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A Sufficient Condition That $J(X^*) = J(X)$ Holds for a Banach Space X

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Abstract. It is shown that the James constant of the space $\mathbf{R}^2$ endowed with a $\pi/2$-rotation invariant norm coincides with that of its dual space. As a corollary, we have the same statement on symmetric absolute norms on $\mathbf{R}^2$.

1. Introduction and the main result

Let X be a Banach space, and let S_X be the unit sphere of X. The symbol X^* stands for the dual space of X. The James constant $J(X)$ of X is given by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

The notion of James constant was introduced in [3], and is sometimes referred as the non-square constant since $J(X) < 2$ if and only if X is uniformly non-square, that is, there exists a $\delta > 0$ such that $\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\delta)$ for each $x, y \in S_X$. Some basic properties of $J(X)$ are as follows:

(i) $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X ([3]).
(ii) $J(H) = \sqrt{2}$ for any Hilbert space H.
(iii) If $\dim X \geq 3$, then $J(X) = \sqrt{2}$ if and only if X is a Hilbert space ([6]).
(iv) If $\dim X = 2$, then there exist various non-Hilbert spaces X satisfying $J(X) = \sqrt{2}$ ([3, 6, 7]).

We recall that another important geometric constant, the so-called von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X, satisfies the equality $C_{NJ}(X^*) = C_{NJ}(X)$ for any Banach space X ([9]). In other words, von Neumann-Jordan constant is invariant under taking the dual space. However, this is not the case for James constant ([4, Example 2]). Hence it is natural to estimate differences between $J(X^*)$ and $J(X)$. In this direction, for example, we

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have $|J(X^*) - J(X)| \leq (\sqrt{2}-1)^2$ by [10, Corollary 4] and

$$|J(X^*) - J(X)| \leq \max\left\{\frac{2J(X) - J(X)^2}{4}, \frac{2J(X^*) - J(X^*)^2}{4}\right\}$$

by [11, Corollary 2.1]; see also [2]. On the other hand, this estimation problem includes the following important part: When does the equality $J(X^*) = J(X)$ hold? In [8], it was shown that if X is a two-dimensional symmetric absolute normed space, then $J(X^*) = J(X)$, where a norm $\|\cdot\|$ on $\mathbf{R}^2$ is said to be *symmetric* if $\|(\alpha, \beta)\| = \|(\beta, \alpha)\|$ for each (α, β), and *absolute* if $\|(\alpha, \beta)\| = \|(|\alpha|, |\beta|)\|$ for each (α, β). Although the proof is complicated, this result provides various examples of normed spaces X satisfying $J(X^*) = J(X)$.

The purpose of this paper is to develop the study on normed spaces X satisfying $J(X^*) = J(X)$ in the two-dimensional setting. More precisely, we aim to extend the above result in [8] to the wider class of $\pi/2$-rotation invariant norms on $\mathbf{R}^2$ by a quite simple argument. For $\theta \in \mathbf{R}$, let $R(\theta)$ be the θ-rotation matrix, that is,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then a norm $\|\cdot\|$ on $\mathbf{R}^2$ is said to be *θ-rotation invariant* if $\|R(\theta)x\| = \|x\|$ for each $x \in \mathbf{R}^2$. From the definition, a norm $\|\cdot\|$ on $\mathbf{R}^2$ is $\pi/2$-rotation invariant if $\|(-\beta, \alpha)\| = \|(\alpha, \beta)\|$ for each (α, β). If $\|\cdot\|$ is a symmetric absolute norm on $\mathbf{R}^2$, then it follows from

$$\|(-\beta, \alpha)\| = \|(|\beta|, |\alpha|)\| = \|(\beta, \alpha)\| = \|(\alpha, \beta)\|$$

for each (α, β) that $\|\cdot\|$ is $\pi/2$-rotation invariant. Hence the class of $\pi/2$-rotation invariant norms contains that of symmetric absolute norms.

The main result in this paper is the following.

THEOREM 1.1. *Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbf{R}^2$. Then $J((\mathbf{R}^2, \|\cdot\|)^*) = J((\mathbf{R}^2, \|\cdot\|))$.*

As an immediate consequence of the main theorem, we have the following.

COROLLARY 1.2 ([8]). *Let $\|\cdot\|$ be a symmetric absolute norm on $\mathbf{R}^2$. Then $J((\mathbf{R}^2, \|\cdot\|)^*) = J((\mathbf{R}^2, \|\cdot\|))$.*

2. Proof of the main theorem

For two elements x, y of a normed space, x is said to be *isosceles orthogonal* to y, denoted by $x \perp_I y$, if $\|x + y\| = \|x - y\|$. Then, as was shown in [3], the James constant can be written as

$$J(X) = \sup\{\|x + y\| (= \|x - y\|) : x, y \in S_X,\ x \perp_I y\}.$$

This is a key ingredient of our argument.

We start this section with the following useful lemma.

LEMMA 2.1 (Gao-Lau [3]; Alonso [1]). *Let X be a two-dimensional normed space. Suppose that $x \in S_X$. Then there exists a unique* (*up to the sign*) *element $y \in S_X$ such that $x \perp_I y$.*

In what follows, let X be a normed space $(\mathbf{R}^2, \|\cdot\|)$ unless otherwise stated. Using the preceding lemma, we can completely determine the isosceles orthogonality in a $\pi/2$-rotation invariant normed space. Although it was already mentioned in Komuro, Saito and Mitani [5], we give its proof only for the sake of completeness.

LEMMA 2.2 ([5]). *Let X be a $\pi/2$-rotation invariant normed space*, *and let $x, y \in S_X$. Then $x \perp_I y$ if and only if $x \perp y$ in the usual sense*, *that is*, $\langle x, y\rangle = 0$. *Equivalently*, *if $x \in S_X$*, *then $x \perp_I y$ if and only if $y = \pm R(\pi/2)x$.*

PROOF. From the preceding lemma, it is enough to show that $x \perp_I R(\pi/2)x$ for each $x \in S_X$. Put $y = R(\pi/2)x \in S_X$. Since $R(\pi/2)$ is an isometry also on the Euclidean space $(\mathbf{R}^2, \|\cdot\|_2)$, it follows that $\|x\|_2 = \|y\|_2$. From this, we have $\langle x + y, x - y\rangle = 0$, which and $\|x + y\|_2 = \|x - y\|_2$ together imply that $x + y = \pm R(\pi/2)(x - y)$. Hence we have $\|x + y\| = \|x - y\|$ since $R(\pi/2)$ is an isometry on X, that is, $x \perp_I y$. □

As a consequence of Lemma 2.2, if X is a $\pi/2$-rotation invariant normed space, then $J(X) = \sup\{\|x + R(\pi/2)x\| : x \in S_X\}$. We now remark that $I + R(\pi/2) = \sqrt{2}R(\pi/4)$. Then one has the following.

THEOREM 2.3. *Let X be a $\pi/2$-rotation invariant normed space. Then $J(X) = \sqrt{2}\|R(\pi/4)\|_X$*, *where $\|\cdot\|_X$ denotes the operator norm with respect to the space X.*

Now, [7, Theorem 3.10] is an easy consequence of the preceding theorem.

COROLLARY 2.4 ([7]). *Let X be a $\pi/2$-rotation invariant normed space. Then $J(X) = \sqrt{2}$ if and only if X is $\pi/4$-rotation invariant.*

PROOF. If X is $\pi/4$-rotation invariant, then $\|R(\pi/4)\| = 1$; and so $J(X) = \sqrt{2}$ by the preceding theorem. Conversely, suppose that $J(X) = \sqrt{2}$. Then, again by the preceding theorem, we have $\|R(\pi/4)\| = 1$. Moreover, since X is $\pi/2$-rotation invariant, one has that $R(\pi/2)$ and its inverse $R(-\pi/2)$ are both isometries on X. Thus it follows that

$$\|R(\pi/4)^{-1}\| = \|R(-\pi/4)\| = \|R(-\pi/2)R(\pi/4)\| \le \|R(-\pi/2)\|\|R(\pi/4)\| = 1\,,$$

which implies that $R(\pi/4)$ is an isometry on X. □

Naturally, the dual space X^* of X can be identified with $(\mathbf{R}^2, \|\cdot\|_*)$ under $X^* \ni f \leftrightarrow (f(1,0), f(0,1)) \in \mathbf{R}^2$. In this manner, the adjoint operator (as a Banach space operator) A^* of a matrix A can be represented by the transpose A^T. Moreover, it is well-known that $\|A^*\|_{X^*} = \|A\|_X$. These facts provide another key ingredient.

LEMMA 2.5. *Let $\theta \in \mathbf{R}$. Suppose that $\|\cdot\|$ is a θ-rotation invariant norm on $\mathbf{R}^2$. Then the dual norm $\|\cdot\|_*$ is also θ-rotation invariant.*

PROOF. Put $X = (\mathbf{R}^2, \|\cdot\|)$. Then $X^* = (\mathbf{R}^2, \|\cdot\|_*)$. We note that the adjoint operator of $R(\theta)$ is given by $R(\theta)^* = R(\theta)^T = R(-\theta)$. Since $\|\cdot\|$ is θ-rotation invariant and $R(-\theta) = R(\theta)^{-1}$, we have $\|R(\theta)\|_{X^*} = \|R(-\theta)\|_{X^*} = 1$. Then it follows that

$$\|R(\theta)x\|_* \leq \|R(\theta)\|_{X^*}\|x\|_* = \|x\|_*$$

for each $x \in X^*$, and that

$$\|x\|_* = \|R(-\theta)R(\theta)x\|_* \leq \|R(-\theta)\|_{X^*}\|R(\theta)x\|_* = \|R(\theta)x\|_*$$

for each $x \in X^*$. Thus $\|R(\theta)x\|_* = \|x\|_*$ for each $x \in X^*$, that is, $\|\cdot\|_*$ is also θ-rotation invariant. □

We are now ready to prove the main theorem.

PROOF OF THEOREM 1.1. Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbf{R}^2$. Put $X = (\mathbf{R}^2, \|\cdot\|)$. Then its dual norm $\|\cdot\|_*$ is also $\pi/2$-rotation invariant by Lemma 2.5. It follows from Theorem 2.3 that $J(X) = \sqrt{2}\|R(\pi/4)\|_X$ and $J(X^*) = \sqrt{2}\|R(\pi/4)\|_{X^*}$. However, since $R(\pi/2)$ is an isometry on X^*, one has that

$$\begin{aligned}\|R(\pi/4)\|_X &= \|R(\pi/4)^*\|_{X^*} = \|R(-\pi/4)\|_{X^*}\\ &= \|R(\pi/2)R(-\pi/4)\|_{X^*} = \|R(\pi/4)\|_{X^*}.\end{aligned}$$

This proves that $J(X^*) = J(X)$, as desired. □

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