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On Sums of Averages of Generalized Ramanujan Sums

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Abstract. We shall consider some formulas for weighted averages of the generalized Ramanujan sum $\sum_{d \mid \gcd(k,n)} f(d) g(k/d) h(n/d)$ for any arithmetical functions f, g and h, with the weights concerning completely multiplicative functions, completely additive functions and others.

1. Introduction

Let $\gcd(k,l)$ be the greatest common divisor of the integers k and l, and let $c_k(n)$ denote, as usual, the Ramanujan sum defined for positive integer k and integer n by $c_k(n) = \sum_{1 \le l \le k,\ \gcd(l,k)=1} e^{2\pi i n l/k} = \sum_{d \mid \gcd(k,n)} d\mu(k/d)$, whose summation is the number of positive integers not exceeding k which are relatively prime to k (see [6], [8], [10], [20], [24]), where μ is the Möbius function. For any fixed positive integer r, E. Alkan [1], [2] established the weighted average of Ramanujan's sums, namely

$$S_r(k) = \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j)\,, \tag{1.1}$$

which was considered in proving exact formulas for certain mean square averages of special values of L-functions. Some mean square averages of L-functions have been obtained by E. Alkan [4], [5], A. Bayad and A. Raouj [9], H. Liu [18], R. Ma, Y. L. Zhang and M. Grützmann [19], and others. E. Alkan [2] used Hölder's formula $c_k(n) = \dfrac{\phi(k)}{\phi(k/\gcd(k,n))}\mu(k/\gcd(k,n))$ to obtain the formula

$$S_r(k) = \frac{\phi(k)}{2k} + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} \prod_{p \mid k} \left(1 - \frac{1}{p^{2m}}\right). \tag{1.2}$$

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Here the function $B_m(x)$ (see [8], [10]) denotes the Bernoulli polynomials defined by the expansion $\frac{ze^{xz}}{e^z-1} = \sum_{m=0}^{\infty} B_m(x)\frac{z^m}{m!}$ with $|z| < 2\pi$, where the number B_m is the Bernoulli number given by $B_m(0)$. Another proof of the identity (1.2) was presented by L. Tóth [26], who established some identities for other weighted averages of Ramanujan's sum with weights concerning certain functions. An asymptotic formula for the average of $S_r(k)$ is reduced to

$$\sum_{k\le x} S_r(k) = \left(\frac{3}{\pi^2} + \frac{1}{r+1}\sum_{m=1}^{[r/2]} \binom{r+1}{2m} \frac{B_{2m}}{\zeta(2m+1)}\right) x + O_r(\log x) \tag{1.3}$$

for any sufficiently large positive number x (≥ 2), where the implied constant depends only on r (≥ 1). Connections between Ramanujan sums and Burgess type zeta functions were investigated in [3]. For any positive integers k and j, D. R. Anderson and T. M. Apostol [6] first introduced the function $s_k(j) = \sum_{d\mid \gcd(k,j)} f(d)g(k/d)$ for a generalization of the Ramanujan sum $c_k(j)$ which is said to be the Anderson–Apostol sum [8]. In [6], they derived the following equality, namely,

Let f be completely multiplicative, and let $g(k) = \mu(k)u(k)$, where u is multiplicative. Assume that $f(p) \neq 0$ and $f(p) \neq u(p)$ for all primes p, and let $F = f * g$. Then, Anderson and Apostol showed that

$$s_k(j) = \frac{F(k)}{F\left(\frac{k}{\gcd(k,j)}\right)} \mu\left(\frac{k}{\gcd(k,j)}\right) u\left(\frac{k}{\gcd(k,j)}\right).$$

This is said to be Hölder's formula for $s_k(j)$, and the studies for the function $s_k(j)$ was considered by T. M. Apostol [7], I. Kiuchi and Y. Tanigawa [16], and many mathematicians and physicists. In particular, K. R. Johnson [12] derived some interesting algebraic properties of a generalized Anderson–Apostol sum.

We denote a generalization of the Anderson–Apostol sum $t_k(j)$ by

$$t_k(j) = \sum_{d\mid \gcd(k,j)} f(d)g\left(\frac{k}{d}\right)h\left(\frac{j}{d}\right),$$

where f, g and h are any arithmetical functions. This sum is a generalization of used definition for weighted averages mentioned by E. Alkan [2], V. A. Liskovet [17], L. Tóth [26], [27]. In a recent paper [15], I. Kiuchi, M. Minamide and M. Ueda considered the weighted averages of $t_k(j)$ with the weight function w concerning completely multiplicative or completely additive, namely

$$T(k) = \sum_{j=1}^{k} w(j)t_k(j).$$

They [15] established that

$$T(k) = (fw * gW)(k) \tag{1.4}$$

with w being a completely multiplicative function and $W(d) = \sum_{m=1}^{d} w(m)h(m)$, and

$$T(k) = (fw * gH)(k) + (f * gW)(k) \tag{1.5}$$

with w being a completely additive function and $H(d) = \sum_{m=1}^{d} h(m)$, who also derived some formulas for the weighted average of $s_k(j)$ with other weight function w concerning logarithms, values of arithmetical functions for gcd's, the Gamma function, the Binomial coefficients and the Bernoulli polynomials.

The first purpose of this paper is to derive some formulas for the sum function $\sum_{k\le x} T(k)$ with weight function w being completely multiplicative and completely additive, and we deduce many useful formulas and some corollaries. Secondly we shall use weight functions w of the Gamma function, the Bernoulli polynomials and values of arithmetical function for gcd's to derive some formulas for the sum function $\sum_{k\le x} \frac{1}{k} \sum_{j=1}^{k} w(j)s_k(j)$. Furthermore, we shall use weight functions w concerning values of arithmetical function for gcd's to derive some formulas for the sum function $\sum_{k\le x} \frac{1}{F(k)} \sum_{j=1}^{k} v(\gcd(k, j))s_k(j)$ with any arithmetical function v. We use Theorems 1, 2, 4–7 (below), the Dirichlet convolution and some Lemmas to derive some useful formulas. Using Lemma 2 and Theorem 2, we shall improve upon the formula (1.3).

Notations. Throughout this paper, we use the following notations to prove Theorems and Corollaries. Let ζ be the Riemann zeta-function, and let f, g and h be any arithmetical functions. For any positive integer n, the functions id, id_q and the unit function $\mathbf{1}$ are given by $\mathrm{id}(n) = n$, $\mathrm{id}_q(n) = n^q$ for any real numbers q and $\mathbf{1}(n) = 1$, respectively. And $*$ denotes the Dirichlet convolution of arithmetical functions, that is, for any arithmetical functions f and g we write $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$. We introduce several arithmetical functions which play an important role in the study for summation formula, whose functions deduce the Jordan function ϕ_m, the generalized Dedekind function ψ_m, the Dedekind function ψ, the divisor functions τ, σ defined by $\phi_m = \mathrm{id}_m * \mu$, $\phi = \phi_1$, $\psi_m = \mathrm{id}_m * |\mu|$, $\psi = \psi_1$, $\tau = \mathbf{1} * \mathbf{1}$, $\sigma = \mathrm{id} * \mathbf{1}$, respectively. In what follows, ε denotes any arbitrarily small positive number, not necessarily the same ones at each occurrence.

2. Some Lemmas

Before the introduction of our results, we prepare the following formulas to prove our Theorems and Corollaries.

LEMMA 1. *For any sufficiently large positive number x and any fixed positive integer $r > 0$, we have*

$$\sum_{n\le x} |\mu(n)| = \frac{1}{\zeta(2)}x + O\left(x^{\frac{1}{2}}\exp\left(-C\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right), \tag{2.1}$$

and

$$\sum_{n\le x}|\mu(n)|n^r = \frac{1}{(r+1)\zeta(2)}x^{r+1} + O_r\left(x^{r+\frac{1}{2}}\exp\left(-C\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right), \tag{2.2}$$

where $C > 0$ is certain positive constant and the implied constant of (2.2) depends only on r.

PROOF. The formula (2.1) follows from (14.24) of A. Ivić [11]. We use partial summation formula and (2.1) to obtain the formula (2.2). □

LEMMA 2. *For any sufficiently large positive number $x > 1$ we have*

$$\sum_{l\le x}\frac{\mu(l)}{l}\theta\left(\frac{x}{l}\right) = O\left((\log x)^{2/3}(\log\log x)^{4/3}\right) \tag{2.3}$$

with $\theta(x) = x - [x] - \frac{1}{2}$, and

$$\sum_{l\le x}\frac{\mu(l)}{l} = O\left(\exp\left(-C\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right), \tag{2.4}$$

where $C > 0$ is certain positive constant.

PROOF. The formulas (2.17) and (2.4) follow from Satz 1 (p.144) in A. Walfisz [29] and R. Q. Jia [13], respectively. □

LEMMA 3. *For any sufficiently large positive number x we have*

$$\sum_{k\le x}\phi(k) = \frac{x^2}{2\zeta(2)} + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right), \tag{2.5}$$

$$\sum_{k\le x}\frac{\phi(k)}{k} = \frac{x}{\zeta(2)} + O\left((\log x)^{2/3}(\log\log x)^{4/3}\right), \tag{2.6}$$

$$\sum_{k\le x}\phi(k)\log k = \frac{1}{2\zeta(2)}x^2\log x - \frac{1}{4\zeta(2)}x^2 + O\left(x(\log x)^{5/3}(\log\log x)^{4/3}\right), \tag{2.7}$$

and

$$\sum_{k\le x}\phi(k)k = \frac{1}{3\zeta(2)}x^3 + O\left(x^2(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{2.8}$$

Furthermore, for any sufficiently large positive number x and any fixed positive integer $m > 1$ we have

$$\sum_{k\le x}\frac{\phi_m(k)}{k^m} = \frac{x}{\zeta(m+1)} + D_m(x) + O_m\left(x^{1-m}\right), \tag{2.9}$$

where $D_m(x)$ denotes

$$-\sum_{l\le x}\frac{\mu(l)}{l^m}\theta\left(\frac{x}{l}\right)-\frac{1}{2\zeta(m)} \tag{2.10}$$

with $\theta(x)=x-[x]-\frac{1}{2}$.

REMARK 1. The upper bound for the first term of (2.10) can be estimated by $\zeta(m)$ for any positive integer $m>1$.

PROOF. The proof of (2.5) is given by (35)(p.144) in A. Walfisz [29]. We use Lemma 2 to obtain

$$\begin{aligned}\sum_{k\le x}\frac{\phi(k)}{k}&=\sum_{l\le x}\frac{\mu(l)}{l}\sum_{d\le x/l}1=\sum_{l\le x}\frac{\mu(l)}{l}\left(\frac{x}{l}-\theta\left(\frac{x}{l}\right)-\frac{1}{2}\right)\\&=\frac{x}{\zeta(2)}+O\left((\log x)^{2/3}(\log\log x)^{4/3}\right),\end{aligned}$$

which completes the proof of (2.6). We use the formula (2.5) and partial summation formula to obtain the formulas (2.7) and (2.8). Now, we have

$$\begin{aligned}\sum_{k\le x}\frac{\phi_m(k)}{k^m}&=\sum_{dl\le x}\frac{\mu(l)}{l^m}=\sum_{l\le x}\frac{\mu(l)}{l^m}\sum_{d\le x/l}1\\&=x\sum_{l\le x}\frac{\mu(l)}{l^{m+1}}-\sum_{l\le x}\frac{\mu(l)}{l^m}\theta\left(\frac{x}{l}\right)-\frac{1}{2}\sum_{l\le x}\frac{\mu(l)}{l^m}\\&=\frac{x}{\zeta(m+1)}-\sum_{l\le x}\frac{\mu(l)}{l^m}\theta\left(\frac{x}{l}\right)-\frac{1}{2\zeta(m)}+O_m\left(x^{1-m}\right),\end{aligned}$$

which completes the proof of (2.9). □

LEMMA 4. *For any sufficiently large positive number x we have*

$$\sum_{n\le x}\psi(n)=\frac{\zeta(2)}{2\zeta(4)}x^2+O\left(x(\log x)^{2/3}\right), \tag{2.11}$$

and

$$\sum_{n\le x}\frac{\psi(n)}{n}=\frac{\zeta(2)}{\zeta(4)}x-\frac{1}{2\zeta(2)}\log x+O\left((\log x)^{2/3}\right). \tag{2.12}$$

PROOF. The proofs of (2.11) and (2.12) are given by Satz 2 (p.100) and Satz 3 (p.102) in A. Walfisz [29], respectively. □

LEMMA 5. *For any sufficiently large positive number $x>1$ we have*

$$\sum_{n\le x}\sigma(n)=\frac{\zeta(2)}{2}x^2+O\left(x(\log x)^{2/3}\right), \tag{2.13}$$

$$\sum_{n\le x}\frac{\sigma(n)}{n}=\zeta(2)\,x-\frac{1}{2}\log x+O\left((\log x)^{2/3}\right), \tag{2.14}$$

$$\sum_{n\le x}\sigma(n)\log n=\frac{\zeta(2)}{2}x^2\log x-\frac{\zeta(2)}{4}x^2+O\left(x(\log x)^{5/3}\right) \tag{2.15}$$

and

$$\sum_{n\le x}\sigma(n)n=\frac{\zeta(2)}{3}x^3+O\left(x^2(\log x)^{2/3}\right). \tag{2.16}$$

PROOF. The formulas (2.13) and (2.14) follow from Satz 4 (p.99) and Satz 1 (p.99) in A. Walfisz [29], respectively. We use partial summation formula and the formula (2.13) to deduce the formulas (2.15) and (2.16). □

LEMMA 6. *For any sufficiently large positive number $x > 1$ and any positive numbers $r > 1$ we have*

$$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+O\left(x^{1/3+\varepsilon}\right), \tag{2.17}$$

$$\sum_{n\le x}\tau(n)n^r=\frac{1}{r+1}x^{r+1}\log x+\frac{1}{r+1}\left(2\gamma-\frac{1}{r+1}\right)x^{r+1}+O\left(x^{r+1/3+\varepsilon}\right), \tag{2.18}$$

$$\sum_{n\le x}\tau(n)\log n=x\log^2 x+2(\gamma-1)x\log x-2(\gamma-1)x+O\left(x^{1/3+\varepsilon}\right), \tag{2.19}$$

and

$$\sum_{n\le x}\frac{\tau(n)}{n}=\frac{1}{2}\log^2 x+2\gamma\log x+\gamma^2-2\gamma_1+O\left(x^{-1/2}\right), \tag{2.20}$$

where γ is the Euler constant and γ_1 is the first Stieltjes constant. Furthermore, we have

$$\sum_{n\le x}\frac{\tau(n)}{n}\log n=\frac{1}{3}\log^3 x+\gamma\log^2 x+O(\log x). \tag{2.21}$$

PROOF. The formula (2.17) follows from Theorem 12.2 in E. C. Titchmarsh [28], whose formula is the classical result of G. F. Voronoi concerning the error term in the Dirichlet divisor problem. We use partial summation formula and the formula (2.17) to deduce the formulas (2.18) and (2.19). The formula (2.20) is given by H. Riesel and R. C. Vaughan [23]. Furthermore, we use partial summation formula and the formula (2.20) to obtain the formula (2.21). □

3. Weighted Averages

3.1. Weighted Averages. We first consider the partial sums for weighted averages of $t_k(j)$ with weight functions w, which is completely multiplicative and completely additive, whose formulas express the following two formulas.

THEOREM 1. *Let the notation be as above. For any positive real number $x > 1$ we have*

$$\sum_{k \le x} T(k) = \sum_{dl \le x} f(d) w(d) g(l) \sum_{m=1}^{l} w(m) h(m) \tag{3.1}$$

if w is a completely multiplicative function, and

$$\sum_{k \le x} T(k) = \sum_{dl \le x} f(d) w(d) g(l) \sum_{m=1}^{l} h(m) + \sum_{dl \le x} f(d) g(l) \sum_{m=1}^{l} w(m) h(m) \tag{3.2}$$

if w is a completely additive function.

PROOF. First of all we consider the case where w is completely multiplicative. From (1.4) and $W(l) = \sum_{m=1}^{l} w(m)h(m)$ we have

$$\begin{aligned}\sum_{k \le x} T(k) &= \sum_{k \le x} \sum_{d|k} f(d) w(d) g\left(\frac{k}{d}\right) W\left(\frac{k}{d}\right) \\ &= \sum_{dl \le x} f(d) w(d) g\,(l) \sum_{m=1}^{l} w(m) h(m)\,,\end{aligned}$$

which completes the proof of (3.1). Similarly, when w is completely additive, we have

$$\begin{aligned}\sum_{k \le x} T(k) &= \sum_{k \le x} \sum_{d|k} f(d) w(d) g\left(\frac{k}{d}\right) H\left(\frac{k}{d}\right) + \sum_{k \le x} \sum_{d|k} f(d) g\left(\frac{k}{d}\right) W\left(\frac{k}{d}\right) \\ &= \sum_{dl \le x} f(d) w(d) g(l) \sum_{m=1}^{l} h(m) + \sum_{dl \le x} f(d) g(l) \sum_{m=1}^{l} w(m) h(m)\end{aligned}$$

since $H(l) = \sum_{m=1}^{l} h(m)$. Hence we obtain the formula (3.2). □

We use the formula (2.5) of Corollary 2.1 in I. Kiuchi, M. Minamide and M. Ueda [15], namely

$$\begin{aligned}&\frac{1}{k^r} \sum_{j=1}^{k} j^r \sum_{d \mid \gcd(k,j)} f(d) g\left(\frac{k}{d}\right) \\ &\qquad = \frac{1}{2}(f * g)(k) + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} (f * \mathrm{id}_{1-2m} \cdot g)(k)\end{aligned} \tag{3.3}$$

to deduce the formula (3.4) of Theorem 2. In particular, applying Theorem 2 we derive some interesting and useful formulas.

THEOREM 2. *Let the notation be as above. For any positive real number $x > 1$ and any fixed positive integer r we have*

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)} f(d)g\left(\frac{k}{d}\right) \tag{3.4}$$

$$= \frac{1}{2}\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l} + \frac{1}{r+1}\sum_{dl\le x}\frac{f(d)}{d}g(l) + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l^{2m}}.$$

PROOF. From (3.3) we have

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)} f(d)g\left(\frac{k}{d}\right)$$

$$= \sum_{k\le x}\frac{1}{2k}(f*g)(k) + \frac{1}{r+1}\sum_{m=0}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{k\le x}\frac{(f*\mathrm{id}_{1-2m}\cdot g)(k)}{k}$$

$$= \frac{1}{2}\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l} + \frac{1}{r+1}\sum_{dl\le x}\frac{f(d)}{d}g(l) + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l^{2m}},$$

which completes the proof of Theorem 2. □

Taking $f = \phi$ and $g = \mathbf{1}$ into (3.4) we obtain

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \gcd(k, j) = \frac{1}{(r+1)\zeta(2)}x\log x + O_r(x) \tag{3.5}$$

by using (2.6), the Gauss formula $\sum_{d\mid \gcd(k,j)}\phi(d) = \gcd(k, j)$ and the formula $\sum_{l\le x}\frac{\phi(l)}{l^2} = \frac{1}{\zeta(2)}\log x + O(1)$ for any positive real number $x > 1$. Taking $f = \mathrm{id}\cdot g$ and $g = f/\mathrm{id}$ into (3.4) we have

$$\sum_{k\le x}\frac{1}{k^{r+2}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)} d^2 g(d) f\left(\frac{k}{d}\right) \tag{3.6}$$

$$= \frac{1}{2}\sum_{dl\le x}\frac{f(d)}{d^2}g(l) + \frac{1}{r+1}\sum_{dl\le x}\frac{f(d)}{d}g(l) + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{dl\le x}\frac{f(d)}{d^{2m+1}}g(l).$$

REMARK 2. From (3.4) and (3.6) we obtain

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)}\left(f(d)g\left(\frac{k}{d}\right) - \frac{d^2}{k}f\left(\frac{k}{d}\right)g(d)\right) \tag{3.7}$$

$$= \frac{1}{2} \sum_{dl \le x} \frac{f(d)}{d} g(l) \left(\frac{1}{l} - \frac{1}{d} \right) + \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{dl \le x} \frac{f(d)}{d} g(l) \left(\frac{1}{l^{2m}} - \frac{1}{d^{2m}} \right) .$$

We take $w(j) = j^r$ for any fixed positive integer r into (1.4), multiply $1/k^{r+1}$ and sum over $k \le x$ to obtain

$$\sum_{k \le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r \sum_{d \mid \gcd(k,j)} f(d) g\left(\frac{k}{d}\right) h\left(\frac{j}{d}\right) = \sum_{dl \le x} \frac{f(d)}{d} \frac{g(l)}{l^{r+1}} \sum_{m=1}^{l} h(m) m^r . \tag{3.8}$$

Substituting $h = \mathbf{1}$ into (3.8) and using the identity

$$\sum_{m=1}^{l} m^r = \frac{l^r}{2} + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} l^{r+1-2m} ,$$

we may prove the formula (3.4) in Theorem 2. Next, we can use Theorem 2 and Lemma 2 to improve slightly the error term $O_r(\log x)$ in the formula (1.3) given by E. Alkan [2] to $O_r\left((\log x)^{2/3} (\log\log x)^{4/3}\right)$, namely

THEOREM 3. *Let the notation be as above. For any positive real number $x > 1$ and any fixed positive integer r we have*

$$\sum_{k \le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r \sum_{d \mid \gcd(k,j)} d \mu\left(\frac{k}{d}\right) \tag{3.9}$$
$$= \left(\frac{1}{2\zeta(2)} + \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} \frac{B_{2m}}{\zeta(2m+1)} \right) x + O_r\left((\log x)^{2/3} (\log\log x)^{4/3} \right) .$$

PROOF. We use Theorem 2 to deduce that

$$\sum_{k \le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r \sum_{d \mid \gcd(k,j)} d \mu\left(\frac{k}{d}\right) \tag{3.10}$$
$$= \frac{1}{2} \sum_{l \le x} \frac{\mu(l)}{l} \left\{ \frac{x}{l} - \theta\left(\frac{x}{l}\right) - \frac{1}{2} \right\} + \frac{1}{r+1} \sum_{k \le x} \sum_{l \mid k} \mu(l)$$
$$+ \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{l \le x} \frac{\mu(l)}{l^{2m}} \left\{ \frac{x}{l} - \theta\left(\frac{x}{l}\right) - \frac{1}{2} \right\} .$$

All terms on the right-hand side of (3.10) give

$$\frac{x}{2\zeta(2)} + \frac{x}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} \frac{B_{2m}}{\zeta(2m+1)} - \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{l \le x} \frac{\mu(l)}{l^{2m}} \theta\left(\frac{x}{l}\right)$$

$$-\frac{1}{2(r+1)}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{l\le x}\frac{\mu(l)}{l^{2m}}+O_r\left((\log x)^{2/3}(\log\log x)^{4/3}\right)$$
$$=\frac{x}{2\zeta(2)}+\frac{x}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}\frac{B_{2m}}{\zeta(2m+1)}+O_r\left((\log x)^{2/3}(\log\log x)^{4/3}\right)$$

by using Lemma 2, where the well-known identity $\sum_{l|k}\mu(l)$ denotes 0 if $k > 1$ and 1 if $k = 1$. □

REMARK 3. As an application of (3.9), this observation may be regarded as an average order of magnitude for the sum (3.9), which is

$$\lim_{x\to\infty}\frac{1}{x}\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r\sum_{d|\gcd(k,j)}d\mu\left(\frac{k}{d}\right)=\frac{1}{2\zeta(2)}+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}\frac{B_{2m}}{\zeta(2m+1)}.$$

Furthermore, we take $w = \log$ into (3.2) to obtain the following formula.

THEOREM 4. *Let the notation be as above. For any positive real number $x > 1$ we have*

$$\sum_{k\le x}\sum_{j=1}^{k}\log j\sum_{d|\gcd(k,j)}f(d)g\left(\frac{k}{d}\right)h\left(\frac{j}{d}\right) \tag{3.11}$$
$$=\sum_{dl\le x}f(d)\log d\ g(l)\sum_{m=1}^{l}h(m)+\sum_{dl\le x}f(d)g(l)\sum_{m=1}^{l}h(m)\log m\,.$$

In particular, taking $h = \mathbf{1}$ into (3.11) we have

$$\sum_{k\le x}\sum_{j=1}^{k}\log j\sum_{d|\gcd(k,j)}f(d)g\left(\frac{k}{d}\right) \tag{3.12}$$
$$=\sum_{dl\le x}f(d)\log d\ g(l)l+\sum_{dl\le x}f(d)g(l)l\log l-\sum_{dl\le x}f(d)g(l)l+\frac{1}{2}\sum_{dl\le x}f(d)g(l)\log l$$
$$+\log\sqrt{2\pi}\sum_{dl\le x}f(d)g(l)+O\left(\sum_{dl\le x}|f(d)|\frac{|g(l)|}{l}\right)$$

by using the formula (see (B.25) in H. L. Montgomery and R. C. Vaughan [21])

$$\sum_{m=1}^{l}\log m=l\log l-l+\frac{1}{2}\log l+\log\sqrt{2\pi}+O\left(\frac{1}{l}\right) \tag{3.13}$$

with l being any positive integer.

3.2. Some Applications of Theorem 2 and (3.8). As for an application of Theorem 2, we deduce some interesting and useful formulas.

COROLLARY 1. *Let the notation be as above. We have*

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r\sum_{d\mid \gcd(k,j)}\mu(d)\phi\left(\frac{k}{d}\right) \tag{3.14}$$
$$=\frac{1}{2(r+1)\zeta(2)\zeta(3)}x^2+O_r\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right),$$

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r\sum_{d\mid \gcd(k,j)}\mu(d)\psi\left(\frac{k}{d}\right) \tag{3.15}$$
$$=\frac{\zeta(2)}{2(r+1)\zeta(3)\zeta(4)}x^2+O_r\left(x(\log x)^{2/3}\right),$$

and

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r\sum_{d\mid \gcd(k,j)}|\mu(d)|\sigma\left(\frac{k}{d}\right)=\frac{\zeta(2)\zeta(3)}{2(r+1)\zeta(6)}x^2+O_r\left(x(\log x)^{2/3}\right). \tag{3.16}$$

PROOF. We take $f=\mu$ and $g=\phi$ into (3.4) to deduce that

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r\sum_{d\mid \gcd(k,j)}\mu(d)\phi\left(\frac{k}{d}\right) \tag{3.17}$$
$$=\frac{1}{2}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{l\le x/d}\frac{\phi(l)}{l}+\frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{l\le x/d}\phi(l)$$
$$+\frac{r}{12}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{l\le x/d}\frac{\phi(l)}{l^2}+\frac{1}{r+1}\sum_{m=2}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{l\le x/d}\frac{\phi(l)}{l^{2m}}.$$

The first term on the right-hand side of (3.17) gives us

$$\frac{x}{2\zeta^2(2)}+O\left((\log x)^{5/3}(\log\log x)^{4/3}\right) \tag{3.18}$$

by using (2.6). Using (2.5), we see that the second term on the right-hand side of (3.17) is evaluated by

$$\frac{x^2}{2(r+1)\zeta(2)\zeta(3)}+O_r\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{3.19}$$

Using the estimates $\sum_{l\le M}\frac{\phi(l)}{l^2}=O(\log M)$ and $\sum_{l\le M}\frac{\phi(l)}{l^{2m}}=\frac{\zeta(2m-1)}{\zeta(2m)}+O\left(M^{2-2m}\right)$ for any positive large number $M>1$ and any fixed positive integer $m\ge 2$, we see that the third and fourth terms on the right-hand side of (3.17) are estimated by $O_r\left(\log^2 x\right)$. Substituting (3.18), (3.19) and the above into (3.17) we obtain (3.14).

Next, we show (3.15) in a similar manner as (3.14). Taking $f = \mu$ and $g = \psi$ into (3.4), we shall investigate

$$\sum_{dl\le x}\frac{\mu(d)\psi(l)}{dl}, \qquad \sum_{dl\le x}\frac{\mu(d)\psi(l)}{d}, \qquad \sum_{dl\le x}\frac{\mu(d)\psi(l)}{dl^{2m}}$$

for $m \ge 1$. We see from (2.12) and (2.11) that the first and second sums are evaluated by $\frac{x}{\zeta(4)} + O\left(\log^2 x\right)$ and $\frac{\zeta(2)}{2\zeta(3)\zeta(4)}x^2 + O_r\left(x(\log x)^{2/3}\right)$, respectively. On the other hand, by the estimate $\sum_{l\le M}\frac{\psi(l)}{l^2} = O(\log M)$ the third sum is $O\left(\log^2 x\right)$. Applying these to (3.4) with $f = \mu$ and $g = \psi$, we obtain (3.15).

Similarly, we apply $f = |\mu|$ and $g = \sigma$ into (3.4) and use (2.13) and (2.14) to obtain the identity (3.16). □

REMARK 4. From (3.14) and (3.15), it follows that

$$\frac{1}{x}\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)}\mu(d)\left(\phi\left(\frac{k}{d}\right) - \frac{2}{5}\psi\left(\frac{k}{d}\right)\right) \tag{3.20}$$
$$= O_r\left((\log x)^{2/3}(\log\log x)^{4/3}\right).$$

As for an application of (3.8), we deduce some interesting and useful formulas.

COROLLARY 2. *Let the notation be as above. We have*

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)} d\phi\left(\frac{k}{d}\right)\left|\mu\left(\frac{j}{d}\right)\right| \tag{3.21}$$
$$= \frac{1}{2(r+1)\zeta(2)}x^2 + O_r\left(x^{3/2}\exp\left(-C'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right)$$

with C' being certain positive constant, and

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)}|\mu(d)|\phi\left(\frac{k}{d}\right)\sigma\left(\frac{j}{d}\right) \tag{3.22}$$
$$= \frac{\zeta(4)}{3(r+2)\zeta(8)}x^3 + O_r\left(x^2(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

Similarly, as in the same manner, the proofs of (3.21) and (3.22) follow easily from (3.8) and is left as an exercise for the reader.

COROLLARY 3. *Let the notation be as above. We have*

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)}\phi(d)\tau\left(\frac{k}{d}\right)\tau\left(\frac{j}{d}\right) \tag{3.23}$$

$$= \frac{1}{3(r+1)\zeta(2)} x \log^3 x + O_r\left(x(\log x)^{8/3}(\log\log x)^{4/3}\right).$$

PROOF. We apply $f = \phi$ and $g = h = \tau$ into (3.8) and use (2.18) to deduce that

$$\sum_{k \le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r \sum_{d \mid \gcd(k,j)} \phi(d)\tau\left(\frac{k}{d}\right)\tau\left(\frac{j}{d}\right) = \sum_{dl \le x} \frac{\phi(d)}{d} \frac{\tau(l)}{l^{r+1}} \sum_{m=1}^{l} \tau(m) m^r$$

$$= \frac{1}{r+1} \sum_{l \le x} \tau(l) \log l \sum_{d \le x/l} \frac{\phi(d)}{d} + \frac{1}{r+1}\left(2\gamma - \frac{1}{r+1}\right) \sum_{l \le x} \tau(l) \sum_{d \le x/l} \frac{\phi(d)}{d} \tag{3.24}$$

$$+ O_r\left(\sum_{l \le x} \frac{\tau(n)}{l^{2/3-\varepsilon}} \sum_{d \le x/l} \frac{\phi(d)}{d}\right).$$

From (2.6), all terms on the right-hand side of (3.24) are

$$\frac{x}{(r+1)\zeta(2)} \sum_{l \le x} \frac{\tau(l)}{l} \log l + O_r\left((\log x)^{2/3}(\log\log x)^{4/3} \sum_{l \le x} \tau(l) \log l\right) \tag{3.25}$$

$$+ \frac{x}{(r+1)\zeta(2)}\left(2\gamma - \frac{1}{r+1}\right) \sum_{l \le x} \frac{\tau(l)}{l} + O_r\left((\log x)^{2/3}(\log\log x)^{4/3} \sum_{l \le x} \tau(l)\right)$$

$$+ O_r\left(x \sum_{l \le x} \frac{\tau(l)}{l^{5/3-\varepsilon}}\right).$$

From (2.19), (2.20) and (2.21), the right-side hand of (3.25) is evaluated as

$$\frac{1}{3(r+1)\zeta(2)} x \log^3 x + O_r\left(x(\log x)^{8/3}(\log\log x)^{4/3}\right).$$

□

Let $s(m,n)$ denote the total number of subgroups of the group $G = \mathbb{Z}_m \times \mathbb{Z}_n$. For any positive integers m and n, the arithmetical function $s(m,n)$ is given by $\sum_{d \mid m, e \mid n} \gcd(d,e)$, whose formula can be written as

$$s(m,n) = \sum_{d \mid \gcd(m,n)} \phi(d)\tau\left(\frac{m}{d}\right)\tau\left(\frac{n}{d}\right).$$

Recently, W. G. Nowak and L. Tóth [22] established an asymptotic formula for the double sum $\sum_{m,n \le x} s(m,n)$, namely

$$\sum_{m,n \le x} s(m,n) = \frac{1}{3\zeta(2)} x^2 \log^3 x + \frac{1}{\zeta(2)}\left(3\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)}\right) x^2 \log^2 x \tag{3.26}$$

$$+ A_1 x^2 \log x + A_0 x^2 + O\left(x^{\frac{1117}{701}+\varepsilon}\right),$$

where A_0 and A_1 are effectively computable constants, and γ is the Euler constant. Comparing (3.23) with (3.26), an average order of magnitude of the formula (3.23) does not coincide with that of the formula (3.26), since the term on the left-hand side of (3.23) can be written as $\sum_{1\le m\le n\le x} \frac{m^r}{n^{r+1}} s(m,n)$ for any fixed positive integer r.

4. Other weighted functions

4.1. Other weighted functions. L. Tóth [26] gave a simpler proof of the identity (1.2) due to Alkan [2] concerning a weighted average of Ramanujan sums. Recently, I. Kiuchi, M. Minamide and M. Ueda [15] established some formulas concerning a generalization of Ramanujan's sums $c_k(j)$ due to the method of L. Tóth. We shall consider the partial sums for weighted averages of the Anderson–Apostol sums

$$s_k(j) = \sum_{d \mid \gcd(k,j)} f(d) g\left(\frac{k}{d}\right)$$

with weights concerning the Gamma function, the Bernoulli polynomials and values of arithmetical function for gcd's. To prove Theorems 5, 6 and 7, our starting point is three formulas (5.1), (5.8) and (5.11) in [15]. That is, for any positive integer k and any fixed positive integer m, we have

$$\sum_{j=1}^{k} s_k(j) \log\Gamma\left(\frac{j}{k}\right) = \log\sqrt{2\pi}\,\{(f * \mathrm{id}\, g)(k) - (f * g)(k)\} - \frac{1}{2}(f * g \log)(k)\,, \tag{4.1}$$

with Γ being the Gamma function,

$$\frac{1}{k}\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) s_k(j) = \frac{B_m}{k^m}\sum_{d\mid k} d^{m-1} f(d) g\left(\frac{k}{d}\right), \tag{4.2}$$

where $B_m(t)$ and B_m are the Bernoulli polynomials and the Bernoulli numbers, respectively, and

$$\sum_{j=1}^{k} v\,(\gcd(k,j))\, s_k(j) = F(k)\left(v * \frac{g\phi}{F}\right)(k) \tag{4.3}$$

for any arithmetical function v, $g = \mu u$ with multiplicative function u and $F = f * g$ with completely multiplicative function f satisfying $f(p) \neq 0$ and $f(p) \neq u(p)$ for each prime number p.

We derive the partial sums for weighted average of $s_k(j)$ concerning the weight $w(j) = \log\Gamma\,(j/k)$.

THEOREM 5. *Let the notation be as above. For any positive real number $x > 1$ we have*

$$\sum_{k\le x}\sum_{j=1}^{k} s_k(j)\log\Gamma\left(\frac{j}{k}\right) \tag{4.4}$$
$$= \log\sqrt{2\pi}\sum_{dl\le x} f(d)g(l)l - \log\sqrt{2\pi}\sum_{dl\le x} f(d)g(l) - \frac{1}{2}\sum_{dl\le x} f(d)g(l)\log l$$

and

$$\sum_{k\le x}\frac{1}{k}\sum_{j=1}^{k} s_k(j)\log\Gamma\left(\frac{j}{k}\right) \tag{4.5}$$
$$= \log\sqrt{2\pi}\sum_{dl\le x}\frac{f(d)}{d}g(l) - \log\sqrt{2\pi}\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l} - \frac{1}{2}\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l}\log l\,.$$

PROOF. The formulas (4.4) and (4.5) follow easily from (4.1). □

We derive the partial sums for weighted average of $s_k(j)$ concerning the Bernoulli polynomials.

THEOREM 6. *Let the notation be as above. For any positive real number $x > 1$ we have*

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right)s_k(j) = B_m\sum_{dl\le x}\frac{f(d)}{d}\frac{g(l)}{l^m}\,. \tag{4.6}$$

PROOF. The formula (4.6) follows easily from (4.2). □

REMARK 5. Taking $f = \mathrm{id}_{1-m}\cdot g$ and $g = \mathrm{id}_{m-1}\cdot f$ into (4.6) we obtain

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right)s_k(j) = \sum_{k\le x}k^{m-2}\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right)\sum_{d\mid \gcd(k,j)} d^{2-2m}g(d)f\left(\frac{k}{d}\right)\,.$$

We derive the partial sums for weighted average of $s_k(j)$ concerning values of arithmetical function for gcd's.

THEOREM 7. *Let f be completely multiplicative, and let $g = \mu\cdot u$, where u is multiplicative. Assume that $f(p)\ne 0$ and $f(p)\ne u(p)$ for all primes p, and let $F = f * g$. For any positive real number $x > 1$ we have*

$$\sum_{k\le x}\frac{1}{F(k)}\sum_{j=1}^{k} v\left(\gcd(k,j)\right)s_k(j) = \sum_{dl\le x} v(d)\frac{g(l)\phi(l)}{F(l)}\,. \tag{4.7}$$

PROOF. The formula (4.7) follows easily from (4.3). □

Now, taking $f = \mathrm{id}$, $g = \mu$ and $f = \mathrm{id}$, $g = |\mu|$ into (4.7) we deduce that

$$\sum_{k \le x} \frac{1}{\phi(k)} \sum_{j=1}^{k} v\left(\gcd(k, j)\right) c_k(j) = \sum_{dl \le x} v(d)\mu(l) \tag{4.8}$$

and

$$\sum_{k \le x} \frac{1}{\psi(k)} \sum_{j=1}^{k} v\left(\gcd(k, j)\right) \sum_{d \mid \gcd(k,j)} d \left|\mu\left(\frac{k}{d}\right)\right| = \sum_{dl \le x} v(d) \frac{|\mu(l)|\phi(l)}{\psi(l)}, \tag{4.9}$$

respectively. The formulas (4.8) and (4.9) follow easily from Proposition 3 in L. Tóth [26] and the formula (5.13) in I. Kiuchi, M. Minamide and M. Ueda [15], respectively.

4.2. Some Applications of Theorem 5.

COROLLARY 4. *Let the notation be as above. We have*

$$\sum_{k \le x} \sum_{j=1}^{k} \log \Gamma\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} \mu(d)\phi\left(\frac{k}{d}\right) \tag{4.10}$$
$$= \frac{\log \sqrt{2\pi}}{3\zeta(2)\zeta(3)} x^3 - \frac{1}{4\zeta^2(2)} x^2 \log x + O\left(x^2 (\log x)^{2/3} (\log\log x)^{4/3}\right),$$

$$\sum_{k \le x} \sum_{j=1}^{k} \log \Gamma\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} \mu(d)\sigma\left(\frac{k}{d}\right) \tag{4.11}$$
$$= \frac{\zeta(2) \log \sqrt{2\pi}}{3\zeta(3)} x^3 - \frac{1}{4} x^2 \log x + O\left(x^2 (\log x)^{2/3}\right)$$

and

$$\sum_{k \le x} \frac{1}{k} \sum_{j=1}^{k} \log \Gamma\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} \mu(d)\phi\left(\frac{k}{d}\right) \tag{4.12}$$
$$= \frac{\log \sqrt{2\pi}}{2\zeta(2)\zeta(3)} x^2 - \frac{1}{2\zeta^2(2)} x \log x + O\left(x (\log x)^{2/3} (\log\log x)^{4/3}\right).$$

PROOF. We shall give a proof of (4.10) only, since (4.11) and (4.12) are similar. We take $f = \mu$ and $g = \phi$ into (4.4) to obtain

$$\sum_{k \le x} \sum_{j=1}^{k} \log \Gamma\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} \mu(d)\phi\left(\frac{k}{d}\right) \tag{4.13}$$
$$= \log \sqrt{2\pi} \sum_{d \le x} \mu(d) \sum_{l \le x/d} \phi(l) l - \log \sqrt{2\pi} \sum_{d \le x} \mu(d) \sum_{l \le x/d} \phi(l)$$
$$- \frac{1}{2} \sum_{d \le x} \mu(d) \sum_{l \le x/d} \phi(l) \log l.$$

Using (2.8), we see that the first term on the right-hand side of (4.13) is

$$\frac{\log \sqrt{2\pi}}{3\zeta(2)\zeta(3)} x^3 + O\left(x^2 (\log x)^{2/3} (\log\log x)^{4/3}\right) .$$

Using (2.5), we see that the second term on the right-hand side of (4.13) is estimated by $O(x^2)$. The third term on the right-hand side of (4.13) gives $-\frac{1}{4\zeta^2(2)} x^2 \log x + O\left(x^2\right)$ by using (2.7). Substituting the above into (4.13) we obtain (4.10). □

4.3. Some Applications of Theorem 6.

COROLLARY 5. *Let the notation be as above. We have*

$$\begin{aligned} \sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_1\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} d\mu\left(\frac{k}{d}\right) & \qquad (4.14) \\ = -\frac{x}{2\zeta(2)} + O\left((\log x)^{2/3} (\log\log x)^{4/3}\right) , & \end{aligned}$$

and

$$\begin{aligned} \sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} d\mu\left(\frac{k}{d}\right) & \qquad (4.15) \\ = \frac{B_m}{\zeta(m+1)} x + B_m D_m(x) + O_m\left(x^{1-m}\right) & \end{aligned}$$

for any fixed positive integer $m > 1$.

PROOF. We take $f = \mathrm{id}$ and $g = \mu$ into (4.6) to obtain

$$\begin{aligned} \sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_1\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} d\mu\left(\frac{k}{d}\right) & \qquad (4.16) \\ = -\frac{x}{2} \sum_{l \le x} \frac{\mu(l)}{l^2} + \frac{1}{2} \sum_{l \le x} \frac{\mu(l)}{l} \theta\left(\frac{x}{l}\right) + \frac{1}{4} \sum_{l \le x} \frac{\mu(l)}{l} . & \end{aligned}$$

From Lemma 2, all terms on the right-hand side of (4.16) are evaluated by

$$-\frac{x}{2\zeta(2)} + O\left((\log x)^{2/3} (\log\log x)^{4/3}\right) ,$$

which completes the proof of (4.14). Similarly, we apply $f = \mathrm{id}$ and $g = \mu$ into (4.6) and use (2.10) to obtain (4.15). □

Similarly, as in the same manner, we have the following formulas, namely

COROLLARY 6. *Let the notation be as above. We have*

$$\sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_1\left(\frac{j}{k}\right) \sum_{d \mid \gcd(k,j)} d \left|\mu\left(\frac{k}{d}\right)\right| = -\frac{\zeta(2)}{2\zeta(4)} x + O\left(\log x\right) , \qquad (4.17)$$

and

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_m\left(\frac{j}{k}\right)\sum_{d\mid \gcd(k,j)} d\left|\mu\left(\frac{k}{d}\right)\right| \tag{4.18}$$
$$=\frac{B_m\zeta(m+1)}{\zeta(2m+2)}x+E_m(x)+O_m\left(x^{1-m}\right),$$

where

$$E_m(x)=-B_m\sum_{l\le x}\frac{|\mu(l)|}{l^m}\theta\left(\frac{x}{l}\right)-\frac{B_m\zeta(m)}{2\zeta(2m)}$$

for any fixed positive integer $m>1$.

REMARK 6. From Corollaries 5, 6 and (4.6), we have

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_{2m+1}\left(\frac{j}{k}\right)\sum_{d\mid \gcd(k,j)} d\left\{\left|\mu\left(\frac{k}{d}\right)\right|-\mu\left(\frac{k}{d}\right)\right\} \tag{4.19}$$
$$=\begin{cases}\dfrac{\zeta(4)-\zeta^2(2)}{2\zeta(2)\zeta(4)}x+O(\log x) & \text{if } m=0\,,\\[2ex] 0 & \text{if } m \text{ is a positive integer}\,,\end{cases}$$

and

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_{2m}\left(\frac{j}{k}\right)\sum_{d\mid \gcd(k,j)} d\left\{\left|\mu\left(\frac{k}{d}\right)\right|-\mu\left(\frac{k}{d}\right)\right\} \tag{4.20}$$
$$=\frac{\zeta^2(2m+1)-\zeta(4m+2)}{\zeta(2m+1)\zeta(4m+2)}B_{2m}x+E_{2m}(x)-B_{2m}D_{2m}(x)+O_m\left(x^{1-2m}\right)$$

for any positive integer $m\ge 1$.

COROLLARY 7. *Let the notation be as above. We have*

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_m\left(\frac{j}{k}\right)\sum_{d\mid \gcd(k,j)} d\sigma(d)\phi_m\left(\frac{k}{d}\right) \tag{4.21}$$
$$=\frac{\zeta^2(2)B_m}{2\zeta(m+2)}x^2+O_m\left(x(\log x)^{5/3}\right)$$

for any fixed positive integer $m>1$.

PROOF. We apply $f=\mathrm{id}\cdot\sigma$ and $g=\phi_m$ into (4.6) and use (2.13) to obtain

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_m\left(\frac{j}{k}\right)\sum_{d\mid \gcd(k,j)} d\sigma(d)\phi_m\left(\frac{k}{d}\right)$$

$$= \frac{B_m \zeta(2)}{2} x^2 \sum_{l \le x} \frac{\phi_m(l)}{l^{m+2}} + O_m \left(x (\log x)^{2/3} \sum_{l \le x} \frac{\phi_m(l)}{l^{m+1}} \right)$$

$$= \frac{B_m \zeta(2)}{2} \frac{\zeta(2)}{\zeta(m+2)} x^2 + O_m \left(x (\log x)^{2/3} \sum_{l \le x} \frac{1}{l} \right),$$

which completes the proof of (4.21). □

We introduce the partial sum of weighted averages for $\gcd(k, j)$ concerning the Bernoulli polynomials. Taking $f = \phi$ and $g = \mathbf{1}$ into (4.6) we get

$$\sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_m \left(\frac{j}{k} \right) \gcd(k, j) = \frac{B_m \zeta(m+1)}{\zeta(2)} x + O_m \left((\log x)^{2/3} (\log \log x)^{4/3} \right) \tag{4.22}$$

for any fixed positive integer $m \ge 2$.

4.4. An Application of (4.8).

COROLLARY 8. *Let the notation be as above. We have*

$$\sum_{k \le x} \frac{1}{\phi(k)} \sum_{j=1}^{k} \phi\left(\gcd(k, j)\right) c_k(j) = \frac{1}{2\zeta^2(2)} x^2 + O \left(x (\log x)^{5/3} (\log \log x)^{4/3} \right) \tag{4.23}$$

and

$$\sum_{k \le x} \frac{1}{\phi(k)} \sum_{j=1}^{k} \psi\left(\gcd(k, j)\right) c_k(j) = \frac{1}{2\zeta(4)} x^2 + O \left(x (\log x)^{5/3} \right). \tag{4.24}$$

PROOF. Using (2.5) and (4.8) we have

$$\sum_{k \le x} \frac{1}{\phi(k)} \sum_{j=1}^{k} \phi\left(\gcd(k, j)\right) c_k(j)$$

$$= \sum_{d \le x} \mu(d) \left(\frac{1}{2\zeta(2)} \frac{x^2}{l^2} + O \left(\frac{x}{l} (\log x)^{2/3} (\log \log x)^{4/3} \right) \right)$$

$$= \frac{1}{2\zeta^2(2)} x^2 + O \left(x (\log x)^{5/3} (\log \log x)^{4/3} \right),$$

which completes the proof of (4.23). Similarly, as in the same manner, the proof of (4.24) follows easily from (2.11). □

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