

Morse–Bott Inequalities for Manifolds with Boundary

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Abstract. In the present paper, we define Morse–Bott functions on manifolds with boundary which are generalizations of Morse functions and show Morse–Bott inequalities for these manifolds.

1. Introduction

In the early 1930s, M. Morse [12] showed the well-known Morse inequalities which describe the relationship between the Betti numbers of manifolds and the number of critical points of functions. Many authors found “Morse inequalities” for manifolds with non-empty boundary. Recently, motivated by the Floer homology, M. Akaho [2] constructed the Morse complex (which is derived from Witten’s work [16]) by setting specific conditions on the boundary. F. Laudenbach [11] constructed the Morse complex by introducing the pseudo-gradient vector fields adapted to the boundary which control flow lines near the boundary and obtained Morse inequalities for manifolds with boundary.

In 1954, R. Bott [5] generalized the Morse theory for functions which have degenerate critical points under some assumptions. Indeed, he [6, 7, 8] established degenerate Morse inequalities (called Morse–Bott inequalities). After that, J.-M. Bismut [4], B. Helffer and J. Sjöstrand [9] gave different proofs of these inequalities. Recently, A. Banyaga and D. Hurtubise [3] published a proof by describing an explicit perturbation of a given Morse–Bott function (called the perturbation technique) under the assumption that negative normal bundles are all orientable. Such functions appear when the domain manifold M is equipped with the action of a compact Lie group G . In that case, the critical set of any G -invariant smooth function is a disjoint union of finitely many closed submanifolds.

In the present paper, by using the definition introduced by F. Laudenbach [11], we define Morse–Bott functions on manifolds with boundary which are generalizations of Morse functions. Then we formulate Morse–Bott inequalities for manifolds with boundary (Theorem 3) and show them by using the perturbation technique.

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2. Preliminaries: Morse homology with local coefficients

In this section, we define the Morse homology with local coefficients and state the Morse homology theorem and the Morse inequalities for closed manifolds. Our main reference is [13, Subsection 7.2].

Let M be an m -dimensional connected closed manifold and $f: M \rightarrow \mathbf{R}$ a Morse function on M . Let R be a ring and \mathcal{L} a local system of R -modules over M . We define the Morse complex $(C_*(f; \mathcal{L}), \partial_*^{f; \mathcal{L}})$ for f with local coefficients in \mathcal{L} . For each $k = 0, 1, \dots, m$, the k -chain group is defined by

$$C_k(f; \mathcal{L}) = \bigoplus_{p \in \text{Crit}_k(f)} \mathcal{L}_p,$$

where \mathcal{L}_p is the fiber of \mathcal{L} over $p \in M$. The boundary operator $\partial_k^{f; \mathcal{L}}: C_k(f; \mathcal{L}) \rightarrow C_{k-1}(f; \mathcal{L})$ is defined by

$$\partial_k^{f; \mathcal{L}}(s \otimes p) = \sum_{q \in \text{Crit}_{k-1}(f)} \left(\sum_{\gamma \in \mathcal{M}_f(p, q)} n_f(\gamma) \Phi_{\tilde{\gamma}}(s) \right) \otimes q,$$

where $p \in \text{Crit}_k(f)$, $s \in \mathcal{L}_p$, the set $\mathcal{M}_f(p, q) = W_f^u(p) \cap W_f^s(q)/\mathbf{R}$ is the moduli space of unparametrized (minus) gradient flow lines of f from p to q , $n_f(\gamma)$ is the sign of γ determined by orientations of the unstable manifolds for f , the path $\tilde{\gamma}: [0, 1] \rightarrow M$ is a reparametrization of $\gamma: \mathbf{R} \rightarrow M$ such that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) = q$, and $\Phi_{\tilde{\gamma}}: \mathcal{L}_p \rightarrow \mathcal{L}_q$ is the R -isomorphism associated with $\tilde{\gamma}$.

THEOREM 1 (Morse homology theorem [13]). *The pair $(C_*(f; \mathcal{L}), \partial_*^{f; \mathcal{L}})$ is a chain complex, i.e., $\partial_*^{f; \mathcal{L}}$ is well-defined and $\partial_{* - 1}^{f; \mathcal{L}} \circ \partial_*^{f; \mathcal{L}} = 0$. Moreover, its homology is isomorphic to the singular homology of M with local coefficients in \mathcal{L} . Namely, for every $k = 0, 1, \dots, m$ we have*

$$H_k(C_*(f; \mathcal{L}), \partial_*^{f; \mathcal{L}}) \cong H_k(M; \mathcal{L}).$$

We state the Morse inequalities for closed manifolds.

DEFINITION 1. The *Morse counting polynomial* of f is defined to be

$$\mathcal{M}_t(f) = \sum_{p \in \text{Crit}(f)} t^{\text{ind}_f p},$$

and the *Poincaré polynomial* $\mathcal{P}_t(M; \mathcal{L})$ of M with local coefficients in \mathcal{L} by

$$\mathcal{P}_t(M; \mathcal{L}) = \sum_{k=0}^m \text{rank}_R H_k(M; \mathcal{L}) t^k.$$

THEOREM 2 (Morse inequalities for closed manifolds). *Let M be a connected closed manifold and f a Morse function on M . Let R be a ring and \mathcal{L} a local system of R -modules over M . Then there exists a polynomial $\mathcal{R}(t)$ with non-negative coefficients such that*

$$\mathcal{M}_t(f) - \frac{1}{\text{rank } \mathcal{L}} \mathcal{P}_t(M; \mathcal{L}) = (1+t)\mathcal{R}(t).$$

The proof is straightforward by applying Theorem 1.

3. Main Theorem

In this section, we define Morse–Bott functions on manifolds with boundary in the sense of Laudenbach and state our main theorem (Theorem 3). Let M be an m -dimensional compact manifold with boundary.

DEFINITION 2. A C^∞ -function $f: M \rightarrow \mathbf{R}$ is called a *Morse–Bott function* if f satisfies the following conditions:

- (1) The set of critical points $\text{Crit}(f) = \{p \in M \mid df(p) = 0\}$ is a disjoint union of connected submanifolds of the interior $\text{Int } M$ and each connected component of $\text{Crit}(f)$ is non-degenerate in the sense of Bott (see [5]).
- (2) The restriction $f|_{\partial M}: \partial M \rightarrow \mathbf{R}$ is a Morse–Bott function on ∂M .

Let $f: M \rightarrow \mathbf{R}$ be a Morse–Bott function. Then the critical submanifolds of $f|_{\partial M}$ are divided into two types:

DEFINITION 3. A connected critical submanifold $C \subset \text{Crit}(f|_{\partial M})$ is said to be of *type N* (resp. *type D*) if for some $p \in C$, hence for all $p \in C$, p is of type N (resp. type D), i.e., $\langle df(p), n(p) \rangle$ is negative (resp. positive) where $n(p) \in T_p M$ is an outward normal vector to the boundary at p (see [11]).

We denote the critical point sets by

$$I(f) = \text{Crit}(f), \quad N(f) = \{p \in \text{Crit}(f|_{\partial M}) \mid p \text{ is of type } N\}$$

and

$$D(f) = \{p \in \text{Crit}(f|_{\partial M}) \mid p \text{ is of type } D\}.$$

Let C_j ($j = 1, \dots, \ell$), Γ_s ($s = 1, \dots, \ell_N$) and Δ_u ($u = 1, \dots, \ell_D$) be the connected components of $I(f)$, $N(f)$ and $D(f)$, respectively. We prepare the following notation: For $j = 1, \dots, \ell$, $s = 1, \dots, \ell_N$ and $u = 1, \dots, \ell_D$, let

$$\begin{aligned} c_j &= \dim C_j, & d_s^N &= \dim \Gamma_s, & d_u^D &= \dim \Delta_u, \\ \lambda_j &= \text{ind}_f C_j, & \mu_s^N &= \text{ind}_{f|_{\partial M}} \Gamma_s, & \mu_u^D &= \text{ind}_{f|_{\partial M}} \Delta_u. \end{aligned}$$

For a non-degenerate critical submanifold C of the Morse–Bott function f , denote by $o(\nu^- C)$ the orientation bundle of the negative normal bundle $\nu^- C$, where $\nu^- C$ is the maximal

subbundle of the normal bundle of C in M such that the eigenvalues of the Hessian of f are all negative. We think of $o(v^-C)$ as a local system of \mathbf{Z} -modules of rank one over C .

DEFINITION 4. The *Morse–Bott counting polynomial of type N* of f is defined to be

$$\mathcal{MB}_t^N(f) = \sum_{j=1}^{\ell} \mathcal{P}_t(C_j; o(v^-C_j)) t^{\lambda_j} + \sum_{s=1}^{\ell_N} \mathcal{P}_t(\Gamma_s; o(v^- \Gamma_s)) t^{\mu_s^N},$$

and the *Morse–Bott counting polynomial of type D* of f is defined to be

$$\mathcal{MB}_t^D(f) = \sum_{j=1}^{\ell} \mathcal{P}_t(C_j; o(v^-C_j)) t^{\lambda_j} + \sum_{u=1}^{\ell_D} \mathcal{P}_t(\Delta_u; o(v^- \Delta_u)) t^{\mu_u^D+1}.$$

Our main result is the following theorem:

THEOREM 3 (Morse–Bott inequalities for manifolds with boundary). *Let M be a compact manifold with boundary and f a Morse–Bott function on M . Then there exists a polynomial $\mathcal{R}(t)$ with non-negative integer coefficients such that*

$$\mathcal{MB}_t^N(f) - \mathcal{P}_t(M; \mathbf{Z}) = (1+t)\mathcal{R}(t).$$

COROLLARY 1. *Suppose that M , the critical submanifolds of f and their negative normal subbundles are all oriented. Then there exists a polynomial $\mathcal{R}(t)$ with non-negative integer coefficients such that*

$$\mathcal{MB}_t^D(f) - \mathcal{P}_t(M, \partial M; \mathbf{Z}) = (1+t)\mathcal{R}(t).$$

PROOF. Since f is Morse–Bott, so is $-f$. The critical point sets of f are decomposed into critical submanifolds as follows:

$$I(f) = \bigsqcup_{j=1}^{\ell} C_j, \quad N(f) = \bigsqcup_{s=1}^{\ell_N} \Gamma_s, \quad D(f) = \bigsqcup_{u=1}^{\ell_D} \Delta_u.$$

Then the critical point sets of $-f$ are

$$I(-f) = \bigsqcup_{j=1}^{\ell} C_j, \quad N(-f) = \bigsqcup_{u=1}^{\ell_D} \Delta_u, \quad D(-f) = \bigsqcup_{s=1}^{\ell_N} \Gamma_s.$$

Moreover, for $j = 1, \dots, \ell$ and $u = 1, \dots, \ell_D$ we have

$$\text{ind}_{-f} C_j = \dim M - \dim C_j - \text{ind}_f C_j = m - c_j - \lambda_j,$$

$$\text{ind}_{-f|_{\partial M}} \Delta_u = \dim \partial M - \dim \Delta_u - \text{ind}_{f|_{\partial M}} \Delta_u = m - d_u^D - (\mu_u^D + 1).$$

Theorem 3 implies that there exists a polynomial $\mathcal{Q}(t)$ with non-negative integer coefficients such that

$$\mathcal{MB}_t^N(-f) - \mathcal{P}_t(M; \mathbf{Z}) = (1+t)\mathcal{Q}(t).$$

Since the negative normal bundles are all oriented, $\mathcal{MB}_t^N(-f)$ is computed as

$$\begin{aligned}
\mathcal{MB}_t^N(-f) &= \sum_{j=1}^{\ell} \mathcal{P}_t(C_j; \mathbf{Z}) t^{\text{ind}_{-f} C_j} + \sum_{u=1}^{\ell^D} \mathcal{P}_t(\Delta_u; \mathbf{Z}) t^{\text{ind}_{-f|_{\partial M}} \Delta_u} \\
&= \sum_{j=1}^{\ell} \mathcal{P}_t(C_j; \mathbf{Z}) t^{m-c_j-\lambda_j} + \sum_{u=1}^{\ell^D} \mathcal{P}_t(\Delta_u; \mathbf{Z}) t^{m-d_u^D-(\mu_u^D+1)} \\
&= t^m \left\{ \sum_{j=1}^{\ell} \mathcal{P}_t(C_j; \mathbf{Z}) t^{-c_j} \left(\frac{1}{t}\right)^{\lambda_j} + \sum_{u=1}^{\ell^D} \mathcal{P}_t(\Delta_u; \mathbf{Z}) t^{-d_u^D} \left(\frac{1}{t}\right)^{\mu_u^D+1} \right\} \\
&= t^m \left\{ \sum_{j=1}^{\ell} \mathcal{P}_{1/t}(C_j; \mathbf{Z}) \left(\frac{1}{t}\right)^{\lambda_j} + \sum_{u=1}^{\ell^D} \mathcal{P}_{1/t}(\Delta_u; \mathbf{Z}) \left(\frac{1}{t}\right)^{\mu_u^D+1} \right\} \\
&= t^m \mathcal{MB}_{1/t}^D(f).
\end{aligned}$$

Here we used the fact that the critical submanifolds are all oriented and hence we have

$$\begin{aligned}
\mathcal{P}_t(C_j; \mathbf{Z}) t^{-c_j} &= \left(\sum_{k=0}^{c_j} \text{rank}_{\mathbf{Z}} H_k(C_j; \mathbf{Z}) t^k \right) t^{-c_j} = \sum_{k=0}^{c_j} \text{rank}_{\mathbf{Z}} H_k(C_j; \mathbf{Z}) \left(\frac{1}{t}\right)^{c_j-k} \\
&= \sum_{k=0}^{c_j} \text{rank}_{\mathbf{Z}} H_{c_j-k}(C_j; \mathbf{Z}) \left(\frac{1}{t}\right)^{c_j-k} = \mathcal{P}_{1/t}(C_j; \mathbf{Z}).
\end{aligned}$$

Similarly we have $\mathcal{P}_t(\Delta_u; \mathbf{Z}) t^{-d_u^D} = \mathcal{P}_{1/t}(\Delta_u; \mathbf{Z})$.

Therefore we obtain

$$\begin{aligned}
t^m \mathcal{MB}_{1/t}^D(f) - \mathcal{P}_t(M; \mathbf{Z}) &= (1+t) \mathcal{Q}(t), \\
\frac{1}{t^m} \mathcal{MB}_t^D(f) - \mathcal{P}_{1/t}(M; \mathbf{Z}) &= \left(1 + \frac{1}{t}\right) \mathcal{Q}\left(\frac{1}{t}\right), \\
\mathcal{MB}_t^D(f) - t^m \mathcal{P}_{1/t}(M; \mathbf{Z}) &= (1+t) t^{m-1} \mathcal{Q}\left(\frac{1}{t}\right).
\end{aligned}$$

Note that the polynomial $t^{m-1} \mathcal{Q}(1/t)$ is a polynomial with non-negative integer coefficients. Since M is oriented, Lefschetz duality shows that

$$\begin{aligned}
t^m \mathcal{P}_{1/t}(M; \mathbf{Z}) &= t^m \sum_{k=0}^m \text{rank}_{\mathbf{Z}} H_k(M; \mathbf{Z}) \left(\frac{1}{t}\right)^k \\
&= \sum_{k=0}^m \text{rank}_{\mathbf{Z}} H^{m-k}(M, \partial M; \mathbf{Z}) t^{m-k} = \mathcal{P}_t(M, \partial M).
\end{aligned}$$

Thus we obtain the desired identity. \square

The perturbation technique of Section 4 is nothing but the *morsification* of f along its critical set. Then, one reduces to the known case of a Morse function. Of course, there is still some work to do, since the main theorem is stated in terms of the Bott critical sets. This is done in the last four pages of the paper.

4. Proof of Main Theorem

Let M be an m -dimensional compact manifold with boundary and f a Morse–Bott function on M .

4.1. Adapted pseudo-gradient vector fields of Bott-type. Let $X \in \mathfrak{X}(M)$ be a vector field on M , i.e., a smooth section $X: M \rightarrow TM$ in the tangent bundle. For any zero point p of X , the differential DX of the smooth map X defines a vector field $DX(p): T_pM \rightarrow T_{X(p)}TM$. Let $\phi: T_{X(p)}TM \rightarrow T_pM$ be an isomorphism. We call $X_p^{\text{lin}} = \phi \circ DX(p): T_pM \rightarrow T_pM$ the *linear part* of X at p . Let C be a submanifold consisting of zero points of X . C is called a *zero submanifold*. The tangent bundle $TM|_C$ splits as $TM|_C = TC \oplus \nu C$ where νC is the normal bundle of C in M . For all $p \in C$ and $V \in T_pC$, we have $DX(p)(V) = 0$. Hence the linear part X_p^{lin} of X at p induces the linear transformation $(X_p^{\text{lin}})^\nu: \nu_p C \rightarrow \nu_p C$.

DEFINITION 5 ([1]). A zero submanifold C of X is said to be *transversely hyperbolic* if for each $p \in C$, the linear transformation $(X_p^{\text{lin}})^\nu$ has no pure imaginary eigenvalues.

DEFINITION 6. A vector field $X \in \mathfrak{X}(M)$ is called a *pseudo-gradient vector field of Bott-type* for the Morse–Bott function f *adapted* to the boundary if X satisfies the following conditions:

- (1) $Xf < 0$ except on the interior critical submanifolds and the boundary critical submanifolds of type N of f .
- (2) X points inwards along the boundary except a neighborhood of the boundary critical submanifolds of type N where it is tangent to ∂M .
- (3) For each interior critical submanifold $C \subset I(f)$, C is a transversely hyperbolic zero submanifold of X , and for all $p \in C$, the Hessian of the Lie derivative $X \cdot f$ restricted to the normal space $\nu_p C$ is negative definite.
- (4) For each boundary critical submanifold $\Gamma \subset N(f)$ of type N and $\gamma \in \Gamma$, there exist local coordinates $x = (r, u, y) \in [0, 1) \times \mathbf{R}^{\dim \Gamma} \times \mathbf{R}^{m - \dim \Gamma - 1}$ of M around γ such that $M = \{r \geq 0\}$ and $f(x) = f(\Gamma) + q(y) + r$ where q is a non-degenerate quadratic form. Moreover, in these coordinates, X is tangent to the boundary, Γ is a transversely hyperbolic zero submanifold of X , and the Hessian of the Lie derivative $X \cdot f$ restricted to the normal bundle $\nu \Gamma$ is negative definite.

REMARK 1. The conditions (1) and (2) are exactly the same as [11]. However, in our setting, since the critical point set might be positive dimensional, the conditions (3) and (4)

are certain generalizations of those of [11].

PROPOSITION 1. *For any Morse–Bott function f , there exists an adapted pseudo-gradient vector field of Bott-type $X \in \mathfrak{X}(M)$ for f .*

PROOF. The proof is similar to that of [11, Proposition of Subsection 2.1]. The existence of local coordinates of (4) follows from the Morse–Bott Lemma [5]. \square

4.2. Perturbation technique [3]. In this subsection, we use the same notation as in Section 3. We identify a collar neighborhood of the boundary ∂M with $[0, 1) \times \partial M$. Let r be the standard coordinate of $[0, 1)$. Fix an adapted pseudo-gradient vector field of Bott-type $X_f \in \mathfrak{X}(M)$ for the Morse–Bott function f (Proposition 1). For all $j = 1, \dots, \ell$ and $s = 1, \dots, \ell_N$, we choose open neighborhoods $U_j \subset \text{Int } M$ of C_j and $U_s^N \subset \partial M$ of Γ_s such that

$$X_f|_{U_j} = -(\text{grad } f)|_{U_j} \quad \text{and} \quad X_f|_{U_s^N} = -(\text{grad } f|_{\partial M})|_{U_s^N}. \quad (4.1)$$

By the condition (4) of Definition 6, we can choose a positive real number $0 < \delta < 1/2$ small enough so that for any $s = 1, \dots, \ell_N$ and $\gamma \in \Gamma_s$, there exist local coordinates $(r, u, y) \in [0, 1) \times \mathbf{R}^{\dim \Gamma_s} \times \mathbf{R}^{m-\dim \Gamma_s-1}$ of M around γ and a non-degenerate quadratic form q_s such that

$$f|_{[0, 2\delta) \times U_s^N} = f(\Gamma_s) + q_s(y) + r, \quad (4.2)$$

$$X_f|_{[0, 2\delta) \times U_s^N} = -(\text{grad } f|_{\partial M})|_{U_s^N} - r \frac{\partial}{\partial r}. \quad (4.3)$$

For all j and s , let $T_j \subset \text{Int } M$ be a tubular neighborhood of C_j of radius δ_j and $T_s^N \subset \partial M$ a tubular neighborhood of Γ_s which satisfy the following conditions:

- (1) For each j and s , we have $T_j \subset U_j$ and $T_s^N \subset U_s^N$.
- (2) For each j and s , T_j and T_s^N are contained in the union of the charts from the Morse–Bott Lemma [5].
- (3) For distinct i and j , we have $T_i \cap T_j = \emptyset$ and $T_i^N \cap T_j^N = \emptyset$.
- (4) For every pseudo-gradient flow line (i.e., flow line of X_f) $\gamma: [0, 1] \rightarrow M$ from T_i to T_j , we have

$$\begin{aligned} & f(\gamma(0)) - f(\gamma(1)) \\ & \geq 3 \max \{ \text{var}(f, T_1), \dots, \text{var}(f, T_\ell), \text{var}(f, T_1^N), \dots, \text{var}(f, T_{\ell_N}^N) \}. \end{aligned}$$

Moreover, the similar conditions hold in the cases of “from T_i to T_j^N ”, “from T_i^N to T_j ” and “from T_i^N to T_j^N ”.

- (5) If $f(C_i) \neq f(C_j)$, then $\text{var}(f, T_i) + \text{var}(f, T_j) < 1/3 |f(C_i) - f(C_j)|$. Moreover, the similar conditions hold in the cases of $f(\Gamma_i) \neq f(\Gamma_j)$ and $f(C_j) \neq f(\Gamma_s)$.

(6) For all j , we have $((0, 1) \times \partial M) \cap T_j = \emptyset$.

Similarly, for all $u = 1, \dots, \ell_D$, let $T_u^D \subset \partial M$ be a tubular neighborhood of Δ_u satisfying the above conditions (2) and (3).

By the Kupka–Smale Theorem [10, 15], for each j, s and u , we can pick positive Morse functions $f_j: C_j \rightarrow \mathbf{R}$, $f_s^N: \Gamma_s \rightarrow \mathbf{R}$ and $f_u^D: \Delta_u \rightarrow \mathbf{R}$ such that their gradient vector fields satisfy the Morse–Smale condition, i.e., all the unstable manifolds and the stable manifolds intersect transversely. We extend these functions to functions on T_j, T_s^N and T_u^D by making constant in the direction normal to C_j, Γ_s and Δ_u respectively. For all $j = 1, \dots, \ell$, let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j of radius $\tilde{\delta}_j$ ($< \delta_j$) in $\text{Int } M$.

Moreover, let $\hat{\rho}_j: [0, \infty) \rightarrow [0, 1]$ be a C^∞ -function satisfying

$$\hat{\rho}_j = \begin{cases} 1 & \text{on } [0, \tilde{\delta}_j), \\ 0 & \text{outside of } [0, \delta_j), \end{cases}$$

$$\hat{\rho}_j > 0 \quad \text{on } [0, \delta_j)$$

and we define a bump function ρ_j by the formula $\rho_j(x) = \hat{\rho}_j(d(x, C_j))$ where d is the distance induced from a metric on M . Now we choose $\varepsilon_1 > 0$ small enough so that for all $j = 1, \dots, \ell$,

$$\sup_{T_j \setminus \tilde{T}_j} \varepsilon_1 \|\text{grad}(\rho_j f_j)\| < \inf_{T_j \setminus \tilde{T}_j} \|\text{grad } f\| \neq 0.$$

Similarly, for every s and u , we choose $\tilde{T}_s^N, \tilde{T}_u^D, \rho_s^N, \rho_u^D, \varepsilon_1^N$ and ε_1^D . For each s and u , we extend f_s^N and f_u^D to functions on $[0, 1) \times T_s^N$ and $[0, 1) \times T_u^D$ by making constant in the r -coordinate, respectively. Similarly, we extend ρ_s^N and ρ_u^D to functions on $[0, 1) \times \partial M$.

On the other hand, let $\hat{\rho}: [0, \infty) \rightarrow [0, 1]$ be a C^∞ -function satisfying

$$\hat{\rho} = \begin{cases} 1 & \text{on } [0, 2\delta), \\ 0 & \text{outside of } [0, 1), \end{cases}$$

$$\hat{\rho} > 0 \quad \text{on } [0, 1)$$

and we define a bump function ρ by the formula $\rho(r, y) = \hat{\rho}(r)$ where $(r, y) \in [0, 1) \times \partial M$. By the condition (1) of Definition 2, we can choose $\varepsilon_2^N > 0$ small enough so that for all $s = 1, \dots, \ell_N$,

$$\sup_{[0, 1) \times T_s^N} \varepsilon_2^N \|\text{grad}(\rho \rho_s^N f_s^N)\| < \inf_{[0, 1) \times T_s^N} \|\text{grad } f\| \neq 0.$$

We can choose ε_2^D satisfying a similar condition.

Lastly, we set $\varepsilon = \min\{\varepsilon_1, \varepsilon_1^N, \varepsilon_1^D, \varepsilon_2^N, \varepsilon_2^D\}$ and define a C^∞ -function on M by

$$h = f + \varepsilon \left\{ \sum_{j=1}^{\ell} \rho_j f_j + \rho \left(\sum_{s=1}^{\ell_N} \rho_s^N f_s^N + \sum_{u=1}^{\ell_D} \rho_u^D f_u^D \right) \right\}.$$

The proof of the following lemma is straightforward:

LEMMA 1. *The function $h: M \rightarrow \mathbf{R}$ is Morse in the sense of Laudenbach [11]. For all $n = 0, 1, \dots, m$,*

$$I_n(h) = \bigsqcup_{\lambda_j+k=n} \text{Crit}_k(f_j), \quad N_n(h) = \bigsqcup_{\mu_s^N+k=n} \text{Crit}_k(f_s^N)$$

and

$$D_n(h) = \bigsqcup_{\mu_u^D+k=n} \text{Crit}_k(f_u^D).$$

Since the number of critical points is finite, we can choose $\tilde{\delta} > 0$ small enough so that for all $s = 1, \dots, \ell_N$ and $\gamma \in N(h) \cap \Gamma_s$, the open ball $B_{4\tilde{\delta}}(\gamma) \subset T_s^N$ of γ of radius $4\tilde{\delta}$ contains no other critical points. Let $\hat{\rho}: [0, \infty) \rightarrow [0, 1]$ be a C^∞ -function satisfying

$$\hat{\rho} = \begin{cases} 0 & \text{on } [0, \tilde{\delta}), \\ 1 & \text{outside of } [0, 2\tilde{\delta}), \\ \hat{\rho} > 0 & \text{on } [0, 2\tilde{\delta}) \end{cases}$$

and we define a bump function $\tilde{\rho}$ by the formula $\tilde{\rho}(y) = \hat{\rho}(d(y, N(h)))$ where $y \in \partial M$.

Then we extend $\tilde{\rho}$ to a function on $[0, 1) \times \partial M$ by making constant in the r -coordinate.

Now let us choose $\varepsilon > 0$ small enough so that for all $s = 1, \dots, \ell_N$,

$$\sup_{[0, 2\tilde{\delta}) \times (T_s^N \setminus B_{\tilde{\delta}}^s)} \varepsilon \delta |(\text{grad}(\tilde{\rho}r)) h| < \inf_{T_s^N \setminus B_{\tilde{\delta}}^s} \|\text{grad } h|_{\partial M}\|^2 \neq 0 \quad (4.4)$$

and

$$\begin{aligned} & \sup_{[2\tilde{\delta}, 1) \times T_s^N} \varepsilon \left| X_f(\rho \rho_s^N f_s^N) - (\text{grad } f)(\rho \rho_s^N f_s^N) + \delta(\text{grad}(\rho \tilde{\rho}r))h \right| \\ & < \inf_{[2\tilde{\delta}, 1) \times T_s^N} |X_f f| \neq 0, \end{aligned} \quad (4.5)$$

where

$$B_{\tilde{\delta}}^s = \bigcup_{\gamma \in N(h) \cap \Gamma_s} B_{\tilde{\delta}}(\gamma).$$

We define a C^∞ -vector field on M by

$$G = X_f - \varepsilon \operatorname{grad} \left\{ \sum_{j=1}^{\ell} \rho_j f_j + \rho \left(\sum_{s=1}^{\ell_N} \rho_s^N f_s^N + \sum_{u=1}^{\ell_D} \rho_u^D f_u^D - \delta \tilde{\rho} r \right) \right\}.$$

LEMMA 2. *The vector field G is an adapted pseudo-gradient vector field for the Morse function h .*

PROOF. We show that G satisfies the conditions (1)–(5) of the definition of the adapted pseudo-gradient vector fields of [11], that is those of Definition 6 for a Morse function.

The condition (3) is verified as follows: For $p \in I(h)$, there exists $j = 1, \dots, \ell$ such that $p \in C_j \subset T_j$. Since $h = f + \varepsilon \rho_j f_j$ and $X_f = -\operatorname{grad} f$ on T_j by (4.1), we have

$$G = X_f - \varepsilon \operatorname{grad}(\rho_j f_j) = -\operatorname{grad} f - \varepsilon \operatorname{grad}(\rho_j f_j) = -\operatorname{grad} h. \quad (4.6)$$

Hence the Hessian of the Lie derivative $G \cdot h$ at p is negative definite since p is a non-degenerate critical point of h by Lemma 1.

The condition (4) is verified as follows: For $\gamma \in N(h)$, there exists $s = 1, \dots, \ell_N$ such that $\gamma \in \Gamma_s \subset T_s^N$. By Lemma 1 and (4.2), on $[0, 2\delta) \times T_s^N$, there exist local coordinates $x = (r, y) \in [0, 1) \times \mathbf{R}^{m-1}$ around γ in M and a non-degenerate quadratic form q such that

$$h(x) = h(\gamma) + q(y) + r. \quad (4.7)$$

By (4.3), on $[0, 2\delta) \times T_s^N$, we obtain

$$\begin{aligned} G &= X_f - \varepsilon \operatorname{grad}(\rho_s^N f_s^N - \delta \tilde{\rho} r) \\ &= \left(-\operatorname{grad} f|_{\partial M} - r \frac{\partial}{\partial r} \right) - \varepsilon \operatorname{grad}(\rho_s^N f_s^N) + \varepsilon \delta \operatorname{grad}(\tilde{\rho} r) \\ &= -\operatorname{grad} h|_{\partial M} - r \frac{\partial}{\partial r} + \varepsilon \delta \operatorname{grad}(\tilde{\rho} r) \end{aligned} \quad (4.8)$$

and in particular, on $[0, 2\delta) \times B_{\tilde{\delta}}(\gamma)$, we have

$$G = -\operatorname{grad} h|_{\partial M} - r \frac{\partial}{\partial r} = -\operatorname{grad} q - r \frac{\partial}{\partial r}. \quad (4.9)$$

Hence the Hessian of the Lie derivative $G \cdot h$ is negative definite.

The condition (2) is verified as follows: By (4.9), since $G = -\operatorname{grad} h|_{\partial M}$ on $\{0\} \times B_{\tilde{\delta}}(\gamma)$, G is tangent to the boundary on the neighborhood $B_{\tilde{\delta}}(\gamma)$ of a boundary critical point γ of type N . On $\partial M \setminus \bigcup_s B_{\tilde{\delta}}^s$, we have

$$G = X_f - \varepsilon \operatorname{grad} \left(\sum_{s=1}^{\ell_N} \rho_s^N f_s^N + \sum_{u=1}^{\ell_D} \rho_u^D f_u^D \right) + \delta \left(r \operatorname{grad} \tilde{\rho} + \tilde{\rho} \frac{\partial}{\partial r} \right).$$

Due to the term $\delta \tilde{\rho} \partial/\partial r \neq 0$, G points inwards along the boundary outside of a neighborhood containing $\bigcup_s B_{\tilde{\delta}}^s$.

The verification of the condition (1) divides into five cases: On a neighborhood $U_p \setminus \{p\}$ of a critical point $p \in I(h) \cap C_j \subset T_j$, we have $G = -\text{grad } h$ by (4.6) and $Gh = -(\text{grad } h)h = -\|\text{grad } h\|^2 < 0$. On the neighborhood $[0, 2\delta) \times B_{\delta}^s(\gamma)$ of a critical point $\gamma \in N(h) \cap \Gamma_s \subset T_s^N$, we have $h = h|_{\partial M} + r$ by (4.7) and

$$Gh = \left(-\text{grad } h|_{\partial M} - r \frac{\partial}{\partial r} \right) (h|_{\partial M} + r) = -\|\text{grad } h|_{\partial M}\|^2 - r < 0$$

by (4.9). On $[0, 2\delta) \times (T_s^N \setminus B_{\delta}^s)$, we have

$$\begin{aligned} Gh &= \left(-\text{grad } h|_{\partial M} - r \frac{\partial}{\partial r} + \varepsilon \delta \text{grad}(\tilde{\rho}r) \right) (h|_{\partial M} + r) \\ &= -\|\text{grad } h|_{\partial M}\|^2 - r + \varepsilon \delta (\text{grad}(\tilde{\rho}r)) h \\ &< -\inf_{T_s^N \setminus B_{\delta}^s} \|\text{grad } h|_{\partial M}\|^2 + \sup_{[0, 2\delta) \times (T_s^N \setminus B_{\delta}^s)} \varepsilon \delta |(\text{grad}(\tilde{\rho}r)) h| < 0 \end{aligned}$$

by (4.7), (4.8) and (4.4). On a neighborhood $[2\delta, 1) \times U_{\gamma}$ of the critical point γ , we have $h = f + \varepsilon \rho \rho_s^N f_s^N$ and

$$\begin{aligned} Gh &= \left(X_f - \varepsilon \text{grad}(\rho \rho_s^N f_s^N) + \varepsilon \delta \text{grad}(\rho \tilde{\rho}r) \right) \left(f + \varepsilon \rho \rho_s^N f_s^N \right) \\ &= X_f f + \varepsilon \left\{ X_f(\rho \rho_s^N f_s^N) - (\text{grad}(\rho \rho_s^N f_s^N)) f \right\} \\ &\quad - \varepsilon^2 \|\text{grad}(\rho \rho_s^N f_s^N)\|^2 + \varepsilon \delta (\text{grad}(\rho \tilde{\rho}r)) h \\ &< -\inf_{[2\delta, 1) \times T_s^N} |X_f f| + \varepsilon \left\{ X_f(\rho \rho_s^N f_s^N) - (\text{grad } f)(\rho \rho_s^N f_s^N) + \delta (\text{grad}(\rho \tilde{\rho}r)) h \right\} \\ &< -\inf_{[2\delta, 1) \times T_s^N} |X_f f| \\ &\quad + \sup_{[2\delta, 1) \times T_s^N} \varepsilon \left\{ X_f(\rho \rho_s^N f_s^N) - (\text{grad } f)(\rho \rho_s^N f_s^N) + \delta (\text{grad}(\rho \tilde{\rho}r)) h \right\} \\ &< 0. \end{aligned}$$

Here we used the fact that $X_f f < 0$ and (4.5). Outside of all the neighborhoods we have considered till here, we have $h = f$ and $G = X_f$. This implies that $Gh = X_f f < 0$.

According to [14], the condition (5) is generically fulfilled for vector fields satisfying conditions (1)–(4). Thus we choose ε small enough so that G satisfies the condition (5) if necessary. \square

4.3. Lemmas. Let ∂_*^h be the boundary operator of the Morse complex for the Morse function h defined in [11]. For $j = 1, \dots, \ell$ and $s = 1, \dots, \ell_N$, let $\partial_*^j = \partial_*^{f_j; o(v^{-C_j})}$ and

$\partial_*^{s,N} = \partial_*^{f_s^N; o(v^- \Gamma_s)}$ be the boundary operators for the Morse functions f_j and f_s^N with local coefficients in $o(v^- C_j)$ and $o(v^- \Gamma_s)$ defined in Section 2, respectively.

Let us denote by $o(v^- C_j)_x \cong \mathbf{Z}$ the fiber of $o(v^- C_j)$ over a point $x \in C_j$. We fix a base point $*$ $\in C_j$. For every $p \in \text{Crit}(f_j)$, let $c_p: [0, 1] \rightarrow C_j$ be a path such that $c_p(0) = *$ and $c_p(1) = p$. Since the interval $[0, 1]$ is contractible and then the pull-back bundle $c_p^* o(v^- C_j) \rightarrow [0, 1]$ is trivial, we can choose bases 1_p of $o(v^- C_j)_p$, $p \in \text{Crit}(f_j)$, so that the associated isomorphism $\Phi_{c_p}: o(v^- C_j)_* \rightarrow o(v^- C_j)_p$ satisfies $\Phi_{c_p}(1_*) = 1_p$. Similarly, we choose bases 1_p of $o(v^- \Gamma_s)_p$, $p \in \text{Crit}(f_s^N)$.

We note that an orientation of the unstable manifold $W_{f_j}^u(p)$ of a critical point p is determined by an orientation of the tangent space $T_p W_{f_j}^u(p)$ at p . We fix orientations of the unstable manifolds $W_G^u(p)$, $p \in \text{Crit}(h)$, where G is the adapted pseudo-gradient vector field for h defined in Subsection 4.2. Since we have the direct sum $T_p W_G^u(p) = T_p W_{f_j}^u(p) \oplus v_p^- C_j$ at p , we get an orientation of $W_{f_j}^u(p)$. In the same manner, we choose orientations of $W_{f_s^N}^u(p)$, $p \in \text{Crit}(f_s^N)$. The following lemmas are generalizations of [3, Lemma 9 and Corollary 10].

LEMMA 3. *Let $j = 1, \dots, \ell$ and $s = 1, \dots, \ell_N$. For all critical points $p, q \in \text{Crit}(f_j)$ (resp. $\text{Crit}(f_s^N)$) of relative index one and all unparametrized (pseudo-)gradient flow lines γ from p to q , we have*

$$n_h(\gamma)1_q = n_{f_j}(\gamma)\Phi_{\tilde{\gamma}}(1_p)$$

(resp. $n_h(\gamma)1_q = n_{f_s^N}(\gamma)\Phi_{\tilde{\gamma}}(1_p)$).

PROOF. We will focus on C_j . By the definition of h (and the choice of ε), the gradient flow lines connecting two critical points in C_j are the same (see the proof of [3, Lemma 9]).

The signs $n_h(\gamma)$ and $n_{f_j}(\gamma)$ are determined by the orientations of the unstable manifolds $W_G^u(p)$, $W_G^u(q)$, $W_{f_j}^u(p)$ and $W_{f_j}^u(q)$ chosen above. We define a loop $l: S^1 = \mathbf{R}/\mathbf{Z} \rightarrow C_j$ by

$$l(t) = \begin{cases} \tilde{\gamma}(3t) & \text{if } 0 \leq t \leq 1/3, \\ c_q(2-3t) & \text{if } 1/3 \leq t \leq 2/3, \\ c_p(3t-2) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

Since we have the direct sum $T_x W_G^u(p) = T_x W_{f_j}^u(p) \oplus v_x^- C_j$ for every $x \in W_G^u(p) \cap C_j$, the orientations of $W_G^u(p)$ and $W_{f_j}^u(p)$ is not compatible along the loop l as long as the pull-back bundle $l^* o(v^- C_j) \rightarrow S^1$ is non-trivial. Hence the ambiguity of these orientations appears as the multiplication by $\Phi_l(1_p)$. Namely, we have

$$n_h(\gamma)1_p = n_{f_j}(\gamma)\Phi_l(1_p).$$

Therefore,

$$\begin{aligned} n_h(\gamma)1_q &= n_h(\gamma)\Phi_{c_q}\left(\Phi_{c_p}^{-1}(1_p)\right) = \Phi_{c_q}\left(\Phi_{c_p}^{-1}(n_h(\gamma)1_p)\right) = \Phi_{c_q}\left(\Phi_{c_p}^{-1}(n_{f_j}(\gamma)\Phi_l(1_p))\right) \\ &= n_{f_j}(\gamma)\Phi_{\tilde{\gamma}}(1_p). \end{aligned}$$

This completes Lemma 3. \square

Now we order $C_1, \dots, C_\ell, \Gamma_1, \dots, \Gamma_{\ell_N}$ by “height”. That is, for

$$\{B_1, \dots, B_{\ell+\ell_N}\} = \{C_1, \dots, C_\ell, \Gamma_1, \dots, \Gamma_{\ell_N}\},$$

we assume that the order of $B_1, \dots, B_{\ell+\ell_N}$ is ascending, i.e., $f(B_i) \leq f(B_j)$ whenever $i \leq j$. On the other hand, by Lemma 1, for each $n = 0, 1, \dots, m$, the n -chain group $F_n^N(h)$ of the Morse complex (see [11] for the definition) of h is of the form:

$$F_n^N(h) = \bigoplus_{j=1}^{\ell} C_{n-\lambda_j}(f_j; \mathbf{Z}) \oplus \bigoplus_{s=1}^{\ell_N} C_{n-\mu_s^N}(f_s^N; \mathbf{Z}),$$

where $C_{n-\lambda_j}(f_j; \mathbf{Z})$ and $C_{n-\mu_s^N}(f_s^N; \mathbf{Z})$ are the free \mathbf{Z} -modules generated by the elements of $\text{Crit}_{n-\lambda_j}(f_j)$ and $\text{Crit}_{n-\mu_s^N}(f_s^N)$, respectively. Therefore any $\beta \in F_n^N(h) \setminus \{0\}$ can be uniquely decomposed into the sum of n -chains of $\text{Crit}_{n-\lambda_j}(f_j)$ and $\text{Crit}_{n-\mu_s^N}(f_s^N)$, $\beta = \beta_{j_1} + \dots + \beta_{j_r}$ where $j_1 < \dots < j_r$ and for $i = 1, \dots, r$ there exists $j = 1, \dots, \ell$ or $s = 1, \dots, \ell_N$ such that $\beta_{j_i} \in C_{n-\lambda_j}(f_j; \mathbf{Z}) \setminus \{0\}$ or $\beta_{j_i} \in C_{n-\mu_s^N}(f_s^N; \mathbf{Z}) \setminus \{0\}$ respectively. We call β_{j_i} the *top chain* of β and denote by $\text{top } \beta$.

LEMMA 4. *Let $n = 0, 1, \dots, m$. If $\beta \in \text{Ker } \partial_n^h$, then we have*

$$\mathbb{1} \otimes \text{top } \beta \in \text{Ker } \partial_{n-\lambda_j}^j \quad \text{or} \quad \mathbb{1} \otimes \text{top } \beta \in \text{Ker } \partial_{n-\mu_s^N}^{s,N},$$

where $\mathbb{1} = \sum_{q \in \text{top } \beta} 1_q \in o(v^-C_j)$ (the sum is taken over generators) and $\text{top } \beta \in C_{n-\lambda_j}(f_j; \mathbf{Z})$ for some $j = 1, \dots, \ell$, or $\mathbb{1} = \sum 1_q \in o(v^- \Gamma_s)$ and $\text{top } \beta \in C_{n-\mu_s^N}(f_s^N; \mathbf{Z})$ for some $s = 1, \dots, \ell_N$, respectively.

PROOF. Let $\beta = \sum_i n_i q_i$ where $n_i \in \mathbf{Z}$ and $q_i \in \text{Crit}_n(h)$. If $\beta \in \text{Ker } \partial_n^h$, we then have

$$\begin{aligned} 0 &= \partial_n^h(\beta) = \sum_i n_i \partial_n^h(q_i) = \sum_i n_i \left\{ \sum_{p \in I_{n-1}(h) \cup N_{n-1}(h)} \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) \right) p \right\} \\ &= \sum_{p \in I_{n-1}(h) \cup N_{n-1}(h)} \left\{ \sum_i n_i \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) \right) \right\} p. \end{aligned}$$

Hence for every $p \in I_{n-1}(h) \cup N_{n-1}(h)$ we have

$$\sum_i n_i \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) \right) = 0.$$

From now on, we will focus on C_j 's. Namely, we assume that $\text{top } \beta \in C_{n-\lambda_j}(f_j; \mathbf{Z})$ for some $j = 1, \dots, \ell$. Then, in particular, for every $p \in I_{n-1}(h) \cap C_j = \text{Crit}_{n-\lambda_j-1}(f_j)$ we have

$$0 = \sum_i n_i \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) \right) = \sum_{q_i \in C_j} n_i \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) \right)$$

since $\mathcal{M}_G(q_i, p) = \emptyset$ for every $q_i \notin C_j$ by the definition of top chains. Therefore, we have

$$\begin{aligned} \partial_{n-\lambda_j}^j(\mathbb{1} \otimes \text{top } \beta) &= \sum_{q_i \in C_j} n_i \partial_{n-\lambda_j}^j(1_{q_i} \otimes q_i) \\ &= \sum_{q_i \in C_j} n_i \left\{ \sum_{p \in \text{Crit}_{n-\lambda_j-1}(f_j)} \left(\sum_{\gamma \in \mathcal{M}_{f_j}(q_i, p)} n_{f_j}(\gamma) \Phi_{\tilde{\gamma}}(1_{q_i}) \right) \otimes p \right\} \\ &= \sum_{p \in \text{Crit}_{n-\lambda_j-1}(f_j)} \left\{ \sum_{q_i \in C_j} n_i \left(\sum_{\gamma \in \mathcal{M}_{f_j}(q_i, p)} n_{f_j}(\gamma) \Phi_{\tilde{\gamma}}(1_{q_i}) \right) \right\} \otimes p \\ &= \sum_{p \in I_{n-1}(h) \cap C_j} \left\{ \sum_{q_i \in C_j} n_i \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) 1_p \right) \right\} \otimes p \\ &= \sum_{p \in I_{n-1}(h) \cap C_j} \left\{ \sum_{q_i \in C_j} n_i \left(\sum_{\gamma \in \mathcal{M}_G(q_i, p)} n_h(\gamma) \right) \right\} (1_p \otimes p) \\ &= 0. \end{aligned}$$

Here we applied Lemma 3 and used the fact that $\mathcal{M}_G(q_i, p)$ coincides with $\mathcal{M}_{f_j}(q_i, p)$ whenever $q_i, p \in C_j$. \square

4.4. Proof of Main Theorem. Under these preparations, the remaining part of the proof is based on a modification of [3].

PROOF OF THEOREM 3. Applying Theorem 2 for the Morse functions f_1, \dots, f_ℓ and $f_1^N, \dots, f_{\ell_N}^N$, there exist polynomials $\mathcal{R}_1(t), \dots, \mathcal{R}_\ell(t)$ and $\mathcal{R}_1^N(t), \dots, \mathcal{R}_{\ell_N}^N(t)$ with non-negative integer coefficients, such that for $j = 1, \dots, \ell$ and $s = 1, \dots, \ell_N$,

$$\mathcal{M}_t(f_j) - \mathcal{P}_t(C_j; o(v^- C_j)) = (1+t)\mathcal{R}_j(t)$$

and

$$\mathcal{M}_t(f_s^N) - \mathcal{P}_t(\Gamma_s; o(v^- \Gamma_s)) = (1+t)\mathcal{R}_s^N(t),$$

respectively. Here we note that $\text{rank } o(v^- C_j) = \text{rank } o(v^- \Gamma_s) = 1$. Moreover, by [11, Corollary A], for the Morse function h , there exists a polynomial $\mathcal{R}_h(t)$ with non-negative integer coefficients such that

$$\mathcal{M}_t^N(h) - \mathcal{P}_t(M; \mathbf{Z}) = (1+t)\mathcal{R}_h(t),$$

where

$$\mathcal{M}_t^N(h) = \sum_{p \in I(h)} t^{\text{ind}_h p} + \sum_{\gamma \in N(h)} t^{\text{ind}_{h|_{\partial M}} \gamma}$$

is the Morse counting polynomial of type N of h . By Lemma 1, $\mathcal{M}_t^N(h)$ can be computed as follows:

$$\begin{aligned} \mathcal{M}_t^N(h) &= \sum_{j,k} \# \text{Crit}_k(f_j) t^{\lambda_j+k} + \sum_{s,k} \# \text{Crit}_k(f_s^N) t^{\mu_s^N+k} \\ &= \sum_{j=1}^{\ell} \mathcal{M}_t(f_j) t^{\lambda_j} + \sum_{s=1}^{\ell_N} \mathcal{M}_t(f_s^N) t^{\mu_s^N}. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathcal{MB}_t^N(f) &= \sum_{j=1}^{\ell} \mathcal{P}_t(C_j; o(v^- C_j)) t^{\lambda_j} + \sum_{s=1}^{\ell_N} \mathcal{P}_t(\Gamma_s; o(v^- \Gamma_s)) t^{\mu_s^N} \\ &= \sum_{j=1}^{\ell} \{ \mathcal{M}_t(f_j) - (1+t)\mathcal{R}_j(t) \} t^{\lambda_j} + \sum_{s=1}^{\ell_N} \{ \mathcal{M}_t(f_s^N) - (1+t)\mathcal{R}_s^N(t) \} t^{\mu_s^N} \\ &= \mathcal{M}_t^N(h) - (1+t) \left(\sum_{j=1}^{\ell} \mathcal{R}_j(t) t^{\lambda_j} + \sum_{s=1}^{\ell_N} \mathcal{R}_s^N(t) t^{\mu_s^N} \right) \\ &= \mathcal{P}_t(M; \mathbf{Z}) + (1+t) \left\{ \mathcal{R}_h(t) - \left(\sum_{j=1}^{\ell} \mathcal{R}_j(t) t^{\lambda_j} + \sum_{s=1}^{\ell_N} \mathcal{R}_s^N(t) t^{\mu_s^N} \right) \right\}. \end{aligned}$$

We set $\mathcal{R}(t) = \mathcal{R}_h(t) - \left(\sum_{j=1}^{\ell} \mathcal{R}_j(t) t^{\lambda_j} + \sum_{s=1}^{\ell_N} \mathcal{R}_s^N(t) t^{\mu_s^N} \right)$. It is enough to show that all the coefficients of the polynomial $\mathcal{R}(t)$ are non-negative.

According to the proof of [3, Theorem 3], $\mathcal{R}_1(t), \dots, \mathcal{R}_\ell(t), \mathcal{R}_1^N(t), \dots, \mathcal{R}_{\ell_N}^N(t)$ and

$\mathcal{R}_h(t)$ have the following specific forms: For all $j = 1, \dots, \ell$ and $s = 1, \dots, \ell_N$,

$$\mathcal{R}_j(t) = \sum_{k=1}^{c_j} (v_k^j - z_k^j) t^{k-1}, \quad \mathcal{R}_s^N(t) = \sum_{k=1}^{d_s^N} (v_k^{s,N} - z_k^{s,N}) t^{k-1}$$

and $\mathcal{R}_h(t) = \sum_{n=1}^m (v_n + v_n^N - z_n) t^{n-1},$

where

$$v_k^j = \# \text{Crit}_k(f_j), \quad z_k^j = \text{rank}_{\mathbf{Z}} \text{Ker } \partial_k^j,$$

$$v_k^{s,N} = \# \text{Crit}_k(f_s^N), \quad z_k^{s,N} = \text{rank}_{\mathbf{Z}} \text{Ker } \partial_k^{s,N},$$

$$v_n = \# I_n(h), \quad v_n^N = \# N_n(h), \quad z_n = \text{rank}_{\mathbf{Z}} \text{Ker } \partial_n^h.$$

Therefore

$$\begin{aligned} \mathcal{R}(t) &= \sum_{n=1}^m (v_n + v_n^N - z_n) t^{n-1} \\ &\quad - \left\{ \sum_{j=1}^{\ell} \sum_{k=1}^{c_j} (v_k^j - z_k^j) t^{\lambda_j+k-1} + \sum_{s=1}^{\ell_N} \sum_{k=1}^{d_s^N} (v_k^{s,N} - z_k^{s,N}) t^{\mu_s^N+k-1} \right\} \\ &= \left(\sum_{j=1}^{\ell} \sum_{k=1}^{c_j} z_k^j t^{\lambda_j+k-1} + \sum_{s=1}^{\ell_N} \sum_{k=1}^{d_s^N} z_k^{s,N} t^{\mu_s^N+k-1} \right) - \sum_{n=1}^m z_n t^{n-1} \\ &= \sum_{n=1}^m \left(\sum_{\lambda_j+k=n} z_k^j + \sum_{\mu_s^N+k=n} z_k^{s,N} - z_n \right) t^{n-1}. \end{aligned}$$

Thus it is enough to show that for all $n = 1, \dots, m$,

$$\sum_{\lambda_j+k=n} z_k^j + \sum_{\mu_s^N+k=n} z_k^{s,N} \geq z_n.$$

Fix $n = 1, \dots, m$. If $z_n = 0$, the inequality holds. Hence we may assume that $z_n > 0$.

If $z_n = 1$, there exists a non-zero element β_1 in $\text{Ker } \partial_n^h$. By Lemma 4, $\mathbb{1} \otimes \text{top } \beta_1 \in \text{Ker } \partial_{k_1}^{j_1}$ (or $\mathbb{1} \otimes \text{top } \beta_1 \in \text{Ker } \partial_{k_1}^{j_1, N}$) where k_1 and j_1 are integers satisfying $\lambda_{j_1} + k_1 = n$ (or $\mu_{j_1}^N + k_1 = n$). Hence the inequality holds.

If $z_n = 2$, we can find an element $\beta_2 \in \text{Ker } \partial_n^h$ which is not in the group generated by β_1 , and by adding a multiple of β_1 to β_2 if necessary, we can choose β_2 whose top part $\mathbb{1} \otimes \text{top } \beta_2 \in \text{Ker } \partial_{k_2}^{j_2}$ (or $\mathbb{1} \otimes \text{top } \beta_2 \in \text{Ker } \partial_{k_2}^{j_2, N}$) is not in the group generated by $\mathbb{1} \otimes \text{top } \beta_1$

where k_2 and j_2 are integers satisfying $\lambda_{j_2} + k_2 = n$ (or $\mu_{j_2}^N + k_2 = n$). Hence the inequality holds.

Repeating this argument finitely many times, we get generators in $\text{Ker } \partial_n^h$ whose the tensor products of $\mathbf{1}$ and top parts are linearly independent in

$$\bigoplus_{\lambda_j+k=n} \text{Ker } \partial_k^j \oplus \bigoplus_{\mu_s^N+k=n} \text{Ker } \partial_k^{s,N}.$$

Thus

$$\sum_{\lambda_j+k=n} \text{rank}_{\mathbf{Z}} \text{Ker } \partial_k^j + \sum_{\mu_s^N+k=n} \text{rank}_{\mathbf{Z}} \text{Ker } \partial_k^{s,N} \geq \text{rank}_{\mathbf{Z}} \text{Ker } \partial_n^h.$$

This completes the proof. \square

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