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Asymptotically Unweighted Shifts, Hypercyclicity, and Linear Chaos

Ayuko NATSUME and Masahiko TANIGUCHI

Nara Women's University

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Abstract. In this paper, we introduce weighted backward shifts, which are asymptotically unweighted, and give several conditions for such operators on the classical ℓ^p spaces to be hypercyclic and chaotic.

1. Introduction

It is widely believed that chaos is linked to nonlinearity. But recently, research on linear chaos becomes very active. Following Devaney (cf. [5]), we say that a bounded linear operator T of a separable Fréchet space X to itself is *chaotic* if T is hypercyclic and X has a dense subset consisting only of periodic vectors with respect to T. Here, we say that T is *hypercyclic* if T has a hypercyclic vector, i.e. a vector with a dense orbit with respect to the action of T.

The research on hypercyclicity has its own history. In 1929, Birkhoff proved that the translation operators

$$T(f)(z) = f(z + c)$$

with $c \in \mathbb{C}$ on the Fréchet space of holomorphic functions on $\mathbb{C}$ are hypercyclic. In 1959, MacLane showed the same result for the differential operator

$$T(f) = f' .$$

Recall that the classical transitivity theorem due to Birkhoff implies that a bounded linear operator T on a separable Fréchet space is hypercyclic if and only if it is *topologically transitive*, i.e. for every pair of non-empty open subsets U and V of X, there is an n such that

$$T^n(U) \cap V \neq \emptyset .$$

Now, epoch-making results were given in the paper [6] of Godefroy and Shapiro. First of all, the authors proved that Birkhoff's operators and MacLane's one are actually chaotic. Moreover, they showed that hypercyclicity only implies the sensitive dependence on initial conditions, usually called the butterfly effect, which is the indispensable phenomenon of the

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chaos in the sense of Lorentz. Also recall that every infinite-dimensional separable Fréchet space admits hypercyclic operators, and hence the butterfly effect. See for instance, [4].

On the other hand, there exist separable Banach spaces which can admit no chaotic bounded linear operators. Such examples were founded firstly in [3]. The recent distinguished result due to Argyros and Haydon in [1] gives a simple proof of this fact. For the backgrounds and other recent results on linear chaos, see for instance [2] and [8].

Typical and classically investigated operators, other than those due to Birkhoff and MacLane, are weighted backward shifts. For the sake of simplicity, here we restrict ourselves to the case of backward shifts on the classical complex Banach space ℓ^p with $1 \leq p < \infty$, which consists of sequences $\mathbf{x} = (x_n)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and is equipped with the ℓ^p norm

$$\|\mathbf{x}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} .$$

In the sequel, we call the bounded linear operator $B : \ell^p \to \ell^p$ defined by

$$B((x_1, \ldots, x_n, \ldots)) = (x_2, \ldots, x_{n+1}, \ldots)$$

for every $(x_n) \in \ell^p$ simply the *shift* on ℓ^p.

DEFINITION 1. We call $\lambda B : \ell^p \to \ell^p$ defined by

$$R((x_1, \ldots, x_n, \ldots)) = (\lambda x_2, \ldots, \lambda x_{n+1}, \ldots)$$

the *Rolewicz operator R*, where $\lambda \in \mathbb{C}$.

PROPOSITION 1 ([6], also cf. [9]). *The Rolewicz operator $R : \ell^p \to \ell^p$ is chaotic if and only if $|\lambda| > 1$.*

In particular, the (unweighted) shift B itself is not chaotic, and moreover we can see that B is not even hypercyclic. To clarify the critical state for the Rolewicz operators, we will consider in this paper the weighted backward shifts which are “asymptotically unweighted" in the following sense.

DEFINITION 2. Let $\{a_n\}$ be a sequence of positive numbers. We say that the weighted shift $S : \ell^p \to \ell^p$ defined by

$$S((x_1, \ldots, x_n, \ldots)) = ((1 + a_1)x_2, \ldots, (1 + a_n)x_{n+1}, \ldots)$$

is *asymptotically unweighted* if $a_n \to 0$ as $n \to \infty$.

Here, it is clear that S is a bounded linear operator if and only if $\{a_n\}$ is a bounded sequence. In the sequel, we always assume that a_n *are bounded*. Also note that, if all a_n are the same constant, then S is a chaotic Rolewicz operator.

Now, in this paper, we give a direct and self-contained proof of the following main theorem.

THEOREM 1. *Fix p with $1 \leq p < \infty$. Let $\{a_n\}$ be a sequence of positive numbers and S be the corresponding weighted shift on ℓ^p as in Definition 2. Assume that S is asymptotically unweighted. Then the following assertions hold.*

1. *S is hypercyclic on ℓ^p if and only if*

$$s_n = \sum_{k=1}^{n} a_k$$

are unbounded.

2. *S is* topologically mixing *on ℓ^p, i.e. for every pair of non-empty open subsets U and V of ℓ^p, there is an N such that*

$$S^n(U) \cap V \neq \emptyset$$

for every $n \geq N$, if and only if $s_n \to \infty$ as $n \to \infty$.

3. *If*

$$\sum_{n=1}^{\infty} s_n^{-p} < \infty ,$$

then S is mixing and chaotic on ℓ^p.

4. *If S is chaotic on ℓ^p, then S is mixing and*

$$\sum_{n=1}^{\infty} e^{-ps_n} < \infty .$$

EXAMPLE 1. For every $p > 1$, take a such that $1/p < a < 1$ and set $s_n = n^a$, or equivalently $a_1 = 1$ and $a_n = n^a - (n-1)^a$ for every $n \geq 2$. Then

$$a_n = \int_{n-1}^{n} ax^{a-1}dx < a(n-1)^{a-1} \to 0$$

as $n \to \infty$ and

$$\sum_{n=1}^{\infty} s_n^{-p} = \sum_{n=1}^{\infty} n^{-ap} < \infty .$$

Thus, the corresponding weighted shift S on ℓ^p is asymptotically unweighted and chaotic.

EXAMPLE 2. Suppose that $p = 1$. First, if we set $a_n = 1/n$, then $s_n \leq 1 + \log n$, and hence

$$\sum_{n=1}^{\infty} e^{-s_n} \geq \sum_{n=1}^{\infty} e^{-1-\log n} = e^{-1} \sum_{n=1}^{\infty} \frac{1}{n} = \infty .$$

Then by Theorem 1(4), the corresponding S cannot be chaotic on ℓ^1.

Next, if we set $a_n = b/n$ with $b > 1$ and take b' with $1 < b' < b$ arbitrarily, then there is an N such that

$$1 + \frac{b}{n} \geq \left(1 + \frac{1}{n}\right)^{b'}$$

for every $n \geq N$. So, we have

$$\prod_{k=N}^{n} (1 + a_k) \geq \prod_{k=N}^{n} \left(1 + \frac{1}{k}\right)^{b'} = \frac{(n+1)^{b'}}{N^{b'}} .$$

In particular, there is a constant M such that

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 + a_k)^{-1} \leq M \sum_{n=1}^{\infty} (n+1)^{-b'} < \infty .$$

Thus, essentially the same argument as in the proof of Lemma 3 below gives that the corresponding S is chaotic on ℓ^1.

REMARK 1. In the case that $p = 1$, if we set $a_n = b/n$ with $b > 1$ as in the previous example, then

$$\sum_{n=1}^{\infty} s_n^{-1} = \infty ,$$

but since $s_n \geq b \log n$

$$\sum_{n=1}^{\infty} e^{-s_n} \leq \sum_{n=1}^{\infty} n^{-b} < \infty .$$

This example might indicate that the condition $\sum_{n=1}^{\infty} e^{-ps_n} < \infty$ in Theorem 1(4) is also necessary for S to be chaotic.

2. Proof

Some parts of Theorem 1 can be proved as corollaries to more general theorems on so-called Fréchet sequence spaces. (For instance, see [7] and [8].) But, for the sake of completeness, we will give a direct and self-contained proof of Theorem 1.

We begin with the following:

LEMMA 1. *If S is hypercyclic on ℓ^p, then s_n are unbounded. Moreover, if S is mixing on ℓ^p, then $s_n \to \infty$ as $n \to \infty$.*

PROOF. Consider two points $\mathbf{0} = (0, \ldots)$ and $\mathbf{e}_1 = (1, 0, \ldots)$. If S is hypercyclic, then for every $\varepsilon > 0$, there are a vector $\mathbf{x} = (x_n)$ and a positive integer n such that

$$\|\mathbf{x} - \mathbf{0}\|_p < \varepsilon\,, \qquad \|S^n(\mathbf{x}) - \mathbf{e}_1\|_p < \varepsilon\,.$$

In particular,

$$|x_{n+1}| < \varepsilon \qquad \left| x_{n+1} \prod_{k=1}^{n} (1 + a_k) - 1 \right| < \varepsilon\,.$$

Hence, we conclude that

$$e^{s_n} > \prod_{k=1}^{n} (1 + a_k) > \frac{1 - \varepsilon}{|x_{n+1}|} > \frac{1 - \varepsilon}{\varepsilon}\,.$$

Since $\varepsilon > 0$ is arbitrarily, s_n can not be bounded.

Next, if S is mixing, then for every $\varepsilon > 0$, there are a vector $\mathbf{x} = (x_n)$ and a positive integer N such that

$$\|\mathbf{x} - \mathbf{0}\|_p < \varepsilon\,, \quad \|S^n(\mathbf{x}) - \mathbf{e}_1\|_p < \varepsilon\,,$$

for every $n > N$. Thus by the same argument as above, we conclude that $s_n \to \infty$ as $n \to \infty$. □

The converse assertions follow from the well-known criteria for a general bounded linear operator T of a separable Fréchet space X to itself stated below. For the sake of completeness, we also include short proofs of them.

PROPOSITION 2 (Hypercyclicity criterion). *Let X_0 and Y_0 be dense subsets of X, $\{n_k\}$ be an increasing sequence of natural numbers, and $\hat{T}_k : Y_0 \to X$ be a map for every k. Suppose that, for every $\mathbf{x} \in X_0$ and every $\mathbf{y} \in Y_0$,*

$$T^{n_k}(\mathbf{x}) \to \mathbf{0}\,, \quad \hat{T}_k(\mathbf{y}) \to \mathbf{0}\,, \quad T^{n_k} \circ \hat{T}_k(\mathbf{y}) \to \mathbf{y}$$

as $k \to \infty$. Then T is hypercyclic.

PROOF. We prove that T satisfying the assumptions of hypercyclicity criterion is topologically transitive, which implies the assertion by Birkhoff's transitivity theorem.

Let U and V be a pair of arbitrarily given non-empty open subsets of X. Take vectors $\mathbf{x}_0 \in X_0 \cap U$ and $\mathbf{y}_0 \in Y_0 \cap V$, which is possible since X_0 and Y_0 are dense in X. Then the assumptions of the hypercyclicity criterion imply that

$$T^{n_k}(\mathbf{x}_0 + \hat{T}_k(\mathbf{y}_0)) = T^{n_k}(\mathbf{x}_0) + T^{n_k} \circ \hat{T}_k(\mathbf{y}_0) \to \mathbf{y}_0 \,.$$

as $k \to \infty$. Since $\hat{T}_k(\mathbf{y}) \to \mathbf{0}$ as $k \to \infty$, we conclude that

$$T^{n_k}(U) \cap V \neq \emptyset$$

for every sufficiently large k. □

REMARK 2. Actually, the hypercyclicity criterion is a criterion for T to be "weakly mixing", i.e., for every two pairs $\{U, V\}$ and $\{U', V'\}$ of non-empty open subsets of X, there is an n such that

$$T^n(U) \cap V \neq \emptyset \,, \quad T^n(U') \cap V' \neq \emptyset \,.$$

Note that a hypercyclic operator need not be weakly mixing. See for instance, [10].

PROPOSITION 3 (Kitai's criterion). *Let X_0 and Y_0 be dense subsets of X and $\hat{T} : Y_0 \to Y_0$ be a map such that*

$$T \circ \hat{T}(\mathbf{y}) = \mathbf{y} \,.$$

Suppose that, for every $\mathbf{x} \in X_0$ and every $\mathbf{y} \in Y_0$,

$$T^n(\mathbf{x}) \to \mathbf{0} \,, \quad \hat{T}^n(\mathbf{y}) \to \mathbf{0}$$

as $k \to \infty$. Then T is mixing.

PROOF. We can prove the assertion similarly as in, but more easily than, the proof of the previous proposition.

Actually, let U and V be a pair of arbitrarily given non-empty open subsets of X. Take vectors $\mathbf{x}_0 \in X_0 \cap U$ and $\mathbf{y}_0 \in Y_0 \cap V$. Then the assumptions of the Kitai's criterion imply that

$$T^n(\mathbf{x}_0 + \hat{T}^n(\mathbf{y}_0)) = T^n(\mathbf{x}_0) + T^n \circ \hat{T}^n(\mathbf{y}_0) = T^n(\mathbf{x}_0) + \mathbf{y}_0 \to \mathbf{y}_0 \,.$$

as $n \to \infty$. Since $\hat{T}^n(\mathbf{y}_0) \to \mathbf{0}$ as $n \to \infty$, we can find an N such that

$$T^n(U) \cap V \neq \emptyset \,.$$

for every $n > N$. □

LEMMA 2. *If s_n are unbounded, then S is hypercyclic on ℓ^p. Moreover, if $s_n \to \infty$ as $n \to \infty$, then S is mixing on ℓ^p.*

PROOF. First suppose that s_n are unbounded. Then there is an increasing sequence $\{n_k\}$ of natural numbers such that $\lim_{k\to\infty} s_{n_k} = \infty$.

Let $X_0 = Y_0$ be the subset of ℓ^p consisting of all elements $\mathbf{x}$ admitting only a finite number of non-zero components. Then X_0 is dense in ℓ^p and it is clear that $\lim_{k\to\infty} S^{n_k}(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in X_0$.

Define $\hat{S}_k : X_0 \to X_0$ by setting

$$\hat{S}_k(\mathbf{x}) = \left(\underbrace{0, \dots, 0,}_{n_k} \frac{x_1}{\prod_{\nu=1}^{n_k}(1+a_\nu)}, \dots, \frac{x_m}{\prod_{\nu=m}^{n_k+m-1}(1+a_\nu)}, \dots\right)$$

where $\mathbf{x} = (x_n)$. Then for every $\mathbf{x} \in X_0$, $S^{n_k} \circ \hat{S}_k(\mathbf{x}) = \mathbf{x}$ by definition.

Now fix $\mathbf{x} \in X_0$ and a natural number N such that $x_n = 0$ for every $n > N$. Then since

$$\sum_{\nu=N}^{n_k} a_\nu = s_{n_k} - s_{N-1}$$

also tend to ∞ as $k \to \infty$, we can conclude that $\lim_{k\to\infty} \hat{S}_k(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in X_0$. Thus by the hypercyclicity criterion implies that S is hypercyclic.

Next, if $s_n \to \infty$ as $n \to \infty$, then by the same argument as above, but by using the Kitai's criterion with

$$\hat{S}(\mathbf{x}) = \left(0, \frac{x_1}{1+a_1}, \dots, \frac{x_n}{1+a_n}, \dots\right)$$

we conclude that S is mixing. □

REMARK 3. For a general sequence $\{a_n\}$ of complex numbers converging to 0, the situation is very delicate. For instance, we can see from Lemma 2 that, if $a_n = 1/n$, then the corresponding S is mixing on ℓ^p.

But for instance, if $a_n = i/n$, then

$$\left|\prod_{k=1}^{n}(1+a_k)\right| = \left(\prod_{k=1}^{n}(1+k^{-2})\right)^{1/2}$$

is bounded, and the same arguments as in the proof of Lemma 1 gives that the corresponding S can not be hypercyclic.

Next, we prove the following

LEMMA 3. *If*

$$\sum_{n=1}^{\infty} s_n^{-p} < \infty$$

then S is mixing and chaotic on ℓ^p.

PROOF. The assumption implies that $s_n \to \infty$ as $n \to \infty$, and S is mixing by Lemma 2.

Next, let X_0 be as in the proof of Lemma 2. Fix $\mathbf{x} = (x_n) \in \ell^p$ arbitrarily. Then for every $\varepsilon > 0$, we can find a positive integer N such that $\mathbf{x}_0 = (x_n^0)$ with $x_n^0 = x_n$ for every $n \le N$ and $x_n^0 = 0$ for every $n > N$ belongs to X_0 and satisfies that $\|\mathbf{x} - \mathbf{x}_0\|_p < \varepsilon$.

By the assumption, and since $s_n \to \infty$ as $n \to \infty$, we can find an M $(> N)$ such that

$$\sum_{n=M}^{\infty} s_n^{-p} < \varepsilon^p$$

and that

$$\sum_{k=N}^{n} a_k \ge \frac{s_n}{2}$$

for every $n > M$.

Set $x'_k = x_k^0$ for every $k \le M$, and

$$x'_{k+\nu M} = \frac{x'_k}{\prod_{m=k}^{k+\nu M-1}(1+a_m)}$$

for every positive integer ν and $k \le M$.

Now, set $\mathbf{x}' = (x'_n)$. Then by definition,

$$\begin{aligned} \|\mathbf{x}' - \mathbf{x}_0\|_p^p &\le \left(\sum_{\nu=1}^{\infty}\left(\min_{k=1,\dots,N}\prod_{m=k}^{k+\nu M-1}(1+a_m)\right)^{-p}\right)\|\mathbf{x}_0\|_p^p \\ &\le \left(\sum_{\nu=1}^{\infty}\prod_{m=N}^{\nu M}(1+a_m)^{-p}\right)\|\mathbf{x}\|_p^p . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{\nu=1}^{\infty}\prod_{m=N}^{\nu M}(1+a_m)^{-p} &\le \sum_{\nu=1}^{\infty}\left(\sum_{k=N}^{\nu M} a_k\right)^{-p} \\ &< \sum_{\nu=1}^{\infty}\left(\frac{s_{\nu M}}{2}\right)^{-p} < 2^p \sum_{n=M}^{\infty} s_n^{-p} < 2^p \varepsilon^p . \end{aligned}$$

In particular, $\mathbf{x}' \in \ell^p$ and

$$\|\mathbf{x}' - \mathbf{x}_0\|_p \le 2\varepsilon\|\mathbf{x}\|_p .$$

Thus, $\mathbf{x}'$ is a periodic vector in ℓ^p (with period M) with respect to the corresponding S such that

$$\|\mathbf{x}' - \mathbf{x}\|_p \leq (2\|\mathbf{x}\|_p + 1)\varepsilon ,$$

which means that we can find periodic vector arbitrarily near to any vector $\mathbf{x}$ in ℓ^p, and we conclude that S is chaotic. □

Finally, the following lemma completes the proof of Theorem 1.

LEMMA 4. *If S is chaotic on ℓ^p, then*

$$\sum_{n=1}^{\infty} e^{-ps_n} < \infty$$

and S is mixing.

PROOF. If S is chaotic, then periodic vectors are dense in ℓ^p, which implies that there exists an M such that periodic vectors with period M is dense on a non-empty open subset of ℓ^p. In particular, for every positive integer k, there is a periodic vector $\mathbf{x}_k = (x_{k,n})$ with period M such that $x_{k,k} \neq 0$.

Since

$$x_{k,k+\nu M} = \frac{x_{k,k}}{\prod_{m=k}^{k+\nu M-1}(1+a_m)}$$

for every positive integer ν, and in particular,

$$\sum_{\nu=1}^{\infty} \frac{|x_{k,k}|^p}{\prod_{m=k}^{k+\nu M-1}(1+a_m)^p} = \sum_{\nu=1}^{\infty} |x_{k,k+\nu M}|^p \leq \|\mathbf{x}_k\|_p^p ,$$

we conclude that

$$\sum_{\nu=1}^{\infty} \frac{1}{\prod_{m=k}^{k+\nu M-1}(1+a_m)^p} < \infty$$

for every k. Hence, we have

$$\sum_{\nu=1}^{\infty} e^{-ps_{k+\nu M-1}} \leq \sum_{\nu=1}^{\infty} \prod_{m=1}^{k+\nu M-1} (1+a_m)^{-p} < \infty .$$

Summing up the terms in the left hand side for $k = 1, \ldots, M$, we conclude that

$$\sum_{n=1}^{\infty} e^{-ps_n} < \infty$$

as desired. In particular, $s_n \to \infty$ as $n \to \infty$, and S is mixing by Lemma 2. □

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Present Addresses:
AYUKO NATSUME
GRADUATE SCHOOL OF HUMANITIES AND SCIENCES,
NARA WOMEN'S UNIVERSITY,
NARA 630–8506, JAPAN.

MASAHIKO TANIGUCHI
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES,
NARA WOMEN'S UNIVERSITY,
NARA 630–8506, JAPAN.
e-mail: tanig@cc.nara-wu.ac.jp