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Common Fixed Points for Nonlinear $(\psi, \varphi)_s$-weakly C-contractive Mappings in Partially Ordered b-metric Spaces

Jamal REZAEI ROSHAN and Mina DINARVAND

Islamic Azad University and K. N. Toosi University of Technology

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Abstract. In this paper, we introduce the concept of $(\psi, \varphi)_s$-weakly C-contractive mappings in the setup of partially ordered b-metric spaces and investigate some fixed point and common fixed point results for such wmappings. Our main results generalize several well-known comparable results in the recent literature. Furthermore, we furnish some suitable examples and an applications of a common solution for a system of integral equations to illustrate the effectiveness and usability of our obtained results.

1. Introduction and Preliminaries

In nonlinear functional analysis, the importance of fixed point theory has been increasing rapidly as an interesting research field. One of the most important reasons for this development is the potential of application of fixed point theory not only in various branches of applied and pure mathematics, but also in many other disciplines such as chemistry, biology, physics, economics, computer science, engineering etc. The Banach contraction principle [10] is one of the fundamental and pivotal results in fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to approximate those fixed points. During the last few decades, several extensions of this famous principle have been established. For more information on this topic see [1, 2, 3, 17, 18, 21, 22, 29, 41].

In 1972, Chatterjea [15] introduced the concept of C-contraction as follows.

DEFINITION 1 ([15]). Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a C-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$d(fx, fy) \leq \alpha\big(d(x, fy) + d(y, fx)\big)$$

holds for all $x, y \in X$.

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In this interesting paper, Chatterjea [15] proved that if X is complete, then every C-contraction has a unique fixed point. In 2009, Choudhury [16] introduced the concept of weakly C-contractive mapping as a generalization of C-contractive mapping.

DEFINITION 2 ([16]). Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be weakly C-contractive if for all $x, y \in X$, the following inequality holds:

$$d(fx, fy) \leq \frac{1}{2}\big(d(x, fy) + d(y, fx)\big) - \varphi\big(d(x, fy), d(y, fx)\big),$$

where $\varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Moreover, Choudhury in [16] proved the existence of a unique fixed point for such mappings on complete metric spaces.

Fixed points theorems in partially ordered metric spaces were firstly obtained in 2004 by Ran and Reurings [40], and then by Nieto and Lopez [36]. In this direction several authors obtained further results under weak contractive conditions. Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in so called spaces. For more details on fixed point results, its applications, comparison of different contractive conditions and related results in ordered metric spaces we refer the interested reader to [5, 23, 25, 26, 35, 39, 46] and the references cited therein.

Khan et al. [34] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

DEFINITION 3 ([34]). A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and nondecreasing;
(2) $\psi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems, which are based on altering distance functions. In [45], Shatanawi, by considering an altering distance function, established some fixed point theorems for a nonlinear weakly C-contraction type mapping in ordered metric spaces.

THEOREM 1 ([45], Theorem 2.1). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi\big(d(fx, fy)\big) \leq \psi\left(\frac{d(x, fy) + d(y, fx)}{2}\right) - \varphi\big(d(x, fy), d(y, fx)\big),$$

for all comparable $x, y \in X$, *where*

1. $\psi : [0,+\infty) \to [0,+\infty)$ *is an altering distance function.*
2. $\varphi : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *is a continuous function such that* $\varphi(x, y) = 0$ *if and only if* $x = y = 0$.

If there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$, *then* f *has a fixed point.*

THEOREM 2 ([45], Theorem 2.2). *Suppose that* X, f, ψ *and* φ *are as in Theorem* 1 *except the continuity of* f. *Let for any nondecreasing sequence* $\{x_n\}$ *in* X *such that* $x_n \to x \in X$, *we have* $x_n \preceq x$ *for all* $n \in \mathbf{N}$. *Then* f *has a fixed point in* X.

The concept of a b-metric space was introduced by Bakhtin in [9], and later extensively used by Czerwik in [19, 20]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces. For further works in this direction, we refer to [4, 8, 11, 12, 13, 14, 24, 27, 30, 31, 37, 38, 42, 43, 44, 47].

Consistent with [20, 24] and [47], the following definitions and results will be needed in the sequel.

DEFINITION 4 ([20]). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbf{R}^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than the class of metric spaces, since a b-metric is a metric, when $s = 1$.

The following example shows that in general a b-metric need not necessarily be a metric (see, also, [47, p. 264]).

EXAMPLE 1. Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a $b-$metric with $s = 2^{p-1}$.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbf{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a $b-$metric on $\mathbf{R}$ with $s = 2$, but is not a metric on $\mathbf{R}$.

Also, the following example of a b-metric space is given in [32].

EXAMPLE 2. Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 dx < +\infty .$$

Define $D : X \times X \to [0, +\infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx .$$

As

$$\left(\int_0^1 |f(x) - g(x)|^2 dx \right)^{\frac{1}{2}}$$

is a metric on X, then, from the previous example, D is a b-metric on X with $s = 2$.

Khamsi [33] also showed that each cone metric space over a normal cone has a b-metric structure.

DEFINITION 5. Let X be a nonempty set. Then $(X, \preceq, d)$ is called a partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X, \preceq)$.

The notions of b-convergent and b-Cauchy sequences, as well as of b-complete b-metric spaces are introduced in an obvious way (see, e.g., [11, 12]).

It should be noted that in general, a b-metric function $d(x, y)$ for $s > 1$ need not be jointly continuous in both variables (see Example 2 in [28]).

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

LEMMA 1. ([4]). *Let (X, d) be a b-metric space with parameter $s \geq 1$, and suppose that the sequences $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \to +\infty} d(x_n, y_n) \leq \limsup_{n \to +\infty} d(x_n, y_n) \leq s^2 d(x, y) .$$

In particular, if $x = y$, then $\lim_{n \to +\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n \to +\infty} d(x_n, z) \leq \limsup_{n \to +\infty} d(x_n, z) \leq s d(x, z) .$$

In this work, we introduce the notion of $(\psi, \varphi)_s$-weakly C-contractive mappings in the framework of partially ordered b-metric spaces and establish some fixed point and common fixed point results for this class of mappings. Our results generalize several well-known comparable results in the existing literature. Moreover, some examples are provided here to illustrate the usability of the obtained results.

2. Main Results

From now on, we assume that

$$\Psi = \Big\{\psi : [0,+\infty) \to [0,+\infty) \,\Big|\, \psi \text{ is an altering distance function}\Big\}$$

and

$$\Phi = \Big\{\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty) \,\Big|\, \varphi(x,y)=0 \Longleftrightarrow x=y=0 \text{ and}$$

$$\varphi\Big(\liminf_{n\to+\infty} a_n, \liminf_{n\to+\infty} b_n\Big) \le \liminf_{n\to+\infty} \varphi(a_n,b_n)\Big\}.$$

DEFINITION 6. Let $(X,d,\preceq)$ be a partially ordered b-metric space with parameter $s \ge 1$. We say that a mapping $f : X \to X$ is a $(\psi,\varphi)_s$-weakly C-contractive mapping if there exist two functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi\big(sd(fx,fy)\big) \le \psi\left(\frac{d(x,fy)+d(y,fx)}{s+1}\right) - \varphi\big(d(x,fy),d(y,fx)\big), \tag{1}$$

for all comparable $x, y \in X$.

THEOREM 3. *Let $(X,d,\preceq)$ be a partially ordered b-complete b-metric space with parameter $s \ge 1$. Let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping. Suppose that f is a $(\psi,\varphi)_s$-weakly C-contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

PROOF. Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. If $fx_0 = x_0$, then there is nothing to prove. Suppose that $x_0 \prec fx_0$. Then, we define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$ for all $n \in \mathbf{N}\cup\{0\}$. Since $x_0 \prec fx_0 = x_1$ and f is nondecreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$. Again, as $x_1 \preceq x_2$ and f is nondecreasing, we have $x_2 = fx_1 \preceq x_3 = fx_2$. By induction on n, we conclude that

$$x_0 \prec x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{2}$$

If $x_n = x_{n+1}$ for some $n \in \mathbf{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \ne x_{n+1}$ for all $n \in \mathbf{N}$.

We complete the proof in the following steps.

STEP 1. We will prove that

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0. \tag{3}$$

Since x_{n-1} and x_n are comparable, by applying the inequality (1), we have

$$\psi\big(sd(x_n,x_{n+1})\big) = \psi\big(sd(fx_{n-1},fx_n)\big)$$

$$\begin{aligned}&\le \psi\left(\frac{d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})}{s+1}\right) - \varphi\big(d(x_{n-1}, fx_n), d(x_n, fx_{n-1})\big)\\&= \psi\left(\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{s+1}\right) - \varphi\big(d(x_{n-1}, x_{n+1}), d(x_n, x_n)\big)\\&= \psi\left(\frac{d(x_{n-1}, x_{n+1})}{s+1}\right) - \varphi\big(d(x_{n-1}, x_{n+1}), 0\big) . \qquad (4)\end{aligned}$$

Therefore, by using the triangular inequality and the properties of ψ and φ, we get

$$\begin{aligned}\psi\big(sd(x_n, x_{n+1})\big) &\le \psi\left(\frac{d(x_{n-1}, x_{n+1})}{s+1}\right)\\&\le \psi\left(\frac{s}{s+1}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big)\right) .\end{aligned}$$

Since ψ is nondecreasing, we have

$$d(x_n, x_{n+1}) \le \frac{1}{s+1}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) ,$$

from which it follows that

$$d(x_n, x_{n+1}) \le \frac{1}{s} d(x_n, x_{n-1}) \le d(x_n, x_{n-1}) .$$

Therefore, $\{d(x_n, x_{n+1}) : n \in \mathbf{N} \cup \{0\}\}$ is a monotone decreasing sequence of nonnegative real numbers. Then there exists $r \ge 0$ such that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = r .$$

From the above argument, we have

$$\begin{aligned}sd(x_n, x_{n+1}) &\le \frac{1}{s+1} d(x_{n-1}, x_{n+1})\\&\le \frac{s}{s+1}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big)\\&\le \frac{s}{2}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) .\end{aligned}$$

By taking limit as $n \to +\infty$ in the above inequality, we obtain

$$\lim_{n\to+\infty} d(x_{n-1}, x_{n+1}) = s(s+1)r .$$

Now, letting $n \to +\infty$ in (4) and using the continuity of ψ and the properties of φ, we obtain

$$\psi(sr) \le \psi(sr) - \varphi(s(s+1)r, 0) .$$

Therefore, $\varphi(s(s+1)r, 0) = 0$ which by our assumptions about φ implies that $r = 0$.

STEP 2. We will show that $\{x_n\}$ is a b-Cauchy sequence in X. Suppose the contrary, that is, $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$n_k > m_k > k, \qquad d(x_{m_k}, x_{n_k}) \geq \varepsilon . \tag{5}$$

This means that

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon . \tag{6}$$

From (5), (6) and by using the triangular inequality, we get

$$\begin{aligned} \varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq sd(x_{m_k}, x_{n_k-1}) + sd(x_{n_k-1}, x_{n_k}) \\ &< s\varepsilon + sd(x_{n_k-1}, x_{n_k}) . \end{aligned}$$

By using (3) and taking the upper limit as $k \to +\infty$, we get

$$\varepsilon \leq \limsup_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq s\varepsilon . \tag{7}$$

On the other hand, from

$$d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{n_k-1}) + sd(x_{n_k-1}, x_{n_k})$$

and by using (3), (5) and (6), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{n_k-1}, x_{m_k}) \leq \varepsilon . \tag{8}$$

Moreover, from

$$d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{m_k-1}) + sd(x_{m_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k}) \leq sd(x_{m_k-1}, x_{m_k}) + sd(x_{m_k}, x_{n_k}) .$$

By taking the upper limit as $k \to +\infty$ in (3) and (6), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{m_k-1}, x_{n_k}) \leq s^2\varepsilon . \tag{9}$$

Similarly, we can show that

$$\frac{\varepsilon}{s} \leq \liminf_{k\to+\infty} d(x_{n_k-1}, x_{m_k}) \leq \varepsilon \tag{10}$$

and

$$\frac{\varepsilon}{s} \leq \liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k}) \leq s^2\varepsilon\,. \tag{11}$$

Since x_{m_k-1} and x_{n_k-1} are comparable, putting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (1) and taking the upper limit as $k \to +\infty$ and applying (7), (8), (9), (10) and (11), we have

$$\begin{aligned}
\psi(s\varepsilon) &\leq \psi\big(s \limsup_{k\to+\infty} d(x_{m_k}, x_{n_k})\big) = \psi\big(s \limsup_{k\to+\infty} d(fx_{m_k-1}, fx_{n_k-1})\big) \\
&\leq \limsup_{k\to+\infty} \psi\left(\frac{d(x_{m_k-1}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{m_k-1})}{s+1}\right) \\
&\quad - \liminf_{k\to+\infty} \varphi\big(d(x_{m_k-1}, fx_{n_k-1}), d(x_{n_k-1}, fx_{m_k-1})\big) \\
&\leq \psi\left(\frac{\limsup_{k\to+\infty} d(x_{m_k-1}, x_{n_k}) + \limsup_{k\to+\infty} d(x_{n_k-1}, x_{m_k})}{s+1}\right) \\
&\quad -\varphi\big(\liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k}), \liminf_{k\to+\infty} d(x_{n_k-1}, x_{m_k})\big) \\
&\leq \psi\left(\frac{s^2\varepsilon + \varepsilon}{s+1}\right) - \varphi\big(\liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k}), \liminf_{k\to+\infty} d(x_{n_k-1}, x_{m_k})\big) \\
&\leq \psi(s\varepsilon) - \varphi\big(\liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k}), \liminf_{k\to+\infty} d(x_{n_k-1}, x_{m_k})\big)
\end{aligned}$$

since $\frac{s^2+1}{s+1} \leq s$. Thus, we have

$$\varphi\big(\liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k}), \liminf_{k\to+\infty} d(x_{n_k-1}, x_{m_k})\big) \leq 0$$

By using the property of φ, we deduce

$$\liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k}) = 0\,, \qquad \liminf_{k\to+\infty} d(x_{n_k-1}, x_{m_k}) = 0\,,$$

which contradicts (10) and (11). So, we conclude that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X, d). Since (X, d) is b-complete, there exists $u \in X$ such that $\lim_{n\to+\infty} x_n = u$ and

$$\lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} fx_n = u\,.$$

By using the triangular inequality, we have

$$d(u, fu) \leq sd(u, fx_n) + sd(fx_n, fu)\,.$$

Now, by taking the upper limit as $n \to +\infty$ in the above inequality and using the continuity of f, we get

$$d(u, fu) \le s \limsup_{n\to+\infty} d(u, fx_n) + s \limsup_{n\to+\infty} d(fx_n, fu) = 0\,.$$

So, we have $fu = u$. Thus, u is a fixed point of f. □

We will show now that the continuity of f in Theorem 3 is not necessary and can be replaced by another assumption.

THEOREM 4. *Under the same hypotheses of Theorem* 3, *without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$, one has $x_n \preceq x$ for all $n \in \mathbf{N}$. Then f has a fixed point in X.*

PROOF. Following similar arguments as those given in the proof of Theorem 3, we construct an increasing sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} x_n = u$ for some $u \in X$. By using the assumption on X we have $x_n \preceq u$ for all $n \in \mathbf{N}$. Putting $x = x_n$ and $y = u$ in (1) and by taking the upper limit as $n \to +\infty$ and using Lemma 1 and the properties of ψ and φ, we obtain

$$\begin{aligned}
\psi\big(d(u, fu)\big) &= \psi\left(s\frac{1}{s}d(u, fu)\right) \le \psi\big(s \limsup_{n\to+\infty} d(x_{n+1}, fu)\big) \\
&= \psi\big(s \limsup_{n\to+\infty} d(fx_n, fu)\big) \\
&\le \psi\left(\limsup_{n\to+\infty} \frac{d(x_n, fu) + d(u, fx_n)}{s+1}\right) \\
&\quad - \liminf_{n\to+\infty} \varphi\big(d(x_n, fu), d(u, fx_n)\big) \\
&\le \psi\left(\frac{\limsup_{n\to+\infty} d(x_n, fu) + \limsup_{n\to+\infty} d(u, x_{n+1})}{s+1}\right) \\
&\quad -\varphi\big(\liminf_{n\to+\infty} d(x_n, fu), \liminf_{n\to+\infty} d(u, x_{n+1})\big) \\
&\le \psi\left(\frac{sd(u, fu) + sd(u, u)}{s+1}\right) - \varphi\big(\liminf_{n\to+\infty} d(x_n, fu), \liminf_{n\to+\infty} d(u, x_{n+1})\big) \\
&\le \psi\big(d(u, fu)\big) - \varphi\big(\liminf_{n\to+\infty} d(x_n, fu), 0\big)\,.
\end{aligned}$$

By the properties of $\varphi \in \Phi$, it follows that $\liminf_{n\to+\infty} d(x_n, fu) = 0$. By applying the triangular inequality, we have

$$d(u, fu) \le sd(u, x_n) + sd(x_n, fu)\,.$$

Now, letting $n \to +\infty$ in the above inequality, we deduce that $d(u, fu) = 0$. Hence, we have $fu = u$. Therefore, u is a fixed point of f. □

By taking $\psi(t) = t$ and $\varphi(t_1, t_2) = (\frac{1}{s+1} - k)(t_1 + t_2)$ where $k \in [0, \frac{1}{s+1})$ in Theorems 3 and 4, we obtain immediately the following two results.

COROLLARY 1. *Let $(X, d, \preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$ and let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping. Suppose that there exists $k \in [0, \frac{1}{s+1})$ such that*

$$d(fx, fy) \leq \frac{k}{s}\left(d(x, fy) + d(y, fx)\right),$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

COROLLARY 2. *Under the same hypotheses of Corollary 1, without the continuity assumption on f, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, let us have $x_n \preceq x$ for all $n \in \mathbf{N}$. Then f has a fixed point in X.*

DEFINITION 7. Let $(X, d, \preceq)$ be a partially ordered b-metric space with parameter $s \geq 1$. We say that a mapping $f : X \to X$ is a $(\psi, \varphi)_s$-weakly C-contractive mapping with respect to a mapping $g : X \to X$ if there exist two functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi\big(s^2 d(fx, gy)\big) \leq \psi\left(\frac{d(x, gy) + d(y, fx)}{s+1}\right) - \varphi\big(d(x, gy), d(y, fx)\big), \tag{12}$$

for all comparable $x, y \in X$.

REMARK 1. If $f : X \to X$ is a $(\psi, \varphi)_s$-weakly C-contractive mapping with respect to f, then f is a $(\psi, \varphi)_s$-weakly C-contractive mapping, to see this, it is enough to put $g = f$ in Definition 7.

DEFINITION 8 ([6]). Let $(X, \preceq)$ be a partially ordered set. Then two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

THEOREM 5. *Let $(X, d, \preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$ and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that f is a $(\psi, \varphi)_s$-weakly C-contractive mapping with respect to g. If either f or g is continuous, then f and g have a common fixed point.*

PROOF. We prove that u is a fixed point of f if and only if u is a fixed point of g. Suppose that u is a fixed point of f, that is, $fu = u$. As $u \preceq u$, by applying (12), we have

$$\psi\big(s^2 d(u, gu)\big) = \psi\big(s^2 d(fu, gu)\big)$$

$$\begin{aligned}
&\le \psi\Big(\frac{d(u,gu)+d(u,fu)}{s+1}\Big) - \varphi\big(d(u,gu),d(u,fu)\big)\\
&\le \psi\Big(\frac{d(u,gu)}{s+1}\Big) - \varphi\big(d(u,gu),0\big)\\
&\le \psi\big(d(u,gu)\big) - \varphi\big(d(u,gu),0\big)\\
&\le \psi\big(s^2 d(u,gu)\big) - \varphi\big(d(u,gu),0\big),
\end{aligned}$$

since ψ is nondecreasing. This yields that $d(u,gu)=0$ by our assumptions about φ. Therefore, $gu=u$. Similarly, we can show that if u is a fixed point of g, then u is a fixed point of f.

Let $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all nonnegative integers. As f and g are weakly increasing with respect to $\preceq$, we have

$$x_1 = fx_0 \preceq gfx_0 = x_2 = gx_1 \preceq fgx_1 = x_3 \preceq \cdots \preceq x_{2n+1} = fx_{2n} \preceq gfx_{2n} = x_{2n+2} \preceq \cdots .$$

If $x_{2n} = x_{2n+1}$ for some $n \in \mathbf{N}$, then $x_{2n} = fx_{2n}$. Thus, x_{2n} is a fixed point of f. By the first part, we conclude that x_{2n} is also a fixed point of g.

If $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbf{N}$, then $x_{2n+1} = gx_{2n+1}$. Thus, x_{2n+1} is a fixed point of g. By the first part, we conclude that x_{2n+1} is also a fixed point of f. Therefore, we assume that $x_n \ne x_{n+1}$ for all $n \in \mathbf{N}$.

Now, we complete the proof in the following steps.

STEP 1. We will show that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{13}$$

Since x_{2n} and x_{2n+1} are comparable, by applying the inequality (12), we have

$$\begin{aligned}
\psi\big(s^2 d(x_{2n+1},x_{2n+2})\big) &= \psi\big(s^2 d(fx_{2n},gx_{2n+1})\big)\\
&\le \psi\Big(\frac{d(x_{2n},gx_{2n+1})+d(x_{2n+1},fx_{2n})}{s+1}\Big)\\
&\quad -\varphi\big(d(x_{2n},gx_{2n+1}),d(x_{2n+1},fx_{2n})\big)\\
&= \psi\Big(\frac{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{s+1}\Big)\\
&\quad -\varphi\big(d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1})\big)\\
&= \psi\Big(\frac{d(x_{2n},x_{2n+2})}{s+1}\Big) - \varphi\big(d(x_{2n},x_{2n+2}),0\big).
\end{aligned} \tag{14}$$

Therefore, by using the triangular inequality and the properties of ψ and φ, we get

$$\begin{aligned}\psi\big(s^2 d(x_{2n+1}, x_{2n+2})\big) &\leq \psi\left(\frac{d(x_{2n}, x_{2n+2})}{s+1}\right)\\ &\leq \psi\left(\frac{s}{s+1}\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big)\right)\\ &\leq \psi\left(\frac{s^2}{s+1}\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big)\right).\end{aligned}$$

Since ψ is nondecreasing, we have

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s+1}\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big),$$

wherefrom

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s} d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, x_{2n+1}). \tag{15}$$

Similarly, we can show that

$$d(x_{2n+1}, x_{2n}) \leq d(x_{2n-1}, x_{2n}). \tag{16}$$

From (15) and (16), we get that $\{d(x_n, x_{n+1}) : n \in \mathbf{N} \cup \{0\}\}$ is a monotone decreasing sequence of nonnegative real numbers. Then there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = r.$$

From the above argument, we have

$$\begin{aligned}s^2 d(x_{2n+1}, x_{2n+2}) &\leq \frac{1}{s+1} d(x_{2n}, x_{2n+2})\\ &\leq \frac{s}{s+1}\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big)\\ &\leq \frac{s^2}{2}\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big).\end{aligned}$$

By taking limit as $n \to +\infty$ in the above inequality, we obtain

$$\lim_{n\to+\infty} d(x_{2n}, x_{2n+2}) = s^2(s+1)r.$$

Now, letting $n \to +\infty$ in (14) and using the continuity of ψ and the properties of φ, we obtain

$$\psi\big(s^2 r\big) \leq \psi\big(s^2 r\big) - \varphi\big(s^2(s+1)r, 0\big).$$

Therefore, $\varphi\big(s^2(s+1)r, 0\big) = 0$ which by our assumptions as regards φ implies that $r = 0$.

STEP 2. We will prove that $\{x_n\}$ is a b-Cauchy sequence in X. To do this, it is sufficient to show that the subsequence $\{x_{2n}\}$ is a b-Cauchy sequence. Assume on the contrary that $\{x_{2n}\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$n_k > m_k > k, \qquad d(x_{2m_k}, x_{2n_k}) \geq \varepsilon\,. \tag{17}$$

This means that

$$d(x_{2m_k}, x_{2n_k-2}) < \varepsilon\,. \tag{18}$$

From (17), (18) and by using the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq d(x_{2m_k}, x_{2n_k}) &\leq sd(x_{2m_k}, x_{2n_k-2}) + sd(x_{2n_k-2}, x_{2n_k})\\ &\leq sd(x_{2m_k}, x_{2n_k-2}) + s^2d(x_{2n_k-2}, x_{2n_k-1}) + s^2d(x_{2n_k-1}, x_{2n_k})\\ &< s\varepsilon + s^2d(x_{2n_k-2}, x_{2n_k-1}) + s^2d(x_{2n_k-1}, x_{2n_k})\,.\end{aligned}$$

By taking the upper limit as $k \to +\infty$ and thanks to (13), we get

$$\varepsilon \leq \limsup_{k\to+\infty} d(x_{2m_k}, x_{2n_k}) \leq s\varepsilon\,. \tag{19}$$

Further, from

$$d(x_{2m_k}, x_{2n_k}) \leq sd(x_{2m_k}, x_{2m_k+1}) + sd(x_{2m_k+1}, x_{2n_k})$$

and thanks to (13) and (19), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k})\,.$$

On the other hand, from

$$d(x_{2m_k+1}, x_{2n_k}) \leq sd(x_{2m_k+1}, x_{2m_k}) + sd(x_{2m_k}, x_{2n_k})$$

and by using (13) and (19), we obtain

$$\limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k}) \leq s^2\varepsilon\,.$$

Consequently, we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k}) \leq s^2\varepsilon\,. \tag{20}$$

Again, by using the triangular inequality, we have

$$\begin{aligned} d(x_{2m_k+1}, x_{2n_k-1}) &\le sd(x_{2m_k+1}, x_{2m_k}) + sd(x_{2m_k}, x_{2n_k-1}) \\ &\le sd(x_{2m_k+1}, x_{2m_k}) + s^2 d(x_{2m_k}, x_{2n_k-2}) + s^2 d(x_{2n_k-2}, x_{2n_k-1}) \\ &< sd(x_{2m_k+1}, x_{2m_k}) + s^2\varepsilon + s^2 d(x_{2n_k-2}, x_{2n_k-1}) . \end{aligned}$$

By taking the upper limit as $k \to +\infty$ in the above inequality and using (13), we get

$$\limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) \le s^2\varepsilon .$$

From (17) and by using the triangular inequality again, we have

$$\begin{aligned} \varepsilon \le d(x_{2m_k}, x_{2n_k}) &\le sd(x_{2m_k}, x_{2m_k+1}) + sd(x_{2m_k+1}, x_{2n_k}) \\ &\le sd(x_{2m_k}, x_{2m_k+1}) + s^2 d(x_{2m_k+1}, x_{2n_k-1}) + s^2 d(x_{2n_k-1}, x_{2n_k}) . \end{aligned}$$

By taking the upper limit as $k \to +\infty$ in the above inequality, we get

$$\frac{\varepsilon}{s^2} \le \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) .$$

Hence, we have

$$\frac{\varepsilon}{s^2} \le \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) \le s^2\varepsilon . \tag{21}$$

Similarly, we can show that

$$\varepsilon \le \liminf_{k\to+\infty} d(x_{2m_k}, x_{2n_k}) \le s\varepsilon . \tag{22}$$

and

$$\frac{\varepsilon}{s^2} \le \liminf_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) \le s^2\varepsilon . \tag{23}$$

By putting $x = x_{2m_k}$ and $y = x_{2n_k-1}$ in (12) and taking the upper limit as $k \to +\infty$ and applying (20), (21), (22) and (23), we have

$$\begin{aligned} \psi\big(s\varepsilon\big) &= \psi\left(s^2\frac{\varepsilon}{s}\right) \le \psi\big(s^2 \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k})\big) \\ &= \psi\big(s^2 \limsup_{k\to+\infty} d(fx_{2m_k}, gx_{2n_k-1})\big) \\ &\le \limsup_{k\to+\infty} \psi\left(\frac{d(x_{2m_k}, gx_{2n_k-1}) + d(x_{2n_k-1}, fx_{2m_k})}{s+1}\right) \end{aligned}$$

$$-\liminf_{k\to+\infty}\varphi\big(d(x_{2m_k}, gx_{2n_k-1}), d(x_{2n_k-1}, fx_{2m_k})\big)$$

$$\leq \psi\left(\frac{\limsup_{k\to+\infty} d(x_{2m_k}, x_{2n_k}) + \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1})}{s+1}\right)$$

$$-\varphi\big(\liminf_{k\to+\infty} d(x_{2m_k}, x_{2n_k}), \liminf_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1})\big)$$

$$\leq \psi\left(\frac{s\varepsilon+s^2\varepsilon}{s+1}\right) - \varphi\big(\liminf_{k\to+\infty} d(x_{2n_k}, x_{2m_k}), \liminf_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1})\big)$$

$$= \psi(s\varepsilon) - \varphi\big(\liminf_{k\to+\infty} d(x_{2n_k}, x_{2m_k}), \liminf_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1})\big)\,.$$

Thus, we have

$$\varphi\big(\liminf_{k\to+\infty} d(x_{2n_k}, x_{2m_k}), \liminf_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1})\big) \leq 0\,.$$

By using the property of φ, we have

$$\liminf_{k\to+\infty} d(x_{2n_k}, x_{2m_k}) = 0, \qquad \liminf_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) = 0\,,$$

which contradicts (22) and (23). So, we deduce that $\{x_n\}$ is a b-Cauchy sequence.

STEP 3. Existence of a common fixed point for f and g.

As $\{x_n\}$ is a b-Cauchy sequence in X which is a b-complete b-metric space, there exists $u \in X$ such that $\lim_{n\to+\infty} x_n = u$ and

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} fx_{2n} = u\,.$$

Now, without any loss of generality, we may assume that f is continuous. By using the triangular inequality, we have

$$d(u, fu) \leq sd(u, fx_{2n}) + sd(fx_{2n}, fu)\,.$$

Now, by taking the upper limit as $n \to +\infty$ in the above inequality and using the continuity of f, we get

$$d(u, fu) \leq s\limsup_{n\to+\infty} d(u, fx_{2n}) + s\limsup_{n\to+\infty} d(fx_{2n}, fu) = 0\,.$$

Thus, we have $fu = u$. Hence, u is a fixed point of f. By the first part, we conclude that u is also a fixed point of g. □

The continuity of one of the functions f or g in Theorem 5 can be replaced by another condition.

THEOREM 6. *Under the same hypotheses of Theorem* 5, *without the continuity assumption on one of the functions* f *or* g, *assume that whenever* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to x \in X$, *one has* $x_n \preceq x$ *for all* $n \in \mathbf{N}$. *Then* f *and* g *have a common fixed point in* X.

PROOF. Reviewing the proof of Theorem 5, we construct an increasing sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} x_n = u$ for some $u \in X$. By using the given assumption on X, we have $x_n \preceq u$ for all $n \in \mathbf{N}$. Now, we show that $fu = gu = u$. Putting $x = x_{2n}$ and $y = u$ in (12) and by taking the upper limit as $n \to +\infty$ and using Lemma 1 and the properties of ψ and φ, we obtain

$$\begin{aligned}
\psi\big(sd(u, gu)\big) &= \psi\left(s^2 \frac{1}{s} d(u, gu)\right) \leq \psi\big(s^2 \limsup_{n\to+\infty} d(x_{2n+1}, gu)\big) \\
&= \psi\big(s^2 \limsup_{n\to+\infty} d(fx_{2n}, gu)\big) \\
&\leq \psi\left(\limsup_{n\to+\infty} \frac{d(x_{2n}, gu) + d(u, fx_{2n})}{s+1}\right) \\
&\quad - \liminf_{n\to+\infty} \varphi\big(d(x_{2n}, gu), d(u, fx_{2n})\big) \\
&\leq \psi\left(\frac{\limsup_{n\to+\infty} d(x_{2n}, gu) + \limsup_{n\to+\infty} d(u, x_{2n+1})}{s+1}\right) \\
&\quad -\varphi\big(\liminf_{n\to+\infty} d(x_{2n}, gu), \liminf_{n\to+\infty} d(u, x_{2n+1})\big) \\
&\leq \psi\left(\frac{sd(u, gu) + sd(u, u)}{s+1}\right) - \varphi\big(\liminf_{n\to+\infty} d(x_{2n}, gu), \liminf_{n\to+\infty} d(u, x_{2n+1})\big) \\
&\leq \psi\big(d(u, gu)\big) - \varphi\big(\liminf_{n\to+\infty} d(x_{2n}, gu), 0\big) \\
&\leq \psi\big(sd(u, gu)\big) - \varphi\big(\liminf_{n\to+\infty} d(x_{2n}, gu), 0\big).
\end{aligned}$$

This yields that $\liminf_{n\to+\infty} d(x_{2n}, gu) = 0$ by our assumptions about φ. By applying the triangular inequality, we have

$$d(u, gu) \leq sd(u, x_{2n}) + sd(x_{2n}, gu).$$

Now, letting $n \to +\infty$ in the above inequality, we conclude that $d(u, gu) = 0$. Thus, we have $u = gu$. Hence, u is a fixed point of g. On the other hand, similar to the first part of the proof of Theorem 1, we can show that $fu = u$. Therefore, u is a common fixed point of f and g. □

By taking $\psi(t) = t$ and $\varphi(t_1,t_2) = (\frac{1}{s+1} - k)(t_1+t_2)$ where $k \in [0, \frac{1}{s+1})$ in Theorems 5 and 6, we obtain immediately the following two results.

COROLLARY 3. *Let $(X, d, \preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$ and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that there exist $k \in [0, \frac{1}{s+1})$ such that*

$$d(fx, gy) \leq \frac{k}{s^2}\big(d(x, gy) + d(y, fx)\big).$$

If either f or g is continuous, then f and g have a common fixed point.

COROLLARY 4. *Under the same hypotheses of Corollary* 3, *without the continuity assumption on one of the functions f or g, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, let us have $x_n \preceq x$ for all $n \in \mathbf{N}$. Then f and g have a common fixed point in X.*

REMARK 2. Recall that a subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable. Note that in Theorems 3 and 4, it can be proved in a standard way that f has a unique fixed point provided that the fixed points of f are comparable. Similarly, in Theorems 5 and 6, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Now, in order to support the usability of our main results, we present the following example.

EXAMPLE 3. Let $X = [0, 1]$ be equipped with the b-metric defined by

$$d(x, y) = \begin{cases} [\max\{x, y\}]^2\,, & \text{if } x \neq y\,,\\ 0\,, & \text{if } x = y\,,\end{cases}$$

where $s = 2^{2-1} = 2$. Define a relation $\preceq$ on X by $y \preceq x$ if and only if $x \leq y$. Consider the mappings $f, g : X \to X$ given by

$$fx = \frac{x}{6} \qquad \text{and} \qquad gx = \frac{x}{7}\,.$$

Take $\psi : [0, +\infty) \to [0, +\infty)$ and $\varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = t^2 \qquad \text{and} \qquad \varphi(t_1, t_2) = \frac{8}{81}(t_1 + t_2)^2\,.$$

Then, we have the following:

(1) $(X, \preceq)$ is a partially ordered set having the b-metric d, where the b-metric space (X, d) is b-complete.

(2) f and g are weakly increasing mappings with respect to $\preceq$.
(3) f and g are continuous.
(4) $\psi \in \Psi$ and $\varphi \in \Phi$
(5) f is a $(\psi, \varphi)_s$-weakly C-contractive mapping with respect to g, that is, for any comparable x and y in X, we have

$$\psi\big(s^2 d(fx, gy)\big) \le \psi\left(\frac{d(x, gy) + d(y, fx)}{s+1}\right) - \varphi\big(d(x, gy), d(y, fx)\big).$$

PROOF. The proof of (1) is clear. To prove (2), given $x \in X$. Since

$$gfx = g\left(\frac{1}{6}x\right) = \frac{1}{42}x \le \frac{1}{6}x = fx\,,$$

we have $fx \preceq gfx$. Similarly, one can show that $gx \preceq fgx$. Therefore, f and g are weakly increasing mappings with respect to $\preceq$. It is easy to see that f and g are continuous. So (3) holds.

To prove (4), it easy to see that $\psi \in \Psi$, now we show that $\varphi \in \Phi$. let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers. First we note that, if f is a continuous and nondecreasing function, then we have

$$f(\liminf_{n\to\infty} a_n) = \liminf_{n\to\infty} f(a_n)\,.$$

In particular for $f(x) = x^2$ we have

$$\left(\liminf_{n\to\infty} a_n\right)^2 = \liminf_{n\to\infty} a_n^2\,.$$

Hence for the function $\varphi(t_1, t_2) = \frac{8}{81}(t_1 + t_2)^2$ we get

$$\begin{aligned}
\varphi\left(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\right) &= \frac{8}{81}\left(\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n\right)^2 \\
&\le \frac{8}{81}\left(\liminf_{n\to\infty}(a_n + b_n)\right)^2 \\
&= \frac{8}{81}\liminf_{n\to\infty}(a_n + b_n)^2 \\
&= \liminf_{n\to\infty} \varphi\,(a_n, b_n)\ .
\end{aligned}$$

To prove (5), given two comparable elements x and y in X. Thus, we have the following cases.

Case 1: If $x \preceq y$, then

$$d(fx, gy) = \left[\max\left\{\frac{x}{6}, \frac{y}{7}\right\}\right]^2 = \frac{x^2}{36},$$

and

$$\begin{aligned} d(x, gy) + d(fx, y) &= d\left(x, \frac{y}{7}\right) + d\left(\frac{x}{6}, y\right) \\ &= \left[\max\left\{x, \frac{y}{7}\right\}\right]^2 + d\left(\frac{x}{6}, y\right) \\ &= x^2 + d\left(\frac{x}{6}, y\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \psi\big(s^2 d(fx, gy)\big) &= 16\big(d(fx, gy)\big)^2 = \frac{x^4}{81} \\ &\le \frac{1}{81}\left(x^2 + d\left(\frac{x}{6}, y\right)\right)^2 \\ &= \frac{1}{9}\left(x^2 + d\left(\frac{x}{6}, y\right)\right)^2 - \frac{8}{81}\left(x^2 + d\left(\frac{x}{6}, y\right)\right)^2 \\ &= \psi\left(\frac{d(x, gy) + d(fx, y)}{3}\right) - \varphi\big(d(x, gy), d(fx, y)\big) \\ &= \psi\left(\frac{d(x, gy) + d(fx, y)}{s+1}\right) - \varphi\big(d(x, gy), d(fx, y)\big). \end{aligned}$$

Case 2: If $y \preceq x$, then

$$d(fx, gy) = \left[\max\left\{\frac{x}{6}, \frac{y}{7}\right\}\right]^2 \le \frac{y^2}{36},$$

and

$$\begin{aligned} d(x, gy) + d(fx, y) &= d\left(x, \frac{y}{7}\right) + d\left(\frac{x}{6}, y\right) \\ &= d\left(x, \frac{y}{7}\right) + \left[\max\left\{\frac{x}{6}, y\right\}\right]^2 \\ &= d\left(x, \frac{y}{7}\right) + y^2. \end{aligned}$$

Hence, we have

$$\begin{aligned}
\psi\big(s^2 d(fx, gy)\big) &= 16\big(d(fx, gy)\big)^2 = \frac{y^4}{81} \\
&\leq \frac{1}{81}\left(d\left(x, \frac{y}{7}\right) + y^2\right)^2 \\
&= \frac{1}{9}\left(d\left(x, \frac{y}{7}\right) + y^2\right)^2 - \frac{8}{81}\left(d\left(x, \frac{y}{7}\right) + y^2\right)^2 \\
&= \psi\left(\frac{d(x, gy) + d(fx, y)}{3}\right) - \varphi\big(d(x, gy), d(fx, y)\big) \\
&= \psi\left(\frac{d(x, gy) + d(fx, y)}{s+1}\right) - \varphi\big(d(x, gy), d(fx, y)\big)\,.
\end{aligned}$$

Combining Cases 1 and 2 together, we conclude that (5) holds. Thus, all the required hypotheses of Theorem 5 are satisfied and hence f and g have a common fixed point. In fact, 0 is the unique common fixed point of f and g. □

Other consequences of our results are the following for the mappings involving contractions of integral type.

Denote by Λ the set of functions $\gamma : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(1) γ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;
(2) for every $\varepsilon > 0$, we have $\int_0^{\varepsilon} \gamma(t)dt > 0$.

COROLLARY 5. *Let $(X, d, \preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$ and let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping. Suppose that there exist $\gamma \in \Lambda$ and $k \in [0, \frac{1}{s+1})$ such that*

$$\int_0^{d(fx, fy)} \gamma(t)dt \leq \frac{k}{s} \int_0^{\big(d(x, fy)+d(y, fx)\big)} \gamma(t)dt\,,$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

COROLLARY 6. *Under the same hypotheses of Corollary 5, without the continuity assumption on f, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, let us have $x_n \preceq x$ for all $n \in \mathbf{N}$. Then f has a fixed point in X.*

COROLLARY 7. *Let $(X,d,\preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$ and let $f,g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that there exist $\gamma \in \Lambda$ and $k \in [0,\frac{1}{s+1})$ such that*

$$\int_0^{d(fx,fy)} \gamma(t)dt \leq \frac{k}{s^2}\int_0^{\left(d(x,gy)+d(y,fx)\right)} \gamma(t)dt\,.$$

If either f or g is continuous, then f and g have a common fixed point.

COROLLARY 8. *Under the same hypotheses of Corollary* 7, *without the continuity assumption on one of the functions f or g, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, let us have $x_n \preceq x$ for all $n \in \mathbf{N}$. Then f and g have a common fixed point in X.*

REMARK 3. Corollary 7 from [7] is a special case of Corollary 1 and Corollary 2, if we take $s = 1$.

REMARK 4. Since a b-metric is a metric when $s = 1$, so our results can be viewed as the generalization and extension of corresponding results in [15, 25, 45] and several other comparable results.

3. Existence of a common solution for a system of integral equations

Consider the following system of integral equations:

$$\begin{aligned} x(t) &= \frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}}\int_a^b K_1(t,r,x(r))dr\,, \\ x(t) &= \frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}}\int_a^b K_2(t,r,x(r))dr\,. \end{aligned} \tag{24}$$

where $b > a \geq 0$. The purpose of this section is to present an existence theorem for a solution to 24 that belongs to $X = C[a,b]$ (the set of continuous real functions defined on [a,b]), by using the obtained result in Theorem 5.

Here, $K_1, K_2, : [a,b]\times[a,b]\times R \to R$. The considered problem can be reformulated in the following manner.

Let $f,g : X \to X$ be the mappings defined by:

$$fx(t) = \frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}}\int_a^b K_1(t,r,x(r))dr\,,$$

$$gx(t) = \frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}} \int_a^b K_2(t,r,x(r))dr\,,$$

for all $x \in X$ and for all $t \in [a,b]$ and for some $p > 1$.

Then the existence of a solution to 24 is equivalent to the existence of a common fixed point of f and g. According to Example 1, X equipped with

$$d(x,y) = \max_{t\in[a,b]} (|x(t)-y(t)|)^p\,,$$

for all $x, y \in X$, is a complete b-metric space with $s = 2^{p-1}$.

We endow X with the partial ordered $\preceq$ given by:

$$x \preceq y \Longleftrightarrow x(t) \le y(t)\,,$$

for all $t \in [a,b]$.

Now, we will prove the following result.

THEOREM 3.1. *Suppose that the following hypotheses hold*:
(i) $K_1, K_2 : [a,b]\times[a,b]\times R \to R$ *are continuous*;
(ii) *for all* $t, r \in [a,b]$ *and* $x \in X$, *we have*,

$$x(t) \le \frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}} \min\left\{\int_a^b K_1(t,r,x(r))dr, \int_a^b K_2(t,r,x(r))dr\right\}$$

(iii) *for all* $r, t \in [a,b]$ *and* $x, y \in X$ *with* $x \preceq y$ *we have*,

$$(|K_1(t,r,x(r))| + |K_2(t,r,y(r))|) \le \xi(t,r)(d(x,gy)+d(y,fx))$$

where ξ *is a continuous function satisfying*

$$\sup_{t\in[a,b]}\left(\int_a^b \xi(t,r)^p dr\right) < \frac{1}{2^{2p^2-2p}(b-a)^{p-1}}\,.$$

Then, *the integral equations* (24) *have a common solution* $x \in X$.

PROOF. From condition (ii), f *and* g *are* weakly increasing *self-maps on* X.
Let $1 \le p, q < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$.
Now, let $x, y \in X$ be such that $x \succeq y$. From condition (iii), for all $t \in [a,b]$ we have,

$$(2^{2p-2}(|fx(t)-gy(t)|))^p = \left(2^{2p-2}\left(\left|\begin{array}{c}\frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}}\int_a^b K_1(t,r,x(r))dr- \\ \frac{1}{\left(2^{p-1}+1\right)^{1+\frac{2}{p}}}\int_a^b K_2(t,r,x(r))dr\end{array}\right|\right)\right)^p$$

$$
\begin{aligned}
&\leq 2^{2p^2-2p}\frac{1}{\left(2^{p-1}+1\right)^{p+2}}\left(\int_a^b(|K_1(t,r,x(r))|+|K_2(t,r,x(r)|)dr\right)^p\\
&\leq 2^{2p^2-2p}\frac{1}{\left(2^{p-1}+1\right)^{p+2}}\left[\left(\int_a^b 1^q dr\right)^{\frac{1}{q}}\left(\int_a^b\begin{pmatrix}|K_1(t,r,x(r))|+\\|K_2(t,r,x(r)|\end{pmatrix}^p dr\right)^{\frac{1}{p}}\right]^p\\
&\leq 2^{2p^2-2p}(b-a)^{\frac{p}{q}}\frac{1}{\left(2^{p-1}+1\right)^{p+2}}\left(\int_a^b\xi(t,r)^p(d(x,gy)+d(y,fx))^p dr\right)\\
&\leq 2^{2p^2-2p}(b-a)^{\frac{p}{q}}\frac{1}{\left(2^{p-1}+1\right)^{p+2}}\left(\int_a^b\xi(t,r)^p dr)(d(x,gy)+d(y,fx)\right)^p\\
&= 2^{2p^2-2p}(b-a)^{p-1}\left(\int_a^b\xi(t,r)^p dr\right)\frac{1}{\left(2^{p-1}+1\right)^{p+2}}(d(x,gy)+d(y,fx))^p\\
&< \frac{1}{\left(2^{p-1}+1\right)^{p+2}}(d(x,gy)+d(y,fx))^p\\
&= \left(\frac{1}{\left(2^{p-1}+1\right)^{p}}-\frac{2^{2p-2}+2^p}{\left(2^{p-1}+1\right)^{p+2}}\right)(d(x,gy)+d(y,fx))^p\\
&= \left(\frac{(d(x,gy)+d(y,fx)}{2^{p-1}+1}\right)^p-\frac{2^{2p-2}+2^p}{\left(2^{p-1}+1\right)^{p+2}}(d(x,gy)+d(y,fx))^p
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(s^2 d(fx,gy))^p &= (s^2\sup_{t\in[a,b]}(|fx(t)-gy(t)|)^p\\
&\leq \left(\frac{(d(x,gy)+d(y,fx)}{2^{p-1}+1}\right)^p-\frac{2^{2p-2}+2^p}{\left(2^{p-1}+1\right)^{p+2}}((d(x,gy)+d(y,fx))^p\,.
\end{aligned}
$$

Taking $\psi(t)=t^p$ and $\varphi(t_1,t_2)=\frac{2^{2p-2}+2^p}{(2^{p-1}+1)^{p+2}}(t_1+t_2)^p$ in Theorem 5 there exists $x\in X$, a common fixed point of f and g, that is, x is a solution for (24). $\square$

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Present Addresses:
JAMAL REZAEI ROSHAN
DEPARTMENT OF MATHEMATICS, QAEMSHAHR BRANCH,
ISLAMIC AZAD UNIVERSITY, QAEMSHAHR, IRAN.
e-mail: jmlroshan@gmail.com, jml.roshan@qaemiau.ac.ir

MINA DINARVAND
FACULTY OF MATHEMATICS,
K. N. TOOSI UNIVERSITY OF TECHNOLOGY,
P.O. BOX 16315–1618, TEHRAN, IRAN.
e-mail: dinarvand_mina@yahoo.com