TOKYO J. MATH.
VOL. 39, NO. 2, 2016

Non-Hopf Hypersurfaces in 2-dimensional Complex Space Forms

Mayuko KON

Shinshu University

(Communicated by K. Ahara)

Abstract. In this paper we give a geometric characterization of non-Hopf hypersurfaces in the complex space form $M^2(c)$ under a condition on the shape operator. We also classify pseudo-parallel real hypersurfaces of $M^2(c)$.

1. Introduction

It is an interesting problem to study real hypersurfaces immersed in the complex space form $M^n(c)$ under a condition on curvature tensor, or the Ricci tensor, or sectional curvature. In this paper we consider the case of the 2-dimensional complex space form $M^2(c)$. In [3], Ivey and Ryan constructed some examples of non-Hopf real hypersurfaces in the non-flat complex space form $M^2(c)$. Let M be a real hypersurface in the complex hyperbolic space $\mathbf{C}H^2$ or the complex projective space $\mathbf{C}P^2$. We denote by (ϕ, ξ, η, g) an almost contact metric structure. At each $p \in M$, we define a subspace $\mathcal{H}_p \subset T_pM$ as the smallest subspace that contains the structure vector field ξ and that is invariant under the shape operator A. We assume that $\mathcal{H} = \sqcup_p \mathcal{H}_p$ is a smooth two-dimensional distribution on M. Then we obtain an adapted orthonormal frame $\{\xi, X, \phi X\}$ with respect to which the shape operator has the form

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}, \tag{1}$$

where $\mathcal{H}$ is spanned by ξ and X at each point.

THEOREM A ([3]). *Let $\alpha(t)$, $h(t)$, $\lambda(t)$, $\nu(t)$ be analytic functions on an open interval*

Received March 26, 2015; revised May 24, 2016
Mathematics Subject Classification: 53C40, 53C55, 53C25
Key words and phrases: non-Hopf hypersurface, shape operator, Ricci tensor, complex space form

$I \subset \mathbf{R}$ satisfying the underdetermined ODE system

$$
\begin{aligned}
\alpha' &= h(\alpha + \lambda - 3\nu)\,, \\
h' &= h^2 + \lambda^2 - 2\lambda\nu + \alpha\nu + c\,, \\
\lambda' &= \frac{(2\lambda + \nu)h^2 + (\nu - \lambda)(\alpha\lambda - \lambda^2 + c)}{h}\,,
\end{aligned}
\tag{2}
$$

with $h(t)$ nowhere zero. Let $\gamma(t)$ be a unit-speed analytic framed curve in $M^2(c)$, defined for $t \in I$, with transverse curvature $\nu(t)$, zero holomorphic curvature and zero torsion. Then there exists a non-Hopf hypersurface M^3 such that

(i) *the distribution $\mathcal{H}$ is rank 2 and integrable*;
(ii) *M has a globally defined frame $\{\xi, X, \phi X\}$ with respect to which the shape operator has the form (1), such that α, h, λ and ν are constant along the leaves of $\mathcal{H}$, and*
(iii) *M contains γ as a principal curve to which the vector field $Y = \phi X$ is tangent, and along which the restricted components of A coincide with the given solution of the ODE system.*

In section 3, we consider a condition on the shape operator that contains the totally η-umbilical condition. We show that some non-Hopf hypersurfaces related to Theorem A also satisfy this condition. We shall prove

THEOREM 1. *Let M be a real hypersurface in $M^2(c)$, $c \neq 0$. Suppose there exists a smooth function $a : M \to \mathbf{R}$ such that $g(AX, Y) = a g(X, Y)$ for any vector fields X and Y orthogonal to the structure vector field ξ. Then M is locally congruent to one of the following*;

(a) *a totally η-umbilical real hypersurface*,
(b) *a ruled real hypersurface*,
(c) *a real hypersurface with the shape operator*

$$
A = \begin{pmatrix} \alpha & h & 0 \\ h & a & 0 \\ 0 & 0 & a \end{pmatrix}
$$

with respect to an orthonormal frame $\{\xi, e_1, \phi e_1\}$, and for a principal curve $\gamma(t)$ ($t \in I$, $\gamma' = \phi e_1$), satisfying

$$
\begin{aligned}
\alpha' &= h\alpha - 2ha, \\
h' &= c - a^2 + a\alpha + h^2, \\
a' &= 3ha.
\end{aligned}
\tag{3}
$$

The corresponding result for a real hypersurface of $M^n(c)$, $n \geq 3$, $c \neq 0$, is given by Ortega [11].

THEOREM B ([11]). *Let M be a real hypersurface of $M^n(c)$, $n \geq 3$, $c \neq 0$. Suppose there exists a smooth function $a : M \to \mathbf{R}$ such that $g(AX, Y) = a g(X, Y)$ for any vector fields X and Y orthogonal to ξ. Then M is locally congruent to one of the following*:

(a) *a totally η-umbilical real hypersurface*,
(b) *a ruled real hypersurface*.

If the curvature tensor R and the Ricci operator S satisfy $R(X, Y) \cdot S = 0$ for any vector fields X and Y, then M is called a *pseudo-Ryan* hypersurface. In [3], as a result of Theorem A, Ivey and Ryan gave an example of a pseudo-Ryan hypersurface in $M^2(c)$.

THEOREM C ([3]). *Let $\alpha(t), h(t), \lambda(t), \nu(t)$ be analytic solutions defined on I of the system* (2)*, such that h is nowhere zero and the equation*

$$h^2\nu^2 + (4c + \lambda\nu)(\alpha(\lambda - \nu) - h^2) = 0$$

holds. Then the hypersurface M constructed by Theorem A is a non-Hopf pseudo-Ryan hypersurface.

In Section 4, we consider a condition that the Ricci operator S is *pseudo-parallel*, that is,

$$R(X, Y) \cdot S = F(X \wedge Y) \cdot S\,,$$

where F is a function, which contains the pseudo-Ryan condition. We define the wedge product $X \wedge Y$ by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$$

for vectors X and Y. It is shown that a ruled real hypersurface in non-flat complex space form $M^2(c)$, $c \neq 0$ cannot have a pseudo-parallel Ricci operator ([2], [6]). In [6], Inoguchi gave a conjecture that real hypersurfaces in a non-flat complex space form $M^2(c)$ with pseudo-parallel Ricci operator are Hopf. We prove the following theorem which gives the negative result.

THEOREM 2. *Let M be a real hypersurface in $M^2(c)$, $c \neq 0$. If the Ricci operator S is pseudo-parallel, then M is a Hopf hypersurface or a non-Hopf hypersurface such that the shape operator has the form*

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$$

with respect to an orthonormal frame $\{\xi, e_1, e_2\}$, and

$$0 = (a_2\alpha - a_1\alpha + h^2)(3c + a_1a_2 - a_1\alpha + h^2) - a_2^2h^2\,,$$

$$F = c + a_1\alpha - h^2 .$$

If there exists a function F such that

$$g((R(X, Y)S)Z, W) = Fg(((X \wedge Y)S)Z, W) ,$$

for all $X,\ Y,\ Z$ and W orthogonal to ξ, then the real hypersurface is said to be pseudo η-parallel, which is a weaker condition than pseudo-parallel. When M is a real hypersurface of $M^n(c)$, $n \geq 3$, $c \neq 0$, the author showed the following.

THEOREM D ([7]). *Let M be a real hypersurface in a complex space form $M^n(c)$, $c \neq 0$, $n \geq 3$. Then the Ricci operator S is pseudo η-parallel if and only if M is pseudo-Einstein.*

We remark that a pseudo-Einstein real hypersurface is a Hopf hypersurface.

2. Preliminaries

Let $M^n(c)$ denote the complex space form of complex dimension n (real dimension $2n$) with constant holomorphic sectional curvature $4c$. We denote by J the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ will be denoted by G.

Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. We denote by g the Riemannian metric induced on M from G. We take the unit normal vector field N of M in $M^n(c)$. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)N , \quad JN = -\xi ,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1,1), η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi , \quad \phi\xi = 0 , \quad \eta(\phi X) = 0$$

for any vector field X tangent to M. Moreover, we have

$$g(\phi X, Y) + g(X, \phi Y) = 0 , \quad \eta(X) = g(X, \xi) ,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) .$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N , \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M. We call A the *shape operator* of M. If the shape operator A of M satisfies $A\xi = \alpha\xi$ for some functions α, then M is called a *Hopf hypersurface*.

For the almost contact metric structure on M, we have

$$\nabla_X \xi = \phi AX\,, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi\,.$$

If the shape operator A of a real hypersurface M is of the form $A = aI$, where I is the identity, then M is said to be totally umbilical. In Tashiro-Tachibana [13], it was proved that any real hypersurface of $M^n(c)$, $c \neq 0$, is not totally umbilical. So we need the notion of totally η-umbilical real hypersurfaces, that is, the shape operator A is of the form $A = aI + b\eta \otimes \xi$.

PROPOSITION E ([12]). *The only totally η-umbilical real hypersurfaces in $\mathbf{C}P^n$, $n \geq 2$, are geodesic hyperspheres.*

PROPOSITION F ([9], [10]). *The only totally η-umbilical real hypersurfaces in $\mathbf{C}H^n$, $n \geq 2$, are horospheres, geodesic hyperspheres and tubes over complex hyperbolic hyperplane.*

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y)Z = c\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY\,, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}\,.$$

From the equation of Gauss, the Ricci operator S of M satisfies

$$\begin{aligned} g(SX, Y) = (2n+1)cg(X, Y) - 3c\eta(X)\eta(Y) \\ + \mathrm{Tr}A g(AX, Y) - g(AX, AY)\,, \end{aligned} \tag{4}$$

where $\mathrm{Tr}A$ is the trace of A. The scalar curvature r is defined by

$$r = \mathrm{Tr}S\,.$$

EXAMPLE ([5], [8]). Let M be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, and let T_0 be the distribution defined by $T_0(x) = \{X \in T_x(M) | X \perp \xi\}$ for $x \in M$. If T_0 is integrable and its integral manifold is a totally geodesic submanifold $M^{n-1}(c)$, then M is called a *ruled real hypersurface*. Let $\gamma(t)$ $(t \in I)$ be an arbitrary (regular) curve in $M^n(c)$. Then for every $t \in I$ there exists a totally geodesic submanifold $M^{n-1}(c)$ in $M^n(c)$ which is orthogonal to the plane τ_t spanned by $\{\gamma'(t), J\gamma'(t)\}$. Here we denote by $M_t^{n-1}(c)$ such a totally geodesic submanifold. Let $M = \{x \in M_t^{n-1}(c) | t \in I\}$. Then the construction of M asserts that M is a ruled real hypersurface in $M^n(c)$. Moreover, the construction of M tells that there are many ruled real hypersurfaces. The *holomorphic sectional curvature* H of a ruled real hypersurface M is $4c$ (see [4]).

3. A condition of shape operator

In this section, we prove Theorem 1. As a consequence of this theorem, we have the following.

COROLLARY 1. *Let M be a real hypersurface in $M^2(c)$, $c \neq 0$. Suppose there exists a constant $a : M \to \mathbf{R}$ such that $g(AX, Y) = ag(X, Y)$ for any vector fields X and Y orthogonal to ξ. Then M is locally congruent to one of the following*:

(a) *a totally η-umbilical real hypersurface*,
(b) *a ruled real hypersurface.*

First we prove the following

LEMMA 1. *Let M be a real hypersurface in $M^2(c)$. Suppose that there exists a smooth function $a : M \to \mathbf{R}$ such that $g(AX, Y) = ag(X, Y)$ for any vector fields X and Y orthogonal to ξ, then M is a Hopf hypersurface or the shape operator A is represented by a matrix*

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a & 0 \\ 0 & 0 & a \end{pmatrix} \tag{5}$$

with respect to a suitable orthonormal frame $\{\xi, u, \phi u\}$, locally.

PROOF. By the assumption, we can take an orthonormal frame $\{\xi, e_1, e_2 = \phi e_1\}$, such that A is represented by a matrix

$$A = \begin{pmatrix} \alpha & k_1 & k_2 \\ k_1 & a & 0 \\ k_2 & 0 & a \end{pmatrix},$$

locally, for suitable functions k_1, k_2 and α. We take a unit vector u that satisfies

$$A\xi = \alpha\xi + hu\,, \quad g(\xi, u) = 0\,,$$

where h is a function. Then $\{\xi, u, \phi u\}$ is another orthonormal frame of $T_x(M)$. We can represent u as

$$u = u_1 e_1 + u_2 e_2\,.$$

Using this, we have

$$\begin{aligned} g(Au, u) &= g(A(u_1 e_1 + u_2 e_2), u_1 e_1 + u_2 e_2) \\ &= u_1^2 g(Ae_1, e_1) + 2u_1 u_2 g(Ae_1, e_2) + u_2^2 g(Ae_2, e_2) \end{aligned}$$

$$= a(u_1^2 + u_2^2) = a\,.$$

Similarly, we also have $g(A\phi u, \phi u) = a$. Moreover, we obtain

$$\begin{aligned} g(Au, \phi u) &= g(A(u_1e_1 + u_2e_2), u_1\phi e_1 + u_2\phi e_2) \\ &= g(A(u_1e_1 + u_2e_2), u_1e_2 - u_2e_1) \\ &= -u_1u_2a + u_2u_1a = 0\,. \end{aligned}$$

From these equations, there exists an orthonormal frame $\{\xi, u, \phi u\}$ of $T_x(M)$ such that the shape operator A is of the form (5). □

Using the equation of Codazzi, we obtain

LEMMA 2. *Let M be a real hypersurface in $M^2(c)$, $c \neq 0$. If there exists an orthonormal frame $\{\xi, e_1, e_2\}$ on a sufficiently small neighborhood $\mathcal{N}$ of $x \in M$ such that the shape operator A can be represented as*

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

then we have

$$(e_1a) = 0\,, \tag{6}$$

$$(-2c + 2a^2 - 2a\alpha) + hg(\nabla_{e_1}e_2, e_1) + (e_2h) = 0\,, \tag{7}$$

$$(e_2a) = 3ha\,, \tag{8}$$

$$(\xi a) = hg(\nabla_{e_2}e_1, e_2)\,, \tag{9}$$

$$(e_2h) = c + a\alpha - a^2 + h^2\,, \tag{10}$$

$$-h(\alpha - 3a) + hg(\nabla_\xi e_2, e_1) + (e_2\alpha) = 0\,, \tag{11}$$

$$(e_1h) - (\xi a) = 0\,, \tag{12}$$

$$(e_1\alpha) - (\xi h) = 0\,. \tag{13}$$

PROOF. By the equation of Codazzi, we have

$$g((\nabla_{e_2}A)e_1 - (\nabla_{e_1}A)e_2, e_2) = 0\,.$$

On the other hand, we have

$$\begin{aligned} &g((\nabla_{e_2}A)e_1 - (\nabla_{e_1}A)e_2, e_2) \\ &\quad = g(\nabla_{e_2}(Ae_1) - A\nabla_{e_2}e_1 - \nabla_{e_1}(Ae_2) + A\nabla_{e_1}e_2, e_2) \\ &\quad = -(e_1a)\,. \end{aligned}$$

Thus we obtain (6). By the similar computation, we have our equations. □

When M is not a Hopf hypersurface, then we can take $x \in M$ and a sufficiently small neighborhood of x, on which $h \neq 0$. In the following, we consider the case that $a \neq 0$ on the neighborhood.

LEMMA 3. *If* $h \neq 0$ *and* $a \neq 0$, *then*,

$$\begin{aligned}
\nabla_{e_1} e_1 &= \frac{-c + a^2 - a\alpha + h^2}{h} e_2 , \\
\nabla_{e_1} e_2 &= \frac{c - a^2 + a\alpha - h^2}{h} e_1 - a\xi , \\
\nabla_{e_2} e_1 &= a\xi, \quad \nabla_{e_2} e_2 = 0 , \\
\nabla_{\xi} e_1 &= a e_2, \quad \nabla_{\xi} e_2 = -a e_1 - h\xi .
\end{aligned}$$

Moreover, *we have*

$$\begin{aligned}
&e_1 a = 0 , \quad e_1 h = 0 , \quad e_1 \alpha = 0 , \\
&e_2 a = 3ha , \quad e_2 h = c - a^2 + a\alpha + h^2 , \quad e_2 \alpha = h\alpha - 2ha , \\
&\xi a = 0 , \quad \xi h = 0 , \quad \xi \alpha = 0 .
\end{aligned}$$

PROOF. First we compute $\nabla_{e_1} e_2$. Using (7) and (10), we have

$$g(\nabla_{e_1} e_2, e_1) = -g(\nabla_{e_1} e_1, e_2) = \frac{c - a^2 + a\alpha - h^2}{h} .$$

Moreover, we obtain $g(\nabla_{e_1} e_2, e_2) = 0$ and

$$g(\nabla_{e_1} e_2, \xi) = -g(e_2, \phi A e_1) = -a .$$

So we have

$$\nabla_{e_1} e_2 = \frac{c - a^2 + a\alpha - h^2}{h} e_1 - a\xi .$$

By the similar computation using Lemma 2, we obtain

$$\begin{aligned}
\nabla_{e_2} e_1 &= \frac{(\xi a)}{h} e_2 + a\xi , \\
\nabla_{e_1} e_1 &= \frac{-c + a^2 - a\alpha + h^2}{h} e_2 , \\
\nabla_{e_2} e_2 &= -\frac{(\xi a)}{h} e_1 .
\end{aligned}$$

We put $g(\nabla_{\xi} e_1, e_2) = P$. Then we have

$$\begin{aligned}
\nabla_{\xi} e_1 &= P e_2 , \\
\nabla_{\xi} e_2 &= -P e_1 - h\xi .
\end{aligned}$$

Since $[X, Y] = \nabla_X Y - \nabla_Y X$ for any X and Y tangent to M, we have

$$\begin{aligned}[e_1, e_2]a &= (\nabla_{e_1} e_2 - \nabla_{e_2} e_1)a \\ &= \frac{c - a^2 + a\alpha - h^2}{h}(e_1 a) - a(\xi a) - \frac{(\xi a)}{h}(e_2 a) - a(\xi a) \\ &= -5a(\xi a)\,.\end{aligned}$$

For the last equality, we use (6) and (8). On the other hand, by (6) and (12), we obtain

$$\begin{aligned}[e_1, e_2]a &= e_1(e_2 a) - e_2(e_1 a) = e_1(3ha) \\ &= 3(e_1 h)a + 3h(e_1 a) = 3(\xi a)a\,.\end{aligned}$$

These equations imply $a(\xi a) = 0$, and hence

$$(\xi a) = (e_1 h) = 0\,. \tag{14}$$

Similarly, we have

$$\begin{aligned}[e_1, \xi]a &= (\nabla_{e_1}\xi - \nabla_\xi e_1)a \\ &= 3ha(a - P)\,.\end{aligned}$$

Using (6) and (14), we obtain

$$[e_1, \xi]a = e_1(\xi a) - \xi(e_1 a) = 0\,.$$

Since $ha \neq 0$, we have $a = P$. Thus, by (11),

$$(e_2\alpha) = h\alpha - 2ha\,.$$

By the similar computation for $[e_2, \xi]a$ and $[e_2, \xi]h$, we also have

$$(\xi h) = 0\,, \quad (e_1\alpha) = 0\,, \quad (\xi\alpha) = 0\,.$$

Combining these results, we have our assertion. □

(Proof of Theorem 1)

When M is a Hopf hypersurface, then we have $AX = aX + b\eta(X)\xi$ for some function b. This means that M is totally η-umbilical.

Next we consider the case that M is not Hopf. Then we can take a point x and a sufficiently small neighborhood of x, on which $h \neq 0$. If $a = 0$ on the neighborhood, we see that the real hypersurface is locally congruent to a ruled real hypersurface.

Finally, we suppose $ha \neq 0$. We can take a unit-speed analytic framed curve $\gamma(t)$ which satisfy $\gamma' = e_2$. Then Lemma 3 shows that a, h and α satisfy (3). We note that the existence of this non-Hopf hypersurface is induced by Theorem A.

Conversely, such hypersurfaces satisfy the condition $g(AX, Y) = ag(X, Y)$ for any vector fields X and Y orthogonal to ξ.

4. 3-dimensional real hypersurfaces with pseudo-parallel Ricci operator

If the Ricci operator S of a real hypersurface M satisfies

$$R(X, Y) \cdot S = F(X \wedge Y) \cdot S\,,$$

where F is a function, then the Ricci operator S is said to be *pseudo-parallel*.

To prove Theorem 2, first we show the following.

LEMMA 4. *Let M be a real hypersurface in $M^2(c)$, $c \neq 0$. If the Ricci operator S is pseudo-parallel, then M is a Hopf hypersurface or the shape operator A is represented by the matrix*

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$$

with respect to an orthonormal frame $\{\xi, e_1, e_2\}$, locally.

PROOF. Suppose that M is not a Hopf hypersurface. We take an orthonormal frame $\{\xi, e_1, e_2\}$, where we have put $e_2 = \phi e_1$. Then there are smooth functions a_1, a_2, h_1 and h_2 such that A is represented by a matrix

$$A = \begin{pmatrix} \alpha & h_1 & h_2 \\ h_1 & a_1 & 0 \\ h_2 & 0 & a_2 \end{pmatrix}$$

with respect to $\{\xi, e_1, e_2\}$, locally. We remark that $h_1 \neq 0$ or $h_2 \neq 0$. From (4), we have

$$\begin{aligned} Se_1 &= (5c + a_1a_2 + a_1\alpha - h_1^2)e_1 - h_1h_2e_2 + a_2h_1\xi\,, \\ Se_2 &= (5c + a_1a_2 + a_2\alpha - h_2^2)e_2 - h_1h_2e_1 + a_1h_2\xi\,, \\ S\xi &= a_2h_1e_1 + a_1h_2e_2 + (2c + a_1\alpha + a_2\alpha - h_1^2 - h_2^2)\xi\,. \end{aligned} \tag{15}$$

Since S is symmetric, there exists an another orthonormal frame $\{v_1, v_2, v_3\}$ that satisfies $Sv_1 = av_1$, $Sv_2 = bv_2$, $Sv_3 = dv_3$ for some functions a, b and d. Since S is pseudo-parallel, we have

$$\begin{aligned} &g(R(X, Y)SZ, W) - g(SR(X, Y)Z, W) \\ &\quad = F\{g(Y, SZ)(X, W) - g(X, SZ)g(Y, W) - g(Y, Z)g(SX, W) \\ &\qquad + g(X, Z)g(SY, W)\}\,. \end{aligned} \tag{16}$$

Putting $X = W = v_1$ and $Y = Z = v_2$, we obtain

$$(b - a)(K(v_1, v_2) - F) = 0,$$

where the sectional curvature K for the plane spanned by v_1 and v_2 is denoted by

$$K(v_1, v_2) = g(R(v_1, v_2)v_2, v_1)\,.$$

By the similar computation, we have

$$(d-a)(K(v_1, v_3) - F) = 0\,,$$
$$(d-b)(K(v_2, v_3) - F) = 0\,.$$

If $a \neq b$, $b \neq c$ and $c \neq a$, then we see that

$$F = K(v_1, v_2) = K(v_1, v_3) = K(v_2, v_3)\,.$$

Thus we obtain

$$\begin{aligned} a = g(Se_1, e_1) &= K(v_1, v_2) + K(v_1, v_3) \\ &= K(v_1, v_2) + K(v_2, v_3) \\ &= g(Se_2, e_2) = b\,. \end{aligned}$$

This is a contradiction. From the fact that no real hypersurfaces of $M^2(c)$ are Einstein, it is sufficient to consider the case that $a = b \neq d$. Then we have

$$F = K(v_1, v_3) = K(v_2, v_3)\,,$$

from which

$$g(Sv_3, v_3) = K(v_1, v_3) + K(v_2, v_3) = d = 2F\,.$$

So the Ricci operator S is represented by a matrix

$$S = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2F \end{pmatrix} \tag{17}$$

with respect to $\{v_1, v_2, v_3\}$.

On the other hand, from the assumption, we have

$$g((R(e_1, e_2)S)e_1, e_1) = Fg(((e_1 \wedge e_2) \cdot S)e_1, e_1).$$

By the equation of Gauss and (15), we obtain

$$\begin{aligned} g((R(e_1, e_2)S)e_1, e_1) &= g(R(e_1, e_2)Se_1, e_1) - g(R(e_1, e_2)e_1, Se_1) \\ &= 2c(g(e_2, Se_1)g(e_1, e_1) - g(\phi e_1, Se_1)(\phi e_2, e_1) \\ &\quad -2g(\phi e_1, e_2)g(\phi Se_1, e_1)) \\ &\quad +2g(Ae_2, Se_1)g(Ae_1, e_1) \\ &= -8ch_1h_2\,. \end{aligned}$$

By (15), we have

$$\begin{aligned} Fg(((e_1 \wedge e_2) \cdot S)e_1, e_1) &= F(g(e_2, Se_1)(e_1, e_1) - g(Se_1, e_1)g(e_2, e_1) - g(e_2, e_1)g(Se_1, e_1) \\ &\quad +g(e_1, e_1)g(Se_2, e_1)) \end{aligned}$$

$$= 2F(Se_1, e_2) = -2Fh_1h_2\,.$$

From these equations, we see that

$$(4c - F)h_1h_2 = 0\,. \tag{18}$$

Similarly, substituting $X = Z = e_1$, $Y = \xi$, $W = e_2$ and $X = Z = e_2$, $Y = \xi$, $W = e_1$, we obtain

$$\begin{aligned} 0 &= h_2\{(c - F)a_1 - a_2h_1^2 + a_1a_2\alpha - a_1h_2^2\}\,, \\ 0 &= h_1\{(c - F)a_2 - a_1h_2^2 + a_1a_2\alpha - a_2h_1^2\}\,, \end{aligned} \tag{19}$$

respectively.

To prove the lemma, it is sufficient to consider the case that $h_1h_2 \neq 0$. From (18) and (19), we have $4c = F$ and

$$(c - F)(a_1 - a_2) = 0\,.$$

Since $c - F = -3c \neq 0$, we obtain $a_1 = a_2$. By Lemma 1, the shape operator A is represented as

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & k & 0 \\ 0 & 0 & k \end{pmatrix} \tag{20}$$

with respect to an orthonormal frame $\{\xi, u, \phi u\}$. Thus we have our assertion. □

(Proof of Theorem 2)

Suppose that M is not a Hopf hypersurface. We put $h_1 = h \neq 0$, locally. Then the Ricci operator S is represented by a matrix

$$S = \begin{pmatrix} 2c + a_1\alpha + a_2\alpha - h^2 & a_2h & 0 \\ a_2h & 5c + a_1a_2 + a_1\alpha - h^2 & 0 \\ 0 & 0 & 5c + a_1a_2 + a_2\alpha \end{pmatrix} \tag{21}$$

with respect to $\{\xi, e_1, e_2\}$. By (17), we see that $5c + a_1a_2 + a_2\alpha = 2F$ or $5c + a_1a_2 + a_2\alpha = a$.

First we suppose $5c + a_1a_2 + a_2\alpha = 2F$. From (17) and (21), taking a trace of S, the scalar curvature r satisfies

$$r = 2(a + F) = 12c + 2a_1a_2 + 2a_1\alpha + 2a_2\alpha - 2h^2\,.$$

So we have

$$a = F + c + a_1\alpha - h^2\,. \tag{22}$$

We put

$$S' = \begin{pmatrix} 2c + a_1\alpha + a_2\alpha - h^2 & a_2h \\ a_2h & 5c + a_1a_2 + a_1\alpha - h^2 \end{pmatrix}\,.$$

Then the eigenvalues of S' are solutions of the equation

$$
\begin{aligned}
0 &= \det(xI - S') \\
&= (x - 5c - a_1a_2 - a_1\alpha + h^2)(x - 2c - a_1\alpha - a_2\alpha + h^2) \qquad (23)\\
&\quad - a_2^2h^2 .
\end{aligned}
$$

Since a is an eigenvalue of S', using (22), we have

$$0 = (F - 4c - a_1a_2)(F - c - a_2\alpha) - a_2^2h^2 .$$

By $a_1a_2 = 2F - 5c - a_2\alpha$, we obtain

$$0 = -(F - c - a_2\alpha)^2 - a_2^2h^2 ,$$

which induces $F - c - a_2\alpha = 0$ and $a_2^2h^2 = 0$. Since $h \neq 0$, we have $a_2 = 0$ and $F = c$. By $5c + a_1a_2 + a_2\alpha = 2F$, we have

$$0 = 2F - 5c = -3c .$$

This is a contradiction.

Next, we suppose $5c + a_1a_2 + a_2\alpha = a$. Then we have

$$r = 2(a + F) = 2(6c + a_1a_2 + a_1\alpha + a_2\alpha - h^2) .$$

From these equations, we have

$$F = c + a_1\alpha - h^2 . \qquad (24)$$

Since a and $2F$ are the solutions of (23), we obtain

$$0 = (a_2\alpha - a_1\alpha + h^2)(3c + a_1a_2 - a_1\alpha + h^2) - a_2^2h^2 . \qquad (25)$$

So we see that if S is pseudo-parallel, then M is a Hopf hypersurface or the shape operator A is represented by

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \qquad (26)$$

with respect to an orthonormal frame $\{\xi, e_1, e_2\}$ and satisfies (24), (25). So we have our theorem. $\square$

In [2], Cho, Hamada and Inoguchi gave a classification of pseudo-parallel Hopf hypersurfaces.

THEOREM G ([2]). *The Hopf hypersurfaces in* $\mathbf{C}P^2(c)$ *or* $\mathbf{C}H^2(c)$ *with pseudo-parallel Ricci operator are locally holomorphically congruent to a horosphere in* $\mathbf{C}H^2(c)$*, a geodesic hypersphere in* $\mathbf{C}P^2(c)$ *or* $\mathbf{C}H^2(c)$*, a homogeneous tube over* $\mathbf{C}H^1(c)$ *in* $\mathbf{C}H^2$*, a* *non-homogeneous real hypersurface which is realized as a tube over a certain holomorphic curve in* $\mathbf{C}P^2(c)$ *with radius* $\pi/\sqrt{4c}$, *or a Hopf hypersurface in* $\mathbf{C}H^2(c)$ *with* $A\xi = 0$.

Using Theorem A, we see the following result (see Corollary 3 in [3]).

COROLLARY 2. *Let* $\alpha(t)$, $h(t)$, $\lambda(t)$, $\nu(t)$ *be analytic solutions defined for* $t \in I$ *of the system* (2), *such that* h *is nowhere zero and*

$$0 = (\nu\alpha - \lambda\alpha + h^2)(3c + \lambda\nu - \lambda\alpha + h^2) - \nu^2 h^2 .$$

Then the hypersurface M *constructed by Theorem A is a non-Hopf pseudo-parallel hypersurface with* $F = c + \lambda\alpha - h^2$.

PROOF. We suppose that A satisfies (24)–(26) and $a_1 = \lambda$, $a_2 = \nu$. It is sufficient to show that

$$g((R(X,Y)S)Z, W) - Fg(((X \wedge Y)S)Z, W) = 0$$

for all $X = e_i$, $Y = e_j$, $Z = e_k$ and $W = e_l$, $1 \le i, j, k, l \le 3$, where $e_3 = \xi$. Using (15), (24)–(26) and the equation of Gauss, we have

$$\begin{aligned}
g((R(e_1,e_2)S)e_1, e_2) &- Fg(((e_1 \wedge e_2)S)e_1, e_2) \\
&= g(R(e_1,e_2)Se_1, e_2) - g(R(e_1,e_2)e_1, Se_2) \\
&\quad - F(-g(Se_1,e_1) + g(Se_2,e_2)) \\
&= -4cg(Se_1,e_1) + g(Ae_2,Se_1)g(Ae_1,e_2) - g(Ae_1,Se_1)g(Ae_2,e_2) \\
&\quad + 4cg(Se_2,e_2) - g(Ae_1,Se_2)g(Ae_2,e_1) + g(Ae_1,e_1)g(Ae_2,Se_2) \\
&\quad - F(-g(Se_1,e_1) + g(Se_2,e_2)) \\
&= (a_2\alpha - a_1\alpha + h^2)(4c - F + a_1a_2) - a_2^2h^2 \\
&= (a_2\alpha - a_1\alpha + h^2)(3c + a_1a_2 - a_1\alpha + h^2) - a_2^2h^2 \\
&= 0 .
\end{aligned}$$

Similarly, for all $X = e_i$, $Y = e_j$, $Z = e_k$ and $W = e_l$, we can show that

$$g((R(X,Y)S)Z, W) - Fg(((X \wedge Y)S)Z, W) = 0$$

by the straightforward computation. □

REMARK. If the shape operator A satisfies (25), (26) and $a_1 = a_2 = 0$, then we have $c = h = 0$ by (10). Thus a ruled real hypersurface is not pseudo-parallel (see [2] and [6]).

References

[1] J. BERNDT, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. Reine Angew. Math. **395** (1989), 132–141.

[2] J. T. CHO, T. HAMADA and J. INOGUCHI, On three dimensional real hypersurfaces in complex space forms, Tokyo J. Math. **33** (2010), 31–47.

[3] T. A. IVEY and P. J. RYAN, The *-Ricci tensor for hypersurfaces in $\mathbf{CP}^n$ and $\mathbf{C}H^n$, Tokyo J. Math. **34** (2011), 445–471.

[4] M. KIMURA, Sectional curvatures of holomorphic planes on a real hypersurface in $P^n(\mathbf{C})$, Math. Ann. **276** (1987), 487–497.

[5] M. KIMURA and S. MAEDA, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299–311.

[6] J. INOGUCHI, Real hypersurfaces in complex space forms with pseudo-parallel Ricci operator, Differ. Geom. Dyn. Syst. **14** (2012), 69–89.

[7] M. KON, Ricci Pseudo η-parallel real hypersurfaces of a complex space form, Nihonkai Math. J. **24** (2013), 45–55.

[8] M. LOHNHERR and H. RECKZIEGEL, On ruled real hypersurfaces in complex space forms, Geom. Dedicata **74** (1999), 267–286.

[9] S. MONTIEL and A. ROMERO, On some real hypersurfaces of a complex hyperbolic space, Geom. Dedicata **20** (1986), 245–261.

[10] R. NIEBERGALL and P. J. RYAN, Real hypersurfaces in complex space forms, *Tight and Taut Submanifolds*, (Eds: T. E. Cecil and S. S. Chern), Math. Sci. Res. Inst. Publ. 32, Cambridge Univ. Press, Cambridge, 1997, 233–305.

[11] M. ORTEGA, Classifications of real hypersurfaces in complex space forms by means of curvature conditions, Bull. Belg. Math. Soc. **9** (2002), 351–360.

[12] R. TAKAGI, Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan **27** (1975), 43–53.

[13] Y. TASHIRO and S. TACHIBANA, On Fubinian and C-Fubinian manifolds, Kōdai Math. Sem. Rep. **15** (1963), 176–183.

Present Address:
FACULTY OF EDUCATION,
SHINSHU UNIVERSITY,
6-RO, NISHINAGANO, NAGANO CITY 380–8544, JAPAN.
e-mail: mayuko_k@shinshu-u.ac.jp