TOKYO J. MATH.
VOL. 39, NO. 3, 2017

Mixed Quantum Double Construction of Subfactors

Dedicated to Professor Ken-ichi SHINODA

Satoshi GOTO

Sophia University

Abstract. We generalize the quantum double construction of subfactors to that from arbitrary flat connections on 4-partite graphs and call it the *mixed quantum double construction*. If all the four graphs of the original 4-partite graph are connected, it is easy to see that this construction produces Ocneanu's asymptotic inclusion of both subfactors generated by the original flat connection horizontally and vertically. The construction can be applied for example to the non-standard flat connections which appear in the construction of the Goodman-de la Harpe-Jones subfactors or to those obtained by the composition of flat part of any biunitary connections as in N. Sato's paper [40]. An easy application shows that the asymptotic inclusions of the Goodman-de la Harpe-Jones subfactors are isomorphic to those of the Jones subfactors of type A_n except for the cases of orbifold type. If two subfactors $A \subset B$ and $A \subset C$ have common A-A bimodule systems, we can construct a flat connection in general. Then by applying our construction to the flat connection, we obtain the asymptotic inclusion of both $A \subset B$ and $A \subset C$. We also discuss the case when the original 4-partite graph contains disconnected graphs and give some such examples. General phenomena when disconnected graphs appear are explained by using bimodule systems.

1. Introduction

Since V. F. R. Jones initiated the index theory of subfactors [21] and found his celebrated polynomial invariant for knots and links [22], substantial progress has been made in studies on relation among subfactor theory, quantum groups, 3-dimensional topology, topological quantum field theory, algebraic quantum field theory and conformal field theory etc.

A. Ocneanu's paragroup theory plays an important role as a part of study on those relations among subfactor theory and other fields of mathematics and mathematical physics. We refer to the book [12] (and references therein) for the basics of subfactor theory, paragroup theory and their relations to other fields mentioned above. See also [10], [24, 26, 27, 28, 29] for surveys in this area of researches.

In his theory of paragroups, A. Ocneanu introduced an *asymptotic inclusion* for a subfactor [34, 35] as an analogue of the *quantum double construction* in the theory of quantum groups [3]. Some other constructions such as central sequence subfactors [35], Longo-Rehren inclusions [31] and symmetric enveloping algebras [39] are also known as an analogue of quantum double construction. The relation among these constructions are studied in [32, 33].

Received March 30, 2015; revised August 8, 2016
2010 *Mathematics Subject Classification*: 46L37 (Primary)

The *asymptotic inclusion* for a subfactor $N \subset M$ is a subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$, where $N \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\infty = \bigvee_{n=0}^{\infty} M_n$ is the Jones tower. The simplest examples of the asymptotic inclusions are subfactors generated by commuting squares of two-sided sequence of Jones projections;

$$\begin{array}{ccc} \langle e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n \rangle & \subset & \langle e_{-n-1}, \ldots, e_{-1}, e_1, \ldots, e_{n+1} \rangle \\ \cap & & \cap \\ \langle e_{-n}, \ldots, e_{-1}, e_0, e_1, \ldots, e_n \rangle & \subset & \langle e_{-n-1}, \ldots, e_{-1}, e_0, e_1, \ldots, e_{n+1} \rangle \,. \end{array}$$

Here Jones projections $\{e_i\}_{i \in \mathbf{Z}}$ satisfy the following relations;

$$\begin{aligned} e_i e_{i\pm 1} e_i &= \beta^{-2} e_i \,, & &\text{for} \quad i \in \mathbf{Z} \,, \\ e_i e_j &= e_j e_i \,, & &\text{whenever} \quad |i-j| \neq 2 \,, \end{aligned}$$

where $\beta = 2\cos(\pi/N)$. The indices of these subfactors were first computed by M. Choda [2]. It is easy to see that the above subfactors are isomorphic to the asymptotic inclusion of Jones subfactor with the principal graph A_n because the commuting square

$$\begin{array}{ccc} M'_{-k} \cap M_0 \vee M'_0 \cap M_k & \subset & M'_{-k-1} \cap M_0 \vee M'_0 \cap M_{k+1} \\ \cap & & \cap \\ M'_{-k} \cap M_k & \subset & M'_{-k-1} \cap M_{k+1} \,, \end{array}$$

generates the asymptotic inclusion for an enough large k. Here we used the notation $M \supset N = M_{-1} \supset M_{-2} \supset \cdots$ for a downward basic construction (tunnel construction).

J. Erlijman [4, 5] generalized the above examples to the case of two-sided sequence of generators $\{g_i\}_{i \in \mathbf{Z}}$ of Hecke algebra of type A, B, C, D and showed that the following commuting squares produce the asymptotic inclusion for the Hecke algebra subfactors of type A, B, C, D of H. Wenzl [42, 43]:

$$\begin{array}{ccc} \langle g_{-n}, \ldots, g_{-1}, g_1, \ldots, g_n \rangle & \subset & \langle g_{-n-1}, \ldots, g_{-1}, g_1, \ldots, g_{n+1} \rangle \\ \cap & & \cap \\ \langle g_{-n}, \ldots, g_{-1}, g_0, g_1, \ldots, g_n \rangle & \subset & \langle g_{-n-1}, \ldots, g_{-1}, g_0, g_1, \ldots, g_{n+1} \rangle \,. \end{array}$$

The author generalized her construction to a fusion algebra with quantum $6j$-symbols which produce periodic commuting squares in [15] and proved that our construction produces the asymptotic inclusion for the subfactor generated by the original periodic commuting square. By applying our result in the case of fusion algebras of $SU(n)_k$ WZW models, which is the same as Hecke algebra subfactors of type A of H. Wenzl [42], we got the same result as in [5].

J. Erlijman and H. Wenzl then generalized quantum double construction to quantum multiple construction for subfactors arising from braid group representations [6, 7] and for braided categories [8]. M. Asaeda [1] further generalized their quantum multiple construction to subfactors whose paragroup satisfies the generalized Yang-Baxter equation which is a weaker condition than braidedness of the bimodule system.

In this paper, we consider a generalization of the quantum double construction which is different from either a generalization to periodic commuting squares or that to the quantum multiple construction as we mentioned above. Here we generalize the quantum double constructions of subfactors to that from arbitrary flat connections on 4-partite graphs. We call it the *mixed quantum double construction*. If all the four graphs of the original 4-partite graph are connected, it is easy to see that this construction produces Ocneanu's asymptotic inclusion of both subfactors generated by the original flat connection horizontally and vertically. The construction can be applied for example to the non-standard flat connections which appear in the construction of Goodman-de la Harpe-Jones subfactors or to those obtained by the composition of flat part of any biunitary connections as in N. Sato's paper [40]. The mixed quantum double construction will also be used for a generalization of quantum multiple construction of M. Asaeda in [18].

The paper is organized as follows.

In section 2 we give a brief review of Ocneanu's asymptotic inclusion. We remark that our definition of asymptotic inclusion is slightly *different* from Ocneanu's original definition. More precisely the original definition coincides with ours for the dual subfactor.

In section 3 we define the mixed quantum double construction from arbitrary flat connections on 4-partite graphs and show that the construction produces Ocneanu's asymptotic inclusion of both subfactors generated by the original flat connection horizontally and vertically if all the four graphs of the original 4-partite graph are connected. The non-standard flat connections which appear in the construction of Goodman-de la Harpe-Jones subfactors are given as the first fundamental examples.

In section 4 we explain that the bimodule systems of the mixed quantum double subfactors are described by Ocneanu's surface bimodules in the same way as usual asymptotic inclusions.

In section 5 we introduce another mixed quantum double construction which can be defined from two subfactors $A \subset B$ and $A \subset C$ with the common A-A bimodule system.

In section 6 we give some examples in the case when all the four graphs of the original 4-partite graph are connected.

In section 7 we discuss the case when the original 4-partite graph contains disconnected graphs and give some such examples. General phenomena when disconnected graphs appear are explained by using bimodule systems.

2. Ocneanu's asymptotic inclusion and quantum double construction of subfactors

Let $N \subset M$ be a subfactor of the hyperfinite II_1 factor M with finite index and finite depth, $N = N_0 \subset M = N_1 \subset N_2 \subset N_2 \subset \cdots \subset N_\infty = \bigvee_{n=0}^{\infty} N_n$ be the Jones tower and $M = N_1 \supset N = N_0 \supset N_{-1} \supset N_{-2} \supset \cdots$ be the downward basic construction.

REMARK 2.1. The notations $N \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$ (resp. $M \supset N = N_0 \supset N_1 \supset N_2 \supset \cdots$) are often used for the *basic construction* (*Jones tower*) (resp. a

$$\begin{array}{ccccccccc}
N' \cap N_\infty & \subset & N'_{-1} \cap N_\infty & \subset & N'_{-2} \cap N_\infty & \subset & \cdots & \subset & N_\infty \\
\cup & & \cup & & \cup & & & & \cup \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
N' \cap N_2 & \subset & N'_{-1} \cap N_2 & \subset & N'_{-2} \cap N_2 & \subset & \cdots & \subset & N_2 \\
\cup & & \cup & & \cup & & & & \cup \\
N' \cap N_1 & \subset & N'_{-1} \cap N_1 & \subset & N'_{-2} \cap N_1 & \subset & \cdots & \subset & N_1 = M \\
\cup & & \cup & & \cup & & & & \cup \\
N' \cap N & \subset & N'_{-1} \cap N & \subset & N'_{-2} \cap N & \subset & \cdots & \subset & N_0 = N
\end{array}$$

FIGURE 1. The commuting squares from the standard invariant of $N \subset M$

downward basic construction (a *tunnel construction*)), but we use $\{N_i\}_{i\geq 0}$ for the tower and $\{N_i\}_{i\leq 0}$ for a choice of tunnels in this paper. We also use the notation $N_\infty = \bigvee_{n=0}^{\infty} N_n$ instead of $M_\infty = \bigvee_{n=0}^{\infty} M_n$.

The commuting squares constructed from the inclusions of finite dimensional C*-algebras $\{N'_k \cap N_l\}$ $(k, l \in \mathbf{Z}, k \leq l)$ are called the *standard invariant* or the *standard λ-lattice* of the subfactor $N \subset M$ [37, 38]. We call the flat connection obtained from the standard invariant the *standard* flat connection. Thanks to Popa's classification theorem [37], the standard invariant generates $N \subset M$ as in Figure 1 and it is a complete invariant for subfactors of the hyperfinite II$_1$ factor with finite index and finite depth. (More generally it is a complete invariant for *strongly amenable* subfactors.)

Ocneanu introduced the *asymptotic inclusion* $M \vee (M' \cap M_\infty) \subset M_\infty$ of a subfactor $N \subset M$ ([34, 35], [11], [12, Section 12.6]). It is known that the asymptotic inclusion becomes a subfactor of the hyperfinite II$_1$ factor with finite index and finite depth (See [12, Theorem 12.24, page 657]).

REMARK 2.2. As we mentioned above, Ocneanu's original definition of the asymptotic inclusion is given by $M \vee (M' \cap M_\infty) \subset M_\infty$. But we refer to $N \vee (N' \cap N_\infty) \subset N_\infty$ as the asymptotic inclusion of a subfactor $N \subset M$ in this paper. Hence the original definition coincides with ours for the dual subfactor $M \subset M_1$.

We now denote the algebras $N'_{-i} \cap N_j$ by $A_{i,j}$, then it is easy to see that the commuting squares

$$\begin{array}{ccccccc}
A_{n,n} & \subset & A_{n+1,n+1} & \subset & \cdots & \longrightarrow & N_\infty \\
\cup & & \cup & & & & \cup \\
A_{n,0} \vee A_{0,n} & \subset & A_{n+1,0} \vee A_{0,n+1} & \subset & \cdots & \longrightarrow & N \vee (N' \cap N_\infty)
\end{array}$$

generates the asymptotic inclusion $N \vee (N' \cap N_\infty) \subset N_\infty$. See also Figure 2.

$$
\begin{array}{ccccccccc}
N' \cap N_\infty & \subset & N'_{-1} \cap N_\infty & \subset & N'_{-2} \cap N_\infty & \subset & \cdots & \subset & N_\infty \\
\cup & & \cup & & \cup & & & & \cup \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
A_{0,2} & \subset & A_{1,2} & \subset & A_{2,2} & \subset & \cdots & \subset & N_2 \\
\cup & & \cup & & \cup & & & & \cup \\
A_{0,1} & \subset & A_{1,1} & \subset & A_{2,1} & \subset & \cdots & \subset & M = N_1 \\
\cup & & \cup & & \cup & & & & \cup \\
A_{0,0} & \subset & A_{1,0} & \subset & A_{2,0} & \subset & \cdots & \subset & N = N_0
\end{array}
$$

FIGURE 2. The commuting squares of the algebras $A_{i,j} = N'_{-i} \cap N_k$

The procedure obtaining the asymptotic inclusion from a subfactor is called the *quantum double construction* of subfactors.

3. Mixed quantum double construction of subfactors

In this section we define the *mixed quantum double construction* of subfactors which is a generalization of quantum double construction of subfactors. We refer readers to [12] for all the basic notions such as subfactors, bimodules, axioms of paragroups, string algebra construction, flat connections, and so on. We use the terminology *4-partite graph* which means four bipartite graphs connected in a square shape as in Figure 3 on which a biunitary connection W is defined. Here we remark that we have to choose and fix a distinguished vertex $*$ of graph G_1 as an initial vertex in order to define commuting squares from the string algebra construction (See Figure 3).

Let W be a biunitary connection which is $*$-*flat* with respect to a given distinguished vertex $*$ on a 4-partite graph as in Figure 3. Then we can construct a double sequence of finite dimensional C*-algebras $B_{i,j}$ $(i, j \geq 0)$ by the string algebra construction as in Figure 4. (See [12, Chapter 11] for the string algebra construction.)

We consider the subfactor $T \subset S$ generated by the following commuting squares.

DEFINITION 3.1. We call a subfactor $T \subset S$ generated by the commuting squares as in Figure 5 the *mixed quantum double subfactor* arising from a $*$-flat connection W. The procedure obtaining the mixed quantum double subfactor from a flat connection W is called the *mixed quantum double construction*.

If a flat connection W is the standard one arising from a subfactor $N \subset M$, then $B_{i,j} = A_{i,j} = N_{-i}{}' \cap N_k$ holds. In this case the mixed quantum double construction coincides with the usual quantum double construction. Hence it is a generalization of quantum double construction of subfactors.

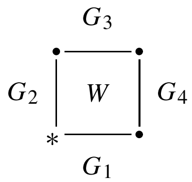


FIGURE 3. A flat connection W on a 4-partite graph

$$
\begin{array}{ccccccccc}
P = P_0 & \subset & Q = P_1 & \subset & P_2 & \subset & \cdots & \subset & P_\infty = N_\infty \\
\cup & & \cup & & \cup & & & & \cup \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
B_{0,2} & \subset & B_{1,2} & \subset & B_{2,2} & \subset & \cdots & \subset & N_2 \\
\cup & & \cup & & \cup & & & & \cup \\
B_{0,1} & \subset & B_{1,1} & \subset & B_{2,1} & \subset & \cdots & \subset & M = N_1 \\
\cup & & \cup & & \cup & & & & \cup \\
B_{0,0} & \subset & B_{1,0} & \subset & B_{2,0} & \subset & \cdots & \subset & N = N_0
\end{array}
$$

FIGURE 4. The commuting squares arising from a flat connection W

$$
\begin{array}{ccccccc}
B_{n,n} & \subset & B_{n+1,n+1} & \subset & \cdots & \longrightarrow & S \\
\cup & & \cup & & & & \cup \\
B_{n,0} \vee B_{0,n} & \subset & B_{n+1,0} \vee B_{0,n+1} & \subset & \cdots & \longrightarrow & T
\end{array}
$$

FIGURE 5. The commuting squares which produce the mixed quantum double subfactor

THEOREM 3.2. *Let W be a $*$-flat connection on a 4-partite graph as in Figure 3 and let $N \subset M$ (resp. $P \subset Q$) be a subfactor obtained by the string algebra construction horizontally (resp. vertically) as in Figure 4. If all the graphs G_i $(i = 1, 2, 3, 4)$ are connected, then the mixed quantum double subfactor arising from the flat connection W is isomorphic to the asymptotic inclusions of the both subfactors $N \subset M$ and $P \subset Q$.*

PROOF. From the flatness assumption, the horizontal algebra $B_{m,0}$ and the vertical algebra $B_{0,n}$ in Figure 4 commute [12, Theorem 11.17]. Then the standard flatness argument (including Ocneanu's compactness argument [12, Section 11.4] and Popa's classification theorem [37, Theorem 5.1.1]) shows that $B_{m,0} = P' \cap P_m$, $B_{0,n} = N' \cap N_n$ (for $n, m \geq 0$), $P = N' \cap N_\infty$ and $N = P' \cap P_\infty$ hold. Hence it is easy to see that the commuting squares in

Figure 5 generates a subfactor isomorphic to $(N \vee (N' \cap N_\infty) \subset N_\infty) = (P \vee (P' \cap P_\infty) \subset P_\infty)$. $\square$

The following corollary is immediate from the theorem.

COROLLARY 3.3. *Let us use the same notations W, $N \subset M$ and $P \subset Q$ as in the above theorem. If all the graphs G_i $(i = 1, 2, 3, 4)$ are connected, then the asymptotic inclusions of $N \subset M$ is isomorphic to that of $P \subset Q$, i.e., $(N \vee (N' \cap N_\infty) \subset N_\infty) \cong (P \vee (P' \cap P_\infty) \subset P_\infty)$.*

EXAMPLE 3.4. The Goodman-de la Harpe-Jones subfactors (abbreviated as GHJ subfactors) [13] are a well-known series of irreducible subfactors most of which have indices greater than 4. They are constructed from the commuting squares arising from the embeddings of type A string algebras into other string algebras of type ADE. (See [12, Chapter 11] for the construction of GHJ subfactors from a viewpoint of string algebra embedding. See also [17], where the (dual) principal graphs and their fusion rules are computed in all cases.) It is easy to see that the biunitary connections corresponding to these commuting squares are $*$-flat [12, Section 11.6, page 593]. Here $*$ represents the endpoint of Dynkin diagram A_n. So it follows from the flatness that if the graph G_2 of the 4-partite graph is connected it becomes the principal graph of the GHJ subfactor. By applying the mixed quantum double construction to these flat connections, we obtain subfactors which are isomorphic to both the asymptotic inclusions of the GHJ subfactors and those of the Jones subfactors of type A_n. Hence the asymptotic inclusions of GHJ subfactors are isomorphic to those of the Jones subfactors of type A_n by Corollary 3.3.

REMARK 3.5. Though it is not explicitly written in [41], Corollary 3.3 can also be proved by using Sato's theorem [41, Theorem 2.3], general theory on the bimodule structure of the asymptotic inclusion ([12, Section 12.6] and [30, Appendix A]) and Popa's classification theorem [37]. But our method is more direct and easier to see. Actually Corollary 3.3 can be proved without using Sato's theorem and it is almost obvious from our construction.

4. The bimodule systems of the mixed quantum double subfactors

In this section we explain that the bimodule systems of the mixed quantum double subfactors are described by Ocneanu's surface bimodules in the same way as usual asymptotic inclusions. We refer readers to [12, Section 12.6] for the notion of Ocneanu's surface bimodules.

Let ${}_N h_M$ (resp. ${}_P k_Q$) be the generator open string bimodule of $N \subset M$ (resp. $P \subset Q$) constructed from a given flat connection W. (See [34, page 133] and [41, Definition 2.1] for the definition of the open string bimodule.) Note that ${}_N h_M$ is constructed in horizontal direction and ${}_P k_Q$ in vertical direction. So the generator bimodule ${}_P k_Q$ corresponds to the opposite subfactor $P^{\mathrm{opp}} \subset Q^{\mathrm{opp}}$ because of the difference of the directions. Then from

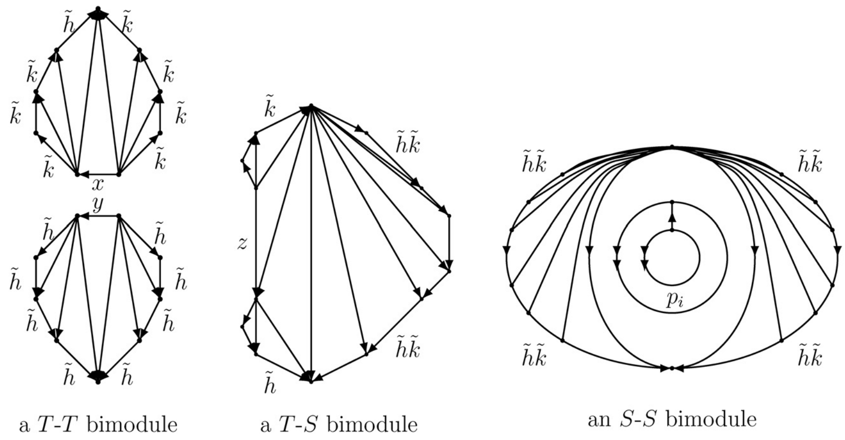


FIGURE 6. The bimodules arising from the mixed quantum double subfactor $T \subset S$

Sato's theorem [41, Theorem 2.3], we have N-N bimodule system ${}_N\mathcal{M}_N$ from ${}_N h_M$ and P-P bimodule system ${}_P\mathcal{M}_P$ from ${}_P k_Q$ in common, i.e., ${}_N\mathcal{M}_N = {}_P\mathcal{M}_P$ holds. Now we put $\tilde{h} = {}_N(h \cdot \bar{h})_N$ and $\tilde{k} = {}_P(k \cdot \bar{k})_P$. Here $\bar{h}$ (resp. $\bar{k}$) is the conjugate bimodule of h (resp. k). Then T-T, T-S and S-S bimodules arising from the mixed quantum double subfactor $T \subset S$ are described by surface bimodules as in Figure 6. Where the bimodules x, y and z are taken from ${}_N\mathcal{M}_N (= {}_P\mathcal{M}_P)$ and p_i is a minimal projection in the tube algebra $\mathrm{Tube}({}_N\mathcal{M}_N)$. See [11], [12, Section 12.6] and [30] for more details of this kind of figures.

Note that the description and figures of M_∞-M_∞ bimodules of the asymptotic inclusions $M \vee (M' \cap M_\infty) \subset M_\infty$ in [11] and [12, Section 12.6] are *incorrect*. The mistakes found in [11] and [12, Section 12.6] were fixed in [30]. See [30, Appendix A] for the details.

From these descriptions of the bimodule systems, we can also confirm that Theorem 3.2 holds, i.e., $(T \subset S) \cong (N \vee (N' \cap N_\infty) \subset N_\infty) \cong (P \vee (P' \cap P_\infty) \subset P_\infty)$ because they have the same paragroup (i.e., standard invariant). See the proof of Theorem 2.3 in [15]. The same argument also works for the proof of Theorem 3.2.

5. From two subfactors $A \subset B$ and $A \subset C$ with the common A-A bimodule system to the mixed quantum double construction

Let $A \subset B$ and $A \subset C$ be two subfactors of the hyperfinite II_1 factor with finite indices and finite depths. In this section, we assume that $A \subset B$ and $A \subset C$ have the common A-A bimodule system and we consider the mixed quantum double construction from such two subfactors. It is known from paragroup theory that we can construct a flat connection W on a 4-partite graph as in Figure 7 from any two subfactors $A \subset B$ and $A \subset C$ with the common A-A bimodule system (See [25, Proposition 4.3]).

We give a brief outline of the general procedure below. For the complete proof, see the

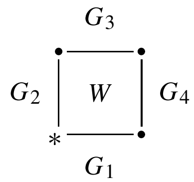


FIGURE 7. A flat connection W on a 4-partite graph

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
{}_AB_A \arrow[u, "{}_BB_A\otimes\cdot"] \arrow[r] & {}_AB\otimes_A C_C \arrow[u] \arrow[r] & {}_AB\otimes_A C_A \arrow[u] \arrow[r] & \cdots \\
{}_BB_A \arrow[u, "{}_AB_B\otimes\cdot"] \arrow[r] & {}_BB\otimes_A C_C \arrow[u] \arrow[r] & {}_BB\otimes_A C_A \arrow[u] \arrow[r] & \cdots \\
{}_A*_A \arrow[u, "{}_BB_A\otimes\cdot"] \arrow[r, "\cdot\otimes_A C_C"'] & {}_AC_C \arrow[u] \arrow[r, "\cdot\otimes_C C_A"'] & {}_AC_A \arrow[u] \arrow[r, "\cdot\otimes_A C_C"'] & \cdots
\end{tikzcd}
$$

FIGURE 8. Double sequence of bimodules (1)

argument in pages 489–491 and Proposition 4.3 in [25].

First we take a trivial bimodule ${}_AA_A$. We denote ${}_AA_A$ by ${}_A*_A$. At each step we draw a new bimodule to the right by tensoring ${}_AC_C$ and ${}_CC_A$ from the right alternately and also draw a bimodule upward by tensoring ${}_BB_A$ and ${}_AB_B$ from the left alternately. Thus we get a double sequence of bimodules as in Figure 8.

Take endomorphism spaces of the bimodules in Figure 8, then we get an increasing double sequence of finite dimensional C*-algebras as follows.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\cup & & \cup & & \cup & & \\
\mathrm{End}({}_AA_A) & \subset & \mathrm{End}({}_AB\otimes_A C_C) & \subset & \mathrm{End}({}_AB\otimes_A C_A) & \subset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathrm{End}({}_BB_A) & \subset & \mathrm{End}({}_BB\otimes_A C_C) & \subset & \mathrm{End}({}_BB\otimes_A C_A) & \subset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathrm{End}({}_AA_A) & \subset & \mathrm{End}({}_AC_C) & \subset & \mathrm{End}({}_AC_A) & \subset & \cdots
\end{array}
$$

This double sequence forms commuting squares with the period two. Thus we get a 4-partite graph and a biunitary connection on the graph. From general paragroup theory, we know that this biunitary connection becomes ${}_A*_A$-flat connection.

Thanks to Popa's classification theorem [37, Theorem 5.1.1] and Proposition 4.3 in [25], these commuting squares generate a subfactor isomorphic to $A \subset B$ in horizontal direction and a subfactor isomorphic to $A^{\mathrm{opp}} \subset C^{\mathrm{opp}}$ in vertical direction. Here A^{opp} represents the opposite von Neumann algebra of A.

Hence by applying the mixed quantum double construction to the flat connection, we obtain the asymptotic inclusions of both subfactors $A \subset B$ and $A \subset C$. So we can apply our construction to any two subfactors $A \subset B$ and $A \subset C$ with the common A-A bimodule system in this way. We call this construction the *mixed quantum double construction from $A \subset B$ and $A \subset C$*.

REMARK 5.1. If two subfactors $A \subset B$ and $A \subset C$ are identical, then the mixed quantum double construction coincides with the usual quantum double construction for the subfactor $(A \subset B) = (A \subset C)$.

EXAMPLE 5.2. Let $N \subset M$ be a subfactor of the hyperfinite II_1 factor M with finite index and finite depth, $N = N_0 \subset M = N_1 \subset N_2 \subset N_2 \subset \cdots \subset N_\infty = \bigvee_{n=0}^{\infty} N_n$ be the Jones tower. Then for any $k, l \geq 1$, the mixed quantum double construction from $N \subset N_k$ and $N \subset N_l$ produces the same subfactor as the asymptotic inclusions of $N \subset M$, i.e., $N \vee (N' \cap N_\infty) \subset N_\infty$.

THEOREM 5.3. *Let $A \subset B$ and $C \subset D$ be inclusions of the hyperfinite II_1 factor with finite indices and finite depths. If A-A bimodule system arising from $A \subset B$ coincides with C-C bimodule system from $C \subset D$ (or C^{opp}-C^{opp} bimodule system from $C^{\mathrm{opp}} \subset D^{\mathrm{opp}}$), then the asymptotic inclusions of $A \subset B$ is isomorphic to that of $C \subset D$.*

PROOF. Applying the mixed quantum double construction for $A \subset B$ and $C \subset D$ (or $C^{\mathrm{opp}} \subset D^{\mathrm{opp}}$), we get the result by Corollary 3.3. □

REMARK 5.4. Theorem 5.3 can be proved again by using general theory on the bimodule structure of the asymptotic inclusion ([12, Section 12.6] and [30, Appendix A]) and Popa's classification theorem ([37]). See also Remark 3.5.

6. Examples in the case of connected graphs

In this section we give some examples of the mixed quantum double subfactors in the case when all the four graphs of the original 4-partite graph are connected.

EXAMPLE 6.1 (The flat connections which generate GHJ subfactors [13], [17]:). As is already mentioned in Example 3.4, the flat connections which generate GHJ subfactors have been known for the first non-trivial examples of non-standard flat connections. Here a *non-standard flat connection* means a flat connection which is not obtained from the

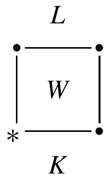


FIGURE 9. a K-L biunitary connection W

standard invariant of any subfactor. In these cases, the mixed quantum double subfactors are isomorphic to the asymptotic inclusions of the Jones subfactors of type A_n.

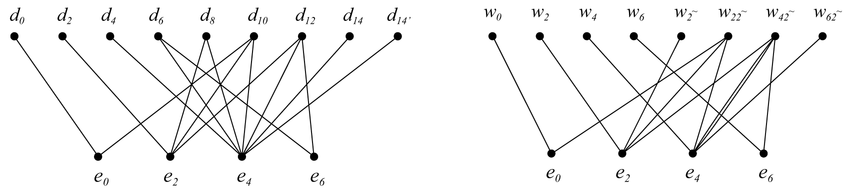
Let us consider a biunitary connection on a 4-partite graph with the upper graph L and the lower graph K as in Figure 9. We call such a connection a *K-L biunitary connection*.

The most fundamental and important cases are those when both K and L are among ADE Dynkin diagrams. Here we call such K-L connections *inter-Dynkin connections* for short. A. Ocneanu has classified all irreducible inter-Dynkin connections and showed many applications of his classification result [36]. (See also [16] for more detailed exposition of Ocneanu's theory of double triangle algebras and its applications.) We remark that GHJ subfactors can be considered as those generated by irreducible A-L inter-Dynkin connections.

EXAMPLE 6.2 (D_{16}-E_8 flat connection [19]:). GHJ subfactors are constructed from the commuting squares arising from the embeddings of type A string algebras into other string algebras of type ADE. So the construction can be generalized by considering embeddings of type D string algebras into type E string algebras. Then the subfactors obtained by the embeddings are constructed from inter-Dynkin connections of type D-E. We call such subfactors *generalized GHJ subfactors of type D-E*. Among D_7-E_6, D_{10}-E_7 and D_{16}-E_8 inter-Dynkin connections, it turned out that only D_{16}-E_8 connections are flat. The author computed the (dual) principal graphs of generalized GHJ subfactor of type D_{16}-E_8 in [19]. The mixed quantum double subfactors from these D_{16}-E_8 flat connections are all isomorphic to the asymptotic inclusions of the D_{16} subfactor.

We give an example of the (dual) principal graph of a generalized GHJ subfactor of type D_{16}-E_8 in Figures 10.

EXAMPLE 6.3 (The composite flat connections [40]:). Let W be a biunitary connection and $N \subset M$ a subfactor generated by the string algebras arising from the connection W as in Figures 11 and 12. N. Sato [40] showed that any connection obtained by the composition of an arbitrary biunitary connection W and its flat part W_f as in Figures 13 and 14 is $*_N$-flat if the subfactor $N \subset M$ as in Figure 11 has finite depth. See also Figure 15 for the commuting squares arising from a connection W and its flat part W_f. Here the vertex $*_N$ of the graph H_1 in Figure 14 corresponds to that of $N' \cap N$. Many non-standard flat connections are obtained in this way. So we can apply the mixed quantum double construction to them.



The principal graph of GHJ $({}_{D_{16}}w_{0+E_8})$ The dual principal graph of GHJ $({}_{D_{16}}w_{0+E_8})$

FIGURE 10. The (dual) principal graph of generalized GHJ$({}_D w_{0+E})$

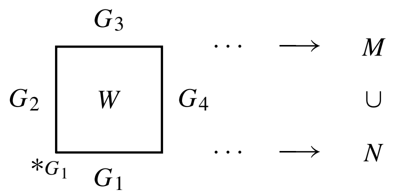


FIGURE 11. A subfactor $N \subset M$ generated by a biunitary connection W

$$
\begin{array}{ccccccccccc}
P = P_0 & \subset & Q = P_1 & \subset & P_2 & \subset & \cdots & & & & \\
\cup & & \cup & & \cup & & & & & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\cup & & \cup & & \cup & & & & \cup & & \\
B_{0,2} & \subset & B_{1,2} & \subset & B_{2,2} & \subset & \cdots & \subset & N_2 & & \\
\cup & & \cup & & \cup & & & & \cup & & \\
B_{0,1} & \subset & B_{1,1} & \subset & B_{2,1} & \subset & \cdots & \subset & M & = & N_1 \\
\cup & & \cup & & \cup & & & & \cup & & \\
B_{0,0} & \subset & B_{1,0} & \subset & B_{2,0} & \subset & \cdots & \subset & N & = & N_0
\end{array}
$$

FIGURE 12. The commuting squares arising from a flat connection W

EXAMPLE 6.4 (D_{10}-E_7 flat connection [40]:). It is known that E_7 biunitary connections are not flat. D. E. Evans and Y. Kawahigashi showed that the flat part of E_7 connection is D_{10} [9]. The composite flat connection is given as an example of an application of N. Sato's theorem [40]. The graphs H_1, G_1 and their composite graph as in Figures 14 are given in Figures 16 and 17, where $H_1 \cong D_{10}$ and $G_1 \cong E_7$. Application of the mixed quantum double construction to this flat connection yield a subfactor isomorphic to the asymptotic inclusions of the D_{10} subfactor.

EXAMPLE 6.5 (The composite flat connections arising from inter-Dynkin connections [20]:). The author computed the flat part of all inter-Dynkin connections in [20]. So

$$
\begin{array}{ccccc}
N' \cap N_{k+1} & \subset & B_{0,k+1} & \subset & B_{1,k+1} \\
\cup & & \cup & & \cup \\
N' \cap N_k & \subset & B_{0,k} & \subset & B_{1,k}
\end{array}
$$

FIGURE 13. Composition of the commuting squares and their flat parts

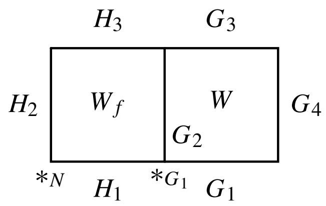


FIGURE 14. Composite flat connection $W_f \cdot W$

$$
\begin{array}{ccccccccccccc}
R & \subset & P = P_0 & \subset & Q = P_1 & \subset & P_2 & \subset & \cdots & & & & \\
\cup & & \cup & & \cup & & \cup & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\cup & & \cup & & \cup & & \cup & & & & \cup & & \\
N' \cap N_2 & \subset & B_{0,2} & \subset & B_{1,2} & \subset & B_{2,2} & \subset & \cdots & \subset & N_2 & & \\
\cup & & \cup & & \cup & & \cup & & & & \cup & & \\
N' \cap N_1 & \subset & B_{0,1} & \subset & B_{1,1} & \subset & B_{2,1} & \subset & \cdots & \subset & M & = & N_1 \\
\cup & & \cup & & \cup & & \cup & & & & \cup & & \\
N' \cap N & \subset & B_{0,0} & \subset & B_{1,0} & \subset & B_{2,0} & \subset & \cdots & \subset & N & = & N_0
\end{array}
$$

FIGURE 15. The commuting squares arising from a connection W and its flat part W_f

a lot of composite flat connections have been obtained. Applying the mixed quantum double construction to these examples yield subfactors isomorphic to the asymptotic inclusions of ADE subfactors and those of dual (generalized) GHJ subfactors. We can determine which asymptotic inclusion is obtained in each case from the computation in [20]. See [20] for more details.

7. Examples in the case of disconnected graphs

We show some examples of the mixed quantum double subfactors in the case when 4-partite graphs contain some disconnected graphs.

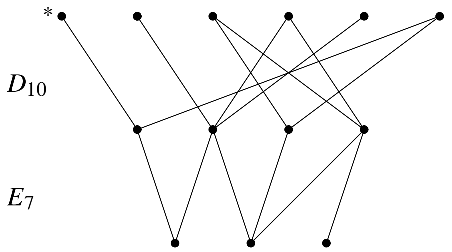


FIGURE 16. The horizontal graph in the composite flat connection of an E_7 connection

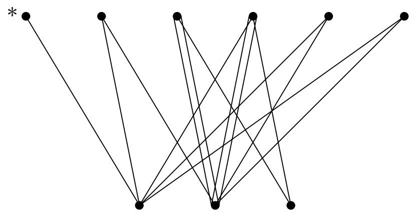


FIGURE 17. The composite horizontal graph

THEOREM 7.1. *Let W be a flat connection on a 4-partite graph as in Figure 3. Suppose that the graphs G_1 and G_3 are connected but that G_2 and G_4 are not necessarily connected. Let $N \subset M$ be a subfactor obtained by the string algebra construction horizontally as in Figure 4. Then the mixed quantum double subfactor arising from the flat connection W is isomorphic to the asymptotic inclusions of the subfactor $N \subset M$, i.e. $N \vee (N' \cap N_\infty) \subset N_\infty$.*

PROOF. We remark that von Neumann algebras P_k $(k \geq 1)$ in Figure 4 do not become factors in general due to the disconnectedness of the graphs G_2 and G_4. But we know $B_{0,n} = N' \cap N_n$ and $P = N' \cap N_\infty$ still hold in this case. Hence the commuting squares in Figure 5 generates a subfactor isomorphic to $N \vee (N' \cap N_\infty) \subset N_\infty$. $\square$

EXAMPLE 7.2 (Orbifold flat connections [23], [14]:). In the case of A_{2n+1}-D_{n+2} inter-Dynkin connections, if we choose the vertex $*_D$ of D_{n+2} as in Figure 18 for the GHJ construction, disconnected graphs appear in graphs G_2 and G_4. All the connected component of G_2 and G_4 are isomorphic to the Dynkin diagram A_3. (See [17] for some examples of the graphs G_2 and G_4.) It is the principal graph of the horizontally generated GHJ subfactor $N \subset M$ with index two. So all the GHJ subfactors in these cases are isomorphic to $R \subset R \rtimes \mathbf{Z}_2$, where R is the hyperfinite II_1 factor. Application of the mixed quantum double construction to these connections yields the asymptotic inclusion of the subfactor $R \subset R \rtimes \mathbf{Z}_2$, which becomes again index two subfactor. Hence the mixed quantum double subfactor is also isomorphic to $R \subset R \rtimes \mathbf{Z}_2$ in these cases.

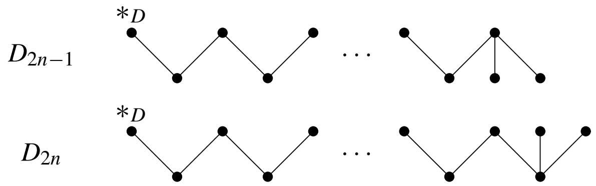


FIGURE 18. The distinguished vertex $*_D$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
{}_C h\bar{h}_C \arrow[u, "{}_B\bar{h}_A\otimes\cdot"] \arrow[r] & {}_C h\bar{h}k_D \arrow[u] \arrow[r] & {}_C h\bar{h}k\bar{k}_C \arrow[u] \arrow[r] & \cdots \\
{}_B\bar{h}_C \arrow[u, "{}_A h_B\otimes\cdot"] \arrow[r] & {}_B\bar{h}k_D \arrow[u] \arrow[r] & {}_B\bar{h}k\bar{k}_C \arrow[u] \arrow[r] & \cdots \\
{}_C *_C \arrow[u, "{}_B\bar{h}_A\otimes\cdot"] \arrow[r, "\cdot\otimes_C k_D"'] & {}_C k_D \arrow[u] \arrow[r, "\cdot\otimes_D \bar{k}_C"'] & {}_C k\bar{k}_C \arrow[u] \arrow[r, "\cdot\otimes_C k_D"'] & \cdots
\end{tikzcd}$$

FIGURE 19. Double sequence of bimodules (2)

EXAMPLE 7.3 (General case: fusion rule subalgebras:). Let $A \subset B$ and $C \subset D$ be two subfactors of the hyperfinite II_1 factor with finite indices and finite depths. Suppose that the A-A bimodule system ${}_A\mathcal{M}_A$ from $A \subset B$ is a subsystem of the C-C bimodule system ${}_C\mathcal{M}_C$ from $C \subset D$, i.e., ${}_A\mathcal{M}_A \subset {}_C\mathcal{M}_C$ holds. We take the generator bimodule ${}_A h_B$ (resp. ${}_C k_D$) of the subfactor $A \subset B$ (resp. $C \subset D$) and consider similar construction of double sequence of bimodules as in Section 5. That is, we take first a trivial bimodule ${}_C C_C$ denoted by ${}_C *_C$, then we draw a new bimodule to the right by tensoring ${}_C k_D$ and ${}_D \bar{k}_C$ from the right alternately and draw a bimodule upward by tensoring ${}_B\bar{h}_A$ and ${}_A h_B$ from the left alternately. Here ${}_B\bar{h}_A$ (resp. ${}_D\bar{k}_C$) is the conjugate bimodule of ${}_A h_B$ (resp. ${}_C k_D$). Thus we get a double sequence of bimodules as in Figure 19.

Take endomorphism spaces of the bimodules in Figure 19, we get a double sequence of finite dimensional C*-algebras which forms commuting squares as in Figures 20 and 21.

Thus we obtain a 4-partite graph and a ${}_C *_C$-flat biunitary connection on the graph in the same way as Section 5. Here we note that the vertical graphs in Figure 19 are disconnected in

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\cup & & \cup & & \cup & & \\
\mathrm{End}({}_C h\bar{h}_C) & \subset & \mathrm{End}({}_C h\bar{h}k_D) & \subset & \mathrm{End}({}_C h\bar{h}k\bar{k}_C) & \subset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathrm{End}({}_B \bar{h}_C) & \subset & \mathrm{End}({}_B \bar{h}k_D) & \subset & \mathrm{End}({}_B \bar{h}k\bar{k}_C) & \subset & \cdots \\
\cup & & \cup & & \cup & & \\
\mathrm{End}({}_C C_C) & \subset & \mathrm{End}({}_C k_D) & \subset & \mathrm{End}({}_C k\bar{k}_C) & \subset & \cdots
\end{array}$$

FIGURE 20. Double sequence of endomorphism spaces

$$\begin{array}{ccccccccc}
P = P_0 & \subset & Q = P_1 & \subset & P_2 & \subset & \cdots & \subset & P_\infty = N_\infty \\
\cup & & \cup & & \cup & & & & \cup \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
B_{0,2} & \subset & B_{1,2} & \subset & B_{2,2} & \subset & \cdots & \subset & N_2 \\
\cup & & \cup & & \cup & & & & \cup \\
B_{0,1} & \subset & B_{1,1} & \subset & B_{2,1} & \subset & \cdots & \subset & M = N_1 \\
\cup & & \cup & & \cup & & & & \cup \\
B_{0,0} & \subset & B_{1,0} & \subset & B_{2,0} & \subset & \cdots & \subset & N = N_0
\end{array}$$

FIGURE 21. The commuting squares arising from endomorphism spaces of the bimodules

general. Actually the connected components of the vertical graphs are the coset fusion graphs by the generator ${}_A h_B$. Hence the von Neumann algebras P_k $(k \geq 1)$ in Figure 21 do not become factors in general.

By applying the mixed quantum double construction to the flat connection, we obtain a subfactor which is isomorphic to the asymptotic inclusions of a subfactor $A \subset B$. We call this construction the *mixed quantum double construction from $A \subset B$ and $C \subset D$*.

EXAMPLE 7.4 ($E_6^{\pm}$ subsystem in the even part of the dual GHJ subfactors of type E_6:). There are two mutually complex conjugate E_6 biunitary connections. Here we denote them by $E_6^{\pm}$. The (dual) principal graph of the GHJ subfactor of type E_6 corresponding to the vertex e_0 as in Figure 22 is given in Figure 24. Let $A \subset B$ be the dual GHJ subfactors of type E_6 and $C \subset D$ be one of the $E_6^{\pm}$ subfactor. The A-A bimodule system arising from $A \subset B$ is given by the even vertices of the left hand side of Figure 26 (see [17]). Therefore the A-A bimodule system contains C-C bimodule system from $C \subset D$ as a subsystem. So we can

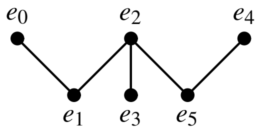


FIGURE 22. The label of vertices of the Dynkin diagram E_6

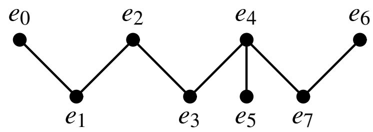


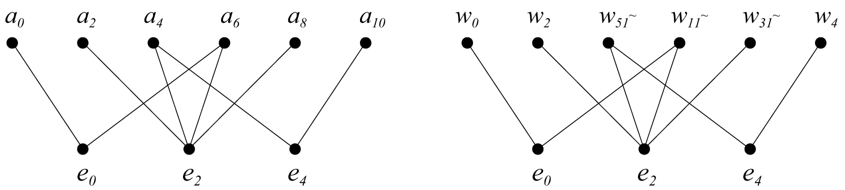
FIGURE 23. The label of vertices of the Dynkin diagram E_8

The adjacency matrix of the principal graph

	e_0	e_2	e_4
a_0	1	0	0
a_2	0	1	0
a_4	0	1	1
a_6	1	1	0
a_8	0	1	0
a_{10}	0	0	1

The adjacency matrix of the dual principal graph

	e_0	e_2	e_4
w_0	1	0	0
w_2	0	1	0
$w_{51}\tilde{}$	0	1	1
$w_{11}\tilde{}$	1	1	0
$w_{31}\tilde{}$	0	1	0
w_4	0	0	1



The principal graph of GHJ (E_6, $* = e_0$) The dual principal graph of GHJ (E_6, $* = e_0$)

FIGURE 24. The (dual) principal graph of GHJ$(E_6, * = e_0)$

apply the previous construction in Example 7.3 and the mixed quantum double subfactor is isomorphic to the asymptotic inclusions of $E_6^{\pm}$ subfactor.

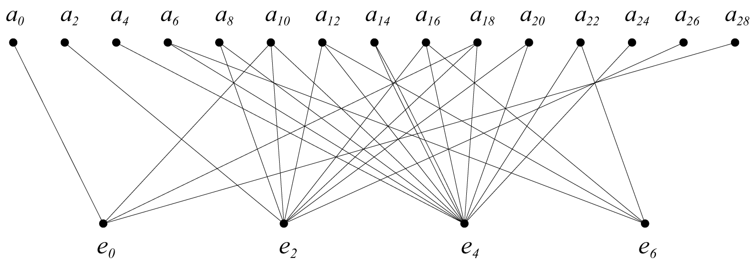
EXAMPLE 7.5 ($E_8^{\pm}$ subsystem in the even part of the dual GHJ subfactors of type E_8:). Similarly to the E_6 connections, there are two mutually complex conjugate E_8 biunitary connections. We denote them by $E_8^{\pm}$. The (dual) principal graph of the GHJ subfactor of type E_8 corresponding to the vertex e_0 as in Figure 23 is given in Figure 25. Let $A \subset B$ be the dual GHJ subfactors of type E_8 and $C \subset D$ be one of the $E_8^{\pm}$ subfactor. The A-A bimodule system arising from $A \subset B$ is given by the even vertices of the right hand side of Figure 26 (see [17]). By applying the construction in Example 7.3, we get the mixed quantum double subfactor which is isomorphic to the asymptotic inclusions of $E_8^{\pm}$ subfactor.

The adjacency matrix of the principal graph

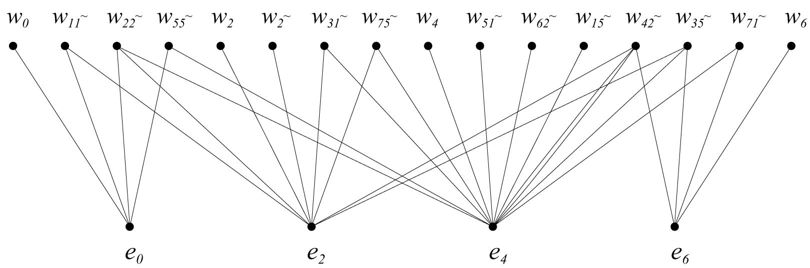
	e_0	e_2	e_4	e_6
a_0	1	0	0	0
a_2	0	1	0	0
a_4	0	0	1	0
a_6	0	0	1	1
a_8	0	1	1	0
a_{10}	1	1	1	0
a_{12}	0	1	1	1
a_{14}	0	0	2	0
a_{16}	0	1	1	1
a_{18}	1	1	1	0
a_{20}	0	1	1	0
a_{22}	0	0	1	1
a_{24}	0	0	1	0
a_{26}	0	1	0	0
a_{28}	1	0	0	0

The adjacency matrix of the dual principal graph

	e_0	e_2	e_4	e_6
w_0	1	0	0	0
$w_{11}^{\sim}$	1	1	0	0
$w_{22}^{\sim}$	1	1	1	0
$w_{55}^{\sim}$	1	0	1	0
w_2	0	1	0	0
$w_2^{\sim}$	0	1	0	0
$w_{31}^{\sim}$	0	1	1	0
$w_{75}^{\sim}$	0	1	1	0
w_4	0	0	1	0
$w_{51}^{\sim}$	0	0	1	0
$w_{62}^{\sim}$	0	0	1	0
$w_{15}^{\sim}$	0	0	1	0
$w_{42}^{\sim}$	0	1	2	1
$w_{35}^{\sim}$	0	1	1	1
$w_{71}^{\sim}$	0	0	1	1
w_6	0	0	0	1



The principal graph of GHJ (E_8, $* = e_0$)



The dual principal graph of GHJ (E_8, $* = e_0$)

FIGURE 25. The (dual) principal graph of the GHJ subfactor corresponding to $(E_8, * = e_0)$

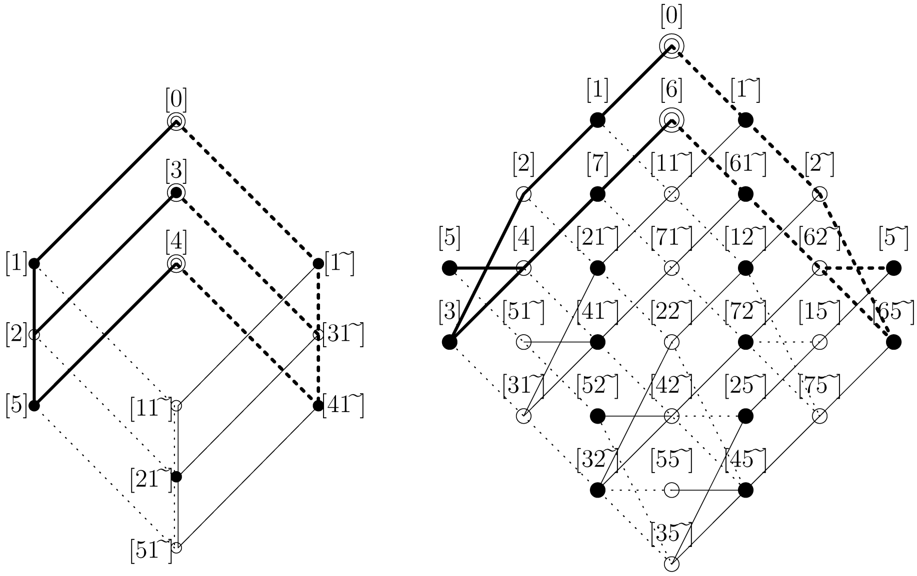


FIGURE 26. The fusion graphs of all K-K connections for $K = E_6$ and E_8

References

[1] M. ASAEDA, Quantum multiple construction of subfactors, Canad. Math. Bull. **51** (2008), 321–333.

[2] M. CHODA, Index for factors generated by Jones' two sided sequence of projections, Pacific J. Math. **139** (1989), 1–16.

[3] V. G. DRINFEL′D, Quantum groups, Proc. ICM-86, Berkeley, 798–820.

[4] J. ERLIJMAN, New subfactors from braid group representations, Trans. Amer. Math. Soc. **350** (1998), 185–211.

[5] J. ERLIJMAN, Two-sided braid groups and asymptotic inclusions, Pacific J. Math. **193** (2000), 57–78.

[6] J. ERLIJMAN, Multi-sided braid type subfactors, Canad. J. Math. **53** (2001), 546–564.

[7] J. ERLIJMAN, Multi-sided braid type subfactors. II., Canad. Math. Bull. **46** (2003), 80–94.

[8] J. ERLIJMAN and H. WENZL, Subfactors from braided C^* tensor categories, Pacific J. Math. **231** (2007), 361–399.

[9] D. E. EVANS and Y. KAWAHIGASHI, The E_7 commuting squares produce D_{10} as principal graph, Publ. RIMS. Kyoto Univ. **30** (1994), 151–166.

[10] D. E. EVANS and Y. KAWAHIGASHI, Subfactors and conformal field theory, in "*Quantum and non-commutative analysis*," Kluwer Academic (1993), 341–369.

[11] D. E. EVANS and Y. KAWAHIGASHI, On Ocneanu's theory of asymptotic inclusions for subfactors, topological quantum field theories and quantum doubles, Internat. J. Math. **6** (1995), 205–228.

[12] D. E. EVANS and Y. KAWAHIGASHI, *Quantum symmetries on operator algebras*, Oxford Mathematical Monographs, Oxford University Press, New York (1998).

[13] F. GOODMAN, P. DE LA HARPE and V. F. R. JONES, *Coxeter graphs and towers of algebras*, MSRI publications 14, Springer (1989).

[14] S. GOTO, Orbifold construction for non-AFD subfactors, Internat. J. Math. **5** (1994), 725–746.

[15] S. GOTO, Quantum double construction for subfactors arising from periodic commuting squares, J. Math. Soc. Japan **52** (2000), 187–198.

[16] S. GOTO, On Ocneanu's theory of double triangle algebras for subfactors and classification of irreducible connections on the Dynkin diagrams, Expo. Math. **28** (2010), 218–253.

[17] S. GOTO, On the fusion algebras of bimodules arising from Goodman-de la Harpe-Jones subfactors, J. Math. Sci. Univ. Tokyo **19** (2012), 409–506.

[18] S. GOTO, A generalization of quantum multiple construction of subfactors, in preparation.

[19] S. GOTO, On generalized Goodman-de la Harpe-Jones subfactors of type D-E, in preparation.

[20] S. GOTO, On the flat part of connections between ADE Dynkin diagrams, in preparation.

[21] V. F. R. JONES, Index for subfactors, Invent. Math. **72** (1983), 1–15.

[22] V. F. R. JONES, A polynomial invariant for knots via von Neumann algebras, Bull. Amer. Math. Soc. **12** (1985), 103–112.

[23] Y. KAWAHIGASHI, On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors, J. Funct. Anal. **127** (1995), 63–107.

[24] Y. KAWAHIGASHI, Quantum doubles and orbifold subfactors, in "*Operator Algebras and Quantum Field Theory*," S. Doplicher, R. Longo, J. Roberts, L. Zsido eds, International Press (1997), 271–283.

[25] Y. KAWAHIGASHI, Quantum Galois correspondence for subfactors, J. Funct. Anal. **167** (1999), 481–497.

[26] Y. KAWAHIGASHI, Conformal quantum field theory and subfactors, Acta Math. Sin. **19** (2003), 557–566.

[27] Y. KAWAHIGASHI, Subfactor theory and its applications—operator algebras and quantum field theory—, in "*Selected Papers on Differential Equations Analysis*," Amer. Math. Soc. Transl. 215, Amer. Math. Soc. (2005), 97–108.

[28] Y. KAWAHIGASHI, Topological quantum field theories and operator algebras, in "*Quantum Field Theory and Noncommutative Geometry*," Lect. Notes in Phys. 662, Springer Verlag (2005), 241–253.

[29] Y. KAWAHIGASHI, Conformal field theory and operator algebras, in "*New Trends in Mathematical Physics*," Springer (2009), 345–356.

[30] Y. KAWAHIGASHI, N. SATO and M. WAKUI, $(2+1)$-dimensional topological quantum field theory from subfactors and Dehn surgery formula for 3-manifold invariants, Adv. Math. **195** (2005), 165–204.

[31] R. LONGO and K.-H. REHREN, Nets of subfactors, Rev. Math. Phys. **7** (1995), 567–597.

[32] T. MASUDA, An analogue of Longo's canonical endomorphism for bimodule theory and its application to asymptotic inclusions, Internat. J. Math. **8** (1997), 249–265.

[33] T. MASUDA, Generalization of Longo-Rehren construction to subfactors of infinite depth and amenability of fusion algebras, J. Funct. Anal. **171** (2000), 53–77.

[34] A. OCNEANU, Quantized group string algebras and Galois theory for algebras, in "*Operator algebras and applications, Vol. 2* (Warwick, 1987)," London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press (1988), 119–172.

[35] A. OCNEANU, Quantum symmetry, differential geometry of finite graphs and classification of subfactors, University of Tokyo Seminary Notes 45, Notes recorded by Y. Kawahigashi (1991).

[36] A. OCNEANU, Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors, in *Lectures on operator theory*, (ed. B. V. Rajarama Bhat et al.), The Fields Institute Monographs, Providence, Rhode Island: AMS Publications (2000), 243–323.

[37] S. POPA, Classification of amenable subfactors of type II, Acta Math. **172** (1994), 253–345.

[38] S. POPA, An axiomatization of the lattice of higher relative commutants of a subfactor, Invent. Math. **120** (1995), 427–445.

[39] S. POPA, Symmetric enveloping algebras, amenability and AFD properties for subfactors, Math. Res. Lett. **1** (1994), 409–425.

[40] N. SATO, Two subfactors arising from a non-degenerate commuting square. An answer to a question raised by V. F. R. Jones, Pacific J. Math. **180** (1997), 369–376.

[41] N. SATO, Two subfactors arising from a non-degenerate commuting square. II. Tensor categories and TQFT's,

Internat. J. Math. **8** (1997), 407–420.
[42] H. WENZL, Hecke algebras of type A and subfactors, Invent. Math. **92** (1988), 345–383.
[43] H. WENZL, Quantum groups and subfactors of type B, C, and D, Comm. Math. Phys. **133** (1990), 383–432.

Present Address:
DEPARTMENT OF INFORMATION AND COMMUNICATION SCIENCES,
FACULTY OF SCIENCE AND TECHNOLOGY,
SOPHIA UNIVERSITY,
7–1 KIOI-CHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.
e-mail: s-goto@sophia.ac.jp