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On the Moduli Space of Pointed Algebraic Curves of Low Genus III —Positive Characteristic—

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Abstract. In his classical work, Pinkham discovered a beautiful theorem on the moduli space of pointed algebraic curves with a fixed Weierstrass gap sequence at the marked point. Namely, the complement of a Weierstrass gap sequence in the set of non-negative integers is a numerical semigroup, and he described such a moduli space in terms of the negative part of the miniversal deformation space of the monomial curve of this semigroup.

Unfortunately, his theorem holds only in characteristic 0 and does not hold in positive characteristic in general. In this paper, we will study his theorem in positive characteristic, and give a fairly sharp condition for his theorem to hold in positive characteristic up to genus 4. As an application, we present a complete analysis of his theorem in positive characteristic in the low genus case.

1. Introduction

In [10], Pinkham developed a beautiful theory on the moduli space of pointed algebraic curves with a given Weierstrass gap sequence at the marked point in characteristic 0. More precisely, let (C, P) $(P \in C)$ be a pointed smooth projective curve of genus g with a Weierstrass gap sequence $J = \{j_1, \ldots, j_g\} \subset \mathbf{N}_0 := \{0, 1, 2, \ldots\}$ at P. Then the complement $N = \mathbf{N}_0 \setminus J$ is a numerical semigroup of genus g.

Let N be any numerical semigroup of genus g and $M_{g,1}^N$ the moduli space of pointed smooth projective curves with a Weierstrass gap sequence $\mathbf{N}_0 \setminus N$ at the marked point. Let X_N be the affine monomial curve defined by N. Pinkham showed that $M_{g,1}^N$ is described by the negative part of miniversal deformation space of the monomial curve X_N if the general fiber of this deformation is smooth. Unfortunately, Pinkham's theorem holds only in characteristic 0 and he said little about positive characteristic case except two short comments (see Remark 3).

With the advance of computer algebra, the algorithm and programs for computing the miniversal deformation of an affine variety have been developed ([5], [6]), and Pinkham's theory has revived as a computational tool for $M_{g,1}^N$. Indeed, in the previous papers under the

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same title [7], [8], we computed $M_{g,1}^N$ explicitly and showed their rationality up to genus 6 in the case where N is generated by 4 or less elements in characteristic 0.

The computation of miniversal deformation spaces needs some hard Groebner bases calculation, and sometimes the appearance of huge coefficients is the obstruction to complete it. Thus in the positive characteristic p case, where huge coefficients do not appear if p is relatively small, we may have a good chance of computing $M_{g,1}^N$ if Pinkham's theorem is available in positive characteristic.

The purpose of this paper is to study Pinkham's theorem in positive characteristic. Our main result is Theorem 4, which gives a fairly sharp sufficient condition for Pinkham's theorem to hold in positive characteristic up to genus 4. Though we have not succeeded in proving a general result which hold for a numerical semigroup of any genus, we present a candidate for the general statement in positive characteristic (Remark 2).

The contents of this paper are as follows. In Section 2, we review the part of Pinkham's theorem which holds in any characteristic. Namely, the definition of Pinkham's map Π and the surjectivity of Π hold in any characteristic, and we will review them in detail.

On the other hand, the injectivity of Π does not hold in positive characteristic in general. Thus, in Section 3, we will study the injectivity of Π in positive characteristic and prove our main results. More precisely, we first present a general guiding problem for the injectivity of Π in positive characteristic (Problem 1). We then show that Problem 1 is affirmatively solved if the numerical semigroup is generated by 2 elements, which is the simplest case (Theorem 3). We next reduce Problem 1 to the inductive assumption of the triviality of the automorphism and the transformation of the generators (Lemma 1). Using Lemma 1, we prove our main result (Theorem 4).

In Section 4, we work out the low genus case ($g = 1, 2$) completely and determine in which characteristic Pinkham's map is bijective. Finally, in Section 4.3, we will report that in the case of semigroup $N = N(4)_7$, which is the semigroup of an ordinary point of a genus 4 curve, we have succeeded in computing the equation of the moduli space $M_{4,1}^N$ in characteristic 7. This is an exciting result for us since, in characteristic 0, the computation of $M_{4,1}^N$ is far beyond our computational capability.

2. Review and confirmation of Pinkham's theorem in any characteristic

In this section, we review Pinkham's theorem on the moduli space of pointed algebraic curves with a given Weierstrass gap sequence at the marked point ([10, Theorem (13.9)]). Pinkham's theorem consists of 3 parts: the definition of Pinkham's map Π, the surjectivity of Π and the injectivity of Π. The definition and the surjectivity of Π in characteristic 0 works also in positive characteristic, whereas the injectivity of Π does not hold in positive characteristic in general. We will recall and confirm the definition and the surjectivity of Π in any characteristic here.

2.1. Monomial Curves of numerical semigroups. Let $\mathbf{N}_0 := \{0, 1, 2, \dots\}$ be the semigroup of non-negative integers with respect to addition. We call a subsemigroup $N \subset \mathbf{N}_0$ *a numerical semigroup of genus* g if the compliment $\mathbf{N}_0 \setminus N$ is finite and consists of g elements. A numerical semigroup N is always finitely generated as a semigroup and we write $N = \langle a_1, \dots, a_n \rangle = \sum_{j=1}^{n} \mathbf{N}_0 a_j$ for a generating set $\{a_1, \dots, a_n\}$ of N. In this note, we always take the minimal canonical generators of N. Namely, a_1 is the least element of $N \setminus \{0\}$, and a_2 is the least element of $N \setminus \langle a_1 \rangle$ and so forth.

Let K be an algebraically closed field of any characteristic $p = \mathrm{char}(K) \geq 0$ and we work over K throughout this note. Let $K[t]$ be a univariate polynomial ring over K with an indeterminate t. For a given numerical semigroup N, we denote by $K[t^N] \subset K[t]$ the subalgebra generated by $\{t^j \mid j \in N\}$ and call $K[t^N]$ *the monomial ring of* N. In case $N = \langle a_1, \dots, a_n \rangle$, we have $K[t^N] = K[t^{a_1}, \dots, t^{a_n}]$. We call the affine curve $X_N := \mathrm{Spec}\, K[t^N]$ *the monomial curve of* N. Since a monomial ring $K[t^N]$ has a natural grading, a monomial curve X_N has an induced $K^\times$-action, where $K^\times$ is a 1-dimensional algebraic torus.

Let $N = \langle a_1, \dots, a_n \rangle$ be a numerical semigroup and $K[x] = K[x_1, \dots, x_n]$ a polynomial ring over K with n indeterminates. Consider a surjective homomorphism $\varphi : K[x] \to K[t^N]$ defined by $\varphi(x_i) := t^{a_i}$. By giving weights $w(x_i) := a_i$ $(1 \leq i \leq n)$ and $w(t) := 1$, φ is homogeneous of degree 0. We call the kernel I_N of φ the defining ideal of the monomial curve X_N since $K[x]/I_N \cong K[t^N]$. By defining a $K^\times$-action on $\mathbf{A}^n = \mathrm{Spec}\, K[x]$ by $\lambda \circ (x_1, \dots, x_n) := (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n)$ $(\lambda \in K^\times)$, the closed embedding $X_N \hookrightarrow \mathbf{A}^n$ induced by $K[x] \to K[t^N]$ is $K^\times$-equivariant.

On the other hand, let (C, P) $(P \in C)$ be a pointed smooth projective curve of genus g over K and set

$$N_P := \{j \in \mathbf{N}_0 \mid \exists \text{ a rational function } f \text{ on } C \text{ which has a pole of order } j \text{ at } P\}.$$

Then N_P is a numerical semigroup of genus g and $\mathbf{N}_0 \setminus N_P$ is the Weierstrass gap sequence at P.

Let $M_{g,1}$ be the coarse moduli space of pointed smooth projective curves of genus g. For a given numerical semigroup N of genus g, we set

$$M_{g,1}^N := \{(C, P) \in M_{g,1} \mid N_P = N\} \subset M_{g,1}.$$

For a given g, let $\{N_1, \dots, N_l\}$ be the set of all the numerical semigroups of genus g (there exist only a finite number of them). Then $M_{g,1}$ is decomposed as $M_{g,1} = \bigcup_{j=1}^{l} M_{g,1}^{N_j}$ (disjoint union) where $M_{g,1}^{N_j}$ is a locally closed subset of $M_{g,1}$.

2.2. Deformation of monomial curves. Pinkham's theorem describes $M_{g,1}^N$ for a semigroup N of genus g by the negative miniversal deformation space of the monomial curve X_N.

Let N be a semigroup of genus g and $X := X_N$ its monomial curve. Since X has a unique singularity at the origin, we have a formal miniversal deformation $\Phi : \chi \to S$ of X.

We may set

$$\chi = \operatorname{Spec} K[[x_1, \dots, x_n, s_1, \dots, s_\tau]]/\langle F \rangle, \quad S = \operatorname{Spec} K[[s_1, \dots, s_\tau]]/\langle J \rangle\,,$$

and Φ is a projection map where $F = (F_1, \dots, F_k) \in K[[x, s]]^k$, $J = (J_1, \dots, J_l) \in K[[s]]^l$, and $\langle F \rangle$ (resp. $\langle J \rangle$) is the ideal generated by F (resp. J) in the formal power series ring $K[[x, s]]$ (resp. $K[[s]]$) . Thus we have the following commutative diagram:

$$\begin{array}{ccc} X \cong \Phi^{-1}(O) & \hookrightarrow & \chi \\ \downarrow & & \downarrow \Phi \\ O = \operatorname{Spec} K & \hookrightarrow & S \end{array}$$

where $O \in S$ is the origin. We note that, since X has a $K^\times$-action, we have an induced $K^\times$-action on χ and S such that Φ is $K^\times$-equivariant ([10, Proposition (2.3)]). We may suppose $K^\times$ acts on χ and S by $\lambda \circ x_i = \lambda^{a_i} x_i$, $\lambda \circ s_j = \lambda^{-e_j} s_j$ $(a_i, e_j \in \mathbf{Z})$. The ideals $\langle F \rangle$ and $\langle J \rangle$ are $K^\times$-invariant (namely homogeneous) ideals.

Pinkham uses the negative weight part of the base space S. Namely, let $\{j_1, \dots, j_m\} \subset \{1, 2, \dots, \tau\}$ be the set of indices such that the weight $e_{j_k} > 0$ and set $S_{(-)} := \{s_{j_1} = \cdots = s_{j_m} = 0\} \cap S$. By restricting Φ to $S_{(-)}$, we have an induced deformation $\Phi_{(-)} : \chi_- \to S_{(-)}$ of X, which we call *the negative miniversal deformation of* X.

The negative miniversal deformation Φ_- can be constructed directly starting from the tangent space of S_- at the origin by Schlessinger's method [11] as follows.

Let $I = I_N$ be the defining ideal of X and $f = (f_1(x), \dots, f_k(x)) \in K[x]^k$ a set of minimal generators of I so that $X = \mathbf{V}(f) \subset \mathbf{A}^n$ ($\mathbf{V}(f)$ is the affine variety defined by f). Then the embedded deformation of $X \subset \mathbf{A}^n$ over the double point $T = \operatorname{Spec} K[t]/\langle t^2 \rangle$ is classified by the normal module $\operatorname{Nor}_{\mathbf{A}^n}(X) := \operatorname{Hom}_{K[x]}(I, K[x]/I)$ of X. We note an element $n \in \operatorname{Nor}_{\mathbf{A}^n}(X)$ is given by $n = (n_1, \dots, n_k) \in (K[x]/I)^k$ such that $\sum_{j=1}^k r_k n_k = 0$ for any relation $r = (r_1, \dots, r_k)$ of $f = (f_1, \dots, f_k)$. We can give a natural grading to $\operatorname{Nor}_{\mathbf{A}^n}(X)$ as follows: $n = (n_1, \dots, n_k)$ is homogeneous of degree d if and only if each n_j is homogeneous of $\deg n_j = \deg f_j + d$.

Let $\Theta_n := \left\{ \sum_{j=1}^n c_j \frac{\partial}{\partial x_j} \mid c_j \in K[x] \right\}$ be the module of derivations on $\mathbf{A}^n$. Then we have a natural map $\Theta_n \to \operatorname{Nor}_{\mathbf{A}^n}(X)$, $\Theta \ni \theta \mapsto (f \mapsto \theta(f))$. We define $T^1(X) := \operatorname{coker}(\Theta_n \to \operatorname{Nor}_{\mathbf{A}^n}(X))$, which is finite-dimensional and classifies the deformations of X over the double point.

Since $\operatorname{Nor}_{\mathbf{A}^n}(X)$ is graded and the map $\Theta_n \to \operatorname{Nor}_{\mathbf{A}^n}(X)$ is homogeneous, $T^1(X)$ has an induced grading and is decomposed as $T^1(X) = \oplus_{j=-\infty}^{\infty} T_j^1(X)$, where $T_j^1(X)$ is the j-th homogeneous part. The subspace $T_j^1(X)$ is the eigenspace of weight j with respect to the $K^\times$-action on $T^1(X)$. Further we set $T_-^1(X) := \oplus_{j \le 0} T_j^1(X)$ (we have $T_0^1(X) = \{0\}$). We note that $T_-^1(X)$ is naturally isomorphic to the Zariski tangent space of S_- at the origin. More precisely, let $\{\overline{g_1}, \dots, \overline{g_p}\}$ be the K-basis of $T_-^1(X)$ where $g_j \in K[x]^k$ and $\overline{g_j}$ is the class of g_j

in $T^1_-(X)$. Then the defining equation F of χ_- satisfies $F(x,s) = f + \sum_{j=1}^{p} s_j g_j \pmod{s^2}$. Starting from this, we can construct $F(x,s)$ degree by degree in s taking obstructions into account (see [10, (2.9)] for a summary). This construction of $\Phi_- : \chi_- \to S_-$ comes to an end in finitely many steps since everything is homogeneous. Thus the formal negative miniversal deformation $\Phi_- : \chi_- \to S_-$ is actually given by polynomials and we can set

$$\Phi_- : \chi_- = \operatorname{Spec} K[x,s]/\langle F\rangle,\ \ S_- = \operatorname{Spec} K[s]/\langle J\rangle\,,$$

$F = (F_1, \ldots, F_k) \in K[x,s]^k$, $J = (J_1, \ldots, J_l) \in K[s]^l$, $x = (x_1, \ldots x_n)$, $s = (s_1, \ldots, s_p)$. We assume $K^\times$ acts on x_i with weight a_i and on s_j with weight $-e_j > 0$.

2.3. Definition of Pinkham's map Π. We finally projectivize each fiber of Φ_- by adding a point. More precisely, introduce a new indeterminate x_{n+1} with weight 1. Then, for each $F_i(x,s) \in F$, replace s_j by $s_j x_{n+1}^{-e_j}$ to get $\overline{F_i} = \overline{F_i}(x_1, \ldots, x_n, x_{n+1}, s_1, \ldots, s_p) \in K[x, x_{n+1}, s]$. Set $\overline{F} = (\overline{F_1}, \ldots, \overline{F_k})$ and $\overline{A} := K[x, x_{n+1}, s]/\langle\overline{F}\rangle$. We define

$$\overline{\Phi_-} : \overline{\chi_-} := \operatorname{Proj}(\overline{A}) \to S_-\,,$$

which is a one-point projectivization of Φ_-. The fiber space $\overline{\Phi_-}$ is a family of projective curves and $\{x_{n+1} = 0\}$ is a section of $\overline{\Phi_-}$ whose intersection with each fiber is a point.

THEOREM 1 (Definition of Pinkham's map). *Suppose the smooth locus $U := \{Q \in S_- \mid \Phi_-^{-1}(Q)$ is smooth$\}$ of Φ_- is non-empty. Then for any $Q \in U$, $(\overline{\Phi_-}^{-1}(Q), \overline{\Phi_-}^{-1}(Q) \cap \{x_{n+1} = 0\})$ is a pointed smooth projective curve of genus g with a numerical group N at the marked point. Thus the restriction of $\overline{\Phi_-}$ to U is a family of pointed smooth projective curves of genus g with a numerical semigroup N at the point. Since all the fibers are isomorphic over a $K^\times$-orbit in U, we have a morphism $\Pi : U/K^\times \to M^N_{g,1}$, which we call Pinkham's map in this note.*

The proof of Theorem 1 in characteristic 0 ([10, Theorem (13.10)]) works without any change in positive characteristic. Since we need some key point in the proof of this theorem later, we outline the proof below.

PROOF OF THEOREM 1. Let $O(Q) \subset U$ be the $K^\times$-orbit of Q and $\mu : T = \mathbf{A}^1 \to \overline{O(Q)}$ the normalization of the closure $\overline{O(Q)}$ of $O(Q)$. Suppose $K^\times$ acts on $T = \mathbf{A}^1$ with weight $w > 0$. Let $\nu : \mathbf{A}^1 \to \mathbf{A}^1$, $t \mapsto t^w$ be a branched covering. We take the pull-back of $\chi_- \to S_-$ by $\mu \circ \nu$ to get a one parameter deformation $\Omega \to \mathbf{A}^1$ of X such that $K^\times$ acts on the base $\mathbf{A}^1$ with weight 1.

On the other hand, consider the affine fiber space

$$\overline{\Phi_-}^{\mathrm{aff}} : W = \operatorname{Spec} \overline{A} \to S_- \quad (\overline{A} = K[x, x_{n+1}, s]/\langle\overline{F}\rangle)\,,$$

whose fiber is an affine surface.

A key observation is that the fiber W_Q of $\overline{\Phi_-}^{\mathrm{aff}}$ over Q is isomorphic to Ω, where $\Omega \to \mathbf{A}^1$ is the one-parameter deformation of X constructed in the previous paragraph. From this, it follows that W_Q is a normal affine surface with a $K^\times$-action and has a unique singular point at the origin.

Then we take a canonical equivariant resolution $Z \to W_Q$, which works in any characteristic by [9]. The affine surface Z is a blow-up of W_Q at the origin and turns out to be smooth. Further Z is a total space of a line bundle L over the smooth projective curve $E = \overline{\Phi_-}^{-1}(Q)$. In fact, let $P = \overline{\Phi_-}^{-1}(Q) \cap \{x_{n+1} = 0\}$ be the marked point of E. Then $L = O_E(-P)$ and by contracting the zero-section of L, we get W_Q. From this description, it follows that E has genus g and the semigroup at P is N. Thus the proof is done. □

2.4. Surjectivity of Π. We here discuss the surjectivity of Π in detail. The proof of surjectivity of Π in characteristic 0 given in [10, (13.11)] works also in positive characteristic. However, since the proof there is too sketchy to confirm that it works in positive characteristic, we will give a detailed proof here for confirmation.

THEOREM 2 (Surjectivity of Π). *Let $\Phi_- : \chi_- \to S_-$ be the negative miniversal deformation of a monomial curve $X = X_N$. Suppose the smooth locus U of Φ_- is non-empty so that Pinkham's map Π is defined. Then $\Pi : U/K^\times \to M_{g,1}^N$ is surjective.*

PROOF. Let (C, P) be a pointed smooth projective curve with a numerical semigroup $N = N_P$ at P. Let $L := O_C(P)$ be the line bundle associated to the one-point divisor P and set $B := \oplus_{j=0}^{\infty} \Gamma(C, L^j)$ where $\Gamma(C, L^j)$ is the vector space of sections of L^j. Since L is ample, B is a graded normal domain of dimension 2.

Suppose $N = \langle a_1, \dots, a_n \rangle$. Then it is easy to show there exists $x_i \in \Gamma(X, L^{a_i})$ $(1 \le i \le n)$ and $x_{n+1} \in \Gamma(X, L)$ with $\operatorname{div}(x_{n+1}) = P$ such that $B = K[x_1, \dots, x_{n+1}]$, where $\operatorname{div}(x_{n+1})$ is the divisor associated to the section x_{n+1} of L.

Set $V := \operatorname{Spec} B$, which is a normal affine surface. From the ring homomorphism $K[x_{n+1}] \hookrightarrow K[x_1, \dots, x_{n+1}] = B$, we have a $K^\times$-equivariant projection $\pi : V = \operatorname{Spec} B \to \operatorname{Spec} K[x_{n+1}] = \mathbf{A}^1$. Since V is irreducible and π is surjective, π is flat by [3, Proposition 9.7].

We first show that $\pi^{-1}(O) \cong X$, where $O \in \mathbf{A}^1$ is the origin. Indeed, consider the injective linear map $x_{n+1}\cdot : \Gamma(C, L^{j-1}) \to \Gamma(C, L^j)$, $u \mapsto x_{n+1}u$ $(u \in \Gamma(C, L^{j-1}))$ for any $j > 0$. Then, by the definition of the semigroup N at P, we have

$$\Gamma(C, L^j) = \begin{cases} x_{n+1} \cdot \Gamma(C, L^{j-1}) & \text{if } j \in \mathbf{N}_0 \setminus N \\ x_{n+1} \cdot \Gamma(C, L^{j-1}) \oplus K \cdot u_j & \text{if } j \in N \end{cases}$$

for some $u_j \in \Gamma(C, L^j) \setminus x_{n+1} \cdot \Gamma(C, L^{j-1})$. Let $\overline{u_j} \in B/\langle x_{n+1} \rangle$ be the class of u_j. Then we have

$$\pi^{-1}(O) = \operatorname{Spec}(B/\langle x_{n+1} \rangle) \cong \operatorname{Spec}(\oplus_{j \in N} K \cdot \overline{u_j}).$$

By the correspondence $\overline{u_j} \leftrightarrow t^j \in K[t]$, we get an isomorphism

$$\pi^{-1}(O) \cong \operatorname{Spec}(\oplus_{j\in N} K \cdot t^j) = X\,.$$

We next show that for any $Q \in \mathbf{A}^1 \setminus \{O\}$, $\pi^{-1}(Q)$ is isomorphic to the affine curve $C \setminus \{P\}$. Since $K^\times$ acts on $\mathbf{A}^1 \setminus O$ transitively, we may take a point $Q = 1 \in K = \mathbf{A}^1$. Suppose $V = \operatorname{Spec} B$ is defined by $g = (g_1(y, y_{n+1}), \dots, g_m(y, y_{n+1})) \in K[y, y_{n+1}]^m$ so that $V = \mathbf{V}(g)$, where $y = (y_1, \dots, y_n)$. Then $\pi^{-1}(Q) \subset \mathbf{A}^n$ is defined by $g(y, 1) = (g_1(y, 1), \dots, g_m(y, 1))$. We have

$$\pi^{-1}(Q) = \operatorname{Spec}(K[y]/\langle g(y, 1)\rangle) \cong \operatorname{Spec} K\left[\frac{x_1}{x_{n+1}^{a_1}}, \dots, \frac{x_n}{x_{n+1}^{a_n}}\right],$$

where the right-hand isomorphism is given by $y_i \longleftrightarrow \frac{x_i}{x_{n+1}^{a_i}}$. Since $K\left[\frac{x_1}{x_{n+1}^{a_1}}, \dots, \frac{x_n}{x_{n+1}^{a_n}}\right]$ is the coordinate ring of the affine curve $C \setminus \{P\}$, we have $\pi^{-1}(Q) \cong C \setminus \{P\}$ as desired.

Thus we have shown that $\pi : V \to \mathbf{A}^1$ is a $K^\times$-equivariant negative smooth 1-parameter deformation of X. Since $\Phi_- : \chi_- \to S_-$ is negative miniversal, we have a $K^\times$-equivariant morphism $g : \mathbf{A}^1 \to S_-$ such that the pull-back of Φ_- by g is isomorphic to π. Let $D = \operatorname{im}(g) \setminus O$ be the orbit. Then as explained in the proof of Theorem 1, $\pi : V \to \mathbf{A}^1$ is isomorphic to the affine surface W_R, which is a fiber of $\overline{\Phi_-}^{\mathrm{aff}} : W = \operatorname{Spec} \overline{A} \to S_-$ over $R \in D$. Since $C \cong \operatorname{Proj} B$, this orbit D goes to (C, P) by Pinkham's map. □

3. Main results

In this section, we will study the injectivity of Π and prove our main results.

3.1. Formulation of injectivity of Π. Let N be a numerical semigroup of genus g and $\Phi_- : \chi_- \to S_-$, $\chi_- = \mathbf{V}(F)$, $F = (F_1, \dots, F_k) \in K[x, s]^k$ the negative miniversal deformation of $X = X_N$. Suppose $\Pi : U/K^\times \to M_{g,1}^N$ is defined. We will formulate the injectivity of Π.

Suppose there are two points $u = (u_1, \dots, u_p)$, $v = (v_1, \dots, v_p) \in U \subset S_- \subset K^p$ such that $\Pi(u) = \Pi(v) = (C, P)$. Let $h_i : \mathbf{A}^1 \to K^p$ $(i = 1, 2)$ be a homogeneous morphism defined by $h_1(t) = (u_1 t^{e_1}, \dots, u_p t^{e_p})$, $h_2(t) = (v_1 t^{e_1}, \dots, v_p t^{e_p})$ where $e_i = w(s_i) > 0$ (in Section 2, we set $-e_i = w(s_i) > 0$ so we change the sign of e_i from now on). Then $\operatorname{Im}(h_1) \setminus \{O\}$ (resp. $\operatorname{Im}(h_2) \setminus \{O\}$) is the $K^\times$-orbit $O(u)$ of u (resp. $O(v)$ of v). The pull-back $\zeta_i : D_i \to \mathbf{A}^1$ of $\chi_- \to S_-$ by h_i $(i = 1, 2)$ are both isomorphic to $\pi : V = \operatorname{Spec} B \to \operatorname{Spec} K[x_{n+1}] = \mathbf{A}^1$, where $B = \oplus_{j=0}^{\infty} \Gamma(C, L^j)$, $L = O_C(P)$ (see the proof of Theorem 2). We note $\zeta_1 : D_1 \to \mathbf{A}^1$ is defined by $G_1(x, t) := F(x_1, \dots, x_n, u_1 t^{e_1}, \dots, u_p t^{e_p}) \in K[x, t]^k$ so that $D_1 = \mathbf{V}(G_1) \subset \mathbf{A}^{n+1}$. The same holds for ζ_2.

Thus we have two isomorphic 1-parameter deformations ζ_i of X over $\mathbf{A}^1$ defined by G_i $(i = 1, 2)$. Then there exist a homogeneous polynomial map $\phi : K[x, t] \to K[x, t]$

and a homogeneous polynomial matrix $\Lambda(x,t) \in M_{k\times k}(K[x,t])$ such that $[\phi(x_j)]_{t=0} = x_j$ $(1 \le j \le n)$, $\phi(t) = t$ and $[\Lambda]_{t=0} = E_k$ (= the identity matrix of degree k), which satisfy the following fundamental equation ([1, Remark 10.2.11]):

$$F(\phi(x_1),\dots,\phi(x_n),u_1t^{e_1},\dots,u_pt^{e_p}) = F(x,v_1t^{e_1},\dots,v_pt^{e_p})\Lambda\,.$$

Here the polynomial map ϕ describes the relative automorphism over $\mathbf{A}^1$ and Λ is the transformation of the generators of the defining ideal. The homogeneity of ϕ means $\phi(x_j)$ is homogeneous of degree $a_j = w(x_j)$, and the homogeneity of Λ means that each i-th element of the row vector $F\Lambda$ is homogeneous of degree equal to $\deg F_i$.

We now consider the following problem:

PROBLEM 1. *Let $\Phi_- : \chi_- \to S_-$, $\chi_- = \mathbf{V}(F)$, $F = (F_1,\dots,F_k) \in K[x,s]^k$ be the negative miniversal deformation of $X = X_N$. Suppose the following equation holds*:

$$F(\phi(x_1),\dots,\phi(x_n),u_1t^{e_1},\dots,u_pt^{e_p}) = F(x,v_1t^{e_1},\dots,v_pt^{e_p})\Lambda \tag{1}$$

where $u = (u_1,\dots,u_p)$, $v = (v_1,\dots,v_p) \in K^p$, $\phi : K[x,t] \to K[x,t]$ is a homogeneous polynomial map such that $[\phi]_{t=0}(x) = x$ and $\phi(t) = t$, $\Lambda(x,t) \in M_{k\times k}(K[x,t])$ is a homogeneous polynomial matrix such that $[\Lambda]_{t=0} = E_k$.

Suppose $p := \mathrm{char}(K)$ does not divide any exponent of the monomials appearing in F (we call this the exponent assumption on F). Then is it true that $u = v$ and ϕ, Λ are trivial (namely, $\phi(x_j) = x_j$ $(1 \le j \le n)$, $\Lambda = E_k$) ? If this is true, then, in particular, Pinkham's map Π is injective.

The exponent assumption on F guarantees that Taylor's expansion is available for F just as in characteristic 0.

3.2. Two generators case. The following result assures that Problem 1 is affirmatively solved in the simplest case where the numerical semigroup is generated by 2 elements.

THEOREM 3. *Let N be a numerical semigroup generated by 2 elements so that the monomial curve $X = X_N$ is a plane curve defined by one equation $f(x) = x_1^{a_2} - x_2^{a_1}$, $x = (x_1,x_2)$, $(w(x_1),w(x_2)) = (a_1,a_2)$. We consider the equation (1) under the exponent assumption on F. Then it holds that $u = v$ and ϕ, Λ are trivial. In particular, Π is injective.*

PROOF. We first note that $\Lambda = 1$ by homogeneity. We may assume $a_1 < a_2$ $(a_1 > 1)$ and $e_1 \le \dots \le e_p$. Then ϕ must be of the form $\phi(x_1) = x_1 + bt^{a_1}$, $\phi(x_2) = x_2 + \sum_{j=1}^{a_2} c_j(x_1)t^j$, where $b \in K$ and $c_j(x_1)$ is a monomial in x_1 of degree $a_2 - j$.

Thus our equation is

$$F(\phi(x_1),\phi(x_2),u_1t^{e_1},\dots,u_pt^{e_p}) = F(x_1,x_2,v_1t^{e_1},\dots,v_pt^{e_p})\,. \tag{2}$$

We will show $u = v$, $b = 0$ and $c_j(x_1) = 0$ $(1 \le j \le a_2)$. First consider the equation (2) modulo t^2. Then we have (mod t^2)

$$F(x_1,x_2+c_1(x_1)t,u_1t,\dots,u_lt,0,\dots,0) = F(x_1,x_2,v_1t,\dots,v_lt,0,\dots,0)\,, \tag{3}$$

where we assume $e_1 = \cdots = e_l = 1 < e_{l+1}$. By Tailor's expansion of (3) at $(x, 0)$, we have (mod t^2)

$$F(x,0) + \frac{\partial F}{\partial x_2}(x,0)c_1(x_1)t + \sum_{j=1}^{l} \frac{\partial F}{\partial s_j}(x,0)u_j t = F(x,0) + \sum_{j=1}^{l} \frac{\partial F}{\partial s_j}(x,0)v_j t\,.$$

We take a K-basis $\{\overline{g_1}, \ldots, \overline{g_p}\}$ of $T^1(X) = \operatorname{coker}(\Theta_n \to \mathrm{Nor}_{\mathbf{A}^n}(X))$ ($g_j \in K[x]^k$). Recall F satisfies $F(x,s) = f + \sum_{j=1}^{p} s_j g_j \pmod{s^2}$. Thus it follows that $F(x,0) = f$, $\frac{\partial F}{\partial x_2}(x,0) = \frac{\partial f}{\partial x_2}$ and $\frac{\partial F}{\partial s_j}(x,0) = g_j$. Hence we have in $K[x]$

$$\frac{\partial f}{\partial x_2}c_1(x_1) + \sum_{j=1}^{l} u_j g_j = \sum_{j=1}^{l} v_j g_j\,.$$

Modulo $\frac{\partial f}{\partial x_2}$, we have in $T^1(X)$, $\sum_{j=1}^{l} u_j\overline{g_j} = \sum_{j=1}^{l} v_j\overline{g_j}$. Since $\{\overline{g_1}, \ldots, \overline{g_l}\}$ is linearly independent, we have $u_j = v_j$ $(1 \le j \le l)$. Then we have $\frac{\partial f}{\partial x_2}c_1(x_1) = 0$. Since $\frac{\partial f}{\partial x_2} = -a_1 {x_2}^{a_1-1} \neq 0$, we have $c_1(x_1) = 0$ (note that $a_1 \neq 0$ by the exponent assumption on F).

Suppose next $e = a_1$ and assume that for any $e_j < e$, $u_j = v_j$ holds, and for any $j < e$, $c_j(x) = 0$ holds. Then we have mod t^{e+1}

$$\begin{aligned}&F(x_1 + bt^e, x_2 + c_e(x_1)t^e, u_1t^{e_1}, \ldots, u_m t^{e_m}, u_{m+1}t^e \ldots, u_q t^e, 0, \ldots, 0)\\ &\quad = F(x_1, x_2, u_1 t^{e_1}, \ldots, u_m t^{e_m}, v_{m+1}t^e, \ldots, v_q t^e, 0, \ldots, 0)\,. \end{aligned}\tag{4}$$

By Tailor's expansion of (4) at $(x, 0)$, we have (mod t^{e+1})

$$\begin{aligned}&F(x,0) + \frac{\partial F}{\partial x_1}(x,0)bt^e + \frac{\partial F}{\partial x_2}(x,0)c_e(x_1)t^e + \left\{\sum_{j=1}^{m}\frac{\partial F}{\partial s_j}(x,0)u_j t^{e_j} + \text{higher terms}\right\}\\ &\qquad + \sum_{j=m+1}^{q}\frac{\partial F}{\partial s_j}(x,0)u_j t^e\\ &= F(x,0) + \left\{\sum_{j=1}^{m}\frac{\partial F}{\partial s_j}(x,0)u_j t^{e_j} + \text{higher terms}\right\} + \sum_{j=m+1}^{q}\frac{\partial F}{\partial s_j}(x,0)v_j t^e. \end{aligned}\tag{5}$$

From (5), we have in $K[x]$

$$\frac{\partial f}{\partial x_1}b + \frac{\partial f}{\partial x_2}c_e(x_1) + \sum_{j=m+1}^{q} u_j g_j = \sum_{j=m+1}^{q} v_j g_j\,.$$

Thus modulo $\langle\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\rangle$, we have $u_j = v_j$ $(m+1 \le j \le q)$. Then we have $\frac{\partial f}{\partial x_1}b + \frac{\partial f}{\partial x_2}c_e(x_1) = 0$. Set $c_e(x_1) = \alpha x^{a_2-a_1}$ $(\alpha \in K)$. Then it follows that $ba_2x_1^{a_2-1} + (-a_1x_2^{a_1-1})\alpha x_1^{a_2-a_1} = 0$. Thus we have $b = \alpha = 0$ since a_1, a_2 are not zero in K.

For e in the range of $1 < e < a_1$ or $a_1 < e$, the same proof works with $b = 0$. Thus by induction on e, we are done. $\square$

3.3. Main theorem. We next analyze the equation (1) under the exponent assumption on F, and reduce Problem 1 to the inductive assumption of the triviality of ϕ and Λ. We assume $a_1 < \cdots < a_n$ and $e_1 \leq \cdots \leq e_p$.

Set $(\phi(x_1), \ldots, \phi(x_n)) = x + \sum_{j \geq 1} c_j(x)t^j$, $c_j(x) \in K[x]^n$ and $\Lambda = \sum_{j \geq 0} \Lambda_j(x)t^j$, $\Lambda_j(x) \in M_{k\times k}(K[x])$ $(\Lambda_0 = E_k)$.

Then the equation (1) mod t^2 gives

$$F(x + c_1(x)t, u_1t, \ldots, u_lt, 0, \ldots, 0) = F(x, v_1t, \ldots, v_lt, 0, \ldots, 0)(E_k + \Lambda_1(x)t)\,. \quad (6)$$

By Taylor's expansion of (6), we get the following equation by a similar computation as in Theorem 2.

$$\sum_{j=1}^{n} c_{1,j}(x)\frac{\partial f}{\partial x_j} + \sum_{i=1}^{l} u_i g_i(x) = f\Lambda_1(x) + \sum_{i=1}^{l} v_i g_i(x)\,,$$

where $c_1(x) = (c_{1,1}(x), \ldots, c_{1,n}(x)) \in K[x]^n$. Modulo f, $\frac{\partial f}{\partial x_j}$ $(1 \leq j \leq n)$, we have in $T^1(X)$, $\sum_{i=1}^{l} u_i\overline{g_i} = \sum_{i=1}^{l} v_i\overline{g_i}$. Since $\{\overline{g_1}, \ldots, \overline{g_p}\}$ is a basis of $T^1(X)$, $u_i = v_i$ $(1 \leq i \leq l)$ holds. Then we have

$$\sum_{j=1}^{n} c_{1,j}(x)\frac{\partial f}{\partial x_j} = f\Lambda_1(x)\,. \quad (7)$$

We here assume that from the equation (7), we can deduce $c_1(x) = 0$ and $\Lambda_1(x) = O_k$ (= the zero matrix of degree k). We call this *the inductive assumption for the triviality* (*IAT for short*) *of* $c(x)$ *and* $\Lambda(x)$ *at order* $e = 1$.

Suppose we have $u_j = v_j$, $c_j(x) = 0$ and $\Lambda_j(x) = O_k$ up to order t^{e-1}. Then mod t^{e+1}, we have

$$\begin{aligned} &F(x + c_e(x)t^e, u_1t^{e_1}, \ldots, u_lt^{e_l}, u_{l+1}t^e, \ldots, u_mt^e, 0, \ldots, 0) \\ &\quad = F(x, u_1t^{e_1}, \ldots, u_lt^{e_l}, v_{l+1}t^e, \ldots, v_mt^e, 0, \ldots, 0)(E_k + \Lambda_e(x)t^e)\,, \end{aligned} \quad (8)$$

where $e_j < e$ for $1 \leq j \leq l$. By Taylor's expansion of (8), we have

$$\sum_{j=1}^{n} c_{e,j}(x)\frac{\partial f}{\partial x_j} + \sum_{i=l+1}^{m} u_i g_i(x) = f\Lambda_e(x) + \sum_{i=l+1}^{m} v_i g_i(x)\,.$$

Hence we have in $T^1(X)$ (namely mod f, $\dfrac{\partial f}{\partial x_j}$), $\sum_{i=l+1}^{m} u_i\overline{g_i} = \sum_{i=l+1}^{m} v_i\overline{g_i}$ so that $u_i = v_i$

$(l+1 \leq i \leq m)$ holds. Then we have the equation:

$$\sum_{j=1}^{n} c_{e,j}(x) \frac{\partial f}{\partial x_j} = f \Lambda_e(x) . \tag{9}$$

If we can deduce $c_e(x) = 0$ and $\Lambda_e(x) = 0$ from the equation (9) (the IAT of ϕ and Λ at order e), then by induction, Problem 1 is affirmatively solved.

Summarizing, we have the following:

LEMMA 1. *Suppose, in the equation* (1), *the exponent assumption on F is satisfied. If the IAT for $c(x)$ and $\Lambda(x)$ holds at any order $e \geq 1$, then $u = v$ and ϕ, Λ are trivial. In particular, Π is injective.*

Now we will show that Problem 1 is affirmatively solved for a numerical semigroup of low genus. The following theorem is the main result of this paper. In this theorem, $N(4)_7 = \langle 5, 6, 7, 8, 9\rangle$ is a numerical semigroup of genus 4, which is the semigroup at an ordinary point (non-Weierstrass point) of a genus 4 curve. For the strange exceptional condition on $N(4)_7$, see the last part of the proof.

THEOREM 4. *Problem* 1 *is affirmatively solved for any numerical semigroup of genus $g \leq 4$ except $N(4)_7$. In the case of $N(4)_7$, under the additional condition $p = \operatorname{char} K \neq 5$, Problem* 1 *is affirmatively solved.*

Thus Pinkham's map Π is bijective if $p = char(K)$ does not divide any exponent of the monomials appearing F in case $g \leq 4$ (in the case of $N(4)_7$, the additional condition $p \neq 5$ is required).

PROOF. For genus $1 \leq g \leq 4$, we have 14 numerical semigroups (see [8, Appendix A, Table 1]). Among them, 6 semigroups are generated by 2 elements so that we are done by Theorem 3 in these cases. For the remaining 8 cases, we check the IAT of ϕ and Λ at any order in each case. Then by Lemma 1, we are done.

We choose $N = N(2)_2 = \langle 3, 4, 5\rangle$ as a typical case and check the IAT of ϕ and Λ. The minimal generators f of the ideal I_N are

$$f = (f_1, f_2, f_3) = (xz - y^2, x^3 - yz, x^2y - z^2) ,$$

where the weights of (x, y, z) are $(3, 4, 5)$ and the degrees of f are $(8, 9, 10)$. We have

$$\frac{\partial f}{\partial x} = (z, 3x^2, 2xy) , \quad \frac{\partial f}{\partial y} = (-2y, -z, x^2) , \quad \frac{\partial f}{\partial z} = (x, -y, -2z) .$$

(i) Suppose $e = 1$. Since the equation (7) is homogeneous, we can set

$$c_1(x) = (0, \alpha_1 x, \alpha_2 y),\ \Lambda_1(x) = \begin{pmatrix} 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\alpha_1, \alpha_2, \delta_1, \delta_2 \in K$ (note $w(t) = 1$). Thus we have by (7)

$$\alpha_1 x(-2y, -z, x^2) + \alpha_2 y(x, -y, -2z) = (0, \delta_1(xz - y^2), \delta_2(x^3 - yz)) .$$

From this, if $p \neq 3$, then $\alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$ follows immediately. Thus the IAT of ϕ and Λ holds at $e = 1$ if $p \neq 3$. We note that $p = 3$ is excluded by the exponent assumption on F since the exponent 3 appears in the monomials of f (thus of F).

(ii) Suppose $e = 2$. Then we can set

$$c_2(x) = (0, 0, cx),\ \Lambda_2(x) = \begin{pmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $c, \delta \in K$. Thus we have by (9)

$$cx(x, -y, -2z) = (0, 0, \delta(xz - y^2)) .$$

From this $c = \delta = 0$ follows. Thus the IAT of ϕ and Λ holds at $e = 2$.

(iii) Suppose $e = 3$. We have $c_3(x) = (c, 0, 0)$ ($c \in K$) and $\Lambda_3(x) = O_3$ by homogeneity. Then we have $c(z, 3x^2, 2xy) = (0, 0, 0)$. Thus $c = 0$ follows and the IAT of ϕ and Λ holds at $e = 3$.

(iv) Suppose $e = 4$. We have $c_4(x) = (0, c, 0)$ ($c \in K$) and $\Lambda_4(x) = O_3$ by homogeneity. Then we have $c(-2y, -z, x^2) = (0, 0, 0)$. Thus $c = 0$ follows and the IAT of ϕ and Λ holds at $e = 4$.

(v) Suppose $e = 5$. We have $c_5(x) = (0, 0, c)$ ($c \in K$) and $\Lambda_5(x) = O_3$ by homogeneity. Then we have $c(x, -y, -2z) = (0, 0, 0)$. Thus $c = 0$ follows and the IAT of ϕ and Λ holds at $e = 5$.

(vi) Suppose $e \geq 6$. Then we have $c_e(x) = (0, 0, 0)$ and $\Lambda_e(x) = O_3$.

Therefore, in the case of $N(2)_2$, we are done. Note that in this case, we need $p \neq 3$ for the IAT of ϕ and Λ to hold. In the following table, we summarize the characteristic condition necessary for the IAT of ϕ and Λ to hold for the numerical semigroups of genus $g \leq 4$ with more than 2 generators.

semigroups	$N(2)_2$	$N(3)_2$	$N(3)_4$	$N(4)_2$	$N(4)_4$	$N(4)_5$	$N(4)_6$	$N(4)_7$
char. cond.	$p \neq 3$	$p \neq 3$	$p \neq 2$	$p \neq 3$	$p \neq 2$	none	$p \neq 2$	$p \neq 5$

These characteristic conditions in this table are included in the exponent assumption on F except $N(4)_7$. In the case of $N(4)_7$, no monomials of f have an exponent divisible by 5, and we have to assume $p \neq 5$ additionally for the IAT of ϕ and Λ to hold. $\square$.

COROLLARY 1. *Let N be a numerical semigroup of genus $g \leq 4$ and $I_N \subset K[x]$ the ideal of the monomial curve X_N. Let $\{f_1, \ldots, f_k\}$ be a minimal set of generators of I_N and set $l := \max\{\deg(f_j) \mid 1 \leq j \leq k\}$. If $p := \mathrm{char}(K) > l$, then Pinkham's map Π is bijective in case it is defined.*

PROOF. Let $\Phi_- : \chi_- \to S_-$, $\chi_- = \mathbf{V}(F)$, $F = (F_1, \dots, F_k) \in K[x,s]^k$ be the negative miniversal deformation of X_N. Since $\deg F_j = \deg f_j$ $(1 \le j \le k)$, if $p > l$, then the exponent assumption on F in Theorem 4 is satisfied. We note that, in the case of $N(4)_7$, we do not need to assume $p \ne 5$ since $l = 18 > 5$ in this case. □

REMARK 1. The converse assertion of Theorem 4 does not hold. For instance, in the case of $g = 2$, $N = N(2)_2 = \langle 3, 4, 5\rangle$ and $p = 2$, some monomial appearing in f (thus in F) has an exponent divisible by 2, but Π is injective (see 4.2).

REMARK 2. We believe that Theorem 4 and Corollary 1 hold for any numerical semigroup of any genus, some characteristic condition being required for the generalization of Theorem 4. However, we have not succeeded in proving this conjecture yet.

REMARK 3. At the end of Introduction of [10, (1.20)], Pinkham stated that if $p = \mathrm{char}(K)$ is sufficiently large compared to g, then Π is injective without proofs. He also stated in the footnote on p.107: "This argument (= the proof of injectivity of Π) breaks down in characteristic $p > 0$." These two are the only comments on positive characteristic in [10].

4. Curves of genus 1 and 2

In this section, we analyze in which characteristic Pinkham's map Π is bijective completely for the genus 1 and 2 cases. For the computation of the miniversal deformation of monomial curves, we use "deform.lib" running on the computer algebra system Singular ([2], [6]), though, in the case of a hypersurface singularity, the miniversal deformation can be computed easily by hands (see [1, Theorem 10.1.7]).

We also note that, in characteristic 0, it is known that for any numerical semigroup of genus ≤ 7, Pinkham's map Π is defined (namely $U \ne \phi$) by [4]. Since we have not confirmed that this holds in positive characteristic, we will check that $U \ne \phi$ for each case to be cautious.

4.1. Elliptic curves. Let N be a numerical semigroup of genus $g = 1$. Then there is only one such semigroup $N = \{0, 2, 3, 4, \dots\} = \mathbf{N}_0 \setminus \{1\} = \langle 2, 3\rangle$. The ideal $I_N \subset K[x, y]$ of the monomial curve X_N is $I_N = \langle x^3 - y^2\rangle$ where $w(x) = 2$, $w(y) = 3$. Since $\deg(x^3 - y^2) = 6$, we know if $p = \mathrm{char}(K) > 5$, then Pinkham's map Π is bijective (when defined) by Corollary 1.

Let us see what happens in the case $p = 2, 3, 5$. Suppose $p = 2$. The miniversal deformation of X_N is given by

$$\Phi : \chi = \mathrm{Spec}\, K[x, y, a, b, c, d]/\langle F\rangle \to \mathrm{Spec}\, K[a, b, c, d] = \mathbf{A}^4$$

where $F := x^3 + y^2 + axy + by + cx + d$. The weights of the variables (x, y, a, b, c, d) are $(2, 3, 1, 3, 4, 6)$ so that F is homogeneous and Φ is $K^\times$-equivariant. In this case, all the weights of the base space are positive so that the negative miniversal deformation Φ_- coincides with Φ.

The singular locus Z of Φ_- is given by $Z = \{b^4 + a^3b^3 + a^4c^2 + a^5bc + a^6d = 0\} \subset S_- = \mathbf{A}^4$ and thus the smooth locus $U = S_- \setminus Z$ is non-empty. Thus $\Pi : U/K^\times \to M_{1,1}^N$ is defined and surjective by Theorem 2. We note $M_{1,1}^N$ is isomorphic to $\mathbf{A}^1$ via the j-invariant of the elliptic curves. Since $U/K^\times$ is a non-empty Zariski open subset of the weighted projective space $\mathbf{P}^3_{(1,3,4,6)}$, $\dim U/K^\times = 3$ and Π is not bijective.

Suppose $p = 3$. The miniversal deformation of X_N is given by

$$\Phi : \chi = \operatorname{Spec} K[x, y, a, b, c]/\langle F\rangle \to \operatorname{Spec} K[a, b, c] = \mathbf{A}^3$$

where $F := x^3 - y^2 + ax^2 + bx + c$. The weights of the variables (x, y, a, b, c) are $(2, 3, 2, 4, 6)$ so that F is homogeneous and Φ is $K^\times$-equivariant. We have $\Phi_- = \Phi$.

The singular locus Z of Φ_- is given by $Z = \{b^3 - a^2b^2 + a^3c = 0\} \subset S_- = \mathbf{A}^3$. Thus the smooth locus $U = \mathbf{A}^3 \setminus Z$ is non-empty and we have a surjective morphism $\Pi : U/K^\times \to M_{1,1}^N \cong \mathbf{A}^1$. Since $U/K^\times$ is a non-empty Zariski open subset of the weighted projective plane $\mathbf{P}^2_{(2,4,6)}$, $\dim U/K^\times = 2$ and Π is not bijective.

Suppose $p = 5$. The miniversal deformation is given by

$$\Phi : \chi = \operatorname{Spec} K[x, y, a, b]/\langle F\rangle \to \operatorname{Spec} K[a, b] = \mathbf{A}^2$$

where $F := -y^2 + x^3 + ax + b$. The weights of the variables (x, y, a, b) are $(2, 3, 4, 6)$ so that F is homogeneous and Φ is $K^\times$-equivariant. We have $\Phi_- = \Phi$.

The singular locus Z of Φ_- is given by $Z = \{b^2 + 2a^3 = 0\} \subset S_- = \mathbf{A}^2$. Hence the smooth locus $U = \mathbf{A}^2 \setminus Z$ is non-empty and $\Pi : U/K^\times \to M_{1,1} \cong \mathbf{A}^1$ is defined and surjective. Since $p = 5$ does not divide any exponent of the monomials appearing in F, $\Pi : U/K^\times \to \mathbf{A}^1$ is bijective by Theorem 4.

For any $p > 5$ and $p = 0$, the negative miniversal deformation of X_N is given by

$$\Phi_- : \chi_- = \operatorname{Spec} K[x, y, a, b]/\langle F\rangle \to \operatorname{Spec} K[a, b] = \mathbf{A}^2$$

where $F := -y^2 + x^3 + ax + b$. The weights of the variables (x, y, a, b) are $(2, 3, 4, 6)$ so that F is homogeneous and Φ is $K^\times$-equivariant. The singular locus Z of $\Phi_- = \Phi$ is given by $Z = \{4a^3 + 27b^2 = 0\} \pmod p$, which is a discriminant of the equation $x^3 + ax + b = 0$. Thus $U = \mathbf{A}^2 \setminus Z$ is always non-empty. Therefore Π is defined and bijective by Theorem 4. Summarizing, we have the following:

THEOREM 5. *For an elliptic curve, Pinkham's map Π is defined in any characteristic. Further, Π is bijective if and only if $p = \operatorname{char}(K) > 3$.*

COROLLARY 2. *For any elliptic curve, if $p = 0$ or $p > 3$, then the moduli space $M_{1,1} = M_{1,1}^N$ is isomorphic to $U/K^\times \subset \mathbf{P}^1_{(4,6)}$, $U := \{4a^3 + 27b^2 \neq 0\} \pmod p$. Further the pointed elliptic curve $C_{(a,b)}$ corresponding to $(a, b) \in U/K^\times \subset \mathbf{P}^1_{(4,6)}$ is given by*

$$C_{(a,b)} = \operatorname{Proj} K[x, y, z]/\langle \overline{F}\rangle \subset \mathbf{P}^2_{(2,3,1)}$$

where $\overline{F} := -y^2 + x^3 + axz^4 + bz^6$ *and the marked point is defined by* $\{z = 0\}$.

4.2. $g = 2$ **case.** Let N be a numerical semigroup of genus 2. Then N is one of the following two semigroups N_1, N_2:

$$N_1 = \mathbf{N}_0 \setminus \{1, 3\} = \langle 2, 5\rangle\,, \quad N_2 = \mathbf{N}_0 \setminus \{1, 2\} = \langle 3, 4, 5\rangle\,.$$

N_2 is the semigroup at the ordinary (non-Weierstrass) point. Now, let I_{N_2} be the ideal of the monomial curve of N_2. Then $I_{N_2} = \langle x^3 - yz, x^2y - z^2, xz - y^2\rangle$ where $w(x) = 3$, $w(y) = 4$, $w(z) = 5$. The degrees of the 3 generators of I_{N_2} are $(9, 10, 8)$. Thus if $p > 7$, Pinkham's map Π is bijective in case it is defined by Corollary 1.

Let us see what happens in the case $p = 2, 3, 5, 7$. Suppose $p = 2$. Then the miniversal deformation of X_{N_2} is given by

$$\Phi : \operatorname{Spec} K[x, y, z, a, b, c, d, e]/\langle F\rangle \to \operatorname{Spec}(K[a, b, c, d, e]) = \mathbf{A}^5$$

where $F = (F_1, F_2, F_3)$, $F_1 := yz + x^3 + bxy + cx^2 + dx$, $F_2 := z^2 + x^2y + az + by^2 + cxy + dy + ex^2 + bey + cex + de$, $F_3 := y^2 + xz + ax + ey$. Since the weights of the variables (a, b, c, d, e) are $(5, 2, 3, 6, 4)$, we have $\Phi_- = \Phi$. The singular locus Z of Φ_- is given by $Z = \{a^6 + a^5bc + a^4b^2d + a^3c^3d + c^4d^3 + a^4c^2e + a^3c^5 + a^2bc^4d + ac^5de + a^2c^4e^2 + abc^5e^2 + b^2c^4de^2 + c^6e^3 = 0\}$ so that $U = \mathbf{A}^5 \setminus Z$ is non-empty.

Thus Pinkham's map $\Pi : U/K^\times \to M_{1,1}^{N_2}$ is defined and surjective, and $\dim U/K^\times = \dim M_{1,1}^{N_2} = 4$. Thus we cannot determine if Π is bijective or not by Theorem 4 or by counting the dimensions since some monomials in F have an exponent divisible by 2. In this case, by using the equation (1) and an explicit formula for F, we can show Π is injective by direct computations. Since this computation is simple but too tedious, we omit the details.

Suppose $p = 3$. Then the miniversal deformation of X_{N_2} is given by

$$\Phi : \operatorname{Spec} K[x, y, z, a, b, c, d, e, f]/\langle F\rangle \to \operatorname{Spec} K[a, b, c, d, e, f] = \mathbf{A}^6$$

where $F = (F_1, F_2, F_3)$, $F_1 := -yz + x^3 + bxz + cxy + dx^2 + ex$, $F_2 := -z^2 + x^2y - az + byz + cy^2 + dxy + ey - fx^2 - bfz - cfy - dfx - ef$, $F_3 := -y^2 + xz + ax + fy$. The weights of the variables (a, b, c, d, e, f) are $(5, 1, 2, 3, 6, 4)$. Thus $\Phi_- = \Phi$ in this case. The smooth locus U of Φ_- is non-empty (we omit the equation of the singular locus since it is lengthy) and $\Pi : U/K^\times \to M_{1,1}^{N_2}$ is surjective. Since $\dim U/K^\times = 5$ and $\dim M_{1,1}^{N_2} = 4$, Π is not bijective in this case.

Suppose $p > 3$ or $p = 0$. In this case, the miniversal deformation of X_{N_2} is given by

$$\Phi : \operatorname{Spec} K[x, y, z, a, b, c, d, e]/\langle F\rangle \to \operatorname{Spec} K[a, b, c, d, e] = \mathbf{A}^5$$

where $F = (F_1, F_2, F_3)$, $F_1 = -yz + x^3 + bxy + cx^2 + dx$, $F_2 = -z^2 + x^2y - az + by^2 + cxy + dy - ex^2 - bey - cex - de$, $F_3 = -y^2 + xz + ax + ey \pmod p$. The weights of the variables (a, b, c, d, e) are $(5, 2, 3, 6, 4)$ so that $\Phi_- = \Phi$. U is non-empty and thus Π

is defined and surjective. Since no exponent of the monomials appearing in F is divisible by $p > 3$, Π is bijective.

We analyze the $N_1 = \langle 2, 5\rangle$ case similarly, and we have the following:

THEOREM 6. *For $g = 2$, we have two numerical semigroups $N_1 = \langle 2, 5\rangle$ and $N_2 = \langle 3, 4, 5\rangle$, where N_2 is the semigroup at an ordinary point.*

(i) *Suppose $N = N_1$. Then Π is defined in any characteristic and Π is bijective if and only if $p \neq 2, 5$.*

(ii) *Suppose $N = N_2$. Then Π is defined in any characteristic and Π is bijective if and only if $p \neq 3$.*

COROLLARY 3. *Suppose $g = 2$.*

(i) *Suppose $N = N_1$ and $p \neq 2, 5$. Then the moduli space $M^N_{2,1} \cong U/K^\times$ is a non-empty Zariski open subset of $\mathbf{P}^3_{(4,6,8,10)}$. The pointed algebraic curve $C_{(a,b,c,d)}$ corresponding to $(a, b, c, d) \in U/K^\times \subset \mathbf{P}^3_{(4,6,8,10)}$ is given by*

$$C_{(a,b,c,d)} = \operatorname{Proj} K[x, y, z]/\langle \overline{F}\rangle \subset \mathbf{P}^2_{(2,5,1)}$$

where $\overline{F} := -y^2 + x^5 + ax^3z^4 + bx^2z^6 + cxz^8 + dz^{10} \pmod p$. The marked point is defined by $z = 0$.

(ii) *Suppose $N = N_2$ and $p \neq 3$. Then the moduli space $M^N_{2,1} \cong U/K^\times$ is a non-empty Zariski open subset of $\mathbf{P}^4_{(5,2,3,6,4)}$. The pointed algebraic curve $C_{(a,b,c,d,e)}$ corresponding to $(a, b, c, d, e) \in U/K^\times \subset \mathbf{P}^4_{(5,2,3,6,4)}$ is given by*

$$C_{(a,b,c,d,e)} = \operatorname{Proj} K[x, y, z, w]/\langle \overline{F}\rangle \subset \mathbf{P}^3_{(3,4,5,1)}$$

where $\overline{F} = (\overline{F_1}, \overline{F_2}, \overline{F_3})$,

$$\overline{F_1} = -yz + x^3 + bxyw^2 + cx^2w^3 + dxw^6\,,$$

$$\overline{F_2} = -z^2 + x^2y - azw^5 + by^2w^2 + cxyw^3 + dyw^6 - ex^2w^4 - beyw^6 - cexw^7 - dew^{10}\,,$$

$$\overline{F_3} = -y^2 + xz + axw^5 + eyw^4 \pmod p\,.$$

The marked point is given by $w = 0$.

REMARK 4. (i) As Corollary 2 and 3 show, the utmost advantage of Pinkham's theorem is that we can describe the moduli space $M^N_{g,1}$ and the pointed algebraic curve corresponding to a point on it explicitly by equations.

(ii) Since $M^N_{g,1}$ is given as a locally closed subset in a weighted projective space, the closure $\overline{M^N_{g,1}}$ in it gives a natural projectivization (compactification) of $M^N_{g,1}$. The (singular) pointed algebraic curve corresponding to a boundary point in $\overline{M^N_{g,1}} \setminus M^N_{g,1}$ gives a degeneration of the family of pointed smooth projective curves of genus g with a given Weierstrass gap sequence, and the study of these degenerations would be an interesting research object in the future.

4.3. An interesting example of genus 4. We finally refer to the case of the numerical semigroup $N(4)_7 = \langle 5, 6, 7, 8, 9 \rangle$ of genus 4. This is the numerical semigroup at an ordinary point of a genus 4 curve. In this case, we have not yet succeeded in computing the equations of the moduli space $M_{4,1}^{N(4)_7}$ in characteristic 0. Indeed, as reported in [8], in the case of the numerical semigroups with more than 4 generators, the computation of the miniversal deformation space in characteristic 0 is very hard and seems almost impossible without a super computer. The reason for this difficulty is probably the appearance of huge coefficients in the Groebner bases computation.

The semigroup $N(4)_7$ is the only one with 5 generators up to genus 4. We have succeeded in computing the equations of $M_{4,1}^{N(4)_7}$ in characteristic 7 by Theorem 4, which is an exciting result for us.

THEOREM 7. *Suppose $p = \mathrm{char}(K) = 7$. Then for $N = N(4)_7$, Π is defined and $\Pi : U/K^\times \to M_{4,1}^N$ is a bijective morphism. The moduli space $M_{4,1}^N \cong U/K^\times$ is a Zariski open subset of a 10-dimensional variety $\mathbf{V}(J) \subset \mathbf{P}^{18}_{(9,8,7,2,3,4,5,10,7,6,7,6,8,6,5,4,5,4,3)}$. The defining equations J of the base space S_- consist of 20 equations, and the defining equations F of the total space χ_- consist of 10 equations. We omit the precise form of the equations F, J since they are too lengthy.*

Using the defining equations J of $M_{4,1}^N$, we might get a computational proof of the rationality of $M_{4,1}^N$, which we will discuss elsewhere. Thus, as we have expected, in the positive characteristic, we have much more possibility for successfully computing $M_{g,1}^N$ in the case of the numerical semigroups with more than 4 generators.

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