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On the Plus and the Minus Selmer Groups for Elliptic Curves at Supersingular Primes

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Abstract. Let p be an odd prime number, and E an elliptic curve defined over a number field. Suppose that E has good reduction at any prime lying above p, and has supersingular reduction at some prime lying above p. In this paper, we construct the plus and the minus Selmer groups of E over the cyclotomic $\mathbb{Z}_p$-extension in a more general setting than that of B.D. Kim, and give a generalization of a result of B.D. Kim on the triviality of finite Λ-submodules of the Pontryagin duals of the plus and the minus Selmer groups, where Λ is the Iwasawa algebra of the Galois group of the $\mathbb{Z}_p$-extension.

1. Introduction

Let p be an odd prime number, F_0 a finite extension of $\mathbb{Q}$, F_∞/F_0 the cyclotomic $\mathbb{Z}_p$-extension and F_n the n-th layer. Denote $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F_0)]]$. Let E be an elliptic curve defined over F_0.

When E has good ordinary reduction at any prime of F_0 lying above p, the Pontryagin dual of the p-primary Selmer group of E over F_∞ is conjectured to be Λ-torsion. This conjecture is proved in several cases now. For example if the p-primary Selmer group of E over F_0 is finite, or if E is defined over $\mathbb{Q}$ and $F_0/\mathbb{Q}$ is abelian, then the conjecture is known to be true (cf. [2] p. 55).

On the contrary, when E has good supersingular reduction at some prime of F_0 lying above p, the Pontryagin dual of the p-primary Selmer group of E over F_∞ is no longer Λ-torsion. S. Kobayashi [11] defined the plus and the minus Selmer groups $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])$ when E is defined over $\mathbb{Q}$, $a_p = 1 + p - \#\widetilde{E}(\mathbb{F}_p) = 0$, and $F_0 = \mathbb{Q}(\mu_p)$, where $\widetilde{E}$ denotes the reduction of E at p, and μ_p denotes the group of p-th roots of unity. He proved that the Pontryagin duals $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ are Λ-torsion. A. Iovita and R. Pollack [6] generalized definitions of the plus and the minus Selmer groups to the case when F_0 is a number field in which p splits completely, E is defined over $\mathbb{Q}$ and $a_p = 0$. Further B.D. Kim [7] generalized them to the case when F_0 is a number field in which p is unramified, E is defined over $\mathbb{Q}$ and $a_p = 0$.

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B.D. Kim proved in [8] the following theorem on the triviality of finite Λ-submodules of $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$.

THEOREM 1.1 ([8] Theorem 1.1). *Let F_0 be a finite extension of $\mathbb{Q}$ in which p is unramified, E an elliptic curve defined over $\mathbb{Q}$, and $a_p = 0$.*

(1) *Assume that $\mathrm{Sel}^{-}(F_\infty, E[p^\infty])^\vee$ is Λ-torsion. Then $\mathrm{Sel}^{-}(F_\infty, E[p^\infty])^\vee$ has no nontrivial finite Λ-submodule.*

(2) *Assume further that p splits completely in F_0, and $\mathrm{Sel}^{+}(F_\infty, E[p^\infty])^\vee$ is Λ-torsion. Then $\mathrm{Sel}^{+}(F_\infty, E[p^\infty])^\vee$ has no nontrivial finite Λ-submodule.*

Throughout this paper, we assume that $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ is Λ-torsion as in the above theorem. The following proposition on the local conditions of the plus and the minus Selmer groups was an important ingredient for Theorem 1.1.

PROPOSITION 1.2 ([7] Proposition 3.13, [8] Proposition 2.2 and Proposition 2.3). *Let k_0 be a finite unramified extension of $\mathbb{Q}_p$, and k_∞ the cyclotomic $\mathbb{Z}_p$-extension of k_0. We denote the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k_0)]]$ by Λ.*

(1) *We have*

$$(E^{-}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \Lambda^{\oplus d}$$

where $d = [k_0 : \mathbb{Q}_p]$.

(2) *Assume further that $k_0 = \mathbb{Q}_p$. Then we have*

$$(E^{+}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \Lambda\,.$$

When B.D. Kim studied the plus Selmer group in [8], he restricted the base field F_0 as in Theorem 1.1 (2) to apply Proposition 1.2, and he got the result only in such a case. Even when he studied the minus Selmer group in [8], the base field F_0 was a finite extension of $\mathbb{Q}$ in which p is unramified. He did not consider the case when $\mu_p \subset F_0$.

In this paper, we consider more general fields for F_0 and k_0, and a more general elliptic curve E, than those of the above known results. In particular, we remove the assumption on the splitting behavior of p in $F_0/\mathbb{Q}$ in Theorem 1.1 (2) and the assumption $k_0 = \mathbb{Q}_p$ in Proposition 1.2 (2). We also note that we also consider the case when $\mu_p \subset F_0$. We get the following result.

MAIN THEOREM 1.3 (Theorem 4.8). *Let F be a finite extension of $\mathbb{Q}$, $F_0 = F(\mu_p)$, F_∞/F_0 the cyclotomic $\mathbb{Z}_p$-extension, and E an elliptic curve defined over a subfield F' of F. Let S_p^{ss} be the set of all primes of F' lying above p where E has supersingular reduction. Assume the following conditions*:

(i) *E has good reduction at any prime of F' lying above p,*

(ii) *S_p^{ss} is nonempty,*

(iii) *any prime $w \in S_p^{\mathrm{ss}}$ is unramified in F,*

(iv) $F'_w = \mathbb{Q}_p$ *for any prime* $w \in S_p^{\mathrm{ss}}$, *where* F'_w *is the completion of* F' *at the prime* w,

(v) $a_w = 1 + p - \#\widetilde{E}_w(\mathbb{F}_p) = 0$ *for any prime* $w \in S_p^{\mathrm{ss}}$, *where* $\widetilde{E}_w$ *is the reduction of* E *at* w, *and*

(vi) *both* $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ *are* Λ*-torsion.*

Then both $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ *have no nontrivial finite* Λ*-submodule.*

REMARK 1.4. (1) We assume the condition (i) since we expect the condition (vi) automatically holds true under this condition.

(2) In the case $S_p^{\mathrm{ss}} = \emptyset$, we have $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty]) = \mathrm{Sel}(F_\infty, E[p^\infty])$. Finite Λ-submodules of $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ was studied by Hachimori and Matsuno [3]. Thus we will be interested in the case (ii).

(3) The conditions (iii) and (iv) on the fields F and F' will be used in applying the local result discussed in Section 3.

(4) The condition (v) is crucial in this paper. In our method, it is important to study the local conditions $E^+(k_n)$ and $E^-(k_n)$, where k_n is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension k_∞/k_0 with k_0 a finite extension of $\mathbb{Q}_p$. In the case when $a_w \neq 0$ for some $w \in S_p^{\mathrm{ss}}$, one might need another submodule of $\mathrm{Sel}(F_\infty, E[p^\infty])$ instead of $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])$. F. Sprung [16] defined $\mathrm{Sel}^{\sharp}(F_\infty, E[p^\infty])$ and $\mathrm{Sel}^{\flat}(F_\infty, E[p^\infty])$ instead of the plus and the minus Selmer groups in the case when $F_0 = \mathbb{Q}(\mu_p)$, and E is defined over $\mathbb{Q}$ which has supersingular reduction at p without assuming $a_p = 0$. He defined $E^{\sharp}(k_\infty)$ and $E^{\flat}(k_\infty)$ in the case when $k_0 = \mathbb{Q}_p(\mu_p)$, however, did not define $E^{\sharp}(k_n)$ nor $E^{\flat}(k_n)$. We cannot yet apply our method in the case when $a_w \neq 0$ for some w.

(5) In some cases, $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ is actually known to be Λ-torsion. For example, let F_0 be a finite abelian extension of $\mathbb{Q}$, and F_∞/F_0 the cyclotomic $\mathbb{Z}_p$-extension. Suppose that E is defined over $\mathbb{Q}$ and has supersingular reduction at p with $a_p = 0$. In this case, one can actually show that $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ is Λ-torsion. Our main theorem implies that $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ has no nontrivial finite Λ-submodule for any finite abelian field F_0. On the other hand, we need a certain assumption on F_0 to apply B.D. Kim's result.

For the proof of Theorem 1.3, we will generalize Proposition 1.2 to the case of our setting. In the study of $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$, we find that its Λ-module structure in our setting is generally different from those in the settings of previous works. We now explain some known results on the Λ-module structure of $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

Takeji [17] considered the case when k_0 is a quadratic unramified extension of $\mathbb{Q}_p$. He generalized Proposition 1.2 to this case, i.e. he proved that $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a free Λ-module of Λ-rank 2. Applying this result, he proved that $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ has no nontrivial finite Λ-submodule in the case when F_0 is a quadratic number field in which p inerts, which is a generalization of Theorem 1.1.

M. Kim proved in his dissertation [10] that $E^{\pm}(k_n)$ are cyclic $\mathbb{Z}_p[\mathrm{Gal}(k_n/\mathbb{Q}_p)]$-modules for all n, where k_n is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension k_∞/k_0, in the case when

k is a general finite unramified extension of $\mathbb{Q}_p$ and $k_0 = k(\mu_p)$, however, he did not notice that the assumption $[k : \mathbb{Q}_p] \not\equiv 0 \pmod 4$ is needed. From this cyclicity, one can show that $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a free Λ-module of Λ-rank $[k_0 : \mathbb{Q}_p]$, which is a generalization of Proposition 1.2.

B.D. Kim [9] independently generalized Proposition 1.2 to the case when k_0 is a finite unramified extension of $\mathbb{Q}_p$ and $[k_0 : \mathbb{Q}_p] \not\equiv 0 \pmod 4$, i.e. he proved in this case that $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a free Λ-module of Λ-rank $[k_0 : \mathbb{Q}_p]$ (cf. [9] Theorem 2.8). Applying this result, we can generalize Theorem 1.1 to the case when F_0 is a finite extension of $\mathbb{Q}$ in which p is unramified and $[F_{0,v} : \mathbb{Q}_p] \not\equiv 0 \pmod 4$ for all primes v of F_0 lying above p, where $F_{0,v}$ is the completion of F_0 at the prime v.

We remark that we consider more general settings than those of all the above known results.

Here we prepare some notations of our settings and explain an obstruction for generalizing Proposition 1.2 to the case of our setting, which we have overcome in this paper. Let k be a finite unramified extension of $\mathbb{Q}_p$ of degree d, $k_0 = k(\mu_p)$, k_∞ the cyclotomic $\mathbb{Z}_p$-extension of k_0, k_n the n-th layer, $\Delta = \mathrm{Gal}(k(\mu_p)/k)$, $\Gamma = \mathrm{Gal}(k_\infty/k_0)$, $\Gamma_n = \mathrm{Gal}(k_\infty/k_n)$, $G_n = \mathrm{Gal}(k_n/\mathbb{Q}_p)$ and $\Lambda = \mathbb{Z}_p[[\Gamma]]$.

An essential property, expected in all previous works [7], [8], [9], [10], and [17] was that $E^{\pm}(k_n)$ are cyclic $\mathbb{Z}_p[G_n]$-modules for all n. From this expected property, we can show that $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a free Λ-module of Λ-rank $[k_0 : \mathbb{Q}_p]$. On the contrary to this expectation, we find that $E^{+}(k_n)$ are not cyclic $\mathbb{Z}_p[G_n]$-modules when $d \equiv 0 \pmod 4$ (cf. Proposition 3.16 and Remark 3.17). An obstruction is that this non-cyclicity makes $(E^{+}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ more complicated. In fact, we find that $(E^{+}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is not a free Λ-module in the case when $d \equiv 0 \pmod 4$ (cf. Remark 3.26). Therefore, the same statement with the conclusion of Proposition 1.2 does not hold in the general setting.

A crucial step for the proof of our main theorem is to investigate the Λ-module structure of $(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ more precisely. We prove the following proposition on the local conditions of the plus and the minus Selmer groups, which is a generalization of Proposition 1.2 and an important ingredient for our main theorem.

PROPOSITION 1.5 (Proposition 3.28). *$(E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ has no nontrivial finite Λ-submodule and its Λ-rank is $[k_0 : \mathbb{Q}_p]$.*

We prove this proposition by calculating the Γ_n-coinvariants of the χ-component $(E^{\pm}(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ for all n, where $\chi : \Delta \to \mathbb{Z}_p^\times$ is a character of Δ. The original idea for such calculations, to calculate the Γ-coinvariants and to apply Nakayama's lemma, was suggested by M. Kurihara. The authors are very grateful to him.

As a consequence of Proposition 1.5, we prove the following proposition which is also an important ingredient for our main theorem.

PROPOSITION 1.6 (Proposition 3.32). *We have*

$$\left(\frac{H^1(k_\infty, E[p^\infty])}{E^{\pm}(k_\infty)\otimes \mathbb{Q}_p/\mathbb{Z}_p}\right)^\vee \cong \Lambda^{\oplus[k_0:\mathbb{Q}_p]}.$$

In comparison with the strategy for the proof of Theorem 1.1, we develop another strategy also for the proof of our main theorem. Our strategy is to prove that the triviality of finite Λ-submodules of $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ is inherited to that of $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$, and use the following theorem.

THEOREM 1.7 (Theorem 4.5). *Assume that both* $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ *are* Λ*-torsion. Then* $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ *has no nontrivial finite* Λ*-submodule.*

The above discussion is enough to prove our main theorem, however, we can determine the explicit structure of the Λ-module $(E^{\pm}(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$, on which we explain here. We prepare some notations concerning the character decomposition. Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character of $\Delta = \mathrm{Gal}(k(\mu_p)/k)$. If M is a $\mathbb{Z}_p[\Delta]$-module, then M is decomposed into

$$M = \bigoplus_\chi \varepsilon_\chi M,$$

where $\varepsilon_\chi = \frac{1}{p-1}\sum_{\sigma\in\Delta}\chi(\sigma)\sigma^{-1} \in \mathbb{Z}_p[\Delta]$. We denote by M^χ the χ-component $\varepsilon_\chi M$. We fix a topological generator $\gamma \in \Gamma$, and identify $\mathbb{Z}_p[[\Gamma]]$ with the ring of formal power series $\mathbb{Z}_p[[X]]$ by identifying γ with $1+X$. We get the following theorem.

THEOREM 1.8 (Theorem 3.34). *Let* $\chi : \Delta \to \mathbb{Z}_p^\times$ *be a character. We have*

$$\left(E^+(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \cong \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus\delta},$$
$$\left(E^-(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \cong \Lambda^{\oplus d}$$

where

$$\delta = \begin{cases} 0 & \text{if } d \not\equiv 0 \pmod 4 \text{ or } \chi \neq \mathbf{1}, \\ 2 & \text{otherwise}. \end{cases}$$

The outline of this paper is as follows. In Section 2, we define the plus and the minus Selmer groups following Kobayashi, and fix a global setting. In Section 3, we study the local conditions in a local setting. Subsection 3.1 is a preparation for the rest of Section 3. In Subsection 3.2, we give a description of $E^{\pm}(k_n)^\chi$ in terms of formal groups and a system of local points. In Subsection 3.3, we study the Λ-modules $(E^{\pm}(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ and $(H^1(k_\infty, E[p^\infty])/(E^{\pm}(k_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p))^\vee$. In Subsection 3.4, we further determine the explicit structure of the Λ-module $(E^{\pm}(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. In Section 4, we study

finite Λ-submodules of the Pontryagin duals of the Selmer groups $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ and $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$. In Subsection 4.1, we prove that the usual p-primary Selmer group has no nontrivial finite Λ-submodule under the same assumption with the main theorem. In this step, it is essential to assume that both $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ are Λ-torsion. In Subsection 4.2, we prove our main theorem.

2. The plus and the minus Selmer groups

Let p be a prime number, F a finite extension of $\mathbb{Q}$, and E an elliptic curve defined over F. For a finite extension K/F, the p-primary Selmer group for E over K is defined by

$$\mathrm{Sel}(K, E[p^\infty]) := \mathrm{Ker}\left(H^1(K, E[p^\infty]) \longrightarrow \prod_v \frac{H^1(K_v, E[p^\infty])}{E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right),$$

where v runs through all places of K, K_v is the completion of K at the place v, and $E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is regarded as a subgroup of $H^1(K_v, E[p^\infty])$ by the Kummer map. For a number field K that is an infinite extension of F, we define the p-primary Selmer group for E over K by

$$\mathrm{Sel}(K, E[p^\infty]) := \varinjlim_{K'} \mathrm{Sel}(K', E[p^\infty]),$$

where K' runs through all the subfields of K which are finite extensions of F, and transition maps are restriction maps between cohomology groups.

We denote $F_n = F(\mu_{p^{n+1}})$, $F_{-1} = F$ and $F_\infty = \bigcup_n F_n$, where μ_{p^n} denotes the group of p^n-th roots of unity. We fix a generator (ζ_{p^n}) of $\mathbb{Z}_p(1)$, namely, for each $n \geq 0$, ζ_{p^n} is a primitive p^n-th root of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$.

Then by definition, we have

$$\mathrm{Sel}(F_\infty, E[p^\infty]) = \varinjlim_n \mathrm{Sel}(F_n, E[p^\infty]).$$

Throughout this paper, we fix the following notations:

- p is an odd prime number,
- F is a finite extension of $\mathbb{Q}$,
- E is an elliptic curve defined over a subfield F' of F.

Denote S_p^{ss} the set of all primes of F' lying above p where E has supersingular reduction.

Throughout this paper, we assume the following:

- E has good reduction at any prime $w|p$ of F',
- S_p^{ss} is nonempty,
- any prime $w \in S_p^{\mathrm{ss}}$ is unramified in F,
- $F'_w = \mathbb{Q}_p$ for any prime $w \in S_p^{\mathrm{ss}}$, where F'_w is the completion of F' at the prime w, and

- $a_w = 1 + p - \#\widetilde{E}_w(\mathbb{F}_p) = 0$ for any prime $w \in S_p^{\mathrm{ss}}$, where $\widetilde{E}_w$ is the reduction of E at w.

When $p \geq 5$, the condition $a_w = 0$ is automatically satisfied since we have $p|a_w$ and $|a_w| \leq 2\sqrt{p}$.

Denote $S_{p,F}^{\mathrm{ss}}$ the set of all primes of F lying above S_p^{ss}.

Following S. Kobayashi [11] we define subgroups $E^+(F_{n,v})$ and $E^-(F_{n,v})$ of $E(F_{n,v})$ for each prime $v \in S_{p,F}^{\mathrm{ss}}$, and define plus and minus Selmer groups $\mathrm{Sel}^{\pm}(F_n, E[p^\infty])$, $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])$ as the following (see also [7] and [10]).

DEFINITION 2.1. (1) For a prime $v \in S_{p,F}^{\mathrm{ss}}$ and $n \geq -1$, let $F_{n,v}$ be the completion of F_n at the unique prime of F_n lying above v. We define

$$E^+(F_{n,v}) = \{P \in E(F_{n,v}) | \operatorname{Tr}_{n/m+1} P \in E(F_{m,v}) \text{ for all even } m, -1 \leq m \leq n-1\},$$

$$E^-(F_{n,v}) = \{P \in E(F_{n,v}) | \operatorname{Tr}_{n/m+1} P \in E(F_{m,v}) \text{ for all odd } m, -1 \leq m \leq n-1\}$$

where $\operatorname{Tr}_{n/m+1} : E(F_{n,v}) \to E(F_{m+1,v})$ is the trace map.

(2) The plus and the minus Selmer groups are defined by

$$\mathrm{Sel}^{\pm}(F_n, E[p^\infty]) := \operatorname{Ker}\left(\mathrm{Sel}(F_n, E[p^\infty]) \longrightarrow \bigoplus_{v \in S_{p,F}^{\mathrm{ss}}} \frac{H^1(F_{n,v}, E[p^\infty])}{E^{\pm}(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}\right),$$

$$\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty]) := \varinjlim_n \mathrm{Sel}^{\pm}(F_n, E[p^\infty]).$$

We denote the Pontryagin dual of a module M by $M^\vee$. Let $\mathcal{G}_n = \mathrm{Gal}(F_n/F)$ and $\mathcal{G}_\infty = \mathrm{Gal}(F_\infty/F)$. Then $\mathbb{Z}_p[\mathcal{G}_n]$ acts naturally on $\mathrm{Sel}^{\pm}(F_n, E[p^\infty])^\vee$ and $\Lambda(\mathcal{G}_\infty) := \mathbb{Z}_p[[\mathcal{G}_\infty]]$ on $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$.

The Pontryagin dual of the usual p-primary Selmer group $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ is not a torsion $\Lambda(\mathcal{G}_\infty)$-module, however, $\mathrm{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee$ is known to be $\Lambda(\mathcal{G}_\infty)$-torsion in the case $F = \mathbb{Q}$ (cf. [11] Theorem 2.2).

3. The formal groups and the norm subgroups

Let $E/\mathbb{Q}_p$ be an elliptic curve with $a_p = 0$ and $\widehat{E}$ the formal group over $\mathbb{Z}_p$ associated with the minimal model of E over $\mathbb{Q}_p$. Let k be a finite unramified extension of $\mathbb{Q}_p$ of degree $d = [k : \mathbb{Q}_p]$ and $\mathcal{O}_k$ the ring of integers of k. For each $n \geq -1$, let $k_n = k(\mu_{p^{n+1}})$ and $\mathfrak{m}_n$ be the maximal ideal of k_n. Let $k_\infty = \bigcup_{n \geq -1} k_n$ and $\mathfrak{m}_\infty = \bigcup_{n \geq -1} \mathfrak{m}_n$. Let $G_n = \mathrm{Gal}(k_n/\mathbb{Q}_p)$, $G_\infty = \mathrm{Gal}(k_\infty/\mathbb{Q}_p)$, $\Gamma_n = \mathrm{Gal}(k_\infty/k_n)$, $\Gamma = \Gamma_0 (= \mathrm{Gal}(k_\infty/k_0))$ and $\Delta = \mathrm{Gal}(k(\mu_p)/k) = \mathrm{Gal}(k_0/k_{-1})$. Let φ be the Frobenius homomorphism in $\mathrm{Gal}(k/\mathbb{Q}_p) = G_{-1}$ characterized by $x^\varphi \equiv x^p \pmod{p\mathcal{O}_k}$. We denote $\Lambda = \mathbb{Z}_p[[\Gamma]]$. We fix a topological generator $\gamma \in \Gamma$. Then

we identify $\mathbb{Z}_p[[\Gamma]]$ with $\mathbb{Z}_p[[X]]$, and $\mathbb{Z}_p[[G_\infty]]$ with $\mathbb{Z}_p[G_0][[X]]$ by identifying γ with $1+X$.

3.1. The formal groups associated to E

PROPOSITION 3.1. *For any n, $\widehat{E}(\mathfrak{m}_n)$ is $\mathbb{Z}_p$-torsion-free.*

PROOF. We can prove this by the same method as the proof of [11] Proposition 8.7. □

The above proposition implies that the formal logarithm $\log_{\widehat{E}}(X)$ induces an injective homomorphism $\log_{\widehat{E}} : \widehat{E}(\mathfrak{m}_n) \to k_n$ for all n, since the kernel of the logarithm of a formal group precisely consists of the elements of finite order.

For such a one-dimensional formal group $\mathscr{F}$ defined over $\mathbb{Z}_p$ with height 2, the formal logarithm $\log_{\mathscr{F}}$ induces isomorphisms as in the following proposition (cf. The proof of Proposition 2.1 in [13], and Lemma 2.4 in [4]).

PROPOSITION 3.2. *Let $\mathscr{F}$ be a one-dimensional formal group defined over $\mathbb{Z}_p$ with height 2. For a finite extension $K/\mathbb{Q}_p$, denote by $\mathfrak{m}_K$ its maximal ideal. Then the logarithm $\log_{\mathscr{F}} : \mathscr{F}(\mathfrak{m}_K) \to K$ induces isomorphisms*

$$\log_{\mathscr{F}} : \mathscr{F}(\mathfrak{m}_K^j) \xrightarrow{\simeq} \mathfrak{m}_K^j$$

for all $j > v_K(p)/(p^2-1)$, where v_K is the normalized valuation of K so that $v_K(\pi_K) = 1$ for a uniformizer π_K of K.

Following [7] and [10] we construct a system of local points $(d_n)_n$.

Fix a generator ζ of the group of roots of unity in k. Then ζ is a primitive (p^d-1)-th root of unity, and we have $k = \mathbb{Q}_p(\zeta)$.

Let $g(X) = (X+\zeta)^p - \zeta^p \in \mathcal{O}_k[X]$, $g^{(m)}(X) = g^{\varphi^{m-1}} \circ g^{\varphi^{m-2}} \circ \cdots \circ g(X) = (X+\zeta)^{p^m} - \zeta^{p^m}$ for $m \geq 1$ and $g^{(0)}(X) = X$. We define a formal power series $\log_{\mathscr{G}}(X)$ by

$$\log_{\mathscr{G}}(X) = \sum_{m=0}^{\infty} (-1)^m \frac{g^{(2m)}(X)}{p^m}.$$

We can check that

$$(\log_{\mathscr{G}}^{\varphi^{-(n+1)}})^{\varphi^2}(X^{p^2}) + p \log_{\mathscr{G}}^{\varphi^{-(n+1)}}(X) \equiv 0 \pmod{p}$$

and $(\log_{\mathscr{G}}^{\varphi^{-(n+1)}})'(X) \in \mathcal{O}_k[[X]]$ for each n. This means that $\log_{\mathscr{G}}^{\varphi^{-(n+1)}}(X)$ is of the Honda type $t^2 + p$ for each n. Hence by Honda theory (cf. [5]) we see that

- there is a formal group $\mathscr{G}_n$ defined over $\mathcal{O}_k$ whose formal logarithm $\log_{\mathscr{G}_n}$ is given by $\log_{\mathscr{G}}^{\varphi^{-(n+1)}}$ for each n, and
- the power series $\exp_{\widehat{E}} \circ \log_{\mathscr{G}_n}$ is contained in $\mathcal{O}_k[[X]]$ and gives an isomorphism $\mathscr{G}_n \to \widehat{E}$ over $\mathcal{O}_k$ for each n.

We fix a generator (ζ_{p^n}) of $\mathbb{Z}_p(1)$, namely, for each $n \geq 0$, ζ_{p^n} is a primitive p^n-th root of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Let $\pi_n = \zeta^{\varphi^{-(n+1)}}(\zeta_{p^{n+1}} - 1) \in \mathfrak{m}_n$ for $n \geq -1$ and $\pi_n = 0$ for $n < -1$. For each n, we can easily show that

$$g^{(m),\varphi^{-(n+1)}}(\pi_n) = \pi_{n-m} \tag{3.1}$$

for any $m \geq 0$ by direct calculation.

Put

$$\begin{aligned}\varepsilon_n &= \zeta^{\varphi^{-(n+3)}} p - \zeta^{\varphi^{-(n+5)}} p^2 + \zeta^{\varphi^{-(n+7)}} p^3 - \cdots \\ &= \sum_{i=1}^{\infty} (-1)^{i-1} \zeta^{\varphi^{-(n+1+2i)}} p^i \in \mathfrak{m}_k\end{aligned}$$

for $n \geq -1$. Since $\log_{\mathscr{G}_n} : \mathscr{G}_n(\mathfrak{m}_k) \to \mathfrak{m}_k$ is an isomorphism for all n (cf. Proposition 3.2), there is $\epsilon_n \in \mathscr{G}_n(\mathfrak{m}_k)$ such that $\log_{\mathscr{G}_n}(\epsilon_n) = \varepsilon_n$ for $n \geq -1$.

DEFINITION 3.3. We define

$$d_n = \exp_{\widehat{E}} \circ \log_{\mathscr{G}_n}(\epsilon_n [+]_{\mathscr{G}_n} \pi_n)$$

for $n \geq -1$, where $[+]_{\mathscr{G}_n}$ is the addition of $\mathscr{G}_n$.

For $n \geq m$, we denote by $\mathrm{Tr}_{n/m} : \widehat{E}(\mathfrak{m}_n) \to \widehat{E}(\mathfrak{m}_m)$ the trace (norm) with respect to the group-law $\widehat{E}(X, Y)$.

PROPOSITION 3.4. *The system of local points* $(d_n)_n \in \prod_{n \geq -1} \widehat{E}(\mathfrak{m}_n)$ *satisfies*
(1) $\mathrm{Tr}_{n/n-1}(d_n) = -d_{n-2}$ *for each* $n \geq 1$,
(2) $\mathrm{Tr}_{0/-1}(d_0) = -(\varphi + \varphi^{-1})d_{-1}$.

PROOF. We prove this by the same method as the proof of Lemma 8.9 in [11]. Since $\log_{\widehat{E}}$ is injective (cf. Proposition 3.1) and commute with the action of G_n on $\widehat{E}(\mathfrak{m}_n)$, it is enough to show that the relation holds after applying $\log_{\widehat{E}}$ to both sides of the equality.

We have

$$\begin{aligned}\log_{\widehat{E}}(d_n) &= \log_{\mathscr{G}_n}(\epsilon_n [+]_{\mathscr{G}_n} \pi_n) \\ &= \log_{\mathscr{G}_n}(\epsilon_n) + \log_{\mathscr{G}_n}(\pi_n) \\ &= \varepsilon_n + \sum_{m=0}^{[\frac{n+1}{2}]} (-1)^m \frac{\pi_{n-2m}}{p^m}.\end{aligned}$$

Here the last equality follows from (3.1) and $\pi_n = 0$ for $n \leq -1$.

For $n \geq 1$, we have

$$\mathrm{Tr}_{n/n-1} \log_{\widehat{E}}(d_n) = p\varepsilon_n - \zeta^{\varphi^{-(n+1)}} p + \sum_{m=1}^{[\frac{n+1}{2}]} (-1)^m \frac{\pi_{n-2m}}{p^{m-1}}$$

$$= -\varepsilon_{n-2} - \sum_{m=0}^{[\frac{n-1}{2}]} (-1)^m \frac{\pi_{n-2-2m}}{p^m}$$
$$= -\log_{\widehat{E}}(d_{n-2})\,.$$

For $n = 0$, we have

$$\mathrm{Tr}_{0/-1} \log_{\widehat{E}}(d_0) = (p-1)\varepsilon_0 - \zeta^{\varphi^{-1}} p$$
$$= -(\varphi + \varphi^{-1})\varepsilon_{-1}$$
$$= -(\varphi + \varphi^{-1}) \log_{\widehat{E}}(d_{-1})\,.$$

□

REMARK 3.5. As long as we define local points as values of certain power series at certain points, the factor $\varphi+\varphi^{-1}$ in the condition (2) always appears (cf. [12] Proposition 3.10, (3.3)). Although B.D. Kim did not mention explicitly in [7], [8], this factor $\varphi + \varphi^{-1}$ was an obstruction. He assumed in [7] and [8] that $k = \mathbb{Q}_p$ when he considered the plus Selmer groups in order to make the situation simpler, i.e. $\varphi + \varphi^{-1} = 2$ in $\mathbb{Z}_p[G_{-1}] = \mathbb{Z}_p$. In this paper, we consider general unramified extension $k/\mathbb{Q}_p$, carefully taking into account this factor $\varphi + \varphi^{-1}$ in $\mathbb{Z}_p[G_{-1}]$.

LEMMA 3.6. *$\varphi + \varphi^{-1}$ is a unit in $\mathbb{Z}_p[G_{-1}]$ if and only if $d \not\equiv 0 \pmod 4$.*

PROOF. First we note that $\varphi + \varphi^{-1} \in \mathbb{Z}_p[G_{-1}]^\times$ if and only if $1 + \varphi^2 \in \mathbb{Z}_p[G_{-1}]^\times$. If d is odd, we have

$$(1+\varphi^2)(1 - \varphi^2 + \varphi^4 - \cdots + (-1)^{\frac{2d-2}{2}} \varphi^{2d-2}) = 1 + (-1)^{d-1}$$
$$= 2\,.$$

If d is even, we have

$$(1+\varphi^2)(1 - \varphi^2 + \varphi^4 - \cdots + (-1)^{\frac{d-2}{2}} \varphi^{d-2}) = 1 + (-1)^{\frac{d-2}{2}}$$
$$= \begin{cases} 2 & \text{if } d \not\equiv 0 \pmod 4\,, \\ 0 & \text{if } d \equiv 0 \pmod 4\,, \end{cases}$$

and $1 - \varphi^2 + \varphi^4 - \cdots + (-1)^{\frac{d-2}{2}} \varphi^{d-2} \neq 0$ in $\mathbb{Z}_p[G_{-1}]$. Since 2 is invertible in $\mathbb{Z}_p[G_{-1}]$, we get the conclusion of Lemma 3.6. □

REMARK 3.7. In the proof of Lemma 3.6, we have proved the following; (1) $\varphi + \varphi^{-1}$ is a unit if $d \not\equiv 0 \pmod 4$, (2) $\varphi + \varphi^{-1}$ is a zero-divisor if $d \equiv 0 \pmod 4$.

Moreover, we can easily check the following lemma.

LEMMA 3.8. *We have*

$$\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1}) = \begin{cases} 0 & \text{if } d \not\equiv 0 \pmod 4, \\ \langle 1-\varphi^2+\varphi^4-\cdots-\varphi^{d-2}\rangle_{\mathbb{Z}_p[G_{-1}]} & \text{if } d \equiv 0 \pmod 4, \end{cases}$$

and $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1}) = 2$ *if* $d \equiv 0 \pmod 4$.

To describe the quotient modules $\widehat{E}(\mathfrak{m}_n)/\widehat{E}(\mathfrak{m}_{n-1})$ using the local points $(d_n)_n$, we prepare the following lemma.

LEMMA 3.9. *We have* $\mathfrak{m}_n/\mathfrak{m}_{n-1} = \langle \pi_n\rangle_{\mathbb{Z}_p[G_n]}$ *for* $n \geq 0$.

PROOF. It is enough to show that $\zeta(\zeta_{p^{n+1}}-1)$ generates $\mathfrak{m}_n/\mathfrak{m}_{n-1}$ as a $\mathbb{Z}_p[G_n]$-module.

We first observe the ring of integers $\mathcal{O}_{k_n}$ of k_n. Let $P_m = \{(\zeta_{p^m}-1)^\tau \mid \tau \in \mathrm{Gal}(k_m/k)\}$ for $m \geq 1$ and $P_0 = \{1\}$. Since $\mathcal{O}_{k_n} = \mathcal{O}_k[\zeta_{p^{n+1}}]$, we have

$$\mathcal{O}_{k_n} = \langle P_0 \cup P_1 \cup \cdots \cup P_{n+1}\rangle_{\mathcal{O}_k}.$$

Thus, for $x \in \mathcal{O}_{k_n}$, we can write $x = a_0 + \sum_{m=1}^{n+1}\sum_{\tau\in\mathrm{Gal}(k_m/k)} a_{m,\tau}(\zeta_{p^m}-1)^\tau$ with $a_0, a_{m,\tau} \in \mathcal{O}_k$. With this notation, since each $(\zeta_{p^m}-1)^\tau$ already has positive valuation, we see that $x \in \mathfrak{m}_n$ if and only if $a_0 \in \mathfrak{m}_k$.

Take any class in $\mathfrak{m}_n/\mathfrak{m}_{n-1}$ with a representative $x \in \mathfrak{m}_n$. Write $x = a_0 + \sum_{m=1}^{n+1}\sum_{\tau\in\mathrm{Gal}(k_m/k)} a_{m,\tau}(\zeta_{p^{n+1}}-1)^\tau$. In this summation, the summands a_0 and $a_{m,\tau}(\zeta_{p^m}-1)^\tau$ with $1 \leq m \leq n$ are contained in $\mathfrak{m}_{n-1}$. Since $\mathcal{O}_k = \mathbb{Z}_p[\zeta] = \langle\zeta\rangle_{\mathbb{Z}_p[G_{-1}]}$, each $a_{n+1,\tau} \in \mathcal{O}_k$ can be written as $a_{n+1,\tau} = \sum_{i=0}^{d-1} b_{\tau,i}\zeta^{\varphi^i}$ with $b_{\tau,i} \in \mathbb{Z}_p$. Therefore we have

$$\begin{aligned} x &\equiv \sum_{\tau\in\mathrm{Gal}(k_{n+1}/k)} a_{n+1,\tau}(\zeta_{p^{n+1}}-1)^\tau \\ &\equiv \sum_{\tau\in\mathrm{Gal}(k_{n+1}/k)}\sum_{i=0}^{d-1} b_{\tau,i}\zeta^{\varphi^i}(\zeta_{p^{n+1}}-1)^\tau \pmod{\mathfrak{m}_{n-1}}. \end{aligned}$$

Here, $\zeta^{\varphi^i}(\zeta_{p^{n+1}}-1)^\tau$ for each i with $0 \leq i \leq d-1$, and each $\tau \in \mathrm{Gal}(k_{n+1}/k)$, is exactly a Galois conjugate of $\zeta(\zeta_{p^{n+1}}-1)$ by $G_n = G_{-1}\times\mathrm{Gal}(k_{n+1}/k)$. This completes the proof. □

PROPOSITION 3.10. *For* $n \geq 0$, *we have* $\log_{\widehat{E}}(\widehat{E}(\mathfrak{m}_n)) \subseteq \mathfrak{m}_n + k_{n-1}$ *and the formal logarithm* $\log_{\widehat{E}}$ *induces canonical isomorphisms of* $\mathbb{Z}_p[G_n]$*-modules*,

$$\widehat{E}(\mathfrak{m}_n)/\widehat{E}(\mathfrak{m}_{n-1}) \xrightarrow{\simeq} \mathfrak{m}_n/\mathfrak{m}_{n-1}.$$

By these isomorphisms, d_n is sent to π_n. In particular, we have

$$\widehat{E}(\mathfrak{m}_n)/\widehat{E}(\mathfrak{m}_{n-1}) = \langle d_n \rangle_{\mathbb{Z}_p[G_n]}\,.$$

PROOF. We prove this by the same method as the proof of Proposition 8.11 in [11] or Proposition 4.9 in [6].

For the first statement, we only note that by the commutative diagram

$$\begin{array}{ccc} \widehat{E}(\mathfrak{m}_n) & \xrightarrow{\log_{\widehat{E}}} & k_n \\ {\scriptstyle \exp_{\mathscr{G}_{-1}} \circ \log_{\widehat{E}}} \downarrow \cong & \nearrow_{\log_{\mathscr{G}_{-1}}} & \\ \mathscr{G}_{-1}(\mathfrak{m}_n) & & \end{array}\,,$$

it is enough to consider $\log_{\mathscr{G}_{-1}} (= \log_{\mathscr{G}})$ on $\mathscr{G}_{-1}(\mathfrak{m}_n)$ instead of $\log_{\widehat{E}}$ on $\widehat{E}(\mathfrak{m}_n)$. Then we can show that $\log_{\mathscr{G}_{-1}}(\mathscr{G}_{-1}(\mathfrak{m}_n)) \subseteq \mathfrak{m}_n + k_{n-1}$ as in [11], [6].

Since we have $\log_{\widehat{E}}(\mathfrak{m}_n) \cap k_{n-1} = \log_{\widehat{E}}(\mathfrak{m}_{n-1})$, the natural map

$$\widehat{E}(\mathfrak{m}_n)/\widehat{E}(\mathfrak{m}_{n-1}) \longrightarrow (\mathfrak{m}_n + k_{n-1})/k_{n-1} \cong \mathfrak{m}_n/\mathfrak{m}_{n-1}$$

is injective. Since $\varepsilon_n \in \mathfrak{m}_{-1}$ and $\pi_{n-2m} \in k_{n-2}$ for $m \geq 1$, we have

$$\begin{aligned} \log_{\widehat{E}}(d_n) &= \varepsilon_n + \pi_n + \sum_{m=1}^{[\frac{n+1}{2}]} (-1)^m \frac{\pi_{n-2m}}{p^m} \\ &\equiv \pi_n \pmod{k_{n-1}}\,. \end{aligned}$$

Since π_n generates $\mathfrak{m}_n/\mathfrak{m}_{n-1}$ as a $\mathbb{Z}_p[G_n]$-module (cf. Lemma 3.9), the above injection is in fact a bijection and d_n generates $\widehat{E}(\mathfrak{m}_n)/\widehat{E}(\mathfrak{m}_{n-1})$ as a $\mathbb{Z}_p[G_n]$-module. □

COROLLARY 3.11. *We have*

$$\widehat{E}(\mathfrak{m}_n) = \begin{cases} \langle d_{-1} \rangle_{\mathbb{Z}_p[G_{-1}]} & \text{if } n = -1\,, \\ \langle d_n, d_{n-1} \rangle_{\mathbb{Z}_p[G_n]} & \text{if } n \geq 0\,. \end{cases}$$

PROOF. The case $n = -1$ follows from $\widehat{E}(\mathfrak{m}_{-1}) \cong \mathfrak{m}_{-1}$ (see Proposition 3.2) and Nakayama's lemma. The case $n \geq 0$ follows easily from Proposition 3.10 and the trace relations satisfied by the d_n (see Proposition 3.4). □

REMARK 3.12. We defined the system of local points $(d_n)_n$ following B.D. Kim [7] and M. Kim [10] in the above. We can take another system of local points instead of $(d_n)_n$. Indeed, what we need for the following discussion is a system of local points $(d_n)_n$ which satisfies the following three conditions

1. $\mathrm{Tr}_{n/n-1}(d_n) = -d_{n-2}$ for each $n \geq 1$ (Proposition 3.4 (1)),
2. $\mathrm{Tr}_{0/-1}(d_0) = -(\varphi + \varphi^{-1})d_{-1}$ (Proposition 3.4 (2)),

3. $\widehat{E}(\mathfrak{m}_n)/\widehat{E}(\mathfrak{m}_{n-1}) = \langle d_n \rangle_{\mathbb{Z}_p[G_n]}$ (Proposition 3.10).

Such a system $(d_n)_n$ obviously admits at least a difference of multiplication by a unit in $\mathbb{Z}_p[G_{-1}]^\times$. S. Kobayashi constructed such a system of local points also in [12] Proof of Proposition 3.12 by using another formal power series $\ell_\epsilon(X)$ and a system $(\zeta_{p^{n+1}}-1)_n$ instead of $\log_{\mathscr{G}}(X)$ and a system $(\epsilon_n[+]_{\mathscr{G}_n}\pi_n)_n$. In our setting, the formal power series $\ell_\epsilon(X)$ is defined for each $\epsilon \in \widehat{E}(\mathfrak{m}_k)$ by

$$\ell_\epsilon(X) = \varepsilon + \sum_{m=0}^{\infty} (-1)^m \frac{f^{(2m)}(\varepsilon' X)}{p^m} \in k[[X]]$$

where $\varepsilon = \log_{\widehat{E}}(\epsilon) \in \mathfrak{m}_k$, $\varepsilon' = (\varphi^2 + p)\varepsilon p^{-1} \in \mathcal{O}_k$, $f(X) = (X+1)^p - 1$ and $f^{(m)}(X)$ is the m-iterated composition of f. By using this formal power series, Kobayashi defined $d_{\epsilon,n}$ for each $n \geq -1$ by

$$d_{\epsilon,n} = \exp_{\widehat{E}} \circ \ell_\epsilon^{\varphi^{-(n+1)}}(\zeta_{p^{n+1}} - 1) \in \widehat{E}(\mathfrak{m}_n)\,.$$

Then the first and the second conditions, which are listed above, are satisfied. If we take $\varepsilon \in \mathfrak{m}_k$ such that $\mathfrak{m}_k = \langle \varepsilon \rangle_{\mathbb{Z}_p[G_n]}$, then the third condition is also satisfied. We also note that we can take such $\varepsilon \in \mathfrak{m}_k$, since $\mathfrak{m}_k$ is known to be a cyclic $\mathbb{Z}_p[G_n]$-module.

3.2. The norm subgroups. Following S. Kobayashi [11] (and M. Kim [10]), we define the n-th plus subgroup $\widehat{E}^+(\mathfrak{m}_n)$, the n-th minus subgroup $\widehat{E}^-(\mathfrak{m}_n)$ and the n-th norm subgroup $\mathscr{C}(\mathfrak{m}_n)$ of $\widehat{E}(\mathfrak{m}_n)$;

DEFINITION 3.13. We define

$$\widehat{E}^+(\mathfrak{m}_n) = \{P \in \widehat{E}(\mathfrak{m}_n) |\, \mathrm{Tr}_{n/m+1}\, P \in \widehat{E}(\mathfrak{m}_m) \text{ for all even } m,\ -1 \leq m \leq n-1\}\,,$$
$$\widehat{E}^-(\mathfrak{m}_n) = \{P \in \widehat{E}(\mathfrak{m}_n) |\, \mathrm{Tr}_{n/m+1}\, P \in \widehat{E}(\mathfrak{m}_m) \text{ for all odd } m,\ -1 \leq m \leq n-1\}\,,$$

for $n \geq 0$. We denote $\widehat{E}^\pm(\mathfrak{m}_\infty) = \bigcup_n \widehat{E}^\pm(\mathfrak{m}_n)$. We also define

$$\mathscr{C}(\mathfrak{m}_n) = \{P \in \widehat{E}(\mathfrak{m}_n) |\, \mathrm{Tr}_{n/m+1}\, P \in \widehat{E}(\mathfrak{m}_m) \text{ for all } m \equiv n \pmod 2,\ -1 \leq m \leq n-1\}$$

for $n \geq 0$ and $\mathscr{C}(\mathfrak{m}_{-1}) = \widehat{E}(\mathfrak{m}_{-1})$.

By the following lemma, it is enough to study $\widehat{E}^\pm(\mathfrak{m}_n)$ instead of $E^\pm(k_n)$ for our purpose.

LEMMA 3.14. *The natural maps* $\widehat{E}^\pm(\mathfrak{m}_n) \to E^\pm(k_n)$ *induce isomorphisms* $\widehat{E}^\pm(\mathfrak{m}_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} E^\pm(k_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ *for all* n, *and thus we have*

$$\widehat{E}^\pm(\mathfrak{m}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p\,.$$

PROOF. We consider the following commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{E}(\mathfrak{m}_n) & \longrightarrow & E(k_n) & \longrightarrow & \widetilde{E}(\mathbb{F}_k) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \iota_n^{\pm}} & & \\ 0 & \longrightarrow & \widehat{E}^{\pm}(\mathfrak{m}_n) & \longrightarrow & E^{\pm}(k_n) & \longrightarrow & A_n^{\pm} & \longrightarrow & 0 \end{array}$$

where $\widetilde{E}$ is the reduction of E modulo p, $\mathbb{F}_k$ is the residue field of k and $A_n^{\pm}$ is the cokernel of $\widehat{E}^{\pm}(\mathfrak{m}_n) \to E^{\pm}(k_n)$. Since $\widehat{E}^{\pm}(\mathfrak{m}_n) = \widehat{E}(\mathfrak{m}_n) \cap E^{\pm}(k_n)$, we see that the right vertical arrows $\iota_n^{\pm}$ are injective. Thus $A_n^{\pm}$ are finite as $\widetilde{E}(\mathbb{F}_k)$ is finite. We also note that $A_n^{\pm}[p^{\infty}] = 0$, since $E/\mathbb{Q}_p$ has supersingular reduction. From the above, our claim will follow immediately. □

By comparing two definitions, we get the following relations between the plus subgroups (the minus subgroups) and the norm subgroups.

LEMMA 3.15. *We have*

$$\widehat{E}^{+}(\mathfrak{m}_n) = \begin{cases} \mathscr{C}(\mathfrak{m}_n) & \text{if } n \text{ is even}, \\ \mathscr{C}(\mathfrak{m}_{n-1}) & \text{if } n \text{ is odd}, \end{cases}$$

$$\widehat{E}^{-}(\mathfrak{m}_n) = \begin{cases} \mathscr{C}(\mathfrak{m}_n) & \text{if } n \text{ is odd}, \\ \mathscr{C}(\mathfrak{m}_{n-1}) & \text{if } n \text{ is even}. \end{cases}$$

We now describe $\mathscr{C}(\mathfrak{m}_n)$ in terms of the system of local points $(d_n)_n$, and thus we get a description of plus and minus subgroups $\widehat{E}^{\pm}(\mathfrak{m}_n)$.

PROPOSITION 3.16. (1) *For each $n \geq -1$, the n-th norm subgroup is generated by d_n and d_{-1} as a $\mathbb{Z}_p[G_n]$-module*;

$$\mathscr{C}(\mathfrak{m}_n) = \langle d_n, d_{-1} \rangle_{\mathbb{Z}_p[G_n]}.$$

(2) *For each $n \geq 0$, we have an exact sequence*

$$0 \longrightarrow \widehat{E}(\mathfrak{m}_{-1}) \longrightarrow \mathscr{C}(\mathfrak{m}_n) \oplus \mathscr{C}(\mathfrak{m}_{n-1}) \longrightarrow \widehat{E}(\mathfrak{m}_n) \longrightarrow 0, \tag{3.2}$$

where the first map is diagonal embedding by inclusions, and the second map is $(a, b) \mapsto a - b$.

PROOF. We will prove this by the same method as the proof of Proposition 8.12 in [11]. The main difference is the element d_{-1} in the first statement.

We can show that $\mathscr{C}(\mathfrak{m}_n) \cap \mathscr{C}(\mathfrak{m}_{n-1}) = \widehat{E}(\mathfrak{m}_{-1})$ for $n \geq 0$ by the completely same way as in [11].

For the moment, let $\mathscr{C}'(\mathfrak{m}_n)$ be the $\mathbb{Z}_p[G_n]$-submodule of $\widehat{E}(\mathfrak{m}_n)$ generated by d_n and d_{-1}. By the trace relations on d_n, clearly we have $\mathscr{C}(\mathfrak{m}_n) \supseteq \mathscr{C}'(\mathfrak{m}_n)$. We now prove

$$\mathscr{C}(\mathfrak{m}_n) = \mathscr{C}'(\mathfrak{m}_n),\ \mathscr{C}(\mathfrak{m}_n) + \mathscr{C}(\mathfrak{m}_{n-1}) = \widehat{E}(\mathfrak{m}_n)$$

for $n \geq 0$, simultaneously by induction.

In the case $n = 0$, we have

$$\mathscr{C}(\mathfrak{m}_0) = \widehat{E}(\mathfrak{m}_0) = \langle d_0, d_{-1}\rangle_{\mathbb{Z}_p[G_0]} = \mathscr{C}'(\mathfrak{m}_0)\,,$$
$$\mathscr{C}(\mathfrak{m}_0) + \mathscr{C}(\mathfrak{m}_{-1}) = \widehat{E}(\mathfrak{m}_0) + \widehat{E}(\mathfrak{m}_{-1}) = \widehat{E}(\mathfrak{m}_0)$$

by Corollary 3.11.

In the case $n \geq 1$, by the induction hypothesis we have

$$\widehat{E}(\mathfrak{m}_{n-1}) = \mathscr{C}(\mathfrak{m}_{n-1}) + \mathscr{C}(\mathfrak{m}_{n-2}),\ \mathscr{C}(\mathfrak{m}_{n-2}) = \mathscr{C}'(\mathfrak{m}_{n-2}) \tag{3.3}$$

and by the trace relation we have $\mathscr{C}'(\mathfrak{m}_{n-2}) \subseteq \mathscr{C}'(\mathfrak{m}_n)$. Therefore, by Proposition 3.10 and (3.3), we have

$$\begin{aligned}\widehat{E}(\mathfrak{m}_n) &= \langle d_n\rangle_{\mathbb{Z}_p[G_n]} + \widehat{E}(\mathfrak{m}_{n-1}) \\ &= (\langle d_n\rangle_{\mathbb{Z}_p[G_n]} + \mathscr{C}'(\mathfrak{m}_{n-2})) + \mathscr{C}(\mathfrak{m}_{n-1}) \\ &\subseteq \mathscr{C}'(\mathfrak{m}_n) + \mathscr{C}(\mathfrak{m}_{n-1})\,.\end{aligned}$$

In particular, we have $\mathscr{C}(\mathfrak{m}_n) \subseteq \mathscr{C}'(\mathfrak{m}_n) + \mathscr{C}(\mathfrak{m}_{n-1})$. This implies that $\mathscr{C}(\mathfrak{m}_n) = \mathscr{C}'(\mathfrak{m}_n)$. Indeed, if $P \in \mathscr{C}(\mathfrak{m}_n)$, then there exist $Q \in \mathscr{C}'(\mathfrak{m}_n)$ and $R \in \mathscr{C}(\mathfrak{m}_{n-1})$ such that $P = Q + R$. Then we see that $R = P - Q \in \mathscr{C}(\mathfrak{m}_n) \cap \mathscr{C}(\mathfrak{m}_{n-1}) = \widehat{E}(\mathfrak{m}_{-1})$. Note that $\widehat{E}(\mathfrak{m}_{-1}) \subseteq \mathscr{C}'(\mathfrak{m}_n)$ since $d_{-1} \in \mathscr{C}'(\mathfrak{m}_n)$. So we get $P = Q + R \in \mathscr{C}'(\mathfrak{m}_n)$ and thus $\mathscr{C}(\mathfrak{m}_n) = \mathscr{C}'(\mathfrak{m}_n)$. It is now clear that $\mathscr{C}(\mathfrak{m}_n) + \mathscr{C}(\mathfrak{m}_{n-1}) = \mathscr{C}'(\mathfrak{m}_n) + \mathscr{C}'(\mathfrak{m}_{n-1}) = \widehat{E}(\mathfrak{m}_n)$. □

REMARK 3.17. We check here that the norm subgroup $\mathscr{C}(\mathfrak{m}_n)$ is not a cyclic $\mathbb{Z}_p[G_n]$-module generated by d_n if and only if $d = [k : \mathbb{Q}_p] \equiv 0 \pmod 4$ and n is even.

(1) When n is odd or $d \not\equiv 0 \pmod 4$, we see that d_{-1} is automatically contained in $\langle d_n\rangle_{\mathbb{Z}_p[G_n]}$. Thus in these cases we see that the norm subgroup $\mathscr{C}(\mathfrak{m}_n)$ is a cyclic $\mathbb{Z}_p[G_n]$-module generated by d_n for each n;

$$\mathscr{C}(\mathfrak{m}_n) = \langle d_n\rangle_{\mathbb{Z}_p[G_n]}\,.$$

Indeed, when n is odd, we have

$$d_{-1} = (-1)^{\frac{n+1}{2}} \operatorname{Tr}_{1/0} \cdots \operatorname{Tr}_{n-2/n-3} \operatorname{Tr}_{n/n-1} d_n \in \langle d_n\rangle_{\mathbb{Z}_p[G_n]}\,.$$

When $d \not\equiv 0 \pmod 4$ and n is even, we have

$$d_{-1} = (-1)^{\frac{n+2}{2}} (\varphi + \varphi^{-1})^{-1} \operatorname{Tr}_{0/-1} \cdots \operatorname{Tr}_{n-2/n-3} \operatorname{Tr}_{n/n-1} d_n \in \langle d_n\rangle_{\mathbb{Z}_p[G_n]}\,,$$

since $\varphi + \varphi^{-1} \in \mathbb{Z}_p[G_{-1}]^\times$ by Lemma 3.6.

(2) When $d \equiv 0 \pmod 4$, d_{-1} cannot be contained in $\langle d_n \rangle_{\mathbb{Z}_p[G_n]}$ for any even n. Thus in this case the norm subgroup $\mathscr{C}(\mathfrak{m}_n)$ is not a cyclic $\mathbb{Z}_p[G_n]$-module generated by d_n for each even n. Indeed, if $\widehat{E}(\mathfrak{m}_0) = \langle d_0 \rangle_{\mathbb{Z}_p[G_0]}$, then $\widehat{E}(\mathfrak{m}_{-1}) = \langle \mathrm{Tr}_{0/-1}(d_0) \rangle_{\mathbb{Z}_p[G_{-1}]}$ since $\mathrm{Tr}_{0/-1} : \widehat{E}(\mathfrak{m}_0) \to \widehat{E}(\mathfrak{m}_{-1})$ is surjective. Since $\widehat{E}(\mathfrak{m}_{-1}) \cong \mathbb{Z}_p[G_{-1}]$, this means that $\mathbb{Z}_p[G_{-1}] = (\varphi + \varphi^{-1})\mathbb{Z}_p[G_{-1}]$, which is impossible by Lemma 3.6.

DEFINITION 3.18. Define $d_n^{\pm}$ by

$$d_n^+ = \begin{cases} (-1)^{\frac{n+2}{2}} d_n & \text{if } n \text{ is even}, \\ (-1)^{\frac{n+1}{2}} d_{n-1} & \text{if } n \text{ is odd}, \end{cases} \qquad d_n^- = \begin{cases} (-1)^{\frac{n+1}{2}} d_n & \text{if } n \text{ is odd}, \\ (-1)^{\frac{n}{2}} d_{n-1} & \text{if } n \text{ is even}. \end{cases}$$

REMARK 3.19. By the relation between $\mathscr{C}(\mathfrak{m}_n)$ and $\widehat{E}^{\pm}(\mathfrak{m}_n)$ (cf. Lemma 3.15), we can translate Proposition 3.16 in terms of the plus and the minus systems of points $(d_n^+)_n$ and $(d_n^-)_n$ such that $\widehat{E}^+(\mathfrak{m}_n) = \langle d_n^+, d_0^- \rangle_{\mathbb{Z}_p[G_n]}$ and $\widehat{E}^-(\mathfrak{m}_n) = \langle d_n^-, d_0^- \rangle_{\mathbb{Z}_p[G_n]}$. As in Remark 3.17, we see that the the plus subgroups $\widehat{E}^+(\mathfrak{m}_n)$ are cyclic $\mathbb{Z}_p[G_n]$-modules generated by d_n^+ for all n if and only if $d \not\equiv 0 \pmod 4$, on the other hand the minus subgroups $\widehat{E}^-(\mathfrak{m}_n)$ are always cyclic $\mathbb{Z}_p[G_n]$-modules generated by d_n^- for all n.

Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character of $\Delta = \mathrm{Gal}(k(\mu_p)/k)$. If M is a $\mathbb{Z}_p[\Delta]$-module, then M is decomposed into

$$M = \bigoplus_{\chi} \varepsilon_\chi M,$$

where $\varepsilon_\chi = \frac{1}{p-1} \sum_{\sigma \in \Delta} \chi(\sigma)\sigma^{-1} \in \mathbb{Z}_p[\Delta]$. We denote by M^χ the χ-component $\varepsilon_\chi M$.

Since we have $G_n \cong G_{-1} \times \Delta \times \mathrm{Gal}(k_n/k_0)$, we can regard a $\mathbb{Z}_p[G_n]$-module as a $\mathbb{Z}_p[\Delta]$-module.

COROLLARY 3.20. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character and $q_n = \sum_{i=0}^n (-1)^i p^{n-i}$. Then we have*

$$\mathrm{rank}_{\mathbb{Z}_p} \mathscr{C}(\mathfrak{m}_n)^\chi = \begin{cases} d(q_n + 1) & \text{if } n : \text{odd and } \chi = \mathbf{1}, \\ dq_n & \text{otherwise}, \end{cases}$$

for each $n \geq 0$ and

$$\mathrm{rank}_{\mathbb{Z}_p} \mathscr{C}(\mathfrak{m}_{-1})^\chi = \begin{cases} d & \text{if } \chi = \mathbf{1}, \\ 0 & \text{if } \chi \neq \mathbf{1}. \end{cases}$$

PROOF. Since $\mathscr{C}(\mathfrak{m}_{-1}) = \widehat{E}(\mathfrak{m}_{-1}) \cong \mathbb{Z}_p[G_{-1}]$, we obtain the latter statement.

From the exact sequence (3.2) we get a recurrence sequence

$$\mathrm{rank}_{\mathbb{Z}_p} \mathscr{C}(\mathfrak{m}_n)^\chi + \mathrm{rank}_{\mathbb{Z}_p} \mathscr{C}(\mathfrak{m}_{n-1})^\chi$$

$$= \operatorname{rank}_{\mathbb{Z}_p} \widehat{E}(\mathfrak{m}_{-1})^{\chi} + \operatorname{rank}_{\mathbb{Z}_p} \widehat{E}(\mathfrak{m}_n)^{\chi} .$$

By the theory of formal groups, we have

$$\begin{cases} \widehat{E}(\mathfrak{m}_n^r)^{\chi} \cong (\mathfrak{m}_n^r)^{\chi} \quad \text{(as } \mathbb{Z}_p[G_n]\text{-modules), and} \\ \#\widehat{E}(\mathfrak{m}_n)^{\chi}/\widehat{E}(\mathfrak{m}_n^r)^{\chi} = \#\mathfrak{m}_n^{\chi}/(\mathfrak{m}_n^r)^{\chi} < \infty \end{cases}$$

for sufficiently large r. Thus we have

$$\begin{aligned} \operatorname{rank}_{\mathbb{Z}_p} \widehat{E}(\mathfrak{m}_n)^{\chi} &= \operatorname{rank}_{\mathbb{Z}_p} \widehat{E}(\mathfrak{m}_n^r)^{\chi} \\ &= \operatorname{rank}_{\mathbb{Z}_p} (\mathfrak{m}_n^r)^{\chi} = \operatorname{rank}_{\mathbb{Z}_p} \mathfrak{m}_n^{\chi} = dp^n \end{aligned}$$

for each $n \geq 0$. Therefore we obtain the former statement. □

We introduce here some notation that will be used throughout the rest of the paper. Let $\omega_n(X) := (1+X)^{p^n} - 1$ and $\Phi_n(X) := \sum_{i=0}^{p-1} X^{ip^{n-1}}$ be the p^n-th cyclotomic polynomial. We define $\widetilde{\omega}_0^{\pm}(X) := 1$ and

$$\widetilde{\omega}_n^{+}(X) = \prod_{1 \leq m \leq n, m:\text{even}} \Phi_m(1+X), \quad \omega_n^{+}(X) = X\widetilde{\omega}_n^{+}(X) ,$$

$$\widetilde{\omega}_n^{-}(X) = \prod_{1 \leq m \leq n, m:\text{odd}} \Phi_m(1+X), \quad \omega_n^{-}(X) = X\widetilde{\omega}_n^{-}(X) .$$

Note that $\omega_n(X) = \widetilde{\omega}_n^{\mp}(X)\omega_n^{\pm}(X)$ for all $n \geq 0$. We write $\omega_n(X)$, $\widetilde{\omega}_n^{\pm}(X)$ and $\omega_n^{\pm}(X)$ simply by ω_n, $\widetilde{\omega}_n^{\pm}$ and $\omega_n^{\pm}$ respectively.

We identify $\mathbb{Z}_p[G_n]$ with $\mathbb{Z}_p[G_0][X]/\langle \omega_n \rangle_{\mathbb{Z}_p[G_0][X]}$ by sending γ_n to $1+X$, where γ_n is the image of γ in $\mathbb{Z}_p[G_n]$.

Set $q_n = \sum_{i=0}^{n} (-1)^i p^{n-i}$ as in Corollary 3.20 and $q_{-1} := 0$. Put

$$q_n^{+} := \begin{cases} q_n & \text{if } n \text{ is even} , \\ q_{n-1} & \text{if } n \text{ is odd} , \end{cases} \qquad q_n^{-} := \begin{cases} q_n & \text{if } n \text{ is odd} , \\ q_{n-1} & \text{if } n \text{ is even} . \end{cases}$$

Note that $q_n^{+} + q_n^{-} = p^n$ for all $n \geq 0$.

For later use, we rephrase Corollary 3.20 in terms of $\widehat{E}^{\pm}(\mathfrak{m}_n)^{\chi}$ and $q_n^{\pm}$ as in the following corollary.

COROLLARY 3.21. *Let $\chi : \Delta \to \mathbb{Z}_p^{\times}$ be a character. Then we have*

$$\operatorname{rank}_{\mathbb{Z}_p}(\widehat{E}^{+}(\mathfrak{m}_n)^{\chi}) = dq_n^{+} ,$$

$$\operatorname{rank}_{\mathbb{Z}_p}(\widehat{E}^{-}(\mathfrak{m}_n)^{\chi}) = \begin{cases} d(q_n^{-}+1) & \text{if } \chi = \mathbf{1} , \\ dq_n^{-} & \text{if } \chi \neq \mathbf{1} . \end{cases}$$

For a character χ of Δ, we define $\mathbb{Z}_p[\chi]$ to be the $\mathbb{Z}_p[\Delta]$-module which is $\mathbb{Z}_p$ as a $\mathbb{Z}_p$-module, and on which Δ acts via χ, namely $\sigma \cdot x = \chi(\sigma)x$ for $\sigma \in \Delta$ and $x \in \mathbb{Z}_p[\chi]$.

PROPOSITION 3.22. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character. We have*

$$\widehat{E}^+(\mathfrak{m}_n)^\chi \cong \begin{cases} \dfrac{\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\widetilde{\omega}_n^+, -(\varphi+\varphi^{-1}))\rangle_{\mathbb{Z}_p[G_{-1}][X]}} & \text{if } \chi = \mathbf{1}\,, \\ \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} \dfrac{\mathbb{Z}_p[G_{-1}][X]}{\langle \omega_n^+ \rangle_{\mathbb{Z}_p[G_{-1}][X]}} & \text{if } \chi \neq \mathbf{1}\,, \end{cases}$$

$$\widehat{E}^-(\mathfrak{m}_n)^\chi \cong \begin{cases} \mathbb{Z}_p[G_{-1}][X]/\langle \omega_n^- \rangle_{\mathbb{Z}_p[G_{-1}][X]} & \text{if } \chi = \mathbf{1}\,, \\ \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} \dfrac{\mathbb{Z}_p[G_{-1}][X]}{\langle \widetilde{\omega}_n^- \rangle_{\mathbb{Z}_p[G_{-1}][X]}} & \text{if } \chi \neq \mathbf{1}\,, \end{cases}$$

as $\mathbb{Z}_p[G_n]$-modules.

PROOF. There is a surjective homomorphism

$$\psi : \frac{\mathbb{Z}_p[G_0][X]}{\langle \omega_n \rangle_{\mathbb{Z}_p[G_0][X]}} \oplus \mathbb{Z}_p[G_{-1}] \longrightarrow \langle d_n^+, d_0^- \rangle_{\mathbb{Z}_p[G_n]} = \widehat{E}^+(\mathfrak{m}_n)$$

obtained by sending $(1,0)$ to d_n^+ and $(0,1)$ to $\frac{d_0^-}{p-1} (= \frac{d_{-1}}{p-1})$. We have a relation

$$\omega_n^+ d_n^+ = \omega_{n-2}^+ d_{n-2}^+ = \cdots = \omega_0^+ d_0^+ = X d_0^+ = 0\,.$$

Since $\varepsilon_{\mathbf{1}} = \frac{1}{p-1}\sum_{\sigma\in\Delta}\sigma^{-1} = \frac{1}{p-1}\,\mathrm{Tr}_{0/-1} \in \mathbb{Z}_p[\Delta]$, we also have another relation

$$\varepsilon_{\mathbf{1}} \widetilde{\omega}_n^+ d_n^+ = \frac{1}{p-1}\,\mathrm{Tr}_{0/-1}\,\widetilde{\omega}_n^+ d_n^+ = \frac{1}{p-1}\,\mathrm{Tr}_{0/-1}\,d_0^+ = (\varphi+\varphi^{-1})\frac{d_0^-}{p-1}\,.$$

Thus the map ψ induces a surjective homomorphism

$$\overline{\psi} : \frac{\frac{\mathbb{Z}_p[G_0][X]}{\langle \omega_n^+ \rangle_{\mathbb{Z}_p[G_0][X]}} \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\varepsilon_{\mathbf{1}}\widetilde{\omega}_n^+, -(\varphi+\varphi^{-1}))\rangle_{\mathbb{Z}_p[G_0][X]}} \longrightarrow \langle d_n^+, d_0^- \rangle_{\mathbb{Z}_p[G_n]} = \widehat{E}^+(\mathfrak{m}_n)\,.$$

This map $\overline{\psi}$ is injective since the source and the target of $\overline{\psi}$ are free $\mathbb{Z}_p$-modules of the same $\mathbb{Z}_p$-rank $d(p-1)q_n^+$ (cf. Corollary 3.21). Thus we have

$$\widehat{E}^+(\mathfrak{m}_n) \cong \frac{\frac{\mathbb{Z}_p[G_0][X]}{\langle \omega_n^+ \rangle_{\mathbb{Z}_p[G_0][X]}} \oplus \mathbb{Z}_p[G_{-1}]}{\left\langle (\varepsilon_{\mathbf{1}}\widetilde{\omega}_n^+, -(\varphi+\varphi^{-1}))\right\rangle_{\mathbb{Z}_p[G_0][X]}}$$

$$\cong \frac{\mathbb{Z}_p[G_0][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\omega_n^+, 0), (\varepsilon_{\mathbf{1}}\widetilde{\omega}_n^+, -(\varphi+\varphi^{-1}))\rangle_{\mathbb{Z}_p[G_0][X]}}$$

$$\cong \bigoplus_{\chi} \frac{\varepsilon_\chi \mathbb{Z}_p[G_0][X] \oplus \varepsilon_\chi \mathbb{Z}_p[G_{-1}]}{\langle (\varepsilon_\chi \omega_n^+, 0), (\varepsilon_\chi \varepsilon_{\mathbf{1}} \widetilde{\omega}_n^+, -\varepsilon_\chi(\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_0][X]}}$$

as $\mathbb{Z}_p[G_n]$-modules, where the last isomorphism is obtained by the character decomposition. Since we have

$$\varepsilon_\chi \mathbb{Z}_p[G_0][X] \oplus \varepsilon_\chi \mathbb{Z}_p[G_{-1}] \cong \begin{cases} \mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}] & \text{if } \chi = \mathbf{1}, \\ \mathbb{Z}_p[G_{-1}][X] & \text{if } \chi \neq \mathbf{1}, \end{cases}$$

and

$$\langle (\varepsilon_\chi \omega_n^+, 0), (\varepsilon_\chi \varepsilon_{\mathbf{1}} \widetilde{\omega}_n^+, -\varepsilon_\chi(\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_0][X]}$$

$$\cong \begin{cases} \langle (\omega_n^+, 0), (\widetilde{\omega}_n^+, -(\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_{-1}][X]} & \text{if } \chi = \mathbf{1}, \\ \langle \omega_n^+ \rangle_{\mathbb{Z}_p[G_{-1}][X]} & \text{if } \chi \neq \mathbf{1} \end{cases}$$

as $\mathbb{Z}_p[G_{-1}][X]$-modules, we have

$$\widehat{E}^+(\mathfrak{m}_n)^\chi \cong \frac{\varepsilon_\chi \mathbb{Z}_p[G_0][X] \oplus \varepsilon_\chi \mathbb{Z}_p[G_{-1}]}{\langle (\varepsilon_\chi \omega_n^+, 0), (\varepsilon_\chi \varepsilon_{\mathbf{1}} \widetilde{\omega}_n^+, -\varepsilon_\chi(\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_0][X]}}$$

$$\cong \begin{cases} \dfrac{\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\omega_n^+, 0), (\widetilde{\omega}_n^+, -(\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_{-1}][X]}} & \text{if } \chi = \mathbf{1}, \\ \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} \dfrac{\mathbb{Z}_p[G_{-1}][X]}{\langle \omega_n^+ \rangle_{\mathbb{Z}_p[G_{-1}][X]}} & \text{if } \chi \neq \mathbf{1} \end{cases}$$

as $\mathbb{Z}_p[G_n]$-modules. Since $(\omega_n^+, 0) = X(\widetilde{\omega}_n^+, -(\varphi + \varphi^{-1}))$, we get the conclusion for $\widehat{E}^+(\mathfrak{m}_n)^\chi$.

Similarly to the above, we have

$$\widehat{E}^-(\mathfrak{m}_n) = \langle d_n^- \rangle_{\mathbb{Z}_p[G_n]}$$

$$\cong \mathbb{Z}_p[G_0][X] / \langle \omega_n^-, (\sigma - 1)\widetilde{\omega_n}^- | \sigma \in \Delta \rangle_{\mathbb{Z}_p[G_0][X]}$$

$$\cong \frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \omega_n^- \rangle_{\mathbb{Z}_p[G_{-1}][X]}} \oplus \bigoplus_{\chi \neq \mathbf{1}} \left(\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} \frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \widetilde{\omega}_n^- \rangle_{\mathbb{Z}_p[G_{-1}][X]}} \right)$$

as $\mathbb{Z}_p[G_n]$-modules. So we get the conclusion for $\widehat{E}^-(\mathfrak{m}_n)^\chi$. □

REMARK 3.23. When $d \not\equiv 0 \pmod 4$ and $\chi = \mathbf{1}$, the description of the Galois module $\widehat{E}^+(\mathfrak{m}_n)^\chi$ in Proposition 3.22 can be made more simpler. Explicitly, we claim that the

homomorphism

$$\mathbb{Z}_p[G_{-1}][X]/\langle\omega_n^+\rangle \xrightarrow{\simeq} \frac{\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle(\widetilde{\omega}_n^+, -(\varphi+\varphi^{-1}))\rangle}$$

given by $x \mapsto (x, 0)$ is an isomorphism. Indeed, since $\varphi + \varphi^{-1} \in \mathbb{Z}_p[G_{-1}]^\times$ in this case (see Lemma 3.6), $(x, y) \in \mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]$ is equivalent to $(x + y(\varphi + \varphi^{-1})^{-1}\widetilde{\omega}_n^+, 0)$ and thus the map is surjective. On the other hand, if $(x, 0) \in \langle(\widetilde{\omega}_n^+, -(\varphi + \varphi^{-1}))\rangle$ for $x \in \mathbb{Z}_p[G_{-1}][X]$, then there exists $a(X) \in \mathbb{Z}_p[G_{-1}][X]$ such that $x = a(X)\widetilde{\omega}_n^+$ and $0 = -a(0)(\varphi+\varphi^{-1})$. Again by Lemma 3.6, we see that $a(0) = 0$. So we get $x = \frac{a(X)}{X}\omega_n^+ \in \langle\omega_n^+\rangle$ and thus the map is injective.

In the rest of this paper, we abbreviate $\mathbb{Z}_p[G_{-1}][X]$-modules $\langle S\rangle_{\mathbb{Z}_p[G_{-1}][X]}$ generated by some set S to $\langle S\rangle$ as in the above remark.

3.3. The plus and the minus local conditions. In this subsection, we study the Λ-module $(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ and prove Proposition 3.28. We also study the Λ-module

$$\left(\frac{H^1(k_\infty, E[p^\infty])}{\widehat{E}^{\pm}(\mathfrak{m}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p}\right)^\vee$$

and prove Proposition 3.32.

We first study $(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

Since $\widehat{E}^{\pm}(\mathfrak{m}_\infty)$ are $\mathbb{Z}_p$-torsion-free, we have an exact sequence

$$0 \longrightarrow \widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \longrightarrow \widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p \longrightarrow \widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0\,.$$

From this exact sequence, we get the Γ_n-invariant-coinvariant exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{E}^{\pm}(\mathfrak{m}_n)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \left(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{\Gamma_n} \\ \longrightarrow \left(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi\right)_{\Gamma_n}[p^\infty] \longrightarrow 0\,, \end{aligned} \tag{3.4}$$

for each $n \geq 0$. We will compute the rightmost modules $\left(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi\right)_{\Gamma_n}[p^\infty]$ for all $n \geq 0$ to study the Λ-module $(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

Define δ by

$$\delta = \begin{cases} 0 & \text{if } d \not\equiv 0 \pmod 4 \text{ or } \chi \neq \mathbf{1}\,, \\ 2 & \text{otherwise}\,. \end{cases}$$

.

PROPOSITION 3.24. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character. Then $\left((\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi)_{\Gamma_n}[p^\infty]\right)^\vee$ are free $\mathbb{Z}_p$-modules for all n, and we have*

$$\operatorname{rank}_{\mathbb{Z}_p}\left((\widehat{E}^{+}(\mathfrak{m}_\infty)^\chi)_{\Gamma_n}[p^\infty]\right)^\vee = dq_n^- + \delta\,,$$

$$\operatorname{rank}_{\mathbb{Z}_p}\left((\widehat{E}^{-}(\mathfrak{m}_\infty)^\chi)_{\Gamma_n}[p^\infty]\right)^\vee = \begin{cases} d(q_n^+ - 1) & \text{if } \chi = \mathbf{1}\,, \\ dq_n^+ & \text{if } \chi \neq \mathbf{1}\,. \end{cases}$$

More precisely, we have

$$\left(\widehat{E}^{+}(\mathfrak{m}_\infty)^\chi\right)_{\Gamma_n}[p^\infty] \cong \begin{cases} \left(\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle\widetilde{\omega}_n^-\rangle} \oplus \operatorname{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1})\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \text{if } \chi = \mathbf{1}\,, \\ \left(\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle\widetilde{\omega}_n^-\rangle}\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \text{if } \chi \neq \mathbf{1}\,, \end{cases}$$

$$\left(\widehat{E}^{-}(\mathfrak{m}_\infty)^\chi\right)_{\Gamma_n}[p^\infty] \cong \begin{cases} \frac{\mathbb{Z}_p[G_{-1}][X]}{\langle\widetilde{\omega}_n^+\rangle} \otimes \mathbb{Q}_p/\mathbb{Z}_p & \text{if } \chi = \mathbf{1}\,, \\ \frac{\mathbb{Z}_p[G_{-1}][X]}{\langle\omega_n^+\rangle} \otimes \mathbb{Q}_p/\mathbb{Z}_p & \text{if } \chi \neq \mathbf{1} \end{cases}$$

as $\mathbb{Z}_p$-modules.

PROOF. We prove the claim for $\left(\widehat{E}^{+}(\mathfrak{m}_\infty)^\chi\right)_{\Gamma_n}[p^\infty]$ in the case where $\chi = \mathbf{1}$. We can prove the rest of the claims similarly.

We have

$$\begin{aligned}\left(\widehat{E}^{+}(\mathfrak{m}_\infty)^\chi\right)_{\Gamma_n}[p^\infty] &= \left(\widehat{E}^{+}(\mathfrak{m}_\infty)^\chi/\omega_n\widehat{E}^{+}(\mathfrak{m}_\infty)^\chi\right)[p^\infty] \\ &\cong \varinjlim_{m\geq n}\left(\widehat{E}^{+}(\mathfrak{m}_m)^\chi/\omega_n\widehat{E}^{+}(\mathfrak{m}_m)^\chi\right)[p^\infty]\,, \end{aligned} \tag{3.5}$$

where transition maps

$$\left(\widehat{E}^{+}(\mathfrak{m}_m)^\chi/\omega_n\widehat{E}^{+}(\mathfrak{m}_m)^\chi\right)[p^\infty] \longrightarrow \left(\widehat{E}^{+}(\mathfrak{m}_{m+1})^\chi/\omega_n\widehat{E}^{+}(\mathfrak{m}_{m+1})^\chi\right)[p^\infty]$$

are multiplication-by-p maps when m is odd and identity maps when m is even. We will calculate $\left(\widehat{E}^{+}(\mathfrak{m}_m)^\chi/\omega_n\widehat{E}^{+}(\mathfrak{m}_m)^\chi\right)[p^\infty]$ for each n, m. Since $\widehat{E}^{+}(\mathfrak{m}_m) = \widehat{E}^{+}(\mathfrak{m}_{m-1})$ if m is odd, we may assume m is even.

We consider the case n is even. By Proposition 3.22 we have

$$\widehat{E}^{+}(\mathfrak{m}_m)^{\chi}/\omega_n\widehat{E}^{+}(\mathfrak{m}_m)^{\chi} \cong \frac{\mathbb{Z}_p[G_{-1}][X]\oplus\mathbb{Z}_p[G_{-1}]}{\left\langle(\widetilde{\omega}_m^{+},-(\varphi+\varphi^{-1})),(\omega_n,0)\right\rangle}. \tag{3.6}$$

We can show that

$$\frac{\mathbb{Z}_p[G_{-1}][X]\oplus\mathbb{Z}_p[G_{-1}]}{\left\langle(\widetilde{\omega}_m^{+},-(\varphi+\varphi^{-1})),(\omega_n,0)\right\rangle}[p^{\infty}] \cong \frac{\mathbb{Z}_p[G_{-1}][X]}{\langle p^{\frac{m-n}{2}},\widetilde{\omega}_n^{-}\rangle}\oplus\frac{\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1})}{\langle p^{\frac{m-n}{2}}\rangle}. \tag{3.7}$$

Indeed, since we have $\omega_m^{+}\equiv p^{\frac{m-n}{2}}\omega_n^{+} \pmod{\omega_n}$ and $\omega_n=\widetilde{\omega}_n^{-}\omega_n^{+}$, there is an exact sequence

$$\begin{aligned}0\longrightarrow\ &\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle p^{\frac{m-n}{2}},\widetilde{\omega}_n^{-}\rangle}\oplus\frac{\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1})}{\langle p^{\frac{m-n}{2}}\rangle}\\ \longrightarrow\ &\frac{\mathbb{Z}_p[G_{-1}][X]\oplus\mathbb{Z}_p[G_{-1}]}{\left\langle(\widetilde{\omega}_m^{+},-(\varphi+\varphi^{-1})),(\omega_n,0)\right\rangle}\\ \longrightarrow\ &\frac{\mathbb{Z}_p[G_{-1}][X]\oplus\mathbb{Z}_p[G_{-1}]}{\langle(\widetilde{\omega}_m^{+},-(\varphi+\varphi^{-1})),(\omega_n^{+},0),(\alpha\widetilde{\omega}_n^{+},0)\rangle}\longrightarrow 0,\end{aligned} \tag{3.8}$$

where the first map is $(x,y)\mapsto(x\omega_n^{+}+y\widetilde{\omega}_n^{+},0)$ and α is a generator of the $\mathbb{Z}_p[G_{-1}]$-module $\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1})$ (cf. Lemma 3.8). There is an another exact sequence

$$\begin{aligned}0\longrightarrow\ &\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle\omega_n^{+},\alpha\widetilde{\omega}_n^{+}\rangle}\\ \longrightarrow\ &\frac{\mathbb{Z}_p[G_{-1}][X]\oplus\mathbb{Z}_p[G_{-1}]}{\langle(\widetilde{\omega}_m^{+},-(\varphi+\varphi^{-1})),(\omega_n^{+},0),(\alpha\widetilde{\omega}_n^{+},0)\rangle}\\ \longrightarrow\ &\frac{\mathbb{Z}_p[G_{-1}]}{\langle\varphi+\varphi^{-1}\rangle}\longrightarrow 0,\end{aligned} \tag{3.9}$$

whose leftmost and rightmost modules are both $\mathbb{Z}_p$-free, where the first map is $x\mapsto(x,0)$ and the second map is $(x,y)\mapsto y$. Thus the rightmost module in (3.8), which is the same as the middle module in (3.9), is $\mathbb{Z}_p$-free. Our claim (3.7) follows from this.

By (3.5), (3.6), and (3.7), we get

$$\begin{aligned}\left(\widehat{E}^{+}(\mathfrak{m}_\infty)^{\chi}\right)_{\Gamma_n}[p^{\infty}] &\cong \varinjlim_{m\geq n}\left(\widehat{E}^{\pm}(\mathfrak{m}_m)^{\chi}/\omega_n\widehat{E}^{\pm}(\mathfrak{m}_m)^{\chi}\right)[p^{\infty}]\\ &\cong \varinjlim_{m\geq n}\frac{\mathbb{Z}_p[G_{-1}][X]\oplus\mathbb{Z}_p[G_{-1}]}{\left\langle(\widetilde{\omega}_m^{+},-(\varphi+\varphi^{-1})),(\omega_n,0)\right\rangle}[p^{\infty}]\\ &\cong \varinjlim_{m\geq n}\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle p^{\frac{m-n}{2}},\widetilde{\omega}_n^{-}\rangle}\oplus\frac{\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi+\varphi^{-1})}{\langle p^{\frac{m-n}{2}}\rangle}\end{aligned}$$

$$\cong \left(\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \widetilde{\omega}_n^- \rangle} \oplus \mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1}) \right) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

when n is even. By replacing $p^{\frac{m-n}{2}}$ with $p^{\frac{m-(n-1)}{2}}$ in the above discussion, we get the statement also in the case where n is odd.

From this description, we see that $\left((\widehat{E}^+(\mathfrak{m}_\infty)^\chi)_{\Gamma_n}[p^\infty]\right)^\vee$ is $\mathbb{Z}_p$-free and

$$\begin{aligned}
&\mathrm{rank}_{\mathbb{Z}_p} \left((\widehat{E}^+(\mathfrak{m}_\infty)^\chi)_{\Gamma_n}[p^\infty]\right)^\vee \\
&= \mathrm{rank}_{\mathbb{Z}_p} \left(\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \widetilde{\omega}_n^- \rangle} \right) + \mathrm{rank}_{\mathbb{Z}_p} \left(\mathrm{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1}) \right) \\
&= dq_n^- + \delta \,.
\end{aligned}$$

□

COROLLARY 3.25. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character. Then the Γ_n-coinvariants $((\widehat{E}^\pm(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee)_{\Gamma_n}$ are free $\mathbb{Z}_p$-modules for all n, and we have*

$$\begin{aligned}
\mathrm{rank}_{\mathbb{Z}_p} \left(\left(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \right)_{\Gamma_n} &= dp^n + \delta \,, \\
\mathrm{rank}_{\mathbb{Z}_p} \left(\left(\widehat{E}^-(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \right)_{\Gamma_n} &= dp^n \,.
\end{aligned}$$

PROOF. It follows from Corollary 3.20, Proposition 3.24 and the exact sequence (3.4). □

REMARK 3.26. From Corollary 3.25, we find that $(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is not a free Λ-module in the case when $\delta = 2$, i.e. the case when $d \equiv 0 \pmod 4$ and $\chi = \mathbf{1}$. Indeed, the $\mathbb{Z}_p$-rank of the Γ_n-coinvariant of a free Λ-module is divisible by p^n for each n. On the other hand, the $\mathbb{Z}_p$-rank of the Γ_n-coinvariant of $(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is not divisible by p^n as in the corollary.

PROPOSITION 3.27. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character. There exist injective homomorphisms of Λ-modules*

$$\begin{aligned}
\left(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee &\longrightarrow \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus \delta} \,, \\
\left(\widehat{E}^-(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee &\longrightarrow \Lambda^{\oplus d}
\end{aligned}$$

with finite cokernels.

PROOF. We prove the claim for $(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. We can prove the rest of the claims similarly.

We first note that $(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ has no nontrivial finite Λ-submodule since its Γ_n-coinvariants are free $\mathbb{Z}_p$-modules for all $n \geq 0$ (see Corollary 3.25). Thus by the structure

theorem for Λ-modules, there exist irreducible distinguished polynomials f_j, nonnegative integers r, s, t, m_i, n_j, and an injective homomorphism

$$f : \left(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \longrightarrow \Lambda^{\oplus r} \oplus \bigoplus_{i=1}^{s} \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^{t} \Lambda/f_j^{n_j} =: \mathcal{E}$$

with finite cokernel Z.

We show that

$$\begin{cases} r = d\,, \\ s = 0 \text{ (in other words } m_i = 0 \text{ for all } i)\,, \\ t = \begin{cases} 0 & \text{if } d \not\equiv 0 \pmod 4 \text{ or } \chi \neq \mathbf{1}\,, \\ 2 & \text{otherwise, and} \end{cases} \\ (f_1^{n_1}, \ldots, f_t^{n_t}) = (X, X) \text{ if } t = 2\,. \end{cases}$$

From the exact sequence $0 \to \left(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \xrightarrow{f} \mathcal{E} \to Z \to 0$, we get the Γ_n-invariant-coinvariant exact sequences

$$\begin{aligned} Z^{\Gamma_n} &\longrightarrow \left((\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\right)_{\Gamma_n} \\ &\longrightarrow \mathcal{E}/\omega_n\mathcal{E} \longrightarrow Z/\omega_n Z \longrightarrow 0 \end{aligned} \tag{3.10}$$

for all n. Note that, the first maps in (3.10) are 0-maps for all n, since $((\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee)_{\Gamma_n}$ is $\mathbb{Z}_p$-free. Then we see that $m_i = 0$ and $f_j^{n_j} | \omega_n$ (and $n_j \leq 1$) for all sufficiently large n since $Z/\omega_n Z$ is bounded as $n \to \infty$. Thus we get $s = 0$ here. We now have

$$\begin{aligned} dp^n + \delta &= \operatorname{rank}_{\mathbb{Z}_p} \left((\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\right)_{\Gamma_n} \\ &= \operatorname{rank}_{\mathbb{Z}_p}(\mathcal{E}/\omega_n\mathcal{E}) = rp^n + \sum_{j=1}^{t} n_j \deg f_j \end{aligned} \tag{3.11}$$

for all sufficiently large n. Thus we get $r = d$.

In the case when $d \not\equiv 0 \pmod 4$ or $\chi \neq \mathbf{1}$, we get $0 = \sum_{j=1}^{t} n_j \deg f_j$ from the above discussion. Thus we get $n_j = 0$ which is the desired result, i.e. $t = 0$.

We finally consider the case when $d \equiv 0 \pmod 4$ and $\chi = \mathbf{1}$. We may assume $n_j = 1$ for all j. In this case, we have

$$2 = \sum_{j=1}^{t} \deg f_j\,, \tag{3.12}$$

$$f_j | \omega_n (= (1+X)^{p^n} - 1) \tag{3.13}$$

for all sufficiently large n. We narrow down the possible combinations of $(t, (f_1, \dots, f_t))$ satisfying these two conditions (3.12) and (3.13). If $p \geq 5$, there is a unique combination $(t, (f_1, \dots, f_t)) = (2, (X, X))$, since $\deg f_j \leq 2$. If $p = 3$, since $\omega_1 = X(X^2 + 3X + 3)$, there are two possible combinations $(t, (f_1, \dots, f_t)) = (2, (X, X)), (1, (X^2 + 3X + 3))$. By showing that the last combination is impossible, we complete the proof. Indeed, we have $\operatorname{rank}_{\mathbb{Z}_p} \mathcal{E}/\omega_0\mathcal{E} = d + 1$ with the combination $(t, (f_1, \dots, f_t)) = (1, (X^2 + 3X + 3))$. On the other hand, from the exact sequence (3.10) for $n = 0$, we must have $\operatorname{rank}_{\mathbb{Z}_p} \mathcal{E}/\omega_0\mathcal{E} = d + 2$ and thus we get the desired conclusion. □

We now get the following proposition which is an important ingredient for the proof of Proposition 3.32.

PROPOSITION 3.28. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character. Then $(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ has no nontrivial finite Λ-submodule and its Λ-rank is d.*

PROOF. This follows from Proposition 3.27. □

In the rest of this subsection, we study the Λ-module

$$\left(\frac{H^1(k_\infty, E[p^\infty])}{\widehat{E}^{\pm}(\mathfrak{m}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p}\right)^\vee .$$

We consider the following exact sequence;

$$0 \to \left(\frac{H^1(k_\infty, E[p^\infty])}{\widehat{E}^{\pm}(\mathfrak{m}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p}\right)^\vee \longrightarrow \left(H^1(k_\infty, E[p^\infty])\right)^\vee \longrightarrow \left(\widehat{E}^{\pm}(\mathfrak{m}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \to 0 . \tag{3.14}$$

We studied the Λ-module structure of the rightmost module. We also know the Λ-module structure of the middle module by the following fact (Proposition 3.29).

PROPOSITION 3.29 (Greenberg [1] §3 Corollary 2). *Let K be a finite extension of $\mathbb{Q}_p$ and K_∞ a $\mathbb{Z}_p$-extension of K. Put $\Lambda_K = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$. If $E(K_\infty)[p^\infty] = 0$, then $H^1(K_\infty, E[p^\infty])^\vee$ is a free Λ_K-module and its Λ_K-rank is $2[K : \mathbb{Q}_p]$;*

$$H^1(K_\infty, E[p^\infty])^\vee \cong \Lambda_K^{\oplus 2[K:\mathbb{Q}_p]} .$$

We can apply Proposition 3.29 in our setting as $K = k_0$, $K_\infty = k_\infty$. Indeed we see that $E(k_\infty)[p^\infty] = 0$ by Proposition 3.1.

Here we recall the following useful lemma on equivalent conditions on freeness of Λ-modules and on triviality of finite Λ-submodules.

LEMMA 3.30. *Let M be a finitely generated Λ-module.*

(1) *M is a free Λ-module if and only if $M^\Gamma = 0$ and M_Γ is a free $\mathbb{Z}_p$-module.*

(2) *M has no nontrivial finite Λ-submodule if and only if M^Γ is a free $\mathbb{Z}_p$-module.*

PROOF. See for example [15] Proposition 5.3.19. □

Applying the following lemma to the exact sequence (3.14), we can now determine the Λ-module structure of $(H^1(k_\infty, E[p^\infty])/(\widehat{E}^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p))^\vee$.

LEMMA 3.31. *Let $f : M \to N$ be a surjective homomorphism of Λ-modules. Suppose that M is a free Λ-module of Λ-rank r, and that N is Λ-module of Λ-rank s which has no non-trivial finite Λ-submodule. Then its kernel* Ker f *is a free Λ-module of rank $r - s$.*

PROOF. We put $M_0 := \mathrm{Ker}\, f$. Then by taking the invariant-coinvariant exact sequence, we have

$$0 \longrightarrow M_0^\Gamma \longrightarrow M^\Gamma \longrightarrow N^\Gamma \longrightarrow M_{0,\Gamma} \longrightarrow M_\Gamma .$$

Since M is a free Λ-module, we have $M^\Gamma = 0$ and M_Γ is a free $\mathbb{Z}_p$-module by Lemma 3.30. Since N has no non-trivial finite Λ-submodule, we see that N^Γ is a free $\mathbb{Z}_p$-module by Lemma 3.30. Hence we have $M_0^\Gamma = 0$ and $M_{0,\Gamma}$ is a free $\mathbb{Z}_p$-module. Thus M_0 is a free Λ-module again by Lemma 3.30. It is easy to see that the Λ-rank of M_0 is $r - s$. □

PROPOSITION 3.32. *We have*

$$\left(\frac{H^1(k_\infty, E[p^\infty])}{E^{\pm}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee \cong \Lambda^{\oplus [k_0:\mathbb{Q}_p]} .$$

PROOF. It follows from Proposition 3.28, Proposition 3.29, and Lemma 3.31 for the exact sequence (3.14). □

3.4. More on the plus and the minus local conditions. The discussion in the previous subsections is enough to prove our main theorem. In this subsection, we proceed to determine the explicit structure of the Λ-module $\left(\widehat{E}^{\pm}(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee$. For that purpose, we show the following lemma.

LEMMA 3.33. *Let $f : M \to N$ be an injective homomorphism of Λ-modules with finite cokernel. Suppose that $M/\omega_n M$ is $\mathbb{Z}_p$-free and $N_{\Lambda\text{-}tors} = \{x \in N | \omega_n x = 0\}$ for all sufficiently large n. Then f induces an isomorphism*

$$\overline{f} : M/M_{\Lambda\text{-}tors} \xrightarrow{\simeq} N/N_{\Lambda\text{-}tors} .$$

PROOF. We regard M as a Λ-submodule of N by f. Since N/M is finite, we see that $\omega_n N \subset M$ for all sufficiently large n. We thus have

$$\begin{aligned} \mathrm{Coker}(\overline{f}) &= N/(M + N_{\Lambda\text{-tors}}) \\ &\xrightarrow[\simeq]{\times \omega_n} \omega_n N/\omega_n M \\ &\hookrightarrow M/\omega_n M \end{aligned}$$

for all sufficiently large n. Since $M/\omega_n M$ is $\mathbb{Z}_p$-free for all sufficiently large n and $\mathrm{Coker}(\overline{f})$ is finite, we get $M/M_{\Lambda\text{-tors}} \cong N/N_{\Lambda\text{-tors}}$. □

THEOREM 3.34. *Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character. We have*

$$\left(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \cong \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus \delta},$$
$$\left(\widehat{E}^-(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \cong \Lambda^{\oplus d}.$$

PROOF. We prove this theorem for $(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. We can prove the rest of the claim similarly.

Let $M = \left(\widehat{E}^+(\mathfrak{m}_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee$, $N = \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus \delta}$, and f be the map obtained in Proposition 3.27. Then the assumptions in Lemma 3.33 are satisfied (see Proposition 3.27, Corollary 3.25). Thus we have $M/M_{\Lambda\text{-tors}} \cong \Lambda^{\oplus d}$ by Lemma 3.33.

We now consider the following commutative diagram;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\Lambda\text{-tors}} & \longrightarrow & M & \longrightarrow & M/M_{\Lambda\text{-tors}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}}\cong & & \\
0 & \longrightarrow & (\Lambda/X)^{\oplus\delta} & \longrightarrow & \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus\delta} & \longrightarrow & \Lambda^{\oplus d} & \longrightarrow & 0.
\end{array}$$

We see that $M_{\Lambda\text{-tors}} \cong (\Lambda/X)^{\oplus\delta}$, since $\mathrm{Coker}(f_0)$ is finite. Therefore, the above horizontal exact sequence splits and thus we get

$$M \cong M/M_{\Lambda\text{-tors}} \oplus M_{\Lambda\text{-tors}} \cong \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus\delta}.$$

□

4. Finite Λ-submodules

We use the notations and the assumptions introduced in Section 2.

Let $\Gamma = \mathrm{Gal}(F_\infty/F_0)$ and $\Lambda = \mathbb{Z}_p[[\Gamma]]$. We fix a topological generator $\gamma \in \Gamma$. Then we identify the completed group ring $\mathbb{Z}_p[[\Gamma]]$ with the ring of power series $\mathbb{Z}_p[[X]]$ by identifying γ with $1 + X$.

4.1. Finite Λ-submodules of $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$. In this subsection, we study finite Λ-submodules of the Pontryagin dual of the p-primary Selmer group. The aim of this subsection is to prove Theorem 4.5.

The following proposition is due to Matsuno [14, Theorem 2.4] (see also Hachimori–Matsuno [3]).

PROPOSITION 4.1. *Let K be a finite extension of $\mathbb{Q}$, K_∞/K a $\mathbb{Z}_p$-extension, K_n its n-th layer, and E an elliptic curve defined over K. Put $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, and $\Lambda_K = \mathbb{Z}_p[[\Gamma_K]]$. Let X_n be the kernel of the restriction map*

$$\mathrm{Sel}(K_n, E[p^\infty]) \longrightarrow \mathrm{Sel}(K_\infty, E[p^\infty])$$

and $X_\infty := \varprojlim X_n$ *where the projective limit is taken with respect to the corestriction maps.*

Assume that the $\mathbb{Z}_p$*-rank of* $\mathrm{Sel}(K_n, E[p^\infty])^\vee$ *is bounded as* $n \to \infty$*. Then the maximal finite* Λ_K*-submodule of* $\mathrm{Sel}(K_\infty, E[p^\infty])^\vee$ *is isomorphic to* X_∞*.*

In particular if we further assume that $E(K)[p] = 0$*, then* $\mathrm{Sel}(K_\infty, E[p^\infty])^\vee$ *has no nontrivial finite* Λ_K*-submodule.*

We check that we can apply the above proposition in our setting $K = F_0$, $K_\infty = F_\infty$.

LEMMA 4.2. *The morphisms* $\mathrm{Sel}^\pm(F_n, E[p^\infty]) \to \mathrm{Sel}^\pm(F_\infty, E[p^\infty])$ *are injective for all* $n \geq 0$*.*

PROOF. We can prove this by the same method as the proof of Lemma 9.1 in [11]. □

We assume from here that both $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ are Λ-torsion. We denote the Iwasawa λ-invariant of $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ by $\lambda^\pm$.

Let

$$\mathrm{Sel}^1(F_n, E[p^\infty]) := \mathrm{Ker}\left(\mathrm{Sel}(F_n, E[p^\infty]) \longrightarrow \prod_{v \in S^{\mathrm{ss}}_{p,F}} \frac{H^1(F_{n,v}, E[p^\infty])}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p}\right),$$

where $S^{\mathrm{ss}}_{p,F}$ is the set of all primes of F lying above p where E has supersingular reduction.

By the exact sequence (3.2), we have an exact sequence

$$0 \longrightarrow \frac{H^1(F_{n,v}, E[p^\infty])}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \frac{H^1(F_{n,v}, E[p^\infty])}{E^+(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \oplus \frac{H^1(F_{n,v}, E[p^\infty])}{E^-(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}$$

$$\longrightarrow \frac{H^1(F_{n,v}, E[p^\infty])}{E(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow 0$$

for each n and for each prime v of F lying above p. Thus, for each n, we get the following exact sequence

$$0 \longrightarrow \mathrm{Sel}^1(F_n, E[p^\infty]) \xrightarrow{\iota} \mathrm{Sel}^+(F_n, E[p^\infty]) \oplus \mathrm{Sel}^-(F_n, E[p^\infty])$$

$$\xrightarrow{\eta} \mathrm{Sel}(F_n, E[p^\infty]) \tag{4.1}$$

where ι is the diagonal embedding by inclusions and η is $(x, y) \mapsto x - y$.

PROPOSITION 4.3. *The cokernel of* η *in the exact sequence* (4.1) *is finite.*

PROOF. We can prove this by the same method as the proof of Lemma 10.1 in [11]. □

PROPOSITION 4.4. *The* $\mathbb{Z}_p$*-rank of* $\mathrm{Sel}(F_n, E[p^\infty])^\vee$ *is bounded as* $n \to \infty$*. More precisely, we have*

$$\mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}(F, E[p^\infty])^\vee + \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}(F_n, E[p^\infty])^\vee \leq \lambda^+ + \lambda^-$$

for every n*.*

PROOF. Since the restriction map $\mathrm{Sel}(F, E[p^\infty]) \to \mathrm{Sel}^1(F_n, E[p^\infty])$ is injective, we have

$$\mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}(F, E[p^\infty])^\vee \leq \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}^1(F_n, E[p^\infty])^\vee .$$

Hence by Lemma 4.2 and Proposition 4.3, we get

$$\begin{aligned} &\mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}(F, E[p^\infty])^\vee + \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}(F_n, E[p^\infty])^\vee \\ &\leq \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}^+(F_n, E[p^\infty])^\vee + \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Sel}^-(F_n, E[p^\infty])^\vee \\ &\leq \lambda^+ + \lambda^- \end{aligned}$$

for every n from (4.1). The boundedness of the $\mathbb{Z}_p$-ranks follows from this immediately. □

From the above argument, we can prove the following theorem.

THEOREM 4.5. *Assume that both* $\mathrm{Sel}^+(F_\infty, E[p^\infty])^\vee$ *and* $\mathrm{Sel}^-(F_\infty, E[p^\infty])^\vee$ *are* Λ*-torsion. Then* $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ *has no nontrivial finite* Λ*-submodule.*

PROOF. The $\mathbb{Z}_p$-rank of $\mathrm{Sel}(F_n, E[p^\infty])^\vee$ is bounded as $n \to \infty$ (cf. Proposition 4.4). Further, we have $E(F_0)[p] = 0$ by Proposition 3.1. Thus we can apply Proposition 4.1 and get the desired result. □

4.2. Finite Λ-submodules of $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$. Finally, we study finite Λ-submodules of the Pontryagin duals of the plus and the minus Selmer groups. The aim of this subsection is to prove our main theorem (Theorem 4.8).

We prove that the triviality of finite Λ-submodules of $\mathrm{Sel}(F_\infty, E[p^\infty])^\vee$ is inherited to that of $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$.

Let us consider the following exact sequence of Λ-modules coming from the definition of the Selmer groups;

$$\begin{aligned} \bigoplus_{v \in S^{\mathrm{ss}}_{p,F}} \left(\frac{H^1(F_{\infty,v}, E[p^\infty])}{E^\pm(F_{\infty,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee &\xrightarrow{\iota^\pm} \mathrm{Sel}(F_\infty, E[p^\infty])^\vee \\ &\longrightarrow \mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee \longrightarrow 0 . \end{aligned} \tag{4.2}$$

PROPOSITION 4.6. *Assume that* $\mathrm{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ *is* Λ*-torsion. Then the map* $\iota^\pm$ *in* (4.2) *is injective.*

PROOF. We have $\mathrm{rank}_\Lambda(\mathrm{Sel}(F_\infty, E[p^\infty])^\vee) \geq \sum_{v \in S^{\mathrm{ss}}_{p,F}} [F_{0,v} : \mathbb{Q}_p]$ (cf. [2] Theorem 1.7). By Proposition 3.32, we have

$$\left(\frac{H^1(F_{\infty,v}, E[p^\infty])}{E^\pm(F_{\infty,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee \cong \Lambda^{\oplus [F_{0,v}:\mathbb{Q}_p]}$$

for each prime $v \in S^{\mathrm{ss}}_{p,F}$. From this and the exact sequence (4.2), we see that

$$\sum_{v \in S^{\mathrm{ss}}_{p,F}} [F_{0,v} : \mathbb{Q}_p] = \mathrm{rank}_\Lambda \left(\bigoplus_{v \in S^{\mathrm{ss}}_{p,F}} \left(\frac{H^1(F_{\infty,v}, E[p^\infty])}{E^{\pm}(F_{\infty,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee \right)$$

$$\geq \mathrm{rank}_\Lambda \left(\mathrm{Sel}(F_\infty, E[p^\infty])^\vee \right) .$$

Thus we get

$$\mathrm{rank}_\Lambda \left(\bigoplus_{v \in S^{\mathrm{ss}}_{p,F}} \left(\frac{H^1(F_{\infty,v}, E[p^\infty])}{E^{\pm}(F_{\infty,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee \right) = \mathrm{rank}_\Lambda \left(\mathrm{Sel}(F_\infty, E[p^\infty])^\vee \right) .$$

From this, we see that the kernel $\mathrm{Ker}\, \iota^{\pm}$ is Λ-torsion. Therefore we get the conclusion since the leftmost direct sum in the exact sequence (4.2) is a torsion-free Λ-module. □

The following proposition is a key tool for the proof of our main theorem.

PROPOSITION 4.7 (Greenberg [2] p.104–105). *Let $f : M \to N$ be an injective homomorphism of Λ-modules. Suppose that N is a finitely generated Λ-module which has no nontrivial finite Λ-submodule, and that M is a free Λ-module. Then the cokernel* $\mathrm{Coker}(f)$ *has no nontrivial finite Λ-submodule.*

THEOREM 4.8. *Assume that both* $\mathrm{Sel}^+(F_\infty, E[p^\infty])^\vee$ *and* $\mathrm{Sel}^-(F_\infty, E[p^\infty])^\vee$ *are Λ-torsion. Then both* $\mathrm{Sel}^+(F_\infty, E[p^\infty])^\vee$ *and* $\mathrm{Sel}^-(F_\infty, E[p^\infty])^\vee$ *have no nontrivial finite Λ-submodule.*

PROOF. It follows from Theorem 4.5, Proposition 4.6 and Proposition 4.7 for $f = \iota^{\pm}$. □

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