TOKYO J. MATH.
VOL. 39, NO. 2, 2016

New Trigonometric Identities and Reciprocity Laws of Generalized Dedekind Sums

Genki SHIBUKAWA

Osaka University

(Communicated by K. Matsuno)

Abstract. In this paper, we prove new trigonometric identities, which are product-to-sum type formulas for the higher derivatives of the cotangent and cosecant functions. Furthermore, from specializations of our formulas, we derive various known and new reciprocity laws of generalized Dedekind sums.

1. Introduction

The so-called *Dedekind sums* and their reciprocity laws have been studied by several famous mathematicians in the past. For an overview of previously defined generalized Dedekind sums, we refer to Section 1 and 2 in [1]. Let $\cot^{(m)}$ denote the m-th derivative of the cotangent function. In [1], for $a_0, a_1, \dots, a_r, m_0, m_1, \dots, m_r \in \mathbb{Z}_{\geq 1}, w_0, w_1, \dots, w_r \in \mathbb{C}$, Beck introduced Dedekind cotangent sums

$$\frac{1}{a_0^{m_0}} \sum_{k \bmod a_0} \prod_{j=1}^{r} \cot^{(m_j-1)} \left(\pi \left(a_j \frac{k+w_0}{a_0} - w_j \right) \right) ,$$

where the sum is taken over $k \bmod a_0$ for which the summand is not singular. The Dedekind cotangent sums include special various generalizations of Dedekind sums expressed by the cotangent functions and their higher derivatives. Moreover, under some conditions for $a_0, \dots, a_r, w_0, \dots, w_r$, Beck computed the residue of

$$\cot^{(m_0-1)}(\pi(a_0 z - w_0)) \prod_{l=1}^{r} \cot^{(m_l-1)}(\pi(a_l z - w_l)) \tag{1.1}$$

and derived various reciprocity laws of the Dedekind cotangent sums, which are not only the already known results by Dedekind, Rademacher, Apostol, Carlitz, Mikolás, Dieter, Zagier, but also truly new ones. However, since his method needs to study certain conditions for singular points of (1.1), we have to prove the reciprocity laws individually.

Received November 7, 2014; revised March 9, 2016
Mathematics Subject Classification: 11L03, 11F20

On the other hand, Fukuhara [5] introduced an analogue of the Dedekind sum which was formed by replacing the cotangent functions in the Dedekind sum by the cosecant functions, and proved its reciprocity laws. For example, Fukuhara treated the following formulas. Let p and q are relatively prime positive integers.

(0) (Proposition 1.3 in [4] or (1.1) in [5]) For any complex number z,

$$pq\cot(pz)\cot(qz) = -\cot^{(1)}(z) - pq + q\sum_{\mu=1}^{p-1}\cot\left(\frac{\pi q\mu}{p}\right)\cot\left(z-\frac{\pi\mu}{p}\right) + p\sum_{\mu=1}^{q-1}\cot\left(\frac{\pi p\mu}{q}\right)\cot\left(z-\frac{\pi\mu}{q}\right). \tag{1.2}$$

(1) ((1.2) in [5]) If q is even, then

$$pq\cot(pz)\csc(qz) = -\cot^{(1)}(z) + q\sum_{\mu=1}^{p-1}\csc\left(\frac{\pi q\mu}{p}\right)\cot\left(z-\frac{\pi\mu}{p}\right) + p\sum_{\mu=1}^{q-1}(-1)^{\mu}\cot\left(\frac{\pi p\mu}{q}\right)\cot\left(z-\frac{\pi\mu}{q}\right). \tag{1.3}$$

(2) ((1.4) in [5]) If q is odd, then

$$pq\cot(pz)\csc(qz) = -\csc^{(1)}(z) + q\sum_{\mu=1}^{p-1}\csc\left(\frac{\pi q\mu}{p}\right)\csc\left(z-\frac{\pi\mu}{p}\right) + p\sum_{\mu=1}^{q-1}(-1)^{\mu}\cot\left(\frac{\pi p\mu}{q}\right)\csc\left(z-\frac{\pi\mu}{q}\right). \tag{1.4}$$

(3) ((1.3) in [5]) If $p+q$ is even, then

$$pq\csc(pz)\csc(qz) = -\cot^{(1)}(z) + q\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\pi q\mu}{p}\right)\cot\left(z-\frac{\pi\mu}{p}\right) + p\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\pi p\mu}{q}\right)\cot\left(z-\frac{\pi\mu}{q}\right). \tag{1.5}$$

(4) ((1.5) in [5]) If $p+q$ is odd, then

$$pq\csc(pz)\csc(qz) = -\csc^{(1)}(z) + q\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\pi q\mu}{p}\right)\csc\left(z-\frac{\pi\mu}{p}\right) + p\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\pi p\mu}{q}\right)\csc\left(z-\frac{\pi\mu}{q}\right), \tag{1.6}$$

where $\csc^{(1)}(z)$ is the derivative of $\csc(z)$.

Fukuhara also pointed out that these formulas can be regarded as a one parameter deformation of the reciprocity laws of some Dedekind sums, or as a generating function of the reciprocity laws of some Dedekind-Apostol sums. Actually, by comparing the coefficients of the Laurent expansion of (1.2) at $z = 0$, we obtain the reciprocity laws of the Dedekind-Apostol sums

$$s_N(q;\,p) := \frac{1}{2^{N+1}p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\pi q\mu}{p}\right)\cot^{(N-1)}\left(\frac{\pi\mu}{p}\right)$$

as follows.

$$s_1(q;\,p) + s_1(p;\,q) = \frac{p^2+q^2+1-3pq}{12pq}\,, \tag{1.7}$$

$$s_{2k+1}(q;\,p) + s_{2k+1}(p;\,q) = \frac{1}{2pq}\frac{B_{2k+2}}{k+1} + \frac{B_{2k+2}}{(2k+1)(2k+2)}(p^{2k+1}q^{-1} + p^{-1}q^{2k+1})$$
$$- (2k)!\sum_{l=1}^{k}\frac{B_{2l}B_{2k+2-2l}}{(2l)!(2k+2-2l)!}p^{2l-1}q^{2k+1-2l}\,. \tag{1.8}$$

Here, k is a positive integer and $\{B_m\}_{m=0,1,\ldots}$ are the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}t^m\,.$$

As described above, from product-to-sum type formulas for some trigonometric functions like (1.2), we easily obtain the reciprocity laws for various generalized Dedekind sums. In this article, taking into account the above investigations, we present a detailed calculation of

$$\prod_{l=1}^{j_I} a_l^{m_l}\cot^{(m_l-1)}(\pi(a_l z - w_l))\prod_{l=j_I+1}^{j_I+j_{II}} a_l^{m_l}\csc^{(m_l-1)}(\pi(a_l z - w_l)) \tag{1.9}$$

and give a sum expression of (1.9) by using the higher derivatives for the cotangent and cosecant functions. It is regarded as a product-to-sum type formula for the higher derivatives of the cotangent and cosecant functions. We prove the formula under completely generic conditions, and use only Liouville's theorem and a limit of some periodic functions at $z \to i\infty$. Thus, our proof is more general than the method of Beck, and simpler than Fukuhara's proof which needs some non-trivial trigonometric identities. Furthermore, from various specializations of our formula, we derive various reciprocity laws of generalized Dedekind sums uniformly, which include the results in [1] and [5] et al.

The content of this paper is as follows. In Section 2, we introduce the main objects $\varphi_N^{(I)}(z)$ and $\varphi_N^{(II)}(z)$ replacing the higher derivatives of the cotangent and cosecant functions, and list their fundamental properties. Section 3 is the main part of this article. Under general situations for the parameters, we provide a product-to-sum type formula for

$$\prod_{l=1}^{j_I} a_l^{m_l} \varphi_{m_l}^{(I)}(a_l z - w_l) \prod_{l=j_I+1}^{j_I+j_{II}} a_l^{m_l} \varphi_{m_l}^{(II)}(a_l z - w_l)$$

and derive new generalized reciprocity laws by writing down some specializations of the main theorem. In Section 4, we give more explicit expressions of our reciprocity laws under some conditions. With these specializations, we show that our main results contain many formulas for generalized Dedekind sums shown by several famous mathematicians. Finally, in Section 5, we present possible future researches on variations of our formulas.

2. Preliminaries

Throughout the paper, we denote the ring of rational integers by $\mathbb{Z}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$ and $i := \sqrt{-1}$. Further we use the notation:

$$\mathfrak{R} := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1\}.$$

From Walker's book [7], we recall two kinds of periodic functions which play central roles in this article. For a positive integer N, the following periodic functions are defined by

$$\varphi_N^{(J)}(z) := \frac{1}{z^N} + \sum_{n=1}^{\infty} (-1)^{n\delta_{J,II}} \left(\frac{1}{(z+n)^N} + \frac{1}{(z-n)^N} \right) \quad (J = I,\ II). \tag{2.1}$$

In [7], $\varphi_N^{(I)}(z)$ and $\varphi_N^{(II)}(z)$ are denoted by $E_N(z)$ and $G_N(z)$ respectively. Here are the main properties of $\varphi_N^{(J)}(z)$ $(J = I, II)$ from [7].

Periodicity For any $\mu \in \mathbb{Z}$,

$$\varphi_N^{(J)}(z+\mu) = (-1)^{\mu\delta_{J,II}} \varphi_N^{(J)}(z). \tag{2.2}$$

Derivation For any $N \in \mathbb{Z}_{\geq 0}$,

$$\varphi_{N+1}^{(J)}(z) = \frac{(-1)^N}{N!} \left(\frac{d}{dz} \right)^N \varphi_1^{(J)}(z). \tag{2.3}$$

In particular,

$$\frac{d\varphi_N^{(J)}}{dz}(z) = -N\varphi_{N+1}^{(J)}(z). \tag{2.4}$$

Laurent expansions Let $\zeta(s)$ be the Riemann zeta function and

$$\alpha_\mu^{(I)} := \begin{cases} 2\zeta(\mu) = (-1)^{\frac{\mu}{2}+1}\frac{B_\mu}{\mu!}(2\pi)^\mu & (\text{if } \mu \text{ is even}) \\ 0 & (\text{if } \mu \text{ is odd}) \end{cases}, \tag{2.5}$$

$$\alpha_\mu^{(II)} := \begin{cases} 2(1-2^{1-\mu})\zeta(\mu) = 2(2^{\mu-1}-1)(-1)^{\frac{\mu}{2}+1}\frac{B_\mu}{\mu!}\pi^\mu & (\text{if } \mu \text{ is even}) \\ 0 & (\text{if } \mu \text{ is odd}) \end{cases}. \tag{2.6}$$

Then, around $z = 0$, we have

$$\varphi_N^{(J)}(z) = \frac{1}{z^N} + (-1)^N \sum_{\nu\ge 0} \binom{N+\nu-1}{N-1} \alpha_{N+\nu}^{(J)} z^\nu , \tag{2.7}$$

where $\binom{N+\nu-1}{N-1}$ is the binomial coefficient. More generally, the following lemma holds. For $X \subset \mathbb{C}$, we put

$$\delta_X(z) := \begin{cases} 1 & (\text{if } z \in X) \\ 0 & (\text{if } z \notin X) \end{cases}$$

and define the signature by

$$\operatorname{sgn}^{(J)}(z_0\,; a, w) := (-1)^{(az_0-w)\delta_{J,II}} \delta_{\mathbb{Z}}(az_0 - w) . \tag{2.8}$$

If $\delta_{\mathbb{Z}}(az_0 - w) = 1$, then

$$\varphi_N^{(J)}(z - (az_0 - w)) = \operatorname{sgn}^{(J)}(z_0\,; a, w)\varphi_N^{(J)}(z)$$

by the periodicity (2.2).

LEMMA 2.1. *For any $a, m \in \mathbb{Z}_{\ge 1}$ and $w, z_0 \in \mathbb{C}$, we have*

$$a^m \varphi_m^{(J)}(az - w) = \operatorname{sgn}^{(J)}(z_0\,; a, w)\,(z - z_0)^{-m} + \sum_{\nu\ge 0} A_\nu^{(J)}(z_0\,; a, m, w)\,(z - z_0)^\nu . \tag{2.9}$$

Here,

$$(m)_\nu := \begin{cases} m(m+1)\cdots(m+\nu-1) & (\textit{if } \nu \ge 1) \\ 1 & (\textit{if } \nu = 0) \end{cases}.$$

$$\begin{aligned} A_\nu^{(J)}(z_0\,; a, m, w) :=& (-1)^m \operatorname{sgn}^{(J)}(z_0\,; a, w) \binom{m+\nu-1}{m-1} \alpha_{m+\nu}^{(J)} a^{m+\nu} \delta_{\mathbb{Z}}(az_0 - w) \\ &+ (-1)^\nu \frac{(m)_\nu}{\nu!} a^{m+\nu} \operatorname*{Res}_{z=z_0} \frac{\varphi_{m+\nu}^{(J)}(az - w)}{z - z_0}\, dz\, (1 - \delta_{\mathbb{Z}}(az_0 - w)) \\ =& \begin{cases} (-1)^m \operatorname{sgn}^{(J)}(z_0\,; a, w)\binom{m+\nu-1}{m-1}\alpha_{m+\nu}^{(J)} a^{m+\nu} & (\textit{if } \delta_{\mathbb{Z}}(az_0 - w) = 1) \\ (-1)^\nu \frac{(m)_\nu}{\nu!} \varphi_{m+\nu}^{(J)}(az_0 - w)\, a^{m+\nu} & (\textit{if } \delta_{\mathbb{Z}}(az_0 - w) = 0) \end{cases}. \end{aligned}$$

PROOF. If $az_0 - w \in \mathbb{C} \setminus \mathbb{Z}$, that means $\delta_{\mathbb{Z}}(az_0 - w) = 0$, then z_0 is not a pole of $a^m \varphi_m^{(J)}(az - w)$ and

$$\left(\frac{d}{dz}\right)^{\nu} a^m \varphi_m^{(J)}(az - w)\Big|_{z=z_0} = (-1)^{\nu}(m)_{\nu}\varphi_{m+\nu}^{(J)}(az_0 - w)a^{m+\nu}.$$

Thus, from the Taylor expansion of $a^m \varphi_m^{(J)}(az - w)$ at $z = z_0$, we have

$$a^m \varphi_m^{(J)}(az - w) = \sum_{\nu \geq 0}(-1)^{\nu}\frac{(m)_{\nu}}{\nu!}\varphi_{m+\nu}^{(J)}(az_0 - w)a^{m+\nu}(z - z_0)^{\nu}.$$

If $az_0 - w \in \mathbb{Z}$, that is the $\delta_{\mathbb{Z}}(az_0 - w) = 1$ case, then there exists $\mu \in \mathbb{Z}$ such that

$$z_0 = \frac{w + \mu}{a}.$$

Hence, by using the periodicity (2.2) and the Laurent expansion at $z_0 = 0$ (2.7), we have

$$\begin{aligned}
a^m \varphi_m^{(J)}(az - w) &= \operatorname{sgn}^{(J)}(z_0\,; a, w)a^m \varphi_m^{(J)}(az - w - (az_0 - w)) \\
&= \operatorname{sgn}^{(J)}(z_0\,; a, w)a^m \varphi_m^{(J)}(a(z - z_0)) \\
&= \operatorname{sgn}^{(J)}(z_0\,; a, w)\,(z - z_0)^{-m} \\
&\quad + (-1)^m \operatorname{sgn}^{(J)}(z_0\,; a, w)\sum_{\nu \geq 0}\binom{m + \nu - 1}{m - 1}\alpha_{m+\nu}^{(J)}a^{m+\nu}(z - z_0)^{\nu}.
\end{aligned}$$

□

Relationship with the cotangent and cosecant functions

$$\varphi_1^{(I)}(z) = \pi\cot(\pi z),\ \ \varphi_1^{(II)}(z) = \pi\csc(\pi z). \tag{2.10}$$

Limit at $z \to i\infty$

LEMMA 2.2.

$$\lim_{z \to i\infty}\varphi_N^{(J)}(z) = -\pi i \delta_{N,1}\delta_{J,I}. \tag{2.11}$$

PROOF. For $N \geq 2$, since $\varphi_N^{(J)}(z)$ is absolutely convergent, $\lim_{z \to i\infty}\varphi_N^{(J)}(z) = 0$. If $N = 1$, then

$$\begin{aligned}
\lim_{z \to i\infty}\varphi_1^{(I)}(z) &= \lim_{z \to i\infty}\pi\cot(\pi z) = -\pi i, \\
\lim_{z \to i\infty}\varphi_1^{(II)}(z) &= \lim_{z \to i\infty}\pi\csc(\pi z) = 0.
\end{aligned}$$

□

3. Main results

Let $r \in \mathbb{Z}_{\geq 2}$, $[r] := \{1, \dots, r\}$. Suppose that $\mathbf{a} := (a_1, \dots, a_r)$, $\mathbf{m} := (m_1, \dots, m_r) \in \mathbb{Z}^r_{\geq 1}$, $\mathbf{w} = (w_1, \dots, w_r) \in \mathfrak{R}^r$. Let $\mathbf{j} = (j_I, j_{II}) \in \mathbb{Z}^2_{\geq 0}$ with $j_I + j_{II} = r$. Further, we put

$$
\begin{aligned}
|\mathbf{m}| &:= m_1 + \cdots + m_r \, , \\
R_\rho &:= (R_\rho(\mathbf{a}, \mathbf{w}) =)\{\Lambda \subset [r] \mid \delta_{\mathbb{Z}}(a_\lambda \rho - w_\lambda) = 1 \text{ (for all } \lambda \in \Lambda)\} \, , \\
\Lambda^c &:= [r] \setminus \Lambda \, , \\
K^{\pm}_{n,\Lambda} &:= \left\{ (\nu_k)_{k \in \Lambda^c} \in \mathbb{Z}^{|\Lambda^c|}_{\geq 0} \middle| n = \pm \left(\sum_{k \in \Lambda^c} \nu_k - \sum_{\lambda \in \Lambda} m_\lambda \right) \right\} , \\
\delta^{(\mathbf{j},l)}_I &:= \sum_{j=1}^{j_I} \delta_{j,l} = \begin{cases} 1 & (\text{if } 1 \leq l \leq j_I) \\ 0 & (\text{otherwise}) \end{cases} \, , \\
\delta^{(\mathbf{j},l)}_{II} &:= \sum_{j=j_I+1}^{j_I+j_{II}} \delta_{j,l} = \begin{cases} 1 & (\text{if } j_I + 1 \leq l \leq j_I + j_{II}) \\ 0 & (\text{otherwise}) \end{cases} \, ,
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{sgn}^{(\mathbf{j},l)}(z_0\,; a, w) :=& \operatorname{sgn}^{(I)}(z_0\,; a, w)\delta^{(\mathbf{j},l)}_I + \operatorname{sgn}^{(II)}(z_0\,; a, w)\delta^{(\mathbf{j},l)}_{II} \\
=& (-1)^{(az_0 - w)\delta^{(\mathbf{j},l)}_{II}} \delta_{\mathbb{Z}}(az_0 - w) \, , \\
\varphi^{(\mathbf{j},l)}_N(z) :=& \varphi^{(I)}_N(z)\delta^{(\mathbf{j},l)}_I + \varphi^{(II)}_N(z)\delta^{(\mathbf{j},l)}_{II} \, , \\
\alpha^{(\mathbf{j},l)}_\nu :=& \alpha^{(I)}_\nu \delta^{(\mathbf{j},l)}_I + \alpha^{(II)}_\nu \delta^{(\mathbf{j},l)}_{II} \, , \\
A^{(\mathbf{j},l)}_\nu(z_0\,; a, m, w) :=& A^{(I)}_\nu(z_0\,; a, m, w)\delta^{(\mathbf{j},l)}_I + A^{(II)}_\nu(z_0\,; a, m, w)\delta^{(\mathbf{j},l)}_{II} \, , \\
\mathscr{A}^{(\mathbf{j},\pm)}_n(z_0\,; \mathbf{a}, \mathbf{m}, \mathbf{w}) :=& \sum_{\Lambda \in R_{z_0}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K^{\pm}_{n,\Lambda}} \prod_{l \in \Lambda} \operatorname{sgn}^{(\mathbf{j},l)}(z_0\,; a_l, w_l) \prod_{u \in \Lambda^c} \{A^{(\mathbf{j},u)}_{\nu_u}(z_0\,; a_u, m_u, w_u)\}.
\end{aligned}
$$

We remark that the set $K^-_{n,\Lambda}$ is finite. For convenience, we consider the following two cases according to $\mathbf{a}$ and $\mathbf{j}$.

$$\text{Case } I : j_{II} = 0, \text{ or } \sum_{l=j_I+1}^{r} a_l \text{ is even.}$$

$$\text{Case } II : \sum_{l=j_I+1}^{r} a_l \text{ is odd.}$$

The following theorem is the main result of this article.

THEOREM 3.1. *Let*

$$\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) := \prod_{l=1}^{r} a_l^{m_l}\varphi_{m_l}^{(\mathbf{j},l)}(a_l z - w_l).$$

Here, if a j_J is zero, we omit the product terms for $\varphi_{m_l}^{(J)}(a_l z - w_l)$. For Case J, we have

$$\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) = \cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} + \sum_{n=1}^{|\mathbf{m}|}\sum_{\rho}\mathscr{A}_n^{(\mathbf{j},-)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w})\varphi_n^{(J)}(z-\rho)\,, \tag{3.1}$$

where ρ runs over all poles of $\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ in $\mathfrak{R}$.

PROOF. We denote $\Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ by the right hand side of (3.1). We claim that for all Case I, II,

$$\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) - \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) = 0.$$

First, we consider the Laurent expansion of $\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ at ρ

$$\begin{aligned}
\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) &= \prod_{l=1}^{r}\left\{\operatorname{sgn}^{(\mathbf{j},l)}(\rho\,;a_l,w_l)(z-\rho)^{-m_l} + \sum_{\nu_l\geq 0} A_{\nu}^{(\mathbf{j},l)}(\rho\,;a_l,m_l,w_l)(z-\rho)^{\nu_l}\right\}\\
&= \prod_{l=1}^{r}\{\operatorname{sgn}^{(\mathbf{j},l)}(\rho\,;a_l,w_l)\}(z-\rho)^{-|\mathbf{m}|} + \sum_{N=1}^{r-1}\sum_{1\leq\lambda_1<\ldots<\lambda_N\leq r}\\
&\quad\cdot\left\{\sum_{\substack{|\mathbf{m}|>\sum_{k=1}^{N}(m_{\lambda_k}+\nu_{\lambda_k}),\\ \nu_{\lambda_1},\ldots,\nu_{\lambda_N}\geq 0}} + \sum_{\substack{|\mathbf{m}|\leq\sum_{k=1}^{N}(m_{\lambda_k}+\nu_{\lambda_k}),\\ \nu_{\lambda_1},\ldots,\nu_{\lambda_N}\geq 0}}\right\}\prod_{l\in[r]\backslash\{\lambda_1,\ldots,\lambda_N\}}\operatorname{sgn}^{(\mathbf{j},l)}(\rho\,;a_l,w_l)\\
&\quad\cdot\prod_{u=1}^{N} A_{\nu_{\lambda_u}}^{(\mathbf{j},\lambda_u)}(\rho\,;a_{\lambda_u},m_{\lambda_u},w_{\lambda_u})(z-\alpha_j)^{-|\mathbf{m}|+\sum_{k=1}^{N}(m_{\lambda_k}+\nu_{\lambda_k})}\\
&= \sum_{n=1}^{|\mathbf{m}|}\mathscr{A}_n^{(\mathbf{j},-)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w})(z-\rho)^{-n} + \sum_{\mu\geq 0}\mathscr{A}_{\mu}^{(\mathbf{j},+)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w})(z-\rho)^{\mu}\,.
\end{aligned} \tag{3.2}$$

Further, from (2.2), we have

$$\Phi^{(\mathbf{j})}(z+\mu\,;\mathbf{a},\mathbf{m},\mathbf{w}) = (-1)^{\mu\delta_{J,II}}\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})\,. \tag{3.3}$$

For Case I, II, $\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ and $\Phi_N^{(J)}(z)$ are thus periodic functions with same period. In addition, $\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) - \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ is entire by (2.9) and (3.2).

We remark that $\varphi_N^{(J)}(z)$ is bounded on the set $\mathfrak{R}_1 := \mathfrak{R} \cap \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \geq 1 + 2\max_\rho |\mathrm{Im}\,\rho|\}$ from (2.11). Thus, $\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) - \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ is also bounded on $\mathfrak{R}_1$. Hence, $\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) - \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})$ is bounded on $\mathbb{C}$ by the periodicity. By the well-known Liouville's theorem, there exists a constant $c^{(\mathbf{j},J)}(\mathbf{a},\mathbf{m},\mathbf{w})$ such that

$$\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) - \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) = c^{(\mathbf{j},J)}(\mathbf{a},\mathbf{m},\mathbf{w})\,.$$

If we restrict $z \in \mathbb{C}$ and $w_1,\ldots,w_r \in \mathfrak{R}$ to $\mathbb{R}$, then $A_{\nu_u}^{(\mathbf{j},u)}(\rho\,;a_u,m_u,w_u)$, $c^{(\mathbf{j},J)}(\mathbf{a},\mathbf{m},\mathbf{w}) \in \mathbb{R}$. In addition, we calculate

$$\begin{aligned}
\lim_{z\to i\infty} \Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) &= \prod_{l=1}^{r} a_l^{m_l}(-\pi i)\delta_{m_l,1}\delta_I^{(\mathbf{j},l)} \\
&= (-i)^r \pi^r \delta_{j_{II},0} \prod_{l=1}^{r} a_l \delta_{m_l,1} \\
&= \left\{\cos\left(\frac{\pi r}{2}\right) - i\sin\left(\frac{\pi r}{2}\right)\right\} \pi^r \delta_{j_{II},0} \prod_{l=1}^{r} a_l\delta_{m_l,1}\,, \qquad (3.4)
\end{aligned}$$

$$\lim_{z\to i\infty} \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) = \cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} - \pi i\delta_{J,I}\sum_\rho \mathscr{A}_1^{(\mathbf{j},-)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w})\,. \qquad (3.5)$$

Thus, we have

$$\begin{aligned}
c^{(\mathbf{j},J)}(\mathbf{a},\mathbf{m},\mathbf{w}) &= \mathrm{Re}\,\{c^{(\mathbf{j},J)}(\mathbf{a},\mathbf{m},\mathbf{w})\} \\
&= \mathrm{Re}\left\{\lim_{z\to i\infty}\{\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) - \Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})\}\right\} = 0\,.
\end{aligned}$$

□

As a corollary of this theorem, we obtain the following theorem immediately.

THEOREM 3.2. (1) *We have*

$$\sum_\rho \mathscr{A}_1^{(\mathbf{j},-)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w}) = \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1}\,. \qquad (3.6)$$

(2) *For any $z_0 \in \mathbb{C}$, $\mu \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{n=1}^{|\mathbf{m}|}\sum_\rho \mathscr{A}_n^{(\mathbf{j},-)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w})A_\mu^{(J)}(z_0\,;1,n,\rho) = -\cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{\mu,0}\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} + \mathscr{A}_\mu^{(\mathbf{j},+)}(z_0\,;\mathbf{a},\mathbf{m},\mathbf{w})\,. \qquad (3.7)$$

PROOF. (1) It follows from (3.4) and (3.5) immediately.

(2) We expand both sides of (3.1) into the Laurent series of $z-z_0$ and compare the coefficients of $(z-z_0)^{\mu}$ of both sides. Indeed, by replacing ρ with z_0 in (3.2), we have

$$\operatorname*{Res}_{z=z_0} \frac{\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})}{(z-z_0)^{\mu+1}}\,dz = \mathscr{A}_{\mu}^{(\mathbf{j},+)}(z_0\,;\mathbf{a},\mathbf{m},\mathbf{w})\,.$$

On the other hand, from (3.1),

$$\operatorname*{Res}_{z=z_0} \frac{\Psi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w})}{(z-z_0)^{\mu+1}}\,dz = \cos\left(\frac{\pi r}{2}\right)\pi^r \delta_{\mu,0}\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} + \sum_{n=1}^{|\mathbf{m}|}\sum_{\rho}\mathscr{A}_n^{(\mathbf{j},-)}(\rho\,;\mathbf{a},\mathbf{m},\mathbf{w})A_{\mu}^{(J)}(z_0\,;1,n,\rho)\,.$$

Therefore, we have the conclusion. □

As we will see later, Theorem 3.2 (1) is a generalization of various reciprocity laws in [1]. Theorem 3.2 (2) means (3.1) is regarded as a generating function of the reciprocity laws (3.7). Hence, this result and proof are generalizations of Theorem 1.2 in [4] and its proof.

REMARK 3.3. For Theorems 3.1 and 3.2, we also have the other expressions that are useful for writing down various specific examples. Let

$$d_l^{(\mu)} := \#\left\{a_j \in \{a_1,\dots,a_r\},\, w_j \in \{w_1,\dots,w_r\},\, \mu_j \in \{0,\dots,a_j-1\} \,\middle|\, \frac{w_l+\mu}{a_l} = \frac{w_j+\mu_j}{a_j}\right\}.$$

Our main result (3.1) becomes

$$\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) = \cos\left(\frac{\pi r}{2}\right)\pi^r \delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} + \sum_{n=1}^{|\mathbf{m}|}\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1}\frac{1}{d_l^{(\mu_l)}}\mathscr{A}_n^{(\mathbf{j},-)}\left(\frac{w_l+\mu_l}{a_l}\,;\mathbf{a},\mathbf{m},\mathbf{w}\right)\varphi_n^{(J)}\left(z-\frac{w_l+\mu_l}{a_l}\right). \tag{3.8}$$

Similarly, (3.6) and (3.7) are

$$\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1}\frac{1}{d_l^{(\mu_l)}}\mathscr{A}_1^{(\mathbf{j},-)}\left(\frac{w_l+\mu_l}{a_l}\,;\mathbf{a},\mathbf{m},\mathbf{w}\right) = \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1}\,. \tag{3.9}$$

and for any $z_0 \in \mathbb{C}$, $\mu \in \mathbb{Z}_{\geq 0}$,

$$\sum_{n=1}^{|\mathbf{m}|}\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1}\frac{1}{d_l^{(\mu_l)}}\mathscr{A}_n^{(\mathbf{j},-)}\left(\frac{w_l+\mu_l}{a_l}\,;\mathbf{a},\mathbf{m},\mathbf{w}\right)A_{\mu}^{(J)}\left(z_0\,;1,n,\frac{w_l+\mu_l}{a_l}\right)$$
$$= -\cos\left(\frac{\pi r}{2}\right)\pi^r \delta_{\mu,0}\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} + \mathscr{A}_{\mu}^{(\mathbf{j},+)}(z_0\,;\mathbf{a},\mathbf{m},\mathbf{w}) \tag{3.10}$$

respectively.

4. Some special cases of the main theorem

By specializing our main results, we derive various reciprocity laws of the generalized Dedekind sums.

4.1. The multiplicity free case. In this subsection, we assume for all distinct $k, l \in [r]$ and $\mu_k = 0, 1, \ldots, a_k - 1$, $\mu_l = 0, 1, \ldots, a_l - 1$,

$$\frac{w_k + \mu_k}{a_k} \neq \frac{w_l + \mu_l}{a_l}.$$

Under this condition, all poles of $a_j^{m_j} \varphi_{m_j}^{(J)}(a_j z - w_j)$ for each $j = 1, \ldots, r$ on $\mathfrak{R}$

$$\frac{w_j + \mu_j}{a_j} \quad (j = 1, \ldots, r, \text{ and } \mu_j = 0, \ldots, a_j - 1)$$

are multiplicity free. That means

$$\delta_{\mathbb{Z}}\left(a_l \frac{w_j + \mu_j}{a_j} - w_l\right) = \begin{cases} 1 & (\text{if } l = j) \\ 0 & (\text{if } l \neq j) \end{cases},$$

and $d_l^{(\mu_l)} = 1$ for all $1 \le l \le r$, $0 \le \mu_l \le a_l - 1$. Hence, we have

$$R_{\frac{w_j+\mu_j}{a_j}} = \{\phi, \{j\}\},$$

$$\operatorname{sgn}^{(\mathbf{j},l)}\left(\frac{w_j + \mu_j}{a_j}; a_l, w_l\right) = \begin{cases} (-1)^{\mu_j \delta_{II}^{(\mathbf{j},j)}} & (\text{if } l = j) \\ 0 & (\text{if } l \neq j) \end{cases},$$

$$\begin{aligned}\mathscr{A}_n^{(\mathbf{j},-)}\left(\frac{w_j + \mu_j}{a_j}; \mathbf{a}, \mathbf{m}, \mathbf{w}\right) &= \sum_{(\nu_k)_{k\in[r]\setminus\{j\}} \in K_{n,\{j\}}^-} (-1)^{\mu_j \delta_{II}^{(\mathbf{j},j)}} \\ &\quad \cdot \prod_{u\in[r]\setminus\{j\}} \left\{ A_{\nu_u}^{(\mathbf{j},u)}\left(\frac{w_j + \mu_j}{a_j}; a_u, m_u, w_u\right)\right\} \\ &= \sum_{\substack{n = m_j - \sum_{1\le k\neq j\le r} \nu_k, \\ \nu_1,\ldots,\nu_r \ge 0}} (-1)^{\mu_j \delta_{II}^{(\mathbf{j},j)}} \\ &\quad \cdot \prod_{1\le u\neq j\le r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(\mathbf{j},u)}\left(a_u \frac{w_j + \mu_j}{a_j} - w_u\right) a_u^{m_u+\nu_u} \right\}.\end{aligned}$$

Therefore, Theorems 3.1, 3.2 become as follows.

THEOREM 4.1. *We have*

$$\Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{w}) = \cos\left(\frac{\pi r}{2}\right)\pi^r \delta_{j_{II},0}\prod_{l=1}^{r} a_l \delta_{m_l,1} + \sum_{n=1}^{\max_{j\in[r]}\{m_j\}} \sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1} \sum_{\substack{n=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\dots,\nu_r\ge 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \cdot \prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi_{m_u+\nu_u}^{(\mathbf{j},u)}\left(a_u\frac{w_l+\mu_l}{a_l}-w_u\right)a_u^{m_u+\nu_u}\right\} \cdot \varphi_n^{(J)}\left(z-\frac{w_l+\mu_l}{a_l}\right). \tag{4.1}$$

THEOREM 4.2. (1) *We have*

$$\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1} \sum_{\substack{1=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\dots,\nu_r\ge 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \cdot \prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi_{m_u+\nu_u}^{(\mathbf{j},u)}\left(a_u\frac{w_l+\mu_l}{a_l}-w_u\right)a_u^{m_u+\nu_u}\right\} = \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1}. \tag{4.2}$$

In particular, for $m_1=\cdots=m_r=1$,

$$\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1}(-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \prod_{1\le u\ne l\le r}\left\{\varphi_1^{(\mathbf{j},u)}\left(a_u\frac{w_l+\mu_l}{a_l}-w_u\right)a_u\right\} = \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\delta_{j_{II},0}\prod_{l=1}^{r} a_l. \tag{4.3}$$

(2) *For any* $\mu\in\mathbb{Z}_{\ge 0}$ *and* $z_0\in\mathfrak{R}$,

$$\sum_{n=1}^{\max_{j\in[r]}\{m_j\}} \sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1} \sum_{\substack{n=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\dots,\nu_r\ge 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \cdot \prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi_{m_u+\nu_u}^{(\mathbf{j},u)}\left(a_u\frac{w_l+\mu_l}{a_l}-w_u\right)a_u^{m_u+\nu_u}\right\} A_\mu^{(J)}\left(z_0\,;1,n,\frac{w_l+\mu_l}{a_l}\right) = -\cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{\mu,0}\delta_{j_{II},0}\prod_{l=1}^{r} a_l\delta_{m_l,1} + \mathscr{A}_\mu^{(\mathbf{j},+)}(z_0\,;\mathbf{a},\mathbf{m},\mathbf{w}). \tag{4.4}$$

EXAMPLE 4.3. When we consider the case of $(j_I, j_{II}) = (r, 0)$, (4.2) is none other than Beck's reciprocity (Theorem 2 in [1])

$$\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1}\sum_{\substack{1=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\dots,\nu_r\ge 0}}\prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi^{(I)}_{m_u+\nu_u}\left(a_u\frac{w_l+\mu_l}{a_l}-w_u\right)a_u^{m_u+\nu_u}\right\}$$

$$= \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\prod_{l=1}^{r}a_l\delta_{m_l,1}\,. \tag{4.5}$$

On the other hand, by putting $(j_I, j_{II}) = (0, r)$, we obtain a cosecant analogue of Beck's result

$$\sum_{l=1}^{r}\sum_{\mu_l=0}^{a_l-1}\sum_{\substack{1=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\dots,\nu_r\ge 0}}(-1)^{\mu_l}\prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi^{(II)}_{m_u+\nu_u}\left(a_u\frac{w_l+\mu_l}{a_l}-w_u\right)a_u^{m_u+\nu_u}\right\}=0\,. \tag{4.6}$$

4.2. The $\mathbf{w} = (0, \dots, 0)$ case. In this subsection, we assume $\mathbf{w} = \mathbf{0} := (0, \dots, 0)$ and that $a_1, \dots, a_r$ are pairwise relatively prime. Under this condition, all poles of $a_j^{m_j}\varphi^{(J)}_{m_j}(a_j z)$ for each $j = 1, \dots, r$ on $\mathfrak{R}$ are

$$\frac{\mu_j}{a_j} \quad (j = 1, \dots, r, \text{ and } \mu_j = 0, \dots, a_j - 1)\,.$$

Further, $\delta_{\mathbb{Z}}(0) = 1$ and for all $l = 1, \dots, r$, and $\mu_j = 1, \dots, a_j - 1$,

$$\delta_{\mathbb{Z}}\left(a_l\frac{\mu_j}{a_j}\right) = \begin{cases}1 & (\text{if } l = j)\\ 0 & (\text{if } l \ne j)\end{cases}, \quad d_v^{(\mu_v)} = \begin{cases} r & (\text{if } \mu_v = 0)\\ 1 & (\text{otherwise})\end{cases}.$$

Hence, we have

$$R_{\frac{\mu_j}{a_j}} = \{\phi, \{j\}\},\ R_0 = 2^{[r]}\,,$$

$$\operatorname{sgn}^{(\mathbf{j},l)}\left(\frac{\mu_j}{a_j}; a_l, 0\right) = \begin{cases}(-1)^{\mu_j\delta_{II}^{(\mathbf{j},j)}} & (\text{if } l = j)\\ 0 & (\text{if } l \ne j)\end{cases}, \ \operatorname{sgn}^{(\mathbf{j},l)}(0; a_l, 0) = 1\,,$$

$$\mathscr{A}_n^{(\mathbf{j},-)}\left(\frac{\mu_j}{a_j}; \mathbf{a}, \mathbf{m}, \mathbf{0}\right) = \sum_{(\nu_k)_{k\in[r]\setminus\{j\}}\in K^-_{n,\{j\}}}(-1)^{\mu_j\delta_{II}^{(\mathbf{j},j)}}\prod_{u\in[r]\setminus\{j\}}\left\{A_{\nu_u}^{(\mathbf{j},u)}\left(\frac{\mu_j}{a_j}; a_u, m_u, 0\right)\right\}$$

$$= \sum_{\substack{n=m_j-\sum_{1\le k\ne j\le r}\nu_k,\\ \nu_1,\dots,\nu_r\ge 0}}(-1)^{\mu_j\delta_{II}^{(\mathbf{j},j)}}$$

$$\cdot \prod_{1\le u\ne j\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi^{(\mathbf{j},u)}_{m_u+\nu_u}\left(a_u\frac{\mu_j}{a_j}\right)a_u^{m_u+\nu_u}\right\},$$

$$\mathscr{A}^{(\mathbf{j},-)}_n(0\,;\mathbf{a},\mathbf{m},\mathbf{0}) = M^{(\mathbf{j})}_n(\mathbf{a},\mathbf{m})\,.$$

Here, we put

$$\begin{aligned} M^{(\mathbf{j})}_n(\mathbf{a},\mathbf{m}) &:= \sum_{\Lambda\in R_0}\sum_{(\nu_k)_{k\in\Lambda^c}\in K^-_{n,\Lambda}}\prod_{u\in\Lambda^c}(-1)^{m_u}\binom{m_u+\nu_u-1}{m_u-1}\alpha^{(\mathbf{j},u)}_{m_u+\nu_u}a_u^{m_u+\nu_u} \\ &= \sum_{N=1}^{r-1}\sum_{1\le\lambda_1<\ldots<\lambda_N\le r}\sum_{\substack{n=|\mathbf{m}|-\sum_{k=1}^{N}(\nu_{\lambda_k}+m_{\lambda_k}),\\ \nu_{\lambda_1},\ldots,\nu_{\lambda_N}\ge 0}}\prod_{u=1}^{N}(-1)^{m_u}\binom{m_u+\nu_u-1}{m_u-1}\alpha^{(\mathbf{j},u)}_{m_u+\nu_u}a_u^{m_u+\nu_u}\,. \end{aligned}$$

Then, Theorems 3.1, 3.2 degenerate to the following results.

THEOREM 4.4. *We obtain*

$$\begin{aligned} \Phi^{(\mathbf{j})}(z\,;\mathbf{a},\mathbf{m},\mathbf{0}) &= \cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{j_{II},0}\prod_{l=1}^{r}a_l\delta_{m_l,1} + \sum_{n=1}^{|\mathbf{m}|}M^{(\mathbf{j})}_n(\mathbf{a},\mathbf{m})\varphi^{(J)}_n(z) \\ &+ \sum_{n=1}^{\max_{j\in[r]}\{m_j\}}\sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1}\sum_{\substack{n=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\ldots,\nu_r\ge 0}}(-1)^{\mu_l\delta^{(\mathbf{j},l)}_{II}} \\ &\cdot\prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi^{(\mathbf{j},u)}_{m_u+\nu_u}\left(a_u\frac{\mu_l}{a_l}\right)a_u^{m_u+\nu_u}\right\}\varphi^{(J)}_n\left(z-\frac{\mu_l}{a_l}\right). \end{aligned} \tag{4.7}$$

THEOREM 4.5. (1) *We have*

$$\begin{aligned} &\sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1}\sum_{\substack{1=m_l-\sum_{1\le k\ne l\le r}\nu_k,\\ \nu_1,\ldots,\nu_r\ge 0}}(-1)^{\mu_l\delta^{(\mathbf{j},l)}_{II}}\prod_{1\le u\ne l\le r}\left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi^{(\mathbf{j},u)}_{m_u+\nu_u}\left(a_u\frac{\mu_l}{a_l}\right)a_u^{m_u+\nu_u}\right\} \\ &= \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\delta_{j_{II},0}\prod_{l=1}^{r}a_l\delta_{m_l,1} - M^{(\mathbf{j})}_1(\mathbf{a},\mathbf{m})\,. \end{aligned} \tag{4.8}$$

In particular, for $\mathbf{m}=\mathbf{1}:=(1,\ldots,1)$,

$$\sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1}(-1)^{\mu_l\delta^{(\mathbf{j},l)}_{II}}\prod_{1\le u\ne l\le r}\left\{\varphi^{(\mathbf{j},u)}_1\left(a_u\frac{\mu_l}{a_l}\right)a_u\right\} = \pi^{r-1}\sin\left(\frac{\pi r}{2}\right)\delta_{j_{II},0}\prod_{l=1}^{r}a_l - M^{(\mathbf{j})}_1(\mathbf{a},\mathbf{1})\,. \tag{4.9}$$

(2) *For any* $\mu \in \mathbb{Z}_{\geq 0}$ *and* $z_0 \in \mathfrak{R}$,

$$\sum_{n=1}^{\max_{j\in[r]}\{m_j\}} \sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1} \sum_{\substack{n=m_l-\sum_{1\leq k\neq l\leq r}\nu_k \\ \nu_1,\ldots,\nu_r\geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}}$$

$$\cdot \prod_{1\leq u\neq l\leq r} \left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi_{m_u+\nu_u}^{(\mathbf{j},u)}\left(a_u\frac{\mu_l}{a_l}\right)a_u^{m_u+\nu_u}\right\}A_{\mu}^{(J)}\left(z_0\,;1,n,\frac{\mu_l}{a_l}\right)$$

$$= -\cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{\mu,0}\delta_{j_{II},0}\prod_{l=1}^{r}a_l\delta_{m_l,1} - \sum_{n=1}^{|\mathbf{m}|}M_n^{(\mathbf{j})}(\mathbf{a},\mathbf{m})A_{\mu}^{(J)}(z_0\,;1,n,0)$$

$$+ \mathscr{A}_{\mu}^{(\mathbf{j},+)}(z_0\,;\mathbf{a},\mathbf{m},\mathbf{w})\,. \tag{4.10}$$

In particular, by taking $z_0 = 0$,

$$\sum_{n=1}^{\max_{j\in[r]}\{m_j\}} \sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1} \sum_{\substack{n=m_l-\sum_{1\leq k\neq l\leq r}\nu_k \\ \nu_1,\ldots,\nu_r\geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}}$$

$$\cdot \prod_{1\leq u\neq l\leq r} \left\{(-1)^{\nu_u}\frac{(m_u)_{\nu_u}}{\nu_u!}\varphi_{m_u+\nu_u}^{(\mathbf{j},u)}\left(a_u\frac{\mu_l}{a_l}\right)a_u^{m_u+\nu_u}\right\}(-1)^{\mu}\frac{(n)_{\mu}}{\mu!}\varphi_{n+\mu}^{(J)}\left(-\frac{\mu_l}{a_l}\right)$$

$$= -\cos\left(\frac{\pi r}{2}\right)\pi^r\delta_{\mu,0}\delta_{j_{II},0}\prod_{l=1}^{r}a_l\delta_{m_l,1} - \sum_{n=1}^{|\mathbf{m}|}M_n^{(\mathbf{j})}(\mathbf{a},\mathbf{m})(-1)^n\binom{n+\mu-1}{n-1}\alpha_{\mu+n}^{(J)}$$

$$+ \sum_{\Lambda\in R_0}\sum_{(\nu_k)_{k\in\Lambda^c}\in K_{\mu,\Lambda}^{+}}\prod_{u\in\Lambda^c}\left\{(-1)^{m_u}\binom{m_u+\nu_u-1}{m_u-1}\alpha_{m_u+\nu_u}^{(\mathbf{j},u)}a_u^{m_u+\nu_u}\right\}. \tag{4.11}$$

EXAMPLE 4.6. By putting $(j_I, j_{II}) = (r, 0)$ in (4.9), we obtain Zagier's result

$$\pi^{r-1}\sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1}\prod_{u\neq l}\left\{\cot\left(\frac{\pi a_u\mu_l}{a_l}\right)a_u\right\} = \sin\left(\frac{\pi r}{2}\right)\pi^{r-1}\prod_{l=1}^{r}a_l$$

$$-\sum_{N=1}^{r-1}\sum_{1\leq\lambda_1<\ldots<\lambda_N\leq r}\sum_{\substack{r-1-N=\sum_{k=1}^{N}\nu_{\lambda_k}, \\ \nu_{\lambda_1},\ldots,\nu_{\lambda_N}\geq 0}}\prod_{u=1}^{N}\alpha_{\nu_u+1}^{(I)}a_u^{\nu_u+1}. \tag{4.12}$$

Further, in the case of $(j_I, j_{II}) = (0, r)$, we have

$$\pi^{r-1}\sum_{l=1}^{r}\sum_{\mu_l=1}^{a_l-1}(-1)^{\mu_l}\prod_{u\neq l}\left\{\csc\left(\frac{\pi a_u\mu_l}{a_l}\right)a_u\right\} = -\sum_{N=1}^{r-1}\sum_{1\leq\lambda_1<\ldots<\lambda_N\leq r}\sum_{\substack{r-1-N=\sum_{k=1}^{N}\nu_{\lambda_k}, \\ \nu_{\lambda_1},\ldots,\nu_{\lambda_N}\geq 0}}$$

$$\cdot \prod_{u=1}^{N} \alpha_{\nu_u+1}^{(II)} a_u^{\nu_u+1}. \tag{4.13}$$

This is a cosecant version of Zagier's reciprocity law.

4.3. The $r = 2, \mathbf{m} = (1,1)$ case. In this subsection, we assume $r = 2, \mathbf{m} = (1,1)$ and that a_1, a_2 are relatively prime. For this simple case, we obtain more explicit expressions of our main results.

THEOREM 4.7. *Let $I \le K_1 \le K_2 \le II$, and A_1, A_2 denote integers for which $A_1 a_2 + A_2 a_1 = 1$ holds. For the following cases*

$$(K_1, K_2, J) = (I, I, I),\ (I, II, I),\ (I, II, II),\ (II, II, I),\ (II, II, II), \tag{4.14}$$

we have

$$\begin{aligned}
&a_1 a_2 \varphi_1^{(K_1)}(a_1 z - w_1)\varphi_1^{(K_2)}(a_2 z - w_2) \\
&= -\pi^2 a_1 a_2 \delta_{K_1,I}\delta_{K_2,I} \\
&\quad + \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1)\operatorname{sgn}_2^{(K_1,K_2)}((a_1,a_2),(w_1,w_2),(A_1,A_2))\varphi_2^{(J)}(z - (A_1 w_2 + A_2 w_1)) \\
&\quad + a_2 \sum_{\mu_1=0}^{a_1-1}{}' (-1)^{\mu_1 \delta_{K_1,II}} \varphi_1^{(K_2)}\left(a_2 \frac{w_1+\mu_1}{a_1} - w_2\right)\varphi_1^{(J)}\left(z - \frac{w_1+\mu_1}{a_1}\right) \\
&\quad + a_1 \sum_{\mu_2=0}^{a_2-1}{}' (-1)^{\mu_2 \delta_{K_2,II}} \varphi_1^{(K_1)}\left(a_1 \frac{w_2+\mu_2}{a_2} - w_1\right)\varphi_1^{(J)}\left(z - \frac{w_2+\mu_2}{a_2}\right).
\end{aligned} \tag{4.15}$$

Here, the sums run over non-singular points and

$$\begin{aligned}
\operatorname{sgn}_2^{(K_1,K_2)}((a_1,a_2),(w_1,w_2),(A_1,A_2)) := {}& \operatorname{sgn}^{(K_1)}(A_1 w_2 + A_2 w_1\,; a_1+a_2, w_1+w_2)\delta_{K_1,K_2} \\
&+ \operatorname{sgn}^{(K_1)}(A_1 w_2 + A_2 w_1\,; a_1, w_1) \\
&\cdot \operatorname{sgn}^{(K_2)}(A_1 w_2 + A_2 w_1\,; a_2, w_2)(1 - \delta_{K_1,K_2}).
\end{aligned}$$

PROOF. The multiplicity free case

$$\text{i.e. } \frac{w_1+\mu_1}{a_1} \ne \frac{w_2+\mu_2}{a_2} \quad (\mu_1 = 0, 1, \dots, a_1 - 1,\ \mu_2 = 0, 1, \dots, a_2 - 1)$$

has been proved by some special cases of (4.1). It is thus enough to show another case. Since a_1, a_2 are relatively prime and $w_1, w_2 \in \mathfrak{R}$, there exist unique integers $\widetilde{\mu_1} \in \{0, 1, \dots, a_1 - 1\}$ and $\widetilde{\mu_2} \in \{0, 1, \dots, a_2 - 1\}$ such that

$$\rho_0 := \frac{w_1 + \widetilde{\mu_1}}{a_1} = \frac{w_2 + \widetilde{\mu_2}}{a_2},$$

and

$$A_1 w_2 + A_2 w_1 = \rho_0 - (A_1 \widetilde{\mu_2} + A_2 \widetilde{\mu_1}),$$

$$a_1(A_1w_2+A_2w_1)-w_1=-a_1(A_1\widetilde{\mu_2}+A_2\widetilde{\mu_1})+\widetilde{\mu_1}\,,$$
$$a_2(A_1w_2+A_2w_1)-w_2=-a_2(A_1\widetilde{\mu_2}+A_2\widetilde{\mu_1})+\widetilde{\mu_2}\,,$$
$$(a_1+a_2)(A_1w_2+A_2w_1)-(w_1+w_2)=-(a_1+a_2)(A_1\widetilde{\mu_2}+A_2\widetilde{\mu_1})+(\widetilde{\mu_1}+\widetilde{\mu_2})\,.$$

Hence, from (3.1), we have

$$\begin{aligned}
&a_1a_2\varphi_1^{(K_1)}(a_1z-w_1)\varphi_1^{(K_2)}(a_2z-w_2)\\
&=-\pi^2a_1a_2\delta_{K_1,I}\delta_{K_2,I}+(-1)^{\widetilde{\mu_1}\delta_{II}^{(\mathbf{j},1)}+\widetilde{\mu_2}\delta_{II}^{(\mathbf{j},2)}}\varphi_2^{(J)}(z-\rho_0)\\
&\quad+a_2\sum_{\mu_1=0}^{a_1-1}{}'(-1)^{\mu_1\delta_{K_1,II}}\varphi_1^{(K_2)}\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)\varphi_1^{(J)}\left(z-\frac{w_1+\mu_1}{a_1}\right)\\
&\quad+a_1\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2\delta_{K_2,II}}\varphi_1^{(K_1)}\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)\varphi_1^{(J)}\left(z-\frac{w_2+\mu_2}{a_2}\right)\,.
\end{aligned}$$

We remark that under the above five cases (4.14) we have

$$(-1)^{\widetilde{\mu_1}\delta_{II}^{(\mathbf{j},1)}+\widetilde{\mu_2}\delta_{II}^{(\mathbf{j},2)}}=(-1)^{\widetilde{\mu_1}\delta_{K_1,II}\delta_{K_2,II}+\widetilde{\mu_2}\delta_{K_2,II}}\,.$$

By the definition of the signature (2.8) and the periodicity of $\varphi_N^{(J)}$ (2.2),

$$\begin{aligned}
&\operatorname{sgn}_2^{(K_1,K_2)}((a_1,a_2),(w_1,w_2),(A_1,A_2))\varphi_2^{(J)}(z-(A_1w_2+A_2w_1))\\
&=\widetilde{\operatorname{sgn}}_2^{((K_1,K_2),J)}((a_1,a_2),(w_1,w_2),(A_1,A_2))(-1)^{\widetilde{\mu_1}\delta_{K_1,II}\delta_{K_2,II}+\widetilde{\mu_2}\delta_{K_2,II}}\varphi_2^{(J)}(z-\rho_0)\,,
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{\operatorname{sgn}}_2^{((K_1,K_2),J)}((a_1,a_2),(w_1,w_2),(A_1,A_2))\\
&:=(-1)^{(A_1\widetilde{\mu_2}+A_2\widetilde{\mu_1})\{(a_1+a_2)\delta_{K_1,II}+\delta_{J,II}\}+\widetilde{\mu_1}\delta_{K_1,II}(1+\delta_{K_2,II})+\widetilde{\mu_2}(\delta_{K_1,II}+\delta_{K_2,II})}\delta_{K_1,K_2}\\
&\quad+(-1)^{(A_1\widetilde{\mu_2}+A_2\widetilde{\mu_1})(a_1\delta_{K_1,II}+a_2\delta_{K_2,II}+\delta_{J,II})+\widetilde{\mu_1}\delta_{K_1,II}(1+\delta_{K_2,II})}(1-\delta_{K_1,K_2})\,.
\end{aligned}$$

Therefore, we claim that for the above five conditions (4.14),

$$\widetilde{\operatorname{sgn}}_2^{((K_1,K_2),J)}((a_1,a_2),(w_1,w_2),(A_1,A_2))=1$$

and obtain the conclusion. □

EXAMPLE 4.8 (0) (Theorem 2.4 in [2]). $(K_1,K_2,J)=(I,I,I)$.

$$\begin{aligned}
&\cot\pi(a_1z-w_1)\cot\pi(a_2z-w_2)\\
&=-1-\frac{1}{a_1a_2}\delta_{\mathbb{Z}}(a_1w_2-a_2w_1)\cot^{(1)}(\pi(z-(A_1w_2+A_2w_1)))\\
&\quad+\frac{1}{a_1}\sum_{\mu_1=0}^{a_1-1}{}'\cot\left(\pi\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)\right)\cot\left(\pi\left(z-\frac{w_1+\mu_1}{a_1}\right)\right)
\end{aligned}$$

$$+\frac{1}{a_2}\sum_{\mu_2=0}^{a_2-1}{}'\cot\left(\pi\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)\right)\cot\left(\pi\left(z-\frac{w_2+\mu_2}{a_2}\right)\right).$$

(1) $(K_1, K_2, J) = (I, II, I)$.

$$\begin{aligned}
&\cot\pi(a_1z-w_1)\csc\pi(a_2z-w_2)\\
&=-\frac{(-1)^{a_2(A_1w_2+A_2w_1)-w_2}}{a_1a_2}\delta_{\mathbb{Z}}(a_1w_2-a_2w_1)\cot^{(1)}(\pi(z-(A_1w_2+A_2w_1)))\\
&+\frac{1}{a_1}\sum_{\mu_1=0}^{a_1-1}{}'\csc\left(\pi\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)\right)\cot\left(\pi\left(z-\frac{w_1+\mu_1}{a_1}\right)\right)\\
&+\frac{1}{a_2}\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2}\cot\left(\pi\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)\right)\cot\left(\pi\left(z-\frac{w_2+\mu_2}{a_2}\right)\right).
\end{aligned}\tag{4.16}$$

(2) $(K_1, K_2, J) = (I, II, II)$.

$$\begin{aligned}
&\cot\pi(a_1z-w_1)\csc\pi(a_2z-w_2)\\
&=-\frac{(-1)^{a_2(A_1w_2+A_2w_1)-w_2}}{a_1a_2}\delta_{\mathbb{Z}}(a_1w_2-a_2w_1)\csc^{(1)}(\pi(z-(A_1w_2+A_2w_1)))\\
&+\frac{1}{a_1}\sum_{\mu_1=0}^{a_1-1}{}'\csc\left(\pi\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)\right)\csc\left(\pi\left(z-\frac{w_1+\mu_1}{a_1}\right)\right)\\
&+\frac{1}{a_2}\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2}\cot\left(\pi\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)\right)\csc\left(\pi\left(z-\frac{w_2+\mu_2}{a_2}\right)\right).
\end{aligned}\tag{4.17}$$

(3) $(K_1, K_2, J) = (II, II, I)$.

$$\begin{aligned}
&\csc\pi(a_1z-w_1)\csc\pi(a_2z-w_2)\\
&=-\frac{(-1)^{(a_1+a_2)(A_1w_2+A_2w_1)-(w_1+w_2)}}{a_1a_2}\delta_{\mathbb{Z}}(a_1w_2-a_2w_1)\cot^{(1)}(\pi(z-(A_1w_2+A_2w_1)))\\
&+\frac{1}{a_1}\sum_{\mu_1=0}^{a_1-1}{}'(-1)^{\mu_1}\csc\left(\pi\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)\right)\cot\left(\pi\left(z-\frac{w_1+\mu_1}{a_1}\right)\right)\\
&+\frac{1}{a_2}\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2}\csc\left(\pi\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)\right)\cot\left(\pi\left(z-\frac{w_2+\mu_2}{a_2}\right)\right).
\end{aligned}\tag{4.18}$$

(4) $(K_1, K_2, J) = (II, II, II)$.

$$\csc\pi(a_1z-w_1)\csc\pi(a_2z-w_2)$$

$$= -\frac{(-1)^{(a_1+a_2)(A_1w_2+A_2w_1)-(w_1+w_2)}}{a_1a_2}\delta_{\mathbb{Z}}(a_1w_2-a_2w_1)\csc^{(1)}(\pi(z-(A_1w_2+A_2w_1)))$$
$$+\frac{1}{a_1}\sum_{\mu_1=0}^{a_1-1}{}'(-1)^{\mu_1}\csc\left(\pi\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)\right)\csc\left(\pi\left(z-\frac{w_1+\mu_1}{a_1}\right)\right)$$
$$+\frac{1}{a_2}\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2}\csc\left(\pi\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)\right)\csc\left(\pi\left(z-\frac{w_2+\mu_2}{a_2}\right)\right). \tag{4.19}$$

The formulas (4.16), (4.17), (4.18) and (4.19) are generalizations of (1.3), (1.4), (1.5) and (1.6) respectively. Actually, by putting $w_1 = w_2 = 0$, our results become Fukuhara's formulas.

THEOREM 4.9. (1)

$$a_2\sum_{\mu_1=0}^{a_1-1}{}'(-1)^{\mu_1\delta_{K_1,II}}\varphi_1^{(K_2)}\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)$$
$$+a_1\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2\delta_{K_2,II}}\varphi_1^{(K_1)}\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)=0. \tag{4.20}$$

(2) *For any* $\mu\in\mathbb{Z}_{\geq 0}$ *and* $z_0\in\Re$,

$$a_2\sum_{\mu_1=0}^{a_1-1}{}'(-1)^{\mu_1\delta_{K_1,II}}\varphi_1^{(K_2)}\left(a_2\frac{w_1+\mu_1}{a_1}-w_2\right)A_\mu^{(J)}\left(z_0;1,1,\frac{w_1+\mu_1}{a_1}\right)$$
$$+a_1\sum_{\mu_2=0}^{a_2-1}{}'(-1)^{\mu_2\delta_{K_2,II}}\varphi_1^{(K_1)}\left(a_1\frac{w_2+\mu_2}{a_2}-w_1\right)A_\mu^{(J)}\left(z_0;1,1,\frac{w_2+\mu_2}{a_2}\right)$$
$$=\pi^2a_1a_2\delta_{K_1,I}\delta_{K_2,I}\delta_{\mu,0}$$
$$-\delta_{\mathbb{Z}}(a_1w_2-a_2w_1)\operatorname{sgn}_2^{(K_1,K_2)}((a_1,a_2),(w_1,w_2),(A_1,A_2))A_\mu^{(J)}(z_0;1,2,A_1w_2+A_2w_1)$$
$$+\operatorname{sgn}^{(K_1)}(z_0;a_1,w_1)A_{\mu+1}^{(K_2)}(z_0;a_2,1,w_2)+\operatorname{sgn}^{(K_2)}(z_0;a_2,w_2)A_{\mu+1}^{(K_1)}(z_0;a_1,1,w_1)$$
$$+\sum_{\nu=0}^{\mu}A_\nu^{(K_1)}(z_0;a_1,1,w_1)A_{\mu-\nu}^{(K_2)}(z_0;a_2,1,w_2). \tag{4.21}$$

5. Concluding remarks

We demonstrated the main theorems (Theorem 3.1 and 3.2) include reciprocity laws of various generalized Dedekind sums as special cases. As a future work, we raise a problem for an elliptic analogue of our main results.

Fix a complex number τ with positive imaginary part. We put

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{\substack{\gamma\in\mathbb{Z}+\mathbb{Z}\tau \\ \gamma\neq 0}} \left\{ \frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right\},$$

$$\varphi(z,\tau) := \sqrt{\wp(z,\tau) - \wp\left(\frac{1}{2},\tau\right)} = \frac{1}{z} - \sum_{\nu\geq 0} \alpha_{\nu+1}(\tau) z^{\nu}.$$

Fukuhara and Yui derived the following formula in [6]. If p and q are relatively prime and $p+q$ is odd, then

$$\begin{aligned}\varphi(pz,\tau)\varphi(qz,\tau) = &-\frac{1}{pq}\varphi'(z,\tau) \\ &+ \frac{1}{p} \sum_{\substack{\mu,\lambda=0 \\ (\mu,\lambda)\neq(0,0)}}^{p-1} \varphi\left(\frac{q(\mu+\lambda\tau)}{p},\tau\right)\varphi\left(z-\frac{\mu+\lambda\tau}{p},\tau\right) \\ &+ \frac{1}{q} \sum_{\substack{\mu,\lambda=0 \\ (\mu,\lambda)\neq(0,0)}}^{q-1} \varphi\left(\frac{p(\mu+\lambda\tau)}{q},\tau\right)\varphi\left(z-\frac{\mu+\lambda\tau}{q},\tau\right). \qquad (5.1)\end{aligned}$$

This formula can be regarded as an elliptic analogue of (1.2).

Further, Egami [3] provided the following reciprocity law which is an elliptic analogue of Zagier's reciprocity laws (4.12). If $a_1,\dots,a_r \in \mathbb{Z}_{\geq 0}$ are relatively prime and $a_1+\cdots+a_r$ is even, then

$$\sum_{l=1}^{r} \sum_{\substack{\mu_l,\lambda_l=0 \\ (\mu_l,\lambda_l)\neq(0,0)}}^{a_l-1} (-1)^{\lambda_l} \prod_{1\leq u\neq l\leq r} \left\{ \varphi\left(a_u \frac{\mu_l+\lambda_l\tau}{a_l},\tau\right) a_u \right\} = -M(\tau;\mathbf{a}), \qquad (5.2)$$

where

$$M(\tau;\mathbf{a}) := \sum_{N=1}^{r-1} \sum_{1\leq\lambda_1<\dots<\lambda_N\leq r} \sum_{\substack{r-1-N=\sum_{k=1}^{N}\nu_{\lambda_k}, \\ \nu_{\lambda_1},\dots,\nu_{\lambda_N}\geq 0}} (-1)^N \prod_{u=1}^{N} \{\alpha_{\nu_u+1}(\tau) a_u^{\nu_u+1}\}.$$

In this article, we obtain a generalization of (1.2) and (4.12). Therefore, we naturally propose the following problem.

PROBLEM 5.1. *Give an elliptic analogue of Theorems* 3.1 *and* 3.2.

ACKNOWLEDGEMENTS. The author thanks the anonymous referee for their helpful advices. The author also thanks Professors K. Matsuno and M. Kurihara for their helpful comments.

References

[1] M. BECK, Dedekind cotangent sums, Acta Arith. **109**-2 (2003), 109–130.

[2] U. DIETER, Cotangent sums, a further generalization of Dedekind sums, J. Number Theory **18**-3 (1984), 289–305.

[3] S. EGAMI, An elliptic analogue of the multiple Dedekind sums, Compositio Math. **99**-1 (1995), 99–103.

[4] S. FUKUHARA, Dedekind symbols associated with J-forms and their reciprocity law, J. Number Theory **98**-2 (2003), 236–253.

[5] S. FUKUHARA, New trigonometric identities and generalized Dedekind sums, Tokyo J. Math. **26**-1 (2003), 1–14.

[6] S. FUKUHARA and N. YUI, Elliptic Apostol sums and their reciprocity laws, Trans. Amer. Math. Soc. **356**-10 (2004), 4237–4254.

[7] P. L. WALKER, *Elliptic functions*, Wiley (1996).

Present Address:
DEPARTMENT OF PURE AND APPLIED MATHEMATICS,
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY,
OSAKA UNIVERSITY,
1–5 YAMADAOKA, SUITA, OSAKA 565–0871, JAPAN.
e-mail: g-shibukawa@math.sci.osaka-u.ac.jp