TOKYO J. MATH.
VOL. 40, NO. 1, 2017

One-parameter Families of Homeomorphisms, Topological Monodromies, and Foliations

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(Communicated by K. Ahara)

Abstract. The homological monodromy of a degeneration whose singular fiber has at most normal crossings was described by C. H. Clemens. In his work, local monodromies were described in detail. It is actually a classical result that the local monodromy around a node is a Dehn twist. For higher-dimensional case, we describe local monodromies alternatively: On a local smooth fiber of dimension $n \geq 2$, we construct $n+1$ singular foliations and then describe the action of the local monodromy on each leaf. Here the ith singular foliation is used for describing its action on the ith face of the boundary of a local smooth fiber.

1. Introduction

Let $\pi : M \to \Delta$ be a degenerating family (a *degeneration*) of complex manifolds over a disk $\Delta = \{z \in \mathbf{C} : |z| < r\}$, that is, $\pi^{-1}(s)$ $(s \neq 0)$ is smooth and $\pi^{-1}(0)$ is singular. The *topological monodromy* of $\pi : M \to \Delta$ is an automorphism of a smooth fiber $\pi^{-1}(s)$ obtained from a 'parallel translation' on $M \setminus \pi^{-1}(0)$ as illustrated in Figure 1. (In [8], a topological monodromy means the isotopy class of this automorphism.) For each $i = 0, 1, 2, \ldots, 2\dim \pi^{-1}(s)$, the automorphism of $H_i(\pi^{-1}(s), \mathbf{C})$ induced from the topological monodromy is the *ith homological monodromy* of $\pi : M \to \Delta$.

C. H. Clemens [3] described each homological monodromy of a degeneration whose singular fiber has at most normal crossings, and showed that it is *quasi-unipotent*, that is, some power is unipotent. On the other hand, Y. Matsumoto and J. M. Montesinos [8] showed that the topological type of a degeneration of Riemann surfaces of genus ≥ 2 is determined by its topological monodromy. A simple example of a degeneration of Riemann surfaces is a Lefschetz fibration whose topological monodromy is a Dehn twist. In higher-dimensional case, there are two different generalizations of a Dehn twist: Consider *additive* and *multiplicative* A-singularities:

$$V := \left\{(z_1, z_2, \ldots, z_n, t) \in \mathbf{C}^{n+1} : z_1^2 + z_2^2 + \cdots + z_n^2 = t\right\},$$

$$W := \left\{(z_1, z_2, \ldots, z_n, t) \in \mathbf{C}^{n+1} : z_1 z_2 \cdots z_n = t\right\}.$$

Received September 18, 2015; revised June 2, 2016
Mathematics Subject Classification: 32G05
Key words: degeneration of complex manifolds, topological monodromy, Dehn twist

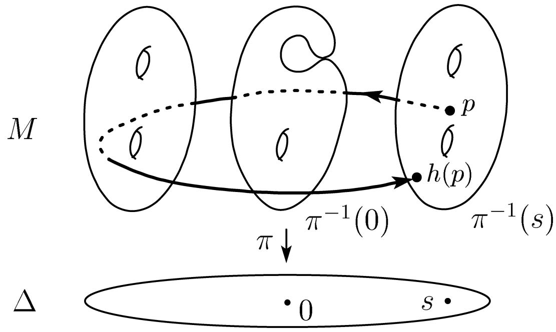


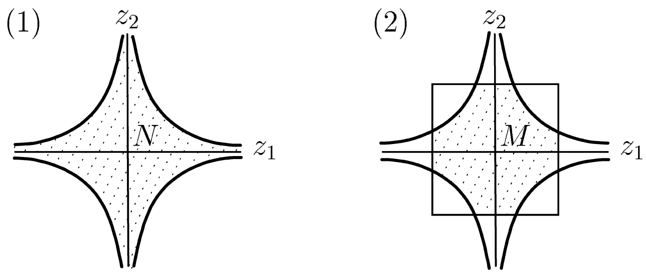
FIGURE 1. A topological monodromy $h : \pi^{-1}(s) \to \pi^{-1}(s)$

Then $f : V \to \mathbf{C}$, $f(z_1, \dots, z_n, t) = t$ and $g : W \to \mathbf{C}$, $g(z_1, \dots, z_n, t) = t$ are degenerations, and their topological monodromies are two different generalizations of a Dehn twist.

REMARK. If $n \geq 3$, V and W are *not* isomorphic (in contrast for $n = 2$, they are isomorphic). In fact, the singularities of the singular fiber of $g : W \to \mathbf{C}$ are not isolated, while that of $f : V \to \mathbf{C}$ is isolated.

The topological monodromy of the degeneration $f : V \to \mathbf{C}$ is described by the *double covering method* (see [2] p. 6). We will describe the topological monodromy of the degeneration $g : W \to \mathbf{C}$. First note that W is isomorphic to $\mathbf{C}^n$ via $(z_1, z_2, \dots, z_n, t) \mapsto (z_1, z_2, \dots, z_n)$, accordingly $g : W \to \mathbf{C}$ is identified with a map $\pi : \mathbf{C}^n \to \mathbf{C}$ given by $\pi(z_1, z_2, \dots, z_n) = z_1 z_2 \cdots z_n$.

Now set $N := \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n : |z_1 z_2 \cdots z_n| \leq 1\}$ and $D := \{s \in \mathbf{C} : |s| \leq 1\}$. The restriction $\pi : N \to D$ of $\pi : \mathbf{C}^n \to \mathbf{C}$ to N is a 'shrinking' of $\pi : \mathbf{C}^n \to \mathbf{C}$. Here N is closed but not compact (see (1) of Schematic figure). To obtain a compact one, take $\rho > 1$ and set $M := \{(z_1, z_2, \dots, z_n) \in N : |z_i| \leq \rho \ (i = 1, 2, \dots, n)\}$. Note that M is compact, indeed M is the intersection of the closed set N and the polydisk of radius ρ (see (2) of Schematic figure). The restriction $\pi : M \to D$ of $\pi : \mathbf{C}^n \to \mathbf{C}$ to M is a local model of a degeneration of complex manifolds.
In order to describe the homological monodromy of a degeneration of complex manifolds, Clemens [3] describes the monodromy of the degeneration $\pi : M \to D$ above. In this paper, we alternatively describe it from a different viewpoint. While for $n = 2$, the topological monodromy of π is known to be a (-1)-Dehn twist of an annulus, for $n \geq 3$, we describe the action of the topological monodromy in terms of n foliations constructed on a smooth fiber of π; precisely speaking, these are *singular* foliations — the dimension of some leaf is less than that of a generic leaf. The ith foliation is used for describing its action on the ith face of the boundary of a smooth fiber.



Schematic figure for $n = 2$

Description of topological monodromies. The description of $\pi^{-1}(s)$ $(s = re^{\mathrm{i}\xi} \neq 0)$ proceeds as follows:

STEP 1. Express $\pi^{-1}(s) = J_r \times K_\xi$, where

$$J_r := \left\{(x_1, \dots, x_n) \in \mathbf{R}^n : 0 \le x_i \le \rho,\ x_1 \cdots x_n = r\right\},$$
$$K_\xi := \left\{(e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_n}) \in T^n : \alpha_1 + \cdots + \alpha_n \equiv \xi \bmod 2\pi\right\}$$

are respectively homeomorphic to the standard $(n-1)$-simplex Δ_{n-1} and an $(n-1)$-dimensional torus T^{n-1}.

STEP 2. We explicitly construct homeomorphisms $\Psi : J_r \to \Delta_{n-1}$ and $\Phi : K_\xi \to T^{n-1}$. The construction of the former is quite involved and based on induction on the dimension. The latter is constructed as follows: Noting that $K_\xi \subset T^n$, let $\mathrm{pr} : T^n \to T^{n-1}$ be a projection given by $\mathrm{pr} : (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_n}) \mapsto (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_{n-1}})$. Then the restriction $\Phi = \mathrm{pr}|_{K_\xi} : K_\xi \to T^{n-1}$ is a homeomorphism.

STEP 3. Using Ψ, we construct a 1-parameter family of homeomorphisms $\{f_\theta : M \to M\}_{0 \le \theta \le 2\pi}$ by $f_\theta : (z_1, \dots, z_n) \mapsto (e^{\mathrm{i}\theta\lambda_1} z_1, \dots, e^{\mathrm{i}\theta\lambda_n} z_n)$, where $(\lambda_1, \dots, \lambda_n) := \Psi(|z_1|, \dots, |z_n|)$. Then $h := f_{2\pi} : \pi^{-1}(s) \to \pi^{-1}(s)$ is the topological monodromy of π. Under the identification of $\pi^{-1}(s)$ with $\Delta_{n-1} \times T^{n-1}$ via $\Psi \times \Phi$, we show that h is given by

$$\begin{aligned}&\big((\lambda_1, \dots, \lambda_n),\ (t_1, \dots, t_{n-1})\big)\\ &\qquad \longmapsto \big((\lambda_1, \dots, \lambda_n),\ (e^{2\pi\mathrm{i}\lambda_1} t_1, \dots, e^{2\pi\mathrm{i}\lambda_{n-1}} t_{n-1})\big).\end{aligned} \tag{1}$$

(See Remark 1 in §4 for another 1-parameter family constructed by Clemens [3].)

STEP 4. As illustrated in Figure 2, we shrink the $(n-1)$-simplex Δ_{n-1} to obtain a family of $(n-1)$-simplexes $\Delta_{n-1|u}$ $(0 \le u \le 1)$ such that $\Delta_{n-1|1} = \Delta_{n-1}$ and $\Delta_{n-1|0}$ is the barycenter b of Δ_{n-1}.
Then $\{\partial\Delta_{n-1|u}\}_{0 \le u \le 1}$ is a singular foliation of Δ_{n-1}. We foliate $\Delta_{n-1} \times T^{n-1}$ by $\partial\Delta_{n-1|u} \times$

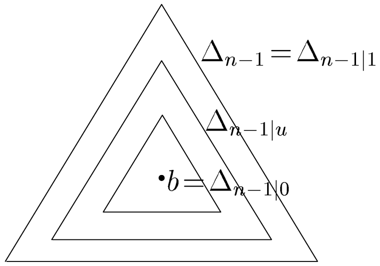


FIGURE 2

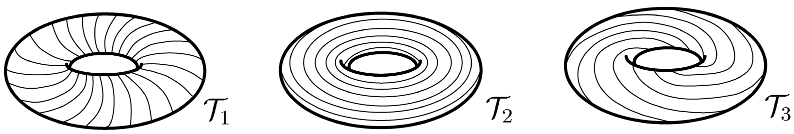


FIGURE 3

T^{n-1} $(0 \le u \le 1)$:

$$\Delta_{n-1} \times T^{n-1} = \coprod_{0 \le u \le 1} (\partial \Delta_{n-1|u} \times T^{n-1}) \quad \text{(disjoint union)}\,.$$

Now identify $\Delta_{n-1} \times T^{n-1}$ with a smooth fiber of $\pi : M \to D$ and regard the topological monodromy of π as a homeomorphism $h : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$. By (1), h maps each leaf $\partial \Delta_{n-1|u} \times T^{n-1}$ to itself. To describe this action, we introduce n foliations of $\partial \Delta_{n-1|u} \times T^{n-1}$. First let $\mathcal{T}_i$ $(i = 1, 2, \dots, n)$ be a foliation of T^{n-1} by parallel subtori $T^{n-2}_{i|v}$ $(v \in S^1)$ as illustrated in Figure 3 (for $n = 3$). Then $\partial \Delta_{n-1|u} \times T^{n-1}$ is foliated by $\partial \Delta_{n-1|u} \times T^{n-2}_{i|v}$ $(v \in S^1)$.

As mentioned above, h maps $\partial \Delta_{n-1|u} \times T^{n-1}$ to itself. We describe this action separately for $u = 0$ and $u \neq 0$.

(D1) Where $b = \partial \Delta_{n-1|0}$ is the barycenter of Δ_{n-1}, h acts on $\{b\} \times T^{n-1}$ as

$$(b, t_1, t_2, \dots, t_{n-1}) \longmapsto (b, e^{2\pi \mathrm{i}/n} t_1, e^{2\pi \mathrm{i}/n} t_1, \dots, e^{2\pi \mathrm{i}/n} t_{n-1})\,.$$

In particular, h maps $\{b\} \times T^{n-2}_{i|v}$ to $\{b\} \times T^{n-2}_{i|ve^{2\pi \mathrm{i}/n}}$.

Now let $\partial \Delta^{(j)}_{n-1}$ $(j = 1, 2, \dots, n)$ denote the jth face of $\partial \Delta_{n-1}$, accordingly $\partial \Delta^{(j)}_{n-1|u}$ denotes the jth face of $\partial \Delta_{n-1|u}$ $(u \neq 0)$. See Figure 4.

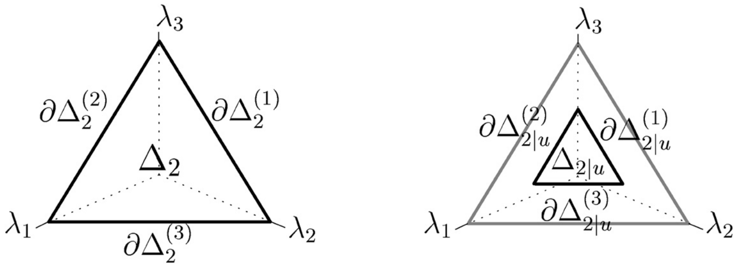


FIGURE 4

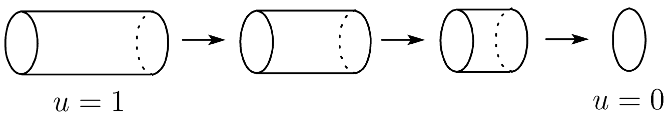


FIGURE 5

(D2) The action of h on $\partial\Delta_{n-1|u} \times T^{n-1}$ ($u \neq 0$) maps each face $\partial\Delta^{(j)}_{n-1|u} \times T^{n-1}$ ($j = 1, 2, \dots, n$) to itself. Setting $\mu := \dfrac{1-u}{n}$, this map is explicitly given by

$$\big((\lambda_1, \dots, \lambda_{j-1}, \mu, \lambda_{j+1}, \dots, \lambda_n), (t_1, t_2, \dots, t_{n-1})\big) \longmapsto$$
$$\big((\lambda_1, \dots, \lambda_{j-1}, \mu, \lambda_{j+1}, \dots, \lambda_n), (e^{2\pi i \lambda_1} t_1, \dots, e^{2\pi i \mu} t_j, \dots, e^{2\pi i \lambda_{n-1}} t_{n-1})\big).$$

Next fixing $u \neq 0$ and i, set $\mathcal{L}_v := \partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ and consider a foliation $\mathcal{F} = \{\mathcal{L}_v\}_{v \in S^1}$ of $\partial\Delta_{n-1|u} \times T^{n-1}$. Then $\mathcal{L}^{(j)}_v := \partial\Delta^{(j)}_{n-1|u} \times T^{n-2}_{i|v}$ is the *jth face* of $\mathcal{L}_v$. The following holds:

(D3) h does not preserve $\mathcal{F}$ — each face $\mathcal{L}^{(j)}_v$ ($j \neq i$) of a leaf $\mathcal{L}_v$ of $\mathcal{F}$ is not mapped to a face of a leaf of $\mathcal{F}$. In contrast, for the case $j = i$, h maps $\mathcal{L}^{(i)}_v$ to $\mathcal{L}^{(i)}_{ve^{2\pi i(1-u)/n}}$. *To describe the action of h on* $\partial\Delta^{(i)}_{n-1|u} \times T^{n-1}$, *we may thus describe its action on* $\mathcal{L}^{(i)}_v$ *for each* $v \in S^1$ *separately.*

For the case $n = 3$, each face of $\partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ ($u \neq 0$) is an annulus, which as $u \to 0$, shrinks to a circle S^1 (Figure 5).
Accordingly the action of the topological monodromy h varies from a (-1)-Dehn twist to the $\frac{2\pi}{3}$-rotation of S^1 (Proposition 2). A similar description is valid for arbitrary dimension.

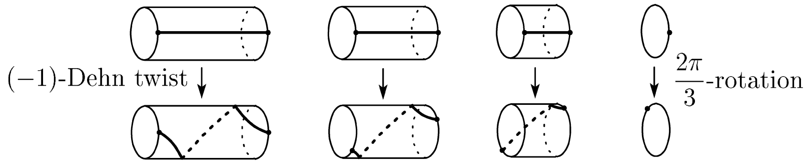


FIGURE 6. The variation of the action of h

2. Local models of degenerations and their fibers

Let $\pi : \mathbf{C}^n \to \mathbf{C}$ be a holomorphic map given by $\pi(z_1, z_2, \ldots, z_n) = z_1 z_2 \cdots z_n$. Its singular fiber $Y := \pi^{-1}(0)$ is a complex analytic variety $z_1 z_2 \cdots z_n = 0$ in $\mathbf{C}^n$. Set $\mathbf{C}^\times := \mathbf{C} \setminus \{0\}$. Each smooth fiber $B_s := \pi^{-1}(s)$ (where $s \neq 0$) is biholomorphic to $(\mathbf{C}^\times)^{n-1}$ via

$$(z_1, z_2, \ldots, z_n) \in B_s \longmapsto (z_1, z_2, \ldots, z_{n-1}) \in (\mathbf{C}^\times)^{n-1}. \qquad (2)$$

Next set $N := \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n : |z_1 z_2 \cdots z_n| \leq 1\}$ and $D := \{s \in \mathbf{C} : |s| \leq 1\}$. Take $\rho > 1$ and set $\Delta := \big\{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n : |z_i| \leq \rho \ (i = 1, 2, \ldots, n)\big\}$. Set $M := N \cap \Delta$, then the restriction $\pi : M \to D$ of $\pi : \mathbf{C}^n \to \mathbf{C}$ to M is a degeneration. Its singular fiber is $X := Y \cap \Delta$ while $C_s := B_s \cap \Delta$ ($s \neq 0$) is a smooth fiber. They are compact. Note that since C_s is a domain in B_s, the positive orientation of B_s naturally defines the positive orientation of the complex manifold C_s.

Write a nonzero complex number s as $re^{\mathrm{i}\xi}$ ($r > 0, 0 \leq \xi < 2\pi$). Then set

$$J_r = \left\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : \begin{array}{l} 0 \leq x_i \leq \rho \ (i = 1, 2, \ldots, n) \\ x_1 x_2 \cdots x_n = r \end{array} \right\},$$

$$K_\xi = \big\{(e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_n}) \in T^n : \alpha_1 + \alpha_2 + \cdots + \alpha_n \equiv \xi \bmod 2\pi\big\},$$

where $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$ is an n-dimensional torus. (Note that K_ξ is homeomorphic to T^{n-1} via $(e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_n}) \in K_\xi \mapsto (e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_{n-1}}) \in T^{n-1}$.) A smooth fiber $C_s = \pi^{-1}(s)$ of $\pi : M \to D$ is homeomorphic to $J_r \times K_\xi$ via

$$(z_1, \ldots, z_n) = (x_1 e^{\mathrm{i}\alpha_1}, \ldots x_n e^{\mathrm{i}\alpha_n}) \mapsto \big((x_1, \ldots, x_n), (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_n})\big).$$

We say that J_r is the *real slice* of C_s, which is a part of a higher dimensional hyperboloid in $\mathbf{R}^n$ (Figure 7).

Note that J_r is homeomorphic to the standard $(n-1)$-simplex $\Delta_{n-1} := \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n : 0 \leq \lambda_i \leq 1, \lambda_1 + \cdots + \lambda_n = 1\}$ (Figure 8). In §3, we explicitly construct a homeomorphism between them.

The *barycentric divisions* of J_r and Δ_{n-1} are given by $J_r = \bigcup_{i=1}^{n} A_i$ and $\Delta_{n-1} = \bigcup_{i=1}^{n} B_i$,

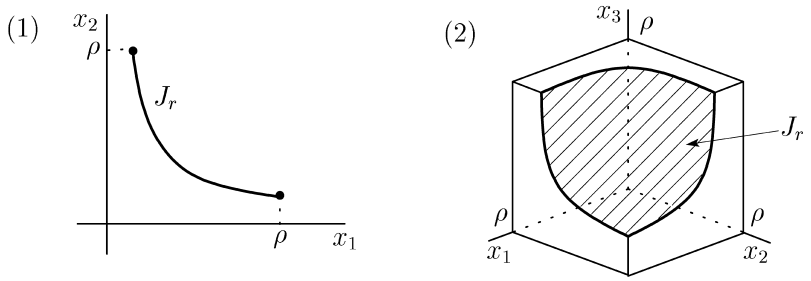


FIGURE 7. (1) $n = 2$. (2) $n = 3$.

FIGURE 8

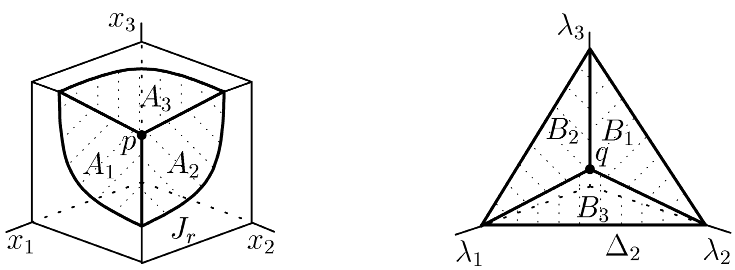


FIGURE 9. The barycentric divisions of J_r and Δ_{n-1} for $n = 3$. The points p and q are the *barycenters* of J_r and Δ_2.

where

$$A_i = \{(x_1, \ldots, x_n) \in J_r \,:\, x_i \geq x_j \text{ for any } j \neq i\}\,,$$

$$B_i = \{(\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1} \,:\, \lambda_i \leq \lambda_j \text{ for any } j \neq i\}\,.$$

The boundary $\partial\Delta_{n-1}$ of Δ_{n-1} consists of n faces $\partial\Delta_{n-1}^{(l)}$ $(l = 1, 2, \ldots, n)$, where $\partial\Delta_{n-1}^{(l)}$ is defined by $\lambda_l = 0$ in Δ_{n-1}. Similarly the boundary ∂J_r of J_r consists of n faces $\partial J_r^{(l)}$ $(l = 1, 2, \ldots, n)$, where $\partial J_r^{(l)}$ is defined by $x_l = \rho$ in J_r.

Subsequently we explicitly construct a homeomorphism from J_r to Δ_{n-1} that maps the

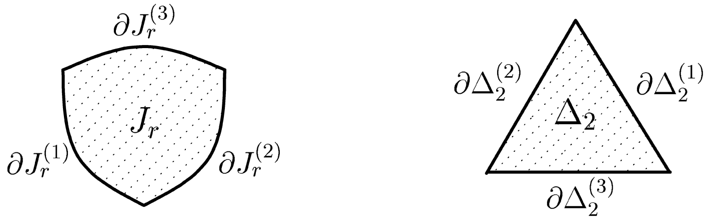


FIGURE 10

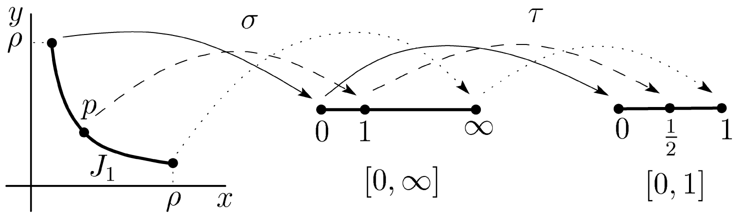


FIGURE 11. $\lambda := \tau \circ \sigma$ maps the midpoint of J_1 to that of [0, 1]

barycentric division of J_r to that of Δ_{n-1} and maps $\partial J_r^{(l)}$ to $\partial \Delta_{n-1}^{(l)}$, $l = 1, 2, \ldots, n$.

3. Construction of a simplicial homeomorphism Ψ

We construct a homeomorphism $\Psi : J_r \to \Delta_{n-1}$. The construction is based on induction on n, so it is convenient to write $n-1$ as m and Ψ, J_r, Δ_{n-1} as Ψ_m, J_m, Δ_m. We first construct Ψ_1 (below, x_1 and x_2 are denoted by x and y). Let $\sigma : J_1 \to [0,\infty]$ and $\tau : [0,\infty] \to [0,1]$ be maps given by $\sigma(x,y) = \dfrac{\rho - y}{\rho - x}$ and $\tau(t) = \begin{cases} \frac{1}{2}t & (t \in [0,1]) \\ 1 - \frac{1}{2t} & (t \in [1,\infty]) \end{cases}$,
then the composite map $\lambda := \tau \circ \sigma : J_1 \to [0,1]$ is a homeomorphism.
Next define a homeomorphism $\varphi : [0,1] \to \Delta_1$ by $\varphi(t) = (1-t, t)$. The composite map $\Psi_1 := \varphi \circ \lambda : J_1 \to \Delta_1$ is the desired homeomorphism:

$$\Psi_1 : (x,y) \in J_1 \longmapsto \big(1 - \lambda(x,y),\ \lambda(x,y)\big) \in \Delta_1 . \tag{3}$$

As illustrated in Figure 12, Ψ_1 maps the barycentric division of J_1 to that of Δ_1.

The construction of $\Psi_m : J_m \to \Delta_m$ ($m = 2, 3, \ldots$) proceeds as follows:

STEP 1. Let $J_m = \bigcup_{i=1}^{m+1} A_i$ and $\Delta_m = \bigcup_{i=1}^{m+1} B_i$ be the barycentric divisions. We then construct homeomorphisms $\psi_i : A_i \to B_i$ ($i = 1, 2, \ldots, m+1$) by using the homeomorphism $\Psi_{m-1} : J_{m-1} \to \Delta_{m-1}$.

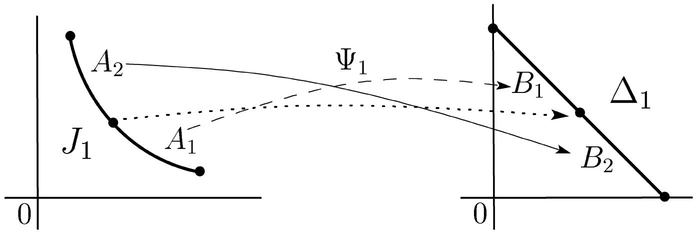


FIGURE 12. Ψ_1 maps A_i to B_i $(i = 1, 2)$

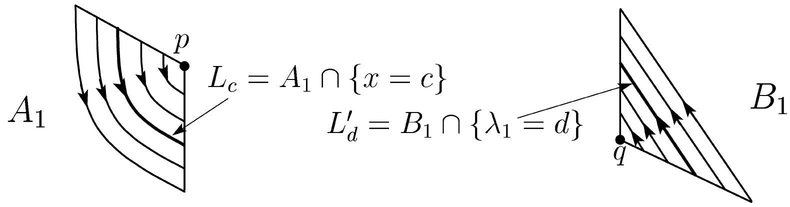


FIGURE 13

STEP 2. We show that $\psi_i = \psi_j$ on $A_i \cap A_j$. Thus $\psi_i : A_i \to B_i$ $(i = 1, 2, \dots, m+1)$ together define a homeomorphism $\Psi_m : J_m \to \Delta_m$ that maps the barycentric division of J_m to that of Δ_m.

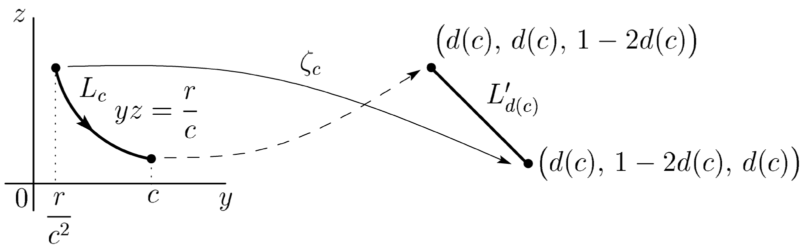
Construction of $\Psi_2 : J_2 \to \Delta_2$**.** Let $J_2 = A_1 \cup A_2 \cup A_3$ and $\Delta_2 = B_1 \cup B_2 \cup B_3$ be the barycentric divisions. We first construct $\psi_1 : A_1 \to B_1$ (below, x_1, x_2, x_3 are denoted by x, y, z). Foliate A_1, B_1 as illustrated in Figure 13: $A_1 = \{L_c\}_{\sqrt[3]{r} \le c \le \rho}$, $B_1 = \{L'_d\}_{0 \le d \le 1/3}$. Note that $L_{\sqrt[3]{r}} = p$, $L'_{1/3} = q$, and L'_d $(d \neq 1/3)$ is homeomorphic to Δ_1 via $\varphi(d, y, z) \mapsto \big(1 - 2d - (1 - 3d)z,\ d + (1 - 3d)z\big)$. For $c \neq \sqrt[3]{r}$, a homeomorphism $\mu : J_1 \to L_c$ is given as follows: First take L_c as subsets of $\mathbf{R}^2$ by ignoring the x coordinate:

$$L_c = \left\{(y, z) \in \mathbf{R}^2 \ : \ \frac{r}{c^2} \le y \le c,\ \frac{r}{c^2} \le z \le c,\ yz = \frac{r}{c}\right\}.$$

Let $\nu : \mathbf{R}^2 \to \mathbf{R}^2$ be a parallel transport given by $\nu(y, z) = (y + \rho - c,\ z + \rho - c)$. For $(y, z) \in J_1$, denote by $\mu'(y, z)$ the intersection of $\nu(L_c)$ and the line connecting (y, z) and (ρ, ρ). Then we have a homeomorphism $\mu' : J_1 \to \nu(L_c)$, and hence the composition $\mu := \nu^{-1} \circ \mu' : J_1 \to L_c$ is a homeomorphism.

Now let $d : [\sqrt[3]{r},\ \rho] \to \left[0,\ \frac{1}{3}\right]$ be a function given by $d(c) := \dfrac{\rho - c}{3(\rho - \sqrt[3]{r})}$, and for each $c \in (\sqrt[3]{r},\ \rho]$, set $\zeta_c := \varphi^{-1} \circ \Psi_1 \circ \mu^{-1} : L_c \to L'_{d(c)}$, where $\Psi_1 : J_1 \to \Delta_1$ is the simplicial homeomorphism in (3):

$$\zeta_c(c, y, z) = \varphi^{-1} \circ \Psi_1 \circ \mu^{-1}(y, z)\,.$$



The homeomorphism $\zeta_{\sqrt[3]{r}} : L_{\sqrt[3]{r}} \to L'_{1/3}$ is naturally defined by $\zeta_{\sqrt[3]{r}}(p) = q$. Then the family of homeomorphisms $\{\zeta_c : L_c \to L_{d(c)}\}_{\sqrt[3]{r} \leq c \leq \rho}$ determines the homeomorphism $\psi_1 : A_1 \to B_1$. In particular,

$$\psi_1(\rho, y, z) = \zeta_\rho(\rho, y, z) = \bigl(0, \Psi_1 \circ \mu^{-1}(y, z)\bigr)\,. \tag{4}$$

Similarly ψ_2 and ψ_3 are constructed. By construction, $\psi_i = \psi_j$ on $A_i \cap A_j$, so they together define a homeomorphism $\Psi_2 : J_2 \to \Delta_2$.

For $m \geq 3$, we can similarly construct $\Psi_m : J_m \to \Delta_m$ from $\Psi_{m-1} : J_{m-1} \to \Delta_{m-1}$. We refer to Ψ_m as a *simplicial homeomorphism.* The positive orientation of Δ_m induces the positive orientation of J_m via $\Psi_m : J_m \to \Delta_m$.

4. One-parameter family of homeomorphisms and topological monodromy

Fix $\rho > 1$ and set

$$M = \bigl\{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n \,:\, |z_i| \leq \rho \ (i = 1, 2, \dots, n),\ |z_1 z_2 \cdots z_n| \leq 1\bigr\}\,.$$

Consider the degeneration $\pi : M \to D$ defined by $\pi(z_1, z_2, \dots z_n) = z_1 z_2 \cdots z_n$. Then $C_s := \pi^{-1}(s)$ for nonzero s is a smooth fiber and $X := \pi^{-1}(0)$ is its singular fiber. The projection $p : M \to J_r$ of M to its real slice J_r is given by $p : (z_1, z_2, \dots, z_n) \mapsto (|z_1|, |z_2|, \dots, |z_n|)$. Let $\Psi : J_r \to \Delta_{n-1}$ be the simplicial homeomorphism constructed in §3, where $\Delta_{n-1} := \bigl\{(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n \,:\, 0 \leq \lambda_i \leq 1,\ \lambda_1 + \cdots + \lambda_n = 1\bigr\}$ is the standard $(n-1)$-simplex. Consider the composite map $F := \Psi \circ p : M \to \Delta_{n-1}$. Set $(\lambda_1, \lambda_2, \dots, \lambda_n) := F(z_1, z_2, \dots, z_n)$, and for any $\theta \in \mathbf{R}$, define a map $f_\theta : M \to M$ by

$$f_\theta(z_1, z_2, \dots, z_n) = \bigl(e^{\mathrm{i}\theta\lambda_1} z_1,\ e^{\mathrm{i}\theta\lambda_2} z_2,\ \dots,\ e^{\mathrm{i}\theta\lambda_n} z_n\bigr)\,. \tag{5}$$

LEMMA 1. *f_θ maps C_s to $C_{se^{\mathrm{i}\theta}}$. In particular, $f_{2\pi}$ maps C_s to itself.*

PROOF. It suffices to show that if $z_1 z_2 \cdots z_n = s$, then $(e^{\mathrm{i}\theta\lambda_1} z_1)(e^{\mathrm{i}\theta\lambda_2} z_2) \cdots (e^{\mathrm{i}\theta\lambda_n} z_n) = s e^{\mathrm{i}\theta}$. Since $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta_{n-1}$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, thus $(e^{\mathrm{i}\theta\lambda_1} z_1)(e^{\mathrm{i}\theta\lambda_2} z_2) \cdots (e^{\mathrm{i}\theta\lambda_n} z_n) = e^{\mathrm{i}\theta} z_1 z_2 \cdots z_n = e^{\mathrm{i}\theta} s$. □

Note that $\{f_\theta : M \to M\}_{\theta \in \mathbf{R}}$ is a 1-parameter family of homeomorphisms: (i) f_0 is the identity map and (ii) $f_{\theta_1+\theta_2} = f_{\theta_1} \circ f_{\theta_2}$. (i) is obvious. (ii) is confirmed as follows: Note first that

$$\begin{aligned}(z_1, z_2, \ldots, z_n) &\xrightarrow{f_{\theta_2}} \left(e^{\mathrm{i}\theta_2\lambda_1} z_1,\ e^{\mathrm{i}\theta_2\lambda_2} z_2,\ \ldots,\ e^{\mathrm{i}\theta_2\lambda_n} z_n\right) \\ &\xrightarrow{f_{\theta_1}} \left(e^{\mathrm{i}\theta_1\lambda_1'} e^{\mathrm{i}\theta_2\lambda_1} z_1,\ e^{\mathrm{i}\theta_1\lambda_2'} e^{\mathrm{i}\theta_2\lambda_2} z_2,\ \ldots,\ e^{\mathrm{i}\theta_1\lambda_n'} e^{\mathrm{i}\theta_2\lambda_n} z_n\right),\end{aligned}$$

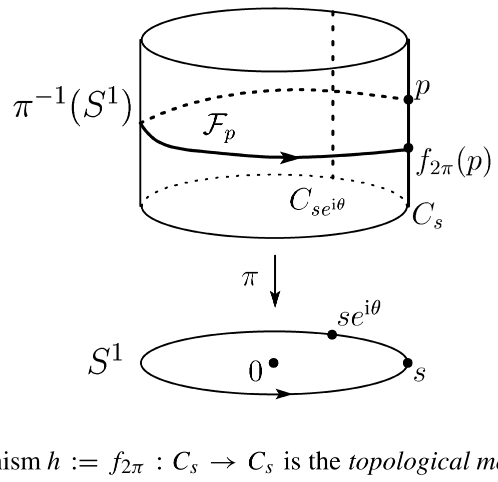
where

$$\begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_n) = \Psi(|z_1|, |z_2|, \ldots, |z_n|), \\ (\lambda_1', \lambda_2', \ldots, \lambda_n') = \Psi(|e^{\mathrm{i}\theta_2\lambda_1} z_1|, |e^{\mathrm{i}\theta_2\lambda_2} z_2|, \ldots, |e^{\mathrm{i}\theta_2\lambda_n} z_n|)\,. \end{cases}$$

Since $|e^{2\pi\mathrm{i}\lambda_i} z_i| = |z_i|$ $(i = 1, 2, \ldots, n)$, we have $(\lambda_1', \lambda_2', \ldots, \lambda_n') = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Hence

$$\begin{aligned} f_{\theta_1} \circ f_{\theta_2}(z_1, z_2, \ldots, z_n) &= \left(e^{\mathrm{i}\theta_1\lambda_1} e^{\mathrm{i}\theta_2\lambda_1} z_1,\ e^{\mathrm{i}\theta_1\lambda_2} e^{\mathrm{i}\theta_2\lambda_2} z_2,\ \ldots,\ e^{\mathrm{i}\theta_1\lambda_n} e^{\mathrm{i}\theta_2\lambda_n} z_n\right) \\ &= \left(e^{\mathrm{i}(\theta_1+\theta_2)\lambda_1} z_1,\ e^{\mathrm{i}(\theta_1+\theta_2)\lambda_2} z_2,\ \ldots,\ e^{\mathrm{i}(\theta_1+\theta_2)\lambda_n} z_n\right) \\ &= f_{\theta_1+\theta_2}(z_1, z_2, \ldots, z_n)\,. \end{aligned}$$

In what follows, we consider $\{f_\theta : M \to M\}_{0 \le \theta \le 2\pi}$. Take a circle $S^1 = \{z \in \mathbf{C} : |z| = r\}$ contained in the unit disk D. For $s \in S^1$, the flow $\mathcal{F}_p := \left\{f_\theta(p) : 0 \le \theta \le 2\pi\right\}$ starting at $p \in C_s$ transversely intersects each smooth fiber $C_{se^{\mathrm{i}\theta}}$ $(0 < \theta < 2\pi)$, which defines a "parallel transport".



The homeomorphism $h := f_{2\pi} : C_s \to C_s$ is the *topological monodromy* of $\pi : M \to D$. For $\left((x_1, \ldots, x_n),\ (w_1, \ldots, w_n)\right) \in J_r \times K_\xi$ $(= C_s)$, set $(\lambda_1, \ldots, \lambda_n) := \Psi(x_1, \ldots, x_n)$, then

$$h\big((x_1,\dots,x_n),\ (w_1,\dots,w_n)\big) \\ = \big((x_1,\dots,x_n),\ (e^{2\pi\mathrm{i}\lambda_1}w_1,\dots,e^{2\pi\mathrm{i}\lambda_n}w_n)\big)\,. \tag{6}$$

Recall that for each $l=1,2,\dots,n$, Ψ maps the lth face $\partial J_r^{(l)}$ of ∂J_r (defined by $x_l=\rho$) to the lth face $\partial\Delta_{n-1}^{(l)}$ of $\partial\Delta_{n-1}$ (defined by $\lambda_l=0$). Hence the action of h on the lth face $\partial J_r^{(l)}\times K_\xi$ of the boundary $\partial J_r\times K_\xi$ of $J_r\times K_\xi$ is given by setting $x_l=\rho$ and $\lambda_l=0$ in (6):

$$h\big((x_1,\dots,x_{l-1},\rho,x_{l+1},\dots,x_n),\ (w_1,\dots,w_n)\big) \\ = \big((x_1,\dots,x_{l-1},\rho,x_{l+1},\dots,x_n), \\ (e^{2\pi\mathrm{i}\lambda_1}w_1,\dots,e^{2\pi\mathrm{i}\lambda_{l-1}}w_{l-1},w_l,e^{2\pi\mathrm{i}\lambda_{l+1}}w_{l+1},\dots,e^{2\pi\mathrm{i}\lambda_n}w_n)\big)\,. \tag{7}$$

We next construct a homeomorphism between C_s and $\Delta_{n-1}\times T^{n-1}$, and regard the topological monodromy $h: C_s\to C_s$ as a homeomorphism $\Delta_{n-1}\times T^{n-1}\to\Delta_{n-1}\times T^{n-1}$, which is more easy to describe.

First let $\mathrm{pr}: T^n\to T^{n-1}$ be the projection given by $\mathrm{pr}: (e^{\mathrm{i}\alpha_1},\dots,e^{\mathrm{i}\alpha_n})\mapsto(e^{\mathrm{i}\alpha_1},\dots,e^{\mathrm{i}\alpha_{n-1}})$. The restriction $\mathrm{pr}: K_\xi\to T^{n-1}$ is a homeomorphism, in fact its inverse is given by

$$(e^{\mathrm{i}\alpha_1},\dots,e^{\mathrm{i}\alpha_{n-1}})\in T^{n-1}\mapsto\big(e^{\mathrm{i}\alpha_1},\dots,e^{\mathrm{i}\alpha_{n-1}},e^{\mathrm{i}(\xi-\alpha_1-\cdots-\alpha_{n-1})}\big)\in K_\xi\,. \tag{8}$$

The positive orientation of T^{n-1} induces the positive orientation of K_ξ via $\mathrm{pr}: K_\xi\to T^{n-1}$. Then $\phi:=\Psi\times\mathrm{pr}: C_s=J_r\times K_\xi\to\Delta_{n-1}\times T^{n-1}$ is an orientation-preserving homeomorphism. We say that ϕ is a *datum homeomorphism*, which gives a trivialization of C_s (this is analogous to the trivialization of vector bundle). Similarly, for $C_{se^{\mathrm{i}\theta}}$ $(0<\theta\le 2\pi)$, define the datum homeomorphism $\phi_\theta: C_{se^{\mathrm{i}\theta}}=J_r\times K_{\xi+\theta}\to\Delta_{n-1}\times T^{n-1}$ by $\phi_\theta=\Psi\times\mathrm{pr}$. Set $F_\theta:=\phi_\theta\circ f_\theta\circ\phi^{-1}$; then $F_{2\pi}=\phi\circ f_{2\pi}\circ\phi^{-1}$ as $\phi_{2\pi}=\phi$. The following diagram commutes, and thus, to describe f_θ, it suffices to describe F_θ.

$$\begin{array}{ccc} C_s & \xrightarrow{\quad f_\theta\quad} & C_{se^{\mathrm{i}\theta}} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi_\theta} \\ \Delta_{n-1}\times T^{n-1} & \xrightarrow{\quad F_\theta\quad} & \Delta_{n-1}\times T^{n-1}\,. \end{array}$$

LEMMA 2. *$F_\theta:\Delta_{n-1}\times T^{n-1}\to\Delta_{n-1}\times T^{n-1}$ is given by*

$$F_\theta\big((\lambda_1,\dots,\lambda_n),\ (t_1,\dots,t_{n-1})\big)=\big((\lambda_1,\dots,\lambda_n),\ (e^{\mathrm{i}\lambda_1\theta}t_1,\dots,e^{\mathrm{i}\lambda_{n-1}\theta}t_{n-1})\big)\,.$$

PROOF. Write t_i as $e^{\mathrm{i}\alpha_i}$ $(i=1,2,\dots,n-1)$. For $(\lambda_1,\dots,\lambda_n)\in\Delta_{n-1}$, set $(x_1,\dots,x_n):=\Psi^{-1}(\lambda_1,\dots,\lambda_n)$. Then

$$F_\theta\big((\lambda_1,\dots,\lambda_n),\ (e^{\mathrm{i}\alpha_1},\dots,e^{\mathrm{i}\alpha_{n-1}})\big)$$

$$
\begin{aligned}
&= \phi_\theta \circ f_\theta \circ \phi^{-1}\big((\lambda_1, \dots, \lambda_n),\ (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_{n-1}})\big)\\
&= \phi_\theta \circ f_\theta\big((x_1, \dots, x_n),\ (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_{n-1}},\ e^{\mathrm{i}(\xi-\alpha_1-\cdots-\alpha_{n-1})})\big)\\
&= \phi_\theta\big((x_1, \dots, x_n),\ (e^{\mathrm{i}\lambda_1\theta}e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\lambda_{n-1}\theta}e^{\mathrm{i}\alpha_{n-1}},\ e^{\mathrm{i}\lambda_n\theta}e^{\mathrm{i}(\xi-\alpha_1-\cdots-\alpha_{n-1})})\big)\\
&= \big((\lambda_1, \dots, \lambda_n),\ (e^{\mathrm{i}\lambda_1\theta}e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\lambda_{n-1}\theta}e^{\mathrm{i}\alpha_{n-1}})\big)\,.
\end{aligned}
$$

□

The lth face $\partial\Delta_{n-1}^{(l)}$ ($l = 1, 2, \dots, n$) of the boundary $\partial\Delta_{n-1}$ of Δ_{n-1} is defined by $\lambda_l = 0$. Hence:

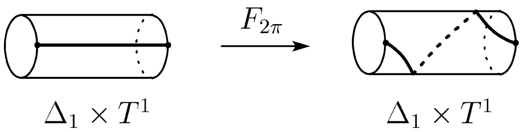
COROLLARY 1. *The action of F_θ on the lth face $\partial\Delta_{n-1}^{(l)} \times T^{n-1}$ of the boundary $\partial\Delta_{n-1} \times T^{n-1}$ of $\Delta_{n-1} \times T^{n-1}$ is given by*

$$
\begin{aligned}
&F_\theta\big((\lambda_1, \dots, \lambda_{l-1}, 0, \lambda_{l+1}, \dots, \lambda_n),\ (t_1, \dots, t_{n-1})\big)\\
&\quad = \big((\lambda_1, \dots, \lambda_{l-1}, 0, \lambda_{l+1}, \dots, \lambda_n),\ (e^{\mathrm{i}\lambda_1\theta}t_1, \dots, t_l, \dots, e^{\mathrm{i}\lambda_{n-1}\theta}t_{n-1})\big)\,.
\end{aligned}
$$

For $n = 2$, $\Delta_{n-1} \times T^{n-1}$ is an annulus $\Delta_1 \times T^1$ and by Lemma 2,

$$
F_{2\pi} : \big((\lambda_1, \lambda_2),\ t_1\big) \in \Delta_1 \times T^1 \ \mapsto\ \big((\lambda_1, \lambda_2),\ e^{2\pi\mathrm{i}\lambda_1}t_1\big) \in \Delta_1 \times T^1. \tag{9}
$$

Here $\lambda_1 + \lambda_2 = 1$, λ_2 varies from 0 to 1 and $\varphi : \lambda_2 \in [0, 1] \mapsto (1 - \lambda_2, \lambda_2) \in \Delta_1$ is an orientation-preserving homeomorphism, so $F_{2\pi} : \big((1 - \lambda_2, \lambda_2),\ t_1\big) \mapsto \big((1 - \lambda_2, \lambda_2),\ e^{-2\pi\mathrm{i}\lambda_2}t_1\big)$, which is a (-1)-Dehn twist of $\Delta_1 \times T^1$.



REMARK 1. Instead of $\pi : M \to D$, Clemens [3] began with the 'whole map' $\pi : \mathbf{C}^n \to \mathbf{C}$; so J_r is replaced with an (infinite) hyperboloid $J_r^{\text{non-cpt}}$ defined by $x_1x_2\cdots x_n = r$ in $\mathbf{R}^n_{>0}$. Then instead of $\Psi : J_r \to \Delta_{n-1}$, he constructed a map $J_r^{\text{non-cpt}} \to \Delta_{n-1}$ that is a composition of a homeomorphism and a retraction as illustrated in Figure 14, and then he restricted this map to a compact domain in $J_r^{\text{non-cpt}}$. Using this map, he constructed a 1-parameter family of homeomorphisms $F'_\theta : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ ($0 \le \theta \le 2\pi$) different from ours.

5. Description of the action of the topological monodromy on foliations

We describe the action of the topological monodromy $F_{2\pi}$ on $\Delta_{n-1} \times T^{n-1}$ ($n \ge 3$) in terms of foliations. As illustrated in Figure 15, we shrink the standard $(n-1)$-simplex Δ_{n-1}

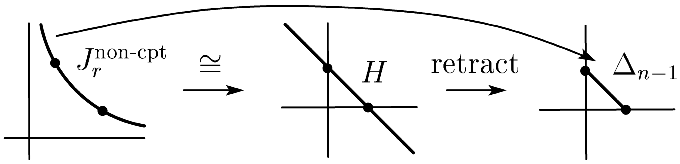


FIGURE 14. H is the hyperplane in $\mathbf{R}^n$ containing Δ_{n-1}.

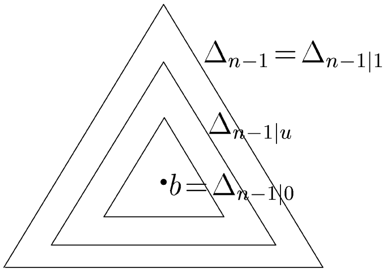


FIGURE 15

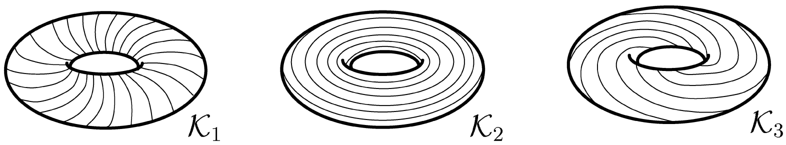


FIGURE 16

to obtain a family of $(n-1)$-simplexes $\Delta_{n-1|u}$ $(0 \leq u \leq 1)$ such that $\Delta_{n-1|1} = \Delta_{n-1}$ and $\Delta_{n-1|0}$ is the barycenter b of Δ_{n-1}.

Then $\{\partial\Delta_{n-1|u}\}_{0\leq u\leq 1}$ is a (singular) foliation of Δ_{n-1}, and $\{\partial\Delta_{n-1|u} \times T^{n-1}\}_{0\leq u\leq 1}$ gives a foliation of a smooth fiber $\Delta_{n-1} \times T^{n-1}$.

The topological monodromy $h := F_{2\pi} : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ maps each leaf $\partial\Delta_{n-1|u} \times T^{n-1}$ to itself. To describe it explicitly, we introduce n foliations on T^{n-1}.

First for each $i = 1, 2, \dots, n$, let $\mathcal{M}_{i|v}$ $(v \in S^1)$ be the $(n-2)$-dimensional subtorus in K_ξ defined by $e^{\mathrm{i}\alpha_i} = v$, then $\mathcal{K}_i = \{\mathcal{M}_{i|v}\}_{v\in S^1}$ is a foliation on K_ξ by parallel subtori (Figure 16 for $n = 3$).

The homeomorphism $\mathrm{pr} : K_\xi\ (\subset T^n) \to T^{n-1}$ given by $(e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_n}) \mapsto (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_{n-1}})$ transforms the foliation $\mathcal{K}_i = \{\mathcal{M}_{i|v}\}_{v\in S^1}$ to a foliation $\mathcal{T}_i = \{T^{n-2}_{i|v}\}_{v\in S^1}$ of T^{n-1}. Then $\{\partial\Delta_{n-1|u} \times T^{n-2}_{i|v}\}_{v\in S^1}$ is a foliation of $\partial\Delta_{n-1|u} \times T^{n-1}$. Here varying u yields a singular foliation of $\Delta_{n-1} \times T^{n-1}$.

The monodromy action on $\{b\} \times T^{n-1}$. The barycenter b ($= \partial\Delta_{n-1|0}$) of Δ_{n-1} is $\left(\frac{1}{n}, \dots \frac{1}{n}\right)$, and the topological monodromy $h : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ acts on $\{b\} \times T^{n-1}$ as

$$(b,\ t_1, t_2, \dots, t_{n-1}) \longmapsto (b,\ e^{2\pi \mathrm{i}/n} t_1, e^{2\pi \mathrm{i}/n} t_2, \dots, e^{2\pi \mathrm{i}/n} t_{n-1})\,.$$

In particular h maps each leaf $\{b\} \times T^{n-2}_{i|v}$ to $\{b\} \times T^{n-2}_{i|ve^{2\pi\mathrm{i}/n}}$.

The monodromy action on $\partial\Delta_{n-1|u} \times T^{n-1}$ ($u \neq 0$). For each $j = 1, 2, \dots, n$, let $\partial\Delta^{(j)}_{n-1|u}$ denote the jth face of $\partial\Delta_{n-1|u} \cong \partial\Delta_{n-1}$; then $\partial\Delta^{(j)}_{n-1|u} \times T^{n-1}$ is the jth face of $\partial\Delta_{n-1|u} \times T^{n-1}$. Fixing $u \neq 0$ and i, set $\mathcal{L}_v := \partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ and consider a foliation $\mathcal{F} = \{\mathcal{L}_v\}_{v \in S^1}$ of $\partial\Delta_{n-1|u} \times T^{n-1}$. Then $\mathcal{L}^{(j)}_v := \partial\Delta^{(j)}_{n-1|u} \times T^{n-2}_{i|v}$ is the jth face of $\mathcal{L}_v$. *While h maps each face of $\partial\Delta_{n-1|u} \times T^{n-1}$ to itself, h does not map a leaf $\mathcal{L}_v$ to itself. Moreover, as we will see below, h maps $\mathcal{L}^{(j)}_v$ to itself only if $j = i$ and $u = 1$.*

We specify the parameter u such that the side length of $\partial\Delta_{n-1|u}$ ($u \neq 0$) is $\sqrt{2}u$. Then for $u \neq 0$, the jth face $\partial\Delta^{(j)}_{n-1|u}$ of $\partial\Delta_{n-1|u}$ is defined by $\lambda_j = \dfrac{1-u}{n}$ in $\partial\Delta_{n-1|u}$. We first consider the case $j = i$; we describe the action of h on $\mathcal{L}^{(i)}_v = \partial\Delta^{(i)}_{n-1|u} \times T^{n-2}_{i|v}$. Set $\mu := \dfrac{1-u}{n}$. Then by Lemma 2, the restriction of h to $\mathcal{L}^{(i)}_v$ is explicitly given by

$$\big((\lambda_1, \dots, \lambda_{i-1}, \mu, \lambda_{i+1}, \dots, \lambda_n), (t_1, \dots, t_{i-1}, v, t_{i+1}, \dots .t_{n-1})\big) \longmapsto$$
$$\big((\lambda_1, \dots, \lambda_{i-1}, \mu, \lambda_{i+1}, \dots, \lambda_n), (e^{2\pi \mathrm{i}\lambda_1} t_1, \dots, e^{2\pi \mathrm{i}\mu} v, \dots, e^{2\pi \mathrm{i}\lambda_{n-1}} t_{n-1})\big)\,.$$

We thus obtain the following:

LEMMA 3. *h maps $\mathcal{L}^{(i)}_v$ to $\mathcal{L}^{(i)}_{ve^{2\pi\mathrm{i}(1-u)/n}}$. In particular when $u = 1$, it maps $\mathcal{L}^{(i)}_v$ to itself, and is explicitly given by*

$$h : \big((\lambda_1, \dots \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_n),\ (t_1, \dots t_{i-1}, v, t_{i+1}, \dots, t_{n-1})\big) \longmapsto$$
$$\big((\lambda_1, \dots \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_n),\ (e^{2\pi \mathrm{i}\lambda_1} t_1, \dots, v, \dots, e^{2\pi \mathrm{i}\lambda_{n-1}} t_{n-1})\big)\,.$$

REMARK 2. While h maps $\mathcal{L}^{(i)}_v$ to $\mathcal{L}^{(i)}_{ve^{2\pi\mathrm{i}(1-u)/n}}$, for $j \neq i$, h does *not* map $\mathcal{L}^{(j)}_v$ ($j \neq i$) to a face of a leaf of $\mathcal{F}$. Indeed,

$$h(\mathcal{L}^{(j)}_v) = \{\big((\lambda_1, \dots, \lambda_n), (t_1, \dots, t_{n-1})\big) \in \partial\Delta^{(j)}_{n-1|u} \times T^{n-1} \ :\ t_i = ve^{2\pi\mathrm{i}\lambda_i}\}$$
$$\cong \partial\Delta^{(j)}_{n-1|u} \times T^{n-2}\,.$$

To emphasize n, write h as $h_n : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$. Here $\partial\Delta^{(i)}_{n-1|1} \times T^{n-2}_{i|v}$

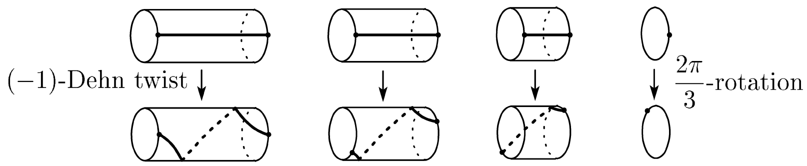


FIGURE 17

is homeomorphic to $\Delta_{n-2} \times T^{n-2}$ via

$$\varphi : \big((\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n),\ (t_1, \ldots, t_{i-1}, v, t_{i+1}, \ldots t_{n-1})\big) \longmapsto \big((\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n),\ (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_{n-1})\big).$$

Identify $\partial\Delta^{(i)}_{n-1|1} \times T^{n-2}_{i|v}$ with $\Delta_{n-2} \times T^{n-2}$ via φ. From Lemma 3,

$$\begin{aligned}
&\varphi \circ h_n \circ \varphi^{-1}\big((\lambda_1, \ldots, \lambda_{n-1}),\ (t_1, \ldots, t_{n-2})\big)\\
&= \varphi \circ h_n\big((\lambda_1, \ldots \lambda_{i-1}, 0, \lambda_i, \ldots, \lambda_{n-1}),\ (t_1, \ldots t_{i-1}, v, t_i, \ldots, t_{n-2})\big)\\
&= \varphi\big((\lambda_1, \ldots \lambda_{i-1}, 0, \lambda_i, \ldots, \lambda_{n-1}),\ (e^{2\pi \mathrm{i}\lambda_1} t_1, \ldots, v, \ldots, e^{2\pi \mathrm{i}\lambda_{n-2}} t_{n-2})\big)\\
&= \big((\lambda_1, \ldots, \lambda_{n-1}),\ (e^{2\pi \mathrm{i}\lambda_1} t_1, \ldots, e^{2\pi \mathrm{i}\lambda_{n-2}} t_{n-2})\big)\\
&= h_{n-1}\big((\lambda_1, \ldots, \lambda_{n-1}),\ (t_1, \ldots, t_{n-2})\big).
\end{aligned}$$

Thus $\varphi \circ h_n \circ \varphi^{-1} = h_{n-1}$, so

PROPOSITION 1. *The restriction of h_n to $\partial\Delta^{(i)}_{n-1|1} \times T^{n-2}_{i|v}$ is h_{n-1}.*

Variation of topological monodromy. As $u \to 0$, the ith face $\mathcal{L}^{(i)}_v = \partial\Delta^{(i)}_{n-1|u} \times T^{n-2}_{i|v}$ of the leaf $\mathcal{L}_v = \partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ shrinks to the $(n-2)$-dimensional torus T^{n-2}. Accordingly $h_u := h\,|_{\mathcal{L}^{(i)}_v}$ varies. In the case $n = 3$, $\mathcal{L}^{(i)}_v$ for $u \neq 0$ is an annulus and for $u = 0$ a circle, and as we see below, h_u varies from a (-1)-Dehn twist to a rotation of S^1, where recall that for each integer k, a k-Dehn twist is a self-homeomorphism of an annulus $[0, 1] \times S^1$ given by $(x, t) \in [0, 1] \times S^1 \to (x, e^{2\pi \mathrm{i} kx} t) \in [0, 1] \times S^1$.

PROPOSITION 2 ($n = 3$). *As u varies from 1 to 0, h_u varies from a (-1)-Dehn twist to the $\frac{2\pi}{3}$-rotation of S^1 as illustrated in Figure 17.*

PROOF. We show the assertion for $i = 1$ (it is similarly shown for other i). Identify $\mathcal{L}^{(1)}_v$ and $\mathcal{L}^{(1)}_{ve^{2\pi \mathrm{i}(1-u)/3}}$ with the annulus $\big[\frac{1-u}{3}, \frac{1+2u}{3}\big] \times S^1$ via the homeomorphism $\big((\lambda_1, \lambda_2, \lambda_3),\ (t_1, t_2)\big) \mapsto (\lambda_3,\ t_2)$. Regard then $h_u : \mathcal{L}^{(1)}_v \to \mathcal{L}^{(1)}_{ve^{2\pi \mathrm{i}(1-u)/3}}$ as a homeomorphism

$$h_u : (x, t) \in \left[\tfrac{1-u}{3}, \tfrac{1+2u}{3}\right] \times S^1 \mapsto (x, e^{-2\pi i\{x+(1-u)/3\}}t) \in \left[\tfrac{1-u}{3}, \tfrac{1+2u}{3}\right] \times S^1 .$$

As u varies from 1 to 0, this varies from a (-1)-Dehn twist of $[0, 1] \times S^1$ to the $\frac{2\pi}{3}$-rotation of S^1 as illustrated in Figure 17. □

ACKNOWLEDGMENT. The author would like to express his deep gratitude to Professor Shigeru Takamura for useful advice.

References

[1] N. A'CAMPO, La fonction zêta d'une monodromie, Comment. Math. Helv. **50** (1975), 233–248.

[2] V. I. ARNOLD, S. M. GUSEIN-ZADE and A. N. VARCHENKO, *Singularities of differentiable maps, II*, Birkhäuser (1988).

[3] C. H. CLEMENS, Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities, Trans. Amer. Math. Soc. **136** (1969), 93–108.

[4] C. H. CLEMENS, Degeneration of Kahler manifolds, Duke Math. J. **44** (2) (1977), 215–290.

[5] C. J. EARLE and P. L. SIPE, Families of Riemann surfaces over the punctured disk, Pacific J. Math. **150** (1991), 79–86.

[6] Y. IMAYOSHI, Holomorphic families of Riemann surfaces and Teichmüller spaces in Riemann surfaces and related topics, Ann. of Math. Studies **97** (1981), 277–300.

[7] Y. MATSUMOTO and J. M. MONTESINOS-AMILIBIA, Pseudo-periodic homeomorphisms and degeneration of Riemann surfaces, Bull. Amer. Math. Soc. **30** (1994), 70–75.

[8] Y. MATSUMOTO and J. M. MONTESINOS-AMILIBIA, *Pseudo-periodic maps and degeneration of Riemann surfaces*, Springer Lecture Notes in Math. 2030 (2011).

[9] H. SHIGA and H. TANIGAWA, On the Maskit coordinates of Teichmüller spaces and modular transformations, Kodai Math. J. **12** (1989), 437–443.

[10] S. TAKAMURA, Towards the classification of atoms of degenerations, II (Geometry of Degenerations of Complex Curves), Preprint (2014).

[11] H. ZOLADEK, *The Monodromy Group*, Birkhäuser (2006).

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