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Homogeneity of Infinite Dimensional Anti-Kaehler Isoparametric Submanifolds II

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Abstract. In this paper, we prove that, if a full irreducible infinite dimensional anti-Kaehler isoparametric submanifold of codimension greater than one has J-diagonalizable shape operators, then it is an orbit of the action of a Banach Lie group generated by one-parameter transformation groups induced by holomorphic Killing vector fields defined entirely on the ambient Hilbert space.

1. Introduction

An *infinite dimensional isoparametric submanifold* is a proper Fredholm submanifold of finite codimension in an infinite dimensional separable Hilbert space over the real number field $\mathbb{R}$ such that its normal holonomy group is trivial and that the shape operator for each parallel normal vector field has constant eigenvalues, where "proper Fredholm" means that the differential of the normal exponential map $\exp^\perp$ of the submanifold is a Fredholm operator and that the restriction of $\exp^\perp$ to unit ball normal bundle is proper. Throughout this paper, all Hilbert spaces mean infinite dimensional separable Hilbert spaces. In 1999, E. Heintze and X. Liu ([13]) proved that all full irreducible infinite dimensional isoparametric submanifolds of codimension greater than one in a Hilbert space are extrinsically homogeneous. In 2002, by using this result of Heintze-Liu, U. Christ ([4]) claimed that all irreducible equifocal submanifolds with flat section of codimension greater than one in a simply connected symmetric space of compact type are extrinsically homogeneous. Let $I(V)$ be the group of all isometries of the Hilbert space V and M a full irreducible isoparametric submanifolds of codimension greater than one in V. Set $H := \{F \in I(V) \mid F(M) = M\}$. The extrinsic homogeneity of M in the result of [13] means that $Hx = M$ $(x \in M)$. Let $I_b(V)$ be the subgroup of $I(V)$ generated by one-parameter transformation groups induced by the Killing vector fields defined entirely on V. Note that $I_b(V)$ is a Banach Lie group. Set $H_b := H \cap I_b(V)$, which is a Banach Lie subgroup of $I(V)$. Recently, C. Gorodski and E. Heintze ([10]) proved that

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$H_b x = M$ holds for any $x \in M$. This improved extrinsic homogeneity theorem closed a gap in the proof of the above extrinsic homogeneity theorem by U. Christ.

In [20], we introduced the notion of a complex equifocal submanifold in a symmetric space of non-compact type. In [21], we showed that the study of complex equifocal C^ω-submanifolds in symmetric spaces of non-compact type is converted to that of anti-Kaehler isoparametric submanifolds in the infinite dimensional anti-Kaehler space, where C^ω means the real analyticity. In this paper, we shall investigate an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators, which was called a *proper* anti-Kaehler isoparametric submanifold in [21]. L. Geatti and C. Gorodski ([9]) introduced the notion of an isoparametric submanifold with diagonalizable Weingarten operators in a finite dimensional pseudo-Euclidean space. Note that anti-Kaehler isoparametric submanifolds with J-diagonalizable shape operators give a subclass of the infinite dimensional version of isoparametric submanifolds with diagonalizable Weingarten operators. Let K be a maximal compact subgroup of a finite dimensional non-compact semi-simple Lie group G and H a symmetric subgroup of G. Define a Hilbert Lie group $P(G^{\mathbb{C}}, H^{\mathbb{C}} \times K^{\mathbb{C}})$ by

$$P(G^{\mathbb{C}}, H^{\mathbb{C}} \times K^{\mathbb{C}}) := \{g \in H^1([0,1], G^{\mathbb{C}}) \mid (g(0), g(1)) \in H^{\mathbb{C}} \times K^{\mathbb{C}}\}.$$

Then any principal orbit of the $P(G^{\mathbb{C}}, H^{\mathbb{C}} \times K^{\mathbb{C}})$-action on $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ is an infinite dimensional anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators. This fact is stated in Remark 1.1 of [22] and shown by Theorem 1.1 (ii) in [21] and Theorem B in [22] because the H-action on G/K is an action of Hermann type. In Example 2 of Section 4, we will state this fact in detail. In addition, for an involutive automorphism σ of G, define a Hilbert Lie group $P(G^{\mathbb{C}}, G(\sigma)^{\mathbb{C}})$ by

$$P(G^{\mathbb{C}}, G(\sigma)^{\mathbb{C}}) := \{g \in H^1([0,1], G^{\mathbb{C}}) \mid (g(0), g(1)) \in G(\sigma)^{\mathbb{C}}\},$$

where $G(\sigma) := \{(g, \sigma(g)) \mid g \in G\}$. Then any principal orbit of $P(G^{\mathbb{C}}, G^{\mathbb{C}}(\sigma))$-action on $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ also is an infinite dimensional anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators. This fact also is shown by Theorem 1.1 in [21] and Theorem B in [22] because the $G(\sigma)$-action on $G = (G \times G)/\triangle G$ is an action of Hermann type. In contrast let $G = KAN$ be the Iwasawa's decomposition of G, where A is the abelian part and N is the nilpotent part. The inverse images of orbits of the natural action $N^{\mathbb{C}} \curvearrowright G^{\mathbb{C}}/K^{\mathbb{C}}$ by $\pi \circ \phi$ are infinite dimensional anti-Kaehler isoparametric submanifolds which do not have J-diagonalizable shape operators, where π is the natural projection of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}/K^{\mathbb{C}}$ and ϕ is the parallel transport map for $G^{\mathbb{C}}$. See [21] (or Example 2 of Section 4) about the definition of ϕ. Assume that a C^ω-submanifold M in G/K has regular complex focal structure satisfying the following two conditions:

$(*_1)$ *The complex focal structure of M is invariant under the parallel translation with respect to the normal connection of M*

and

$(*_2)$ *The complex focal set of* M *at any point* $x(\in M)$ *consists of infinitely many complex hyperplanes in the complexified normal space* $(T_x^\perp M)^{\mathbf{c}}$ *and the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete. Also, for any unit normal vector* v *of* M*, the nullity spaces of complex focal radii along the normal geodesic* γ_v *with* $\gamma_v'(0) = v$ *span* $\left((\mathrm{Ker}\,A_v \cap \mathrm{Ker}\,R(v))^{\mathbb{C}}\right)^\perp$.

Then each connected component of $(\pi \circ \phi)^{-1}(M^{\mathbb{C}})$ is an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators.

Recently we have proved the following extrinsic homogeneity theorem ([26]):

Let M *be a full irreducible anti-Kaehler isoparametric* C^ω*-submanifold with* J*-diagonalizable shape operators of codimension greater than one in an infinite dimensional anti-Kaehler space. Then* M *is extrinsically homogeneous.*

Let $I_h(V)$ be the group of all holomorphic isometries of an infinite dimensional anti-Kaehler space V and set $H := \{F \in I_h(V) \mid F(M) = M\}$. The extrinsic homogeneity of M in the above result means $Hx = M$ $(x \in M)$. Let $I_h^b(V)$ be the subgroup of $I_h(V)$ generated by one-parameter transformation groups induced by holomorphic Killing vector fields defined entirely on V. Note that $I_h^b(V)$ is a Banach Lie group. Set $H_b := H \cap I_h^b(V)$, which is a Banach Lie subgroup of $I_h^b(V)$. In this paper, we prove the following extrinsic homogeneity theorem similar to the result of [10].

THEOREM A. *Let* M *be a full irreducible anti-Kaehler isoparametric* C^ω*-submanifold with* J*-diagonalizable shape operators of codimension greater than one in the infinite dimensional anti-Kaehler space* V*. Then* $M = H_b x$ *holds for any* $x \in M$.

The assumption of the J-diagonalizability of shape operators is essential in our method to prove Theorem A. It is still an open problem whether any submanifold in the statement of Theorem A is given as a principal orbit of the above $P(G^{\mathbb{C}}, H^{\mathbb{C}} \times K^{\mathbb{C}})$-action or $P(G^{\mathbb{C}}, G(\sigma)^{\mathbb{C}})$-action for some $G,\ H,\ K$ or some $G,\ \sigma$.

2. Basic notions and facts

In this section, we shall recall some basic notions and facts.

2.1. Some notions associated with anti-Kaehler isoparametric submanifolds. Let $(V, \langle\ ,\ \rangle, J)$ be an infinite dimensional anti-Kaehler space and M an anti-Kaehler isoparametric submanifold in V. See [21] and [26] about the definitions of an infinite dimensional anti-Kaehler space and an anti-Kaehler isoparametric submanifold. Denote by $(\langle\ ,\ \rangle, J)$ the anti-Kaehler structure of M and A the shape tensor of M. Fix a unit normal vector v of M. If there exists $X(\neq 0) \in TM$ with $A_v X = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a *J-eigenvalue of* A_v (or a *J-principal curvature of direction* v) and call X a *J-eigenvector* *for* $a+b\sqrt{-1}$. Also, we call the space of all J-eigenvectors for $a+b\sqrt{-1}$ a *J-eigenspace for* $a+b\sqrt{-1}$. The J-eigenspaces are orthogonal to one another and they are J-invariant, respectively. We call the set of all J-eigenvalues of A_v the *J-spectrum of* A_v and denote it by $\mathrm{Spec}_J A_v$. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal system of T_xM. If $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$ is an orthonormal base of T_xM, then we call $\{e_i\}_{i=1}^\infty$ (rather than $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$) a *$J$-orthonormal base*. If there exists a J-orthonormal base consisting of J-eigenvectors of A_v, then we say that A_v *is diagonalized with respect to a J-orthonormal base* (or A_v *is J-diagonalizable*). If, for each $v \in T^\perp M$, the shape operator A_v is J-diagonalizable, then we say that M *has J-diagonalizable shape operators*. Let M be an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators. The shape operators A_v's ($v \in T_x^\perp M$) are simultaneously diagonalized with respect to a J-orthonormal base. Let $\{E_0\} \cup \{E_i \mid i \in I\}$ be the family of distributions on M such that, for each $x \in M$, $\{(E_0)_x\} \cup \{(E_i)_x \mid i \in I\}$ is the set of all common J-eigenspaces of A_v's ($v \in T_x^\perp M$), where $(E_0)_x = \bigcap_{v \in T_x^\perp M} \mathrm{Ker}\, A_v$. For each $x \in M$, T_xM is equal to the closure $\overline{(E_0)_x \oplus \left(\bigoplus_{i\in I} (E_i)_x\right)}$ of $(E_0)_x \oplus \left(\bigoplus_{i\in I} (E_i)_x\right)$. We regard $T_x^\perp M$ ($x \in M$) as a complex vector space by $J_x|_{T_x^\perp M}$ and denote the dual space of the complex vector space $T_x^\perp M$ by $(T_x^\perp M)^{*\mathbf{c}}$. Also, denote by $(T^\perp M)^{*\mathbf{c}}$ the complex vector bundle over M having $(T_x^\perp M)^{*\mathbf{c}}$ as the fibre over x. Let λ_i ($i \in I$) be the section of $(T^\perp M)^{*\mathbf{c}}$ such that $A_v = \mathrm{Re}(\lambda_i)_x(v)\mathrm{id} + \mathrm{Im}(\lambda_i)_x(v)J_x$ on $(E_i)_x$ for any $x \in M$ and any $v \in T_x^\perp M$. We call λ_i ($i \in I$) *J-principal curvatures* of M and E_i ($i \in I$) *J-curvature distributions* of M. The distribution E_i is integrable and each leaf of E_i is a complex sphere. Each leaf of E_i is called a *complex curvature sphere*. It is shown that there uniquely exists a normal vector field n_i of M with $\lambda_i(\cdot) = \langle n_i, \cdot\rangle - \sqrt{-1}\langle Jn_i, \cdot\rangle$. We call n_i ($i \in I$) the *J-curvature normals* of M. Set $l_i^x := (\lambda_i)_x^{-1}(1)$. Then the tangential focal set of M at x is equal to $\bigcup_{i\in I} l_i^x$ ([[21], Theorem 2 (i)]). We call each l_i^x a *complex focal hyperplane of M at x*. Let $\widetilde{v}$ be a parallel normal vector field of M. If $\widetilde{v}_x$ belongs to at least one l_i, then it is called a *focal normal vector field* of M. For a focal normal vector field $\widetilde{v}$, the focal map $f_{\widetilde{v}}$ is defined by $f_{\widetilde{v}}(x) := x + \widetilde{v}_x$ ($x \in M$). The image $f_{\widetilde{v}}(M)$ is called a *focal submanifold* of M, which we denote by $F_{\widetilde{v}}$. For each $x \in F_{\widetilde{v}}$, the inverse image $f_{\widetilde{v}}^{-1}(x)$ is called a focal leaf of M. Denote by T_i^x the complex reflection of order 2 with respect to l_i^x (i.e., the rotation of angle π having l_i^x as the axis), which is an affine transformation of $T_x^\perp M$. Let $\mathcal{W}_x$ be the group generated by T_i^x's ($i \in I$), which is an affine Weyl group. This group $\mathcal{W}_x$ is independent of the choice of $x \in M$ (up to group isomorphicness). Hence we simply denote it by $\mathcal{W}$. We call this group the *complex Coxeter group associated with M*. According to Lemma 3.8 of [23], $\mathcal{W}$ is decomposable (i.e., it is decomposed into a non-trivial product of two discrete complex reflection groups) if and only if there exist two J-invariant linear subspaces P_1 ($\neq \{\mathbf{0}\}$) and P_2 ($\neq \{\mathbf{0}\}$) of $T_x^\perp M$ such that $T_x^\perp M = P_1 \oplus P_2$ (orthogonal direct sum), $P_1 \cup P_2$ contains all J-curvature normals of M at x and that P_i ($i = 1, 2$) contains at least one J-curvature normal of M at x, where $\mathbf{0}$ is the zero vector of $T^{\perp}_x M$. Also, M is irreducible if and only if $\mathcal{W}$ is not decomposable ([[23], Theorem 1]).

We note that the notions described in this subsection are defined also for a finite dimensional anti-Kaehler space similarly.

2.2. Aks-representation. Let L/H be an irreducible anti-Kaehler symmetric space and $(\mathfrak{l}, \tau)$ the anti-Kaehler symmetric Lie algebra associated with L/H. See [24] and [26] about the definitions of these notions. Also, set $\mathfrak{p} := \mathrm{Ker}(\tau + \mathrm{id})$. The space $\mathrm{Ker}(\tau - \mathrm{id})$ is equal to the Lie algebra $\mathfrak{h}$ of H and $\mathfrak{p}$ is identified with $T_{eK}(L/H)$. Denote by Ad_L be the adjoint representation of L. Define $\rho : H \to \mathrm{GL}(\mathfrak{p})$ by $\rho(h) := \mathrm{Ad}_L(h)|_{\mathfrak{p}}$ $(h \in H)$. We call this representation ρ an *aks-representation* (*associated with L/H*). Denote by $\mathrm{ad}_{\mathfrak{h}}$ the adjoint representation of $\mathfrak{h}$. Let $\mathfrak{a}_s$ be a maximal split abelian subspace of $\mathfrak{p}$ (see [35] or [31] about the definition of a maximal split abelian subspace) and $\mathfrak{p} = \mathfrak{p}_0 + \sum_{\alpha \in \triangle_+} \mathfrak{p}_\alpha$ the root space decomposition with respect to $\mathfrak{a}_s$, where the space $\mathfrak{p}_\alpha$ is defined by $\mathfrak{p}_\alpha := \{X \in \mathfrak{p} \mid \mathrm{ad}_{\mathfrak{l}}(a)^2(X) = \alpha(a)^2 X \text{ for all } a \in \mathfrak{a}_s\}$ $(\alpha \in \mathfrak{a}_s^*)$ and $\triangle_+$ is the positive root system of the root system $\triangle := \{\alpha \in \mathfrak{a}_s^* \mid \mathfrak{p}_\alpha \neq \{0\}\}$ under some lexicographic ordering of $\mathfrak{a}_s^*$. Set $\mathfrak{a} := \mathfrak{p}_0$ $(\supset \mathfrak{a}_s)$, $j := J_{eK}$ and $\langle\ ,\ \rangle_0 := \langle\ ,\ \rangle_{eH}$. Note that $(\mathfrak{p}, j, \langle\ ,\ \rangle_0)$ is a (finite dimensional) anti-Kaehler space. It is shown that $\langle\ ,\ \rangle_0|_{\mathfrak{a}_s \times \mathfrak{a}_s}$ is positive (or negative) definite, $\mathfrak{a} = \mathfrak{a}_s \oplus j\mathfrak{a}_s$ and $\langle\ ,\ \rangle_0|_{\mathfrak{a}_s \times j\mathfrak{a}_s} = 0$. Note that $\mathfrak{p}_\alpha = \{X \in \mathfrak{p} \mid \mathrm{ad}_{\mathfrak{l}}(a)^2(X) = \alpha^{\mathbb{C}}(a)^2 X \text{ for all } a \in \mathfrak{a}\}$ holds for each $\alpha \in \triangle_+$, where $\alpha^{\mathbb{C}}$ is the complexification of $\alpha : \mathfrak{a}_s \to \mathbf{R}$ (which is a complex linear function over $\mathfrak{a}_s^{\mathbb{C}} = \mathfrak{a}$) and $\alpha^{\mathbb{C}}(a)^2 X$ means $\mathrm{Re}(\alpha^{\mathbb{C}}(a)^2)X + \mathrm{Im}(\alpha^{\mathbb{C}}(a)^2)jX$. Let $l_\alpha := (\alpha^{\mathbb{C}})^{-1}(0)$ $(\alpha \in \triangle)$ and $D := \mathfrak{a} \setminus \bigcup_{\alpha \in \triangle_+} l_\alpha$. Elements of D are said to be *regular*. Take $x \in D$ and let M be the orbit of the aks-representation ρ through x. From $x \in D$, M is a principal orbit of this representation. Denote by A the shape tensor of M. Take $v \in T^{\perp}_x M (= \mathfrak{a})$. Then we have $T_x M = \sum_{\alpha \in \triangle_+} \mathfrak{p}_\alpha$ and $A_v|_{\mathfrak{p}_\alpha} = -\frac{\alpha^{\mathbb{C}}(v)}{\alpha^{\mathbb{C}}(x)}\mathrm{id}$ $(\alpha \in \triangle_+)$. Let $\widetilde{v}$ be the parallel normal vector field of M with $\widetilde{v}_x = v$. Then we can show that $A_{\widetilde{v}_{\rho(h)(x)}}|_{\rho(h)_{*x}(\mathfrak{p}_\alpha)} = -\frac{\alpha^{\mathbb{C}}(v)}{\alpha^{\mathbb{C}}(x)}\mathrm{id}$ for any $h \in H$. Hence M is an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators.

3. Homogeneity theorem

In this section, we shall recall the extrinsic homogeneity theorem for an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators, which was obtained in [26], and the outline of its proof. Let M be an irreducible anti-Kaehler isoparametric submanifold of codimension greater than one in an infinite dimensional anti-Kaehler space $(V, \langle\ ,\ \rangle, J)$. Denote by the same symbol $(\langle\ ,\ \rangle, J)$ the anti-Kaehler structure of M. Assume that M has J-diagonalizable shape operators. We use the notations in Subsection 2.1. Denote by l_i^x the complex focal hyperplane $(\lambda_i)_x^{-1}(1)$ of M at x. Also set $(l_i^x)' := (\lambda_i)_x^{-1}(0)$.

Fix $x_0 \in M$. Set $l_i := l_i^{x_0}$ and $l_i' := (l_i^{x_0})'$. Let $Q(x_0)$ be the set of all points of M connected with x_0 by a piecewise smooth curve in M each of whose smooth segments is contained in some complex curvature sphere (which may depend on the smooth segment). By using the generalized Chow theorem (see Theorem D of [13]), we showed the following fact.

LEMMA 3.1 ([28]). *The set $Q(x_0)$ is dense in M.*

Here we note that the generalized Chow's theorem is valid because the base manifold M is a Hilbert manifold even if the metric of M is a pseudo-Riemannian metric. For each complex affine subspace P of $T^{\perp}_{x_0}M$, define I_P by

$$I_P := \begin{cases} \{i \in I \mid (n_i)_{x_0} \in P\} & (\mathbf{0} \notin P) \\ \{i \in I \mid (n_i)_{x_0} \in P\} \cup \{0\} & (\mathbf{0} \in P)\,. \end{cases}$$

Define a distribution D_P on M by $D_P := \underset{i \in I_P}{\oplus} E_i$, which is integrable. Denote by L^P_x the leaf through x of the foliation given by D_P, and L^i_x the leaf through x of the foliation given by E_i. According to Lemma 4.3 of [26], if $\mathbf{0} \notin P$, then I_P is finite and $(\underset{i \in I_P}{\cap} l_i) \setminus (\underset{i \in I \setminus I_P}{\cup} l_i) \neq \emptyset$, and, if $\mathbf{0} \in P$, then I_P is infinite or $I_P = \{0\}$ and $(\underset{i \in I_P \setminus \{0\}}{\cap} l_i') \setminus (\underset{i \in I \setminus I_P}{\cup} l_i') \neq \emptyset$, where $\underset{i \in I_P \setminus \{0\}}{\cap} l_i'$ means $T^{\perp}_{x_0}M$ when $I_P = \{0\}$. Set $(W_P)_x := x + (D_P)_x \oplus \mathrm{Span}_{\mathbb{C}}\{(n_i)_x \mid i \in I_P \setminus \{0\}\}$ $(x \in M)$. Let $\gamma : [0,1] \to M$ be a piecewise smooth curve. Throughout this section, we assume that the domains of all piecewise smooth curves are equal to $[0,1]$. If $\dot\gamma(t) \perp (D_P)_{\gamma(t)}$ for each $t \in [0,1]$, then γ is said to be *perpendicular to D_P* (or *D_P-perpendicular*). Fix $i_0 \in I \cup \{0\}$ and $x_0 \in M$. For each geodesic $\gamma : [0,1] \to L^{i_0}_{x_0}$ in $L^{i_0}_{x_0}$, we ([26]) constructed a one-parameter family $\{F_{\gamma|_{[0,t]}}\}_{t \in [0,1]}$ of holomorphic isometries of V satisfying $F_{\gamma|_{[0,t]}}(\gamma(0)) = \gamma(t)$ and $(F_{\gamma|_{[0,t]}})_{*\gamma(0)}|_{T^{\perp}_{\gamma(0)}M} = \tau^{\perp}_{\gamma|_{[0,t]}}$ $(t \in [0,1])$, where $\tau^{\perp}_{\gamma|_{[0,t]}}$ is the parallel translation along $\gamma|_{[0,t]}$ with respect to the normal connection of M. From Proposition 4.6 of [26], the following fact holds.

LEMMA 3.2. *The holomorphic isometry $F_{\gamma|_{[0,t]}}$ preserves M invariantly* (*i.e.*, *$F_{\gamma|_{[0,t]}}(M) = M$*). *Furthermore, it preserves E_i $(i \in I)$ invariantly* (*i.e.*, *$(F_{\gamma|_{[0,t]}})_*(E_i) = E_i$*).

By using Lemmas 3.1 and 3.2, we can prove the following fact (see the proof of Theorem A in [26]).

THEOREM 3.3. *The submanifold M is extrinsically homogeneous*, *that is*, *$Hx = M$ $(x \in M)$ holds*, *where $H := \{F \in I_h(V) \mid F(M) = M\}$.*

4. The affine root system associated with an irreducible anti-Kaehler isoparametric submanifold

In this section, we shall first recall the notions of the Weyl group, the affine Weyl group and the root system associated with a certain kind of family of the affine hyperplanes in a finite dimensional Euclidean affine space $\mathbb{E}$. Denote by $(\mathbb{V}, \langle\ ,\ \rangle)$ the Euclidean vector space associated with $\mathbb{E}$. Let $\mathcal{H}$ be a family of affine hyperplanes in $\mathbb{E}$ and $\mathcal{W}_{\mathcal{H}}$ the group generated by the (orthogonal) reflections with respect to members of $\mathcal{H}$. Assume that unit normal vectors of the members of $\mathcal{H}$ span $\mathbb{V}$ and that $\mathcal{H}$ is invariant under $\mathcal{W}_{\mathcal{H}}$. Then $\mathcal{H}$ is a finite family of affine hyperplanes having a common point or a finite family of equidistant infinite parallel families of affine hyperplanes. In the first case, $\mathcal{W}_{\mathcal{H}}$ is a Weyl group and hence $\mathcal{H}$ is described as

$$\mathcal{H} = \{\alpha^{-1}(0) \mid \alpha \in \triangle\} \tag{4.1}$$

for some root system $\triangle(\subset \mathbb{V}^*)$ by translating $\mathcal{H}$ suitably. In the second case, $\mathcal{W}$ is an affine Weyl group and hence $\mathcal{H}$ is described as

$$\mathcal{H} = \{\alpha^{-1}(ka_\alpha) \mid \alpha \in \triangle \ \&\ k \in \mathbb{Z}\} \tag{4.2}$$

for some root system $\triangle(\subset \mathbb{V}^*)$ and some positive constants a_α by translating and homothetically transforming $\mathcal{H}$ suitably. Set $l_{\alpha,k} := \alpha^{-1}(ka_\alpha)$ $((\alpha, k) \in \triangle \times \mathbb{Z})$. Define a system $\mathcal{R}$ by

$$\begin{aligned} \mathcal{R} := &\{(v_\alpha, l_{\alpha,k}) \in \mathbb{V} \times \mathcal{H} \mid (\alpha, k) \in \triangle \times \mathbb{Z}\} \\ &\cup \left\{ \left(\frac{1}{2}v_\alpha, l_{\alpha,k}\right) \in \mathbb{V} \times \mathcal{H} \,\middle|\, (\alpha, k) \in \triangle' \times \mathbb{Z} \right\}, \end{aligned} \tag{4.3}$$

where v_α is the vector of $\mathbb{V}$ defined by $\alpha(\bullet) = \langle v_\alpha, \bullet\rangle$ and $\triangle'$ is a subset of $\triangle$. If $\mathcal{R}$ is $\mathcal{W}$-invariant, then $\mathcal{R}$ is a root system in the sense of I.G. Macdonald [27] (see Definition 7.3 of [10] also). This root system $\mathcal{R}$ is called a *root system associated with* $\mathcal{H}$. In particular, if $\mathcal{W}$ is infinite, then it is called an *affine root system associated with* $\mathcal{H}$. If $\triangle' = \emptyset$ (resp. $\triangle' \neq \emptyset$), then $\mathcal{R}$ is said to be *reduced* (resp. *non-reduced*). Also, if $\mathcal{W}$ is irreducible (resp. reducible), then $\mathcal{R}$ is said to be *irreducible* (resp. *reducible*). Assume that $\mathcal{R}$ is a reduced irreducible affine root system of rank greater than one. Then the Dynkin diagram of $\mathcal{R}$ is defined as follows. Let Π be the simple root system of $\triangle$ with respect to some lexicographic ordering of V^* and δ be the highest root of $\triangle$ with respect to the lexicographic ordering. If $\mathcal{W}$ is finite (resp. infinite), then the family $\{l_{\alpha,0} \mid \alpha \in \Pi\}$ (resp. $\{l_{\alpha,0} \mid \alpha \in \Pi\} \cup \{l_{\delta,1}\}$) is the whole of walls of an alcove C of $\mathcal{W}$-action. For any element $(v_\alpha, l_{\alpha,k})$ and $(v_{\alpha'}, l_{\alpha',k'})$ of $\mathcal{R}$, $\dfrac{||v_\alpha||}{||v_{\alpha'}||} = 1,\ 2,\ \dfrac{1}{2},\ 3$ or $\dfrac{1}{3}$ holds. We assign a white circle to each $\alpha \in \Pi$ or $\Pi \cup \{\delta\}$ and link the white circles corresponding to α and α' $(\alpha, \alpha' \in \Pi$ or $\Pi \cup \{\delta\})$ by $1, 2$ or 3 edges in correspondence to $\dfrac{||v_\alpha||}{||v_{\alpha'}||} = 1,\ 2^{\pm 1}$ or $3^{\pm 1}$. Also, in the case where $\dfrac{||v_\alpha||}{||v_{\alpha'}||} = 2^{\pm 1}$ or $3^{\pm 1}$, we add the arrow pointing to the white circle corresponding to the shorter length one of α and α' to the 2 or 3 edges. The diagram obtained thus is called the *Dynkin diagram* of $\mathcal{R}$. All of reduced irreducible affine root systems of rank greater than one are $(\widetilde{A}_r)\ (r \geq 2)$, $(\widetilde{B}_r)\ (r \geq 3)$, $(\widetilde{B}^v_r)\ (r \geq 3)$, $(\widetilde{C}_r)\ (r \geq 2)$, $(\widetilde{C}^v_r)\ (r \geq 2)$, $(\widetilde{D}_r)\ (r \geq$

4), $(\widetilde{E}_6)$, $(\widetilde{E}_7)$, $(\widetilde{E}_8)$, $(\widetilde{F}_4)$, $(\widetilde{F}_4^v)$, $(\widetilde{G}_2)$ and $(\widetilde{G}_2^v)$. See Table 1 of [10] in detail. Assume that $\mathcal{R}$ (given by (4.3)) is a non-reduced irreducible affine root system of rank greater than one. Define subsystems $\mathcal{R}_{\mathrm{red}}$ and $\mathcal{R}_{\mathrm{red}'}$ by

$$\begin{aligned}\mathcal{R}_{\mathrm{red}} &:= \{(v_\alpha, l_{\alpha,k}) \in \mathbb{V} \times \mathcal{H} \mid (\alpha, k) \in (\triangle \setminus \triangle') \times \mathbb{Z}\} \\ &\cup \left\{\left(\frac{1}{2}v_\alpha, l_{\alpha,k}\right) \in \mathbb{V} \times \mathcal{H} \,\middle|\, (\alpha, k) \in \triangle' \times \mathbb{Z}\right\}\end{aligned} \tag{4.4}$$

and

$$\mathcal{R}_{\mathrm{red}'} := \{(v_\alpha, l_{\alpha,k}) \in \mathbb{V} \times \mathcal{H} \mid (\alpha, k) \in \triangle \times \mathbb{Z}\}\,. \tag{4.5}$$

Then the Dynkin diagram of $\mathcal{R}$ is defined as follows. We add the second smaller concentric white circles to the white circles corresponding to α's ($\alpha \in \Pi \cap \triangle'$ or $(\Pi \cup \{\delta\}) \cap \triangle'$) in the Dynkin diagram of $\mathcal{R}_{\mathrm{red}}$. The diagram obtained thus is called the *Dynkin diagram* of $\mathcal{R}$. All of non-reduced irreducible affine root systems of rank greater than one are $(\widetilde{B}_r, \widetilde{B}_r^v)$ $(r \geq 3)$, $(\widetilde{C}_r^v, \widetilde{C}_r')$ $(r \geq 2)$, $(\widetilde{C}_r', \widetilde{C}_r)$ $(r \geq 2)$, $(\widetilde{C}_r^v, \widetilde{C}_r)$ $(r \geq 2)$ and $(\widetilde{C}_2, \widetilde{C}_2^v)$, where these notations denote the pairs of types of $\mathcal{R}_{\mathrm{red}}$ and $\mathcal{R}_{\mathrm{red}'}$. See Table 2 of [10] in detail.

Next we shall introduce the notion of the root system associated with an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators. Let M be an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators in an anti-Kaehler space V, where V may be of finite dimension. We use the notations in the previous section. Let $V = V_- \oplus V_+$ be the orthogonal decomposition of V such that $\langle\ ,\ \rangle|_{V_-\times V_-}$ (resp. $\langle\ ,\ \rangle|_{V_+\times V_+}$) is negative (resp. positive) definite and that $JV_- = V_+$. Note that such a decomposition is unique. Denote by ∇ and $\widetilde{\nabla}$ the Riemannian connections of M and V, respectively. Since the complex Coxeter group associated with M permutes $\{l_i^x \mid i \in I\}$ and it is discrete, there exist a finite family $\{\mu_\beta^x \mid \beta \in B\}$ of complex linear functions over the normal space $T_x^\perp M$ (regarded as a complex linear space by J_x) and a finite family $\{b_\beta \mid \beta \in B\}$ of complex numbers such that $\{(\mu_\beta^x)^{-1}(1 + b_\beta j) \mid \beta \in B,\ j \in \mathbb{Z}\}$ is equal to $\{l_i^x \mid i \in I\}$. Set $\lambda^x_{(\beta,j)} := \frac{1}{1+b_\beta j}\mu_\beta^x$. Note that $(\lambda^x_{(\beta,j)})^{-1}(1) = (\mu_\beta^x)^{-1}(1 + b_\beta j)$. Define sections $\lambda_{(\beta,j)}$ of $(T^\perp M)^{*\mathbb{C}}$ by assigning $\lambda^x_{(\beta,j)}$ to each $x \in M$. Set $B_0 := \{\beta \in B \mid b_\beta = 0\}$. Then the set of all J-principal curvatures of M is equal to

$$\{\lambda_{(\beta,j)} \mid (\beta, j) \in (B \setminus B_0) \times \mathbb{Z}\} \cup \{\lambda_{(\beta,0)} \mid \beta \in B_0\}\,.$$

Hence, we have $I = (B_0 \times \{0\}) \cup ((B \setminus B_0) \times \mathbb{Z})$. Note that $B = B_0$ when V is of finite dimension. Let TM_+ be the half-dimensional subdistribution of the tangent bundle TM such that $\langle\ ,\ \rangle|_{TM_+\times TM_+}$ is positive definite and that $\langle TM_+, JTM_+\rangle = 0$, and set $TM_- := JTM_+$. Note that such subdistributions are determined uniquely. Similarly, we define the half-dimensional subdistributions $T^\perp M_\pm$ (resp. $(E_i)_\pm$) of the normal bundle $T^\perp M$

(resp. J-curvature distributions E_i's ($i \in I \cup \{0\}$)). Clearly we have

$$TM_- = \overline{(E_0)_- \oplus \left(\underset{i \in I}{\oplus} (E_i)_- \right)}$$

and

$$TM_+ = \overline{(E_0)_+ \oplus \left(\underset{i \in I}{\oplus} (E_i)_+ \right)}\,.$$

Fix $x_0 \in M$. Set $\mathfrak{b} := T^{\perp}_{x_0}M$ and $\mathfrak{b}_{\pm} := (T^{\perp}M_{\pm})_{x_0}$. Clearly we have $\mathfrak{b}_- = J_{x_0}\mathfrak{b}_+$ and $\mathfrak{b} = \mathfrak{b}_+ + \mathfrak{b}_-(\approx \mathfrak{b}_+^{\mathbb{C}})$.

LEMMA 4.1. *Let i_1 and i_2 be elements of I such that $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ are linearly independent over $\mathbb{C}$. Set $\mathfrak{b}' := \mathrm{Span}_{\mathbb{R}}\{(n_{i_1})_{x_0}, (n_{i_2})_{x_0}\}$. Then we have $J_{x_0}\mathfrak{b}' \cap \mathfrak{b}' = \{0\}$.*

PROOF. Since $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ are linearly independent over $\mathbb{C}$, there exists a complex affine line P of $T^{\perp}_{x_0}M$ which passes through $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ but does not pass through 0. Then $L^P_{x_0}(\subset (W_P)_{x_0})$ is a (finite dimensional) anti-Kaehler isoparametric submanifold with J-digonalizable shape operators of complex codimension greater two. Since the complex codimension of $L^P_{x_0}$ is equal to two, it is irreducible or the product of two irreducible anti-Kaehler isoparametric submanifolds $L^{P_i}_{x_0}(\subset (W_{P_i})_{x_0})$ ($i = 1, 2$) with J-diagonalizable shape operators of complex codimension one, where we note that $(W_P)_{x_0} = (W_{P_1})_{x_0} \oplus (W_{P_2})_{x_0}$. Also, note that $L^{P_i}(\subset (W_{P_i})_{x_0})$ ($i = 1, 2$) are complex spheres because they are of complex codimension one.

First we consider the case where $L^P_{x_0}$ is irreducible. Then, according to Theorem 4.4 of [26], $L^P_{x_0}$ is a principal orbit of the aks-representation associated with an irreducible anti-Kaehler symmetric space of complex rank greater than one. Denote by L/H this irreducible anti-Kaehler symmetric space. We use the notations in Subsection 2.2. Let $L^P_{x_0} = \rho(H) \cdot w$, where ρ is the aks-representation associated with L/H and w is the element of $\mathfrak{p}$ identified with x_0. Let $\mathfrak{a}_s$ be the maximal split abelian subspace of $\mathfrak{p}$ containing w and $\mathfrak{a}$ the Cartan subspace of $\mathfrak{p}$ containing $\mathfrak{a}_s$. The space $\mathfrak{a}$ is identified with the normal space of $T^{\perp}_{x_0}L^P_{x_0}$ of $L^P_{x_0}(\subset (W_P)_{x_0})$ at x_0. Let $\triangle_+$ be the positive root system of the root system $\triangle$ (with respect to $\mathfrak{a}_s$) under some lexicographic ordering of $\mathfrak{a}_s^*$. For each $\alpha \in \triangle_+$, define the section λ_α of the **C**-dual bundle $(T^{\perp}L^P_{x_0})^*$ of $T^{\perp}L^P_{x_0}$ by

$$(\lambda_\alpha)_{\rho(h)(w)} := -\frac{\alpha^{\mathbb{C}} \circ \rho(h)^{-1}_{*w}}{\alpha^{\mathbb{C}}(w)} \quad (h \in H)\,.$$

The set of all J-principal curvatures of $L^P_{x_0}$ is equal to $\{\lambda_\alpha \mid \alpha \in \triangle_+\}$. Let n_α be the J-curvature normal corresponding to λ_α. Since $(\lambda_\alpha)_w = -\frac{\alpha^{\mathbb{C}}}{\alpha^{\mathbb{C}}(w)}$, we have $(n_\alpha)_{x_0} \in \mathfrak{a}_s$ for any $\alpha \in \triangle_+$. This fact implies that $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ belong to $\mathfrak{a}_s$. Hence we obtain

$J_{x_0}\mathfrak{b}' \cap \mathfrak{b}' = \{0\}$.

Next we consider the case of $L^P_{x_0} = L^{P_1}_{x_0} \times L^{P_2}_{x_0} (\subset (W_{P_1})_{x_0} \oplus (W_{P_2})_{x_0})$. Then one of $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ belongs to $T^\perp_{x_0} L^{P_1}_{x_0}$ and another belongs to $T^\perp_{x_0} L^{P_2}_{x_0}$. From this fact, it follows that $J_{x_0}\mathfrak{b}' \cap \mathfrak{b}' = \{0\}$. This completes the proof. $\square$

Define a linear subspace $\mathfrak{b}_{\mathbb{R}}$ of $\mathfrak{b}$ by

$$\mathfrak{b}_{\mathbb{R}} := \mathrm{Span}_{\mathbb{R}}\{(n_i)_{x_0} \mid i \in I\}\,.$$

From Lemma 4.1, it follows that $J_{x_0}\mathfrak{b}_{\mathbb{R}} \cap \mathfrak{b}_{\mathbb{R}} = \{0\}$. Furthermore, since M is full, $\mathfrak{b}_{\mathbb{R}}$ is a real form of $\mathfrak{b}$. For simplicity denote $l_i^{x_0}$ by l_i. It is easy to show that $l_i \cap \mathfrak{b}_{\mathbb{R}} = ((\lambda_i)_{x_0}|_{\mathfrak{b}_{\mathbb{R}}})^{-1}(1)$. Denote by $l_i^{\mathbb{R}}$ this affine hyperplane $l_i \cap \mathfrak{b}_{\mathbb{R}}$ of $\mathfrak{b}_{\mathbb{R}}$. Let $\mathcal{W}_{\mathbb{R}}$ be the group generated by the reflections with respect to $l_i^{\mathbb{R}}$'s $(i \in I)$. It is clear that

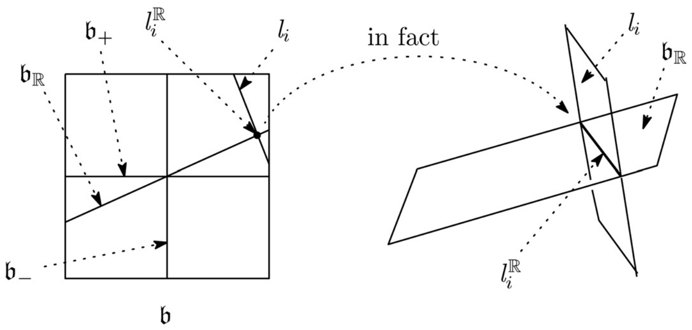

FIGURE 1. Generators of the affine Weyl group associated to M

$\mathcal{W}_{\mathbb{R}}$ is isomorphic to $\mathcal{W}$. Hence, $\mathcal{W}_{\mathbb{R}}$ is an affine Weyl group. Let B' be the set of all elements β's of B satisfying the following condition:

> There exists $\hat\beta \in B$ such that $(n_{(\beta,0)})_{x_0}$ and $(n_{(\hat\beta,0)})_{x_0}$ are linearly independent over $\mathbb{C}$, for the complex affine line P through $(n_{(\beta,0)})_{x_0}$ and $(n_{(\hat\beta,0)})_{x_0}$, the root system associated with $L^P_{x_0} (\subset W_P)$ is of type (BC_2) and the $\frac{1}{2}$-multiple of the root $\alpha \in \triangle_+$ ($\triangle_+$: as in the proof of Lemma 4.1) corresponding to β also belongs to $\triangle_+$.

Fix $Z_0 \in \bigcap_{\beta\in B} l^{\mathbb{R}}_\beta$. There exists a root system $\triangle_M$ $(\subset (\mathfrak{b}_{\mathbb{R}})^*)$ such that

$$\begin{aligned}&\left\{-\frac{\alpha}{\alpha(Z_0)}\,\middle|\,\alpha \in (\triangle_M)_+\right\} \cup \left\{-\frac{\alpha}{2\alpha(Z_0)}\,\middle|\,\alpha \in (\triangle_M)_+ \text{ such that } \frac{\alpha}{2} \in (\triangle_M)_+\right\}\\ &= \{\lambda_{(\beta,0)}|_{\mathfrak{b}_{\mathbb{R}}} \mid \beta \in B\} \cup \left\{\frac{1}{2}\lambda_{(\beta,0)}|_{\mathfrak{b}_{\mathbb{R}}}\,\middle|\,\beta \in B'\right\}\,,\end{aligned}$$

where $(\triangle_M)_+$ is the positive root system of $\triangle_M$ under a lexicographic ordering of $(\mathfrak{b}_{\mathbb{R}})^*$. When $\alpha(\in(\triangle_M)_+)$ corresponds to $\beta\in B$ (i.e., $-\frac{\alpha}{\alpha(Z_0)}=\lambda_{(\beta,0)}|_{\mathfrak{b}_{\mathbb{R}}}$), we denote $\lambda_{(\beta,j)}$, $n_{(\beta,j)}$, $l_{(\beta,j)}$ and b_β by $\lambda_{(\alpha,j)}$, $n_{(\alpha,j)}$, $l_{(\alpha,j)}$ and b_α, respectively. Hence we may denote $(\triangle_M)_+\times\mathbb{Z}$ by I. In the sequel, I denotes $(\triangle_M)_+\times\mathbb{Z}$. Define a system $\mathcal{R}_M$ by

$$\mathcal{R}_M:=\{((n_{(\alpha,0)})_{x_0},\ l^{\mathbb{R}}_{(\alpha,j)})\mid\alpha\in(\triangle_M)_+,\ j\in\mathbb{Z}\}.$$

This root system $\mathcal{R}_M$ is a root system associated with $\mathcal{H}$. In particular, if $B_0\neq B$, then it is an affine root system associated with $\mathcal{H}$.

DEFINITION. We call $\mathcal{R}_M$ *the root system associated with* M. In particular, if $B\neq B_0$, then we call $\mathcal{R}_M$ *the affine root system associated with* M.

For $\mathcal{R}_M$, the following fact holds.

PROPOSITION 4.2. *If M is irreducible, then $\mathcal{W}$ is infinite and hence $\mathcal{R}_M$ is the affine root system.*

PROOF. To show this statement, we suffice to show that $B\neq B_0$. Suppose that $B=B_0$. Then we have

$$T_{x_0}M=(E_0)_{x_0}\oplus\left(\underset{\beta\in B}{\oplus}(E_{(\beta,0)})_{x_0}\right).$$

This implies that M is the cylinder over a finite dimensional anti-Kaehler isoparametric submanifold of J-diagonalizable shape operators. This contradicts the fact that M is irreducible. Hence we obtain $B\neq B_0$. □

EXAMPLE 1. Let (L,H) be an anti-Kaehler symmetric pair and $\rho:H\to\mathrm{GL}(\mathfrak{p})$ the aks-representation associated with (L,H), where $\mathfrak{p}$ is as in Subsection 2.2. We use the notations in Subsection 2.2. Let M be the orbit of $\rho(H)$-action through a regular element $x_0(\in\mathfrak{a})$ and V an infinite dimensional anti-Kaehler space. Then the cylinder $M\times V(\subset\mathfrak{p}\times V)$ over M is a (reducible) anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators. The set $\mathcal{J}PC_{M\times V}$ of all J-principal curvatures of $M\times V$ is given by

$$\mathcal{J}PC_{M\times V}=\left\{-\frac{\widetilde{\alpha^{\mathbb{C}}}}{\alpha(x_0)}\,\middle|\,\alpha\in\triangle_+\right\},$$

where $\widetilde{\alpha^{\mathbb{C}}}$ is the parallel section of $(T^\perp M)^{*\mathbb{C}}$ with $\left(\widetilde{\alpha^{\mathbb{C}}}\right)_{x_0}=\alpha^{\mathbb{C}}$. Hence we have

$$\mathcal{H}=\{\alpha^{-1}(-\alpha(x_0))\mid\alpha\in\triangle_+\},$$

and

$$\mathcal{R}_M=\{((n_\alpha)_{x_0},\ \alpha^{-1}(-\alpha(x_0)))\mid\alpha\in\triangle_+\},$$

where $(n_\alpha)_{x_0}$ is the element of $\mathfrak{a}_s$ with $\alpha(\bullet)=\langle(n_\alpha)_{x_0},\bullet\rangle$. Also, we have $\triangle_M=\triangle$. Thus both the types of $\triangle_M$ and $\mathcal{R}_M$ are equal to that of $\triangle$.

EXAMPLE 2. Let G/K be a symmetric space of non-compact type and $H \curvearrowright G/K$ a Hermann type action (i.e., H is a symmetric subgroup of G). Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H, and θ (resp. σ) the involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$ (resp. $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$). Denote by the same symbols the involutions of $\mathfrak{g}$ induced from θ and σ. Set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$ and $\mathfrak{q} := \mathrm{Ker}(\sigma + \mathrm{id})$. Assume that θ and σ commute. Then we have $\mathfrak{p} = \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$. Take a maximal abelian $\mathfrak{b}'$ of $\mathfrak{p} \cap \mathfrak{q}$. Let $\mathfrak{p} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}') + \sum_{\alpha \in \triangle'_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{b}'$, where $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}')$ is the centralizer of $\mathfrak{b}'$ in $\mathfrak{p}$, $\triangle'_+$ is the positive root system of the root system $\triangle' := \{\alpha \in \mathfrak{b}'^* \mid \exists X (\neq 0) \in \mathfrak{p}$ such that $\mathrm{ad}(b)^2(X) = \alpha(b)^2 X \ (\forall b \in \mathfrak{b}')\}$ under some lexicographic ordering of $\mathfrak{b}'^*$ and $\mathfrak{p}_\alpha := \{X \in \mathfrak{p} \mid \mathrm{ad}(b)^2(X) = \alpha(b)^2 X \ (\forall b \in \mathfrak{b}')\}$ $(\alpha \in \triangle'_+)$. Also, let $\triangle'^V_+ := \{\alpha \in \triangle'_+ \mid \mathfrak{p}_\alpha \cap \mathfrak{q} \neq \{0\}\}$ and $\triangle'^H_+ := \{\alpha \in \triangle'_+ \mid \mathfrak{p}_\alpha \cap \mathfrak{h} \neq \{0\}\}$. Also, let $\phi : H^0([0,1], \mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ be the parallel transport map for $G^{\mathbb{C}}$ and $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ the natural projection. See [K2] about the definition of the parallel transport map for $G^{\mathbb{C}}$. Let $H^{\mathbb{C}} \curvearrowright G^{\mathbb{C}}/K^{\mathbb{C}}$ be the complexified action of the H-action, M the principal orbit of the $H^{\mathbb{C}}$-action through $\mathrm{Exp}\, Z_0$ and $\widetilde{M}$ a connected component of $(\pi \circ \phi)^{-1}(M)$, where Z_0 is a point of $\mathfrak{b} := \mathfrak{b}'^{\mathbb{C}} (= T^\perp_{eK^{\mathbb{C}}} M)$ (e : the identity element of $G^{\mathbb{C}}$) and Exp is the exponential map of $G^{\mathbb{C}}/K^{\mathbb{C}}$ at $eK^{\mathbb{C}}$. Note that $\widetilde{M}$ is a principal orbit of the $P(G^{\mathbb{C}}, H^{\mathbb{C}} \times K^{\mathbb{C}})$-action stated in Introduction. This submanifold $\widetilde{M}$ is an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators in $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$. In particular, if G/K is irreducible, then $\widetilde{M}$ is (extrinsically) irreducible. Fix $u_0 \in (\pi \circ \phi)^{-1}(x_0) \cap \widetilde{M}$. By the similar argument to Section 4 of [K6], it is shown that the set $\mathcal{J}PC_{\widetilde{M}}$ of all J-principal curvatures of $\widetilde{M}$ is given by

$$
\begin{aligned}
\mathcal{J}PC_{\widetilde{M}} = & \left\{ -\frac{\widetilde{\alpha^{\mathbb{C}}}}{\alpha(Z_0) + k\pi\sqrt{-1}} \,\middle|\, \alpha \in \triangle'^V_+, \ k \in \mathbb{Z} \right\} \\
& \cup \left\{ -\frac{\widetilde{\alpha^{\mathbb{C}}}}{\alpha(Z_0) + (k+\frac{1}{2})\pi\sqrt{-1}} \,\middle|\, \alpha \in \triangle'^H_+, \ k \in \mathbb{Z} \right\},
\end{aligned} \tag{4.6}
$$

where $\widetilde{\alpha^{\mathbb{C}}}$ is the parallel section of $(T^\perp \widetilde{M})^{*\mathbb{C}}$ with $\left(\widetilde{\alpha^{\mathbb{C}}}\right)_{u_0} = \alpha^{\mathbb{C}}$. Here the normal space $T^\perp_{u_0}\widetilde{M}$ of $\widetilde{M}$ at u_0 is identified with $T^\perp_{x_0} M (= \mathfrak{b})$ through $(\pi \circ \phi)_{*u_0}$. Define a complex linear function $\lambda_{(\alpha,0)}$ over $\mathfrak{b} (= \mathfrak{b}'^{\mathbb{C}})$ by $\lambda_{(\alpha,0)} := -\frac{\widetilde{\alpha^{\mathbb{C}}}}{\alpha(Z_0)}$, which is a J-principal curvature of $\widetilde{M}$. Let $n_{(\alpha,0)}$ be the J-curvature normal of $\widetilde{M}$ corresponding to $\lambda_{(\alpha,0)}$. From (4.6), we have

$$
\begin{aligned}
\mathcal{H} = & \left\{ \alpha^{-1}(-\alpha(Z_0) + k\pi\sqrt{-1}) \,\middle|\, \alpha \in \triangle'^V_+, \ k \in \mathbb{Z} \right\} \\
& \cup \left\{ \alpha^{-1}(-\alpha(Z_0) + (k+\frac{1}{2})\pi\sqrt{-1}) \,\middle|\, \alpha \in \triangle'^H_+, \ k \in \mathbb{Z} \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_M = &\left\{\left((n_{(\alpha,0)})_{u_0},\ \alpha^{-1}(-\alpha(Z_0)+k\pi\sqrt{-1})\right)\,\middle|\,\alpha\in\triangle'^{V}_{+},\ k\in\mathbb{Z}\right\}\\
\cup&\left\{\left((n_{(\alpha,0)})_{u_0},\ \alpha^{-1}(-\alpha(Z_0)+(k+\frac{1}{2})\pi\sqrt{-1})\right)\,\middle|\,\alpha\in\triangle'^{H}_{+},\ k\in\mathbb{Z}\right\},\\
\cup&\left\{\left(\frac{1}{2}(n_{(\alpha,0)})_{u_0},\ \alpha^{-1}(-\alpha(Z_0)+k\pi\sqrt{-1})\right)\,\middle|\,\alpha\in(\triangle'^{V}_{+})',\ k\in\mathbb{Z}\right\},\\
\cup&\left\{\left(\frac{1}{2}(n_{(\alpha,0)})_{u_0},\ \alpha^{-1}(-\alpha(Z_0)+(k+\frac{1}{2})\pi\sqrt{-1})\right)\,\middle|\,\alpha\in(\triangle'^{H}_{+})',\ k\in\mathbb{Z}\right\},
\end{aligned}
$$

where $(\triangle'^{V}_{+})' := \{\alpha\in\triangle'^{V}_{+}\mid\frac{1}{2}\alpha\in\triangle'_{+}\}$ and $(\triangle'^{H}_{+})' := \{\alpha\in\triangle'^{H}_{+}\mid\frac{1}{2}\alpha\in\triangle'_{+}\}$. Also, we have $\triangle_M = \triangle'$.

5. Proof of Theorem A

Let $M(\subset V)$ be as in Theorem A. We use the notations in Sections 3 and 4. Note that $I = (\triangle_M)_+\times\mathbb{Z}$. For simplicity denote $\mathcal{R}_M$ by $\mathcal{R}$. Let P be a complex affine subspace of $\mathfrak{b} = T^{\perp}_{x_0}M$ and D_P a distribution on M defined in Section 3. Then it is easy to show that D_P is a totally geodesic distribution on M. We call the integral manifold L^P_x of D_P through x a *slice* of M. Denote by $\mathbf{0}$ the origin of $\mathfrak{b}$. If $\mathbf{0}\notin P$, then L^P_x is a focal leaf. Then, since $L^P_{x_0}$ is a finite dimensional anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators of codimension greater than one in $(W_P)_{x_0}$, it is the product of principal orbits of the aks-representations associated with some irreducible anti-Kaehler symmetric spaces by Theorem 4.4 in [26], where we use also the fact that a finite dimensional anti-Kaehler isoparametric (complex) hypersurface is a complex sphere (i.e., a principal orbit of the aks-representation associated with an anti-Kaehler symmetric space of complex rank one). If $\mathbf{0}\in P$, then the slice $L^P_{x_0}$ is an infinite dimensional anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators in $(W_P)_{x_0}$. Take any $w_0\in(E_i)_{x_0}$ $(i\in I)$. Let $\gamma:[0,1]\to L^i_{x_0}$ be the geodesic in $L^i_{x_0}$ with $\gamma'(0)=w_0$ and $\{F_{\gamma|_{[0,t]}}\}_{t\in\mathbb{R}}$ the one-parameter family of holomorphic isometries of V stated in Section 3. For simplicity set $F^{w_0}_t := F_{\gamma_{[0,t]}}$. Let X^{w_0} be the holomorphic Killing field associated with the one-parameter transformation group $\{F^{w_0}_t\}_{t\in\mathbf{R}}$, that is, $X^{w_0}_x := \left.\frac{d}{dt}\right|_{t=0} F^{w_0}_t(x)$, where x moves over the set (which we denote by U) of all elements x's where $\left.\frac{d}{dt}\right|_{t=0} F^{w_0}_t(x)$ exists. Set $A^{w_0} := \left.\frac{d}{dt}\right|_{t=0}(F^{w_0}_t)_{*x_0}$ and $b^{w_0} := (X^{w_0})_{\mathbf{0}}$, where $\mathbf{0}$ in $(X^{w_0})_{\mathbf{0}}$ is the zero element of V (i.e., $(X^{w_0})_x = A^{w_0}x + b^{w_0}$). Clearly we have

$$\left(\underset{i\in I\cup\{0\}}{\oplus}(E_i)_{x_0}\right)\oplus\mathfrak{b}\subset U,$$

where we regard the left-hand side as a subspace of V under the identification of $T_{x_0}V$ and V. However, U does not necessarily coincide with the whole of V. For simplicity we set $V'_{x_0} := \left(\underset{i\in I\cup\{0\}}{\oplus}(E_i)_{x_0}\right)\oplus\mathfrak{b}$ and $(V'_{x_0})_T := \underset{i\in I\cup\{0\}}{\oplus}(E_i)_{x_0}$. Define a map $\overline{\Gamma}_{w_0} : (V'_{x_0})_T \to V$ by $\overline{\Gamma}_{w_0}(w) := \left.\dfrac{d}{dt}\right|_{t=0}(F_t^{w_0})_{*x_0}(w)(= A^{w_0}w)$ $(w\in (V'_{x_0})_T)$ and a map $\Gamma_{w_0} : (V'_{x_0})_T \to T_{x_0}M$ by $\Gamma_{w_0}w := (\overline{\Gamma}_{w_0}w)^T$ $(w \in (V'_{x_0})_T)$, where $(\cdot)^T$ is the $T_{x_0}M$-component of $(\cdot)$. Also, by using $\overline{\Gamma}_w$'s $(w\in \underset{i\in I}{\cup}(E_i)_{x_0})$, we define a map $\overline{\Gamma}^{x_0} : \left(\underset{i\in I}{\oplus}(E_i)_{x_0}\right)\times (V'_{x_0})_T \to V$ by setting $\overline{\Gamma}^{x_0}_{w_1}w_2 := \overline{\Gamma}_{w_1}(w_2)$ $(w_1\in \underset{i\in I}{\cup}(E_i)_{x_0},\ w_2\in (V'_{x_0})_T)$ and extending linearly with respect to the first component. Similarly, by using Γ_w's $(w\in \underset{i\in I}{\cup}(E_i)_{x_0})$, we define a map $\Gamma^{x_0} : \left(\underset{i\in I}{\oplus}(E_i)_{x_0}\right)\times (V'_{x_0})_T \to T_{x_0}M$. This map Γ^{x_0} is called the *homogeneous structure* of M at x_0.

In this section, we prove the following fact.

THEOREM 5.1. *The holomorphic Killing field X^{w_0} is defined on the whole of V.*

For simplicity we denote the extrinsically homogeneous structure Γ^{x_0} by Γ. Denote by h the second fundamental form of M. It is clear that $\overline{\Gamma}_{w_0}w = \Gamma_{w_0}w + h(w_0, w)$ $(w \in V'_T)$ and that $h(w_0,\cdot)$ is defined on the whole of $T_{x_0}M$. Hence, in order to show this theorem, we suffice to show that $\Gamma_{w_0}(: (V'_{x_0})_T \to T_{x_0}M)$ is defined (continuously) on the whole of $T_{x_0}M$. Since $(T_{x_0}M, \langle\ ,\ \rangle)$ is an anti-Kaehler space, $(T_{x_0}M, -\mathrm{pr}^*_{(T_{x_0}M)_-}\langle\ ,\ \rangle + \mathrm{pr}^*_{(T_{x_0}M)_+}\langle\ ,\ \rangle)$ is a Hilbert space, where $\mathrm{pr}_{(T_{x_0}M)_\pm}$ is the orthogonal projection of $T_{x_0}M$ onto $(T_{x_0}M)_\pm$. Set $\langle\ ,\ \rangle_\pm := -\mathrm{pr}^*_{(T_{x_0}M)_-}\langle\ ,\ \rangle + \mathrm{pr}^*_{(T_{x_0}M)_+}\langle\ ,\ \rangle$. Denote by $||\bullet||$ the norm of a vector of $T_{x_0}M$ with respect to $\langle\ ,\ \rangle_\pm$ and the operator norm of a linear transformation from $(V'_{x_0})_T$ to $T_{x_0}M$ with respect to $\langle\ ,\ \rangle_\pm$. To show that $\Gamma_{w_0}(: (V'_{x_0})_T \to T_{x_0}M)$ is defined (continuously) on the whole of $T_{x_0}M$, we suffice to show that it is bounded with respect to $||\bullet||$. In the sequel, we shall prove the boundedness of Γ_{w_0} with respect to $||\bullet||$ by the similar argument to [10]. Even if the proof is similar to that of [10], we need to discuss it carefully. For the domain of Γ is an anti-Kaehler space but there exist some parts discussed on a special real form of the space. Some of facts corresponding to lemmas and propositions in Sections 3–6 and 8 of [10] are shown in the same methods as their proofs in [10]. We shall state the facts as lemmas without the proof.

For Γ, we can show the following fact.

LEMMA 5.2. *Let $i_1\in I$ and $i_2, i_3 \in I\cup\{0\}$.*
(i) *For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2, 3)$, we have*

$$\langle\Gamma_{w_1}w_2, w_3\rangle + \langle w_2, \Gamma_{w_1}w_3\rangle = 0\,,$$

(ii) *For any* $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$ *and any holomorphic isometry* f *of* V *preserving* M *invariantly*, *we have*

$$f_* \Gamma_{w_1} w_2 = \Gamma_{f_* w_1} f_* w_2 \, .$$

Also, for $F_t^{w_0}$, we have the following fact.

LEMMA 5.3. *Let* L *be a slice of* M, i_0 *an element of* $I \cup \{0\}$ *with* $(E_{i_0})_{x_0} \subset T_{x_0} L$ *and* W *the complex affine span of* L. *If* $w_0 \in (E_{i_0})_{x_0}$, *then* $F_t^{w_0}(L) = L$ *holds for all* $t \in [0, 1]$ *and* X^{w_0} *is tangent to* W *along* W. *Furthermore*, *if* L *is irreducible and is of rank greater than one, then* $F_t^{w_0}|_W = {}^L F_t^{w_0}$ *holds for all* $t \in [0, 1]$, *where* ${}^L F_t^{w_0}$ *is the one-parameter transformation group of* W *defined for* L *in similar to* $F_t^{w_0}$, *and hence the extrinsically homogeneous structure of* $L(\subset W)$ *at* x_0 *is the restriction of* Γ.

These lemmas are proved in the methods of the proofs of Lemmas 3.4 and 3.5 of [10], respectively. Let $\widetilde{v}$ be a (non-focal) parallel normal vector field of M, $\eta_{\widetilde{v}} : M \to V$ the end-point map for $\widetilde{v}$ (i.e., $\eta_{\widetilde{v}}(u) := \exp^{\perp}(\widetilde{v}_u)$ $(u \in M)$) and $M_{\widetilde{v}}$ the parallel submanifold for $\widetilde{v}$ (i.e., the image of $\eta_{\widetilde{v}}$). Denote by ${}^{\widetilde{v}}\Gamma$ the extrinsically homogeneous structure of $M_{\widetilde{v}}$ at $\eta_{\widetilde{v}}(x_0)$. Then we have the following fact.

LEMMA 5.4. *For any* $w_1 \in (E_{i_1})_{x_0}$ $(i_1 \in I)$ *and any* $w_2 \in (E_{i_2})_{x_0}$ $(i_2 \in I \cup \{0\})$, *we have*

$${}^{\widetilde{v}}\Gamma_{(\eta_{\widetilde{v}})_* w_1} w_2 = (\eta_{\widetilde{v}})_* (\Gamma_{w_1} w_2) \, ,$$

where we note that $T_{x_0} M = T_{\eta_{\widetilde{v}}(x_0)} M_{\widetilde{v}}$ *under the parallel translation in* V. *Also*, *we have* $(\eta_{\widetilde{v}})_* w_1 = (1 - (\lambda_{i_1})_{x_0}(\widetilde{v}_0)) w_1$.

PROOF. From $(\eta_{\widetilde{v}})_{*x_0} = \mathrm{id} - A_{\widetilde{v}_0}$, the second relation follows directly, where A is the shape tensor of M. Since $(\eta_{\widetilde{v}})_{*x_0}$ maps the J-curvature distributions of M to those of $M_{\widetilde{v}}$, $\eta_{\widetilde{v}}$ maps the complex curvature spheres of M through x_0 to those of $M_{\widetilde{v}}$ through $\eta_{\widetilde{v}}(x_0)$. On the other hand, since $F_t^{w_1}$ preserves M inavariantly and its differential at a point of M induces the parallel translation with respect to the normal connection of M, we have $\eta_{\widetilde{v}} \circ F_t^{w_1}|_M = F_t^{w_1} \circ \eta_{\widetilde{v}}$. By using these facts and the properties of $F_t^{w_1}$, we can show that $F_t^{w_1}$ coincides with $F_t^{(\eta_{\widetilde{v}})_* w_1}$. From this fact, the first relation follows. □

We have the following fact for a principal orbit of an aks-representation of complex rank greater than one.

LEMMA 5.5. *Let* N *be a principal orbit of an aks-representation of complex rank greater than one*, $\{n_i \mid i \in I\}$ *the set of all* J*-curvature normals of* N, E_i *the* J*-curvature distribution corresponding to* n_i *and* Γ *the extrinsically homogeneous structure of* N *at* x. *If the* 2*-dimensional complex affine subspace* P *through* n_{i_1}, n_{i_2} *and* n_{i_3} *which does not pass through* $\mathbf{0}$, *then*, *for any* $w_k \in (E_{i_k})_x$ $(k = 1, 2, 3)$, *we have*

$$\Gamma_{w_1} \Gamma_{w_2} w_3 - \Gamma_{w_2} \Gamma_{w_1} w_3 = \Gamma_{(\Gamma_{w_1} w_2 - \Gamma_{w_2} w_1)} w_3 \, .$$

PROOF. Let L/H be an irreducible anti-Kaehler symmetric space and $(\mathfrak{l},\tau)$ the anti-Kaehler symmetric Lie algebra associated with L/H. We use the notations in Subsection 2.2. Note that $I=\triangle_+\times\{0\}(=\triangle_+)$. Let N be the principal orbit of the aks-representation $\rho:=\mathrm{Ad}_L|_{\mathfrak{p}}:H\to GL(\mathfrak{p})$ through a regular element $x(\in D)$. Take any $\alpha\in\triangle_+$ and any $w\in(E_\alpha)_x(=\mathfrak{p}_\alpha)$. Then, according to the proof of Lemma 4.6.3 of [26], the holomorphic isometry F_t^w is equal to $\rho(\exp_L(t\overline{w}))$, where $\overline{w}$ is the element of $\mathfrak{h}_\alpha$ such that $\mathrm{ad}_{\mathfrak{l}}(a)(\overline{w})=w$ for all $a\in\mathfrak{a}$, where $\mathfrak{h}_\alpha:=\{X\in\mathfrak{h}\mid\mathrm{ad}_{\mathfrak{l}}(a)^2(X)=\alpha^{\mathbb{C}}(a)^2X\ \text{for all}\ a\in\mathfrak{a}\}$. Hence we have

$$\varGamma_w=\mathrm{ad}_{\mathfrak{l}}(\overline{w})\,. \tag{5.1}$$

Therefore we can derive the desired relation in the method of the proof of Proposition 3.8 of [10]. □

For each $i\in I$, denote by W_i the complex affine subspace $x_0+((E_i)_{x_0}\oplus\mathrm{Span}_{\mathbb{C}}\{(n_i)_{x_0}\})$ of V. Also, let f_i be the focal map having $L_u^{E_i}$'s $(u\in M)$ as fibres, $\varPhi_i$ the normal holonomy group of the focal submanifold $f_i(M)$ at $f_i(x_0)$ and $(\varPhi_i)_{x_0}$ the isotropy group of $\varPhi_i$ at x_0. This group $(\varPhi_i)_{x_0}$ preserves $(E_i)_{x_0}$ invariantly. The irreducible decomposition of the action $(\varPhi_i)_{x_0}\curvearrowright(E_i)_{x_0}$ is given by the form $(E_i)_{x_0}=(E_i)'_{x_0}\oplus(E_i)''_{x_0}$, where $\dim_{\mathbb{C}}(E_i)''_{x_0}=0,1$ or 3, and $\dim_{\mathbb{C}}(E_i)'_{x_0}$ is even in case of $\dim_{\mathbb{C}}(E_i)''_{x_0}=1$ or 3. Set $m_i:=\dim_{\mathbb{C}}E_i$. Note that $\varPhi_i$ is orbit equivalent to the aks-representation associated with one of the following irreducible complex rank one anti-Kaehler symmetric spaces:

$$SO(m_i+2,\mathbb{C})/SO(m_i+1,\mathbb{C}),\ \ SL(\tfrac{m_i+1}{2}+1,\mathbb{C})/SL(\tfrac{m_i+1}{2},\mathbb{C})\cdot\mathbb{C}_*\,,$$
$$Sp(\tfrac{m_i+1}{4}+1,\mathbb{C})/Sp(1,\mathbb{C})\times Sp(\tfrac{m_i+1}{4},\mathbb{C})$$

and that

$$\dim_{\mathbb{C}}(E_i)''_{x_0}=\begin{cases}0 & ((\varPhi_i)_{x_0}=SO(m_i+1,\mathbb{C}))\\ 1 & ((\varPhi_i)_{x_0}=SL(\frac{m_i+1}{2},\mathbb{C})\cdot\mathbb{C}_*)\\ 3 & ((\varPhi_i)_{x_0}=Sp(1,\mathbb{C})\times Sp(\frac{m_i+1}{4},\mathbb{C}))\,.\end{cases}$$

By using Lemma 5.3 and (5.1), we can derive the following fact corresponding to Proposition 3.11 of [10].

LEMMA 5.6. *Let $i\in I$. Then we have*

$$\varGamma_{(E_i)''_{x_0}}(E_i)''_{x_0}=0,\quad \varGamma_{(E_i)'_{x_0}}(E_i)''_{x_0}\subset(E_i)'_{x_0}\,,$$
$$\varGamma_{(E_i)''_{x_0}}(E_i)'_{x_0}\subset(E_i)'_{x_0}\quad\text{and}\quad\varGamma_{(E_i)'_{x_0}}(E_i)'_{x_0}\subset(E_i)''_{x_0}\,.$$

Also, we have the following facts corresponding to Propositions 3.12 and 3.13 of [10].

LEMMA 5.7. *For $i_1\in I$ and $i_2\in I\cup\{0\}$ with $i_2\neq i_1$, we have $\langle\varGamma_{(E_{i_1})_{x_0}}(E_{i_2})_{x_0},(E_{i_2})_{x_0}\rangle=0$.*

LEMMA 5.8. *Let $i_1\in I$ and $i_2,i_3\in I\cup\{0\}$. For $w_k\in(E_{i_k})_{x_0}$ $(k=1,2,3)$, we have $(\overline{\nabla}_{w_1}\widetilde{h})(w_2,w_3)=\langle\varGamma_{w_1}w_2,w_3\rangle((n_{i_2})_{x_0}-(n_{i_3})_{x_0})$ and $\varGamma_{w_1}w_2=\widetilde{\nabla}_{w_1}\widetilde{w}_2\pmod{(E_{i_2})_{x_0}}$,* *where $\overline{\nabla}$ is the connection of the tensor bundle $T^*M \otimes T^*M \otimes T^{\perp}M$ induced from ∇ and the normal connection $\nabla^{\perp}$ of M, and $\widetilde{w}_2$ is a local section of E_{i_2} with $(\widetilde{w}_2)_{x_0} = w_2$.*

Let $i_1, i_2, i_3 \in I \cup \{0\}$ with $i_2 \neq i_3$. Then we define $\dfrac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}}$ by

$$\frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}} := \begin{cases} b & \left(\begin{array}{c} \text{(when } (n_{i_1})_{x_0} - (n_{i_3})_{x_0} = b((n_{i_2})_{x_0} - (n_{i_3})_{x_0}) \\ \text{for some } b \in \mathbb{C} \end{array} \right) \\ 0 & \left(\begin{array}{c} \text{when } (n_{i_1})_{x_0} - (n_{i_3})_{x_0} \text{ and } (n_{i_2})_{x_0} - (n_{i_3})_{x_0}) \\ \text{are linearly independent over } \mathbb{C} \end{array} \right) \end{cases} .$$

Note that this value is independent of the choice of $x_0 \in M$. Denote by w^k the $(E_k)_{x_0}$-component of $w \in T_{x_0}M$. We can derive the following fact corresponding to Proposition 3.15 of [10] from the first relation in Lemma 5.8 and the Codazzi equation.

LEMMA 5.9. *Let $i_1, i_2 \in I$ and $i_3 \in I \cup \{0\}$ with $i_3 \neq i_2$. For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$, we have*

$$(\Gamma_{w_1} w_2)^{i_3} = \frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}} (\Gamma_{w_2} w_1)^{i_3} .$$

Also, we have the following fact corresponding to Lemma 3.16 of [10].

LEMMA 5.10. (i) *Let $i_1 \in I$ and $i_2, i_3 \in I \cup \{0\}$. If $(\Gamma_{w_1} w_2)^{i_3} \neq 0$ for some $w_1 \in (E_{i_1})_{x_0}$ and $w_2 \in (E_{i_2})_{x_0}$, then $(n_{i_1})_{x_0}, (n_{i_2})_{x_0}$ and $(n_{i_3})_{x_0}$ are contained in a complex affine line.*

(ii) *Let $i_1, i_2, i_3 \in I$. The condition $(\Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0})^{i_3} \neq 0$ is symmetric in i_1, i_2, i_3.*

Also, we have the following fact corresponding to Theorem 4.1 of [10].

LEMMA 5.11. $\sum_{i_1, i_2 \in I \text{ s.t. } i_1 \neq i_2} \Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0}$ *is dense in $T_{x_0}M$ and includes $\sum_{i \in I} (E_i)_{x_0}$.*

By using this lemma, we can derive the following fact corresponding to Corollary 4.2 of [10].

LEMMA 5.12. (i) *For each $i_1 \in I$, we have*

$$\sum_{i_2, i_3 \in I \text{ s.t. } n_{i_2}, n_{i_3} \notin \mathrm{Span}_{\mathbb{C}}\{n_{i_1}\}} (\Gamma_{(E_{i_2})_{x_0}} (E_{i_3})_{x_0})^{i_1} = (E_{i_1})_{x_0} .$$

(ii) $\sum_{i_1, i_2 \in I \text{ s.t. } n_{i_1}, n_{i_2} : \text{lin. dep.}} (\Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0})^0$ *is dense in $(E_0)_{x_0}$, where "lin. dep." means "linearly dependent".*

Notation. In the sequel, for $w \in (E_i)_{x_0}$ $(i \in I \cup \{0\})$, $\widetilde{w}$ denotes a local section of E_i with $\widetilde{w}_{x_0} = w$.

For $w_1 \in (E_{i_1})_{x_0}$ and $w_2 \in (E_{i_2})_{x_0}$ $(i_1, i_2 \in I \cup \{0\})$, define $\nabla'_{\widetilde{w}_1} \widetilde{w}_2$ by $(\nabla'_{\widetilde{w}_1} \widetilde{w}_2)_x := (\nabla_{\widetilde{w}_1} \widetilde{w}_2)_x - \Gamma^x_{(\widetilde{w}_1)_x} (\widetilde{w}_2)_x$, where x moves over the common domain of $\widetilde{w}_1$ and $\widetilde{w}_2$. Denote by

R the curvature tensor of M. Let $i_1, i_2, i_3 \in I$, $i_4 \in I \cup \{0\}$ and $w_k \in (E_{i_k})_{x_0}$ $(k = 1, \dots, 4)$. According to the Gauss equation, we have

$$\langle R(w_1, w_2)w_3, w_4\rangle = (\langle w_1, w_4\rangle\langle w_2, w_3\rangle - \langle w_1, w_3\rangle\langle w_2, w_4\rangle)\langle n_{i_1}, n_{i_2}\rangle\,. \tag{5.2}$$

Also, from the definition of ∇', we have

$$\begin{aligned}\langle R(w_1, w_2)w_3, w_4\rangle = {} & \langle \Gamma_{w_1} w_3, \Gamma_{w_2} w_4\rangle - \langle \Gamma_{w_2} w_3, \Gamma_{w_1} w_4\rangle - \langle(\nabla_{[\widetilde{w}_1,\widetilde{w}_2]}\widetilde{w}_3)_{x_0}, w_4\rangle \\ & + w_1\langle(\nabla_{\widetilde{w}_2}\widetilde{w}_3)_{x_0}, w_4\rangle - \langle(\nabla'_{\widetilde{w}_2}\widetilde{w}_3)_{x_0}, (\nabla_{\widetilde{w}_1}\widetilde{w}_4)_{x_0}\rangle - \langle \Gamma_{w_2} w_3, (\nabla'_{\widetilde{w}_1}\widetilde{w}_4)_{x_0}\rangle \\ & - w_2\langle(\nabla_{\widetilde{w}_1}\widetilde{w}_3)_{x_0}, w_4\rangle + \langle(\nabla'_{\widetilde{w}_1}\widetilde{w}_3)_{x_0}, (\nabla_{\widetilde{w}_2}\widetilde{w}_4)_{x_0}\rangle + \langle \Gamma_{w_1} w_3, (\nabla'_{\widetilde{w}_2}\widetilde{w}_4)_{x_0}\rangle\,.\end{aligned} \tag{5.3}$$

For ∇' and Γ, we have the following relations.

LEMMA 5.13. *Let $i_1, i_2, i_3 \in I$ and $i_4 \in I \cup \{0\}$.*
(i) *For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2, 3)$, we have*

$$w_1\langle\widetilde{w}_2, \widetilde{w}_3\rangle = \langle(\nabla'_{\widetilde{w}_1}\widetilde{w}_2)_{x_0}, \widetilde{w}_3\rangle + \langle w_2, (\nabla'_{\widetilde{w}_1}\widetilde{w}_3)_{x_0}\rangle\,.$$

(ii) *If $i_1 \neq i_2$, then we have $\nabla'_{\widetilde{w}_1}\widetilde{w}_2 = (\nabla_{\widetilde{w}_1}\widetilde{w}_2)^{i_2}$ for any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$.*
(iii) *For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2, 3)$, we have*

$$\left(\nabla'_{\widetilde{w}_1}(\widetilde{(\Gamma_{w_2} w_3)^{i_3}})\right)_{x_0} = \left(\Gamma_{(\nabla'_{\widetilde{w}_1}\widetilde{w}_2)_{x_0}} w_3\right)^{i_3} + \left(\Gamma_{w_2}(\nabla'_{\widetilde{w}_1}\widetilde{w}_3)_{x_0}\right)^{i_3}\,.$$

PROOF. The relations in (i) and (ii) are trivial. From (ii) of Lemma 5.2, the relation in (iii) is shown in the method of the proof of Lemma 5.2 of [10]. □

Let $i_1 \in I$ and $i_2 \in I \cup \{0\}$. For $w \in T_{x_0}M$, $w_1 \in (E_{i_1})_{x_0}$ and $w_2 \in (E_{i_2})_{x_0}$, we define $\langle \Gamma_w w_1, w_2\rangle$ by

$$\begin{aligned}&\langle \Gamma_w w_1, w_2\rangle := -\sum_{i\in I}\left\langle \Gamma_{w_1} w_2, \frac{n_i - n_{i_2}}{n_{i_1} - n_{i_2}} w^i\right\rangle \\ &\left(= \lim_{m\to\infty}\sum_{i\in I \text{ s.t. } |w^i| > \frac{1}{m}}\left\langle \Gamma_{w_1} w_2, \frac{n_i - n_{i_2}}{n_{i_1} - n_{i_2}} w^i\right\rangle\right)\,.\end{aligned} \tag{5.4}$$

According to (i) of Lemma 5.2 and Lemma 5.9, this definition is valid. From the relation in (iii) of Lemma 5.13, we can show the following fact in the method of the proof of Theorem 5.7 of [10].

LEMMA 5.14. *Let $i_1, i_2, i_3 \in I$ and $i_4 \in I \cup \{0\}$ with $i_4 \neq i_3$. For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, \dots, 4)$, we have*

$$\begin{aligned}&\left\langle\left([\Gamma_{w_1}, \Gamma_{w_2}] - \Gamma_{\Gamma_{w_1} w_2 - \Gamma_{w_2} w_1}\right) w_3, w_4\right\rangle \\ &= -(\langle w_1, w_4\rangle\langle w_2, w_3\rangle - \langle w_1, w_3\rangle\langle w_2, w_4\rangle)\langle n_{i_1}, n_{i_2}\rangle\,.\end{aligned}$$

By using Lemmas 5.9 and 5.14, we can show the following fact.

LEMMA 5.15. *Let (i_1, i_2, i_3) be an element of $I^2 \times (I \cup \{0\})$ such that there exists no complex affine line containing $(n_{i_1})_{x_0}$, $(n_{i_2})_{x_0}$ and $(n_{i_3})_{x_0}$, and i_4 an element of I. For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, \ldots, 4)$, we have*

$$\langle \Gamma_{w_1} w_2, \Gamma_{w_4} w_3 \rangle = \langle \Gamma_{w_4} w_2, \Gamma_{w_1} w_3 \rangle + c \langle \Gamma_{w_1} w_4, \Gamma_{w_2} w_3 \rangle ,$$

where c is a constant. Furthermore, if $i_1 = i_4$ or the intersection of the complex affine line through $(n_{i_1})_{x_0}$ and $(n_{i_4})_{x_0}$ and the complex affine line through and $(n_{i_2})_{x_0}$ and $(n_{i_3})_{x_0}$ contains no J-curvature normal, then we have $c = 0$. On the other hand, if their intersection contains a J-curvature normal $(n_{i_5})_{x_0}$, then we have

$$c = \frac{n_{i_3} - n_{i_5}}{n_{i_2} - n_{i_3}} \times \frac{n_{i_1} - n_{i_4}}{n_{i_1} - n_{i_5}} .$$

We can show the following fact in the method of the proof of Corollary 5.11 of [10].

LEMMA 5.16. *Let $i_1, i_2, i_3 \in I$ satisfying $i_3 \neq i_1, i_2$ and $\frac{n_{i_2}}{n_{i_3}} \neq -\frac{n_{i_1} - n_{i_2}}{n_{i_1} - n_{i_3}}$. Assume that $\langle (\Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0})^{i_4}, \Gamma_{(E_{i_1})_{x_0}} (E_{i_3})_{x_0} \rangle = 0$ for any $i_4 \in I$ and $(\Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0})^{i_3} = 0$ (these conditions hold if $\Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0} \subset (E_0)_{x_0}$). Then we have $\langle \Gamma_{(E_{i_1})_{x_0}} (E_{i_2})_{x_0}, \Gamma_{(E_{i_1})_{x_0}} (E_{i_3})_{x_0} \rangle = 0$.*

Also, we can derive the following fact.

LEMMA 5.17. *Let $i_1, i_2 \in I$ with $i_1 \neq i_2$. For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$, we have*

$$\sum_{i_3 \in (I \cup \{0\}) \setminus \{i_1\}} \mathrm{Re} \left(\frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} \right) ||(\Gamma_{w_1} w_2)^{i_3}||^2 = \frac{1}{2} \langle n_{i_1}, n_{i_2} \rangle \langle w_1, w_1 \rangle ||w_2||^2 .$$

PROOF. Let $w_2 = (w_2)_- + (w_2)_+$ $((w_2)_- \in ((E_{i_2})_-)_{x_0},\ (w_2)_+ \in ((E_{i_2})_+)_{x_0})$. In similar to Corollary 5.13 of [10], we can show

$$\begin{aligned} &\sum_{i_3 \in (I \cup \{0\}) \setminus \{i_1\}} \left\langle (\Gamma_{w_1} (w_2)_\varepsilon)^{i_3}, \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} (\Gamma_{w_1} (w_2)_\varepsilon)^{i_3} \right\rangle \\ &= \frac{1}{2} \langle n_{i_1}, n_{i_2} \rangle \langle w_1, w_1 \rangle \langle (w_2)_\varepsilon, (w_2)_\varepsilon \rangle , \end{aligned} \tag{5.5}$$

where $\varepsilon = -$ or $+$. On the other hand, since $F_t^{w_1}$'s preserve E_i's invariantly and they are holomorphic isometries, Γ_{w_1} preserves $((E_i)_-)_{x_0}$'s and $((E_i)_+)_{x_0}$ invariantly, respectively. Hence we have $\Gamma_{w_1} (w_2)_\varepsilon = (\Gamma_{w_1} w_2)_\varepsilon$. Also, it is clear that $((\Gamma_{w_1} w_2)_\varepsilon)^{i_3} = ((\Gamma_{w_1} w_2)^{i_3})_\varepsilon$. From these relations, we have

$$\begin{aligned} &\left\langle (\Gamma_{w_1} (w_2)_\varepsilon)^{i_3}, \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} (\Gamma_{w_1} (w_2)_\varepsilon)^{i_3} \right\rangle \\ &= \mathrm{Re} \left(\frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} \right) \langle ((\Gamma_{w_1} w_2)^{i_3})_\varepsilon, ((\Gamma_{w_1} w_2)^{i_3})_\varepsilon \rangle . \end{aligned}$$

By summing the (-1)-multiples of (5.5)'s for $\varepsilon = \pm$ and using this relation, we have the desired relation. $\square$

By using Lemmas 5.3, 5.7, 5.10 and 5.17, we can show the following fact.

LEMMA 5.18. *Assume that the complex Coxeter group $\mathcal{W}$ associated with M is of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$. Let i_1 and i_2 be elements of I such that n_{i_1} and n_{i_2} are linearly independent.*

(i) *If n_{i_1} and n_{i_2} are orthogonal, then we have $\Gamma_{w_1} w_2 = 0$ for any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$.*

(ii) *If n_{i_1} and n_{i_2} are not orthogonal, then we have $||\Gamma_{w_1} w_2|| \leq \frac{1}{2}||w_1||\,||w_2||\,||n_{i_1}||$ for any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$.*

PROOF. Let P be the complex affine line in $\mathfrak{b}$ through $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$. Since n_{i_1} and n_{i_2} are linearly independent, we have $0 \notin P$. Hence the slice $L^P_{x_0}$ is a finite dimensional anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators (of codimension two in $(W_P)_{x_0}$). Hence, since $\mathcal{W}$ is isomorphic to an affine Weyl group of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$, the root system (which we denote by $\triangle_P$) associated with $L^P_{x_0}$ is of type $A_1 \times A_1$ or A_2. First we shall show the statement (i). Assume that $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ are orthogonal. Then $\triangle_P$ is of type $A_1 \times A_1$ and hence P contains no other J-curvature normal. By using this fact and Lemma 5.3, we can show $\Gamma_{w_1} w_2 = {}^{L^P_{x_0}}\Gamma_{w_1} w_2 = 0$ for any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$, where ${}^{L^P_{x_0}}\Gamma$ is the extrinsically homogeneous structure of $L^P_{x_0}$. Next we shall show the statement (ii). Assume that $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ are not orthogonal. Then $\triangle_P$ is of type A_2 and hence there exists $i_3 \in I \setminus \{i_1, i_2\}$ with $(n_{i_3})_{x_0} \in P$. The set $l_{i_1} \cap l_{i_2} \cap l_{i_3} \cap \mathrm{Span}_{\mathbb{C}}\{(n_{i_1})_{x_0}, (n_{i_2})_{x_0}\}$ consists of the only one point. Denote by p_0 this point. Let e_1, e_2 and e_3 be a unit normal vector of l_{i_1}, l_{i_2} and l_{i_3}, respectively. Since $\triangle_P$ is of type (A_2), we may assume that $e_3 = e_1 + e_2$ by replacing some of these vectors to the (-1)-multiples of them if necessary. Since $\frac{(n_{i_1})_{x_0}}{\langle (n_{i_1})_{x_0}, (n_{i_1})_{x_0}\rangle} \in l_{i_1}$, we have $(n_{i_1})_{x_0} = \frac{e_1}{\langle \overrightarrow{0p_0}, e_1\rangle}$, where 0 is the origin of $\mathfrak{b}$. Similarly we have $(n_{i_2})_{x_0} = \frac{e_2}{\langle \overrightarrow{0p_0}, e_2\rangle}$ and $(n_{i_3})_{x_0} = \frac{e_3}{\langle \overrightarrow{0p_0}, e_3\rangle}$. By using these facts, Lemmas 5.7, 5.10 and 5.17, we can show

$$\begin{aligned} ||\Gamma_{w_1} w_2||^2 = ||(\Gamma_{w_1} w_2)^{i_3}||^2 &\leq \frac{1}{2}\mathrm{Re}\left(\frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}}\right)|\langle n_{i_1}, n_{i_2}\rangle|\,||w_1||^2\,||w_2||^2 \\ &\leq \frac{1}{4}||w_1||^2\,||w_2||^2\,||n_{i_1}||^2 . \end{aligned}$$

Thus we obtain the desired relation. $\square$

By using Lemmas 5.3, 5.4, 5.7 and 5.10, we can show the following fact.

LEMMA 5.19. *We have*

$$\sup_{i \in I}\ \sup_{P \in \mathcal{H}_i}\ \sup_{(w_1, w_2) \in (E_i)_{x_0} \times (D_P)_{x_0}} \frac{||\Gamma_{w_1} w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||} < \infty ,$$

where $\mathcal{H}_i$ is the set of all complex affine subspaces P in $T_{x_0}M$ with $0 \notin P$ and $(n_i)_{x_0} \in P$.

PROOF. Let $\mathcal{H}_i^{\mathrm{irr}}$ be the set of all elements P of $\mathcal{H}_i$ such that $L^P_{x_0}(\subset (W_P)_{x_0})$ is irreducible. First we shall show

$$\sup_{i\in I}\ \sup_{P\in\mathcal{H}_i^{\mathrm{irr}}}\ \sup_{(w_1,w_2)\in(E_i)_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||} < \infty . \tag{5.6}$$

Fix $i_0 \in I$ and $P_0 \in \mathcal{H}_{i_0}^{\mathrm{irr}}$. If the complex codimension of $L^{P_0}_{x_0}(\subset (W_{P_0})_{x_0})$ is equal to one, then we can take $P_0' \in \mathcal{H}_{i_0}^{\mathrm{irr}}$ such that $P_0 \subset P_0'$ and that the complex codimension of $L^{P_0'}_{x_0}(\subset (W_{P_0'})_{x_0})$ is greater than one. Then we have

$$\begin{aligned}
&\sup_{(w_1,w_2)\in(E_{i_0})_{x_0}\times(D_{P_0})_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_{i_0})_{x_0}||}\\
\leq\ &\sup_{(w_1,w_2)\in(E_{i_0})_{x_0}\times(D_{P_0'})_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_{i_0})_{x_0}||} .
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{i\in I}\ \sup_{P\in\mathcal{H}_i^{\mathrm{irr}}}\ \sup_{(w_1,w_2)\in(E_i)_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||}\\
&= \sup_{i\in I}\ \sup_{P\in\mathcal{H}_i^{\mathrm{irr},\geq 2}}\ \sup_{(w_1,w_2)\in(E_i)_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||} ,
\end{aligned}\tag{5.7}$$

where $\mathcal{H}_i^{\mathrm{irr},\geq 2}$ is the set of all elements P's of $\mathcal{H}_i^{\mathrm{irr}}$ such that the complex codimension of $L^P_{x_0}(\subset (W_P)_{x_0})$ is greater than one. Fix $\alpha_1 \in (\triangle_M)_+$ and $P_1 \in \mathcal{H}^{\mathrm{irr},\geq 2}_{(\alpha_1,0)}$. Take any $j_1 \in \mathbb{Z}$. For each $P \in \mathcal{H}^{\mathrm{irr}}_{(\alpha_1,0)}$, there exists $P' \in \mathcal{H}^{\mathrm{irr}}_{(\alpha_1,j_1)}$ such that $\{\alpha \in (\triangle_M)_+ \mid \exists\, j \in \mathbb{Z} \text{ s.t. } (n_{(\alpha,j)})_{x_0} \in P\} = \{\alpha \in (\triangle_M)_+ \mid \exists\, j \in \mathbb{Z} \text{ s.t. } (n_{(\alpha,j)})_{x_0} \in P'\}$. Then, since $\dim_{\mathbb{C}}(W_P)_{x_0} = \dim_{\mathbb{C}}(W_{P'})_{x_0}$, and since the root systems associated with $L^P_{x_0}$ and $L^{P'}_{x_0}$ coincide, they are regarded as principal orbits of the aks-representation of the same irreducible anti-Kaehler symmetric space. That is, $L^{P'}_{x_0}$ is regarded as a parallel submanifold of $L^P_{x_0}$ under a suitable identification of $(W_P)_{x_0}$ and $(W_{P'})_{x_0}$. Therefore, by using Lemmas 5.3 and 5.4, we can show

$$\begin{aligned}
&\sup_{P\in\mathcal{H}^{\mathrm{irr}}_{(\alpha_1,0)}}\ \sup_{(w_1,w_2)\in(E_{(\alpha_1,0)})_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_{(\alpha_1,0)})_{x_0}||}\\
=\ &\sup_{P\in\mathcal{H}^{\mathrm{irr}}_{(\alpha_1,j_1)}}\ \sup_{(w_1,w_2)\in(E_{(\alpha_1,j_1)})_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_{(\alpha_1,j_1)})_{x_0}||} .
\end{aligned}$$

Hence it follows from the arbitrariness of j_1 that

$$\sup_{i\in I}\ \sup_{P\in\mathcal{H}_i^{\mathrm{irr}}}\ \sup_{(w_1,w_2)\in(E_i)_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||}$$
$$= \sup_{\alpha\in(\triangle_M)_+}\ \sup_{P\in\mathcal{H}_{(\alpha,0)}^{\mathrm{irr}}}\ \sup_{(w_1,w_2)\in(E_{(\alpha,0)})_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_{(\alpha,0)})_{x_0}||} < \infty\,.$$

Thus we obtain (5.6). For simplicity set

$$C := \sup_{i\in I}\ \sup_{P\in\mathcal{H}_i^{\mathrm{irr}}}\ \sup_{(w_1,w_2)\in(E_i)_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||}\,.$$

Fix $i_0\in I$ and $P_0\in\mathcal{H}_{i_0}\setminus\mathcal{H}_{i_0}^{\mathrm{irr}}$. Let $L_{x_0}^{D_{P_0}}=L_1\times\cdots\times L_k$ be the irreducible decomposition of $L_{x_0}^{D_{P_0}}$. Take any $i_1,i_2\in I$ with $(n_{i_1})_{x_0},(n_{i_2})_{x_0}\in P_0$. If $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ are not orthogonal, then $(E_{i_1})_{x_0}\oplus(E_{i_2})_{x_0}\subset T_{x_0}L_a$ for some $a\in\{1,\dots,k\}$. Hence we have

$$\sup_{(w_1,w_2)\in(E_{i_1})_{x_0}\times(E_{i_2})_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_{i_1})_{x_0}||}\leq C\,.$$

If $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ are orthogonal, then the complex affine line through $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$ does not contain other J-curvature normal. Hence it follows from Lemma 5.7 and (i) of Lemma 5.10 that $\Gamma_{(E_{i_1})_{x_0}}(E_{i_2})_{x_0}=0$. Therefore, we obtain

$$\sup_{i\in I}\ \sup_{P\in\mathcal{H}_i}\ \sup_{(w_1,w_2)\in(E_i)_{x_0}\times(D_P)_{x_0}} \frac{||\Gamma_{w_1}w_2||}{||w_1||\,||w_2||\,||(n_i)_{x_0}||}=C\,.$$

This completes the proof. □

By using Lemma 5.19, we can show the following fact.

LEMMA 5.20. *Let $i_0=(\alpha_0,j_0)\in I$ and $w\in(E_{i_0})_{x_0}$. Then Γ_w can be extended continuously to $T_{x_0}M$ if and only if the restriction of Γ_w to $\bigoplus_{j\in\mathbb{Z}}(E_{(\alpha_0,j)})_{x_0}$ can be extended continuously to $\overline{\bigoplus_{j\in\mathbb{Z}}(E_{(\alpha_0,j)})_{x_0}}$.*

PROOF. Set $V_0:=(E_0)_{x_0}$, $V_1:=\bigoplus_{i\in I\setminus\{(\alpha_0,j)\,|\,j\in\mathbb{Z}\}}(E_i)_{x_0}$ and $V_2:=\bigoplus_{j\in\mathbb{Z}}(E_{(\alpha_0,j)})_{x_0}$. Clearly we have $T_{x_0}M=V_0\oplus\overline{V}_1\oplus\overline{V}_2$. Since Γ_w is a closed operator by the definition and since $(E_0)_{x_0}$ is closed in the domain of Γ_w, $\Gamma_w|_{(E_0)_{x_0}}$ also is a closed operator. Hence, according to the closed graph theorem, $\Gamma_w|_{(E_0)_{x_0}}$ is bounded (hence continuous). Easily we can show

$$V_1=\bigoplus_l\left(\bigoplus_{i\in I\setminus\{(\alpha_0,j)\,|\,j\in\mathbb{Z}\}\text{ s.t. }(n_i)_{x_0}\in l}(E_i)_{x_0}\right),$$

where l runs over the set of all complex affine lines in $\mathfrak{b} \setminus \{0\}$ through $(n_{i_0})_{x_0}$. For simplicity set

$$V_{1,l} := \bigoplus_{i \in I \setminus \{(\alpha_0, j) | j \in \mathbb{Z}\} \text{ s.t. } (n_i)_{x_0} \in l} (E_i)_{x_0} .$$

According to Lemma 5.19, for each l, we have

$$\sup_{w' \in V_{1,l}} \frac{||\Gamma_w w'||}{||w'||} \leq C||(n_{i_0})_{x_0}|| \, ||w|| ,$$

where C is the positive constant as in the proof of Lemma 5.19, and hence

$$\sup_{w' \in V_1} \frac{||\Gamma_w w'||}{||w'||} \leq C||(n_{i_0})_{x_0}|| \, ||w|| .$$

Therefore the restriction of Γ_w to V_1 is bounded and hence it can be extended continuously to $\overline{V}_1$. From these facts, the statement of this lemma follows. □

According to Lemma 6.4 of [10], we have the following fact.

LEMMA 5.21. *Let W be a Hilbert space, $W = \overline{\bigoplus_{i \in \mathbb{Z}} W_i}$ the orthogonal decomposition of W and f a linear map from $\bigoplus_{i \in \mathbb{Z}} W_i$ to W. Assume that there exists a positive constant C such that $||f(w)|| \leq C||w||$ for all $w \in \bigcup_{i \in \mathbb{Z}} W_i$ and that there exist injective maps $\mu_a : \mathbb{Z} \to \mathbb{Z}$ $(a = 1, \ldots, r)$ such that $\langle f(W_i), f(W_j) \rangle = 0$ for any $j \notin \{\mu_1(i), \ldots, \mu_r(i)\}$. Then we have $||f|| \leq \sqrt{r} C$ and hence f can be extended continuously to W.*

Easily we can show that

$$\frac{n_{(\alpha, j_1)} - n_{(\alpha, j_3)}}{n_{(\alpha, j_2)} - n_{(\alpha, j_3)}} = \frac{j_1 - j_3}{j_2 - j_3} \times \frac{1 + j_2 b_\alpha \mathbf{i}}{1 + j_1 b_\alpha \mathbf{i}} . \tag{5.8}$$

By using (5.8) and Lemma 5.17, we can show the following fact.

LEMMA 5.22. *Let $\alpha \in (\triangle_M)_+$ and $j_1, j_2 \in \mathbb{Z}$. For any $w_1 \in (E_{(\alpha, j_1)})_{x_0}$ and any $w_2 \in (E_{(\alpha, j_2)})_{x_0}$, we have*

$$\begin{aligned} &\sum_{j \in \mathbb{Z} \setminus \{j_1\}} \frac{j - j_2}{j - j_1} ||(\Gamma_{w_1} w_2)^{(\alpha, j)}||^2 + ||(\Gamma_{w_1} w_2)^0||^2 \\ &= \frac{1}{2} \left(\mathrm{Re} \left(\frac{1 + j_1 b_\alpha \mathbf{i}}{1 + j_2 b_\alpha \mathbf{i}} \right) \right)^{-1} \mathrm{Re} \left(\frac{1}{(1 + j_1 b_\alpha \mathbf{i})(1 + j_2 b_\alpha \mathbf{i})} \right) \\ &\quad \times \langle (n_{(\alpha,0)})_{x_0}, (n_{(\alpha,0)})_{x_0} \rangle \, \langle w_1, w_1 \rangle \, ||w_2||^2 . \end{aligned}$$

Also, we can show the following fact.

LEMMA 5.23. *Let P be the complex affine line through $\mathbf{0}$ and $(n_{(\alpha_0,0)})_{x_0}$ for some $\alpha_0 \in (\triangle_M)_+$.*

(i) *If the affine root system* $\mathcal{R}$ *is of type* $(\widetilde{A}_m)$ $(m \geq 2)$, $(\widetilde{D}_m)$ $(m \geq 4)$, $(\widetilde{E}_m)$ $(m = 6, 7, 8)$ *or* $(\widetilde{F}_4)$, *then there exists a* (*complex*) 2-*dimensional complex affine subspace* P' *including* P *such that the affine root system associated with* $L^{P'}_{x_0}(\subset (W_{P'})_{x_0})$ *is of type* $(\widetilde{A}_2)$.

(ii) *If the affine root system* $\mathcal{R}$ *is of type* $(\widetilde{B}_m)$, $(\widetilde{B}^v_m)$ *or* $(\widetilde{B}_m, \widetilde{B}^v_m)$ $(m \geq 2)$, *then there exists a* (*complex*) 2-*dimensional complex affine subspace* P' *including* P *such that the affine root system associated with* $L^{P'}_{x_0}(\subset (W_{P'})_{x_0})$ *is of type "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}_2)$*", "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}^v_2)$*" or "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}_2, \widetilde{C}^v_2)$*", respectively.*

(iii) *If the affine root system* $\mathcal{R}$ *is of type* $(\widetilde{C}_m)$, $(\widetilde{C}^v_m)$, $(\widetilde{C}'_m)$, $(\widetilde{C}^v_m, \widetilde{C}'_m)$, $(\widetilde{C}'_m, \widetilde{C}_m)$, $(\widetilde{C}^v_m, \widetilde{C}_m)$ *or* $(\widetilde{C}_m, \widetilde{C}^v_m)$ $(m \geq 2)$, *then there exists a* (*complex*) 2-*dimensional complex affine subspace* P' *including* P *such that the affine root system associated with* $L^{P'}_{x_0}(\subset (W_{P'})_{x_0})$ *is of type "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}_2)$*", "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}^v_2)$*", "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}'_2)$*", "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}^v_2, \widetilde{C}'_2)$*", "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}'_2, \widetilde{C}_2)$*", "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}^v_2, \widetilde{C}_2)$*" or "*$(\widetilde{A}_2)$ *or* $(\widetilde{C}_2, \widetilde{C}^v_2)$*", respectively.*

PROOF. First we shall show the statement (i). Let $\Pi(\subset (\triangle_M)_+)$ be a simple root system of $\triangle_M$. Without loss of generality, we may assume that α_0 is one of the elements of Π. Since $\mathcal{R}$ is of $(\widetilde{A}_m)$ $(m \geq 2)$, $(\widetilde{D}_m)$ $(m \geq 4)$, $(\widetilde{E}_m)$ $(m = 6, 7, 8)$ or $(\widetilde{F}_4)$, it follows from their Dynkin diagrams that there exists $\alpha_1 \in \Pi$ such that the angle between $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$ is equal to $\frac{2\pi}{3}$. Let P_1 be the complex affine line through $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$, and P' the (complex) 2-dimensional complex affine subspace through $\mathbf{0}$, $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$. It is clear that $P_1 \subset P'$. Also, it is easy to show that the root system associated with $L^{P_1}_{x_0}$ is of type (A_2) and hence the affine root system associated with $L^{P'}_{x_0}$ is of type $(\widetilde{A}_2)$. This completes the proof of the statement (i).

Next we shall show the statement (ii). Since $\triangle_M$ is of type (B_m), the positive root system $(\triangle_M)_+$ is described as

$$(\triangle_M)_+ = \{\theta_a \,|\, 1 \leq a \leq m\} \cup \{\theta_a \pm \theta_b \,|\, 1 \leq a < b \leq m\}$$

for an orthonormal base $\theta_1, \ldots, \theta_m$ of the dual space $\mathfrak{b}^*$ of $\mathfrak{b}$, the simple root system Π is equal to $\{\theta_i - \theta_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\theta_n\}$ and the highest root is equal to $\theta_1 + \theta_2$, where we need to replace the inner product $\langle\ ,\ \rangle|_{\mathfrak{b}_{\mathbb{R}} \times \mathfrak{b}_{\mathbb{R}}}$ to its suitable constant-multiple. Without loss of generality, we may assume that α_0 is one of the elements of Π. In the case where α_0 is other than θ_n, there exists $\alpha_1 \in \Pi$ such that the angle between $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$ is equal to $\frac{2\pi}{3}$. Let P_1 be the complex affine line through $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$, and P' the (complex) 2-dimensional complex affine subspace through $\mathbf{0}$, $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$. Then it is shown that the root system associated with $L^{P_1}_{x_0}$ is of type (A_2) and hence the affine root system associated with $L^{P'}_{x_0}$ is of type $(\widetilde{A}_2)$. In the case where α_0 is equal to θ_n, we can take $\alpha_1 \in \Pi$ such that the angle between $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$ is equal to $\frac{3\pi}{4}$. Let P_1 be the complex affine line through $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$, and P' the (complex) 2-dimensional complex affine subspace through $\mathbf{0}$, $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$. Then it is shown that, in correspondence to $\mathcal{W}$ is of type $(\widetilde{B}_m)$, $(\widetilde{B}^v_m)$ or $(\widetilde{B}_m, \widetilde{B}^v_m)$ $(m \geq 2)$, the root system associated with $L^{P_1}_{x_0}$ is of type (C_2), (C^v_2) or (C_2, C^v_2) and hence the affine root system associated with $L^{P'}_{x_0}$ is of type $(\widetilde{C}_2)$, $(\widetilde{C}^v_2)$ or $(\widetilde{C}_2, \widetilde{C}^v_2)$.

Next we shall show the statement (iii). Since $\triangle_M$ is of type (C_m), the positive root system $(\triangle_M)_+$ is described as

$$(\triangle_M)_+ = \{2\theta_a \,|1 \leq a \leq m\} \cup \{\theta_a \pm \theta_b \,|1 \leq a < b \leq m\}$$

for an orthonormal base $\theta_1, \ldots, \theta_m$ of the dual space $\mathfrak{b}^*$, the simple root system Π is equal to $\{\theta_i - \theta_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2\theta_n\}$ and the highest root is equal to $2\theta_1$, where we need to replace the inner product $\langle\ ,\ \rangle|_{\mathfrak{b}_{\mathbb{R}} \times \mathfrak{b}_{\mathbb{R}}}$ to its suitable constant-multiple. Without loss of generality, we may assume that α_0 is one of the elements of Π. In the case where α_0 is other than $2\theta_n$, there exists $\alpha_1 \in (\triangle_M)_+$ such that the angle between $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$ is equal to $\frac{2\pi}{3}$. Let P_1 be the complex affine line through $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$, and P' the (complex) 2-dimensional complex affine subspace through $\mathbf{0}$, $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$. Then it is shown that the root system associated with $L^{P_1}_{x_0}$ is of type (A_2) and hence the affine root system associated with $L^{P'}_{x_0}$ is of type $(\widetilde{A}_2)$. In the case where α_0 is equal to $2\theta_n$, we can take $\alpha_1 \in (\triangle_M)_+$ such that the angle between $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$ is equal to $\frac{3\pi}{4}$. Let $P_1(\subset \mathfrak{b}^{\mathbb{C}})$ be the complex affine line through $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$, and P' the (complex) 2-dimensional complex affine subspace through $\mathbf{0}$, $(n_{(\alpha_0,0)})_{x_0}$ and $(n_{(\alpha_1,0)})_{x_0}$. Then it is shown that, in correspondence to $\mathcal{W}$ is of type $(\widetilde{C}_m)$, $(\widetilde{C}^v_m)$, $(\widetilde{C}'_m)$, $(\widetilde{C}^v_m, \widetilde{C}'_m)$, $(\widetilde{C}'_m, \widetilde{C}_m)$, $(\widetilde{C}^v_m, \widetilde{C}_m)$ or $(\widetilde{C}_m, \widetilde{C}^v_m)$ $(m \geq 2)$, the root system associated with $L^{P_1}_{x_0}$ is of type (C_2), (C^v_2), (C'_2), (C^v_2, C'_2), (C'_2, C_2), (C^v_2, C_2) or (C_2, C^v_2) and hence the affine root system associated with $L^{P'}_{x_0}$ is of type $(\widetilde{C}_2)$, $(\widetilde{C}^v_2)$, $(\widetilde{C}'_2)$, $(\widetilde{C}^v_2, \widetilde{C}'_2)$, $(\widetilde{C}'_2, \widetilde{C}_2)$, $(\widetilde{C}^v_2, \widetilde{C}_2)$ or $(\widetilde{C}_2, \widetilde{C}^v_2)$. □

Also, we can show the following fact.

LEMMA 5.24. *If the affine root system $\mathcal{R}$ is of type $(\widetilde{G}_2)$ and if $\langle n_{i_1}, n_{i_2}\rangle = 0$, then $\Gamma_{w_{i_1}} w_{i_2} = 0$ for any $w_{i_1} \in (E_{i_1})_{x_0}$ and $w_{i_2} \in (E_{i_2})_{x_0}$.*

PROOF. Let $i_k = (\alpha_k, j_k)$ $(k = 1, 2)$. Let P be the complex affine line through $(n_{i_1})_{x_0}$ and $(n_{i_2})_{x_0}$. Since $\langle n_{i_1}, n_{i_2}\rangle = 0$, we have $\langle (n_{i_1})_{x_0}, (n_{i_2})_{x_0}\rangle = 0$. If there does not exist further $i_3 \in I$ with $(n_{i_3})_{x_0} \in P$, then the root system associated with the slice $L^P_{x_0}$ is of type $(A_1 \times A_1)$. Hence we have $\Gamma_{(E_{i_1})_{x_0}}(E_{i_2})_{x_0} = 0$. Otherwise, it is shown that $\{i \in I \mid (n_i)_{x_0} \in P\}$ consists of exactly six elements because $\triangle_M$ is of type (G_2), where we note that $\{i \in I \mid (n_i)_{x_0} \in P\} = \{i \in I \mid (n_i)_{x_0} \in P \cap \mathfrak{b}_{\mathbb{R}}\}$ and that each $P \cap \mathfrak{b}_{\mathbb{R}}$ is a real affine line in $\mathfrak{b}_{\mathbb{R}}$. The root system $\triangle_P$ associated with the slice $L^P_{x_0}(\subset (W_P)_{x_0})$ is of type (G_2). The slice $L^P_{x_0}$ is regarded as a principal orbit of the isotropy action of an anti-Kaehler symmetric space L/H whose root system is of type (G_2). Let $\mathfrak{l} = \mathfrak{h} + \mathfrak{p}$ be the canonical decomposition of the Lie algebra $\mathfrak{l}$ of L associated with the symmetric pair (L, H). The space $\mathfrak{p}$ is identified with $(W_P)_{x_0}$ and the normal space of $L^P_{x_0}(\subset (W_P)_{x_0})$ at x_0 is identified with a maximal abelian subspace $\mathfrak{b}'$ of $\mathfrak{p}$. Denote by $\mathfrak{p}_{\overline{\alpha}}(\subset \mathfrak{p})$ and $\mathfrak{h}_{\overline{\alpha}}(\subset \mathfrak{h})$ be the root spaces for $\overline{\alpha} \in \triangle_P$. The restriction $\overline{\alpha}_k := \alpha_k|_{\mathfrak{b}'}$ of α_k to $\mathfrak{b}'$ $(k = 1, 2)$ are elements of $\triangle_P$, where $\mathfrak{b}'$ is regarded as a linear subspace of $\mathfrak{b}$ under the identification of $\mathfrak{b}'$ and the normal space $T^{\perp}_{x_0} L^P_{x_0}$ of $L^P_{x_0}$ in $(W_P)_{x_0}$. For any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$, we have

$$\Gamma_{w_1} w_2 \in [\mathfrak{h}_{\overline{\alpha}_1}, \mathfrak{p}_{\overline{\alpha}_2}] \subset \mathfrak{p}_{\overline{\alpha}_1+\overline{\alpha}_2} + \mathfrak{p}_{\overline{\alpha}_1-\overline{\alpha}_2} .$$

Since $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are orthogonal and $\triangle_P$ is of type (G_2), we have $\overline{\alpha}_1 \pm \overline{\alpha}_2 \notin \triangle_P$. Hence we have $\Gamma_{w_1} w_2 = 0$. This completes the proof. $\square$

By using Lemmas 5.6, 5.7, 5.10, 5.11, 5.14, 5.23, 5.24 and Lemma 8.3 of [10], we can show the following fact.

THEOREM 5.25. *If $\mathcal{R}$ is of type $(\widetilde{A}_m)$ $(m \geq 2)$, $(\widetilde{D}_m)$ $(m \geq 4)$, $(\widetilde{E}_6)$, $(\widetilde{E}_7)$, $(\widetilde{E}_8)$, $(\widetilde{F}_4)$ or $(\widetilde{G}_2)$, then $\Gamma_{(E_{(\alpha,j_1)})_{x_0}} (E_{(\alpha,j_2)})_{x_0} \subset (E_0)_{x_0}$ holds for any $\alpha \in (\triangle_M)_+$ and $j_1, j_2 \in \mathbb{Z}$.*

PROOF. According to Lemma 5.23, we may assume that $\mathcal{R}$ is of type $(\widetilde{A}_2)$ or $(\widetilde{G}_2)$. Furthermore, according to Lemma 5.6, we may assume that $j_1 \neq j_2$. Set $i_k := (\alpha, j_k)$ $(k = 1, 2)$. Suppose that $(\Gamma_{w_1} w_2)^{i_3} \neq 0$ for some $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$ and some $i_3 \in I$. Take $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$ with $(\Gamma_{w_1} w_2)^{i_3} \neq 0$. Let P be the complex affine line through $\mathbf{0}$ and $(n_{i_1})_{x_0}$. Since $L^P_{x_0}$ is totally geodesic in M, we have $(E_{i_3})_{x_0} \subset T_{x_0} M$ and hence $(n_{i_3})_{x_0} \in P$. Hence i_3 is expressed as $i_3 = (\alpha, j_3)$ for some $j_3 \in \mathbb{Z}$. According to Lemma 5.7, we have $j_3 \neq j_1, j_2$. According to Lemma 5.11, there exists $i_4, i_5 \in I$ such that $(n_{i_4})_{x_0}$ and $(n_{i_5})_{x_0}$ are $\mathbb{C}$-linearly independent and that $\langle (\Gamma_{w_1} w_2)^{i_3}, \Gamma_{w_5} w_4 \rangle \neq 0$ for some $w_4 \in (E_{i_4})_{x_0}$ and some $w_5 \in (E_{i_5})_{x_0}$. Since $\langle (\Gamma_{w_1} w_2)^{i_3}, \Gamma_{w_5} w_4 \rangle \neq 0$, we have $(\Gamma_{w_5} w_4)^{i_3} \neq 0$. Hence it follows from Lemma 5.10 that $(n_{i_3})_{x_0}$, $(n_{i_4})_{x_0}$ and $(n_{i_5})_{x_0}$ are contained in a complex affine line P_1. Since $P \cap P_1 = \{(n_{i_3})_{x_0}\}$, it follows from Lemma 5.10 that $\langle \Gamma_{w_1} w_2, \Gamma_{w_5} w_4 \rangle = \langle (\Gamma_{w_1} w_2)^{i_3}, (\Gamma_{w_5} w_4)^{i_3} \rangle \neq 0$. Also, it is clear that arbitrarily chosen three of $(n_{i_1})_{x_0}$, $(n_{i_2})_{x_0}$, $(n_{i_4})_{x_0}$ and $(n_{i_5})_{x_0}$ are not contained in any complex affine line. Hence, it follows from Lemma 5.15 that

$$\langle \Gamma_{w_1} w_2, \Gamma_{w_5} w_4 \rangle = \langle \Gamma_{w_5} w_2, \Gamma_{w_1} w_4 \rangle + c \langle \Gamma_{w_1} w_5, \Gamma_{w_2} w_4 \rangle ,$$

where c is as in Lemma 5.15. Hence we have

$$\text{(I)} \quad \langle \Gamma_{w_5} w_2, \Gamma_{w_1} w_4 \rangle \neq 0 \qquad \text{or} \qquad \text{(II)} \quad \langle \Gamma_{w_1} w_5, \Gamma_{w_2} w_4 \rangle \neq 0 .$$

We consider the case of (I). According to Lemma 5.10, this fact implies that the complex affine line through $(n_{i_2})_{x_0}$ and $(n_{i_5})_{x_0}$ intersects with the complex affine line through and $(n_{i_1})_{x_0}$ and $(n_{i_4})_{x_0}$ and the only intersection point is equal to $(n_{i_6})_{x_0}$ for some $i_6 \in I$. Then, since $(n_{i_1})_{x_0}$, $(n_{i_2})_{x_0}$ and $(n_{i_3})_{x_0}$ are $\mathbb{C}$-linearly dependent pairwisely, the complex focal hyperplanes l_{i_1}, l_{i_2} and l_{i_3} are mutually parallel. Note that they are complex lines because we assume that $\mathcal{R}$ is of type $(\widetilde{A}_2)$ or $(\widetilde{G}_2)$. Hence the (real) lines $l^{\mathbb{R}}_{i_1}$, $l^{\mathbb{R}}_{i_2}$ and $l^{\mathbb{R}}_{i_3}$ (in $\mathfrak{b}_{\mathbb{R}}$) are mutually parallel. Also, since $(n_{i_3})_{x_0}$, $(n_{i_4})_{x_0}$ and $(n_{i_5})_{x_0}$ are contained in a complex line which does not pass 0, we have l_{i_3}, l_{i_4} and l_{i_5} have a common point. Hence the lines $l^{\mathbb{R}}_{i_3}$, $l^{\mathbb{R}}_{i_4}$ and $l^{\mathbb{R}}_{i_5}$ have a common point. Denote by p_{345} this common point. Similarly, since $(n_{i_2})_{x_0}$, $(n_{i_5})_{x_0}$ and $(n_{i_6})_{x_0}$ are contained in a complex line which does not pass 0, we have l_{i_2}, l_{i_5} and l_{i_6} have a common point. Hence the lines $l^{\mathbb{R}}_{i_2}$, $l^{\mathbb{R}}_{i_5}$ and $l^{\mathbb{R}}_{i_6}$ have a common point. Denote by p_{256} this common point. Also, since $(n_{i_1})_{x_0}$, $(n_{i_4})_{x_0}$ and $(n_{i_6})_{x_0}$ are contained in a complex line which does not pass 0, l_{i_1}, l_{i_4} and l_{i_6} have a common point. Hence the lines $l^{\mathbb{R}}_{i_1}$, $l^{\mathbb{R}}_{i_4}$ and $l^{\mathbb{R}}_{i_6}$ have a common point. Denote by p_{146} this common point. These three intersection points p_{345}, p_{256} and p_{146} lie in no line in $\mathfrak{b}_-$ because of $i_4 \neq i_5$. On the other hand, in the case where $\mathcal{R}$ is of type $(\widetilde{A}_2)$, it is clear that the angle between arbitrarily chosen two of $l^{\mathbb{R}}_{i_k}$ $(k = 1, \ldots, 6)$ is equal to an integer-multiple of $\frac{\pi}{6}$ other than $\frac{\pi}{2}$. Also, in the case where $\mathcal{R}$ is of type $(\widetilde{G}_2)$, it follows from Lemmas 5.10 and 5.24 that the angle between arbitrarily chosen two of $l^{\mathbb{R}}_{i_k}$ $(k = 1, \ldots, 6)$ is equal to an integer-multiple of $\frac{\pi}{6}$ other than $\frac{\pi}{2}$. Hence, it follows from (i) of Lemma 5.25 that p_{345}, p_{256} and p_{146} lie in a line in $\mathfrak{b}_{\mathbb{R}}$. Thus a contradiction arises. Similarly, in case of (II), we can drive a contradiction. Therefore we obtain $(\Gamma_{w_1} w_2)^{i_3} = 0$. It follows from the arbitrariness of i_3 that $\Gamma_{w_1} w_2 \in (E_0)_{x_0}$. This completes the proof. $\square$

From Lemmas 5.17 and 5.21 and Theorem 5.25, we have the following fact.

PROPOSITION 5.26. *If $\mathcal{R}$ is one of the following types*:

$$(\widetilde{A}_m)\ (m \geq 2),\quad (\widetilde{D}_m)\ (m \geq 4),\quad (\widetilde{E}_6),\quad (\widetilde{E}_7),\quad (\widetilde{E}_8),\quad (\widetilde{F}_4),\quad (\widetilde{F}^v_4),\quad (\widetilde{G}_2),\quad (\widetilde{G}^v_2)\,,$$

then Γ_w can be extended continuously to $T_{x_0}M$ for any $w \in \underset{i \in I}{\cup} E_i$.

PROOF. Let $\alpha \in (\triangle_M)_+$ and $j_1, j_2 \in \mathbb{Z}$. Set $i_k := (\alpha, j_k)$ $(k = 1, 2)$. From Lemma 5.17 and Theorem 5.25, we have

$$||\Gamma_{w_1} w_2||^2 = \frac{1}{2}\mathrm{Re}\left(\frac{n_{i_1} - 0}{n_{i_2} - 0}\right) \langle n_{i_1}, n_{i_2}\rangle \langle w_1, w_1\rangle\, ||w_2||^2$$

for any $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$. Clearly we have

$$\sup_{j \in \mathbb{Z}} \left| \mathrm{Re}\left(\frac{n_{i_1} - 0}{n_{(\alpha, j)} - 0}\right) \langle n_{i_1}, n_{(\alpha, j)}\rangle \right| < \infty\,.$$

Denote by C this supremum. Then we have

$$||\Gamma_{w_1} w_2|| \leq \sqrt{\frac{C}{2}}\, |||w_1||\, ||w_2||\,.$$

Hence, it follows from the arbitrarinesses of w_2 and j_2 that

$$||\Gamma_{w_1} w|| \leq \sqrt{\frac{C}{2}}\, |||w_1||\, ||w||$$

for any $w \in \bigcup_{j\in\mathbb{Z}}(E_{(\alpha,j)})_{x_0}$. On the other hand, since $\varGamma_{(E_{i_1})_{x_0}}(E_{(\alpha,j)})_{x_0} \subset (E_0)_{x_0}$ $(j \in \mathbb{Z})$ by Theorem 5.25, it follows from Lemma 5.16 that

$$\langle \varGamma_{(E_{i_1})_{x_0}}(E_{(\alpha,j)})_{x_0},\ \varGamma_{(E_{i_1})_{x_0}}(E_{(\alpha,j')})_{x_0}\rangle = 0$$

for any $j' \in \mathbb{Z}$ satisfying $j' \neq j_1, j, 2j_1 - j$. Therefore, by using Lemma 5.21, we can show that

$$||\varGamma_{w_1} w|| \leq \sqrt{\frac{3C}{2}}|||w_1||\,||w||$$

for any $w \in \bigoplus_{j\in\mathbb{Z}}(E_{(\alpha,j)})_{x_0}$. Thus the restriction of $\varGamma_{w_1}$ to $\bigoplus_{j\in\mathbb{Z}}(E_{(\alpha,j)})_{x_0}$ is bounded and hence it can be extended continuously to $\overline{\bigoplus_{j\in\mathbb{Z}}(E_{(\alpha,j)})_{x_0}}$. Therefore, according to Lemma 5.20, $\varGamma_{w_1}$ can be extended continuously to $T_{x_0}M$. $\square$

From Lemmas 5.10, 5.11, 5.15 5.21, 5.23, Theorem 5.25 and Lemma 8.3 of [10], we have the following fact.

LEMMA 5.27. *For any $\alpha \in (\triangle_M)_+$ and any $j_1, j_2 \in \mathbb{Z}$, we have*

$$\varGamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0} \subset (E_0)_{x_0} \oplus (E_{(\alpha,2j_1-j_2)})_{x_0} \oplus (E_{(\alpha,2j_2-j_1)})_{x_0} \oplus (E_{(\alpha,\frac{j_1+j_2}{2})})_{x_0}\,,$$

where the last term is omitted in the case where $j_1 + j_2$ is odd.

PROOF. For simplicity set $i_k := (\alpha, j_k)$ $(k = 1, 2)$. According to Lemma 5.23 and Theorem 5.25, we suffice to show in the case where $(\mathcal{R})$ is of type $(\widetilde{C}_2)$, $(\widetilde{C}_2^v)$, $(\widetilde{C}_2')$, $(\widetilde{C}_2^v, \widetilde{C}_2')$, $(\widetilde{C}_2', \widetilde{C}_2)$, $(\widetilde{C}_2^v, \widetilde{C}_2)$ or $(\widetilde{C}_2, \widetilde{C}_2^v)$. Let P be the complex affne line through $\mathbf{0}$ and $(n_{(\alpha,0)})_{x_0}$. Since $L^P_{x_0}$ is totally geodesic in M, we have

$$\varGamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0} \subset \overline{(E_0)_{x_0} \oplus \left(\bigoplus_{j\in\mathbb{Z}}(E_{(\alpha,j)})_{x_0}\right)}\,.$$

Assume that $(\varGamma_{w_1}w_2)^{(\alpha,j_3)} \neq 0$ for some $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$ and some $j_3 \in \mathbb{Z}$. Set $i_3 := (\alpha, j_3)$. Then it follows from Lemma 5.7 that $j_3 \neq j_1, j_2$. According to Lemma 5.11, there exist $i_k = (\alpha_k, j_k)$ $(k = 4, 5)$ such that $\langle(\varGamma_{w_1}w_2)^{i_3}, \varGamma_{w_4}w_5\rangle \neq 0$ for some $w_k \in (E_{i_k})_{x_0}$ $(k = 4, 5)$. As in the proof of Theorem 5.25, we can show

$$\text{(I)}\quad \langle\varGamma_{w_5}w_2, \varGamma_{w_1}w_4\rangle \neq 0 \qquad \text{or} \qquad \text{(II)}\quad \langle\varGamma_{w_1}w_5, \varGamma_{w_2}w_4\rangle \neq 0$$

in terms of Lemmas 5.10 and 5.15. We consider the case of (I). According to Lemma 5.10, this fact implies that the complex affine line through $(n_{i_2})_{x_0}$ and $(n_{i_5})_{x_0}$ intersects with the complex affine line through $(n_{i_1})_{x_0}$ and $(n_{i_4})_{x_0}$ and the only intersection point is equal to $(n_{i_6})_{x_0}$ for some $i_6 \in I$. Then, as in the proof of Theorem 5.25, we can show that $l^{\mathbb{R}}_{i_1}$, $l^{\mathbb{R}}_{i_2}$ and $l^{\mathbb{R}}_{i_3}$ are mutually parallel, that $l^{\mathbb{R}}_{i_3}$, $l^{\mathbb{R}}_{i_4}$ and $l^{\mathbb{R}}_{i_5}$ have the common point (which we denote by

p_{345}), that $l^{\mathbb{R}}_{i_2}$, $l^{\mathbb{R}}_{i_5}$ and $l^{\mathbb{R}}_{i_6}$ have the common point (which we denote by p_{256}) and that $l^{\mathbb{R}}_{i_1}$, $l^{\mathbb{R}}_{i_4}$ and $l^{\mathbb{R}}_{i_6}$ have the common point (which we denote by p_{146}). These three intersection points p_{345}, p_{256} and p_{146} are lie in no line in $\mathfrak{b}_{\mathbb{R}}$ because of $i_4 \neq i_5$. Hence, it follows from (ii) of Lemma 5.25 that one of $l^{\mathbb{R}}_{i_1}$, $l^{\mathbb{R}}_{i_2}$, $l^{\mathbb{R}}_{i_3}$ lies in the half way distant between the other two, that is, one of j_1, j_2, j_3 is equal to the half of the sum of the other two (i.e., $j_3 = \frac{j_1+j_2}{2}$, $2j_1 - j_2$ or $2j_2 - j_1$). Thus we obtain the desired relation. Similarly, in case of (II), we can derive the desired relation. $\square$

By using Lemmas 5.16, 5.21 and 5.27, we can show the following fact in the method of the proof of Corollary 8.7 of [10].

LEMMA 5.28. *Let $\alpha \in (\triangle_M)_+$ and $j_k \in \mathbb{Z}$ $(k = 1, 2, 3)$ with $j_1 \neq j_2$. Then we have $\langle \Gamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0}, \Gamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_3)})_{x_0}\rangle = 0$ if j_3 is not one of*

$$4j_2 - 3j_1,\ 2j_2 - j_1,\ j_2,\ \frac{j_1 + j_2}{2},\ \frac{3j_1 + j_2}{4},\ \frac{3j_1 - j_2}{2},\ 2j_1 - j_2,\ 3j_1 - 2j_2\,.$$

Let P be a complex affine line in $\mathfrak{b}$ containing exactly four J-curvature normals $(n_{(\alpha_k,j_k)})_{x_0}$ $(k = 1, \ldots, 4)$ at x_0 and $\mathfrak{b}'$ the (complex) 2-dimensional complex linear subspace of $\mathfrak{b}$ spanned by $(n_{(\alpha_k,j_k)})_{x_0}$ $(k = 1, \ldots, 4)$. Set $i_k := (\alpha_k, j_k)$ $(k = 1, \ldots, 4)$. Then the root system (which we denote by $\triangle_P$) of the slice $L^P_{x_0}$ is of type (B_2) or (BC_2). Hence $\triangle_P$ is given by

$$\triangle_P = \begin{cases} \{\pm\alpha_k|_{\mathfrak{b}'\cap\mathfrak{b}_{\mathbb{R}}} \mid k = 1, \ldots, 4\} & (\text{when } \triangle_P : (B_2)\text{-type}) \\ \{\pm\alpha_k|_{\mathfrak{b}'\cap\mathfrak{b}_{\mathbb{R}}} \mid k = 1, \ldots, 4\} & \\ \cup\{\pm 2\alpha_k|_{\mathfrak{b}'\cap\mathfrak{b}_{\mathbb{R}}} \mid k = 1, 2\} & (\text{when } \triangle_P : (BC_2)\text{-type})\,, \end{cases}$$

where we need to permute $i_1, \ldots, i_4$ suitably if necessary. If $\triangle_P$ is of type (B_2), then E_{i_k} $(k = 1, \ldots, 4)$ are irreducible with respect to $(\Phi_{i_k})_{x_0}$, respectively, where Φ_{i_k} is the normal holonomy group of the focal submanifold $f_{i_k}(M)$ corresponding to E_{i_k} at x_0 and $(\Phi_{i_k})_{x_0}$ is the isotropy group of Φ_{i_k} at x_0. Also, if $\triangle_P$ is of type (BC_2), then E_{i_k} $(k = 1, 2)$ are reducible with respect to $(\Phi_{i_k})_{x_0}$, respectively, and E_{i_k} $(k = 3, 4)$ are irreducible with respect to $(\Phi_{i_k})_{x_0}$, respectively. We can show the following lemma in the method of the proof of Lemma 8.8 of [10].

LEMMA 5.29. *Let P be as above and $(E_{i_k})_{x_0} = (E'_{i_k})_{x_0} \oplus (E''_{i_k})_{x_0}$ the irreducible decomposition of the action $(\Phi_{i_k})_{x_0} \curvearrowright (E_{i_k})_{x_0}$, where $\dim_{\mathbb{C}}(E''_{i_k})_{x_0} = 0, 1$ or 3.*

(i) *If $\triangle_P$ is of type (B_2), then we have $\Gamma_{(E_{i_3})_{x_0}}(E_{i_4})_{x_0} = 0$.*

(ii) *If $\triangle_P$ is of type (BC_2), then the $(E_{i_k})'_{x_0}$-component of $\Gamma_{(E_{i_3})_{x_0}}(E_{i_4})_{x_0}$ vanishes, where $k = 1, 2$.*

(iii) *If $\triangle_P$ is of type (BC_2), then we have $\Gamma_{(E_{i_1})''_{x_0}}(E_{i_2})_{x_0} = \Gamma_{(E_{i_1})_{x_0}}(E_{i_2})''_{x_0} = 0$.*

By using Lemmas 5.10, 5.23, 5.27, 5.29, Theorem 5.25 and Lemma 8.3 of [10], we can show the following fact corresponding to Theorem 8.12 and Proposition 8.13 of [10].

LEMMA 5.30. (i) *If $E_{(\alpha,j_1)}$ is irreducible and if $j_1 - j_2$ is divisible by* 4 *or the affine root system $\mathcal{R}$ associated with M is not of type $(\widetilde{C}_n)$ $(n \geq 2)$, then we have $\Gamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0} \subset (E_0)_{x_0}$.*

(ii) *If $E_{(\alpha,j_1)}$ is irreducible and if $j_1 - j_2$ is even, then we have $\Gamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0} \subset (E_0)_{x_0} \oplus (E_{(\alpha,\frac{j_1+j_2}{2})})_{x_0}$.*

(iii) *If $E_{(\alpha,j_1)}$ is reducible and if $j_1 - j_2$ is even $(j_1 \neq j_2)$, then we have*

$$(\Gamma_{(E''_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0})^{(\alpha,\frac{j_1+j_2}{2})} = 0\,.$$

Furthermore, if $j_1 - j_2$ is divisible by 4, *then $E_{(\alpha,\frac{j_1+j_2}{2})}$ is reducible and the $(E'_{(\alpha,\frac{j_1+j_2}{2})})_{x_0}$-component of each element of $\Gamma_{(E_{(\alpha,j_1)})_{x_0}}(E_{(\alpha,j_2)})_{x_0}$ vanishes.*

For $\alpha \in (\triangle_M)_+$, we set

$$C_\alpha := \sup_{j,j'\in\mathbb{Z}}\left|\mathrm{Re}\left(\frac{1+j'b_\alpha\mathbf{i}}{1+jb_\alpha\mathbf{i}}\right)^{-1} \times \mathrm{Re}\left(\frac{1}{(1+jb_\alpha\mathbf{i})(1+j'b_\alpha\mathbf{i})}\right)\right|^{\frac{1}{2}}.$$

Clearly we have $C_\alpha < \infty$. By using Lemmas 5.22 and 5.27, we can show the following fact.

LEMMA 5.31. *Let $i_k = (\alpha, j_k)$ $(k = 1, 2)$ and $w_k \in (E_{i_k})_{x_0}$ $(k = 1, 2)$. If $j_1 - j_2$ is not divisible by 2^m, then we have*

$$||\Gamma_{w_1}w_2|| \leq 2^{m-1}C_\alpha\,||(n_{(\alpha,0)})_{x_0}||\;||w_1||\;||w_2||\,,$$

where m is a positive integer.

PROOF. From Lemmas 5.22 and 5.27, we have

$$\begin{aligned}
&2||(\Gamma_{w_1}w_2)^{(\alpha,2j_1-j_2)}||^2 + \frac{1}{2}||(\Gamma_{w_1}w_2)^{(\alpha,2j_2-j_1)}||^2\\
&-||(\Gamma_{w_1}w_2)^{(\alpha,\frac{j_1+j_2}{2})}||^2 + ||(\Gamma_{w_1}w_2)^0||^2\\
=&\frac{1}{2}\mathrm{Re}\left(\frac{1+j_2b_\alpha\mathbf{i}}{1+j_1b_\alpha\mathbf{i}}\right)^{-1}\mathrm{Re}\left(\frac{1}{(1+j_1b_\alpha\mathbf{i})(1+j_2b_\alpha\mathbf{i})}\right)\\
&\times\langle(n_{(\alpha,0)})_{x_0},(n_{(\alpha,0)})_{x_0}\rangle\langle w_1,w_1\rangle||w_2||^2\,.
\end{aligned}$$

By multiplying 2 to both sides and adding $3||(\Gamma_{w_1}w_2)^{(\alpha,\frac{j_1+j_2}{2})}||^2$ to both sides, we obtain

$$\begin{aligned}
||\Gamma_{w_1}w_2||^2 \leq &\left|\mathrm{Re}\left(\frac{1+j_2b_\alpha\mathbf{i}}{1+j_1b_\alpha\mathbf{i}}\right)^{-1}\mathrm{Re}\left(\frac{1}{(1+j_1b_\alpha\mathbf{i})(1+j_2b_\alpha\mathbf{i})}\right)\right|\\
&\times||(n_{(\alpha,0)})_{x_0}||^2||w_1||^2||w_2||^2\\
&+3||(\Gamma_{w_1}w_2)^{(\alpha,\frac{j_1+j_2}{2})}||^2\\
\leq &\,C_\alpha^2||(n_{(\alpha,0)})_{x_0}||^2||w_1||^2||w_2||^2 + 3||(\Gamma_{w_1}w_2)^{(\alpha,\frac{j_1+j_2}{2})}||^2\,.
\end{aligned} \tag{5.9}$$

We use the induction on m. In case of $m = 1$, the statement of this lemma is derived from (5.9) directly. Now we assume that the statement of this lemma holds for $m(\geq 1)$ and that $j_1 - j_2$ is not divisible by 2^{m+1}. Set $w := (\Gamma_{w_1} w_2)^{(\alpha, \frac{j_1+j_2}{2})}$. Since $F_t^{w_1}$'s are holomorphic isometries, Γ_{w_1} preserves $(T_{x_0}M)_-$ and $(T_{x_0}M)_+$ invariantly, respectively. Hence we have $\Gamma_{w_1}((w_2)_\varepsilon) = (\Gamma_{w_1} w_2)_\varepsilon$ ($\varepsilon = -$ or $+$). Also, it follows from the definitions of $(T_{x_0}M)_\varepsilon$ ($\varepsilon = -$ or $+$) that $((\Gamma_{w_1} w_2)_\varepsilon)^{(\alpha, \frac{j_1+j_2}{2})} = ((\Gamma_{w_1} w_2)^{(\alpha, \frac{j_1+j_2}{2})})_\varepsilon$ ($\varepsilon = -$ or $+$). From (i) of Lemma 5.2 and these relations, we have

$$\langle (\Gamma_{w_1} w_2)_\varepsilon, w_\varepsilon \rangle = \langle \Gamma_{w_1} w_2, w_\varepsilon \rangle = -\langle (w_2)_\varepsilon, (\Gamma_{w_1} w)_\varepsilon \rangle \,.$$

Hence we have

$$\langle \Gamma_{w_1} w_2, w \rangle_\pm = -\langle w_2, \Gamma_{w_1} w \rangle_\pm \,. \tag{5.10}$$

Since $j_1 - \frac{j_1+j_2}{2}$ is not divisible by 2^m, it follows from (5.10) and the assumption in the induction that

$$\begin{aligned} &||(\Gamma_{w_1} w_2)^{(\alpha, \frac{j_1+j_2}{2})}||^2 = \langle \Gamma_{w_1} w_2, w \rangle_\pm = -\langle w_2, \Gamma_{w_1} w \rangle_\pm \\ &\leq ||w_2||\ ||\Gamma_{w_1} w|| \leq 2^{m-1} C_\alpha ||(n_{(\alpha,0)})_{x_0}||\ ||w_1||\ ||w||\ ||w_2||\,, \end{aligned}$$

that is,

$$||(\Gamma_{w_1} w_2)^{(\alpha, \frac{j_1+j_2}{2})}|| \leq 2^{m-1} C_\alpha ||(n_{(\alpha,0)})_{x_0}||\ ||w_1||\ ||w_2||\,.$$

From this inequality and (5.9), we obtain

$$||\Gamma_{w_1} w_2|| \leq 2^m C_\alpha ||(n_{(\alpha,0)})_{x_0}||\ ||w_1||\ ||w_2||\,.$$

Thus the statement of this lemma holds for $m + 1$. Therefore the statement of this lemma is true for all $m \in \mathbb{Z}$. □

By using Lemmas 5.7, 5.19, 5.21, 5.22, 5.27, 5.28, 5.30 and 5.31, we shall prove Theorem 5.1.

PROOF OF THEOREM 5.1. Let $i = (\alpha, j) \in I$ and $w \in (E_i)_{x_0}$. We suffice to show that Γ_w is bounded in order to show that X^w is defined on the whole of V.

(Step I) First we shall show that, in the case where j' is an integer with $j' \neq j$ such that $j' - j$ is divided by 4, there exists a positive constant $\bar{C}_\alpha$ depending on only α such that

$$||(\Gamma_w w')^{(\alpha, \frac{j+j'}{2})}|| \leq \bar{C}_\alpha ||(n_{(\alpha,0)})_{x_0}||\ ||w||\ ||w'|| \tag{5.11}$$

holds for any $w' \in (E_{(\alpha, j')})_{x_0}$. If $(E_i)_{x_0}$ is irreducible with respect to $(\Phi_i)_{x_0}$ or "$(E_i)_{x_0}$ is reducible with respect to $(\Phi_i)_{x_0}$ and $w \in (E_i'')_{x_0}$", then the left-hand side of (5.11) vanishes by (i) and (iii) of Lemma 5.30. In the sequel, we consider the case where $(E_i)_{x_0}$ is reducible and where $w \in (E_i')_{x_0}$. Set $i' := (\alpha, j')$, $i'' := (\alpha, \frac{j+j'}{2})$ and $w'' := (\Gamma_w w')^{i''}$. According to

(iii) of Lemma 5.30, we have $w'' \in (E''_{i''})_{x_0}$. In similar to (5.10), we have

$$\langle \Gamma_w w', w'' \rangle_{\pm} = -\langle w', \Gamma_w w'' \rangle_{\pm} \,. \tag{5.12}$$

From this relation, we have

$$\|(\Gamma_w w')^{i''}\|^2 = \langle \Gamma_w w', w'' \rangle_{\pm} = -\langle w', (\Gamma_w w'')^{i'} \rangle_{\pm} \le \|w'\| \, \|(\Gamma_w w'')^{i'}\| \,. \tag{5.13}$$

On the other hand, it follows from Lemma 5.27 that

$$\Gamma_w w'' = (\Gamma_w w'')^0 + (\Gamma_w w'')^{i'} + (\Gamma_w w'')^{(\alpha,(3j-j')/2)} + (\Gamma_w w'')^{(\alpha,(3j+j')/4)} \,.$$

Hence, by using Lemma 5.22, we can show

$$\begin{aligned}
&\frac{1}{2}\|(\Gamma_w w'')^{i'}\|^2 + 2\|(\Gamma_w w'')^{(\alpha,(3j-j')/2)}\|^2 \\
&-\|(\Gamma_w w'')^{(\alpha,(3j+j')/4)}\|^2 + \|(\Gamma_w w'')^0\|^2 \\
&\le \frac{1}{2}\left| \mathrm{Re}\left(\frac{1+jb_\alpha\mathbf{i}}{1+((j+j')/2)b_\alpha\mathbf{i}}\right)^{-1} \mathrm{Re}\left(\frac{1}{(1+((j+j')/2)b_\alpha\mathbf{i})(1+jb_\alpha\mathbf{i})}\right)\right| \\
&\times\|(n_{(\alpha,0)})_{x_0}\|^2\|w''\|^2\|w\|^2 \,.
\end{aligned}$$

Also, it follows from (iii) of Lemma 5.30 that $(\Gamma_{w''} w)^{(\alpha,(3j+j')/4)} = 0$. Hence we obtain

$$\begin{aligned}
&\|(\Gamma_w w'')^{i'}\| \\
&\le \left| \mathrm{Re}\left(\frac{1+jb_\alpha\mathbf{i}}{1+((j+j')/2)b_\alpha\mathbf{i}}\right)^{-1} \mathrm{Re}\left(\frac{1}{(1+((j+j')/2)b_\alpha\mathbf{i})(1+jb_\alpha\mathbf{i})}\right)\right|^{\frac{1}{2}} \\
&\times\|(n_{(\alpha,0)})_{x_0}\| \, \|w''\| \, \|w\| \,.
\end{aligned} \tag{5.14}$$

Easily we can show

$$\sup_{j,j'\in\mathbb{Z}} \left| \mathrm{Re}\left(\frac{1+jb_\alpha\mathbf{i}}{1+((j+j')/2)b_\alpha\mathbf{i}}\right)^{-1} \mathrm{Re}\left(\frac{1}{(1+((j+j')/2)b_\alpha\mathbf{i})(1+jb_\alpha\mathbf{i})}\right)\right|^{\frac{1}{2}} < \infty \,.$$

Denote by $\bar{C}_\alpha$ this supremum. From (5.13) and (5.14), it follows that the inequality (5.11) holds for this constant $\bar{C}_\alpha$.

(Step II) From the fact shown in (Step I), Lemmas 5.19, 5.21, 5.28, 5.30 and 5.31, it follows that there exists a positive constant $\widehat{C}_\alpha$ depending on only α such that

$$\|\Gamma_w w'\| \le \widehat{C}_\alpha \|w\| \, \|w'\|$$

for any $w' \in (E_0)^\perp_{x_0}$. Assume that $w' \in (E_0)_{x_0}$. Then, since $\Gamma_w w' \in (E_0)^\perp_{x_0}$ by Lemma 5.7, we can find a sequence $\{w''_k\}$ in $\underset{\hat{i}\in I}{\oplus} (E_{\hat{i}})_{x_0}$ with $\lim\limits_{k\to\infty} w''_k = \Gamma_w w'$ (with respect to $\|\cdot\|$). Then we have

$$\begin{aligned}||\Gamma_w w'||^2 &= \lim_{k\to\infty}\langle \Gamma_w w', w''_k\rangle_{\pm} = -\lim_{k\to\infty}\langle w', \Gamma_w w''_k\rangle_{\pm} \\ &\leq \lim_{k\to\infty}||w'||\,||\Gamma_w w''_k|| \leq \widehat{C}_\alpha ||w||\,||w'||\,||\Gamma_w w'||\,,\end{aligned}$$

that is,

$$||\Gamma_w w'|| \leq \widehat{C}_\alpha ||w||\,||w'||\,,$$

where $\widehat{C}_\alpha$ is as above. Thus Γ_w is bounded. Therefore, X^w is defined on the whole of V. □

By using Theorem 3.3, its proof (see the proof of Theorem A in [26]) and Theorem 5.1, we shall prove Theorem A.

PROOF OF THEOREM A. Take any $i \in I$ and any $w_0 \in (E_i)_{x_0}$. According to Theorem 5.1, X^{w_0} is defined over the whole of V, that is, $F_1^{w_0} \in I_h^b(V)$. On the other hand, $F_1^{w_0}$ preserves M invariantly. Hence we have $F_1^{w_0} \in H_b$. Since the holomorphic isometries f_k's in the proof of Theorem A in [26] are given as the composition of the holomorphic isometries of $F_1^{w_0}$-type, it is then shown that f_k's are elements of H_b and hence so is also the holomorphic isometry $\widehat{f}$ in Step IV of the proof of Theorem A in [26] (see the construction of $\widehat{f}$ in Step IV). Therefore we obtain $H_b \cdot x = M$ for any $x \in M$. □

Appendix

In this Appendix, we give examples of elements of $I_h(V) \setminus I_h^b(V)$. Denote by $\mathcal{K}^h$ the Lie algebra of all holomorphic Killing fields on the whole of V. Also, denote by $\mathfrak{o}_{AK}(V)$ the Lie algebra of all continuous skew-symmetric complex linear maps from V to oneself. Any $X \in \mathcal{K}^h$ is described as $X_u = Au + b$ ($u \in V$) for some $A \in \mathfrak{o}_{AK}(V)$ and some $b \in V$. Hence $\mathcal{K}^h$ is identified with $\mathfrak{o}_{AK}(V) \times V$. Give $\mathfrak{o}_{AK}(V)$ the operator norm (which we denote by $||\cdot||_{\mathrm{op}}$) associated with $\langle\ ,\ \rangle_{\pm}$ and $\mathcal{K}^h$ the product norm of this norm $||\cdot||_{\mathrm{op}}$ of $\mathfrak{o}_{AK}(V)$ and the norm $||\cdot||$ of V. The space $\mathcal{K}^h$ is a Banach Lie algebra with respect to this norm. The group $I_h^b(V)$ is a Banach Lie group consisting of all holomorphic isometry f's of V which admit a one-parameter transformation group $\{f_t \mid t \in \mathbb{R}\}$ of V such that each f_t is a holomorphic isometry of V, that $f_1 = f$ and that $\left.\frac{d}{dt}\right|_{t=0}(f_t)_*$ is an element of $\mathfrak{o}_{AK}(V)$. Note that, for a general holomorphic isometry f of V, $\left.\frac{d}{dt}\right|_{t=0}(f_t)_*$ is not necessarily defined on the whole of V (but it can be defined on a dense linear subspace of V). It is clear that the Lie algebra of this Banach Lie group $I_h^b(V)$ is equal to $\mathcal{K}^h$.

EXAMPLE. We shall give an example of an element of $I_h(V) \setminus I_h^b(V)$. Let V be a complex linear topological space consisting of all complex number sequences $\{z_k\}_{k=1}^{\infty}$'s satisfying

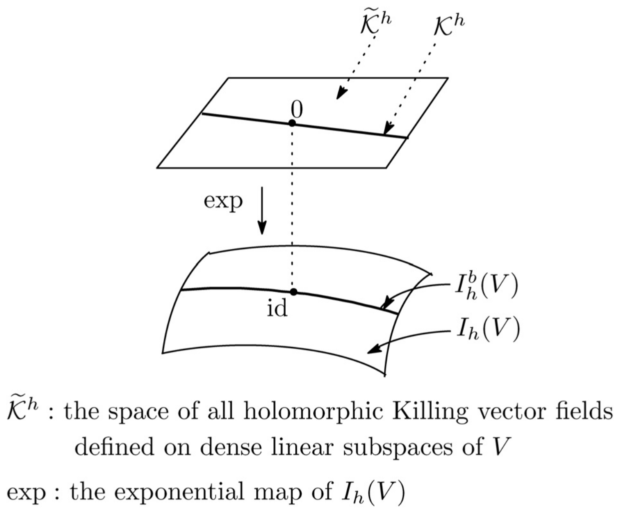

FIGURE 2. The Banach Lie subgroup of the isometry group of V

$\sum_{k=1}^{\infty}|z_k|^2 < \infty$, and $\langle\ ,\ \rangle$ a non-degenerate inner product of V defined by

$$\langle \{z_k\}_{k=1}^{\infty},\ \{w_k\}_{k=1}^{\infty}\rangle := 2\mathrm{Re}\left(\sum_{k=1}^{\infty} z_k w_k\right) \qquad (\{z_k\}_{k=1}^{\infty},\ \{w_k\}_{k=1}^{\infty} \in V)\,.$$

The pair $(V, \langle\ ,\ \rangle)$ is an infinite dimensional anti-Kaehler space. Define a complex linear transformation A_t $(t \in \mathbb{R})$ of V by assigning $\{w_k\}_{k=1}^{\infty}$ defined by

$$\begin{pmatrix} w_{2k-1} \\ w_{2k} \end{pmatrix} := \begin{pmatrix} \cos 2k\pi t & -\sin 2k\pi t \\ \sin 2k\pi t & \cos 2k\pi t \end{pmatrix} \begin{pmatrix} z_{2k-1} \\ z_{2k} \end{pmatrix} \qquad (k \in \mathbb{N})$$

to each $\{z_k\}_{k=1}^{\infty} \in V$. It is clear that each A_t is a holomorphic linear isometry of V. Define $f_t \in I_h(V)$ by $f_t(u) := A_t u + b_t$ $(u \in V)$, where b_t is a curve in V with $b_0 = 0$. Set

$$B := \left.\frac{d}{dt}\right|_{t=0} f_{t*} = \left.\frac{d}{dt}\right|_{t=0} A_t\,.$$

It is easy to show that B is a skew-symmetric complex linear map from a dense linear subspace U of V to V assigning $\{w_k\}_{k=1}^{\infty}$ defined by

$$\begin{pmatrix} w_{2k-1} \\ w_{2k} \end{pmatrix} := \begin{pmatrix} 0 & -2k\pi \\ 2k\pi & 0 \end{pmatrix} \begin{pmatrix} z_{2k-1} \\ z_{2k} \end{pmatrix} \qquad (k \in \mathbb{N})\,,$$

to each $\{z_k\}_{k=1}^{\infty} \in U$, where U is the set of all elements $\{z_k\}_{k=1}^{\infty}$'s of V satisfying $B(\{z_k\}_{k=1}^{\infty}) \in V$. Let $\{a_k\}_{k=1}^{\infty}$ be an element of V defined by $a_k := \frac{1}{[\frac{k+1}{2}]}$ $(k \in \mathbb{N})$, where $[\cdot]$ is the

Gauss's symbol of $\cdot$. Then we can show $B(\{a_k\}_{k=1}^{\infty}) \notin V$, that is, $\{a_k\}_{k=1}^{\infty} \notin U$. Thus B is not an element of $\mathfrak{o}_{AK}(V)$ and hence f_t does not belong to $I_h^b(V)$ for positive numbers t's sufficiently close to 0, where we note that $f_1 = \mathrm{id} \in I_h^b(V)$.

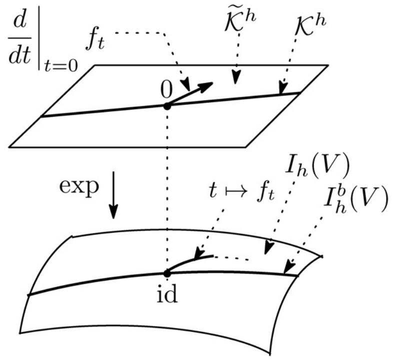

FIGURE 3. An example of an element of $I_h(V) \setminus I_h^b(V)$

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