TOKYO J. MATH.
VOL. 41, NO. 2, 2018
DOI: 10.3836/tjm/1502179260

Maximal Operator and its Commutators on Generalized Weighted Orlicz-Morrey Spaces

Fatih DERINGOZ, Vagif S. GULIYEV and Sabir G. HASANOV

Ahi Evran University, Institute of Mathematics and Mechanics of NAS of Azerbaijan and Ganja State University

(Communicated by Y. Sawano)

Abstract. In the present paper, we shall give necessary and sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator and its commutators on generalized weighted Orlicz-Morrey spaces $M_w^{\Phi,\varphi}(\mathbb{R}^n)$. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators and we do not need Δ_2-condition for the boundedness of the maximal operator. We also consider the vector-valued boundedness of the Hardy-Littlewood maximal operator.

1. Introduction

The classical Morrey spaces were introduced by Morrey [32] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Mizuhara [31] and Nakai [33] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [14]); Komori and Shirai [27] defined weighted Morrey spaces $L^{p,\kappa}(w)$; Guliyev [15] gave a concept of the generalized weighted Morrey spaces $M_w^{p,\varphi}(\mathbb{R}^n)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(w)$.

The spaces $M_w^{p,\varphi}(\mathbb{R}^n)$ defined by the norm

$$\|f\|_{M_w^{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L_w^p(B(x,r))},$$

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w is a non-negative measurable function on $\mathbb{R}^n$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$.

The Orlicz space was first introduced by Orlicz in [35, 36] as generalizations of Lebesgue spaces $L^p(\mathbb{R}^n)$. Since then, the theory of Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

Received July 19, 2016; revised August 15, 2016
Mathematics Subject Classification: 42B20, 42B25, 42B35, 46E30
Key words: generalized weighted Orlicz-Morrey space, maximal operator, commutator, vector-valued inequalities

In [7], the generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [34] and [42]. In words of [19], our generalized Orlicz-Morrey space is the third kind and the ones in [34] and [42] are the first kind and the second kind, respectively. According to the examples in [10], one can say that the generalized Orlicz-Morrey spaces of the first kind and the second kind are different and that second kind and third kind are different. However, we do not know the relation between the first and the third kind.

As based on the results of [2], the following conditions were introduced in [7] (see, also [17]) for the boundedness of the maximal operators and the singular integral operators on $M^{\Phi,\varphi}(\mathbb{R}^n)$, respectively,

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi_2(x,r)\,, \tag{1.1}$$

$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \right) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\,\varphi_2(x,r)\,, \tag{1.2}$$

where C does not depend on x and r. It was also shown in [7], the condition (1.1) is weaker than (1.2).

Various versions of generalized weighted Orlicz-Morrey spaces were introduced in [29], [24], [37] and [18]. The spaces in [29] and [24] can be seen as the weighted version of generalized Orlicz-Morrey spaces of the first kind and the spaces in [37] can be seen as the weighted version of generalized Orlicz-Morrey spaces of the second kind. We used the definition of [18] which can be seen as the weighted version of generalized Orlicz-Morrey spaces of the third kind.

In this paper, we shall investigate the boundedness of the maximal operator M and its commutators M^b on generalized weighted Orlicz-Morrey spaces. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators and we do not need to Δ_2-condition for the boundedness of the maximal operator. We also consider the vector-valued boundedness of the Hardy-Littlewood maximal operator.

The following results are the fundamental theorems in this paper:

THEOREM 1.1. *Φ be a Young function and φ_1, φ_2 positive measurable functions on $\mathbb{R}^n \times (0,\infty)$.*

1. *If $\Phi \in \nabla_2$ and $w \in A_{i_\Phi}$, then the condition*

$$\sup_{r<t<\infty} \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(w(B(x,s))^{-1}\big)} \right) \Phi^{-1}\big(w(B(x,t))^{-1}\big) \le C\,\varphi_2(x,r)\,, \tag{1.3}$$

where C does not depend on x and r, is sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

2. *If* $\varphi_1 \in \mathcal{G}_w^\Phi$, *then the condition*

$$\varphi_1(x,r) \le C\varphi_2(x,r)\,, \tag{1.4}$$

where C *does not depend on* x *and* r, *is necessary for the boundedness of* M *from* $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$ *and* $w \in A_{i_\Phi}$. *If* $\varphi_1 \in \mathcal{G}_w^\Phi$, *then the condition* (1.4) *is necessary and sufficient for the boundedness of* M *from* $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

THEOREM 1.2. *Let* $b \in BMO(\mathbb{R}^n)$, Φ *be a Young function and* φ_1, φ_2 *positive measurable functions on* $\mathbb{R}^n \times (0,\infty)$.

1. *Let* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $w \in A_1$, *then the condition*

$$\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(w(B(x,t))^{-1}\big)\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(w(B(x,s))^{-1}\big)} \le C\,\varphi_2(x,r)\,,$$

where C *does not depend on* x *and* r, *is sufficient for the boundedness of* M^b *from* $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

2. *If* $\Phi \in \Delta_2$, $\varphi_1 \in \mathcal{G}_w^\Phi$ *and* $w \in A_1$, *then the condition* (1.4) *is necessary for the boundedness of* M^b *from* $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $w \in A_1$. *If* $\varphi_1 \in \mathcal{G}_w^\Phi$ *satisfies the condition*

$$\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\varphi_1(x,t) \le C\varphi_1(x,r)\,,$$

where C *does not depend on* x *and* r, *then the condition* (1.4) *is necessary and sufficient for the boundedness of* M^b *from* $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definitions and Preliminary Results

We recall the definition of Young functions.

DEFINITION 2.1. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function, if Φ is convex, left-continuous, $\lim\limits_{r\to 0^+}\Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty}\Phi(r) = \infty$.

The convexity and the condition $\Phi(0) = 0$ force any Young function to be increasing. In particular, if there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then it follows that $\Phi(r) = \infty$ for $r \ge s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty\,.$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

We recall an important pair of indices used for Young functions. For any Young function Φ, write

$$h_\Phi(t) = \sup_{s>0} \frac{\Phi(st)}{\Phi(s)}, \quad t > 0\,.$$

The lower and upper dilation indices of Φ are defined by

$$i_\Phi = \lim_{t\to 0^+} \frac{\log h_\Phi(t)}{\log t} \text{ and } I_\Phi = \lim_{t\to\infty} \frac{\log h_\Phi(t)}{\log t}\,,$$

respectively.

Even though the A_p class is well known, for completeness, we offer the definition of A_p weight functions.

Definition 2.2. For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p'}{p}} < \infty\,.$$

A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|}\int_B w(y)dy \le Cw(x), \qquad a.e.\ x \in B$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\ge 1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.

It is well known that if $w \in A_p,\ 1 \le p < \infty$, then there exist a constant C such that

$$w(Q)\left(\frac{|S|}{|Q|}\right)^p \le Cw(S) \tag{2.1}$$

for measurable sets $S \subset Q$. See, for example [8].

Definition 2.3. For a Young function Φ and $w \in A_\infty$, the set

$$L^\Phi_w(\mathbb{R}^n) \equiv \left\{ f\text{-measurable} : \int_{\mathbb{R}^n} \Phi(k|f(x)|)w(x)dx < \infty \text{ for some } k > 0 \right\}$$

is called the weighted Orlicz space. The local weighted Orlicz space $L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^\Phi_w(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that $L^\Phi_w(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi_w(\mathbb{R}^n)} \equiv \|f\|_{L^\Phi_w} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)w(x)dx \le 1\right\}$$

and

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi_w}}\Big) w(x)dx \le 1\,. \tag{2.2}$$

For a Young function Φ [38] and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) \equiv \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty)\,.$$

We also note that [38, Proposition 13]

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r))\,, \qquad 0 \le r < \infty\,.$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r)\,, \qquad r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr)\,, \qquad r \ge 0$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition and it fails the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but it fails the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) \equiv \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & \text{if} \quad r \in [0,\infty)\,, \\ \infty & \text{if} \quad r = \infty\,. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and it satisfies $\widetilde{\widetilde{\Phi}} = \Phi$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$.

It is also known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r\,, \qquad r \ge 0\,. \tag{2.3}$$

The following analogue of the Hölder inequality is known.

$$\left|\int_{\mathbb{R}^n} f(x)g(x)w(x)dx\right| \le 2\|f\|_{L^\Phi_w}\|g\|_{L^{\widetilde{\Phi}}_w}\,. \tag{2.4}$$

For the proof of (2.3) and (2.4), see, for example [38].

We can easily prove the following by a direct calculation:

$$\|\chi_B\|_{L^\Phi_w} = \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)}\,, \qquad B \in \mathcal{B}\,, \tag{2.5}$$

where χ_B denotes the characteristic function of the B.

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy\,, \qquad x \in \mathbb{R}^n$$

for a locally integrable function f on $\mathbb{R}^n$.

THEOREM 2.4 ([25, Theorem 1]). *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$. Assume in addition $w \in A_{i_\Phi}$. Then, there is a constant $C \geq 1$ such that*

$$\int_{\mathbb{R}^n} \Phi\left(Mf(x)\right) w(x)dx \leq C \int_{\mathbb{R}^n} \Phi\left(|f(x)|\right) w(x)dx \tag{2.6}$$

for any locally integrable function f.

With [7, Remark 2.5] and [12, Remark 6.1.3] taken into account, the better boundedness result which was proved in [13] runs as follows.

THEOREM 2.5 ([13]). *Let Φ be a Young function with $\Phi \in \nabla_2$. Assume in addition $w \in A_{i_\Phi}$. Then the modular inequality* (2.6) *holds.*

REMARK 2.6. Note that the strong modular inequality (2.6) implies the corresponding norm inequality. Indeed, let (2.6) hold. Then, using the sublinearity of M, convexity of Φ and (2.2) we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{C\|f\|_{L^\Phi_w}}\right) w(x)dx &= \int_{\mathbb{R}^n} \Phi\left(M\Big(\frac{f}{C\|f\|_{L^\Phi_w}}\Big)(x)\right) w(x)dx \\ &\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{C\|f\|_{L^\Phi_w}}\right) w(x)dx \leq 1\,,\end{aligned}$$

where C is the constant in (2.6). This implies $\|Mf\|_{L^\Phi_w} \lesssim \|f\|_{L^\Phi_w}$.

LEMMA 2.7. *Let Φ be a Young function with $\Phi \in \nabla_2$. Let $f \in L^\Phi_{w,\mathrm{loc}(\mathbb{R}^n)}$. Assume in addition $w \in A_{i_\Phi}$. For a ball B, the following inequality is valid*:

$$\|f\|_{L^1(B)} \lesssim |B|\Phi^{-1}\left(w(B)^{-1}\right) \|f\|_{L^\Phi_w(B)}\,,$$

where $\|f\|_{L^\Phi_w(B)} := \|f\chi_B\|_{L^\Phi_w}$.

PROOF. Let

$$\mathfrak{M}f(x) = \sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{|B|} \int_B |f(y)|dy\,, \quad x \in \mathbb{R}^n$$

and $\tilde{f}$ denotes the extension of f from B to $\mathbb{R}^n$ by zero. It is well known that $\mathfrak{M}f(x) \leq 2^n Mf(x)$ for all $x \in \mathbb{R}^n$. Then using Theorem 2.5, we have

$$\frac{\|f\|_{L^1(B)}}{|B|}\|\chi_B\|_{L^\Phi_w(B)} = \frac{\|\tilde{f}\|_{L^1(B)}}{|B|}\|\chi_B\|_{L^\Phi_w(B)} \lesssim \|\mathfrak{M}\tilde{f}\|_{L^\Phi_w(B)}$$

$$\lesssim \|M\tilde{f}\|_{L^\Phi_w(B)} \le \|M\tilde{f}\|_{L^\Phi_w} \lesssim \|\tilde{f}\|_{L^\Phi_w} = \|f\|_{L^\Phi_w(B)}.$$

So, Lemma 2.7 is proved. □

Let v be a weight. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)|g(t)|$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ and $\overline{S}_u^*$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(r) := \|u(t)g(t)\|_{L_\infty(r,\infty)}, \quad r \in (0,\infty).$$

$$(\overline{S}_u^* g)(r) := \left\|\left(1+\ln\frac{t}{r}\right) u(t)g(t)\right\|_{L_\infty(r,\infty)}, \quad r \in (0,\infty).$$

THEOREM 2.8 ([5, Lemma 5.2]). *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\|v_2\overline{S}_u\left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\right)\right\|_{L_\infty(0,\infty)} < \infty.$$

The following theorem can be proved analogously to Theorem 2.8.

THEOREM 2.9. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u^*$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\|v_2\overline{S}_u^*\left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\right)\right\|_{L_\infty(0,\infty)} < \infty.$$

3. Generalized weighted Orlicz-Morrey spaces

In this section, we give the definition of the generalized weighted Orlicz-Morrey spaces $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ and investigate the fundamental structure of $M^{\Phi,\varphi}_w(\mathbb{R}^n)$. In the sequel we use the notation $\varphi(B(x,r)) \equiv \varphi(x,r)$.

DEFINITION 3.1. Let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, let w be a non-negative measurable function on $\mathbb{R}^n$ and Φ any Young function. Denote by $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized weighted Orlicz-Morrey space, the space of all functions $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_w^{\Phi,\varphi}(\mathbb{R}^n)} \equiv \|f\|_{M_w^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\big(w(B(x,r))^{-1}\big)\, \|f\|_{L_w^\Phi(B(x,r))}$$
$$\equiv \sup_{B\in\mathcal{B}} \varphi(B)^{-1}\, \Phi^{-1}\big(w(B)^{-1}\big)\, \|f\|_{L_w^\Phi(B)} < \infty .$$

Notice that there is another family of generalized weighted Orlicz-Morrey spaces studied in [24]. The generalized weighted Orlicz-Morrey spaces given in this paper can be viewed as the weighted Morrey spaces generated by the norm while the one used in [24] is defined via the modular.

EXAMPLE. Let $1 \le p < \infty$ and $0 < \kappa < 1$.

- If $\Phi(r) = r^p$ and $\varphi(x,r) = w(B(x,r))^{-1/p}$, then $M_w^{\Phi,\varphi}(\mathbb{R}^n) = L_w^p(\mathbb{R}^n)$.
- If $\Phi(r) = r^p$ and $\varphi(x,r) = w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_w^{\Phi,\varphi}(\mathbb{R}^n) = L^{p,\kappa}(w)$.
- If $\Phi(r) = r^p$, then $M_w^{\Phi,\varphi}(\mathbb{R}^n) = M_w^{p,\varphi}(\mathbb{R}^n)$.
- If $\varphi(x,r) = \Phi^{-1}\big(w(B(x,r))^{-1}\big)$, then $M_w^{\Phi,\varphi}(\mathbb{R}^n) = L_w^\Phi(\mathbb{R}^n)$.

For a Young function Φ and a non-negative measurable function w, we denote by $\mathcal{G}_\Phi^w$ the set of all almost decreasing functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that

$$\inf_{B\in\mathcal{B};\, r_B \le r_{B_0}} \varphi(B) \gtrsim \varphi(B_0) \quad \text{for all } B_0 \in \mathcal{B}$$

and

$$\inf_{B\in\mathcal{B};\, r_B \ge r_{B_0}} \frac{\varphi(B)}{\Phi^{-1}\big(w(B)^{-1}\big)} \gtrsim \frac{\varphi(B_0)}{\Phi^{-1}\big(w(B_0)^{-1}\big)} \quad \text{for all } B_0 \in \mathcal{B},$$

where r_B and r_{B_0} denote the radius of the balls B and B_0, respectively.

LEMMA 3.2. *Let $B_0 := B(x_0,r_0)$. If $\varphi \in \mathcal{G}_\Phi^w$, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(x_0,r_0)} \le \|\chi_{B_0}\|_{M_w^{\Phi,\varphi}} \le \frac{C}{\varphi(x_0,r_0)} .$$

PROOF. Let $B = B(x,r)$ denote an arbitrary ball in $\mathbb{R}^n$. By the definition and (2.5), it is easy to see that

$$\|\chi_{B_0}\|_{M_w^{\Phi,\varphi}} = \sup_{B\in\mathcal{B}} \varphi(B)^{-1}\Phi^{-1}(w(B)^{-1}) \frac{1}{\Phi^{-1}(w(B\cap B_0)^{-1})}$$
$$\ge \varphi(B_0)^{-1}\Phi^{-1}(w(B_0)^{-1}) \frac{1}{\Phi^{-1}(w(B_0\cap B_0)^{-1})} = \frac{1}{\varphi(B_0)} .$$

Now if $r \le r_0$, then $\varphi(B_0) \le C\varphi(B)$ and

$$\varphi(B)^{-1}\Phi^{-1}(w(B)^{-1})\|\chi_{B_0}\|_{L^{\Phi}_w(B)} \le \frac{1}{\varphi(B)} \le \frac{C}{\varphi(B_0)}.$$

On the other hand if $r \ge r_0$, then $\frac{\varphi(B_0)}{\Phi^{-1}(w(B_0)^{-1})} \le C\frac{\varphi(B)}{\Phi^{-1}(w(B)^{-1})}$ and

$$\varphi(B)^{-1}\Phi^{-1}(w(B)^{-1})\|\chi_{B_0}\|_{L^{\Phi}_w(B)} \le \frac{C}{\varphi(B_0)}.$$

This completes the proof. □

4. Maximal Operator in the spaces $M^{\Phi,\varphi}_w(\mathbb{R}^n)$

In this section necessary and sufficient conditions for the boundedness of the operator M in generalized weighted Orlicz-Morrey spaces will be obtained.

LEMMA 4.1. *Let Φ be a Young function with $\Phi \in \nabla_2$, $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$, $B = B(x,r)$ and $2B = B(x,2r)$. Assume in addition $w \in A_{i_\Phi}$. Then*

$$\|Mf\|_{L^{\Phi}_w(B)} \lesssim \|f\|_{L^{\Phi}_w(2B)} + \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} t^{-n}\|f\|_{L_1(B(x,t))}.$$

PROOF. We put $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{2B}}$ and have

$$\|Mf\|_{L^{\Phi}_w(B)} \le \|Mf_1\|_{L^{\Phi}_w(B)} + \|Mf_2\|_{L^{\Phi}_w(B)}.$$

By the boundedness of the operator M on $L^{\Phi}_w(\mathbb{R}^n)$ by Theorem 2.5 we have

$$\|Mf_1\|_{L^{\Phi}_w(B)} \lesssim \|f\|_{L^{\Phi}_w(2B)}.$$

Moreover we know that estimate

$$Mf_2(x) \le 2^n \sup_{t>2r} \frac{1}{|B(x,t)|}\int_{B(x,t)} |f(z)|dz, \text{ for all } x \in B \tag{4.1}$$

holds, see [7].

Thus

$$\|Mf\|_{L^{\Phi}_w(B)} \lesssim \|f\|_{L^{\Phi}_w(2B)} + \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)}\left(\sup_{t>2r}\frac{1}{|B(x,t)|}\int_{B(x,t)}|f(z)|dz\right).$$

□

LEMMA 4.2. *Let Φ be a Young function with $\Phi \in \nabla_2$, $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ and $B = B(x,r)$. Assume in addition $w \in A_{i_\Phi}$. Then*

$$\|Mf\|_{L^{\Phi}_w(B)} \lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x,t))^{-1}\big)\|f\|_{L^{\Phi}_w(B(x,t))}. \tag{4.2}$$

PROOF. Denote

$$M_1 := \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)| dz \right),$$

$$M_2 := \|f\|_{L^{\Phi}_w(2B)} .$$

By Lemma 2.7, we get

$$M_1 \lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \|f\|_{L^{\Phi}_w(B(x,t))} .$$

Meanwhile, since

$$\frac{1}{\Phi^{-1}\big(w(2B)^{-1}\big)} = \|\chi_{2B}\|_{L^{\Phi}_w} \le C \|M\chi_B\|_{L^{\Phi}_w} \le \|\chi_B\|_{L^{\Phi}_w} = \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)}$$

from the well-known pointwise estimate $\chi_{2B}(z) \le 2^n M\chi_B(z)$, for all $z \in \mathbb{R}^n$ and Theorem 2.5, we have

$$\frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \|f\|_{L^{\Phi}_w(B(x,t))}$$

$$\gtrsim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \|f\|_{L^{\Phi}_w(2B)} \gtrsim M_2 .$$

Since $\|Mf\|_{L^{\Phi}_w(B)} \le M_1 + M_2$ by Lemma 4.1, we arrive at (4.2). □

PROOF OF THEOREM 1.1. The first part of the theorem follows from Lemma 4.2 and Theorem 2.8. We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. It is easy to see that $M\chi_{B_0}(x) = 1$ for every $x \in B_0$. Therefore, by (2.5) and Lemma 3.2

$$1 = \Phi^{-1}(w(B_0)^{-1}) \|M\chi_{B_0}\|_{L^{\Phi}_w(B_0)} \le \varphi_2(B_0) \|M\chi_{B_0}\|_{M^{\Phi,\varphi_2}_w}$$

$$\le C\varphi_2(B_0) \|\chi_{B_0}\|_{M^{\Phi,\varphi_1}_w} \le C \frac{\varphi_2(B_0)}{\varphi_1(B_0)} .$$

Since this is true for every $B_0 > 0$, we are done.

The third statement of the theorem follows from the other statements of the theorem. □

REMARK 4.3. Note that the result of Theorem 1.1 is stronger than the Euclidean version of a result for the maximal operator in generalized Morrey spaces (the case $\Phi(r) = r^p$) obtained in [41] over the quasi-metric measure space.

5. Commutators

DEFINITION 5.1. Given a measurable function b the maximal commutator is defined by

$$M^b f(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |b(x) - b(y)| |f(y)| dy , \qquad x \in \mathbb{R}^n$$

for a locally integrable function f on $\mathbb{R}^n$.

It is well known that the operator M^b plays an important role in the study of commutators of singular integral operators with BMO symbols, (see, for instance, [11, 28, 43, 44]). Garcia-Cuerva et al. [11] proved that M^b is bounded in $L^p(\mathbb{R}^n)$ for any $p \in (1, \infty)$ if and only if $b \in BMO(\mathbb{R}^n)$, and Alphonse [3] proved that M^b enjoys the weak-type $L(\log L)$ estimate. The maximal operator M^b was studied intensively and there exist plenty of results about it.

In this section necessary and sufficient conditions for the boundedness of M^b in generalized weighted Orlicz-Morrey spaces have been obtained. For proving our main results, we need the following estimate.

LEMMA 5.2. *If $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $B_0 = B(x_0, r_0)$, then $|b(x) - b_{B_0}| \le C M^b \chi_{B_0}(x)$ for every $x \in B_0$.*

PROOF. It is well known that

$$\mathfrak{M}^b f(x) \le 2^n M^b f(x)\,, \tag{5.1}$$

where $\mathfrak{M}^b f(x) = \sup\limits_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)|dy$.

Now let $x \in B_0$. By using (5.1), we get

$$\begin{aligned} M^b \chi_{B_0}(x) &\ge C\mathfrak{M}^b f(x) = C \sup_{B \ni x} |B|^{-1} \int_{B \cap B_0} |b(x) - b(y)|dy \\ &\ge \Big|C|B_0|^{-1} \int_{B_0} (b(x) - b(y))dy\Big| = C|b(x) - b_{B_0}|\,. \end{aligned}$$

□

THEOREM 5.3 ([1, Theorem 1.13]). *Let $b \in BMO(\mathbb{R}^n)$. Suppose that X is a Banach space of measurable functions defined on $\mathbb{R}^n$. Moreover, assume that X satisfies the lattice property, that is*

$$0 \le g \le f \quad \Rightarrow \quad \|g\|_X \lesssim \|f\|_X\,.$$

Assume that M is bounded on X. Then the operator M^b is bounded on X, and the inequality

$$\|M^b f\|_X \le C\|b\|_* \|f\|_X$$

holds with constant C independent of f.

Combining Theorems 2.5 and 5.3, we obtain the following statement.

COROLLARY 5.4. *Let Φ be a Young function with $\Phi \in \nabla_2$ and $b \in BMO(\mathbb{R}^n)$. Assume in addition $w \in A_{i_\Phi}$, then M^b is bounded on $L^\Phi_w(\mathbb{R}^n)$.*

LEMMA 5.5 ([23]). *Let $w \in A_1$, $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then,*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(w(B(x,r))^{-1}\big) \left\|b - b_{B(x,r)}\right\|_{L^\Phi_w(B(x,r))}.$$

Note that for $b \in BMO(\mathbb{R}^n)$

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{5.2}$$

where C does not depend on b, x, r and t.

LEMMA 5.6. *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function with $\Delta_2 \cap \nabla_2$ and $w \in A_1$, then the inequality*

$$\|M^b f\|_{L^\Phi_w(B(x_0,r))} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \sup_{t>r}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\|f\|_{L^\Phi_w(B(x_0,t))}$$

holds for any ball $B(x_0,r)$ and for all $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$.

PROOF. For $B = B(x_0,r)$, write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$, so that

$$\left\|M^b f\right\|_{L^\Phi_w(B)} \le \left\|M^b f_1\right\|_{L^\Phi_w(B)} + \left\|M^b f_2\right\|_{L^\Phi_w(B)}.$$

By Corollary 5.4, we obtain

$$\|M^b f_1\|_{L^\Phi_w(B)} \le \|M^b f_1\|_{L^\Phi_w(\mathbb{R}^n)} \lesssim \|b\|_* \|f_1\|_{L^\Phi_w(\mathbb{R}^n)} = \|b\|_* \|f\|_{L^\Phi_w(2B)}. \tag{5.3}$$

For $x \in B$ we have

$$M^b f_2(x) = \sup_{t>0}\frac{1}{|B(x,t)|}\int_{B(x,t)\cap\,\complement_{(2B)}} |b(y)-b(x)||f(y)|dy.$$

Note that if $B(x,t)\cap\{\complement(2B)\} \ne \emptyset$, then $t > r$. Indeed, if $y \in B(x,t)\cap\{\complement(2B)\}$, then $t > |x-y| \ge |x_0-y| - |x_0-x| > 2r - r = r$.

On the other hand, $B(x,t)\cap\{\complement(2B)\} \subset B(x_0,2t)$. Indeed, if $y \in B(x,t)\cap\{\complement(2B)\}$, then we get $|x_0-y| \le |x-y| + |x_0-x| < t + r < 2t$.

Hence

$$\begin{aligned} M^b f_2(x) &\le \sup_{t>r}\frac{1}{|B(x_0,t)|}\int_{B(x_0,2t)} |b(y)-b(x)||f(y)|dy \\ &= 2^n \sup_{t>2r}\frac{1}{|B(x_0,t)|}\int_{B(x_0,t)} |b(y)-b(x)||f(y)|dy. \end{aligned}$$

Then

$$\|M^b f_2\|_{L^\Phi_w(B)} \lesssim \left\|\sup_{t>2r}\frac{1}{|B(x_0,t)|}\int_{B(x_0,t)} |b(y)-b(\cdot)||f(y)|dy\right\|_{L^\Phi_w(B)}$$

$$\lesssim J_1 + J_2 = \left\| \sup_{t>2r} \frac{1}{|B(x_0,t)|} \int_{B(x_0,t)} |b(y) - b_B||f(y)|dy \right\|_{L_w^\Phi(B)}$$

$$+ \left\| \sup_{t>2r} \frac{1}{|B(x_0,t)|} \int_{B(x_0,t)} |b(\cdot) - b_B||f(y)|dy \right\|_{L_w^\Phi(B)} .$$

For the term J_1 by (2.5) we obtain

$$J_1 \approx \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \frac{1}{|B(x_0,t)|} \int_{B(x_0,t)} |b(y) - b_B||f(y)|dy$$

and split it as follows:

$$J_1 \lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \int_{B(x_0,t)} \frac{w(B(x_0,t))}{|B(x_0,t)|w(B(x_0,t))} |b(y) - b_{B(x_0,t)}||f(y)|dy$$

$$+ \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \frac{1}{|B(x_0,t)|} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|dy .$$

By the definition of the A_1 class we have

$$J_1 \lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \int_{B(x_0,t)} \frac{1}{w(B(x_0,t))} |b(y) - b_{B(x_0,t)}||f(y)|w(y)dy$$

$$+ \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \frac{1}{|B(x_0,t)|} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|dy .$$

Applying Hölder's inequality, by Lemmas 2.7 and 5.5 and from the inequalities (5.2), (2.3) we get

$$J_1 \lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \frac{1}{w(B(x_0,t))} \left\| b(\cdot) - b_{B(x_0,t)} \right\|_{L_w^{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L_w^\Phi(B(x_0,t))}$$

$$+ \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} |b_{B(x_0,r)} - b_{B(x_0,t)}| \Phi^{-1}\big(w(B(x_0,t)^{-1}\big) \|f\|_{L_w^\Phi(B(x_0,t))}$$

$$\lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \frac{\|b\|_*}{\widetilde{\Phi}^{-1}\big(w(B(x_0,t))^{-1}\big) w(B(x_0,t))} \|f\|_{L_w^\Phi(B(x_0,t))}$$

$$+ \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \ln \frac{t}{r} \Phi^{-1}\big(w(B(x_0,t)^{-1}\big) \|f\|_{L_w^\Phi(B(x_0,t))}$$

$$\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x_0,t)^{-1}\big)\Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_w^\Phi(B(x_0,t))} .$$

For J_2 we obtain from Lemmas 2.7 and 5.5

$$J_2 \approx \|b(\cdot) - b_B\|_{L_w^\Phi(B)} \sup_{t>2r} \frac{1}{|B(x_0,t)|} \int_{B(x_0,t)} |f(y)|dy$$

$$\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \|f\|_{L_w^\Phi(B(x_0,t))} .$$

Gathering the estimates for J_1 and J_2, we get

$$\|M^b f_2\|_{L^\Phi_w(B)} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^\Phi_w(B(x_0,t))}. \quad (5.4)$$

To unite (5.4) with (5.3), observe that

$$\frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\|f\|_{L^\Phi_w(B(x,t))} \ge \|f\|_{L^\Phi_w(B(x,2r))},$$

which completes the proof. □

PROOF OF THEOREM 1.2. The first part is follows from Lemma 5.6 and Theorem 2.9.

We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 5.2 we have $|b(x) - b_{B_0}| \le C M^b \chi_{B_0}(x)$. Therefore, by Lemmas 3.2 and 5.5

$$\begin{aligned} 1 &\le C\frac{\|M^b\chi_{B_0}\|_{L^\Phi_w(B_0)}}{\|b(\cdot)-b_{B_0}\|_{L^\Phi_w(B_0)}} \le \frac{C}{\|b\|_*}\|M^b\chi_{B_0}\|_{L^\Phi_w(B_0)}\Phi^{-1}(w(B_0)^{-1}) \\ &\le \frac{C}{\|b\|_*}\varphi_2(B_0)\|M^b\chi_{B_0}\|_{M^{\Phi,\varphi_2}_w} \le C\varphi_2(B_0)\|\chi_{B_0}\|_{M^{\Phi,\varphi_1}_w} \le C\frac{\varphi_2(B_0)}{\varphi_1(B_0)}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem. □

6. Weak-type results

In this section necessary and sufficient conditions for the weak-type boundedness of the operator M in generalized weighted Orlicz-Morrey spaces have been obtained.

For a weight w, a measurable function f and $t > 0$, let

$$m(w,\ f,\ t) = w(\{x \in \mathbb{R}^n : |f(x)| > t\}).$$

DEFINITION 6.1. The weak weighted Orlicz space

$$WL^\Phi_w(\mathbb{R}^n) = \{f\text{-measurable} : \|f\|_{WL_\Phi} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^\Phi_w(\mathbb{R}^n)} \equiv \|f\|_{WL^\Phi_w} = \inf\Big\{\lambda > 0\ :\ \sup_{t>0}\Phi(t)m\Big(w,\ \frac{f}{\lambda},\ t\Big)\ \le 1\Big\}.$$

We can prove the following by a direct calculation:

$$\|\chi_B\|_{WL^\Phi_w} = \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)}, \quad B \in \mathcal{B}, \quad (6.1)$$

In [13] the following weak-type result was also proved.

THEOREM 6.2 ([13]). *Let Φ be a Young function. Assume in addition $w \in A_{i_\Phi}$. Then, there is a constant $C > 1$ such that*

$$\Phi(t)m(w,\ Mf,\ t) \le C \int_{\mathbb{R}^n} \Phi\left(C|f(x)|\right) w(x)dx \tag{6.2}$$

for every locally integrable f and every $t > 0$.

REMARK 6.3. The weak modular inequality (6.2) implies the corresponding norm inequality. Indeed, let (6.2) holds. Then, processing as in Remark 2.6 we have

$$\begin{aligned}\Phi(t)w\Big(\Big\{x \in \mathbb{R}^n : \frac{Mf(x)}{C^2\|f\|_{L^\Phi_w}} > t\Big\}\Big) &= \Phi(t)w\Big(\Big\{x \in \mathbb{R}^n : M\Big(\frac{f}{C^2\|f\|_{L^\Phi_w}}\Big)(x) > t\Big\}\Big) \\ &\le C \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{C\|f\|_{L^\Phi_w}}\Big) w(x)dx \le 1\,,\end{aligned}$$

which implies $\|Mf\|_{WL^\Phi_w} \lesssim \|f\|_{L^\Phi_w}$.

We denote by $WM^{\Phi,\varphi}_w(\mathbb{R}^n)$ the weak generalized weighted Orlicz-Morrey space, the space of all functions $f \in WL^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ such that

$$\|f\|_{WM^{\Phi,\varphi}_w(\mathbb{R}^n)} \equiv \|f\|_{WM^{\Phi,\varphi}_w} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\big(w(B(x,r))^{-1}\big)\, \|f\|_{WL^\Phi_w(B(x,r))} < \infty\,.$$

LEMMA 6.4. *Let Φ be a Young function, $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ and $B = B(x,r)$. Assume in addition $w \in A_{i_\Phi}$. Then*

$$\|Mf\|_{WL^\Phi_w(B)} \lesssim \|f\|_{L^\Phi_w(2B)} + \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} t^{-n}\|f\|_{L_1(B(x,t))}\,. \tag{6.3}$$

PROOF. Let Φ be an arbitrary Young function. It is obvious that

$$\|Mf\|_{WL^\Phi_w(B)} \le \|Mf_1\|_{WL^\Phi_w(B)} + \|Mf_2\|_{WL^\Phi_w(B)}$$

for every B. By the boundedness of the operator M from $L^\Phi_w(\mathbb{R}^n)$ to $WL^\Phi_w(\mathbb{R}^n)$, by Theorem 6.2, we have

$$\|Mf_1\|_{WL^\Phi_w(B)} \lesssim \|f\|_{L^\Phi_w(2B)}\,.$$

Then by (4.1) we get the inequality (6.3). □

LEMMA 6.5. *Let Φ be a Young function, $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ and $B = B(x,r)$. Assume in addition $w \in A_{i_\Phi}$. Then*

$$\|Mf\|_{WL^\Phi_w(B)} \lesssim \frac{1}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}\big(w(B(x,t))^{-1}\big)\, \|f\|_{L^\Phi_w(B(x,t))}\,. \tag{6.4}$$

PROOF. Processing as in the proof of Lemma 4.2, the inequality (6.4) directly follows from (6.3). □

THEOREM 6.6. *Let Φ be a Young function and φ_1, φ_2 positive measurable functions on $\mathbb{R}^n \times (0,\infty)$.*

1. *If $w \in A_{i_\Phi}$, then the condition* (1.3) *is sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.*
2. *If $\varphi_1 \in \mathcal{G}_w^{\Phi}$, then the condition* (1.4) *is necessary for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.*
3. *Let $w \in A_{i_\Phi}$. If $\varphi_1 \in \mathcal{G}_w^{\Phi}$, then the condition* (1.4) *is necessary and sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.*

PROOF. The first part of the theorem follows from Lemma 6.5 and Theorem 2.8. We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. It is easy to see that $M\chi_{B_0}(x) = 1$ for every $x \in B_0$. Therefore, by (6.1) and Lemma 3.2

$$\begin{aligned} 1 &= \Phi^{-1}(w(B_0)^{-1})\|M\chi_{B_0}\|_{WL_w^{\Phi}(B_0)} \le \varphi_2(B_0)\|M\chi_{B_0}\|_{WM_w^{\Phi,\varphi_2}} \\ &\le C\varphi_2(B_0)\|\chi_{B_0}\|_{M_w^{\Phi,\varphi_1}} \le C\frac{\varphi_2(B_0)}{\varphi_1(B_0)}. \end{aligned}$$

Since this is true for every $B_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

7. Vector-valued maximal inequalities

The study of vector-valued maximal inequalities was initiated by Fefferman and Stein in [9]. After Fefferman and Stein proved vector-valued maximal inequalities in [9], a passage to a number of important function spaces in harmonic analysis is done by many people.

In [4], the vector-valued maximal inequalities on the weighted Lebesgue spaces were obtained. The vector-valued maximal inequalities on Orlicz spaces, in term of modular and norm, were established in [26, Theorem 1.3.3] and [26, Theorem 1.3.5], respectively. In [45], the vector-valued maximal inequalities was generalized to Morrey spaces.

In fact, the result in [45] provides an access to solve a conjecture proposed by Mazzucato [30] for the study of Morrey type Triebel-Lizorkin spaces [40, 46]. It further inspired the study in [21] which showed that, roughly speaking, the validity of the vector-valued maximal inequalities on a Banach function space X can guarantee that the Triebel-Lizorkin type space on X is well defined and possesses atomic and molecular decompositions [21, 22]. Furthermore, the extension of the vector-valued maximal inequalities to rearrangement-invariant (r.-i.) quasi-Banach spaces and the corresponding Morrey type space were obtained in [21].

In this section, we further generalize the vector-valued maximal inequalities to generalized weighted Orlicz-Morrey spaces.

DEFINITION 7.1. Let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, w be a non-negative measurable function on $\mathbb{R}^n$, Φ be any Young function and $1 \le q \le \infty$. The generalized vector-valued weighted Orlicz-Morrey spaces $M_w^{\Phi,\varphi}(l_q) = M_w^{\Phi,\varphi}(l_q,\mathbb{R}^n)$ is defined as the set of all sequences $F = \{f_j\}_{j=1}^{\infty}$ of Lebesgue measurable functions on $\mathbb{R}^n$ such that

$$\|F\|_{M_w^{\Phi,\varphi}(l_q)} = \big\|\{f_j\}_{j=1}^{\infty}\big\|_{M_w^{\Phi,\varphi}(l_q)} := \big\|\big\|\{f_j(\cdot)\}_{j=1}^{\infty}\big\|_{l_q}\big\|_{M_w^{\Phi,\varphi}} < \infty\,.$$

The proof of the following vector-valued modular inequality for the Hardy-Littlewood maximal operator in weighted Orlicz spaces can be found in [6, Chapter 4].

PROPOSITION 7.2. *Let $1 < q < \infty$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$. Assume in addition $w \in A_{i_\Phi}$. Then, for any family of locally integrable functions $F = \{f_j\}_{j=1}^{\infty}$,*

$$\int_{\mathbb{R}^n} \Phi\left(\|MF(x)\|_{l_q}\right) w(x)dx \le C \int_{\mathbb{R}^n} \Phi\left(\|F(x)\|_{l_q}\right) w(x)dx$$

for some $C > 0$ independent of F, where $MF = \{Mf_j\}_{j=1}^{\infty}$.

The following lemma is well known and for the proof, see [39, 45].

LEMMA 7.3. *For any ball B, we have*

$$M[\chi_{\mathbb{R}^n\setminus 2B} f](x) \lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(y)|dy$$

for all $x \in B$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^{\infty} g(s)w(s)ds, \quad 0 < t < \infty\,,$$

where w is a weight.

THEOREM 7.4 ([16]). *Let v_1, v_2 and w be weights on $(0,\infty)$ and assume that v_1 is bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^{\infty} \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty\,.$$

The following lemma is true.

LEMMA 7.5. *Let $1 < q < \infty$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$. Assume in addition $w \in A_{i_\Phi}$. Then*

$$\|\,\|MF(\cdot)\|_{l_q}\|_{L_w^{\Phi}(B)} \lesssim \|\,\|F(\cdot)\|_{l_q}\|_{L_w^{\Phi}(2B)} + \frac{1}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \frac{\|\,\|F(\cdot)\|_{l_q}\|_{L^1(B(x,t))}}{t^{n+1}}\,dt \tag{7.1}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \subset L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^n)$ *and for any ball* $B = B(x,r)$.

PROOF. We split $F = \{f_j\}_{j=1}^{\infty}$ with

$$F = F_1 + F_2\,, \quad F_1 = \{f_{j,1}\}_{j=1}^{\infty}\,, \quad F_2 = \{f_{j,2}\}_{j=1}^{\infty}\,,$$
$$f_{j,1}(y) = f_j(y)\chi_{B(x,2r)}(y)\,, \quad f_{j,2}(y) = f_j(y)\chi_{\mathbb{R}^n\setminus B(x,2r)}(y)\,, \quad r > 0\,.$$

It is obvious that for any ball $B = B(x,r)$

$$\| \, \|MF\|_{l_q} \|_{L_w^{\Phi}(B)} \le \| \, \|MF_1\|_{l_q} \|_{L_w^{\Phi}(B)} + \| \, \|MF_2\|_{l_q} \|_{L_w^{\Phi}(B)}\,.$$

At first estimate $\| \, \|MF_1\|_{l_q} \|_{L_w^{\Phi}(B)}$. By Proposition 7.2 we have

$$\begin{aligned} \| \, \|MF_1\|_{l_q} \|_{L_w^{\Phi}(B)} &\le \| \, \|MF_1\|_{l_q} \|_{L_w^{\Phi}} \\ &\le C \| \, \|F_1\|_{l_q} \|_{L_w^{\Phi}} = C \| \, \|F\|_{l_q} \|_{L_w^{\Phi}(2B)}\,, \end{aligned} \tag{7.2}$$

where $C > 0$ is independent of the vector-valued function f.

Let $z \in B$ be fixed. Inspired by the ideas of [20], from Lemma 7.3, we have

$$\begin{aligned} \|MF_2(z)\|_{l_q} &\lesssim \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f_j(y)| dy \right)^q \right)^{1/q} \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_j}{|2^{k+1}B|} \int_{2^{k+1}B} |f_j(y)| dy \end{aligned}$$

where a_j is a positive constant satisfying $\|a_j\|_{l^{q'}} = 1$.

We use Hölder's inequality to obtain

$$\begin{aligned} \|MF_2(z)\|_{l_q} &\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \sum_{j=1}^{\infty} a_j |f_j(y)| dy \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \|f_j(y)\|_{l^q} dy \\ &\lesssim \int_{2r}^{\infty} \frac{\| \, \|F(\cdot)\|_{l_q} \|_{L^1(B(x,t))}}{t^{n+1}} dt\,. \end{aligned}$$

Hence

$$\| \, \|MF_2\|_{l_q} \|_{L_w^{\Phi}(B)} \lesssim \frac{1}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \frac{\| \, \|F(\cdot)\|_{l_q} \|_{L^1(B(x,t))}}{t^{n+1}} dt\,. \tag{7.3}$$

Then we obtain (7.1) from (7.2) and (7.3). $\square$

LEMMA 7.6. *Let* $1 < q < \infty$, Φ *be a Young function with* $\Phi \in \Delta_2 \cap \nabla_2$. *Assume in addition* $w \in A_1$. *Then*

$$\| \|MF\|_{l_q} \|_{L^{\Phi}_w(B)} \lesssim \frac{1}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \Phi^{-1}(w(B(x,t))^{-1}) \| \|F\|_{l_q} \|_{L^{\Phi}_w(B(x,t))} \frac{dt}{t} \tag{7.4}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \subset L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ *and for any ball* $B = B(x,r)$.

PROOF. Denote

$$M_1 := \frac{1}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \frac{\| \|F\|_{l_q} \|_{L^1(B(x,t))}}{t^{n+1}} \, dt \,,$$
$$M_2 := \| \|F\|_{l_q} \|_{L^{\Phi}_w(2B)} \,.$$

By Lemma 2.7, we get

$$M_1 \lesssim \frac{1}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \Phi^{-1}(w(B(x,t))^{-1}) \| \|F\|_{l_q} \|_{L^{\Phi}_w(B(x,t))} \frac{dt}{t} \,.$$

On the other hand by (2.3),

$$\begin{aligned}
\Phi^{-1}\big(w(B(x,r))^{-1}\big) &\approx \Phi^{-1}\big(w(B(x,r))^{-1}\big) r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&= \Phi^{-1}\big(w(B(x,r))^{-1}\big) w(B(x,r)) w(B(x,r))^{-1} r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\approx \frac{1}{\widetilde{\Phi}^{-1}\big(w(B(x,r))^{-1}\big)} w(B(x,r))^{-1} r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\lesssim w(B(x,r))^{-1} r^n \int_{2r}^{\infty} \frac{1}{\widetilde{\Phi}^{-1}\big(w(B(x,t))^{-1}\big)} \frac{dt}{t^{n+1}} \\
&= w(B(x,r))^{-1} r^n \int_{2r}^{\infty} \frac{w(B(x,t))}{\widetilde{\Phi}^{-1}\big(w(B(x,t))^{-1}\big) w(B(x,t))} \frac{dt}{t^{n+1}} \\
&\approx \frac{r^n}{w(B(x,r))} \int_{2r}^{\infty} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \frac{w(B(x,t))}{t^n} \frac{dt}{t} \,.
\end{aligned}$$

By using (2.1) for $p = 1$, we get

$$\begin{aligned}
M_2 = \| \|F\|_{l_q} \|_{L^{\Phi}_w(2B)} &\lesssim \frac{\| \|F\|_{l_q} \|_{L^{\Phi}_w(2B)}}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \Phi^{-1}(w(B(x,t))^{-1}) \frac{dt}{t} \\
&\lesssim \frac{1}{\Phi^{-1}(w(B)^{-1})} \int_{2r}^{\infty} \Phi^{-1}(w(B(x,t))^{-1}) \| \|F\|_{l_q} \|_{L^{\Phi}_w(B(x,t))} \frac{dt}{t} \,.
\end{aligned}$$

Since $\| \|MF\|_{l_q} \|_{L_{\Phi}(B)} \leq M_1 + M_2$ by Lemma 7.5, we arrive at (7.4). □

THEOREM 7.7. *Let $1 < q < \infty$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_1$ and $(\Phi, \varphi_1, \varphi_2)$ satisfies the condition*

$$\int_r^{\infty} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(w(B(x,s))^{-1}\big)} \Big) \Phi^{-1}(w(B(x,t))^{-1}) \frac{dt}{t} \le C \, \varphi_2(x,r) \,, \tag{7.5}$$

where C does not depend on x and r. Then the maximal operator M is bounded from $M_w^{\Phi,\varphi_1}(l_q)$ to $M_w^{\Phi,\varphi_2}(l_q)$, i.e., there is a constant $C > 0$ such that

$$\|MF\|_{M_w^{\Phi,\varphi_2}(l_q)} \le C\|F\|_{M_w^{\Phi,\varphi_1}(l_q)} \tag{7.6}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \in M_w^{\Phi,\varphi_1}(l_q)$.

PROOF. This follows from Lemma 7.6 and Theorem 7.4. □

Note that, for $q = \infty$, we have the following more general result.

THEOREM 7.8. *Let $w \in A_{i_\Phi}$, $\Phi \in \nabla_2$ and $(\Phi, \varphi_1, \varphi_2)$ satisfies the condition* (1.3). *Then the maximal operator M is bounded from $M_w^{\Phi,\varphi_1}(l_\infty)$ to $M_w^{\Phi,\varphi_2}(l_\infty)$, i.e., there is a constant $C > 0$ such that*

$$\|MF\|_{M_w^{\Phi,\varphi_2}(l_\infty)} \le C\|F\|_{M_w^{\Phi,\varphi_1}(l_\infty)}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \in M_w^{\Phi,\varphi_1}(l_\infty)$.

PROOF. We know that the following pointwise estimate

$$\|MF(x)\|_{l_\infty} \le M(\|F\|_{l_\infty})(x) \,, \qquad x \in \mathbb{R}^n \tag{7.7}$$

holds, see [20, p.72].

By using the pointwise estimate (7.7) and Theorem 1.1, we obtain the inequality (7.6). □

REMARK 7.9. Note that, for $w \in A_1$, the condition (1.3) weaker than the condition (7.5). We refer to [7, Remark 5.6] for details.

ACKNOWLEDGMENT. The authors would like to express their gratitude to the referees for his (her) very valuable comments and suggestions.

References

[1] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA and R. MUSTAFAYEV, A note on maximal commutators and commutators of maximal functions, J. Math. Soc. Japan. **67** (2) (2015), 581–593.

[2] A. AKBULUT, V. S. GULIYEV and R. MUSTAFAYEV, On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces, Math. Bohem. **137** (1) (2012), 27–43.

[3] A. M. ALPHONSE, An end point estimate for maximal commutators, J. Fourier Anal. Appl. **6** (4) (2000), 449–456.

[4] K. ANDERSEN and R. JOHN, Weighted inequalities for vector-valued maximal functions and singular integrals, Studia Math. **69** (1980), 19–31.

[5] V. BURENKOV, A. GOGATISHVILI, V. S. GULIYEV and R. MUSTAFAYEV, Boundedness of the fractional maximal operator in local Morrey-type spaces, Complex Var. Elliptic Equ. **55** (8–10) (2010), 739–758.

[6] D. CRUZ-URIBE, J. MARTELL and C. PEREZ, *Weights, extrapolation and the theory of Rubio de Francia*. Operator Theory: Advances and Applications, 215. Birkhauser/Springer Basel, 2011.

[7] F. DERINGOZ, V. S. GULIYEV and S. SAMKO, Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces. Operator Theory, Operator Algebras and Applications, Series: Operator Theory: Advances and Applications **242** (2014), 139–158.

[8] J. DUOANDIKOETXEA, *Fourier analysis*. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, 2001.

[9] C. FEFFERMAN and E. STEIN, Some maximal inequalities, Amer. J. Math. **93** (1971), 107–115.

[10] S. GALA, Y. SAWANO and H. TANAKA, A remark on two generalized Orlicz-Morrey spaces, J. Approx. Theory **98** (2015), 1–9.

[11] J. GARCIA-CUERVA, E. HARBOURE, C. SEGOVIA and J. L. TORREA, Weighted norm inequalities for commutators of strongly singular integrals, Indiana Univ. Math. J. **40** (4) (1991), 1397–1420.

[12] I. GENEBASHVILI, A. GOGATISHVILI, V. KOKILASHVILI and M. KRBEC, *Weight theory for integral transforms on spaces of homogeneous type*, Longman, Harlow, 1998.

[13] A. GOGATISHVILI and V. KOKILASHVILI, Criteria of weighted inequalities in Orlicz classes for maximal functions defined on homogeneous type spaces, Georgian Math. J. **1** (6) (1994), 641–673.

[14] V. S. GULIYEV, Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal. Appl. Art. ID 503948, 20 pp (2009).

[15] V. S. GULIYEV, Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J. **3** (3) (2012), 33–61.

[16] V. S. GULIYEV, Generalized local Morrey spaces and fractional integral operators with rough kernel, J. Math. Sci. (N. Y.) **193** (2) (2013), 211–227.

[17] V. S. GULIYEV, F. DERINGOZ and J. J. HASANOV, Φ-admissible singular operators and their commutators on vanishing generalized Orlicz-Morrey spaces, J. Inequal. Appl. (2014), 2014:143, 18 pp.

[18] V. S. GULIYEV, M. N. OMAROVA and Y. SAWANO, Boundedness of intrinsic square functions and their commutators on generalized weighted Orlicz-Morrey spaces, Banach J. Math. Anal. **9** (2) (2015), 44–62.

[19] V. S. GULIYEV, S. G. HASANOV, Y. SAWANO and T. NOI, Non-smooth atomic decompositions for generalized Orlicz-Morrey spaces of the third kind, Acta Appl. Math. **145** (2016), 133–174.

[20] D. I. HAKIM, E. NAKAI and Y. SAWANO, Generalized fractional maximal operators and vector-valued inequalities on generalized Orlicz-Morrey spaces, Rev. Mat. Complut. **29** (1) (2016), 59–90.

[21] K.-P. HO, Littlewood-Paley spaces, Math. Scand. **108** (2011), 77–102.

[22] K.-P. HO, Wavelet bases in Littlewood-Paley spaces, East J. Approx. **17** (2011), 333–345.

[23] K.-P. HO, Characterizations of BMO by A_p weights and p-convexity, Hiroshima Math. J. **41** (2) (2011), 153–165.

[24] K.-P. HO, Vector-valued maximal inequalities on weighted Orlicz-Morrey spaces, Tokyo J. Math. **36** (2) (2013), 499–512.

[25] R. A. KERMAN and A. TORCHINSKY, Integral inequalities with weights for the Hardy maximal function, Studia Math. **71** (3) (1981/82), 277–284.

[26] J. KOKILASHVILLI and M. KRBEC, *Weighted Inequalities in Lorentz and Orlicz spaces*, World Scientific, 1991.

[27] Y. KOMORI and S. SHIRAI, Weighted Morrey spaces and a singular integral operator, Math. Nachr. **282** (2) (2009), 219–231.

[28] D. LI, G. HU and X. SHI, Weighted norm inequalities for the maximal commutators of singular integral operators, J. Math. Anal. Appl. **319** (3) (2006), 509–521.

[29] Y. LIANG, E. NAKAI, D. YANG and J. ZHANG, Boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz Morrey and Campanato spaces, Banach J. Math. Anal. **8** (1) (2014), 221–268.

[30] A. MAZZUCATO, Decomposition of Besov-Morrey spaces, Contemp. Math. **320** (2003), Amer. Math. Soc. 279–294.

[31] T. MIZUHARA, Boundedness of some classical operators on generalized Morrey spaces, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo (1991), 183–189.

[32] C. B. MORREY, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[33] E. NAKAI, Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, Math. Nachr. **166** (1994), 95–103.

[34] E. NAKAI, Generalized fractional integrals on Orlicz-Morrey spaces, In: Banach and Function Spaces. (Kitakyushu, 2003), Yokohama Publishers, Yokohama, 323–333 (2004).

[35] W. ORLICZ, Über eine gewisse Klasse von Räumen vom Typus B, Bull. Acad. Polon. A (1932), 207–220. ; reprinted in: Collected Papers, PWN, Warszawa (1988), 217–230.

[36] W. ORLICZ, Über Räume (L^M), Bull. Acad. Polon. A (1936), 93–107. ; reprinted in: Collected Papers, PWN, Warszawa (1988), 345–359.

[37] J. POELHUIS and A. TORCHINSKY, Weighted local estimates for singular integral operators, Trans. Amer. Math. Soc. **367** (11) (2015), 7957–7998.

[38] M. M. RAO and Z. D. REN, *Theory of Orlicz Spaces*, M. Dekker, Inc., New York, 1991.

[39] Y. SAWANO and H. TANAKA, Morrey spaces for non-doubling measures. Acta Math. Sin. (Engl. Ser.) **21** (6) (2005), 1535–1544.

[40] Y. SAWANO and H. TANAKA, Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, Math. Z. **257** (2007), 871–905.

[41] Y. SAWANO, Generalized Morrey spaces for non-doubling measures, NoDEA Nonlinear Differential Equations Appl. **15** (2008), 413–425.

[42] Y. SAWANO, S. SUGANO and H. TANAKA, Orlicz-Morrey spaces and fractional operators, Potential Anal. **36** (4) (2012), 517–556.

[43] C. SEGOVIA and J. L. TORREA, Weighted inequalities for commutators of fractional and singular integrals, Publ. Mat. **35** (1) (1991), 209–235. Conference on Mathematical Analysis (El Escorial, 1989)

[44] C. SEGOVIA and J. L. TORREA, Higher order commutators for vector-valued Calderón-Zygmund operators, Trans. Amer. Math. Soc. **336** (2) (1993), 537–556.

[45] L. TANG and J. XU, Some properties of Morrey type Besov-Triebel spaces, Math. Nachr. **278** (2005), 904–917.

[46] H. WANG, Decomposition for Morrey type Besov-Triebel spaces, Math. Nachr. **282** (2009), 774–787.

Present Addresses:
FATIH DERINGOZ
DEPARTMENT OF MATHEMATICS,
AHI EVRAN UNIVERSITY,
KIRSEHIR, TURKEY.
e-mail: fderingoz@ahievran.edu.tr

VAGIF S. GULIYEV
INSTITUTE OF MATHEMATICS AND MECHANICS OF NAS OF AZERBAIJAN,
BAKU, AZERBAIJAN.
DEPARTMENT OF MATHEMATICS,
AHI EVRAN UNIVERSITY,
KIRSEHIR, TURKEY.
e-mail: vagif@guliyev.com

SABIR G. HASANOV
GANJA STATE UNIVERSITY,
GANJA, AZERBAIJAN.
e-mail: sabhasanov@gmail.com