TOKYO J. MATH.
VOL. 40, NO. 1, 2017

On the Finiteness Properties of Local Cohomology Modules for Regular Local Rings

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(Communicated by H. Kaji)

Abstract. Let $\mathfrak{a}$ denote an ideal in a regular local (Noetherian) ring R and let N be a finitely generated R-module with support in $V(\mathfrak{a})$. The purpose of this paper is to show that all homomorphic images of the R-modules $\mathrm{Ext}^j_R(N, H^i_{\mathfrak{a}}(R))$ have only finitely many associated primes, for all $i, j \geq 0$, whenever $\dim R \leq 4$ or $\dim R/\mathfrak{a} \leq 3$ and R contains a field. In addition, we show that if $\dim R = 5$ and R contains a field, then the R-modules $\mathrm{Ext}^j_R(N, H^i_{\mathfrak{a}}(R))$ have only finitely many associated primes, for all $i, j \geq 0$.

1. Introduction

In the present paper we continue the study of the finiteness properties of local cohomology modules for regular local rings. An interesting problem in commutative algebra is determining when the set of associated primes of the i-th local cohomology module $H^i_{\mathfrak{a}}(R)$ of a Noetherian ring R with support in an ideal $\mathfrak{a}$ of R, is finite. This question was raised by Huneke in [11] at the Sundance Conference in 1990. Examples given by A. Singh [25] (in the non-local case) and M. Katzman [16] (in the local case) show there exist local cohomology modules of Noetherian rings with infinitely many associated primes. However, in recent years there have been several results showing that this conjecture is true in many situations. The first result were obtained by Huneke and Sharp. In fact, Huneke and Sharp [12] (in the case of positive characteristic) have shown that, if R is a regular local ring containing a field, then $H^i_{\mathfrak{a}}(R)$ has only finitely many associated primes for all $i \geq 0$. Subsequently, G. Lyubeznik in [13] and [14] showed this result for unramified regular local rings of mixed characteristic and in characteristic zero. Further, Lyubeznik posed the following conjecture:

CONJECTURE. *If R is a regular ring and $\mathfrak{a}$ an ideal of R, then the local cohomology modules $H^i_{\mathfrak{a}}(R)$ have finitely many associated prime ideals for all $i \geq 0$.*

Received September 28, 2015; revised August 26, 2016
2010 *Mathematics Subject Classification*: 13D45, 14B15, 13H05
Key words and phrases: Associated prime, cofinite module, local cohomology, minimax module, regular ring, weakly Laskerian module
This research was in part supported by a grant from Azarbaijan Shahid Madani University (No. 217/d/917).

While this conjecture remains open in this generality, several nice results are now available, see [3, 10, 19]. In lower dimensional cases, Marley in [19] showed that the set of associated prime ideals of the local cohomology modules $H^i_{\mathfrak{a}}(M)$ is finite if R is a local ring of dimension d and $\mathfrak{a}$ an ideal of R, M is a finitely generated R-module, in the following cases: (1) $d \leq 3$; (2) $d \leq 4$ and R has an isolated singularity; (3) $d = 5$ and R is an unramified regular local ring and M is torsion-free.

For a survey of recent developments on finiteness properties of local cohomology modules, see Lyubeznik's interesting paper [15].

The purpose of this paper is to provide some results concerning the set of associated primes of the local cohomology modules for a regular local ring, that almost results are extensions of Marley's results on local cohomology modules over the strong ring (i.e. the regular local ring containing a field). Namely, we show that, for a finitely generated module N over a regular local ring R with support in $V(\mathfrak{a})$, the R-modules $\operatorname{Ext}^j_R(N, H^i_{\mathfrak{a}}(R))$ are weakly Laskerian, for all $i, j \geq 0$, whenever $\dim R \leq 4$ or $\dim R/\mathfrak{a} \leq 3$ and R contains a field. In addition, we show that if $\dim R = 5$, then, for all $i, j \geq 0$, the R-module $\operatorname{Ext}^j_R(N, H^i_{\mathfrak{a}}(R))$ has only finitely many associated primes, when R contains a field.

We say that an R-module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite [6]. Our main result in Section 2 is to establish some finiteness results of local cohomology modules for a regular local ring R with respect to an ideal $\mathfrak{a}$ with $\dim R/\mathfrak{a} \leq 3$. More precisely, we prove the following.

THEOREM 1.1. *Let R be a regular local ring containing a field with* $\dim R = d \geq 3$, *and let $\mathfrak{a}$ be an ideal of R such that* $\dim R/\mathfrak{a} \leq 3$. *Then* $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(R))$ *is weakly Laskerian, for all $i, j \geq 0$.*

The result of Theorem 1.1 is proved in Section 2. Pursuing this point of view further, we obtain the following consequence of Theorem 1.1, which is an extension of Marley's result in [19].

COROLLARY 1.2. *Let R be a regular local ring of dimension $d \leq 4$ containing a field and $\mathfrak{a}$ an ideal of R. Then, for any finitely generated R-module N with* $\operatorname{Supp}(N) \subseteq V(\mathfrak{a})$, *the R-module* $\operatorname{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ *is weakly Laskerian, for all integers $n, i \geq 0$.*

It will be shown in Section 3 that the subjects of Section 2 can be used to prove a finiteness result of local cohomology modules for a regular local ring of dimension 5. In fact, we will generalize the main result of Marley for a regular local ring of dimension 5. More precisely, we shall show that:

THEOREM 1.3. *Let R be a regular local ring containing a field with* $\dim R = 5$ *and let $\mathfrak{a}$ be an ideal of R. Then the set* $\operatorname{Ass}_R \operatorname{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ *is finite, for each finitely generated R-module N with support in $V(\mathfrak{a})$ and for all integers $n, i \geq 0$.*

The proof of Theorem 1.3 is given in Section 3. The following proposition will be one of our main tools for proving Theorem 1.3.

PROPOSITION 1.4. *Let R be a regular local ring of dimension d, and let $\mathfrak{a}$ be an ideal of R with* height $\mathfrak{a} = 1$. *Then the set* $\operatorname{Ass}_R \operatorname{Ext}^i_R(N, H^1_{\mathfrak{a}}(R))$ *is finite for each finitely generated R-module N with support in $V(\mathfrak{a})$ and for all integers $i \geq 0$.*

As a consequence of Theorem 1.3 we derive the following result.

THEOREM 1.5. *Let R be a regular local ring containing a field such that* $\dim R \leq 5$, *and let $\mathfrak{a}$ be an ideal of R. Then, for all integers $n, i \geq 0$ and any finitely generated R-module N with support in $V(\mathfrak{a})$, the set* $\operatorname{Ass}_R \operatorname{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ *is finite.*

Finally, in this section we will show that, if M is a finitely generated module over a regular local ring R with $\dim R \leq 4$, then for any finitely generated R-module N with support in $V(\mathfrak{a})$, the R-module $\operatorname{Ext}^n_R(N, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian, for all integers $i, n \geq 0$. In particular the set $\operatorname{Ass}_R H^i_{\mathfrak{a}}(M)$ is finite, for all integers $i, n \geq 0$.

Hartshorne [9] introduced the notion of a cofinite module, answering in negative a question of Grothendieck [8, Exposé XIII, Conjecture 1.1]. In fact, Grothendieck asked if the modules $\operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ always are finitely generated for any ideal $\mathfrak{a}$ of R and any finitely generated R-module M. This is the case when $\mathfrak{a} = \mathfrak{m}$, the maximal ideal in a local ring, since the modules $H^i_{\mathfrak{m}}(M)$ are Artinian. Hartshorne defined an R-module M to be $\mathfrak{a}$-*cofinite* if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$.

In [26], H. Zöschinger, introduced an interesting class of minimax modules, and he has in [26, 27] given many equivalent conditions for a module to be minimax. An R-module N is said to be a minimax module, if there is a finitely generated submodule L of N, such that N/L is Artinian. The class of minimax modules thus includes all finitely generated and all Artinian modules. Also, an R-module M is called $\mathfrak{a}$-*cominimax* if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is minimax for all $i \geq 0$. The concept of the $\mathfrak{a}$-cominimax modules is introduced in [2].

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ will be an ideal of R. For an R-module M, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)\,.$$

We refer the readers to [7] and [5] for more details on local cohomology.

We shall use $\operatorname{Min}(\mathfrak{a})$ to denote the set of all minimal primes of $\mathfrak{a}$. For each R-module L, we denote by $\operatorname{Assh}_R L$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. Also, for any ideal $\mathfrak{b}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$. Finally, for any ideal $\mathfrak{c}$ of R, *the radical of* $\mathfrak{c}$, denoted by $\operatorname{Rad}(\mathfrak{c})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{c}$ for some $n \in \mathbf{N}\}$. For any unexplained notation and terminology we refer the readers to [5] and [21].

2. Finiteness of local cohomology modules for ideals of small dimension

The purpose of this section is to study the finiteness properties of local cohomology modules for a regular local ring R with respect to an ideal $\mathfrak{a}$ of R with $\dim R/\mathfrak{a} \leq 3$. The main goal is Theorem 2.7, which plays a key role in Section 3. This result extends a main result of T. Marley [19].

The following lemmas and proposition will be needed in the proof of Theorem 2.7.

THEOREM 2.1 (Huneke-Sharp and Lyubeznik). *Let $(R, \mathfrak{m})$ be a regular local ring containing a field. Then for each ideal $\mathfrak{a}$ of R and all integers $i \geq 0$, the set of associated primes of the local cohomology modules $H^i_{\mathfrak{a}}(R)$ are finite.*

PROOF. See [12, 13, 14]. □

LEMMA 2.2. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field with $\dim R = d \geq 3$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of R such that $\dim R/\mathfrak{p}_i = 3$ for all $i = 1, \dots, n$. Then $H^d_{\mathfrak{a}}(R) = 0$ and $\operatorname{Supp}(H^j_{\mathfrak{a}}(R))$ is finite, for $j = d-1, d-2$, where $\mathfrak{a} := \bigcap_{i=1}^n \mathfrak{p}_i$.*

PROOF. It follows easily from Lichtenbaum-Hartshorne and Grothendieck's vanishing theorems that $H^d_{\mathfrak{a}}(R) = 0$, $\operatorname{Supp}(H^{d-1}_{\mathfrak{a}}(R)) \subseteq \{\mathfrak{m}\}$ and for each $\mathfrak{q} \in \operatorname{Supp}(H^{d-2}_{\mathfrak{a}}(R))$ we have $\dim R/\mathfrak{q} \leq 1$. Therefore $\dim \operatorname{Supp}(H^{d-2}_{\mathfrak{a}}(R)) \leq 1$, and so

$$\operatorname{Supp}(H^{d-2}_{\mathfrak{a}}(R)) \subseteq \operatorname{Assh}_R H^{d-2}_{\mathfrak{a}}(R) \cup \{\mathfrak{m}\}.$$

Now it follows from Lemma 2.1 that $\operatorname{Supp}(H^{d-2}_{\mathfrak{a}}(R))$ is a finite set, as required. □

COROLLARY 2.3. *Let $(R, \mathfrak{m})$ and $\mathfrak{a}$ be as in Lemma 2.2. Then $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(R))$ is a weakly Laskerian R-module for all $i \geq 0$ and for $j = d-1, d-2$.*

PROOF. The result follows easily from Lemma 2.2. □

The next lemma was proved by Melkersson for $\mathfrak{a}$-cofiniteness. The proof given in [22, Proposition 3.11] can be easily carried for weakly Laskerian modules.

LEMMA 2.4. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R, and M an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all i. If t is a non-negative integer such that the R-module $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is weakly Laskerian, for all i and all $j \neq t$, then this is the case also when $j = t$.*

PROPOSITION 2.5. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field with $\dim R = d \geq 3$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of R such that $\dim R/\mathfrak{p}_i = 3$, for all $i = 1, \dots, n$, and let $\mathfrak{a} := \bigcap_{i=1}^n \mathfrak{p}_i$. Then $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(R))$ is a weakly Laskerian R-module for all $i, j \geq 0$.*

PROOF. If $H^i_{\mathfrak{a}}(R) \neq 0$, then it follows from Lemma 2.2 and [5, Theorems 6.1.2 and 6.2.7] that $i \in \{d-3, d-2, d-1\}$. Whence, the assertion follows from Corollary 2.3 and Lemma 2.4. □

The next result was proved by Kawasaki for finitely generated modules. The proof given in [17, Lemma 1] can be easily carried for weakly Laskerian modules.

LEMMA 2.6. *Let R be a Noetherian ring, T an R-module, and $\mathfrak{a}$ an ideal of R. Then the following conditions are equivalent*:

(i) $\operatorname{Ext}^n_R(R/\mathfrak{a}, T)$ *is weakly Laskerian for all* $n \geq 0$.

(ii) *For any finitely generated R-module N with support in* $V(\mathfrak{a})$, $\operatorname{Ext}^n_R(N, T)$ *is weakly Laskerian for all* $n \geq 0$.

We now are prepared to prove the main theorem of this section, which shows that when R is a regular local ring contains a field and $\mathfrak{a}$ an ideal of R such that $\dim R/\mathfrak{a} \leq 3$, then all homomorphic images of the R-modules $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(R))$ have only finitely many associated primes.

THEOREM 2.7. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field with* $\dim R = d \geq 3$, *and let $\mathfrak{a}$ be an ideal of R such that* $\dim R/\mathfrak{a} \leq 3$. *Then the R-module* $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(R))$ *is weakly Laskerian, for all* $i, j \geq 0$.

PROOF. In view of [4, Corollaries 2.7 and 3.2], we may assume that $\dim R/\mathfrak{a} = 3$. Then we have $\operatorname{grade} \mathfrak{a} = d - 3$. Hence, by virtue of [5, Theorems 6.1.2 and 6.2.7], $H^i_{\mathfrak{a}}(R) = 0$, whenever $i \notin \{d-3, d-2, d-1\}$. Moreover, by [23, Corollary 3.3], the set $\operatorname{Supp}(H^{d-1}_{\mathfrak{a}}(R))$ is finite, in view of Proposition 2.5, it is enough for us to show that the R-module $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^{d-3}_{\mathfrak{a}}(R))$ is weakly Laskerian. To this end, let

$$X := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height} \mathfrak{p} = d - 3\} \quad \text{and} \quad Y := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height} \mathfrak{p} \geq d - 2\}.$$

Then $X \neq \emptyset$. Set $\mathfrak{b} = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. If $Y = \varnothing$, then $\operatorname{Rad}(\mathfrak{a}) = \mathfrak{b}$, and so the assertion follows from Proposition 2.5. Therefore we may assume that $Y \neq \emptyset$. Let $\mathfrak{c} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. Then $\operatorname{Rad}(\mathfrak{a}) = \mathfrak{b} \cap \mathfrak{c}$ and $\operatorname{height} \mathfrak{c} \geq d-2$. Next, we show that $\operatorname{height} (\mathfrak{b}+\mathfrak{c}) \geq d-1$. Suppose the contrary is true. Then there exists a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{b} + \mathfrak{c} \subseteq \mathfrak{q}$ and $\operatorname{height} \mathfrak{q} = d - 2$. Therefore there exist $\mathfrak{p}_1 \in X$ and $\mathfrak{p}_2 \in Y$ such that $\mathfrak{p}_1 + \mathfrak{p}_2 \subseteq \mathfrak{q}$. Since $\operatorname{height} \mathfrak{p}_2 \geq d - 2$, it follows that $\mathfrak{q} = \mathfrak{p}_2$, and so $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2$, which is a contradiction (note that $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Min}(\mathfrak{a})$).

Consequently, $\operatorname{grade} (\mathfrak{b} + \mathfrak{c}) \geq d - 1$ and $\operatorname{grade} \mathfrak{c} \geq d - 2$. Therefore, it follows from [5, Theorem 6.2.7], that

$$H^{d-3}_{\mathfrak{b}+\mathfrak{c}}(R) = 0 = H^{d-2}_{\mathfrak{b}+\mathfrak{c}}(R) \quad \text{and} \quad H^{d-3}_{\mathfrak{c}}(R) = 0\,.$$

It now follows from $\operatorname{Rad}(\mathfrak{a}) = \mathfrak{b} \cap \mathfrak{c}$ and the Mayer-Vietoris sequence (see [5, Theorem 3.2.3]), that $H^{d-3}_{\mathfrak{a}}(R) \cong H^{d-3}_{\mathfrak{b}}(R)$, and so by Proposition 2.5, the R-module $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^{d-3}_{\mathfrak{a}}(R))$ is weakly Laskerian, for all $j \geq 0$.

On the other hand, since $\dim R/(\mathfrak{b} + \mathfrak{c}) \leq 1$, it is easy to see that

$$V(\mathfrak{b} + \mathfrak{c}) = \operatorname{Assh}_R R/(\mathfrak{b} + \mathfrak{c}) \cup \{\mathfrak{m}\}\,.$$

Now, as

$$\mathrm{Supp}(\mathrm{Ext}^i_R(R/\mathfrak{c}, H^{d-3}_{\mathfrak{b}}(R))) \subseteq V(\mathfrak{b}+\mathfrak{c})\,,$$

it follows that $\mathrm{Ext}^i_R(R/\mathfrak{c}, H^{d-3}_{\mathfrak{b}}(R))$ is a weakly Laskerian R-module, for all $i \geq 0$. Also, as

$$\mathrm{Supp}(\mathrm{Ext}^i_R(R/\mathfrak{b}+\mathfrak{c}, H^{d-3}_{\mathfrak{b}}(R))) \subseteq V(\mathfrak{b}+\mathfrak{c})\,,$$

analogous to the preceding, we see that the R-module

$$\mathrm{Ext}^i_R(R/\mathfrak{b}+\mathfrak{c}, H^{d-3}_{\mathfrak{b}}(R))$$

is also weakly Laskerian for all $i \geq 0$. Now, the exact sequence

$$0 \longrightarrow R/\operatorname{Rad}(\mathfrak{a}) \longrightarrow R/\mathfrak{b} \oplus R/\mathfrak{c} \longrightarrow R/\mathfrak{b}+\mathfrak{c} \longrightarrow 0\,,$$

induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{b}, H^{d-3}_{\mathfrak{a}}(R)) \oplus \mathrm{Ext}^i_R(R/\mathfrak{c}, H^{d-3}_{\mathfrak{a}}(R)) \longrightarrow \mathrm{Ext}^i_R(R/\operatorname{Rad}(\mathfrak{a}), H^{d-3}_{\mathfrak{a}}(R))$$

$$\longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{b}+\mathfrak{c}, H^{d-3}_{\mathfrak{a}}(R)) \longrightarrow \cdots\,,$$

which shows that the R-module $\mathrm{Ext}^i_R(R/\operatorname{Rad}(\mathfrak{a}), H^{d-3}_{\mathfrak{a}}(R))$ is weakly Laskerian, for all $i \geq 0$, and so it follows from Lemma 2.6 that, the R-module $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^{d-3}_{\mathfrak{a}}(R))$ is weakly Laskerian, for all $i \geq 0$, as required. □

COROLLARY 2.8. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 3$ containing a field, and let $\mathfrak{a}$ be an ideal of R such that* $\dim R/\mathfrak{a} \leq 3$. *Then, for any finitely generated R-module N with* $\mathrm{Supp}(N) \subseteq V(\mathfrak{a})$, *the R-module* $\mathrm{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ *is weakly Laskerian, for all integers $n, i \geq 0$.*

PROOF. The result follows from Theorem 2.7 and Lemma 2.6. □

COROLLARY 2.9. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \leq 4$ containing a field, and $\mathfrak{a}$ an ideal of R. Then, for any finitely generated R-module N with* $\mathrm{Supp}(N) \subseteq V(\mathfrak{a})$, *the R-module* $\mathrm{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ *is weakly Laskerian, for all integers $n, i \geq 0$.*

PROOF. Since R is regular local, so $\dim R/\mathfrak{a} = 4$ if and only if $\mathfrak{a} = 0$. Thus, the assertion is clear in the case of $\dim R/\mathfrak{a} = 4$. Moreover, the case $\dim R/\mathfrak{a} = 3$ follows from Corollary 2.8. Also, the case $\dim R/\mathfrak{a} = 2$ follows from [4, Theorem 3.1] and Lemma 2.4. Finally, if $\dim R/\mathfrak{a} \leq 1$, then the assertion follows from [4, Theorem 2.6] and [17, Lemma 1]. □

3. Finiteness of local cohomology modules for regular local rings of small dimension

It will be shown in this section that the subjects of the previous section can be used to prove the finiteness of local cohomology modules for a regular local ring R with $\dim R = 5$.

The main result is Theorem 3.4. The following proposition will serve to shorten the proof of that theorem. The following easy lemma will be used in Proposition 3.2.

LEMMA 3.1. *Let R be a Noetherian ring and let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence of R-modules such that M' is weakly Laskerian and M'' has only finitely many associated primes. Then M has only finitely many associated primes.*

PROOF. The assertion follows from the exact sequence

$$0 \longrightarrow M'/\operatorname{Im} g \longrightarrow M \longrightarrow \operatorname{Im} g \longrightarrow 0\,,$$

by applying [21, Theorem 6.3]. □

PROPOSITION 3.2. *Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring, and $\mathfrak{a}$ an ideal of R such that* $\operatorname{height}\mathfrak{a} = 1$. *Then the set $\operatorname{Ass}_R \operatorname{Ext}^i_R(N, H^1_{\mathfrak{a}}(R))$ is finite, for each finitely generated R-module N with support in $V(\mathfrak{a})$ and for all integers $i \geq 0$.*

PROOF. Let

$$X := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height}\mathfrak{p} = 1\} \quad \text{and} \quad Y := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height}\mathfrak{p} \geq 2\}\,.$$

Then $X \neq \emptyset$. Let $\mathfrak{b} = \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$. Since R is a UFD, it follows from [21, Exercise 20.3] that $\mathfrak{b}$ is a principal ideal, and so there is an element $x \in R$ such that $\mathfrak{b} = Rx$. Hence, if $Y = \varnothing$, then $\operatorname{Rad}(\mathfrak{a}) = Rx$, and so it follows from [18, Theorem 1] that, the R-module $\operatorname{Ext}^i_R(N, H^1_{\mathfrak{a}}(R))$ is finitely generated. Thus the set $\operatorname{Ass}_R \operatorname{Ext}^i_R(N, H^1_{\mathfrak{a}}(R))$ is finite.

Therefore, we may assume that $Y \neq \emptyset$. Then $\operatorname{Rad}(\mathfrak{a}) = Rx \cap \mathfrak{c}$, where $\mathfrak{c} = \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}$. It is easy to see that $\operatorname{height}\mathfrak{c} \geq 2$ and $\operatorname{height}(\mathfrak{b} + \mathfrak{c}) \geq 3$. Whence, by using the Mayer-Vietoris sequence it yields that $H^1_{\mathfrak{a}}(R) \cong H^1_{\mathfrak{b}}(R)$. Therefore, by [18, Theorem 1], the R-module $H^1_{\mathfrak{a}}(R)$ is $\mathfrak{b}$-cofinite.

On the other hand, according to Artin-Rees lemma, there exists a positive integer n such that

$$\mathfrak{b}^n N \cap \Gamma_{\mathfrak{b}}(N) = 0 = \mathfrak{c}^n N \cap \Gamma_{\mathfrak{c}}(N)\,.$$

We claim that $\mathfrak{b}^n N \cap \mathfrak{c}^n N = 0$. To this end, suppose that $\mathfrak{q} \in \operatorname{Ass}_R(\mathfrak{b}^n N \cap \mathfrak{c}^n N)$. Then $\mathfrak{q} = (0 :_R y)$, for some $y(\neq 0) \in \mathfrak{b}^n N \cap \mathfrak{c}^n N$. As $\operatorname{Supp}(N) \subseteq V(\mathfrak{a})$, it follows that $\mathfrak{a} \subseteq \mathfrak{q}$, and so $\mathfrak{b} \cap \mathfrak{c} \subseteq \mathfrak{q}$. Thus $\mathfrak{b} \subseteq \mathfrak{q}$ or $\mathfrak{c} \subseteq \mathfrak{q}$, and so $y \in \Gamma_{\mathfrak{b}}(N)$ or $y \in \Gamma_{\mathfrak{c}}(N)$. Furthermore, since $y \in \mathfrak{b}^n N \cap \mathfrak{c}^n N$, it follows that $y = 0$, which is a contradiction.

Now, since $\mathfrak{b}^n N \cap \mathfrak{c}^n N = 0$, the exact sequence

$$0 \longrightarrow N \longrightarrow N/\mathfrak{b}^n N \oplus N/\mathfrak{c}^n N \longrightarrow N/(\mathfrak{b}^n + \mathfrak{c}^n)N \longrightarrow 0\,,$$

induces the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(N/\mathfrak{b}^n N, H^1_{\mathfrak{a}}(R)) \oplus \operatorname{Ext}^i_R(N/\mathfrak{c}^n N, H^1_{\mathfrak{a}}(R)) \longrightarrow \operatorname{Ext}^i_R(N, H^1_{\mathfrak{a}}(R))$$

$$\longrightarrow \operatorname{Ext}^{i+1}_R(N/(\mathfrak{b}^n + \mathfrak{c}^n)N, H^1_{\mathfrak{a}}(R)) \longrightarrow \cdots\,. \qquad (\dagger)$$

Since $H^1_{\mathfrak{a}}(R)$ is $\mathfrak{b}$-cofinite and

$$\operatorname{Supp}(N/(\mathfrak{b}^n+\mathfrak{c}^n)N) \subseteq \operatorname{Supp}(N/\mathfrak{b}^n N) \subseteq V(\mathfrak{b}),$$

it follows from [17, Lemma 1] that the R-modules

$$\operatorname{Ext}^i_R(N/\mathfrak{b}^n N, H^1_{\mathfrak{a}}(R)) \text{ and } \operatorname{Ext}^i_R(N/(\mathfrak{b}^n+\mathfrak{c}^n)N, H^1_{\mathfrak{a}}(R))$$

are finitely generated for all $i \geq 0$.

Next, let $T := N/\mathfrak{c}^n N$ and we show that $\operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R))$ is also $\mathfrak{b}$-cofinite for all $i \geq 0$. To do this, since $\operatorname{Supp}(\Gamma_{\mathfrak{b}}(T)) \subseteq V(\mathfrak{b})$ it follows from [17, Lemma 1] that the R-module $\operatorname{Ext}^i_R(\Gamma_{\mathfrak{b}}(T), H^1_{\mathfrak{a}}(R))$ is finitely generated for each $i \geq 0$. Hence it is enough to show that the R-module $\operatorname{Ext}^i_R(T/\Gamma_{\mathfrak{b}}(T), H^1_{\mathfrak{a}}(R))$ is finitely generated. As $T/\Gamma_{\mathfrak{b}}(T)$ is $\mathfrak{b}$-torsion-free, we therefore make the additional assumption that T is a $\mathfrak{b}$-torsion-free R-module. Then in view of [5, Lemma 2.11], x is a non-zerodivisor on T, and so the exact sequence

$$0 \longrightarrow T \xrightarrow{x} T \longrightarrow T/xT \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(T/xT, H^1_{\mathfrak{a}}(R)) \longrightarrow \operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R)) \xrightarrow{x} \operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R))$$

$$\longrightarrow \operatorname{Ext}^{i+1}_R(T/xT, H^1_{\mathfrak{a}}(R)) \longrightarrow \cdots. \qquad (\dagger\dagger)$$

Since $\operatorname{Supp}(T/xT) \subseteq V(\mathfrak{b})$ it follows from [17, Lemma 1] that $\operatorname{Ext}^i_R(T/xT, H^1_{\mathfrak{a}}(R))$ is a finitely generated R-module, for all $i \geq 0$, (note that $\mathfrak{b} = Rx$). Consequently, it follows from the exact sequence $(\dagger\dagger)$ that the R-modules

$$(0 :_{\operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R))} x) \text{ and } \operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R))/x\operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R))$$

are finitely generated and hence $\mathfrak{b}$-cofinite. Therefore it follows from Melkersson's result [22, Corollary 3.4] that the R-module $\operatorname{Ext}^i_R(T, H^1_{\mathfrak{a}}(R))$ is $\mathfrak{b}$-cofinite for all $i \geq 0$.

Now, let $\Omega := \operatorname{Ext}^i_R(N/\mathfrak{b}^n N, H^1_{\mathfrak{a}}(R)) \oplus \operatorname{Ext}^i_R(N/\mathfrak{c}^n N, H^1_{\mathfrak{a}}(R))$. Then for every finitely generated submodule L of Ω the R-module Ω/L is also $\mathfrak{b}$-cofinite, and so the set $\operatorname{Ass}_R \Omega/L$ is finite. Now, it follows from the exact sequence $(\dagger)$ and Lemma 3.1 that the set $\operatorname{Ass}_R \operatorname{Ext}^i_R(N, H^1_{\mathfrak{a}}(R))$ is finite, as required. □

The next lemma was proved in [1] in the case R is local. The proof given in [1, Lemma 2.5] can be easily carried over Noetherian rings, so that we omit the proof.

LEMMA 3.3. *Let R be a Noetherian ring, x an element of R and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a} \subseteq \operatorname{Rad}(Rx)$. Then, for any finitely generated R-module M, the R-homomorphism $H^j_{\mathfrak{a}}(M) \xrightarrow{x} H^j_{\mathfrak{a}}(M)$ is an isomorphism, for each $j \geq 2$.*

We are now ready to state and prove the main theorem of this section.

THEOREM 3.4. *Let $(R,\mathfrak{m})$ be a five-dimensional regular local ring containing a field and $\mathfrak{a}$ an ideal of R. Then the set $\operatorname{Ass}_R \operatorname{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ is finite, for each finitely generated R-module N with support in $V(\mathfrak{a})$ and for all integers $n, i \geq 0$.*

PROOF. Since $\dim R/\mathfrak{a} = 5$ if and only if $\mathfrak{a} = 0$, the assertion is clear in this case. Hence we consider the case when $\dim R/\mathfrak{a} \leq 4$. If $\dim R/\mathfrak{a} \leq 3$, then the result follows from the proof of Corollary 2.8. Therefore, we may assume that $\dim R/\mathfrak{a} = 4$. Then $\operatorname{height}(\mathfrak{a}) = 1$ and in view of the Lichtenbaum-Hartshorne vanishing theorem $H^5_{\mathfrak{a}}(R) = 0$. Whence $H^i_{\mathfrak{a}}(R) = 0$ whenever $i \notin \{1,2,3,4\}$. Also, since by [23, Corollary 3.3], the set $\operatorname{Supp}(H^4_{\mathfrak{a}}(R))$ is finite, it follows from

$$\operatorname{Supp}(\operatorname{Ext}^n_R(N, H^4_{\mathfrak{a}}(R))) \subseteq \operatorname{Supp}(H^4_{\mathfrak{a}}(R))$$

that the set $\operatorname{Ass}_R \operatorname{Ext}^n_R(N, H^4_{\mathfrak{a}}(R))$ is also finite. Consequently, in view of Proposition 3.2, we may consider the cases $i = 2, 3$.

CASE 1. $i = 2$. Suppose that

$$X := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height}\mathfrak{p} = 1\} \quad \text{and} \quad Y := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height}\mathfrak{p} \geq 2\}.$$

Then $X \neq \emptyset$ and $Y \neq \emptyset$. Let $\mathfrak{b} = \bigcap_{\mathfrak{p}\in X}\mathfrak{p}$ and $\mathfrak{c} = \bigcap_{\mathfrak{p}\in Y}\mathfrak{p}$. Since R is a UFD, it follows from [21, Ex. 20.3] that $\mathfrak{b}$ is a principal ideal, and so there is an element $x \in R$ such that $\mathfrak{b} = Rx$. Moreover, $\operatorname{Rad}(\mathfrak{a}) = Rx \cap \mathfrak{c}$, $\operatorname{height}\mathfrak{c} \geq 2$, and in view of Proposition 3.2, we have $H^1_{\mathfrak{a}}(R) \cong H^1_{\mathfrak{b}}(R)$. Thus, by [18, Theorem 1], $H^1_{\mathfrak{a}}(R)$ is a $\mathfrak{b}$-cofinite module. As $\mathfrak{c} \nsubseteq \bigcup_{\mathfrak{p}\in X}\mathfrak{p}$, it follows that there is an element $y \in \mathfrak{c}$ such that $y \notin \bigcup_{\mathfrak{p}\in X}\mathfrak{p}$. Now, in view of [24, Corollary 3.5], there exists the exact sequence

$$0 \longrightarrow H^1_{Ry}(H^1_{Rx}(R)) \longrightarrow H^2_{Rx+Ry}(R) \longrightarrow H^0_{Ry}(H^2_{Rx}(R)) \longrightarrow 0\,,$$

and so $H^2_{Rx+Ry}(R) \cong H^1_{Ry}(H^1_{Rx}(R))$, (note that $H^2_{Rx}(R) = 0$).

Using again [24, Corollary 3.5], to show that there exists the exact sequence

$$0 \longrightarrow H^1_{Ry}(H^1_{\mathfrak{a}}(R)) \longrightarrow H^2_{\mathfrak{a}+Ry}(R) \longrightarrow H^0_{Ry}(H^2_{\mathfrak{a}}(R)) \longrightarrow 0\,, \qquad (\dagger)$$

and so it follows from $H^1_{\mathfrak{a}}(R) \cong H^1_{\mathfrak{b}}(R)$ that

$$H^1_{Ry}(H^1_{\mathfrak{a}}(R)) \cong H^1_{Ry}(H^1_{\mathfrak{b}}(R)) = H^1_{Ry}(H^1_{Rx}(R)) \cong H^2_{Rx+Ry}(R)\,.$$

Also, since $xy \in \operatorname{Rad}(\mathfrak{a})$, it follows that the R-module $H^2_{\mathfrak{a}}(R)$ is $R(yx)$-torsion. Hence using Lemma 3.3, it is easy to see that the R-module $H^2_{\mathfrak{a}}(R)$ is Ry-torsion. That is $H^0_{Ry}(H^2_{\mathfrak{a}}(R)) = H^2_{\mathfrak{a}}(R)$. Consequently, from the exact sequence $(\dagger)$ we get the following exact sequence,

$$0 \longrightarrow H^2_{Rx+Ry}(R) \longrightarrow H^2_{\mathfrak{a}+Ry}(R) \longrightarrow H^2_{\mathfrak{a}}(R) \longrightarrow 0\,. \qquad (\dagger\dagger)$$

Furthermore, in view of Proposition 3.2, there exists a positive integer n such that the sequence

$$0 \longrightarrow N \longrightarrow N/\mathfrak{b}^n N \oplus N/\mathfrak{c}^n N \longrightarrow N/(\mathfrak{b}^n + \mathfrak{c}^n)N \longrightarrow 0$$

is exact, and so we obtain the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(N/\mathfrak{b}^n N, H^2_{\mathfrak{a}}(R)) \oplus \operatorname{Ext}^i_R(N/\mathfrak{c}^n N, H^2_{\mathfrak{a}}(R)) \longrightarrow \operatorname{Ext}^i_R(N, H^2_{\mathfrak{a}}(R))$$

$$\longrightarrow \operatorname{Ext}^{i+1}_R(N/(\mathfrak{b}^n + \mathfrak{c}^n)N, H^2_{\mathfrak{a}}(R)) \longrightarrow \cdots . \qquad (\dagger\dagger\dagger)$$

Since

$$\operatorname{Supp}(N/(\mathfrak{b}^n + \mathfrak{c}^n)N) \subseteq \operatorname{Supp}(N/\mathfrak{b}^n N) \subseteq V(\mathfrak{b}),$$

it follows from Lemma 3.3 that

$$\operatorname{Ext}^j_R(N/\mathfrak{b}^n N, H^2_{\mathfrak{a}}(R)) = 0 = \operatorname{Ext}^j_R(N/(\mathfrak{b}^n + \mathfrak{c}^n)N, H^2_{\mathfrak{a}}(R)),$$

for all $j \geq 0$. Whence, the exact sequence $(\dagger\dagger\dagger)$ implies that

$$\operatorname{Ext}^j_R(N, H^2_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{\mathfrak{a}}(R)).$$

Next, it is easy to see that $\operatorname{height}(Rx + Ry) = 2$, and so $\operatorname{height}(\mathfrak{a} + Ry) = 2$. Thus $\dim R/(\mathfrak{a} + Ry) = 3$, and hence as

$$\operatorname{Supp}(N/\mathfrak{c}^n N) \subseteq V(\mathfrak{c}) \subseteq V(\mathfrak{a} + Ry),$$

it follows from Corollary 2.8 that the R-module $\operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{\mathfrak{a}+Ry}(R))$ is weakly Laskerian, for all $j \geq 0$.

Also, as $\operatorname{grade}(Rx + Ry) = 2$, it follows from [5, Theorem 3.3.1] and [22, Proposition 3.11] that, the R-module $H^2_{Rx+Ry}(R)$ is $Rx + Ry$-cofinite. Now, by modifying the argument of the proof of Proposition 3.2, one can see that the R-module $\operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{Rx+Ry}(R))$ is Rx-cofinite for all $j \geq 0$. In particular, $\operatorname{Ass}_R \operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{Rx+Ry}(R))$ is a finite set, for all $j \geq 0$. Moreover, from the exact sequence $(\dagger\dagger)$, we deduce the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{\mathfrak{a}+Ry}(R)) \longrightarrow \operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{\mathfrak{a}}(R))$$

$$\longrightarrow \operatorname{Ext}^{j+1}_R(N/\mathfrak{c}^n N, H^2_{Rx+Ry}(R)) \longrightarrow \cdots .$$

Now using Lemma 3.1 and the above long exact sequence induced, it follows from the isomorphism

$$\operatorname{Ext}^j_R(N, H^2_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^2_{\mathfrak{a}}(R)),$$

that the set $\operatorname{Ass}_R \operatorname{Ext}^j_R(N, H^2_{\mathfrak{a}}(R))$ is finite.

CASE 2. $i = 3$. Let $X \neq \emptyset$, $Y \neq \emptyset$ and y be as in the case 1. Then using the same argument, it follows from Lemma 3.3 that the R-modules $H^2_{\mathfrak{a}}(R)$ and $H^3_{\mathfrak{a}}(R)$ are Ry-torsion. Thus

$$H^1_{Ry}(H^2_{\mathfrak{a}}(R)) = 0 \quad \text{and} \quad H^0_{Ry}(H^3_{\mathfrak{a}}(R)) = H^3_{\mathfrak{a}}(R)\,.$$

Therefore, using the exact sequence

$$0 \longrightarrow H^1_{Ry}(H^2_{\mathfrak{a}}(R)) \longrightarrow H^3_{\mathfrak{a}+Ry}(R) \longrightarrow H^0_{Ry}(H^3_{\mathfrak{a}}(R)) \longrightarrow 0\,, \qquad (\dagger)$$

(see [24, Corollary 3.5]), we obtain that $H^3_{\mathfrak{a}}(R) \cong H^3_{\mathfrak{a}+Ry}(R)$.

Moreover, it follows easily from the exact sequence

$$0 \longrightarrow N \longrightarrow N/\mathfrak{b}^n N \oplus N/\mathfrak{c}^n N \longrightarrow N/(\mathfrak{b}^n + \mathfrak{c}^n)N \longrightarrow 0\,,$$

that

$$\operatorname{Ext}^j_R(N, H^3_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^3_{\mathfrak{a}}(R)),$$

for all $j \geq 0$. Hence, for all $j \geq 0$ the R-module $\operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^3_{\mathfrak{a}+Ry}(R))$ is weakly Laskerian, and so $\operatorname{Ext}^j_R(N/\mathfrak{c}^n N, H^3_{\mathfrak{a}}(R))$ is also a weakly Laskerian R-module. Therefore $\operatorname{Ext}^j_R(N, H^3_{\mathfrak{a}}(R))$ is a weakly Laskerian R-module, and hence it has finitely many associated primes, as required. □

THEOREM 3.5. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field such that* $\dim R \leq 5$, *and let $\mathfrak{a}$ be an ideal of R. Then, for all integers $n, i \geq 0$ and any finitely generated R-module N with support in $V(\mathfrak{a})$, the set* $\operatorname{Ass}_R \operatorname{Ext}^n_R(N, H^i_{\mathfrak{a}}(R))$ *is finite.*

PROOF. The assertion follows from Corollary 2.8 and Theorem 3.4. □

The final theorem of this section shows that if R is a regular local ring with $\dim R \leq 4$, then the R-module $H^i_{\mathfrak{a}}(M)$ has finitely many associated primes, for all $i \geq 0$ and for any finitely generated M over R. Recall that, an R-module L is called an $\mathfrak{a}$-*cominimax module* [2] if $\operatorname{Supp}(L) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, L)$ is minimax, for all $i \geq 0$.

THEOREM 3.6. *Let $(R, \mathfrak{m})$ be a regular local ring such that* $\dim R \leq 4$. *Suppose that $\mathfrak{a}$ is an ideal of R and M a finitely generated R-module. Then for all integers $i, j \geq 0$, the R-module* $\operatorname{Ext}^j(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

PROOF. If $\dim R \leq 3$, then by virtue of [1, Theorem 2.12], the R-modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax, and so the R-module $\operatorname{Ext}^j(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian. Hence we may assume that $\dim R = 4$.

Now if $\dim R/\mathfrak{a} = 4$, then $\mathfrak{a} = 0$ and so the result holds. Also, case $\dim R/\mathfrak{a} \leq 2$ follows from [4, Corollary 3.2]. Hence we may assume that $\dim R/\mathfrak{a} = 3$. Then we have $\operatorname{height}(\mathfrak{a}) = 1$.

On the other hand, if $\dim M \leq 2$ then in view of [4, Corollary 2.7] and [22, Proposition 5.1], the R-module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, and so the result is clear. Also, in the case of $\dim M = 3$, the assertion follows from [22, Proposition 5.1], [23, Corollary 3.3], the Grothendieck Vanishing Theorem and Lemma 2.4.

Therefore we may assume that $\dim M = 4$. Then in view of [22, Proposition 5.1], [23, Corollary 3.3], the Grothendieck Vanishing Theorem and Lemma 2.4, it is enough to show that the R-module $\operatorname{Ext}^j(R/\mathfrak{a}, H^2_{\mathfrak{a}}(M))$ is weakly Laskerian, for all $j \geq 0$. To this end, let

$$X := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height} \mathfrak{p} = 1\} \quad \text{and} \quad Y := \{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a}) \mid \operatorname{height} \mathfrak{p} \geq 2\}.$$

Then $X \neq \emptyset$. Let $\mathfrak{b} = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. If $Y = \emptyset$ then $\operatorname{Rad}(\mathfrak{a}) = \mathfrak{b}$, and so $H^2_{\mathfrak{a}}(M) = 0$, as required. Therefore we may assume that $Y \neq \emptyset$. Then $\operatorname{Rad}(\mathfrak{a}) = \mathfrak{b} \cap \mathfrak{c}$, $\operatorname{height} \mathfrak{c} \geq 2$, and in view of Theorem 2.7, $\operatorname{height}(\mathfrak{b} + \mathfrak{c}) \geq 3$, where $\mathfrak{c} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$.

Now, since $\operatorname{Rad}(\mathfrak{a}) = \mathfrak{b} \cap \mathfrak{c}$, in view of the Mayer-Vietoris sequence the sequence

$$H^2_{\mathfrak{b}+\mathfrak{c}}(M) \longrightarrow H^2_{\mathfrak{c}}(M) \xrightarrow{f} H^2_{\mathfrak{a}}(M) \xrightarrow{g} H^3_{\mathfrak{b}+\mathfrak{c}}(M) \qquad (\dagger)$$

is exact.

Since $\dim R/(\mathfrak{b} + \mathfrak{c}) \leq 1$ we have $V(\mathfrak{b} + \mathfrak{c}) = \operatorname{Min}(\mathfrak{b} + \mathfrak{c})$, and so as

$$\operatorname{Supp}(H^i_{\mathfrak{b}+\mathfrak{c}}(M)) \subseteq V(\mathfrak{b} + \mathfrak{c}),$$

it follows that the set $\operatorname{Supp}(H^i_{\mathfrak{b}+\mathfrak{c}}(M))$ is finite, for $i = 2, 3$. Therefore, it follows from the exact sequence $(\dagger)$ that the R-modules $\operatorname{Ker} f$ and $\operatorname{Im} g$ are weakly Laskerian. Thus, as $\dim R/\mathfrak{c} \leq 2$ it follows from [4, Corollary 3.3] that the R-module $\operatorname{Ext}^i_R(R/\mathfrak{c}, H^2_{\mathfrak{c}}(M)/\operatorname{Ker} f)$ is weakly Laskerian for all $i \geq 0$. Now, the exact sequence

$$0 \longrightarrow H^2_{\mathfrak{c}}(M)/\operatorname{Ker} f \longrightarrow H^2_{\mathfrak{c}}(M) \longrightarrow \operatorname{Im} g \longrightarrow 0,$$

induces the long exact sequence

$$\operatorname{Ext}^i_R(R/\mathfrak{c}, H^2_{\mathfrak{c}}(M)/\operatorname{Ker} f) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{c}, H^2_{\mathfrak{c}}(M)) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{c}, \operatorname{Im} g),$$

for all $i \geq 0$. Consequently, in view of Lemma 3.1, $\operatorname{Ext}^i_R(R/\mathfrak{c}, H^2_{\mathfrak{c}}(M))$ is a weakly Laskerian R-module.

Finally, by using the exact sequence

$$0 \longrightarrow R/\operatorname{Rad}(\mathfrak{a}) \longrightarrow R/\mathfrak{b} \oplus R/\mathfrak{c} \longrightarrow R/\mathfrak{b} + \mathfrak{c} \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{b} + \mathfrak{c}, H^2_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{b} \oplus R/\mathfrak{c}, H^2_{\mathfrak{a}}(M))$$

$$\longrightarrow \operatorname{Ext}^i_R(R/\operatorname{Rad}(\mathfrak{a}), H^2_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^{i+1}_R(R/\mathfrak{b} + \mathfrak{c}, H^2_{\mathfrak{a}}(M)) \longrightarrow \cdots.$$

Now, by applying Lemma 3.3, we obtain that

$$\operatorname{Ext}^i_R(R/\mathfrak{b}+\mathfrak{c}, H^2_{\mathfrak{a}}(M)) = 0 = \operatorname{Ext}^i_R(R/\mathfrak{b}, H^2_{\mathfrak{a}}(M)),$$

for each $i \geq 0$, and therefore

$$\operatorname{Ext}^i_R(R/\operatorname{Rad}(\mathfrak{a}), H^2_{\mathfrak{a}}(M)) \cong \operatorname{Ext}^i_R(R/\mathfrak{c}, H^2_{\mathfrak{a}}(M)),$$

for each $i \geq 0$.

Hence, the R-module $\operatorname{Ext}^i_R(R/\operatorname{Rad}(\mathfrak{a}), H^2_{\mathfrak{a}}(M))$ is weakly Laskerian, for each $i \geq 0$. Thus, it follows from Lemma 2.6 that, the R-module $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^2_{\mathfrak{a}}(M))$ is also weakly Laskerian, for all $i \geq 0$, and this completes the proof. □

We end this section with a result which is a generalization of Corollary 2.8.

COROLLARY 3.7. *Let the situation be as in Theorem* 3.6. *Then for any finitely generated R-module N with support in $V(\mathfrak{a})$, the R-module* $\operatorname{Ext}^j_R(N, H^i_{\mathfrak{a}}(M))$ *is weakly Laskerian, for all integers $i, j \geq 0$. In particular the set* $\operatorname{Ass}_R \operatorname{Ext}^j_R(N, H^i_{\mathfrak{a}}(M))$ *is finite, for all integers $i, j \geq 0$.*

PROOF. The assertion follows from Theorem 3.6 and Lemma 2.6. □

ACKNOWLEDGMENTS. The authors are deeply grateful to the referee for a very careful reading of the manuscript and many valuable suggestions in improving the quality of the paper. We also would like to thank from the Azarbaijan Shahid Madani University for the financial support (No. 217/d/917).

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