = \big(f(x)\big)^r$, where

$$f(x) = \frac{\sum_{i=N}^{\infty} \frac{x^{2i}}{(2i)!}}{\frac{x^{2N}}{(2N)!}} = \sum_{j=0}^{\infty} \frac{(2N)!}{(2N+j)!} \frac{1+(-1)^j}{2} x^j\,.$$

Since

$$H^{(i)}(f)\Big|_{x=0} = \sum_{j=i}^{\infty} \frac{(2N)!}{(2N+j)!} \frac{1+(-1)^j}{2} \binom{j}{i} x^{j-i} \Bigg|_{x=0}$$

$$= \frac{(2N)!}{(2N+i)!} \frac{1+(-1)^i}{2}\,,$$

we have for $i \ge 0$

$$H^{(2i)}(f)\Big|_{x=0} = \frac{(2N)!}{(2N+2i)!} \quad \text{and} \quad H^{(2i+1)}(f)\Big|_{x=0} = 0\,.$$

Hence, by the product rule of the Hasse-Teichmüller derivative in Lemma 1, we get

$$H^{(2e)}(h)\Big|_{x=0} = \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r \ge 0}} H^{(2i_1)}(f)\Big|_{x=0} \cdots H^{(2i_r)}(f)\Big|_{x=0}$$

$$= \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r \ge 0}} \frac{(2N)!}{(2N+2i_1)!} \cdots \frac{(2N)!}{(2N+2i_r)!}$$

$$= \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r \ge 0}} \frac{\big((2N)!\big)^r}{(2N+2i_1)! \cdots (2N+2i_r)!} = R_r(2e)$$

and $H^{(2e+1)}(h)\big|_{x=0} = 0$. Hence, by the quotient rule of the Hasse-Teichmüller derivative in Lemma 2 (12), we have

$$\frac{E_{N,2n}^{(r)}}{(2n)!} = \sum_{k=1}^{n} \frac{(-1)^k}{h^{k+1}}\bigg|_{x=0} \sum_{\substack{e_1+\cdots+e_k=n \\ e_1,\dots,e_k\ge 1}} H^{(2e_1)}(h)\Big|_{x=0} \cdots H^{(2e_k)}(h)\Big|_{x=0}$$
$$= \sum_{k=1}^{n}(-1)^k \sum_{\substack{e_1+\cdots+e_k=n \\ e_1,\dots,e_k\ge 1}} R_r(2e_1)\cdots R_r(2e_k)\,.$$

□

Now, we can show a determinant expression of $E_{N,2n}^{(r)}$.

THEOREM 2. *For $N, n \ge 1$, we have*

$$E_{N,2n}^{(r)} = (-1)^n (2n)! \begin{vmatrix} R_r(2) & 1 & & & \\ R_r(4) & R_r(2) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R_r(2n-2) & R_r(2n-4) & \cdots & R_r(2) & 1 \\ R_r(2n) & R_r(2n-2) & \cdots & R_r(4) & R_r(2) \end{vmatrix},$$

where $R_r(2e)$ are given by (11).

REMARK 1. When $r = 1$ in Theorem 2, we get (5).

PROOF. For simplicity, put $F_{N,2n}^{(r)} = (-1)^n E_{N,2n}^{(r)}/(2n)!$. Then, we shall prove that for any $n \ge 1$

$$F_{N,2n}^{(r)} = \begin{vmatrix} R_r(2) & 1 & & & \\ R_r(4) & R_r(2) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R_r(2n-2) & R_r(2n-4) & \cdots & R_r(2) & 1 \\ R_r(2n) & R_r(2n-2) & \cdots & R_r(4) & R_r(2) \end{vmatrix}. \tag{14}$$

When $n = 1$, (14) is valid because

$$R_r(2) = \frac{r}{(2N+1)(2N+2)} = F_{N,2}^{(r)}\,.$$

Assume that (14) is valid up to $n-1$. Notice that by (10), we have

$$F_{N,2n}^{(r)} = \sum_{l=1}^{n} (-1)^{l-1} F_{N,2n-2l}^{(r)} R_r(2l)\,.$$

Thus, by expanding the first row of the right-hand side of (14), it is equal to

$$
R_r(2)F_{N,2n-2}^{(r)}-\begin{vmatrix} R_r(4) & 1 & & & \\ R_r(6) & R_r(2) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R_r(2n-2) & R_r(2n-6) & \cdots & R_r(2) & 1 \\ R_r(2n) & R_r(2n-4) & \cdots & R_r(4) & R_r(2) \end{vmatrix}
$$

$$
\begin{aligned}
&= R_r(2)F_{N,2n-2}^{(r)}-R_r(4)F_{N,2n-4}^{(r)} \\
&\quad +\begin{vmatrix} R_r(6) & 1 & & & \\ R_r(8) & R_r(2) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R_r(2n-2) & R_r(2n-8) & \cdots & R_r(2) & 1 \\ R_r(2n) & R_r(2n-6) & \cdots & R_r(4) & R_r(2) \end{vmatrix} \\
&= R_r(2)F_{N,2n-2}^{(r)}-R_r(4)F_{N,2n-4}^{(r)}+\cdots \\
&\quad +(-1)^{n-3}R_r(2n-4)F_{N,4}^{(r)}+(-1)^{n-2}\begin{vmatrix} R_r(2n-2) & 1 \\ R_r(2n) & R_r(2) \end{vmatrix} \\
&= \sum_{l=1}^{n}(-1)^{l-1}R_r(2l)F_{N,2n-2l}^{(r)}=F_{N,2n}^{(r)}\,.
\end{aligned}
$$

Note that $F_{N,2}^{(r)}=R_r(2)$ and $F_{N,0}^{(r)}=1$. □

4. Multiple hypergeometric Euler numbers of the second kind

For positive integers N and r, define the *higher order hypergeometric Euler numbers of the second kind* $\widehat{E}_{N,n}^{(r)}$ by the generating function

$$
\begin{aligned}
\frac{1}{\big({}_1F_2(1;N+1,(2N+3)/2;x^2/4)\big)^r} &= \left(\frac{x^{2N+1}/(2N+1)!}{\sinh x-\sum_{i=0}^{N-1}x^{2i+1}/(2i+1)!}\right)^r \\
&= \sum_{n=0}^{\infty}\widehat{E}_{N,n}^{(r)}\frac{x^n}{n!}\,. \qquad (15)
\end{aligned}
$$

When $r=1$, $\widehat{E}_{N,n}=\widehat{E}_{N,n}^{(1)}$ are the *hypergeometric Euler numbers of the second kind*, introduced and studied in [16]. When $r=1$ and $N=0$, $\widehat{E}_n=\widehat{E}_{0,n}^{(1)}$ are the *Euler numbers of the second kind*, given by

$$
\frac{x}{\sinh x}=\sum_{n=0}^{\infty}\widehat{E}_n\frac{x^n}{n!}\,.
$$

From the definition (15), as an analogue of (7), for $n \ge 1$, we have the following relation.

PROPOSITION 2.

$$\sum_{m=0}^{n} \sum_{\substack{i_1+\cdots+i_r=n-m \\ i_1,\dots,i_r\ge 0}} \frac{\widehat{E}_{N,2m}^{(r)}}{(2m)!(2N+2i_1+1)!\cdots(2N+2i_r+1)!}=0$$

with $\widehat{E}_{N,0}^{(r)}=1$.

Notice that by the definition (9), $\widehat{E}_{N,n}^{(r)}=0$ for odd n.

By using Proposition 2 or

$$\begin{aligned}\widehat{E}_{N,2n}^{(r)}&=-(2n)!\big((2N+1)!\big)^r\\ &\quad\times\sum_{m=0}^{n-1} \sum_{\substack{i_1+\cdots+i_r=n-m \\ i_1,\dots,i_r\ge 0}} \frac{\widehat{E}_{N,2m}^{(r)}}{(2m)!(2N+2i_1+1)!\cdots(2N+2i_r+1)!}\end{aligned} \tag{16}$$

with $\widehat{E}_{N,0}^{(r)}=1$ ($N\ge 1$), some values of $\widehat{E}_{N,2n}^{(r)}$ ($0\le n\le 4$) are explicitly given (see Appendix).

As an analogue of Theorem 1, we have an explicit expression of $\widehat{E}_{N,n}^{(r)}$.

THEOREM 3. *For $N\ge 0$ and $n\ge 1$, we have*

$$\widehat{E}_{N,2n}^{(r)}=(2n)!\sum_{k=1}^{n}(-1)^k \sum_{\substack{e_1+\cdots+e_k=n \\ e_1,\dots,e_k\ge 1}} \widehat{R}_r(2e_1+1)\cdots\widehat{R}_r(2e_k+1)\,,$$

where

$$\widehat{R}_r(2e+1)=\sum_{\substack{i_1+\cdots+i_r=e \\ i_1,\dots,i_r\ge 0}} \frac{\big((2N+1)!\big)^r}{(2N+2i_1+1)!\cdots(2N+2i_r+1)!}\,. \tag{17}$$

REMARK 2. When $r=1$, the expression in Theorem 3 is reduced to an explicit expression of $\widehat{E}_{N,n}$ for each even n ([20]). Namely, for $N\ge 0$ and $n\ge 1$ we have

$$\widehat{E}_{N,2n}=(2n)!\sum_{r=1}^{n}(-1)^r \sum_{\substack{i_1+\cdots+i_r=n \\ i_1,\dots,i_r\ge 1}} \frac{\big((2N+1)!\big)^r}{(2N+2i_1+1)!\cdots(2N+2i_r+1)!}\,. \tag{18}$$

The first few values of $\widehat{R}_r(2e+1)$ are given in Appendix.

Similarly to Theorem 2, we can show a determinant expression of $\widehat{E}_{N,2n}^{(r)}$.

THEOREM 4. *For $N, n \ge 1$, we have*

$$\widehat{E}^{(r)}_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \widehat{R}_r(3) & 1 & & & \\ \widehat{R}_r(5) & \widehat{R}_r(3) & & & \\ \vdots & \vdots & \ddots & 1 & \\ \widehat{R}_r(2n-1) & \widehat{R}_r(2n-3) & \cdots & \widehat{R}_r(3) & 1 \\ \widehat{R}_r(2n+1) & \widehat{R}_r(2n-1) & \cdots & \widehat{R}_r(5) & \widehat{R}_r(3) \end{vmatrix},$$

where $\widehat{R}_r(2e+1)$ are given by (17).

REMARK 3. When $r = 1$ in Theorem 4, we have Theorem 1 in [16]:

$$\widehat{E}_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N+1)!}{(2N+3)!} & 1 & & \\ \frac{(2N+1)!}{(2N+5)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N+1)!}{(2N+2n+1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} \end{vmatrix}.$$

When $r = 1$ and $n = 0$ in Theorem 4, we have a determinant expression of $\widehat{E}_{2n}$ in [16, Corollay 1].

5. Different expressions of multiple hypergeometric Euler numbers

In this section, by using a little different method, we shall give expressions of $E^{(r)}_{N,n}$ and $\widehat{E}^{(r)}_{N,n}$, which are different from Theorem 1 and Theorem 3, respectively. Their expressions in this section can be recognized simpler because of fewer numbers of the summations.

THEOREM 5. *For $N \ge 0$ and $n \ge 1$, we have*

$$E^{(r)}_{N,2n} = (2n)! \sum_{k=1}^{2n} (-1)^k \binom{r+k-1}{r-1} \binom{r+2n}{r+k} R_k(2n),$$

where $R_k(2n)$ is given by (11).

PROOF. Put $1 + w = {}_1F_2(1; N+1, (2N+1)/2; x^2/4)$. By the definition (9),

$$\begin{aligned} E^{(r)}_{N,n} &= \frac{d^n}{dx^n}(1+w)^{-r}\bigg|_{x=0} \\ &= \frac{d^n}{dx^n}\left(\sum_{l=0}^{\infty}\binom{r+l-1}{l}(-w)^l\right)\bigg|_{x=0} \\ &= \sum_{l=0}^{n}\binom{r+l-1}{l}\frac{d^n}{dx^n}(-w)^l\bigg|_{x=0} \end{aligned}$$

$$= \sum_{l=0}^{n} \binom{r+l-1}{l} \sum_{k=0}^{l} (-1)^k \binom{l}{k} \frac{d^n}{dx^n} \big({}_1F_2(1; N+1, (2N+1)/2; x^2/4)\big)^k \Big|_{x=0} .$$

By the proof of Theorem 1, we know that if $k \ge 1$, then

$$\frac{d^n}{dx^n} \big({}_1F_2(1; N+1, (2N+1)/2; x^2/4)\big)^k \Big|_{x=0} = \begin{cases} n!R_k(n) & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

It is easily checked that the relation above holds also in the case of $k = 0$. Since $R_k(2n) = 0$ if $k = 0$, we have

$$\begin{aligned} E_{N,2n}^{(r)} &= \sum_{l=0}^{2n} \binom{r+l-1}{l} \sum_{k=0}^{l} (-1)^k \binom{l}{k} (2n)! R_k(2n) \\ &= (2n)! \sum_{k=0}^{2n} (-1)^k \sum_{l=k}^{2n} \binom{r+l-1}{l} \binom{l}{k} R_k(2n) \\ &= (2n)! \sum_{k=1}^{2n} (-1)^k R_k(2n) \sum_{l=k}^{2n} \binom{r+l-1}{l} \binom{l}{k} . \end{aligned}$$

Since

$$\begin{aligned} \sum_{l=k}^{2n} \binom{r+l-1}{l} \binom{l}{k} &= \binom{r+k-1}{r-1} \sum_{l=0}^{2n-k} \binom{r+l+k-1}{r+k-1} \\ &= \binom{r+k-1}{r-1} \binom{r+2n}{r+k} , \end{aligned}$$

we get

$$E_{N,2n}^{(r)} = (2n)! \sum_{k=1}^{2n} (-1)^k \binom{r+k-1}{r-1} \binom{r+2n}{r+k} R_k(2n) .$$

□

When $r = 1$ in Theorem 5, we have a new expression of hypergeometric Euler numbers.

COROLLARY 1. *For $N \ge 0$ and $n \ge 1$, we have*

$$E_{N,2n} = (2n)! \sum_{k=1}^{2n} (-1)^k \binom{2n+1}{k+1} R_k(2n) .$$

Similarly to Theorem 5, we have an expression of $\widehat{E}_{N,n}^{(r)}$, which is different from Theorem 3.

THEOREM 6. *For $N \geq 0$ and $n \geq 1$, we have*

$$\widehat{E}_{N,2n}^{(r)} = (2n)! \sum_{k=1}^{2n} (-1)^k \binom{r+k-1}{r-1} \binom{r+2n}{r+k} \widehat{R}_k(2n+1)\,,$$

where $\widehat{R}_k(2n+1)$ is given by (17).

When $r = 1$ in Theorem 6, we have a new expression of hypergeometric Euler numbers of the second kind.

COROLLARY 2. *For $N \geq 0$ and $n \geq 1$, we have*

$$\widehat{E}_{N,2n} = (2n)! \sum_{k=1}^{2n} (-1)^k \binom{2n+1}{k+1} \widehat{R}_k(2n+1)\,.$$

6. Applications from Trudi's formula

In the final section, as one type of applications, we shall give still different explicit expressions and converse relations for the numbers $E_{N,2n}^{(r)}$ and $\widehat{E}_{N,2n}^{(r)}$. Once we get the determinantal expressions seen in Theorem 2 and Theorem 4, by using the Trudi's formula, we can obtain the related interesting expressions and relations.

LEMMA 3 (Trudi's formula [18, 24]). *For a positive integer m, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & & & \vdots \\ \vdots & & \ddots & & 0 \\ \vdots & & & a_1 & a_0 \\ a_m & \cdots & \cdots & a_2 & a_1 \end{vmatrix}$$

$$= \sum_{t_1+2t_2+\cdots+mt_m=m} \binom{t_1+\cdots+t_m}{t_1,\dots,t_m} (-a_0)^{m-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m}\,,$$

where $\binom{t_1+\cdots+t_m}{t_1,\dots,t_m} = \frac{(t_1+\cdots+t_m)!}{t_1!\cdots t_m!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [21, Vol. 3, p. 214], [24] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [2], [21, Vol. 3, pp. 208–209].

In addition, we also introduce the following inversion formula (see, e.g. [18]).

LEMMA 4. *If* $\{\alpha_n\}_{n\ge 0}$ *is a sequence defined by* $\alpha_0 = 1$ *and for* $n \ge 1$

$$\alpha_n = \begin{vmatrix} D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix}, \text{ then } D(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

Moreover, if

$$A = \begin{pmatrix} 1 & & & \\ \alpha_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_n & \cdots & \alpha_1 & 1 \end{pmatrix}, \text{ then } A^{-1} = \begin{pmatrix} 1 & & & \\ D(1) & 1 & & \\ \vdots & \ddots & \ddots & \\ D(n) & \cdots & D(1) & 1 \end{pmatrix}.$$

PROOF. The proof of Lemma 4 is based upon the relation

$$\sum_{k=0}^{n}(-1)^k \alpha_{n-k} D(k) = 0 \quad (n \ge 1)$$

with $D(0) = 1$. □

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n}.$$

By applying these lemmata to Theorem 2 we obtain an explicit expression for the higher order hypergeometric Euler numbers.

THEOREM 7. *For* $n \ge 1$

$$E_{N,2n}^{(r)} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{t_1+\cdots+t_n} R_r(2)^{t_1} R_r(4)^{t_2} \cdots R_r(2n)^{t_n}.$$

Moreover,

$$R_r(2n) = \begin{vmatrix} -\frac{E_{N,2}^{(r)}}{2!} & 1 & & \\ \frac{E_{N,4}^{(r)}}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n E_{N,2n}^{(r)}}{(2n)!} & \cdots & \frac{E_{N,4}^{(r)}}{4!} & -\frac{E_{N,2}^{(r)}}{2!} \end{vmatrix}$$

and

$$\begin{pmatrix} 1 & & & & \\ -\frac{E_{N,2}^{(r)}}{2!} & 1 & & & \\ \frac{E_{N,4}^{(r)}}{4!} & -\frac{E_{N,2}^{(r)}}{2!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^n E_{N,2n}^{(r)}}{(2n)!} & \cdots & \frac{E_{N,4}^{(r)}}{4!} & -\frac{E_{N,2}^{(r)}}{2!} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & \\ R_r(2) & 1 & & & \\ R_r(4) & R_r(2) & 1 & & \\ \vdots & & \ddots & & \\ R_r(2n) & \cdots & R_r(4) & R_r(2) & 1 \end{pmatrix}.$$

If $r = 1$ in Theorem 7, we have an expression for hypergeometric Euler numbers $E_{N,2n} = E_{N,2n}^{(1)}$, which is different from (8).

COROLLARY 3. *For $n \geq 1$*

$$E_{N,2n} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{t_1+\cdots+t_n} \times \left(\frac{(2N)!}{(2N+2)!}\right)^{t_1} \left(\frac{(2N)!}{(2N+4)!}\right)^{t_2} \cdots \left(\frac{(2N)!}{(2N+2n)!}\right)^{t_n}.$$

Moreover,

$$\frac{(2N)!}{(2N+2n)!} = \begin{vmatrix} -\frac{E_{N,2}}{2!} & 1 & & \\ \frac{E_{N,4}}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n E_{N,2n}}{(2n)!} & \cdots & \frac{E_{N,4}}{4!} & -\frac{E_{N,2}}{2!} \end{vmatrix}.$$

By applying two lemmata to Theorem 4 we obtain an explicit expression for the higher order hypergeometric Euler numbers of the second kind.

THEOREM 8. *For $n \geq 1$*

$$\widehat{E}_{N,2n}^{(r)} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} \times (-1)^{t_1+\cdots+t_n} \widehat{R}_r(3)^{t_1} \widehat{R}_r(5)^{t_2} \cdots \widehat{R}_r(2n+1)^{t_n}.$$

Moreover,

$$\widehat{R}_r(2n+1)=\begin{vmatrix} -\frac{\widehat{E}^{(r)}_{N,2}}{2!} & 1 & & \\ \frac{\widehat{E}^{(r)}_{N,4}}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n\widehat{E}^{(r)}_{N,2n}}{(2n)!} & \cdots & \frac{\widehat{E}^{(r)}_{N,4}}{4!} & -\frac{\widehat{E}^{(r)}_{N,2}}{2!} \end{vmatrix}$$

and

$$\begin{pmatrix} 1 & & & & \\ -\frac{\widehat{E}^{(r)}_{N,2}}{2!} & 1 & & & \\ \frac{\widehat{E}^{(r)}_{N,4}}{4!} & -\frac{\widehat{E}^{(r)}_{N,2}}{2!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^n\widehat{E}^{(r)}_{N,2n}}{(2n)!} & \cdots & \frac{\widehat{E}^{(r)}_{N,4}}{4!} & -\frac{\widehat{E}^{(r)}_{N,2}}{2!} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & \\ \widehat{R}_r(3) & 1 & & & \\ \widehat{R}_r(5) & \widehat{R}_r(3) & 1 & & \\ \vdots & & \ddots & & \\ \widehat{R}_r(2n+1) & \cdots & \widehat{R}_r(5) & \widehat{R}_r(3) & 1 \end{pmatrix}.$$

If $r=1$ in Theorem 8, we have an expression for hypergeometric Euler numbers of the second kind $\widehat{E}_{N,2n}=\widehat{E}^{(1)}_{N,2n}$, which is different from (18).

COROLLARY 4. *For $n\ge 1$*

$$\widehat{E}_{N,2n}=(2n)!\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}(-1)^{t_1+\cdots+t_n}$$
$$\times\left(\frac{(2N+1)!}{(2N+3)!}\right)^{t_1}\left(\frac{(2N+1)!}{(2N+5)!}\right)^{t_2}\cdots\left(\frac{(2N+1)!}{(2N+2n+1)!}\right)^{t_n}.$$

Moreover,

$$\frac{(2N+1)!}{(2N+2n+1)!}=\begin{vmatrix} -\frac{\widehat{E}_{N,2}}{2!} & 1 & & \\ \frac{\widehat{E}_{N,4}}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n\widehat{E}_{N,2n}}{(2n)!} & \cdots & \frac{\widehat{E}_{N,4}}{4!} & -\frac{\widehat{E}_{N,2}}{2!} \end{vmatrix}.$$

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Appendix

Here we list several initial values of $E_{N,2n}$, $E^{(r)}_{N,2n}$, $R_r(2e)$, $\widehat{E}^{(r)}_{N,2n}$ and $\widehat{R}_r(2e+1)$.

$$E_{N,2} = -\frac{2}{(2N+1)(2N+2)}\,,$$

$$E_{N,4} = \frac{2\cdot 4!(4N+5)}{(2N+1)^2(2N+2)^2(2N+3)(2N+4)}\,,$$

$$E_{N,6} = \frac{4\cdot 6!(8N^3-2N^2-65N-61))}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)(2N+5)(2N+6)}\,,$$

$$E_{N,8} = \frac{16\cdot 8!(16N^6-44N^5-516N^4-667N^3+1283N^2+3126N+1662)}{(2N+1)^4(2N+2)^4(2N+3)^2(2N+4)^2(2N+6)(2N+7)(2N+8)}\,.$$

$$E_{N,0}^{(r)} = 1\,,$$

$$E_{N,2}^{(r)} = -\frac{2r}{(2N+1)(2N+2)}\,,$$

$$E_{N,4}^{(r)} = 4!\left(\frac{2r(4N+5)}{(2N+1)^2(2N+2)^2(2N+3)(2N+4)} + \binom{r}{2}\frac{1}{(2N+1)^2(2N+2)^2}\right)\,,$$

$$E_{N,6}^{(r)} = 6!\left(\frac{4r(8N^3-2N^2-65N-61)}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)(2N+5)(2N+6)} - 4\binom{r}{2}\frac{4N+5}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)} - \binom{r}{3}\frac{1}{(2N+1)^3(2N+2)^3}\right)\,,$$

$$\begin{aligned} E_{N,8}^{(r)} &= \frac{16\cdot 8!r(16N^6-44N^5-516N^4-667N^3+1283N^2+3126N+1662)}{(2N+1)^4(2N+2)^4(2N+3)^2(2N+4)^2(2N+6)(2N+7)(2N+8)} \\ &\quad + \binom{r}{2}\frac{8\cdot 8!(256N^4+1760N^3+4466N^2+4939N+2007)}{(2N+1)^4(2N+2)^4(2N+3)^2(2N+4)^2(2N+5)(2N+6)} \\ &\quad + \binom{r}{2}\frac{2\cdot 8!(r-4)(4N+5)}{(2N+1)^4(2N+2)^4(2N+3)(2N+4)} \\ &\quad + \binom{r}{4}\frac{8!}{(2N+1)^4(2N+2)^4}\,. \end{aligned}$$

$$R_r(2) = \frac{r}{(2N+1)(2N+2)}\,,$$

$$R_r(4) = \frac{r}{(2N+1)(2N+2)(2N+3)(2N+4)} + \binom{r}{2}\frac{1}{(2N+1)^2(2N+2)^2}\,,$$

$$R_r(6) = \frac{r}{(2N+1)(2N+2)(2N+3)(2N+4)(2N+5)(2N+6)}$$

$$+\frac{r(r-1)}{(2N+1)^2(2N+2)^2(2N+3)(2N+4)}$$
$$+\binom{r}{3}\frac{1}{(2N+1)^3(2N+2)^3},$$
$$R_r(8)=\frac{r}{(2N+1)(2N+2)\cdots(2N+8)}$$
$$+\frac{r(r-1)}{(2N+1)^2(2N+2)^2(2N+3)\cdots(2N+6)}$$
$$+\binom{r}{2}\frac{1}{(2N+1)^2(2N+2)^2(2N+3)^2(2N+4)^2}$$
$$+r\binom{r-1}{2}\frac{1}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)}$$
$$+\binom{r}{4}\frac{1}{(2N+1)^4(2N+2)^4}.$$

$$\widehat{E}_{N,0}^{(r)}=1,$$
$$\widehat{E}_{N,2}^{(r)}=-\frac{2r}{(2N+2)(2N+3)},$$
$$\widehat{E}_{N,4}^{(r)}=4!\left(\frac{2r(4N+7)}{(2N+2)^2(2N+3)^2(2N+4)(2N+5)}+\binom{r}{2}\frac{1}{(2N+2)^2(2N+3)^2}\right),$$
$$\widehat{E}_{N,6}^{(r)}=6!\left(\frac{4r(8N^3+10N^2-61N-93)}{(2N+2)^3(2N+3)^3(2N+4)(2N+5)(2N+6)(2N+7)}-4\binom{r}{2}\frac{4N+7}{(2N+2)^3(2N+3)^3(2N+4)(2N+5)}-\binom{r}{3}\frac{1}{(2N+2)^3(2N+3)^3}\right),$$
$$\widehat{E}_{N,8}^{(r)}=\frac{8\cdot 8!r(32N^6+8N^5-1132N^4-3538N^3-1063N^2+7280N+6858)}{(2N+2)^4(2N+3)^4(2N+4)^2(2N+5)^2(2N+7)(2N+8)(2N+9)}$$
$$+\binom{r}{2}\frac{8\cdot 8!(256N^4+2272N^3+7490N^2+10853N+5829)}{(2N+2)^4(2N+3)^4(2N+4)^2(2N+5)^2(2N+6)(2N+7)}$$
$$+\binom{r}{2}\frac{2\cdot 8!(r-4)(4N+7)}{(2N+2)^4(2N+3)^4(2N+4)(2N+5)}$$
$$+\binom{r}{4}\frac{8!}{(2N+2)^4(2N+3)^4}.$$

$$\widehat{R}_r(3)=\frac{r}{(2N+2)(2N+3)},$$

$$\widehat{R}_r(5) = \frac{r}{(2N+2)(2N+3)(2N+4)(2N+5)} + \binom{r}{2}\frac{1}{(2N+2)^2(2N+3)^3},$$

$$\widehat{R}_r(7) = \frac{r}{(2N+2)(2N+3)(2N+4)(2N+5)(2N+6)(2N+7)} + \frac{r(r-1)}{(2N+2)^2(2N+3)^2(2N+4)(2N+5)} + \binom{r}{3}\frac{1}{(2N+2)^3(2N+3)^3},$$

$$\widehat{R}_r(9) = \frac{r}{(2N+2)(2N+3)\cdots(2N+9)} + \frac{r(r-1)}{(2N+2)^2(2N+3)^2(2N+4)\cdots(2N+7)} + \binom{r}{2}\frac{1}{(2N+2)^2(2N+3)^2(2N+4)^2(2N+5)^2} + r\binom{r-1}{2}\frac{1}{(2N+2)^3(2N+3)^3(2N+4)(2N+5)} + \binom{r}{4}\frac{1}{(2N+2)^4(2N+3)^4}.$$

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